Random processes and its application in real life

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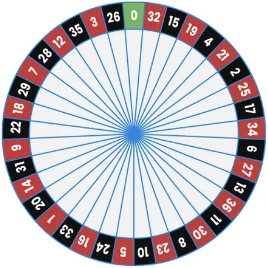
Problem Statement

Context: You are at a roulette table with **1000 units** of money.

Objective: Make more money !!!

Conditions:

- Bet on 37 numbers (1-36 and 0).
- Payouts vary by bet type:
  - Single number bet: 35 to 1
  - Red/Black, Even/Odd: 1 to 1
  - Other structured bets available (e.g., splits, corners).

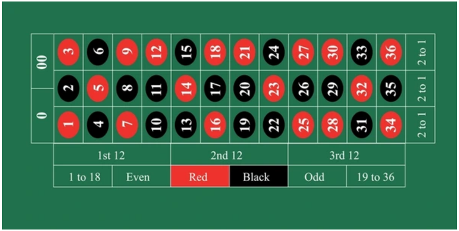


Example of an roulette wheel

Problem:
What is the optimal way to use the 1000 units?

Types of bets

Bet Type	Payout	Probability	Description
Straight Up	35:1	$\frac{1}{37} \approx 2.70\%$	Single number (e.g., 7)
Street	11:1	$\frac{3}{37} \approx 8.11\%$	Three numbers in a row (e.g., 1–2–3)
Corner (Square)	8:1	$\frac{4}{37} \approx 10.81\%$	Four numbers forming a square (e.g., 1, 2, 4, 5)
Six Line	6:1	$\frac{6}{37} \approx 16.22\%$	Two adjacent rows of three numbers (e.g., 1–2–3 and 4–5–6)
Column / Dozen	2:1	$\frac{12}{37} \approx 32.43\%$	Group of 12 numbers
Even/Odd, Red/Black	1:1	$\frac{18}{37} \approx 48.65\%$	18-number bets: even vs. odd, red vs. black, etc.



Change in Wealth After a Bet

Let

- X_t — Wealth at time t
- U_t — Bet amount at time t
- b — Payout ratio (e.g., 35 for 35:1)
- I_t — Indicator variable: 1 if the bet wins, 0 if it loses

Then

$$\begin{aligned} X_{t+1} &= X_t + (U_t(b-1))I_t - U_t(1-I_t) \\ &= X_t + U_t b I_t - U_t \end{aligned}$$

Interpretation:

- If you win ($I_t = 1$): $\Delta X_t = U_t \cdot (b-1)$
- If you lose ($I = 0$): $\Delta X_t = -U_t$

Sequence vs. Random Process

$$X_{t+1} = X_t + (U_t(b-1))I_t - U_t(1-I_t)$$

Deterministic Sequence

- A **sequence** is an ordered list of values, usually generated by a fixed rule.
- Examples: arithmetic sequence ($x_n = a + nd$), Fibonacci sequence, geometric sequence ($x_n = ar^n$).
- Entirely predictable.
- **Problem:** This assumes we know the outcome of each bet ahead of time — unrealistic in most real-world settings.

Sequence vs. Random Process

$$X_{t+1} = X_t + (U_t(b-1))I_t - U_t(1-I_t)$$

Random Process

- A **random process** is a sequence of random variables indexed by time. [1]
- Each value is **uncertain** and governed by a **probability distribution**.
- Examples: temperature over time, noise in a signal, stock prices.

Possible betting strategies

We have the **wealth update** equation

$$\begin{aligned} X_{t+1} &= X_t + (U_t(b-1))I_t - U_t(1-I_t) \\ &= X_t + bU_tI_t - U_t \end{aligned}$$

What are all the possible ways of making bet ? We know

$$0 \leq U_t \leq X_t$$

- $U_t = X_t$: All in
- $U_t = constant$: constant bet
- $U_t = g_t(t, X_t)$: As a function of time
- $U_t = g(X_t)$

What is Kelly Betting?

- PROPORTIONAL BETTING

$$U_t = g(X_t) = fX_t, \quad 0 < f < 1$$

- Maximize long-term capital growth.

Derivation of the Kelly Betting Strategy

Setup:

- Initial wealth: X_t
- Fraction of wealth bet: f
- Odds ratio: b
- Indicator of winning: $I_t = \begin{cases} 1 & \text{with probability } p \text{ (win)} \\ 0 & \text{with probability } 1 - p \text{ (loss)} \end{cases}$
- I_t - **Bernoulli** I.I.D random variable.

Wealth update equation:

$$X_{t+1} = X_t + fX_t(b-1)I_t - fX_t(1-I_t)$$

$$\frac{X_{t+1}}{X_t} = 1 + f(b-1)I_t - f(1-I_t)$$

$$\frac{X_n}{X_0} = \prod_{t=0}^{n-1}(1 + f(b-1)I_t - f(1-I_t))$$

Strong Law of Large Numbers (SLLN)

Theorem: Let $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. (independent and identically distributed) random variables with expected value $\mathbb{E}[X_i] = \mu$ and $\mathbb{E}[|X_i|] < \infty$. Then,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\text{a.s.}} \mu \quad \text{as } n \to \infty$$

The *sample average* converges **almost surely** (with probability 1) to the *expected value* μ.

Derivation of the Kelly Betting Strategy (Single bet)

Taking log:

$$\log(\frac{X_n}{X_0}) = \sum_{t=0}^{n-1} \log(1 + f(b-1)I_t - f(1-I_t))$$

The wealth at time n is of the form

$$X_n = X_0 e^{nS_n}$$

Where,

$$S_n = \frac{1}{n} \sum_{t=0}^{n-1} \log(1 + f(b-1)I_t - f(1-I_t))$$

Derivation of the Kelly Betting Strategy (Single bet)

Using SNNL, we can infer that

$$S_n \xrightarrow{\text{a.s.}} \mathbb{E}[Y_t] \quad \text{as } n \to \infty$$

as $Y_t = \log(1 + f(b-1)I_t - f(1-I_t))$ is I.I.D.

$$\mathbb{E}[Y_t] = p\log(1 + f(b-1)) + (1-p)\log(1-f) = \mu$$

If $\mu > 0$ then $X_n \to \infty \quad as \quad n \to \infty$ else, $X_n \to 0$.
Let

$$G(f) = p\log(1 + f(b-1)) + (1-p)\log(1-f)$$

$$G(f) \to \text{Expected growth rate}$$

The Kelly Formula (Single Bet)

First-order condition:

$$\frac{p(b-1)}{1+f^*(b-1)} - \frac{1-p}{1-f^*} = 0$$

Solve for f^*:

$$f^* = p - \frac{1-p}{b-1}$$

This is the Kelly Criterion for a single binary bet:

- p: probability of winning
- b: odds (Total payout of b for every 1 unit bet)
- $q = 1 - p$

The Kelly fraction is:

$$f^* = p - \frac{q}{b-1}$$

Bet $f^* \times$ your capital.

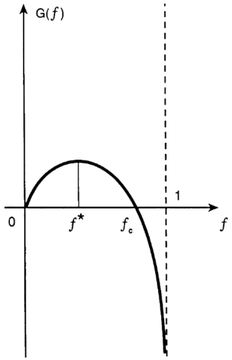


Figure: Illustrative plot of $G(f)$ [2]

Six Line Bet

$$f^* = p - \frac{q}{b-1}$$

Six Line Bet:

- Covers 6 out of 37 numbers $\rightarrow p = \frac{6}{37}$, $q = \frac{31}{37}$
- $b = 6$

$$\begin{aligned} f^* &= \frac{6}{37} - \frac{\frac{31}{37}}{6-1} \\ &= \frac{-1}{185} = -0.0054 \end{aligned}$$

Conclusion: The optimal Kelly bet is negative $\rightarrow$ you should not take this bet.

Kelly for Multiple Simultaneous Bets

Multiple **mutually exclusive** outcomes: only one outcome can occur,

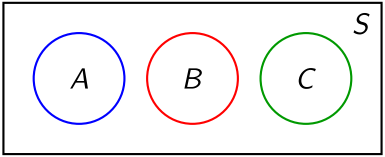
- Derive $G(\boldsymbol{f}) = \mathbb{E}[\log(X_{t+1}/X_t)]$
- Find f^* by maximizing $G(\boldsymbol{f})$, subject to: $\sum_i f_i \leq 1,\ f_i \geq 0$ [3]

Wealth update equation:

$$X_{t+1} = X_t + \sum_{i=1}^{m} f_i X_t (b_i - 1) I_{it} - \sum_{i=1}^{m} f_i X_t (1 - I_{it})$$

$$\frac{X_n}{X_0} = \prod_{t=0}^{n-1} (1 + \sum_{i=1}^{m} f_i (b_i - 1) I_{it} - \sum_{i=1}^{m} f_i (1 - I_{it}))$$

Kelly for Multiple Simultaneous Bets



SNNL:

$$Y_t = \log(1 + \sum_{i=1}^{m} f_i(b_i - 1)I_{it} - \sum_{i=1}^{m} f_i(1 - I_{it}))$$

$$S_n = \frac{1}{n}\sum_{t=0}^{n-1} Y_t = \frac{1}{n}\sum_{t=0}^{n-1} \log(1 + \sum_{i=1}^{m} f_i(b_i - 1)I_{it} - \sum_{i=1}^{m} f_i(1 - I_{it}))$$

$$G(\boldsymbol{f}) = \mathbb{E}[Y_t] = \sum_{i=1}^{m} p_i \log(1 + f_i(b_i - 1) - \sum_{j \neq i} f_j)$$

$$+ (1 - \sum_{i=1}^{m} p_i) \log(1 - \sum_{j=1}^{m} f_j)$$

Kelly for Multiple Simultaneous Bets

Obtaining first order derivative:

$$\frac{\partial G(\boldsymbol{f})}{\partial f_k} = \frac{\partial}{\partial f_k}\left\{\sum_{i=1}^{m} p_i \prod_{j\neq i}(1-p_j)\log(1+f_i(b_i-1)-\sum_{j\neq i} f_j)\right\}$$
$$+\frac{\partial}{\partial f_k}\left\{\prod_{i=1}^{m}(1-p_i)\log(1-\sum_{j=1}^{m} f_j)\right\} = 0$$

We get m linear equations in $f_1, f_2 ..., f_m$.
Also, Hessian must be negative for maxima.

Kelly for Multiple Simultaneous Bets

For m = 3:

$$f_1 = \frac{-b_1 b_2 b_3 p_1 + b_1 b_2 p_1 + b_1 b_3 p_1 - b_2 b_3 p_2 - b_2 b_3 p_3 + b_2 b_3}{-b_1 b_2 b_3 + b_1 b_2 + b_1 b_3 + b_2 b_3}$$

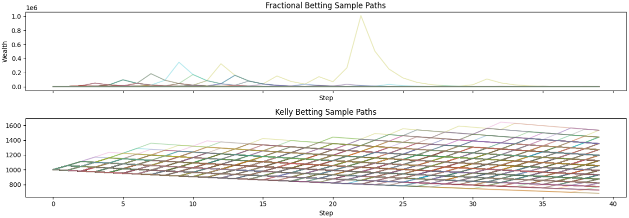
$$f_2 = \frac{-b_1 b_2 b_3 p_2 + b_1 b_2 p_2 - b_1 b_3 p_1 - b_1 b_3 p_3 + b_1 b_3 + b_2 b_3 p_2}{-b_1 b_2 b_3 + b_1 b_2 + b_1 b_3 + b_2 b_3}$$

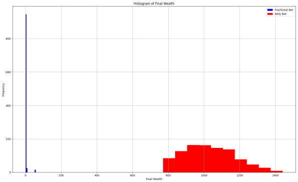
$$f_3 = \frac{-b_1 b_2 b_3 p_3 - b_1 b_2 p_1 - b_1 b_2 p_2 + b_1 b_2 + b_1 b_3 p_3 + b_2 b_3 p_3}{-b_1 b_2 b_3 + b_1 b_2 + b_1 b_3 + b_2 b_3}$$

Simulation Parameters

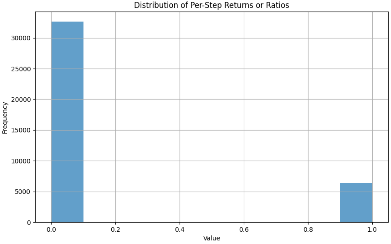
- **Number of Steps**: $n_{\text{steps}} = 40$
- **Number of Samples**: $n_{\text{samples}} = 1000$
- **Initial Wealth**: $x_0 = 1000$
- **Win Probability**: $p = 0.162$
- **Payoff (Net Odds)**: $b = 6.5$
- **Wager Fraction (Kelly)**: $\text{frac} = 0.00982$
- **Wager Fraction** : $\text{frac} = 0.5$
- **Expected growth rate at Kelly fraction** : 0.00026
- **Expected growth rate at f = 0.5** : -0.366

Kelly bet - single





Outcomes between the 1st and 99th percentiles



Kelly Bet: Multiple Simultaneous Bets

Double Street Bet (Six Line Bet):

- Covers 6 consecutive numbers on the roulette table.
- Probability of winning: $\frac{6}{37}$ (European roulette).
- Payout: 6 to 1 (total payout).

Multiple Bets:

- Three non-overlapping double street bets cover 18 unique numbers.
- Probability of winning at least one bet:

$$\frac{18}{37} \approx 0.4865$$

- Overall risk is diversified.

Kelly Bet: Multiple Simultaneous Bets

Bet Parameters:

- $p_1 = 0.162$ (Probability of winning bet 1)
- $b_1 = 6$ (Total payout for bet 1)
- $p_2 = 0.162$ (Probability of winning bet 2)
- $b_2 = 6$ (Total payout for bet 2)
- $p_3 = 0.162$ (Probability of winning bet 3)
- $b_3 = 6$ (Total payout for bet 3)

Subtituting it in the Kelly bet fraction formulae we found [4]

- $f_1 \approx 0 \quad \& \quad < 0$
- $f_2 \approx 0 \quad \& \quad < 0$
- $f_3 \approx 0 \quad \& \quad < 0$

Pros and Cons of Kelly Strategy

Pros

- Maximizes long-term capital growth
- Avoids bankruptcy in the long run
- Disciplines risk-taking

Cons

- Requires accurate estimation of probabilities
- High short-term volatility - mitigated by **fractional kelly** : $f^*/\alpha, \quad \alpha > 1$

From Roulette to Investment Strategy

Application to Investment

- **Systematic Investment Plan (SIP):** regular fraction of income is invested consistently to build wealth over time.
- **Diversification:** Spreading investments across different assets (e.g., equity, gold, bonds) helps reduce overall risk.
- **Expected Value:** successful investing focuses on identifying opportunities with a positive expected growth rate over time.

Smart investing, like smart betting, is not about luck—it's about disciplined, probability-informed decisions.

Note: This discussion is for educational purposes only and does not constitute financial advice.

Reference I

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