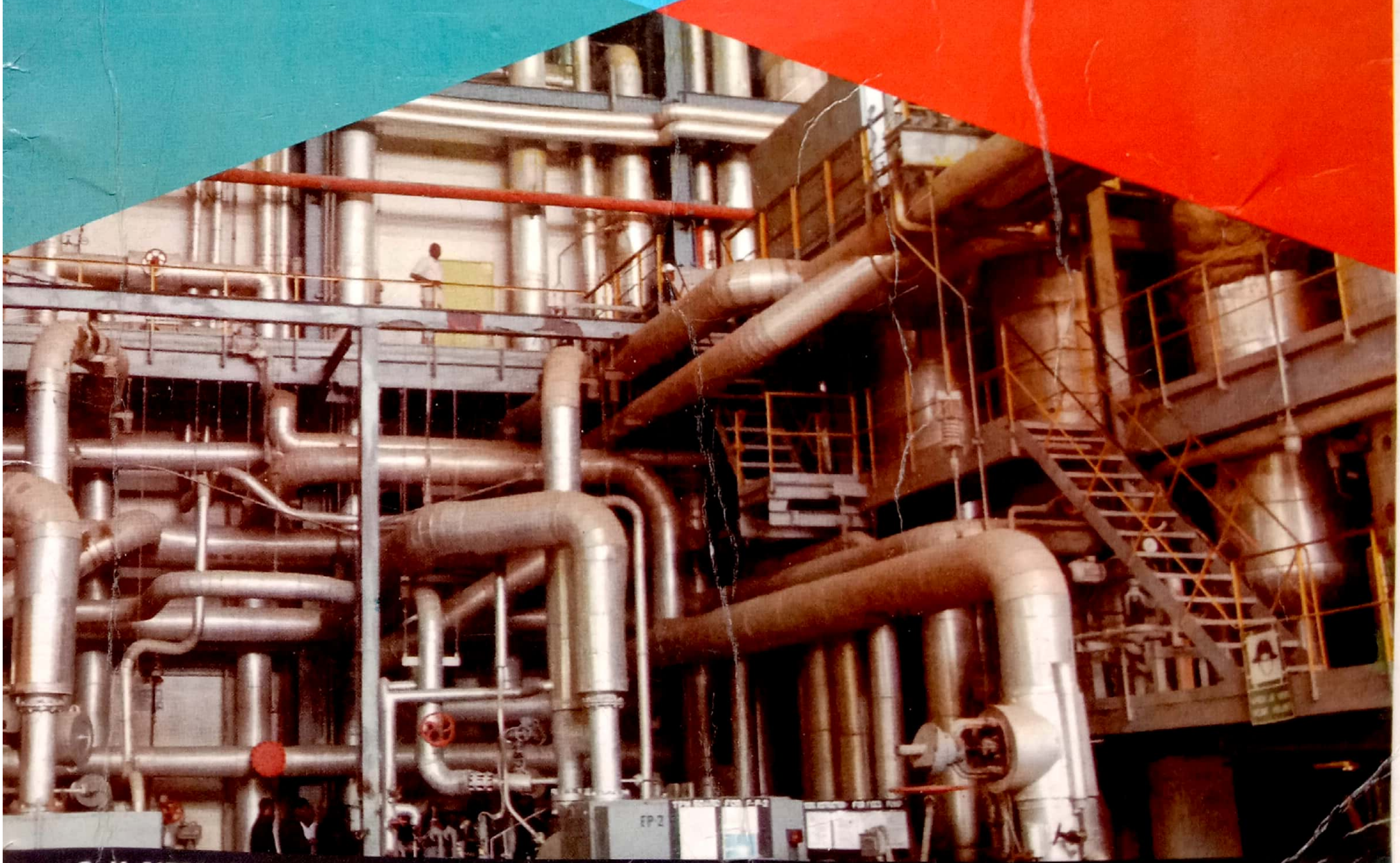
ADVANCED ENGINEERING ANALYSIS



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Advanced Engineering Analysis

(First Edition)

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Preface to the First Edition

We are very happy to have come up with yet an important edition known as "***Advanced Engineering Analysis.***" This book presents a course of mathematics intended for undergraduates of Engineering and physical sciences. The material has been designed to enable the student apply the techniques acquired for the analysis, modelling and solution of realistic Engineering problems often encountered at undergraduate level.

The book satisfies the requirements of the NUC and NBTE approved course on Engineering Analysis for Nigerian Universities and Polytechniques. The book is an outgrowth of many lectures on the course Engineering Mathematics II given by the authors at various Engineering disciplines in the institutions of higher learning, and is essentially developed to serve as a semester course in Engineering Mathematics II [ENG 202] for the Engineering students of Federal University Oye-Ekiti.

Due to the universality and objectivity of Engineering Analysis, the book is also suitable for use both as learning and instructional material on Engineering Mathematics in African, European, Asian, American, etc. Universities and Polytechniques.

The importance of detailed knowledge of mathematical tools for Engineering Analysis cannot be over-emphasised, as these would equip the Engineer and Physicist with the requisite skills for analysing and solving engineering problems.

A course on First-Order Differential Equations, Algebra, Applications of Sequences & Series as well as Differential and Integral Calculus is a prerequisite for this course. Hence, we make bold to refer the reader to a book titled **Fundamentals of Engineering Analysis II** written by G.K. IJEMARU & O. AKINSANMI to serve the purpose. The reader may wish to grab that copy in order to enhance his/her understanding of this text.

One attractive aspect of the book is the establishment of differential equations for given physical and engineering situations. The description of a physical or engineering situation in terms of mathematical expressions is called modelling, and is of great practical significance to the Engineer and Physicist. Due to the great importance of modelling, illustrations are made from all branches of engineering of the development of mathematical models.

The book – **Advanced Engineering Analysis** consists of ten chapters.

Chapter 1 deals with Complex Analysis, while **Chapter 2** looks at Partial Differentiation. **Chapters 3 & 4** explain Power series expansion of functions and Second-Order Differential Equations respectively. These open discussions on Differential Operator as well as Series Solution of O.D.E which make up **Chapters 5 & 6** respectively. **Chapter 7** explains the application of second-order differential equations, where an attempt is made to present modelling of engineering systems leading to second-order differential equations. **Chapter 8** looks into Matrix Algebra, while **Chapters 9 & 10** are devoted to the analyses of Fourier and Laplace transforms respectively.

While carefully presenting the topics in the book, much emphasis is placed on the development of mathematical tools and techniques for Engineering Analysis and Applications. In doing this, we have made extensive use of the fundamental theories and principles in order to enable clearer understanding.

A very interesting aspect of the book is the presentation of myriads of solutions and exercises in order to afford the reader the much fundamental rudiments and analytical tools for a resounding engineering analysis.

We hope that the book will fulfil the need of the targeted audience and wish our esteemed readers happy reading as they make their sojourn in the book. We would also welcome any suggestions towards the improvement of the book.

Ijemaru Gerald et al.
May, 2017.

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$$u(x,y) = x^3 - 3xy^2 = \text{Re}z^3$$
$$v(x,y) = 3x^2y - y^3 = \text{Im}z^3$$

Hence, $f(z) = z^3 = u(x,y) + jv(x,y)$

$$= (x^3 - 3xy^2 + j(3x^2y - y^3)$$

Example 6: Split the following expressions into their real and imaginary parts making use of the rectangular coordinates:

(i) $z.\bar{z}$ (ii) $\frac{z-j}{1-\bar{z}j}$, $z \neq j$

Solutions

(i) $z.\bar{z}$

In rectangular coordinate representation, $z = x + jy$, and the conjugate $\bar{z} = x - jy$

$\therefore z.\bar{z} = (x + jy)(x - jy)$

$= x^2 - jxy + jxy - j^2y^2$

$= x^2 + y^2$

hence, $\mathbf{Re}(z.\bar{z}) = x^2 + y^2$ and $\mathbf{Im}(z.\bar{z}) = 0$

(ii) $\frac{z-j}{1-\bar{z}j}$

substituting for z and $\bar{z}$,

$$\frac{z-j}{1-\bar{z}j} = \frac{x+jy-j}{1-(x-jy)j}$$
$$= \frac{x+j(y-1)}{1-jx-y}$$
$$= \frac{x+j(y-1)}{(1-y)-jx}$$

Note that when dividing complex numbers, both the numerator and the denominator of the complex fraction are multiplied by the complex conjugate of the denominator. Thus,

$$\frac{x+j(y-1)}{(1-y)-jx} \times \frac{(1-y)+jx}{(1-y)+jx}$$
$$= \frac{x-xy+jx^2+j(y-y^2-1+y)+j^2x(y-1)}{(1-y)^2-(jx)^2}$$
$$= \frac{x-xy+jx^2+j(2y-y^2-1)-x(y-1)}{(1-y)^2-(jx)^2}$$
$$= \frac{x-xy+jx^2+j(2y-y^2-1)-xy+x}{(1-y)^2-(jx)^2}$$
$$= \frac{2x-2xy+j(x^2+2y-y^2-1)}{(1-y)^2-(jx)^2}$$
$$= \frac{2x(1-y)+j(x^2+2y-y^2-1)}{x^2+(1-y)^2}$$

$$\Rightarrow \mathbf{Re}\left(\frac{z-j}{1-\bar{z}j}\right) = \frac{2x(1-y)}{x^2+(1-y)^2} \text{ while } \mathbf{Im}\left(\frac{z-j}{1-\bar{z}j}\right) = \frac{x^2+2y-y^2-1}{x^2+(1-y)^2}$$

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1.3 MAPPING &TRANSFORMATION

The transformation of A in the z-plane onto A′ in the w-plane is said to be a mapping of A onto A′ under the transformation equation $w = f(z)$ *and* A′ is referred to as the image of A.

Note – Transformation is the relationship between z and w

$\Rightarrow w = f(z)$ where, $z = x + jy$ and $w = u + jv$

Example 7: Determine the image of the point A, $z = 5 + j3$ on the w-plane under the transformation

$w = 4z + 3 - 2j$

Solution:

$w = u + jv = f(z) = 4z + 3 - j2$

But $z = x + jy \Rightarrow w = 4(x + jy) + 3 - j2$

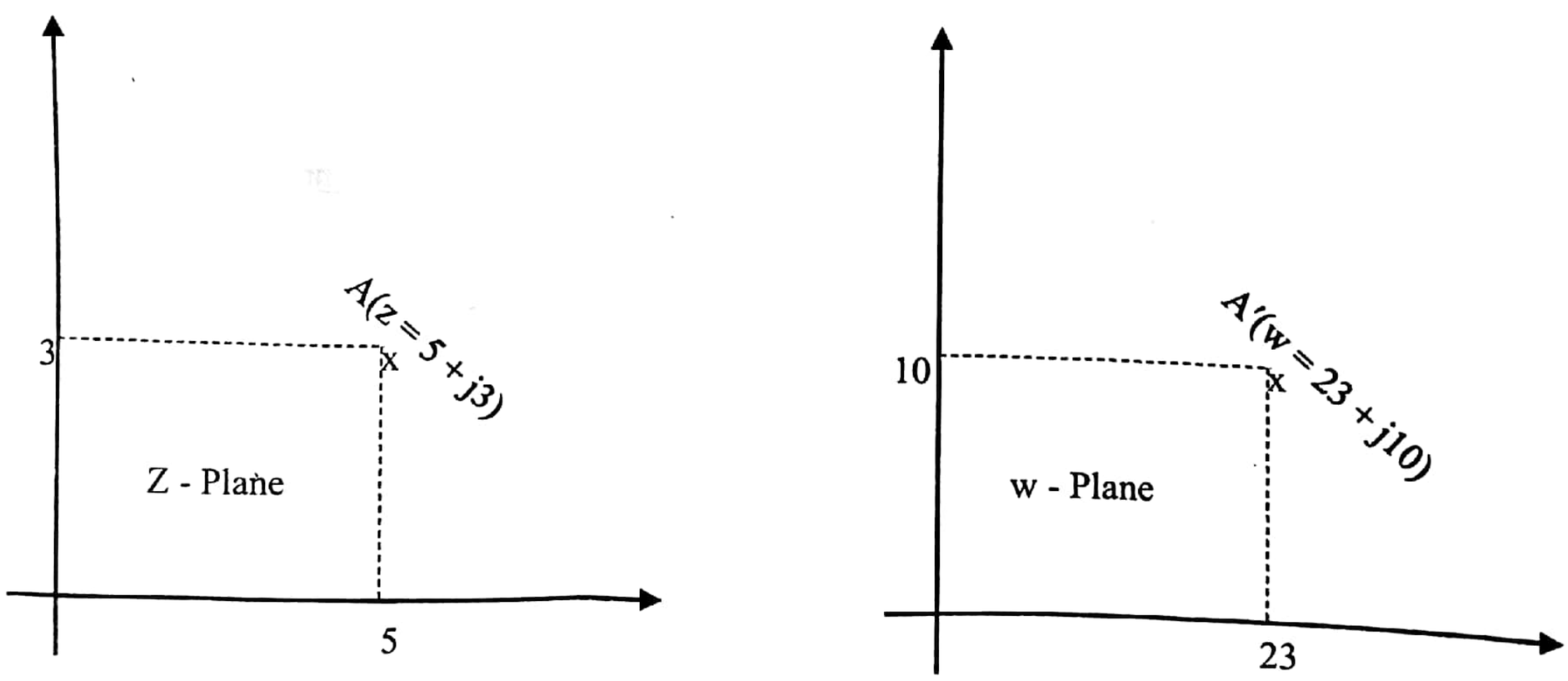
$\therefore w = 4x + 3 + j(4y - 2)$

Then the point $A(z = 5 + j3)$ transforms onto $w = 4x + 3 + j(4y - 2)$

$x = 5, y = 3 \therefore w = 4x5 + 3 + j(4x3 - 2) = 23 + j10$

Hence, the image of A is $A'(23 + j10)$

We can illustrate the transformation in the diagram, as shown:



Example 8: Map the point $Z = 4 - j2$ onto the w-plane under the transformation $w = j3Z + j2$ and illustrate the transformation on a diagram.

Solution:

$w = u + jv = f(z) = j3Z + j2.$ But $z = x + jy$

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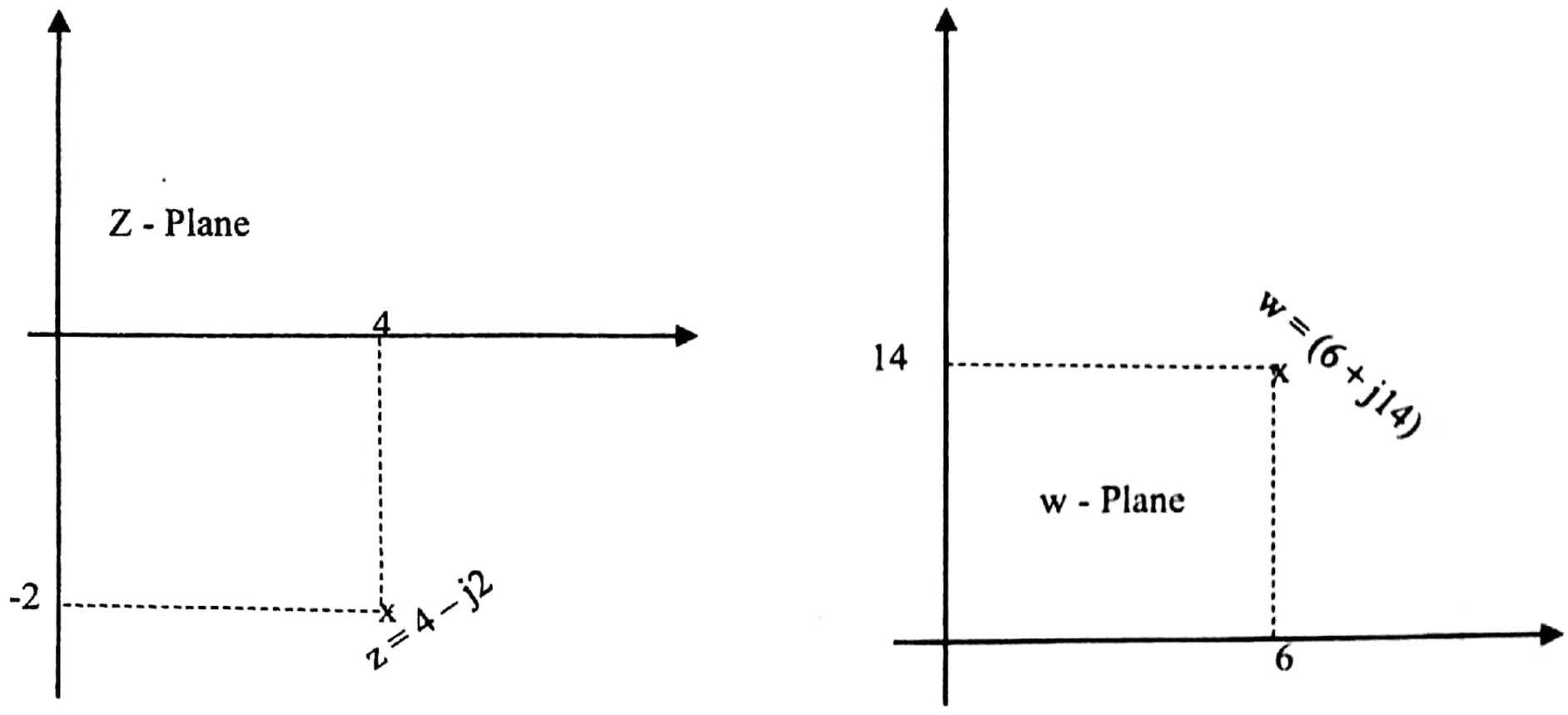
$\therefore w = j3(x + jy) + j2 = j3x - 3y + j2 = -3y + j(3x + 2)$

for $z = 4 - j2$ where $x = 4, y = -2$, then, $w = -3y + j(3x + 2) = -3x - 2 + j(3x4 + 2) = 6 + j14$



Example 9: A triangle in the Z-plane has vertices at p(z = 0), Q(z = 6) and R(z = 6+j4). Determine the image of this triangle in the w-plane under the transformation equation: w = (2+j) Z

Solution:

$w = u + jv = f(z) = (2 + j)Z$. but $z = x + jy$

$\therefore w = (2 + j)(x + jy) = 2x - y + j(2y + x)$

For the image P′, $z = 0 + j0 \Rightarrow x = 0, y = 0$

$\therefore w = 2(0) - 0 + j(2x6) + 0) = 0$

For the image Q′, $z = 6 + j0 \Rightarrow x = 6, y = 0$

$\therefore w = 2(6) - 0 + j(2x6 + 0 = 12 + j6$

For the image R′, $z = 6 + j4 \Rightarrow x = 6, y = 4$

$\therefore w = 2(6) - 4 + j(2x4 + 6) = 8 + j14$

Hence, the image of the triangle is

$P'(w = 0), Q'(w = 12 + j6)$ and $R'(w = 8 + j14)$

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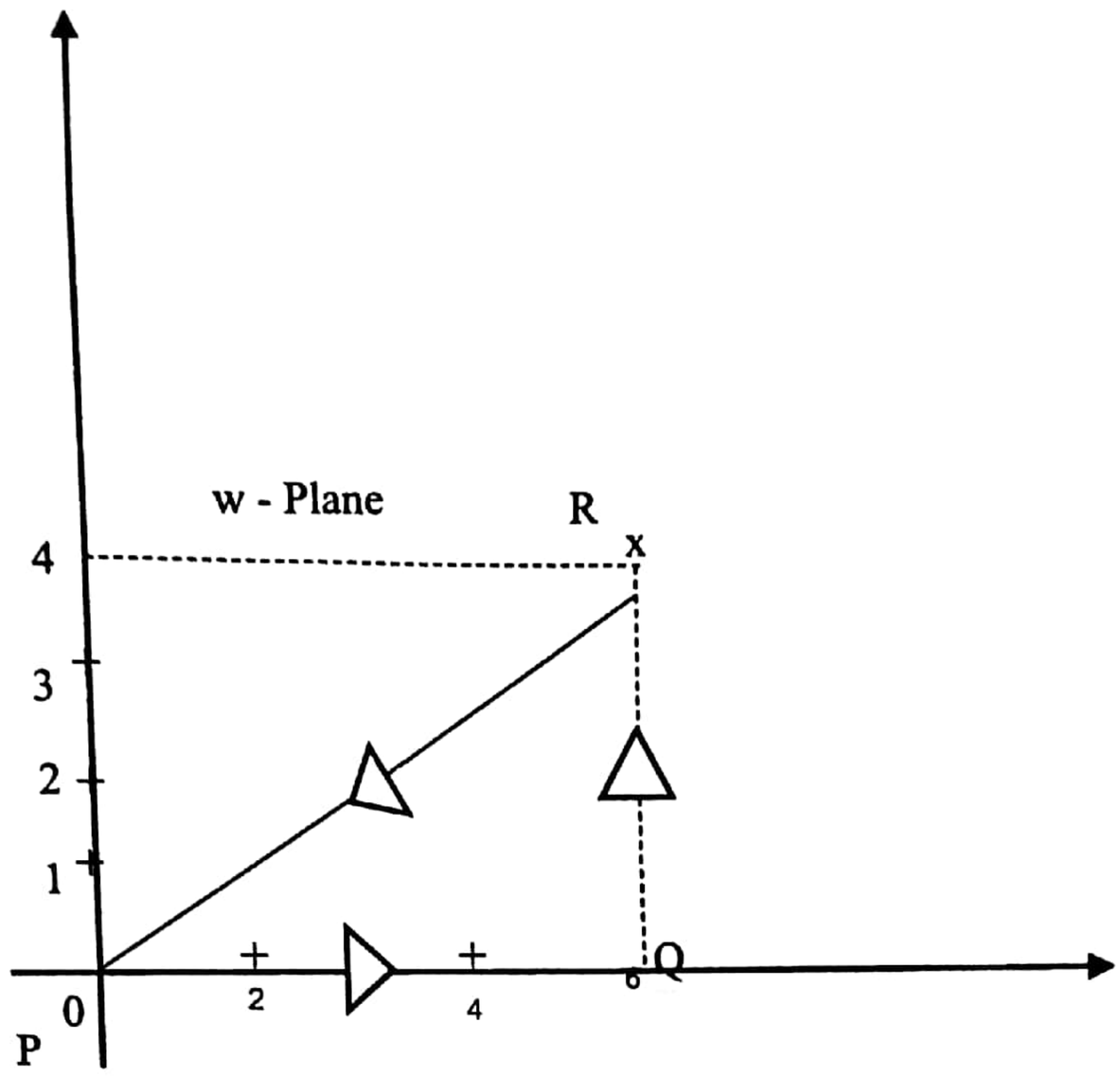

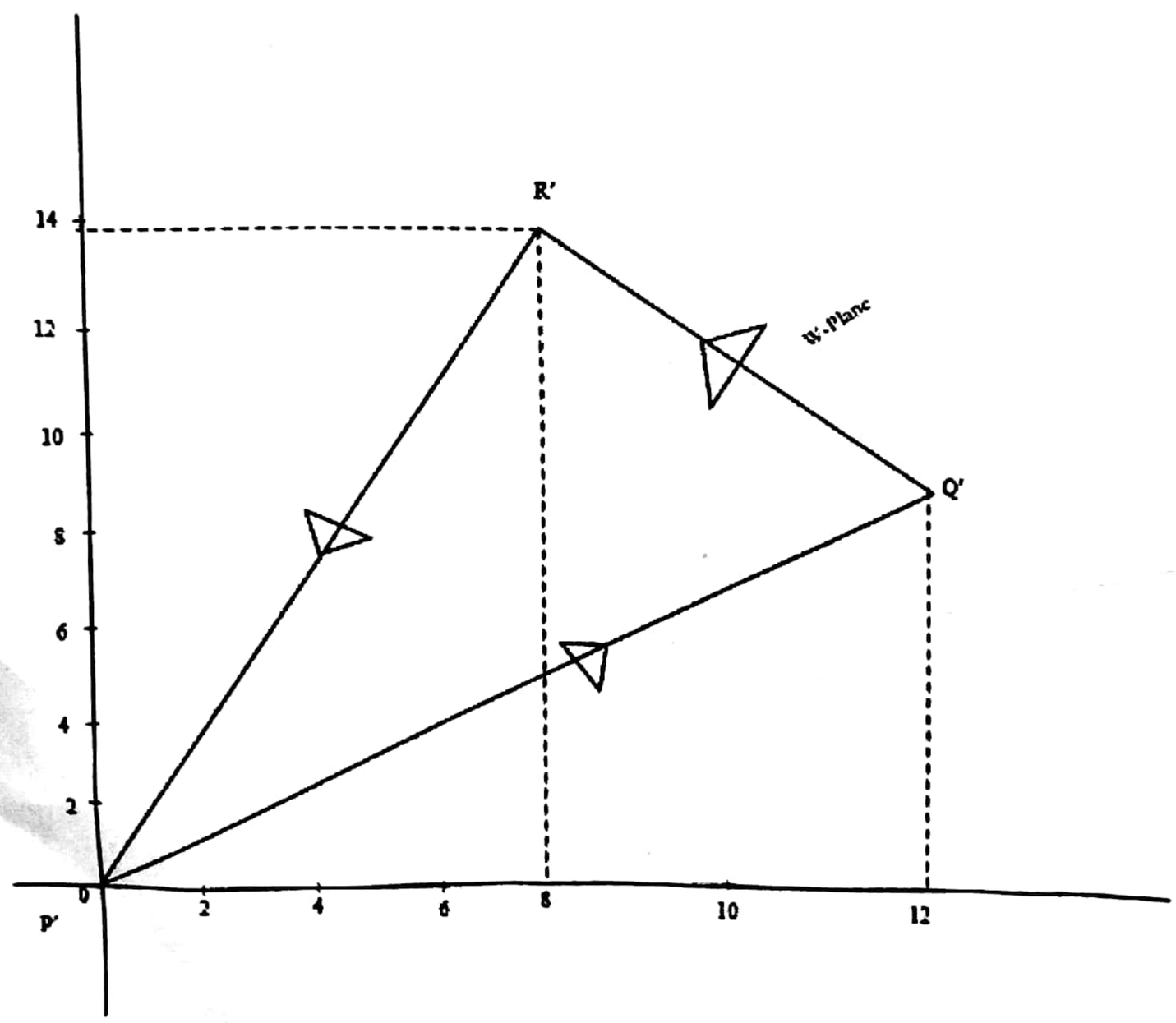

Assignment: A triangle ABC in the Z-plane with vertices A(-1 - j), B(2 +j2), C(-1 +j2) is mapped under the transformation $w = (1 - j)Z + (1 + j2)$. Determine the image A'B'C' in the w-plane and illustrate the mapping onto the w-plane in a diagram.

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1.4 MAGNIFICATION, ROTATION & TRANSLATION

In a linear transformation, $w = az + b$, where a and b are each real or complex, a straight line in the z-plane maps onto a corresponding straight line in the w-plane.

In general, if $w = az + b$ represents a transformation equation, we can calculate the magnification, rotation and translation. Hence,

Magnification (or magnitude) = |a|

Rotation (Rad) = arg a

Translation = b

By extension, if $w = (a + jb)z + c - jd$

Mag = $|a + jb| = \sqrt{a^2 + b^2}$

Rotation = $\arg(a + jb) = arc\tan\left(\frac{b}{a}\right)$

Translation = $c - jd$

Example 10: Given the transformation equation $w = (2 - j3)z - 4 + j5$, map the straight line joining $p(1 + j2)$ and $Q(4 + j)$ in the z-plane onto the w-plane and state the resulting magnification, rotation and translation involved.

Solution:

$w = (2 - j3)Z - 4 + j5$ But $Z = x + jy$

$\Rightarrow w = (2 - j3)(x + jy) - 4 + j5 = 2x + j2y - j3x + 3y - 4 + j5$

$\therefore w = 2x + 3y - 4 + j(2y - 3x + 5)$

For $P': z = 1 + j2 \Rightarrow x = 1, y = 2.$

$\Rightarrow w = 2 + 6 - 4 + j(4 - 3 + 5) = 4 + j6$

For $Q': z = 1 + j \Rightarrow x = 4, y = 1$, so $w = 8 + 3 - 4 + j(2 - 4 + 15) = 7 - j15$

Hence, $P'(w = 4 + j6), Q'(w = 7 - j5)$

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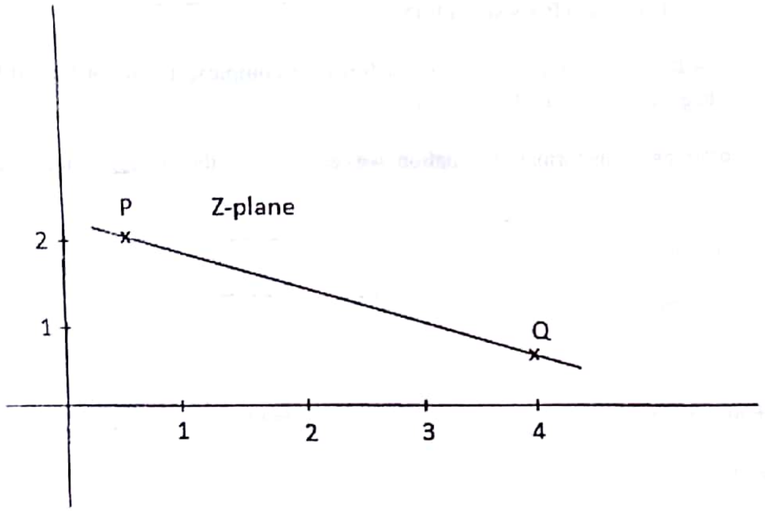
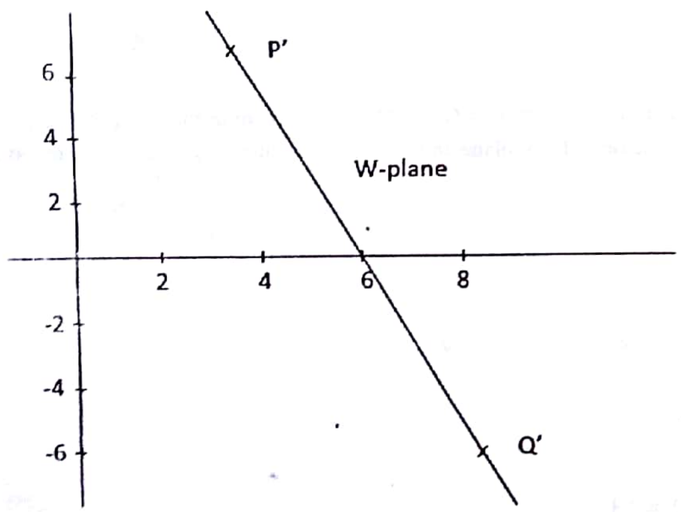


(a)



(b) Magnification = $|2 - j3| = \sqrt{2^2 + 3^2} = \sqrt{13} = 3.606$

(c) Rotation = $\arg(2 - j3) = tan^{-1}\left(\frac{-3}{2}\right) = -56° = 0.9828rad$

(d) Translation = $-4 + j5$ that is 4 units to the left, 5 units upwards

Example 11: Given the transformation equation: $w = (1 - j2)Z + 2 - j3$, calculate the magnification, Rotation and Translation involved.

Solution:

$w = (1 - j2)Z + 2 - j3$

(a) Magnification = $|1 - j2| = \sqrt{1^2 + 2^2}\ \sqrt{5} = 2.236$

(b) Rotation = $\arg(1 - j2) = tan^{-1}\left(\frac{-2}{1}\right) = -36° = 1.107rads$

(c) Translation = $2 - j3$ $i.e.$ 2 units to the right and 3 units downwards

Assignment: The straight line joining $A(-2 + j3)$ & $B(1 + j2)$ in the z-plane is mapped onto the w-plane by the transformation equation: $w = (-3 + j)Z + 2 + j4$. Determine;

(a) The image of A and B

(b) The magnification, rotation & translation involved.

The above is an assignment for the student. When you are done, confirm your answer with the one in the box.

Answer: (a) Straight line joining $A'(-2 + j3)$ to $B'(1 + j2)$;
(b) Mag = 3.162, Rotation = 161°, Translation = 2 to right, 4 upwards.

1.5 DIFFERENTIATION OF COMPLEX NUMBERS

To understand the differentiation of complex numbers, the concept of limits and continuity have to be known.

A function $f(x)$ is said to be continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

On the other hand, a function is discontinuous at $x = a$ if $\lim_{x \to a} f(x) \neq f(a)$

Example 12: Determine if the function $f(x) = \frac{x^3 - 27}{x - 3}$ $for\ x = 3\ is\ continuos$

Solution:

$$f(x) = \frac{x^3 - 27}{x - 3}$$

$$\lim_{x \to 3} \frac{(x - 3)(x^2 + 3x + 9)}{x - 3} = \lim_{x \to 3}(x^2 + 3x + 9) = 45$$

Hence, $f(x)$ is continuous at $x = 3$

Example 13: Determine if the function $f(x) = \frac{\sqrt{x} - 2}{x - 4}$ for $x = 4$ is continuous.

Solution:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{x - 4} = \frac{x^{\frac{1}{2}} - 2}{x - 4} = \frac{\left(x^{\frac{1}{2}} - 2\right)}{\left(x^{\frac{1}{2}}\right)^2 - 2^2}$$

$$= \frac{\left(x^{\frac{1}{2}} - 2\right)}{\left(x^{\frac{1}{2}} - 2\right)\left(x^{\frac{1}{2}} + 2\right)} = \frac{1}{\left(x^{\frac{1}{2}} + 2\right)} \text{ thus, } f(x) \to \frac{1}{(4)^{\frac{1}{2}} + 2} = \frac{1}{2 + 2} = \frac{1}{4} \text{ as } x \to 4$$

Hence, it is continuous at x = 4

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1.6 DIFFERENTIABILITY OF A COMPLEX FUNCTION

A complex function $f(z)$ is differentiable at a point z_0 if and only if the limiting value exists. Hence, we say that a complex function $f(z)$ is differentiable at z_0 if;

$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

OR

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \quad where \quad \Delta z = z - z_0$$

That is, if $w = f(z), then \frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$

1.7 REGULAR & ANALYTIC FUNCTION

A function $f(z)$ is said to be regular or analytic at a point z_0 if it is infinitely differentiable at every point at and around z_0

Example 14: Determine whether the function $f(x) = z^2$ is analytic and differentiable.

Solution:

For the function $f(z)$ to be analytic, it must be differentiable and for the function to be differentiable, its limit must exist. Hence, we begin by finding the limit of the function.

$f(z) = z^2, f(z_0) = z_0^2$

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z \to z_0} \frac{z^2-z_0^2}{z-z_0} = \lim_{z \to z_0} \frac{(z-z_0)(z+z_0)}{z-z_0} = \lim_{z \to z_0} z + z_0 = z_0 + z_0 = 2z_0.$$

Since the limit exists, the function $f(z)$ is analytic and differentiable.

Example 15: Given that $f(z) = \frac{z-1}{2z+1}, z \neq 1, find\ f'(z)$

Solution:

Recall quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$Let\ u = z - 1 \Rightarrow \frac{du}{dz} = 1,$

$while\ v = 2z + 1 \Rightarrow \frac{dv}{dz} = 2$

$\therefore f'(z) = \frac{(2z+1)-(z-1)2}{(2z+1)^2}$

$= \frac{2z+1-2z+2}{(2z+1)^2} = \frac{3}{(2z+1)^2}$

1.8 CAUCHY-RIEMANN EQUATIONS

If $f(z) = u + jv$, we need to establish condition for $f(z)$ to be differentiable at z_0. These conditions are stated as Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ and \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

A complex function is differentiable if it satisfies Cauchy-Riemann equations. Hence, a necessary condition for a function $f(z) = u + jv$ to be regular or analytic at z_0 is that it must satisfy Cauchy-Riemann equations

(i.e. u, v and their partial derivatives are continuous at z_0)

Example 16: Show that $f'(x)$ does not exist at any point if $f(z) = 4x + jy^2$

Solution:

$f(z) = u(x,y) + jv(x,y)$. But $f(z) = 4x + jy^2$

$\therefore u(x,y) = 4x \ and \ v(x,y) = y^2.$

Using Cauchy-Riemann equation:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ and \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial x} = 4, \frac{\partial v}{\partial y} = 2y; \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

Similarly,

$\frac{\partial v}{\partial x} = 0, \frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial x} \neq -\frac{\partial u}{\partial y}$

Hence, $f'(x)$ does not exist.

Example 17: If $f(z) = z^3 = (x^3 - 3xy^2) + j(3x^2y - y^3)$, show whether the function satisfies Cauchy-Riemann equations. What is the implication?

Solution:

$\text{Re}\,(z^3) = u(x,y) = (x^3 - 3xy^2)$

while $\text{Im}\,(z^3) = v(x,y) = (3x^2y - y^3)$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 - 3xy^2) = 3x^2 - 3y^2$

$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(3x^2y - y^3) = 3x^2 - 3y^2 \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Also,

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3xy^2) = -6xy$

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$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(3x^2y - y^3) = 6xy \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence, the function satisfies Cauchy-Riemann equations and so the function is differentiable.

Example 18: Determine whether the function $f(z) = z^2$ is analytic using Cauchy-Riemann equations.

Solution:

$f(z) = z^2$ But $z = x + jy$

$F(z) = (x + jy)(x + jy) = (x^2 - y^2) + j2xy$

$u = x^2 - y^2, v = 2xy$

$\frac{\partial u}{\partial x} = 2x, \frac{\partial v}{\partial y} = 2x, \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Also,

$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial u}{\partial y} = -2y, \Rightarrow \Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Hence the function satisfies Cauchy-Riemann equations and is therefore analytic.

1.9 COMPLEX INTEGRATION

If $z = x + jy$ and $f(z) = u(x, y) + jv(x, y)$ where u and v are functions of x & y also,

$dz = dx + jdy$ and $dw = du + jdv$

Hence, $\int wdz = \int f(z)dz = \int(u + jv)(dx + jdy) = \int[(udx - v\,dy) + j(vdx + udy)$

$$\therefore \int \mathbf{f(z)dz = I = \int (udx - vdy) + j\int (vdx + udy)}$$

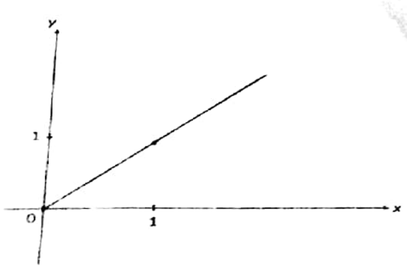
Contour Integration

The evaluation of line integral in the complex plane is known as contour integration. In general, the value of a line integral depends on the path of integration between the end points.

Example 19: Compute the integration of the function $f(z) = \bar{z}^2 = (x^2 - y^2) - j(2xy)$ over a straight line and along a parabola $y = x^2$, from the point $A(z = 0)$ to the point $B(z = 1 + j)$

Solution: Case1: Considering a straight line $y = x$ from $z = 0$ to $z = 1 + j$,

$z = 0, \quad z = 1 + j \begin{cases} x = 1 \\ y = 1 \end{cases}$



Equation of a straight line $y = mx + c, y = 1, x = 1$ *and* $c = 0$

Hence, $1 = m, \therefore m = 1$ *since* $c = 0$

$y = x$ (1)

Hence, the equation of the straight line is $y = x$

$f(z) = \bar{z}^2 = (x^2 - y^2) - j(2xy)$ (2)

$z = x + jy$ (3)

$I = \int f(z)dz$ (4)

From (3)

$dz = dx + jdy$ (5)

From (1), $dy = dx$ (6)

Putting equation (1) in (2) and (3)

$f(z) = x^2 - x^2 - j2x^2 = -j2x^2$ and $z = x + jx$

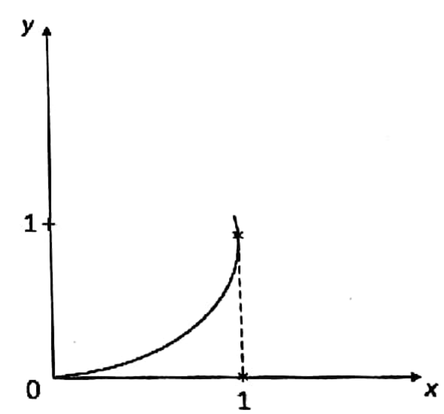
$d(z) = (1 + j)dx$

$\therefore I = \int f(z)dz \Rightarrow I = \int_0^1(-j2x^2)(1 + j)dx = \int_0^1(-j2x^2 + 2x^2)dx = \int_0^1(2x^2 - j2x^2)dx = \frac{2x^3}{3} - j\frac{2x^3}{3}\Big|_0^1$

$I = \frac{2}{3} - j\frac{2}{3} = \frac{2}{3}(1 - j)$

Case2:

Considering an arc of parabola $y = x^2$ from $z = 0$ to $z = 1 + j$



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$f(x) = x^2 - y^2 - j(2xy)$ (1)

$z = x + jy$ (2)

Here, $y = x^2$ (3)

Putting equation (3) into (2) and (1),

$f(z) = x^2 - x^4 - j(2x^3)$ (4)

$z = x + jx^2$ (5)

From (5), $dz = (1 + j2x)dx$

$$I = \int f(z)dz = \int_0^1 (x^2 - x^4 - j2x^3)(1 + j2x)dx$$

$$= \int_0^1 (x^2 + j2x^3 - x^4 - j2x^5 - j2x^3 + 4x^4)dx$$

$$= \int_0^1 (x^2 + 3x^4 - j2x^5)dx$$

$$\frac{x^3}{3} + \frac{3x^5}{5} - j\frac{x^6}{3}\Big|_0^1$$

$$= \frac{1}{3} + \frac{3}{5} - j\frac{1}{3} = \frac{14}{15} - j\frac{1}{3}$$

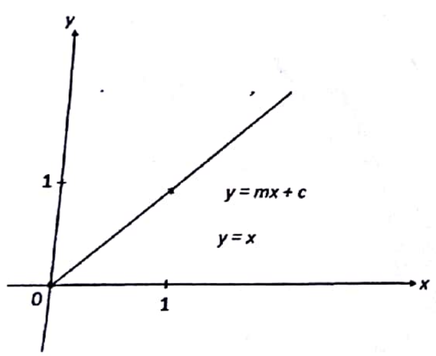
Example 20: Find the integral of the complex function $f(z) = (x^2 + y) + j(xy)$ $from\ a = 0\ to\ b = 1 + j$ along:

(i) A straight line

(ii) A parabola $y = x^2$ connecting the two points

Solution:

Taking the points $a = 0, b = 1 + j$



$f(z) = x^2 + y + j(xy)$ (1)

$z = x + jy$ (2)

$y = x$ (3)

Putting y = x in (2) and (1), we have

$f(z) = x^2 + x + jx^2$ (4)

$z = x + jx$ (5)

From (5) $dz = (1 + j)dx$

But $I = \int f(z)dz$

$$I = \int_0^1 (x^2 + x + jx^2)(1 + j)dx$$

$$I = \int_0^1 (x^2 + jx^2 + x + jx + jx^2 - x^2)dx$$

$$I = \int_0^1 [x + j(2x^2 + x)]dx$$

$$= \frac{x^2}{2} + j\left(\frac{2x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1$$

$$= \frac{1}{2} + j(\frac{2}{3} + \frac{1}{2})$$

$$= \frac{1}{2} + j\frac{7}{6}$$

(iii) $y = x^2$ (1)

$f(z) = x^2 + y + j(xy)$ (2)

$z = x + jy$ (3)

Putting (1) in (2) and (3)

$f(z) = x^2 + x^2 + j(x^3)$

$= 2x^2 + jx^3$

Also, $z = x + jx^2$

$dz = (1 + j2x)dx$

$$I = \int f(z)dz$$

$$= \int_0^1 (2x^2 + jx^3)(1 + j2x)dx$$

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$$= \int_0^1 (2x^2 + j4x^3 + jx^3 - 2x^4)dx$$

$$= \int_0^1 (2x^2 - 2x^4 - j5x^3)dx$$

$$I = \frac{2x^3}{3} - \frac{2x^5}{5} + j\frac{5x^4}{4} = \frac{2}{3} - \frac{2}{5} + j\frac{5}{4} = \frac{14}{15} + j\frac{5}{4}$$

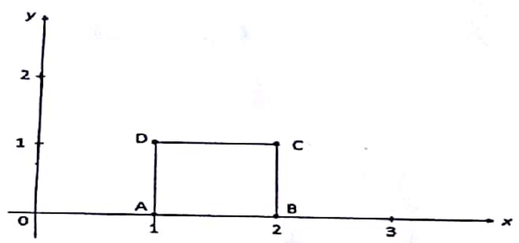
1.10 CAUCHY's THEOREM

This theorem states that if $f(z)$ is regular at every point within and on a closed curve C, then,

$$\oint f(z)dz = 0$$

Example 21: Evaluate the integral $\oint f(z)dz$, where $f(z) = z^2$ around the square formed by joining the points $z = 1, z = 2, z = 2 + j$ and $z = 1 + j$. Hence use your solutions to verify Cauchy's theorems.

Solution:



$f(z) = z^2$ but $z = x + jy$

$z^2 = x^2 - y^2 + j2xy$

$dz = dx + jdy$

$$\oint_C f(z)dz = \oint_C z^2 dz = \oint_C (x^2 - y^2 + j2xy)dz$$

$$= \oint_C (x^2 - y^2 + j2xy)(dx + jdy)$$

$$\oint_C [(x^2 - y^2)dx - 2xydy] + j\oint_C [2xydx + (x^2 - y^2)dy]$$

Taking each of the sides of the square in turn, we have:-

(i) $AB \Rightarrow y = 0 \therefore dy = 0$

$$\int_{AB} f(z)dz = \int_1^2 x^2 dx = \frac{x^3}{3}\Big|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

(ii) $BC: x = 2 \therefore dx = 0$

$$\int_{BC} f(z)dz = \int_0^1 (-4ydy) + j\int_0^1 (4 - y^2)dy$$

$$= -2y^2\Big|_0^1 + j[4y - \frac{y^3}{3}\Big|_0^1$$

$$= -2 + j\left(4 - \frac{1}{3}\right) = -2 + j\frac{11}{3}$$

(iii) $CD: y = 1, dy = 0$

$$\int_{CD} f(z)dz = \int_2^1 (x^2 - 1)dx + j\int_2^1 2xdx$$

$$= \frac{x^2}{3} - x\Big|_2^1 + j[x^2]_2^1 = \frac{-4}{3} - j3$$

(iv) $DA: x = 1 \therefore dx = 0$

$$\therefore \int_{DA} f(z) = \int_1^0 (-2ydy) + j\int_1^0 (1 - y^2)dy$$

$$= -y^2]_1^0 + j(y - \frac{y^3}{3})\Big|_1^0 = 1 - j\frac{2}{3}$$

Collecting the four results,

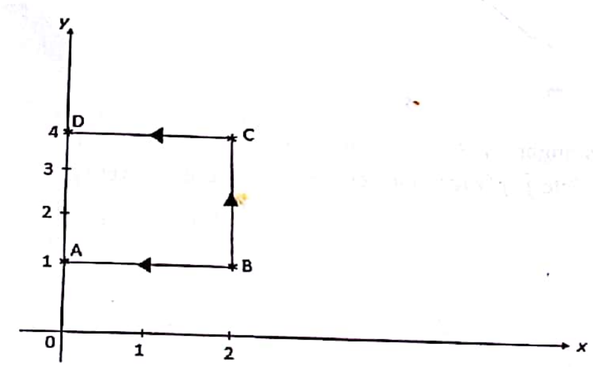
$$\oint_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz$$

$$= \frac{7}{3} + \left(-2 + j\frac{11}{3}\right) + \left(\frac{-4}{3} - j3\right) + \left(1 - j\frac{2}{3}\right) = 0$$

$\oint_C f(z)dz = 0$, Hence Cauchy's Theorem Verified.

Example 22: Evaluate $\oint_C f(z)dz$ where $f(z) = z^2$ round the rectangle formed by joining the points $z = 2 + j, z = 2 + j4,\ z = j4, and\ z = j$. hence, use your results to verify Cauchy's theorem. (Assignment)

Solution:



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$f(z) = z^2$ But $z = x + jy$

$z^2 = x^2 - y^2 + j(2xy), dz = dx + jdy$

$$\oint_C f(z)dz = \oint_C [(x^2 - y^2) + j2xy][dx + jdy]$$

$$\oint_C [(x^2 - y^2)dx - 2xydy] + j\oint_C [2xydx + (x^2 - y^2)dy$$

Taking each of the sides of the rectangle, we have:-

(i) AB $\Rightarrow y = 1, \therefore dy = 0$

$$\int_{AB} f(z)dz = \int_0^2 (x^2 - 1)dx + j\int_0^2 2xdx = \frac{x^3}{3} - x|_0^2 + jx^2|_0^2 = \frac{8}{3} - 2 + j4 = \frac{2}{3} + j4$$

(ii) BC: $x = 2 \quad \therefore dx = 0$

$$\oint_{BC} f(z)dz = \int_1^4 -4ydy + j\int_1^4 (4 - y^2)dy = -2y^2|_1^4 + j\left(4y - \frac{y^3}{3}\right)|_1^4 = -32 + 2 + j\left(\frac{-16}{3} - \frac{11}{3}\right)$$

$$= -30 - j9$$

(iii) CD: $y = 4, \therefore dy = 0$

$$\oint_{CD} f(z)dz \int_2^0 (x^2 - 16)dx + j\int_2^0 8xdx = \frac{x^3}{3} - 16x|_2^0 + j4x^2|_2^0 = \frac{88}{3} - j16$$

(iv) DA: $x = 0 \therefore dx = 0$

$$\oint_{DA} f(z)dz = j\int_4^1 -y^2dy = -j\frac{y^3}{3}|_4^1 = -j\left(\frac{1}{3} - \frac{64}{3}\right) = j21$$

Collecting the four results,

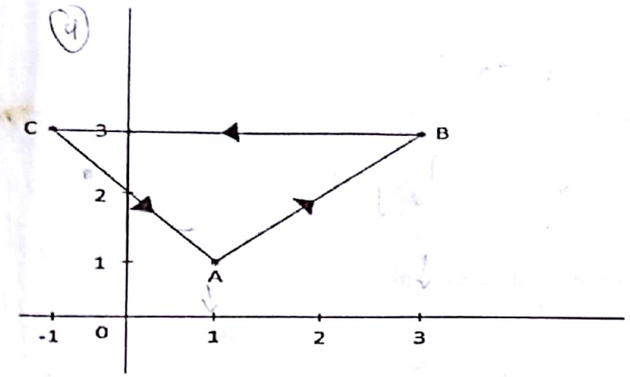
$$\oint_C f(z)dz = \oint_{AB} f(z)dz + \oint_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz$$

$$= \frac{2}{3} + j4 - 30 - j9 + \frac{88}{3} - j16 + j21 = 0$$

$\oint_C f(z)dz = 0$ Hence, Cauchy's theorem verified.

Example 23: A closed path C consists of three straight lines joining the points $A(1 + j), B(3 + j3)$ *and* $C(-1 + j3)$. Given that $f(z) = z - 1 + j$, evaluate $\oint_C f(z)dz$, and hence use your results to verify Cauchy's theorem.

Solution:



$f(z) = z - 1 + j, \quad z = x + jy = x + jy - 1 + j = x - 1 + j(y + 1)$

$dz = dx + jdy$

$$\oint_C f(z)dz = \oint_C [x - 1 + j(y + 1)][dx + jdy]$$

$$\oint_C [(x - 1)dx - (y + 1)dy + j\oint_C [(y + 1)dx + (x - 1)dy$$

Taking each of the sides of the triangle, we have:-

AB: $x = 1, \therefore dx = 0$

$$\oint_{AB} f(z)dz = \oint_1^3 -(y + 1)dy = \frac{-y^2}{2} - y|_1^3 = \left(\frac{-9}{2} - 3\right) - \left(\frac{-1}{2} - 1\right) = -6$$

BC: $x = 3, dx = 0$

$$\int_{BC} f(z)dz = \int_3^{-1} -(y + 1)dy + j\int_3^{-1} 2dy = \frac{-y^2}{2} - y|_3^{-1} = \left(\frac{-1}{2} + 1\right) - \left(\frac{-9}{2} - 3\right) + j(-2 - 6)$$

$$= 8 - j8$$

CA: $y = 3 \therefore dy = 0$

$$\int_{CA} f(z)dz = \int_{-1}^1 (x - 1)dx + j\int_{-1}^1 4dx$$

$$= \left(\frac{1}{2} - 1\right) - \left(\frac{1}{2} + 1\right) + j(4 + 4) = -2 + j8$$

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Collecting the three results,

$$\oint_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CA} f(z)dz$$

$$= -6 + 8 - j8 - 2 + j8 = 0$$

$\oint_C f(z)dz = 0$, Hence, Cauchy's theorem verified.

EXERCISE 1

1. Determine whether the following functions are continuous or not
   (i) $f(x) = \frac{x^3-16}{x-2}$, at $x = 2$
   (ii) $f(x) = \frac{x^3-1}{x-1}$, at $x = 1$
   (iii) $f(x) = \frac{\sqrt{x}-\sqrt{2}}{x^2-2x}$, at $x = 2$
   (iv) $f(x) = \frac{x^2-4}{x^3-8}$, at $x = 2$
2. Determine the analyticity of the function $f(z) = z^2$, using Cauchy's theorem.
3. A closed path C consists of a rectangle formed by joining the points $A(z = 2 + j), B(z = 2 + j4)$, $C(z = j4)$, and $D(z = j)$ given that $f(z) = z^2$, evaluate $\oint_C f(z)dz$, and hence use your results to verify Cauchy's theorem.
4. A closed path C consists of three straight lines joining the points $P(z = 1 + j)$, $Q(z = 3 + j3)$, *and* $R(z = -1 + j3)$. If $f(z) = z - 1 + j$, evaluate $\oint_C f(z)dz$, use your results to verify Cauchy's theorem.
5. Compute the integral of the complex function $A(z = -j)$ to $B(z = 1 + j)$ along
   (i) A straight line
   (ii) A parabola $y = 2x^2$, connecting the two points.
   $f(z) = x^2 + y + j(xy)$
6. A closed path C consists of a square formed by joining the points $P(z = 1)$, $Q(z = 2)$, $R(z = 2 + j)$ and $S(z = 1 + j)$. Given that $f(z) = z^2$, evaluate $\oint_C F(z)dz$, and hence use your results to verify Cauchy's theorem.

Chapter 2: Partial Differentiation

2.1 INTRODUCTION

Let $z = f(x, y)$ be function of two independent variables x and y. If we keep y constant and x varies, then z becomes a function of x only. The derivative of z with respect to x, keeping y constant is called partial derivative of "z", w.r.t. "x" and is denoted by symbols $\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), etc.$

Example 1: If $z = 4x^2 + 3xy + 6y^2$, *find* $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$

Solution:

(i) $\frac{\partial z}{\partial x} = 8x + 3y$ (ii) $\frac{\partial z}{\partial y} = 3x + 12y$ (iii) $\frac{\partial^2 z}{\partial x^2} = 8$

Example 2: If $z = \tan(10x + 2y)$, *find* $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Solution:

(i) $\frac{\partial z}{\partial x} = 10sec^2(10x + 2y)$ (ii) $\frac{\partial z}{\partial y} = 2sec^2(10x + 2y)$

Example 3: If $z = \frac{5x + y}{3x - 4y}$, *find* $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Solution:

(i) $\frac{\partial z}{\partial x} = \frac{(3x-4y)5-(5x+y)3}{(3x-4y)^2} = \frac{15x-20y-15x-3y}{(3x-4y)^2} = \frac{-23y}{(3x-4y)^2}$

(ii) $\frac{\partial z}{\partial y} = \frac{(3x-4y)-(-4)(5x+y)}{(3x-4y)^2} = \frac{23x}{(3x-4y)^2}$

Example 4: If $z = \ln(e^{3x} + e^{3y})$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 3$

Solution:

$\frac{\partial z}{\partial x} = \frac{3e^{3x}}{e^{3x}+e^{3y}}$ and $\frac{\partial z}{\partial y} = \frac{3e^{3y}}{e^{3x}+e^{3y}}$

$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{3e^{3x}}{e^{3x}+e^{3y}} + \frac{3e^{3y}}{e^{3x}+e^{3y}} = \frac{3(e^{3x}+e^{3y})}{e^{3x}+e^{3y}} = 3$

$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 3$ **ans.**

Note the following:

1. $\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial^2 z}{\partial x^2} = f_{xx}$
2. $\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}$
3. $\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$
4. $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

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Example 5: If $z = 4x^3 - 5xy^2 + 3y^3$, *find* (i) $\frac{\partial^2 z}{\partial x^2}$ (ii) $\frac{\partial^2 z}{\partial x \partial y}$ (iii) $\frac{\partial^2 z}{\partial y \partial x}$

Solution:

(i) $\frac{\partial z}{\partial x} = 12x^2 - 5y^2$

$\therefore \frac{\partial^2 z}{\partial x^2} = 24x$ **ans.**

(ii) $\frac{\partial z}{\partial y} = -10xy + 9y^2$

$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}(-10xy + 9y^2) = -10y$ **ans.**

(iii) $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial}{\partial y}(12x^2 - 5y^2) = -10y$

$\therefore \frac{\partial^2 z}{\partial y \partial x} = -10y$ **ans.**

Example 6: Prove that, if $z = \ln(x^2 + y^2)$, then $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

Solution:

$\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2} \Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{2(x^2+y^2)-2x(2x)}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$

$\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2} \Rightarrow \frac{\partial^2 z}{\partial y^2} = \frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$

$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = 0$ **ans.**

EXERCISE 2.1

1. If $z = \cos(2x + 3y)$, *find* $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}$
2. If $z = x^2 \sin(2x + 3y)$, *find* $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}$
3. If $z = e^{(x^2 - y^2)}$, *find* $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x \partial y}$
4. If $z = \frac{1}{x^2 + y^2 - 1}$, *show that* $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -2z(1 + z)$
5. If $u = \frac{x + y + z}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$, *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$
6. If $z = e^x(x \cos y - y \sin y)$, *show that* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$
7. If $u = (1 + x)\sinh(5x - 2y)$, *show that* $4\frac{\partial^2 u}{\partial x^2} + 20\frac{\partial^2 u}{\partial x \partial y} + 25\frac{\partial^2 u}{\partial y^2} = 0$
8. If $z^3 - 3yz - 3x = 0$, show that (i) $z\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, (ii) $z\left(\frac{\partial^2 z}{\partial x \partial y} + \left(\frac{\partial z}{\partial x}\right)^2\right) = \frac{\partial^2 z}{\partial y^2}$
9. If $z = x\ln(x^2 + y^2) + 2y\tan^{-1}\left(\frac{y}{x}\right)$, *show that* $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + 2x$
10. If $z = \frac{xy}{(x^2+y^2)^2}$, *show that* $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$
11. If $z = \ln\sqrt{(x^2 + y^2)}$, *show that* $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$

2.2 TOTAL DIFFERENTIALS & DERIVATIVES

If $z = f(x, y)$, then the total differential, dz, is given by the sum of the separate partial differentials of z; that is $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + \cdots$

Similarly, if $z = f(u, v, w)$, then $dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw + \cdots$

Thus, dz is called the **total differential** of z.

Example 7: If $z = x^3y + x^2y^2 + xy^3$, determine the total differential dz.

Solution:

$z = x^3y + x^2y^2 + xy^3$

$\frac{\partial z}{\partial x} = 3x^2y + 2xy^2 + y^3$

$\frac{\partial z}{\partial y} = x^3 + 2x^2y + 3xy^2$

$\therefore dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (3x^2y + 2xy^2 + y^3)dx + (x^3 + 2x^2y + 3xy^2)dy$

Example 8: If $z = x \sin y - y \sin x$, find the total differential dz.

Solution:

$z = x \sin y - y \sin x$

$\frac{\partial z}{\partial x} = \sin y - y \cos x$

$\frac{\partial z}{\partial y} = x \cos y - \sin x$

$\therefore dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (\sin y - y \cos x)dx + (x \cos y - \sin x)dy$

2.3 APPLICATIONS OF DIFFERENTIAL CALCULUS

Jacobians – Inverse Functions

If $z = f(x, y)$ and $u = g(x, y)$ and $v = h(x, y)$ then

(i) $\frac{\partial x}{\partial u} = \frac{\frac{\partial v}{\partial y}}{J}$, (ii) $\frac{\partial x}{\partial v} = -\frac{\frac{\partial u}{\partial y}}{J}$, (iii) $\frac{\partial y}{\partial u} = -\frac{\frac{\partial v}{\partial x}}{J}$, (iv) $\frac{\partial y}{\partial v} = \frac{\frac{\partial u}{\partial x}}{J}$, where in each case

$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u, v, w.r.t x & y and is written as $\frac{\partial(u, v)}{\partial(x, y)}$ or $J\left(\frac{u, v}{x, y}\right)$

That is, $J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$ or $J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

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Similarly, the Jacobian of u, v, w, w.r.t x, y, z is $J = \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Example 9: If $z = 4x^2 - 3xy + 2y^2$, $u = x^2 - y^2$ and $v = 3x + 6y$, determine:

(i) $\frac{\partial x}{\partial u}$ (ii) $\frac{\partial x}{\partial v}$ (iii) $\frac{\partial y}{\partial u}$ (iv) $\frac{\partial y}{\partial v}$ (v) $\frac{\partial z}{\partial u}$ (vi) $\frac{\partial z}{\partial v}$

Solution:

$u = x^2 - y^2;\ v = 3x + 6y$

$\frac{\partial u}{\partial x} = 2x;\ \frac{\partial u}{\partial y} = -2y;\ \frac{\partial v}{\partial x} = 3;\ \frac{\partial v}{\partial y} = 6$

$$J = \frac{\partial(u,\ v)}{\partial(x,\ y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 3 \\ -2y & 6 \end{vmatrix} = 12x + 6y = 6(2x + y)$$

(i) $\frac{\partial x}{\partial u} = \frac{\frac{\partial v}{\partial y}}{J} = \frac{6}{6(2x+y)} = \frac{1}{(2x+y)}$

(ii) $\frac{\partial x}{\partial v} = -\frac{\frac{\partial u}{\partial y}}{J} = -\frac{-2y}{6(2x+y)} = \frac{y}{3(2x+y)}$

(iii) $\frac{\partial y}{\partial u} = -\frac{\frac{\partial v}{\partial x}}{J} = -\frac{3}{6(2x+y)} = -\frac{1}{2(2x+y)}$

(iv) $\frac{\partial y}{\partial v} = \frac{\frac{\partial u}{\partial x}}{J} = \frac{2x}{6(2x+y)} = \frac{x}{3(2x+y)}$

(v) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$

From $z = 4x^2 - 3xy + 2y^2 \Rightarrow \frac{\partial z}{\partial x} = 8x - 3y;\ \frac{\partial z}{\partial y} = -3x + 4y$

$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = (8x - 3y)\frac{1}{(2x+y)} + (-3x + 4y)\left(-\frac{1}{2(2x+y)}\right) = \frac{19x - 10y}{2(2x + y)}$

(vi) $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$

$\therefore \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v} = (8x - 3y)\frac{y}{3(2x+y)} + (-3x + 4y)\frac{x}{3(2x+y)} = \frac{4xy - x^2 - y^2}{2x + y}$

Example 10: If $z = 2x^2 - 3y$, $u = x^2 \sin y$ and $v = 2y \cos x$, determine expressions for:

(i) $\frac{\partial z}{\partial u}$ and (ii) $\frac{\partial z}{\partial v}$

Solution:

$z = 2x^2 - 3y,\ u = x^2 \sin y,\ v = 2y \cos x$

(i) $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$

We first find the values of the following:

$\frac{\partial u}{\partial x} = 2x \sin y,\ \frac{\partial u}{\partial y} = x^2 \cos y,\ \frac{\partial v}{\partial x} = -2y \sin x,\ \frac{\partial v}{\partial y} = 2 \cos x$

$$J = \frac{\partial(u,\ v)}{\partial(x,\ y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x\sin y & -2y\sin x \\ x^2 \cos y & 2 \cos x \end{vmatrix} = 4x \sin y \cos x + 2x^2 \sin x \cos y$$

$\frac{\partial x}{\partial u} = \frac{\frac{\partial v}{\partial y}}{J} = \frac{2 \cos x}{J};\ \frac{\partial z}{\partial x} = 4x;\ \frac{\partial y}{\partial u} = -\frac{\frac{\partial v}{\partial x}}{J} = \frac{2y \sin x}{J};\ \frac{\partial z}{\partial y} = -3$

$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u} = \frac{4x(2 \cos x)}{J} + \frac{-3(2y \sin x)}{J} = \frac{8x \cos x - 6y \sin x}{J} = \frac{8x \cos x - 6y \sin x}{4x \sin y \cos x + 2x^2 \sin x \cos y}$

$\therefore \frac{\partial z}{\partial u} = \frac{8x \cos x - 6y \sin x}{4x \sin y \cos x + 2x^2 \sin x \cos y}$ **ans.**

(ii) $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$

$\frac{\partial x}{\partial v} = -\frac{\frac{\partial u}{\partial y}}{J} = \frac{-x^2 \cos y}{J};\ \frac{\partial y}{\partial v} = \frac{\frac{\partial u}{\partial x}}{J} = \frac{2x \sin y}{J}$

$\frac{\partial z}{\partial v} = \frac{4x(-x^2 \cos y)}{J} + \frac{-3(2x \sin y)}{J} = \frac{-4x^3 \cos y - 6x \sin y}{J} = \frac{-4x^3 \cos y - 6x \sin y}{4x \sin y \cos x + 2x^2 \sin x \cos y}$

$\therefore \frac{\partial z}{\partial v} = \frac{-(4x^3 \cos y + 6x \sin y)}{4x \sin y \cos x + 2x^2 \sin x \cos y}$ **ans.**

Example 11: Given that $u = x^2 - y^2$, $v = 2xy$, determine $\frac{\partial(x,\ y)}{\partial(u,\ v)}$.

Solution:

$\frac{\partial u}{\partial x} = 2x;\ \frac{\partial u}{\partial y} = -2y;\ \frac{\partial v}{\partial x} = 2y;\ \frac{\partial v}{\partial y} = 2x$

$$J = \frac{\partial(u,\ v)}{\partial(x,\ y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -2y & 2x \end{vmatrix} = 4x^2 + 4y^2$$

$\therefore \frac{\partial(x,\ y)}{\partial(u,\ v)} = \frac{1}{J} = \frac{1}{4x^2 + 4y^2} = \frac{1}{4(x^2 + y^2)}$ **ans.**

EXERCISE 2.2

12. If $z = f(x, y)$, where $x = uv$ and $y = u^2 - v^2$, show that:
    (i) $2x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$
    (ii) $2\frac{\partial z}{\partial y} = \frac{1}{u^2 + v^2}\left[u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v}\right]$
13. If $u = x + y + z$, $uv = y + z$, $uvw = z$, evaluate $\frac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)}$
14. If $u = xyz$, $v = xy + yz + xz$, $w = x + y + z$, find $\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}$
15. If $u = x^2 + y^2$ and $v = 4xy$, determine $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$
16. If $x = uv$, $y = \frac{u + v}{u - v}$, find $\frac{\partial(u,\ v)}{\partial(x,\ y)}$

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2.4 LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

CASE I – Two Independent Variables

Let $u = f(x, y)$ be a function of two variables x and y with constraint $\emptyset(x, y) = 0$, then $\frac{\partial u}{\partial x} = K\frac{\partial \emptyset}{\partial x}$ and $\frac{\partial u}{\partial y} = K\frac{\partial \emptyset}{\partial y}$ so that

$$\boxed{\frac{\partial u}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y} = 0}$$

Where $\lambda = -K$ is called a Lagrange Multiplier and the equations in block coupled with the constraint equation $\emptyset(x, y) = 0$ gives us three relationships from which the values of x and y at the extremal points and also the value of λ $(if\ required)$ can be found. Quite often, the value of λ is not important and hence is neglected.

Example 12: Use Lagrange's method of undetermined multipliers to obtain the stationary values of $u = x^2 + y^2$ subject to the constraint $\emptyset = 4x^2 + 6xy + 4y^2 = 9$

Solution:

$u = x^2 + y^2$, $\emptyset = 4x^2 + 6xy + 4y^2 = 9$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial \emptyset}{\partial x} = 8x + 6y$, $\frac{\partial \emptyset}{\partial y} = 6x + 8y$

$\frac{\partial u}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x} = 0$ .....(i)

$2x + \lambda(8x + 6y) = 0$ .....(ii)

$\frac{\partial u}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y} = 0$ .....(iii)

$2y + \lambda(6x + 8y) = 0$ .....(iv)

Eqn (ii) $\div$ eqn (iv), we have:

$\frac{2x}{2y} = \frac{-\lambda(8x + 6y)}{-\lambda(6x + 8y)} \Rightarrow \frac{x}{y} = \frac{4x + 3y}{3x + 4y}$

$3x^2 + 4xy = 4xy + 3y^2$

$x^2 = y^2 \Rightarrow x = \pm y$

When $x = y$, we substitute in $\emptyset$

$\emptyset = 4x^2 + 6xy + 4y^2 - 9 = 0 \Rightarrow \emptyset = 4x^2 + 6x^2 + 4x^2 - 9 = 0$

$14x^2 - 9 = 0$

$\therefore x = \pm\frac{3}{\sqrt{14}}$

Hence, $x = y = \pm\frac{3}{\sqrt{14}}$

When $x = -y$, we substitute in $\emptyset$

$\emptyset = 4x^2 + 6xy + 4y^2 - 9 = 0 \Rightarrow \emptyset = 4x^2 - 6x^2 + 4x^2 - 9 = 0$

$2x^2 - 9 = 0$

$\therefore x = \pm\frac{3}{\sqrt{2}}$

Since, $x = -y$, $\therefore y = -x = \mp\frac{3}{\sqrt{2}}$

Hence, $x = y = \pm\frac{3}{\sqrt{14}}$, and $x = \pm\frac{3}{\sqrt{2}}$, $y = \mp\frac{3}{\sqrt{2}}$ ans.

CASE II – THREE INDEPENDENT VARIABLES

Let $u = f(x, y, z)$, we can find the stationary points of u subject to the constraint

$\emptyset(x, y, z) = 0$ .....(i)

At stationary points,

$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$ .....(ii)

From eqn (i), we obtain total differentiation as

$\frac{\partial \emptyset}{\partial x}dx + \frac{\partial \emptyset}{\partial y}dy + \frac{\partial \emptyset}{\partial z}dz = 0$ .....(iii)

Multiplying eqn (iii) by λ and adding to eqn (ii), we have:

$\left(\frac{\partial u}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z} + \lambda\frac{\partial \emptyset}{\partial z}\right)dz = 0$ .....(iv)

From eqn (iv), we obtain:

$\frac{\partial u}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x} = 0$ .....(v)

$\frac{\partial u}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y} = 0$ .....(vi)

$\frac{\partial u}{\partial z} + \lambda\frac{\partial \emptyset}{\partial z} = 0$ .....(vii)

Now, to find the values of x, y, z and λ for which the function $u = f(x, y, z)$ has stationary points, we solve eqns (v), (vi), (vii) as well as the constraint equation.

Example 13: Use Lagrange's method of undetermined multipliers to obtain the stationary values of $u = x^2 + y^2 + z^2$ subject to the constraint $\emptyset = 3x - 2y + z - 4$.

Solution:

$u = x^2 + y^2 + z^2$ .....(i)

$\emptyset = 3x - 2y + z - 4$ .....(ii)

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial u}{\partial z} = 2z$

$\frac{\partial \emptyset}{\partial x} = 3$, $\frac{\partial \emptyset}{\partial y} = -2$, $\frac{\partial \emptyset}{\partial z} = 1$

Now,

$\frac{\partial u}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x} = 0 \Rightarrow 2x + 3\lambda = 0 \Rightarrow \therefore x = -\frac{3\lambda}{2}$ .....(iii)

$\frac{\partial u}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y} = 0 \Rightarrow 2y - 2\lambda = 0 \Rightarrow \therefore y = \lambda$ .....(iv)

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$\frac{\partial u}{\partial z} + \lambda\frac{\partial \emptyset}{\partial z} = 0 \Rightarrow 2z + \lambda = 0 \Rightarrow \therefore z = -\frac{\lambda}{2}$ .....(v)

Putting the values of x, y, z in eqn (ii), we obtain:

$\emptyset = 3x - 2y + z - 4 = 3\left(-\frac{3\lambda}{2}\right) - 2(\lambda) - \frac{\lambda}{2} - 4 = 0$

$-9\lambda - 4\lambda - \lambda - 8 = 0 \therefore \lambda = -\frac{4}{7}$

Hence, from eqns (iii), (iv), & (v) we obtain:

$x = -\frac{3\lambda}{2} = -\frac{3}{2} \times -\frac{4}{7} = \frac{6}{7}$

$y = \lambda = -\frac{4}{7}$

$z = -\frac{\lambda}{2} = -\frac{1}{2} \times -\frac{4}{7} = \frac{2}{7}$

$\therefore x = \frac{6}{7};\ y = -\frac{4}{7};\ z = \frac{2}{7}$ **ans.**

Chapter 3: Power Series Expansion of Functions

3.1 INTRODUCTION: Power series in x may be generated in various ways. In the expression $\frac{1}{1-x}$, if we continue to divide indefinitely, we shall obtain: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + \cdots$

The above equation is satisfied only on the interval $|x| < 1$ or $-1 < x < 1$. There are other methods of generating power series. But in this chapter, we shall be considering a general method for expanding a function in a power series in x and in (x – m).

3.2 GENERAL METHODS OF EXPANSION

A. **By Maclaurin's Series:** If a given series can be represented by a power series in x, that series is necessarily of the form of Maclaurin's series:

$$f(x) = f(0) + \frac{xf^1(0)}{1!} + \frac{x^2f^{11}(0)}{2!} + \frac{x^3f^{111}(0)}{3!} + \frac{x^4f^{iv}(0)}{4!} + \cdots + \frac{x^{n-1}f^{n-1}}{(n-1)!} + \cdots$$

Proof of Maclaurin's Series

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$

$$f(0) = a_0 \Rightarrow a_0 = f(0)$$

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4$

$$f^1(0) = a_1 \Rightarrow a_1 = f^1(0)$$

$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots$

$$f^{11}(0) = 2a_2 \Rightarrow a_2 = \frac{f^{11}(0)}{2!}$$

$f'''(x) = 6a_3 + 24a_4x + 60a_5x^2 + \cdots$

$$f^{111}(0) = 6a_3 \Rightarrow a_3 = \frac{f^{111}(0)}{6} = \frac{f^{111}(0)}{3!}$$

$f^{iv}(x) = 24a_4 + 120a_5x + \cdots$

$$f^{iv}(0) = 24a_4 \Rightarrow a_4 = \frac{f^{iv}(0)}{24} = \frac{f^{iv}(0)}{4!}$$

Substituting the values of $a_0, a_1, \ldots etc$, we have:

$$f(x) = f(0) + \frac{xf^1(0)}{1!} + \frac{x^2f^{11}(0)}{2!} + \frac{x^3f^{111}(0)}{3!} + \frac{x^4f^{iv}(0)}{4!} + \cdots$$

The above is known as the **Maclaurin's series.**

Conditions for Maclaurin's Series

Maclaurin's series may be used to represent any function, say f(x), as a power series provided that at $x = 0$, the following three conditions are satisfied:

a. $f(0) \neq \infty$. For instance, if $f(x) = \cos x$, $f(0) = \cos 0 = 1$. Thus $\cos x$ meets the condition. But if $f(x) = \ln x$, *or* if $f(x) = \frac{1}{x}$, $f(0) = \ln 0 = -\infty$ and $f(0) = \frac{1}{0} = \infty$; thus, $\ln x$ and $\frac{1}{x}$ do not meet the condition.

b. $f^1(0), f^{11}(0), f^{111}(0), \ldots \neq \infty$

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c. The resultant Maclaurin's series must be convergent. This means that the value of the terms, or groups of terms, must get progressively smaller and the sum of the terms must reach a limiting value.

B. By Taylor's Series: If a given function can be represented by a power series in (x – a), that series is necessarily of the form of Taylor's series, stated as:

$$f(x) = f(a) + \frac{f^{1}(a)(x-a)}{1!} + \frac{f^{11}(a)(x-a)^2}{2!} + \frac{f^{111}(a)(x-a)^3}{3!} + \cdots$$

Example 1: Determine the power series for $\sin x$ using the Maclaurin's series.

Solution:

$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f^{1}(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f^{11}(0) = 0$

$f'''(x) = -\cos x \Rightarrow f^{111}(0) = -1$

$f^{iv}(x) = \sin x \Rightarrow f^{iv}(0) = 0$

$f^{v}(x) = \cos x \Rightarrow f^{v}(0) = 1$

Using the Maclaurin's series, $f(x) = f(0) + \frac{xf'(0)}{1!} + \frac{x^2f''(0)}{2!} + \frac{x^3f'''(0)}{3!} + \cdots$

$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Example 2: Determine the first four terms of the power series for $\cos x$

Solution:

We can use the power series of $\sin x$ to find $\cos x$ by differentiating $\sin x$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Example 3: Determine the power series for $\sin^2 2x$ as far as the term in x^6

Solution:

From double-angle formula, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$. From which, we obtain

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\sin^2 2x = \frac{1}{2}(1 - \cos 4x)$

But $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Hence, $\cos 4x = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!} + \cdots$

$= 1 - \frac{16x^2}{2!} + \frac{256x^4}{4!} - \frac{4096x^6}{6!} + \cdots$

$= 1 - 8x^2 + \frac{32x^4}{3} - \frac{256x^6}{45} + \cdots$

Thus, $\sin^2 2x = \frac{1}{2}(1 - \cos 4x) = \frac{1}{2}[1 - (1 - 8x^2 + \frac{32x^4}{3} - \frac{256x^6}{45} + \cdots)]$

$\sin^2 2x = \frac{1}{2}\left(8x^2 - \frac{32x^4}{3} + \frac{256x^6}{45} - \cdots\right)$

$\therefore \sin^2 2x = 4x^2 - \frac{16x^4}{3} + \frac{128x^6}{45} - \cdots$

Example 4: Determine the power series for $\cos^2 x$ as far as the term in x^6

Solution:

Form the double-angle formula, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

But $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ and $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots$

$\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots$

Hence, $\cos^2 x = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2}\left(1 + 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots\right)$

$\therefore \cos^2 x = \frac{1}{2}\left(2 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots\right) = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots$

Alternatively,

$\cos^2 x = (\cos x)^2 = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$

$\cos^2 x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{48} + \frac{x^4}{24} - \frac{x^6}{48} - \frac{x^6}{720} + \cdots$

Collecting like terms, we have:

$\therefore \cos^2 x = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots$

Example 5: Find the power series expansion for $\ln\left(\frac{1+x}{1-x}\right)$

Solution:

$f(x) = \ln(1+x)$, $f(0) = \ln 1 = 0$; $f^{1}(x) = \frac{1}{1+x}$, $f^{1}(0) = 1$;

$f''(x) = \frac{-1}{(1+x)^2}$, $f''(0) = -1$; $f'''(x) = \frac{2}{(1+x)^3}$; $f'''(0) = 2$

$f^{iv}(x) = \frac{-6}{(1+x)^4}$, $f^{iv}(x) = -6$; $f^{v}(x) = \frac{24}{(1+x)^5}$, $f^{v}(0) = 24$;

Substituting these values in the Maclaurin's series formula, that is,

$f(x) = f(0) + \frac{xf'(0)}{1!} + \frac{x^2f''(0)}{2!} + \frac{x^3f'''(0)}{3!} + \cdots$

$f(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

Hence, for $\ln(1-x)$, we simply replace x by $-x$ in the series for $\ln(1+x)$, that is

$\ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \cdots$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$

Now, $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\right)$

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$$\therefore \ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{3} + \cdots$$

$$\therefore \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{3} + \cdots\right)$$

3.3 BINOMIAL SERIES

A binomial expression is one which involves two terms connected by a plus or minus sign. Thus, $(p+q)$, $(a+x)^2$, $(2x+y)^3$ are examples of binomial expression. Maclaurin's series can also be applied to obtain a power series for $(1+x)^n$. Let us obtain the power series of $(1+x)^n$.

$f(x) = (1+x)^n$, $f(0) = 1$; $f'(x) = n(1+x)^{n-1}$, $f'(0) = n$

$f''(x) = n(n-1)(1+x)^{n-2}$, $f''(0) = n(n-1)$;

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$, $f'''(0) = n(n-1)(n-2)$

$f^{iv}(x) = n(n-1)(n-2)(n-3)(1+x)^{n-4}$, $f^{iv}(0) = n(n-1)(n-2)(n-3)$; etc.

Substituting these values in the general Maclaurin's formula, that is:

$$f(x) = f(0) + \frac{xf^{1}(0)}{1!} + \frac{x^2f^{11}(0)}{2!} + \frac{x^3f^{111}(0)}{3!} + \cdots$$

$$f(x) = (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \frac{n(n-1)(n-2)(n-3)x^4}{4!} + \ldots$$

The above expansion is valid for $|x| < 1$.

Similarly, the Maclaurin's series expansion of $(1-x)^n$ can be obtained in the same way or from the series expansion of $(1+x)^n$. Here, we shall replace x by $-x$, that is

$$(1-x)^n = 1 + n(-x) + \frac{n(n-1)(-x)^2}{2!} + \frac{n(n-1)(n-2)(-x)^3}{3!} + \frac{n(n-1)(n-2)(n-3)(-x)^4}{4!} + \cdots$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \frac{n(n-1)(n-2)(n-3)(x)^4}{4!} + \ldots$$

The general binomial expansion of $(a+x)^n$ is:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots$$

From the foregoing, we can observe that $(a+b)^n$ will have n factors when expanded. That is $(a+b)^n = (a+b)(a+b)(a+b)\ldots$

The power of the binomial is n. If we expand binomially, we shall obtain terms consisting of the product $a^t b^r$ such that $r + t = n$ or $t = n - r$. So the terms of the expansion are of the form $a^{n-r}b^r$. This term is obtained by picking b from r factors out of the n factors and picking a from the remaining $(n-r)$ factors. Selecting r factors out of n factors can be done in ${}^nC_r$ ways. In other words, the term $a^{n-r}b^r$ occurs ${}^nC_r$ times in the expansion of $(a+b)^n$. That is, the coefficient of $a^{n-r}b^r$ is ${}^nC_r = \frac{n!}{(n-r)!\,r!}$. Hence, the complete expansion of $(a+b)^n$ consists of the sum of all the terms ${}^nC_r a^{n-r}b^r$ as r ranges from 0 to n.

Theorem:

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1 a^{n-1}b^1 + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}b^3 + \ldots + {}^nC_r\, a^{n-r}b^r$$

Example 6: Use the binomial series to determine the expansion of $(2+x)^7$

Solution:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots$$

$$(2+x)^7 = 2^7 + 7(2)^6x + \frac{7(6)2^5x^2}{2!} + \frac{(7)(6)(5)2^4x^3}{3!} + \frac{(7)(6)(5)(4)2^3x^4}{4!} + \frac{(7)(6)(5)(4)(3)2^2x^5}{5!}$$
$$+ \frac{(7)(6)(5)(4)(3)(2)2^1x^6}{6!} + \frac{(7)(6)(5)(4)(3)(2)(1)x^7}{7!}$$

$$\therefore (2+x)^7 = 128 + 448x + 672x^2 + 560x^3 + 280x^4 + 84x^5 + 14x^6 + x^7$$

Example 7: Use the binomial expansion to determine the value of $(0.98)^7$ correct to 5 s.f.s

Solution:

$$(1-x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \frac{n(n-1)(n-2)(n-3)(x)^4}{4!} + \ldots$$

Here, $x = 0.02$, $n = 7$ so that $0.98 = 1 - 0.02$

$$(0.98)^7 = (1-0.02)^7 = 1 - 7(0.02) + \frac{7(6)(0.02)^2}{2!} - \frac{7(6)(5)(0.02)^3}{3!} + \frac{7(6)(5)(4)(0.02)^4}{4!} -$$
$$\frac{7(6)(5)(4)(3)(0.02)^5}{5!} + \frac{7(6)(5)(4)(3)(2)(0.02)^6}{6!} - \frac{7(6)(5)(4)(3)(2)(1)(0.02)^7}{7!}$$

$$(0.98)^7 = 1 - 0.14 + 0.0084 - 0.00028 + 0.0000056 - 0.0000000672 + 0.000000000448$$
$$- 0.00000000000128$$

$(0.98)^7 = 0.868125 \approx 0.86813$ correct to 5 s.f.s.

Example 8: Evaluate $(4.044)^6$ correct to one decimal place, using the binomial expansion.

Solution:

$$(4.044)^6 = 4^6(1.011)^6 = 4^6(1+0.011)^6$$

Here, $x = 0.011$, $n = 6$

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \frac{n(n-1)(n-2)(n-3)x^4}{4!} + \ldots$$

$$(1+0.011)^6 = 1 + 6(0.011) + \frac{6(5)(0.011)^2}{2!} + \frac{6(5)(4)(0.011)^3}{3!} + \frac{6(5)(4)(3)(0.011)^4}{4!} +$$
$$\frac{6(5)(4)(3)(2)(0.011)^5}{5!} + \frac{6(5)(4)(3)(2)(1)(0.011)^6}{6!}$$

$$(1+0.011)^6 = 1 + 0.066 + 1.85 \times 10^{-3} + 2.662 \times 10^{-5} + 2.19615 \times 10^{-7} +$$
$$9.66306 \times 10^{-10} + 1.771561 \times 10^{-12}$$

$(1+0.011)^6 = 1.067841841$

$\therefore 4^6(1+0.011)^6 = 4^6(1.067841841) = 4373.88 \cong 4373.9$ to 1 d.p.

Example 9: Expand $\frac{1}{(1-2x)(1+3x)}$ in a series of ascending powers of x, and state the validity of the expansion.

Solution:

$$\frac{1}{(1-2x)(1+3x)} = (1-2x)^{-1}(1+3x)^{-1}$$

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$(1-2x)^{-1} = 1 + 2x + 4x^3 + 16x^4 + \cdots$

$(1+3x)^{-1} = 1 - 3x + 9x^2 - 27x^3 + 81x^4$

$(1-2x)^{-1}(1+3x)^{-1} = (1 + 2x + 4x^3 + 16x^4 + \cdots)(1 - 3x + 9x^2 - 27x^3 + 81x^4)$

$\frac{1}{(1-2x)(1+3x)} = 1 - 3x + 9x^2 - 27x^3 + 81x^4 + 2x - 6x^2 + 18x^3 - 54x^4 + 162x^5 + 4x^2$

$- 12x^3 + 36x^4 - 108x^5 + 144x^6 - 432x^7 + 1296x^8 - \cdots$

Collecting like terms, we obtain:

$\frac{1}{(1-2x)(1+3x)} = 1 - x + 7x^2 - 13x^3 + 55x^4 + 78x^5 + \cdots$

The expansion is valid provided $|2x| < 1$ and $|3x| < 1$

i.e. $|x| < \frac{1}{2}$ and $|x| < \frac{1}{3}$ OR $-\frac{1}{2} < x < \frac{1}{2}$ and $-\frac{1}{3} < x < \frac{1}{3}$

Hence, the condition for validity of the series is $-\frac{1}{2} < x < \frac{1}{2}$ and $-\frac{1}{3} < x < \frac{1}{3}$

Alternatively,

$\frac{1}{(1-2x)(1+3x)} = \frac{2}{5(1-2x)} + \frac{3}{5(1+3x)}$ on splitting into partial fractions.

$\frac{2}{5(1-2x)} + \frac{3}{5(1+3x)} = \frac{2}{5}(1-2x)^{-1} + \frac{3}{5}(1+3x)^{-1}$

Applying the binomial expansion, we have:

$\frac{2}{5}\left[1 + 2x + \frac{(-1)(-2)(2x)^2}{2!} - \frac{(-1)(-2)(-3)(2x)^3}{3!} + \cdots\right] +$

$\frac{3}{5}\left[1 - 3x + \frac{(-1)(-2)(3x)^2}{2!} + \frac{(-1)(-2)(-3)(3x)^3}{3!} + \cdots\right]$

$= \frac{2}{5}[1 + 2x + 4x^2 + 8x^3 + \cdots] + \frac{3}{5}[1 - 3x + 9x^2 - 27x^3 + \cdots]$

$\frac{2}{5} + \frac{4}{5}x + \frac{8}{5}x^2 + \frac{16}{5}x^3 + \cdots + \frac{3}{5} - \frac{9}{5}x + \frac{27}{5}x^2 - \frac{81}{5}x^3 + \cdots$

Collecting like terms, we obtain:

$= 1 - x + 7x^2 - 13x^3 + \cdots$

The expansion is valid at $-\frac{1}{2} < x < \frac{1}{2}$ and $-\frac{1}{3} < x < \frac{1}{3}$

Example 10: Compute $\sqrt{18}$ to four significant figures without tables or calculator. If x is so small that the second and higher powers of x may be neglected, find an approximation for $(2-x)^3(1 + \frac{x}{2})^{\frac{1}{2}}\left(\frac{1}{4-3x}\right)$.

Solution:

$\sqrt{18} = 18^{\frac{1}{2}} = (16+2)^{\frac{1}{2}} = 16^{\frac{1}{2}}(1 + \frac{2}{16})^{\frac{1}{2}} = 4(1 + \frac{1}{8})^{\frac{1}{2}} = 4(1 + 0.125)^{\frac{1}{2}}$

Using binomial series:

$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \frac{n(n-1)(n-2)(n-3)x^4}{4!} + \ldots$

$4\left[1 + \frac{1}{2}(0.125) + \frac{\frac{1}{2}(-\frac{1}{2})(0.125)^2}{2!} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(0.125)^3}{3!} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(0.125)^4}{4!} + \cdots\right]$

$= 4(1 + 0.0625 - 0.001953125 + 0.0001220703125 - 0.000009536743164 + \cdots)$

$= 4(1.060659409) = 4.242637634$

$\therefore \sqrt{18} = 4.2426$ correct to 4 s.f.s

$(2-x)^3(1 + \frac{x}{2})^{\frac{1}{2}}\left(\frac{1}{4-3x}\right) = (2-x)^3(1 + \frac{x}{2})^{\frac{1}{2}}(4-3x)^{-1}$

Using binomial series: (neglecting the 2nd & higher powers of x)

$(2-x)^3 = 2^3\left(1 - \frac{x}{3}\right)^3 = 8\left(1 - \frac{3x}{2} + \cdots\right) = 8 - 12x$

$(1 + \frac{x}{2})^{\frac{1}{2}} = 1 + \frac{1}{2}\left(\frac{x}{2}\right) = 1 + \frac{x}{4} + \cdots$

$(4-3x)^{-1} = 4^{-1}\left(1 - \frac{3x}{4}\right)^{-1} = 4^{-1}\left[1 - (-1)\left(\frac{3x}{4}\right) + \cdots\right] = 4^{-1}\left(1 + \frac{3x}{4}\right) = \frac{1}{4} + \frac{3x}{16} + \cdots$

Hence, $(2-x)^3(1 + \frac{x}{2})^{\frac{1}{2}}(4-3x)^{-1} = (8 - 12x)\left(1 + \frac{x}{4}\right)\left(\frac{1}{4} + \frac{3x}{16}\right)$

$= 2 + \frac{3x}{2} - \frac{5x}{2} = 2 - x$

$\therefore (2-x)^3(1 + \frac{x}{2})^{\frac{1}{2}}(4-3x)^{-1} = 2 - x$ (neglecting the 2nd & higher powers of x)

3.4 EXPONENTIAL SERIES

We can also use the Maclaurin's series to derive expression for exponential series e^x. See how it goes!

$f(x) = e^x,\ f(0) = e^0 = 1;\quad f'(x) = e^x, f'(0) = 1;\quad f''(x) = e^x, f''(0) = 1$

$f'''(x) = e^x, f'''(0) = 1;\quad f^{iv}(x) = e^x,\ f^{iv}(0) = 1;$ etc.

Maclaurin's states: $f(x) = f(0) + \frac{xf^1(0)}{1!} + \frac{x^2f^{11}(0)}{2!} + \frac{x^3f^{111}(0)}{3!} + \cdots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Similarly, we can find e^{-x} from e^x by simply replacing x by $-x$. Thus,

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$

Hence,

$$\boxed{e^{kx} = 1 + kx + \frac{(kx)^2}{2!} + \frac{(kx)^3}{3!} + \frac{(kx)^4}{4!} + \cdots}$$

Example 11: Expand $e^{\frac{3}{2}x}$ in a power series as far as the term in x^4

Solution:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$e^{\frac{3}{2}x} = 1 + \left(\frac{3}{2}x\right) + \frac{\left(\frac{3}{2}x\right)^2}{2!} + \frac{\left(\frac{3}{2}x\right)^3}{3!} + \frac{\left(\frac{3}{2}x\right)^4}{4!} + \cdots$

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$$1 + \frac{3}{2}x + \frac{9}{8}x^2 + \frac{9}{16}x^3 + \frac{27}{128}x^4 + \cdots$$

Example 12: Expand $\frac{1 + 2x + 3x^2}{e^x}$ as a power series in x and write down the coefficient of x^n.

Solution:

$$\frac{1 + 2x + 3x^2}{e^x} = (1 + 2x + 3x^2)e^{-x} = (1 + 2x + 3x^2)\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)$$

Expanding, we obtain:

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^{n-1}\frac{x^{n-1}}{(n-1)!} + \cdots$$

$$+ 2x - 2x^2 + \frac{2x^3}{2!} - \frac{2x^4}{3!} + \cdots + 2(-1)^{n-1}\frac{x^n}{(n-1)!} + \cdots$$

$$+ 3x^2 - 3x^3 + \frac{3x^4}{2!} - \frac{3x^5}{3!} + \cdots + 3(-1)^{n-1}\frac{x^{n+1}}{(n-1)!} + \cdots$$

$$1 + x + \frac{3x^2}{2} - \frac{13x^3}{6} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}(x^{n-1} + 2x^n + 3x^{n+1}) + \cdots$$

$$1 + x + \frac{3x^2}{2} - \frac{13x^3}{6} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{x^n}{x} + 2x^n + 3xx^n\right) + \cdots$$

$$= 1 + x + \frac{3x^2}{2} - \frac{13x^3}{6} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{1}{x} + 2 + 3x\right)x^n + \cdots$$

$$= 1 + x + \frac{3x^2}{2} - \frac{13x^3}{6} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{1 + 2x + 3x^2}{x}\right)x^n + \cdots$$

Hence, the coefficient of x^n is $\frac{(-1)^{n-1}}{(n-1)!}\left(\frac{1 + 2x + 3x^2}{x}\right)$

Example 13: Use power series expansion to evaluate $e^{2x}\ln(1 + x)$ up to four terms.

Solution:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\therefore e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

$$e^{2x}\ln(1 + x) = \left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

Expanding the bracket, we obtain:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + 2x^2 - x^3 + \frac{2x^4}{3} - \frac{x^5}{2} + \cdots + 2x^3 - x^4 + \frac{2x^5}{3} - \cdots + \frac{4x^4}{3} - \frac{2x^5}{3}$$

Collecting like terms, we have:

$$x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{3}{4}x^4 - \frac{3}{10}x^5$$

$$\therefore e^{2x}\ln(1 + x) = x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{3}{4}x^4$$

3.5 FURTHER PROBLEMS

Example 14: Use power series expansion to evaluate $\lim_{x\to 0}\frac{\sin^2 x}{x^2}$

Solution:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\therefore \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots \text{ that is } \cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots$$

But $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Hence $\sin^2 x = \frac{1}{2}\left[1 - \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots\right)\right] = \frac{1}{2}\left(2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 - \cdots\right)$

$$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots$$

$$\lim_{x\to 0}\frac{\sin^2 x}{x^2} = \lim_{x\to 0}\left\{\frac{x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \cdots}{x^2}\right\} = \lim_{x\to 0}\left\{1 - \frac{x^2}{3} + \frac{2x^4}{45} - \cdots\right\} = 1$$

Example 15: Prove that, if x is sufficiently small, $\text{Log}_e(1 + x + x^2) = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} - \cdots$

Solution:

Let $y = x + x^2$ so that $\text{Log}_e(1 + x + x^2) = \text{Log}_e(1 + y)$

But $\text{Log}_e(1 + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \cdots$

Putting $y = x + x^2$ in the above equation, we obtain

$$\text{Log}_e(1 + x + x^2) = x + x^2 - \frac{(x + x^2)^2}{2} + \frac{(x + x^2)^3}{3} - \frac{(x + x^2)^4}{4} + \cdots$$

$$= x + x^2 - \frac{(x^2 + 2x^3 + x^4)}{2} + \frac{x^3 + 3x^4 + 3x^5 + x^6}{3} - \frac{(x^4 + 4x^5 + 6x^6 + \cdots)}{4} + \cdots$$

$$= x + x^2 - \frac{x^2}{2} - x^3 - \frac{x^4}{2} + \frac{x^3}{3} + x^4 + \cdots - \frac{x^4}{4} - \cdots$$

Collecting like terms together, we have:

$$= x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} - \cdots$$

$$\therefore \text{Log}_e(1 + x + x^2) = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} - \cdots$$

Example 16: Prove that $\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots$

EXAM

Solution:

We recall that: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$\frac{x}{e^x - 1} = \frac{x}{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots} = \frac{x}{x\left[1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right]} = \frac{1}{\left[1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right]}$$

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$$= \left[1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots\right]^{-1} = \left[1 + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \cdots\right)\right]^{-1}$$

Recall that $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \frac{n(n-1)(n-2)(n-3)x^4}{4!} + \cdots$

Here, we replace x by $\left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \cdots\right)$ and $n = -1$, and so we obtain:

$$\left[1 + \left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \cdots\right)\right]^{-1} = 1 + -\left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \cdots\right) +$$

$$\frac{-1(-2)\left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \right)^2}{2!} + \frac{-1(-2)(-3)\left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \right)^3}{3!} +$$

$$\frac{-1(-2)(-3)(-4)\left(\frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \right)^4}{4!} + \cdots$$

If we expand the brackets while neglecting powers of x above x^4 (i.e. from x^5 upwards), we shall obtain:

$$= 1 - \frac{x}{2} - \frac{x^2}{6} - \frac{x^3}{24} - \frac{x^4}{120} - \cdots + \frac{x^2}{4} + \frac{x^3}{6} + \frac{5x^4}{72} + \cdots - \frac{x^3}{8} - \frac{x^4}{24} - \frac{x^4}{12} - \cdots + \frac{x^4}{16} + \cdots$$

Grouping like terms together, we have:

$$= 1 - \frac{x}{2} + \left(\frac{x^2}{4} - \frac{x^2}{6}\right) + \left(\frac{x^3}{6} - \frac{x^3}{24} - \frac{x^3}{8}\right) + \left(\frac{5x^4}{72} + \frac{x^4}{16} - \frac{x^4}{120} - \frac{x^4}{24} - \frac{x^4}{12}\right) + \cdots$$

Simplifying the terms in the brackets, we obtain:

$$= 1 - \frac{x}{2} + \frac{x^2}{12} + 0 - \frac{x^4}{720} + \cdots$$

$$\therefore \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots \quad \text{(Proved)}$$

Example 17: Use binomial theorem to expand $\frac{9}{(3+x)^3}$ in ascending powers of x, and determine the limits of x for which the series is valid.

Solution:

$$\frac{9}{(3+x)^3} = 9\left[3\left(1+\frac{x}{3}\right)\right]^{-3} = 9\left[3^{-3}\left(1+\frac{x}{3}\right)^{-3}\right] = \frac{9}{27}\left(1+\frac{x}{3}\right)^{-3} = \frac{1}{3}\left(1+\frac{x}{3}\right)^{-3}$$

But $(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \frac{n(n-1)(n-2)(n-3)x^4}{4!} + \cdots$

Replacing x by $\frac{x}{3}$ and $n = -3$, we obtain:

$$\frac{1}{3}\left(1+\frac{x}{3}\right)^{-3} = \frac{1}{3}\left[1 - 3\left(\frac{x}{3}\right) + \frac{-3(-4)\left(\frac{x}{3}\right)^2}{2!} + \frac{-3(-4)(-5)\left(\frac{x}{3}\right)^3}{3!} + \frac{-3(-4)(-5)(-6)\left(\frac{x}{3}\right)^4}{4!} + \cdots\right]$$

$$= \frac{1}{3}\left[1 - x + \frac{2x^2}{3} - \frac{10x^3}{27} + \frac{5x^4}{27} - \cdots\right]$$

$$\therefore \frac{9}{(3+x)^3} = \frac{1}{3}\left(1+\frac{x}{3}\right)^{-3} = \frac{1}{3}\left[1 - x + \frac{2x^2}{3} - \frac{10x^3}{27} + \frac{5x^4}{27} - \cdots\right]$$

The expansion is valid when $\left|\frac{x}{3}\right| < 1$,

i.e. $|x| < 3$ or $-3 < x < 3$

Now, if $y = Ae^{mx}$ .....(i)

$y' = Ame^{mx}$ .....(ii)

$y'' = Am^2e^{mx}$ .....(iii)

Substituting eqns. (ii), (iii), & (iv) into eqn. (i), we obtain:

$aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0$ .....(v)

Dividing through by Ae^{mx}, we obtain:

$am^2 + bm + c = 0$ .....(vi)

Eqn. (vi) above is a quadratic equation whose roots are given as m_1 and m_2. The quadratic equation is called the **auxiliary equation or the characteristic equation**, and is obtained directly from the second-order equation by noting that: $m^2 = y''$, $m = y'$, $1 = y$

For instance,

For the equation, $4y'' + 3y' + 2 = 0$, the **auxiliary equation** is $4m^2 + 3m + 2 = 0$ Also, for the equation, $y'' - 5y' + 3 = 0$, the **auxiliary equation** is $m^2 - 5m + 3 = 0$.

If $y_1 = k_1e^{m_1x}$ and $y_2 = k_2e^{m_2x}$ are two solutions of the given equations, and are called the basis of solutions of the homogeneous differential equation with constant coefficients, then $y = y_1 + y_2 = k_1e^{m_1x} + k_2e^{m_2x}$, where k_1 and k_2 are arbitrary constants for a second order equation, and m_1 and m_2 are the roots of the quadratic equation (auxiliary equation).

The type of solution of the second order equation depends on the nature of the roots of the auxiliary equation, since the roots may be:

a. two distinct real roots
b. real and equal roots
c. complex roots

This further leads us to the three types of the auxiliary equation.

TYPE A: Two Distinct Real Roots.

If $b^2 - 4ac > 0$, and $m_1 \neq m_2$, and the roots are real and distinct, then the basis of solution is $y_1 = k_1e^{m_1x}$, $y_2 = k_2e^{m_2x}$ and the corresponding general solution is

$y = y_1 + y_2 = k_1e^{m_1x} + k_2e^{m_2x}$ .....(vii)

Eqn. (vii) also represents the **complementary function** of the equation and may be written as

$C.F = y_c = k_1e^{m_1x} + k_2e^{m_2x}$ .....(viii)

$$\boxed{y = y_c = k_1e^{m_1x} + k_2e^{m_2x}}$$

Example 3: Solve the differential equation: $y'' - y = 0$

Solution:

Aux. eqn. $m^2 - 1 = 0$

$\therefore m = \pm 1 \Rightarrow m_1 = 1,\ m_2 = -1$

$\therefore$ *the general solution is* $y = k_1e^x + k_2e^{-x}$

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Example 4: Solve the equation: $2y'' - 14y' + 24y = 0$

Solution:

Aux. eqn. $2m^2 - 14m + 24 = 0 \Rightarrow m^2 - 7m + 12 = 0$

$\therefore m = 4 \text{ or } 3 \Rightarrow m_1 = 4, m_2 = 3$

$\therefore y = k_1e^{4x} + k_2e^{3x}$

Example 5: Solve the differential equation: $y'' - y' = 0$

Solution:

Aux. eqn. $m^2 - m = 0 \Rightarrow m(m-1) = 0$

$\therefore m = 0 \text{ or } 1 \Rightarrow m_1 = 0,\ m_2 = 1$

$\therefore y = k_1 + k_2e^x$

TYPE B: Real and Equal (Double) Roots

If $b^2 - 4ac = 0, m_1 = m_2 = m$, and the roots are real and equal, then the general solution is

$y = e^{mx}(k_1 + k_2x)$.....(ix)

That is,

$$\boxed{y = y_c = e^{mx}(k_1 + k_2x)}$$

Example 6: Solve the differential equation: $y'' - 12y' + 36y = 0$

Solution:

Aux. eqn. $m^2 - 12m + 36 = 0 \Rightarrow m = 6(\text{twice})$

$\therefore y = e^{6x}(k_1 + k_2x)$

Example 7: Solve the differential equation: $y''' - 5y'' + 8y' - 4 = 0$

Solution:

Aux. eqn. $m^3 - 5m^2 + 8m - 4 = 0 \Rightarrow (m-2)^2(m-1)$

$\therefore m = 2(twice), m = 1 \Rightarrow m_1 = m_2 = m = 2,\ m_3 = 1$

$\therefore$ the general solution is $y = e^{mx}(k_1 + k_2x) + k_3e^{m_3x}$

$\therefore y = e^{2x}(k_1 + k_2x) + k_3e^x$

TYPE C: Complex Conjugate Roots

If $b^2 - 4ac < 0$, and $m = a \pm jb \Rightarrow m_1 = a + jb$, and $m_2 = a - jb$, then the general solution is

$y = e^{ax}(k_1cosbx + k_2sinbx)$.....(x)

That is,

$$\boxed{y = y_c = e^{ax}(k_1cosbx + k_2sinbx)}$$

Proof of TYPE C:

Suppose the auxiliary equation has roots of the form: $m = a \pm jb$ where

$m_1 = a + jb$, and $m_2 = a - jb$, then from CASE I, the solution would be

$y = y_c = Ae^{m_1x} + Be^{m_2x} = Ae^{(a+jb)x} + Be^{(a-jb)x}$

$y = Ae^{ax}.e^{jbx} + Be^{ax}.e^{-jbx} = e^{ax}(Ae^{jbx} + Be^{-jbx})$ .....(xi)

From the Euler formula, we know that $e^{jx} = cosx + jsinx$ and $e^{-jx} = cosx - jsinx$.

Putting these values in eqn. (xi) we obtain:

$y = e^{ax}[A(cosbx + jsinbx) + B(cosbx - jsinbx)]$

$y = e^{ax}[Acosbx + Ajsinbx + Bcosbx - Bjsinbx] = e^{ax}[(A+B)cosbx + (A-B)jsinbx$

If we let $k_1 = (A+B)$ and $k_2 = j(A-B)$, then the solution becomes:

$y = e^{ax}(k_1cosbx + k_2sinbx)$, where k_1 and k_2 are arbitrary constants.

From the foregoing, it means that if $m = -5 \pm j6$, then the solution is $y = e^{-5x}(k_1cos6x + k_2sin6x)$ and if $m = 3 \pm j5$, then the solution is $y = e^{3x}(k_1cos5x + k_2sin5x)$. Now let us consider a few examples:

Example 8: Solve the differential equation: $2y'' + 4y' + 3y = 0$

Solution:

Aux. eqn: $2m^2 + 4m + 3 = 0$

$m = -1 \pm j\frac{\sqrt{2}}{2} \equiv a \pm jb \Rightarrow a = -1,\ b = \frac{\sqrt{2}}{2}$

$\therefore$ the solution is $y = e^{ax}(k_1cosbx + k_2sinbx) = e^{-x}(k_1cos\frac{\sqrt{2}}{2}x + k_2\sin\frac{\sqrt{2}}{2}x)$

Example 9: Solve the differential equation: $y'' + y = 0$

Solution:

$m^2 + 1 = 0 \Rightarrow m = \pm\sqrt{-1}$

$m = 0 \pm j \equiv a \pm jb \Rightarrow a = 0,\ b = 1$

$\therefore y = e^{ax}(k_1cosbx + k_2sinbx) = e^0(k_1cosx + k_2sinx) = k_1cosx + k_2sinx$

Example 10: Solve the differential equation: $y'' - 6y' + 12y = 0$

Solution:

Aux. eqn: $m^2 - 6m + 12 = 0 \Rightarrow m = 3 \pm j\sqrt{3} \Rightarrow a = 3,\ b = \sqrt{3}$

$\therefore y = e^{ax}(k_1cosbx + k_2sinbx) = e^{3x}(k_1cos\sqrt{3}x + k_2sin\sqrt{3}x)$

Summary of the Types:

For Homogeneous Second Order Differential Equations of the form: $ay'' + by' + cy = 0$, where Auxiliary Eqn. is $am^2 + bm + c = 0$, we have the following three types:

(1) Real & Different Roots ($m_1 \neq m_2$): $y = k_1e^{m_1x} + k_2e^{m_2x}$
(2) Real & Equal Roots ($m_1 = m_2$): $y = e^{mx}(k_1 + k_2x)$
(3) Complex Roots ($m = a \pm jb$): $y = e^{ax}(k_1cosbx + k_2sinbx)$

Remember that in each of the three types above, we simply solve the Auxiliary Equation to establish the values of m and substitute in the appropriate form of the result.

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EXERCISE 4.2

Solve the following differential equations:

7. $y'' - 5y' + 6y = 0$
8. $2y'' - 3y' - 2y = 0$
9. $3y'' - 2y' + y = 0$
10. $y'' - 3y'' + 2y' = 0$
11. $y''' + 3y' = 0$
12. $2y'' + 3y' + 2y = 0$
13. $y'' + 9y' + 16y = 0$
14. $5y'' + 4y' + y = 0$
15. $4y'' - 4y' + y = 0$
16. $y'' + 25y = 0$
17. $y''' - 2y' = 0$
18. $y''' + 4y'' + 4y' = 0$

4.4 NON-HOMOGENEOUS SECOND-ORDER O.D.E WITH CONSTANT COEFFICIENTS.

In general, a non-homogeneous second-order linear differential equation can be expressed in the form of eqn. (10.1) as: $y'' + f(x)y' + g(x)y = h(x)$.

Theorem:

Now, consider the non-homogeneous second order eqn:

$y'' + a(x)y' + b(x) = h(x)$ .....(4.3)

Eqn. (10.3) consists of two parts – the **homogeneous part** & the **non-homogeneous part.** The homogeneous part is obtained when $h(x) = 0$, thus we have:

$y'' + a(x)y' + b(x) = 0$ .....(4.4)

Whereas the non-homogeneous part is obtained when $h(x) \neq 0$. The solution obtained by solving the homogeneous part is called the **Complementary Function (C.F)**, while that obtained from the non-homogeneous part is known as the **Particular Integral (P.I).** Hence, the complete **General Solution (G.S)** of the linear non-homogeneous differential equation is the sum of the solution of the corresponding homogeneous equation and the particular integral. Thus, **General Solution = complementary function + particular integral**

If y_T denotes the complete general solution, y_c denotes the complementary function, and y_p denotes the particular integral, then $y_T = y_c + y_p$ .....(4.5)

Thus, to determine the general solution of a non-homogeneous second-order linear differential equation, we first of all solve the corresponding homogeneous equation, employing the methods described in the previous sections. Thereafter, we solve to obtain the particular integral.

In this section, our main focus will be how to find the particular integral for non-homogeneous linear differential equations. We shall consider several methods of determining the particular integral.

4.5 SOLUTIONS FOR NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS

In this book, we shall consider two known methods of solution. One of which is the method of using determined coefficients, also known as the method of "undetermined coefficients", the second method is

known as the "Differential Operator method" also known as D-Operator method. The third method, known as the use of variation of parameters, is beyond the scope of this book. We shall consider the first method under this chapter. The Differential Operator shall be discussed extensively in a separate chapter.

METHOD OF UNDETERMINED COEFFICIENTS

The function $h(x)$ on the right-hand side could be one of the following kinds– Polynomial, Exponential or Trigonometric function. This will lead us to the following cases of solution.

CASE 1 –Application to the Polynomial Function

If $h(x) = x$, then we let the P.I to be $y_p = ax + b$. If $h(x) = x^2$, $y_p = ax^2 + bx + c$. If $h(x) = x^3$, $y_p = ax^3 + bx^2 + cx + d$, and so on.

Example 11: Solve the differential equation: $y'' + y = x^2 + x$

Solution:

$y'' + y = x^2 + x$ .....(i)

General Solution, $y_T = y_c + y_p$ where $y_c = complementary\ function$, $y_p = particular\ integral$.

LHS→ Aux. eqn: $m^2 + 1 = 0 \Rightarrow m = \pm\sqrt{-1} = \pm j = 0 \pm j$

$\therefore\ y_c = k_1 cosx + k_2 sinx$

RHS→ P.I: $y_p = ax^2 + bx + c$ .....(ii)

Where $a, b, and\ c$ are undetermined coefficients

$y'_p = 2ax + b$ .....(iii)

$y''_p = 2a$ .....(iv)

Substituting $(ii), (iii), (iv)\ in\ (i), gives$:

$ax^2 + bx + 2a + c \equiv x^2 + x$

Equating coefficients of powers of x, we have:

$a = 1,\ \ b = 1,\ \ c = -2$

$\therefore\ y_p = x^2 + x - 2$

Hence, $y_T = y_c + y_p = k_1 cosx + k_2 sinx + x^2 + x - 2$

Example 12: Solve the differential equation: $y'' + 6y = x^3 + 2x$

Solution:

$y'' + 6y = x^3 + 2x$ .....(i)

LHS → Aux. eqn: $m^2 + 6 = 0 \Rightarrow m = \pm\sqrt{-6} = \pm j\sqrt{6} = 0 \pm j\sqrt{6}$

$\therefore\ y_c = k_1 cos\sqrt{6}x + k_2 sin\sqrt{6}x$

RHS →P.I: $y_p = ax^3 + bx^2 + cx + d$.....$(ii)$

Where $a, b, c, and\ d$ are undetermiend coefficients.

$y'_p = 3ax^2 + 2bx + c$ .....(iii)

$y''_p = 6ax + 2b$ .....(iv)

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Substituting *eqns* $(ii), (iii), (iv)$ *in eqn* (i), we have:

$6ax + 2b + 6(ax^3 + bx^2 + cx + d) \equiv x^3 + 2x$

$6ax^3 + 6bx^2 + (6a + 6c)x + (2b + 6d) \equiv x^3 + 2x$

Equating coefficients of powers of x, we obtain:

$a = \frac{1}{6},\ b = 0,\ c = \frac{1}{6},\ d = 0$

$\therefore\ y_p = ax^3 + bx^2 + cx + d = \frac{x^3}{6} + \frac{x}{6}$

Hence, $y_T = y_c + y_p = k_1 cos\sqrt{6}\,x + k_2 sin\sqrt{6}x + \frac{x^3}{6} + \frac{x}{6}$

$y_T = k_1 cos\sqrt{6}\,x + k_2 sin\sqrt{6}x + \frac{1}{6}(x^3 + x)$

Example 13: Solve the differential equation: $y'' + 4y = 2x^3 + 4x^2$

Solution:

$y'' + 4y = 2x^3 + 4x^2$ .....(i)

LHS → Aux. eqn: $m^2 + 4 = 0 \Rightarrow m = \pm\sqrt{-4} = \pm j2 = 0 \pm j2$

$y_c = k_1 cos2x + k_2 sin2x$

RHS →P.I: $y_p = ax^3 + bx^2 + cx + d$.....$(ii)$

$y'_p = 3ax^2 + 2bx + c$ .....(iii)

$y''_p = 6ax + 2b$ .....(iv)

Substituting *eqns* $(ii), (iii), (iv)$ *in eqn* (i), we have:

$6ax + 2b + 4(ax^3 + bx^2 + cx + d) \equiv 2x^3 + 4x^2$

$4ax^3 + 4bx^2 + (6a + 4c)x + (2b + 4d) \equiv 2x^3 + 4x^2$

Equating coefficients of powers of x, we obtain:

$a = \frac{1}{2},\ b = 1,\ c = -\frac{3}{4},\ d = -\frac{1}{2}$

$\therefore\ y_p = ax^3 + bx^2 + cx + d = \frac{x^3}{2} + x^2 - \frac{3}{4}x - \frac{1}{2}$

Hence, $y_T = y_c + y_p = k_1 cos2x + k_2 sin2x + \frac{x^3}{2} + x^2 - \frac{3}{4}x - \frac{1}{2}$

CASE 2 –Application to the Exponential Function

Generally, if $h(x) = e^{px}$, then we assume $y_p = Ce^{px}$ for the particular integral (P.I). However, we should note that the form of solution with the exponential functions on the RHS is usually determined by the roots of the auxiliary equations. This further gives rise to the following three categories of solutions:

Type 2.1: ***If*** $h(x) = e^{px}$, **and none of the roots from the auxiliary equation is equal to p, then we let**

$y_p = Ce^{px}$.

Example 14: Solve the differential equation: $y'' + y' - 6y = 28e^{4x}$

Solution:

$y'' + y' - 6y = 28e^{4x}$ .....(i)

LHS → Aux. eqn: $m^2 + m - 6 = 0 \Rightarrow m = 2\ or - 3$

$\therefore\ y_c = k_1 e^{2x} + k_2 e^{-3x}$

RHS →P.I: $y_p = Ce^{4x}$, since none of the roots from the **aux. eqn** is equal to p (i.e. 4).

$y_p = Ce^{4x}$ .....(ii)

$y'_p = 4Ce^{4x}$ .....(iii)

$y''_p = 16Ce^{4x}$ .....(iv)

Substituting these values in eqn (i), we have:

$16Ce^{4x} + 4Ce^{4x} - 6Ce^{4x} = 28e^{4x}$

$14Ce^{4x} = 28e^{4x} \Rightarrow C = 2$

$\therefore\ y_p = 2e^{4x}$

Hence, $y_T = y_c + y_p = k_1 e^{2x} + k_2 e^{-3x} + 2e^{4x}$

Type 2.2: ***If*** $h(x) = e^{px}$, **and one of the roots from the auxiliary equation equals p, then we let** $y_p = Cxe^{px}$, **that is** $m_1 \neq m_2 = p$

Example 15: Solve the differential equation: $y'' - 4y' + 3y = 2e^{3x}$

Solution:

$y'' - 4y' + 3y = 2e^{3x}$ .....(i)

LHS → Aux. eqn: $m^2 - 4m + 3 = 0 \Rightarrow m = 1\ or\ 3$

$\therefore y_c = k_1 e^{x} + k_2 e^{3x}$

RHS →P.I: $y_p = Cxe^{3x}$, since one of the roots from the **aux. eqn** equals p (i.e. 3)

$y_p = Cxe^{3x}$ .....(ii)

$y^1{}_p = 3Cxe^{3x} + Ce^{3x}$ .....(iii)

$y^{11}{}_p = 9Cxe^{3x} + 6Ce^{3x}$ .....(iv)

Substituting these values in eqn (i), we obtain:

$9Cxe^{3x} + 6Ce^{3x} - 4(3Cxe^{3x} + Ce^{3x}) + 3Cxe^{3x} \equiv 2e^{3x}$

$2Ce^{3x} \equiv 2e^{3x} \Rightarrow C = 1$

$\therefore\ y_p = Cxe^{3x} = xe^{3x}$

Hence, $y_T = y_c + y_p = k_1 e^{x} + k_2 e^{3x} + xe^{3x}$

Type 2.3: ***If*** $h(x) = e^{px}$, **and the two roots from the auxiliary equation are equal to p, then we let** $y_p = Cx^2 e^{px}$, **that is** $m_1 = m_2 = p$.

Example 16: Solve the differential equation: $y'' - 6y' + 9y = 2e^{3x}$

Solution:

$y'' - 6y' + 9y = 2e^{3x}$ .....(i)

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LHS → Aux. eqn: $m^2 - 6m + 9 = 0 \Rightarrow m = 3(twice)$

$\therefore\ y_c = e^{3x}(k_1 + k_2x)$

RHS →P.I: $y_p = Cx^2e^{3x}$, since both roots are equal to p

$y_p = Cx^2e^{3x}$ .....(ii)

$y'_p = 3Cx^2e^{3x} + 2cxe^{3x}$ .....(iii)

$y'' = 6Cxe^{3x} + 9Cx^2e^{3x} + 6Cxe^{3x} + 2Ce^{3x}$ .....(iv)

Substituting these values in eqn (i), we obtain:

$6Cxe^{3x} + 9Cx^2e^{3x} + 6Cxe^{3x} + 2Ce^{3x} - 6(3Cx^2e^{3x} + 2cxe^{3x}) + 9(Cx^2e^{3x}) \equiv 2e^{3x}$

$2Ce^{3x} \equiv 2e^{3x} \Rightarrow C = 1$

$\therefore\ y_p = Cx^2e^{3x} = x^2e^{3x}$

$y_T = y_c + y_p = e^{3x}(k_1 + k_2x) + x^2e^{3x}$

The above solutions obtained are known as the **general solution** (G.S). The **particular solution** (P.S) is obtained, if boundary values are given. With boundary values, the values of k_1 and k_2 can be obtained. It is instructive to note that the values of k_1 and k_2 can only be obtained from the complete general solution and not from the C.F as soon as you obtain it. This is a common error, so do not be caught by it! Ensure that you obtain the complete general solution before substituting for the values of k_1 and k_2. A few examples would explain!

Example 17: Use the method of undetermined coefficients to obtain the solution of the differential equation: $y'' + 4y' + 3y = e^{-3x}; y(0) = \frac{1}{2}, y'(0) = -2$

Solution:

$y'' + 4y' + 3y = e^{-3x}$ .....(i)

Aux. eqn: $m^2 + 4m + 3 = 0 \Rightarrow m = -1\ or - 3$

$\therefore\ y_c = k_1e^{-x} + k_2e^{-3x}$

$y_p = Cxe^{-3x}$ .....(ii)

$y'_p = -3Cxe^{-3x} + Ce^{-3x}$ .....(iii)

$y''_p = 9Cxe^{-3x} - 6Ce^{-3x}$ .....(iv)

Substituting eqns (ii), (iii), (iv) in (i), we have:

$9Cxe^{-3x} - 6Ce^{-3x} - 12Cxe^{-3x} + 4Ce^{-3x} + 3Cxe^{-3x} \equiv e^{-3x}$

$-2Ce^{-3x} \equiv e^{-3x}$

$\therefore\ C = -\frac{1}{2} \Rightarrow y_p = -\frac{1}{2}xe^{-3x}$

Hence, the complete general solution is

$y_T = y_c + y_p = k_1e^{-x} + k_2e^{-3x} - \frac{1}{2}xe^{-3x}$ .....(v)

When $x = 0, y = \frac{1}{2}$ and $k_1 + k_2 = \frac{1}{2}$ .....(vi)

From (v), $y' = -k_1e^{-x} - 3k_2e^{-3x} - \frac{1}{2}[-3xe^{-3x} + e^{-3x}]$

That is, $y' = -k_1e^{-x} - 3k_2e^{-3x} + \frac{3}{2}xe^{-3x} - \frac{1}{2}e^{-3x}$ .....(vii)

When $x = 0, y' = -2$ and $-k_1 - 3k_2 = -2$ .....(viii)

Solving eqns (vi) & (viii) simultaneously gives $k_1 = 0$ and $k_2 = \frac{1}{2}$

Hence, the particular solution is $y = \frac{1}{2}e^{-3x} - \frac{1}{2}xe^{-3x} = \frac{1}{2}e^{-3x}(1 - x)$

Good! When the values of k_1 and k_2 are known, the resulting equation is known as the **particular solution.** In the above example, the particular solution is $y = \frac{1}{2}e^{-3x}(1 - x)$, whereas the general solution is $y = k_1e^{-x} + k_2e^{-3x} - \frac{1}{2}xe^{-3x}$

Example 18: Use the method of undetermined coefficients to obtain the solution of the differential equation: $y'' + 4y' + 5y = 13e^{3x}; y(0) = \frac{5}{2}$, and $y^1(0) = \frac{1}{2}$

Solution:

$y'' + 4y' + 5y = 13e^{3x}$ .....(i)

Aux eqn: $m^2 + 4m + 5 = 0 \Rightarrow m = -2 \pm j$

$y_c = e^{-2x}(k_1cosx + k_2sinx)$

$y_p = Ce^{3x}$ .....(ii)

$y'_p = 3Ce^{3x}$ .....(iii)

$y''_p = 9Ce^{3x}$ .....(iv)

Putting eqns (ii), (iii) & (iv) in eqn (i) gives $C = \frac{1}{2}$

$\therefore\ y_p = \frac{1}{2}e^{3x}$

$\therefore\ y_T = y_c + y_p = e^{-2x}(k_1cosx + k_2sinx) + \frac{1}{2}e^{3x}$, where k_1 and k_2 are undetermined coefficients.

Taking boundary conditions, when $x = 0, y = \frac{5}{2}$, *so that* $k_1 = 2$

Also when $x = 0, y' = \frac{1}{2}$, so that $k_2 = 3$

$\therefore\ y = e^{-2x}(2cosx + 3sinx) + \frac{1}{2}e^{3x}$

CASE 3 –Application to Trigonometric Functions

In this case, $h(x)$ is of the form **sinx** or **cosx** or sines or cosines of multiple angles or other forms. So if $h(x) = asinx\ or\ acosx$, assume the particular integral to be: $\boldsymbol{y_p = C_1cosx + C_2sinx}$

Example 19: Solve the differential equation: $y'' + 3y' + 2y = 20sinx$

Solution: $y'' + 3y' + 2y = 20sinx$ .....(i)

Aux eqn: $m^2 + 3m + 2 = 0 \Rightarrow m = -1\ or - 2$

$\therefore\ y_c = k_1e^{-x} + k_2e^{-2x}$

$y_p = C_1cosx + C_2sinx$ .....(ii)

$y'_p = -C_1sinx + C_2cosx$ .....(iii)

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$y''_p = -C_1 cosx - C_2 sinx$ .....(iv)

Putting eqns (ii), (iii) & (iv) in eqn (i), we have:

$-C_1 cosx - C_2 sinx - 3C_1 sinx + 3C_2 cosx + 2C_1 cosx + 2C_2 sinx = 20sinx$

$(C_1 + 3C_2)cosx + (C_2 - 3C_1)sinx \equiv 20sinx$

Comparing identities & coefficients, we obtain:

$C_1 + 3C_2 = 0$ .....(v)

$C_2 - 3C_1 = 20$ .....(vi)

Solving eqns (v) & (vi) gives $C_1 = -6$ & $C_2 = 2$

$\therefore y_p = -6cosx + 2sinx$

Hence the complete general solution is $y_T = k_1 e^{-x} + k_2 e^{-2x} - 6cosx + 2sinx$

Note: The above method of solution is generally true for all cases of h(x) = $sinx$ or $cosx$ with just one unique exception. That is, when **h(x) = sinpx or cospx such that m = ±jp or m = ±p is the root of the auxiliary eqn.** Examples include $y'' \pm y = sinx$, $y'' \pm 9y = sin3x, y'' \pm 4y = sin2x, etc.$

In such cases, we assume the **particular integral** to be $y_p = xc_1 cospx + xc_2 sinpx$

So, generally, if $h(x) = sinpx$ or $cospx$ such that $m = \pm jp$ or $m = \pm p$, then $y_p = xc_1 cospx + xc_2 sinpx$

Example 20: Solve the differential equation: $y'' + 9y = 5sin3x$

Solution:

$y'' + 9y = 5sin3x$ .....(i)

Aux eqn: $m^2 + 9 = 0 \Rightarrow m = \pm j3$

$\therefore y_c = e^{ax}(k_1 cosbx + k_2 sinbx) = k_1 cos3x + k_2 sin3x$

$y_p = xc_1 cos3x + xc_2 sin3x$ .....(ii)

$y'_p = -3xc_1 sin3x + c_1 cos3x + 3xc_2 cos3x + c_2 sin3x$.....(iii)

$y''_p = -9xc_1 cos3x - 6c_1 sin3x - 9xc_2 sin3x + 6c_2 cos3x$.....(iv)

Putting eqns (ii) & (iv) in eqn (i), we obtain:

$-9xc_1 cos3x - 6c_1 sin3x - 9xc_2 sin3x + 6c_2 cos3x + 9xc_1 cos3x + 9xc_2 sin3x \equiv 5sin3x$

$6c_2 cos3x - 6c_1 sin3x \equiv 5sin3x$

Comparing coefficients and identities, we have:

$\therefore c_2 = 0$, and $c_1 = -\frac{5}{6}$

$\therefore y_p = -\frac{5}{6} xcos3x$

$\therefore y_T = y_c + y_p = k_1 cos3x + k_2 sin3x - \frac{5}{6} xcos3x$

EXERCISE 4.3

Using the method of Undetermined Coefficients, obtain the solutions of the differential equations:

19. $y'' - 6y' + 9y = 2e^{2x}$ — **Ans:** $y = e^{3x}(k_1 + k_2 x) + 2e^{2x}$
20. $y'' + 9y = 17e^{-5x}$ — **Ans:** $y = k_1 cos3x + k_2 sin3x + \frac{1}{2} e^{-5x}$

21. $y'' - 4y = 2cos3x$ — **Ans:** $y = k_1 e^{2x} + k_2 e^{-2x} - \frac{2}{13} cos3x$
22. $y'' + 9y = 5sin2x$ — **Ans:** $y = k_1 cos3x + k_2 sin3x + sin2x$
23. $y'' + y = 4sinx$ — **Ans:** $y = k_1 cosx + k_2 sinx - 2xcosx$
24. $y'' - 4y' + 3y = x + e^{2x}$ — **Ans:** $y = k_1 e^{3x} + k_2 e^{x} + \frac{1}{9}(3x + 4) - e^{2x}$
25. $y'' - 9y = e^{3x} + sin3x$ — **Ans:** $y = k_1 e^{3x} + k_2 e^{-3x} + \frac{1}{6} xe^{3x} - \frac{1}{18} sin3x$
26. $y'' + 3y' + 2y = 3sinx; y(0) = -0.9$, and $y^1(0) = -0.7$ — **Ans:** $e^{-2x} - e^{-x} + \frac{3}{10}(sinx - 3cosx)$
27. $y'' + 4y' + 4y = 2cos^2 x$ — **Ans:** $e^{-2x}(k_1 + k_2 x) + \frac{1}{4} + \frac{1}{8} sin2x$
28. $y'' - y' - 2y = 8$
29. $y'' - 4y = 10e^{3x}$
30. $y'' + 2y' + y = e^{-2x}$
31. $y'' + 25y = 5x^2 + x$
32. $y'' - 2y^1 + y = 4sinx$
33. $y'' + 4y' + 5y = 2e^{-2x}; y(0) = 1$ and $y'(0) = -2$
34. $3y'' - 2y' - y = 2x - 3$
35. $y'' - 6y' + 8y = 8e^{4x}$
36. $y'' + 4y = 2cosx - sinx; y(0) = 1, y'(0) = -1$
37. $y'' + 6y = secx$
38. $y'' + 4y' + 4y = e^{-x} \ln x$
39. $y'' + 4y = cosecx$
40. $y'' + 4y = x$
41. $x^2 y'' - 2y = 2x^4$
42. $2y'' - 7y' - 4y = e^{3x}$
43. $y'' - 6y' + 9y = 54x + 18$
44. $y'' + 2y' + y = 4sinhx$
45. $y'' + y' - 2y = 2coshx$
46. $y'' - 2y' + 3y = x^2 - 1$
47. $y'' + 6y' + 10y = 50x$
48. $y'' - 5y' + 4y = \frac{1}{x^3} e^{-x}$
49. $y'' - 4y' + 2y = \frac{1}{x^2} e^{3x}$
50. $y'' - 4y = x^2$
51. $y'' + 4y = sin2x$
52. $y'' + 2y' + y = cosx$
53. $y'' + 9y = sec3x$
54. $y'' + 3y = e^{-x} sinx$
55. $y'' - 2y = sin3x$
56. $y'' + 3y' - 6y = e^{x} cosx$
57. $y'' - 2y' + 4y = x^2 - 2x;\ y(0) = 3, y'(0) = 1$
58. $y'' - y' + 3y = 2e^{3x};\ y(0) = 1, y'(0) = -2$
59. $y'' + y = x + x^2;\ y(0) = 2, y'(0) = 1$

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Chapter 5: Differential Operator

5.1 DEFINITIONS – An operator is a transformation of one function to another, so that the characteristics of the newly transformed function could be used to solve problems, which are difficult with the characteristics of the original function. The D-Operator method is used to obtain the particular integral (P.I) of a non-homogeneous equation, and can be applied with greater ease than the method of undetermined coefficients. It is particularly suitable for problems where h(x) is either a polynomial, a trigonometric or an exponential function.

Let $D = \frac{d}{dx}$ then $\frac{dy}{dx} \equiv Dy$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}.\frac{d}{dx}.y = D^2y$

$$\boxed{\frac{d}{dx}y = Dy = y', \qquad \frac{d^2y}{dx^2} = D^2y = y'', \qquad \frac{d^3y}{dx^3} = D^3y = y''', \qquad \frac{d^ny}{dx^n} = D^ny}$$

Hence, $\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$ can be written as $D^2y + aDy + by = 0$ or $(D^2 + aD + b)y = 0$

From the foregoing, if $y = cosx$, then $Dy = D(cosx) = -sinx$ and if $y = e^{5x}$, then $Dy = 5e^{5x}$

Also, if $y = 3x^2 + 5x - 12$, then $Dy = 6x + 5$, and so on! Naturally, all the rules of differentiation still hold good.

For instance, if $y = \frac{cos10x}{2x+1}$, then $Dy = \frac{(2x+1)(-10sin10x)-2cos10x}{(2x+1)^2}$

The above is the quotient rule, which is the same with the normal differentiation. The only difference is the use of the symbol "D" to represent the same operators. A special advantage of this single letter "D" as an operator is the ease with which it can be manipulated algebraically. For example,

Example 1: Evaluate (D+6)(sin3x)

Solution:

D(sin3x) + 6sin3x = 3cos3x + 6sin3x **ans.**

Example 2: Evaluate $(D + 2)^2(cosx)$

Solution:

$(D^2 + 4D + 4)(cosx) = D^2(cosx) + 4D(cosx) + 4cosx = -\cos x - 4\sin x + 4\cos x$ **ans.**

Now, try this one. When you are done, check your answer with the one in the box.

Example 3: Evaluate $(D^2 - 3D + 4)(e^{-3x} + 2\sin 4x)$

$$\boxed{22e^{-3x} - 24\sin 4x - 24\cos 4x}$$

Because, $D^2(e^{-3x} + 2\sin 4x) = 9e^{-3x} - 32\sin 4x$

$3D(e^{-3x} + 2\sin 4x) = -9e^{-3x} + 24\cos 4x$

$4(e^{-3x} + 2\sin 4x) = 4e^{-3x} + 8\sin 4x$

$\therefore (D^2 - 3D + 4)(e^{-3x} + 2\sin 4x) = 9e^{-3x} - 32\sin 4x + 9e^{-3x} - 24\cos 4x + 4e^{-3x} + 8\sin 4x$

$= 22e^{-3x} - 24\sin 4x - 24\cos 4x$ **ans.**

the D-operator method. In using the D-operator method, the C.F is found from the auxiliary equation and the solution is the same as discussed in the preceding chapter.

Remember that in finding the P.I, knowledge of the various theorems already discussed will be of paramount importance! The reader, therefore, is expected to have mastered the theorems properly. Let us illustrate with examples.

Example 22: Solve the equation: $y'' - 3y' + 2y = e^{3x}$, using the D-operator method.

Solution:

Aux eqn: $m^2 - 3m + 2 = 0 \Rightarrow m = 2 \text{ or } 1$

C.F: $y_c = k_1e^x + k_2e^{2x}$

For the P.I, we write the equation in terms of the D-operator.

$(D^2 - 3D + 2)y_p = e^{3x} \Rightarrow y_p = \frac{1}{D^2-3D+2}e^{3x}$. Now applying theorem 2, we obtain:

$y_p = \frac{1}{3^2-3(3)+2}e^{3x} = \frac{e^{3x}}{2}$

$\therefore y_T = y_c + y_p = k_1e^x + k_2e^{2x} + \frac{e^{3x}}{2}$ **ans.**

Example 23: Solve the differential equation by operator-D method: $y'' + y' - 6y = 28e^{4x}$

Solution:

Aux eqn: $m^2 + m - 6 = 0 \Rightarrow m = 2 \text{ or } -3$

$y_c = k_1e^{2x} + k_2e^{-3x}$

$y_p = \frac{1}{D^2+D-6}28e^{4x} = \frac{1}{4^2+4-6}28e^{4x} = \frac{28e^{4x}}{14} = 2e^{4x}$

$\therefore y_T = y_c + y_p = k_1e^{2x} + k_2e^{-3x} + 2e^{4x}$ **ans.**

Example 24: Solve the differential equation by operator-D method: $y'' - 6y' + 9y = 2e^{2x}$

Solution:

Aux eqn: $m^2 - 6m + 9 = 0 \Rightarrow m = 3(twice)$

$y_c = e^{3x}(k_1 + k_2x)$

$y_p = \frac{1}{D^2-6D+9}2e^{2x} = \frac{1}{2^2-6(2)+9}2e^{2x} = 2e^{2x}$

$\therefore y_T = e^{3x}(k_1 + k_2x) + 2e^{2x}$ **ans.**

Example 25: Use the D-operator method to solve the equation: $y'' - 5y' + 6y = e^{3x}$

Solution:

Aux eqn: $m^2 - 5m + 6 = 0 \Rightarrow m = 3 \text{ or } 2$

$y_c = k_1e^{2x} + k_2e^{3x}$

$y_p = \frac{1}{D^2-5D+6}e^{3x} = \frac{1}{3^2-5(3)+6}e^{3x} = \frac{e^{3x}}{0}$. Theorem 2 fails. We apply theorem 3. That is $\frac{1}{F(D)}e^{ax} = x.\frac{1}{f^1(a)}e^{ax}$.

$y_p = x\frac{1}{2D-5}e^{3x} = x\frac{1}{2(3)-5}e^{3x} = xe^{3x}$

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$\therefore y_T = k_1e^{2x} + k_2e^{3x} + xe^{3x}$ **ans.**

Example 26: Use the D-operator method to solve the equation: $y'' - 6y' + 9y = 2e^{3x}$

Solution:

Aux eqn: $m^2 - 6m + 9 = 0 \Rightarrow m = 3(twice)$

$y_c = e^{3x}(k_1 + k_2x)$

$y_p = \frac{1}{D^2-6D+9}2e^{3x} = \frac{1}{3^2-6(3)+9}2e^{3x} = \frac{2e^{3x}}{0}$. Theorem 2 fails. We apply theorem 3. Thus

$\frac{1}{F(D)}e^{ax} = x.\frac{1}{f^1(a)}e^{ax}$

$y_p = x\frac{1}{2D-6}2e^{3x} = x\frac{1}{2(3)-6}2e^{3x} = x\frac{2e^{3x}}{0}$ Theorem fails. We apply it again.

$y_p = x^2\frac{1}{2}2e^{3x} = x^2e^{3x}$

$\therefore y_T = e^{3x}(k_1 + k_2x) + x^2e^{3x}$ **ans.**

Example 27: Use the D-operator method to solve the equation: $y'' + 3y' + 2y = 20\sin x$

Solution:

Aux eqn: $m^2 + 3m + 2 = 0 \Rightarrow m = -2\ or - 1$

$y_c = k_1e^{-x} + k_2e^{-2x}$

$y_p = \frac{1}{D^2+3D+2}20\sin x$. Now, applying theorem 7 – replacing D^2 by $-a^2$,

$y_p = \frac{1}{D^2+3D+2}20\sin x = \frac{1}{-1^2+3D+2}20\sin x = \frac{20\sin x}{3D+1}$

Multiplying top & bottom of the function by $3D - 1$, we have

$y_p = \frac{(3D-1)(20\sin x)}{(3D+1)(3D-1)} = \frac{(3D-1)(20\sin x)}{9D^2-1}$ Applying theorem 7 again to the D^2 at the denominator,

$y_p = \frac{(3D-1)(20\sin x)}{-9-1} = \frac{(3D-1)(20\sin x)}{-10} = 2(1-3D)\sin x = (2-6D)\sin x$

$\therefore y_p = (2-6D)\sin x = 2\sin x - 6D\sin x = 2\sin x - 6\cos x$

$\therefore y_T = k_1e^{-x} + k_2e^{-2x} + 2\sin x - 6\cos x$ **ans.**

Example 28: Use the D-operator method to solve the equation: $y'' - 16y = 3\cos 5x$

Solution:

Aux eqn: $m^2 - 16 = 0 \Rightarrow m = \pm 4$

$y_c = k_1e^{4x} + k_2e^{-4x}$

$y_p = \frac{1}{D^2-16}3\cos 5x = \frac{1}{-5^2-16}3\cos 5x = \frac{3\cos 5x}{-41} = \frac{3\cos 5x}{-41}$

$\therefore y_T = k_1e^{4x} + k_2e^{-4x} - \frac{3\cos 5x}{41}$ **ans.**

Example 29: Solve the differential equation by D-operator method: $D^2y + 2Dy + 2y = \cos 2x$

Solution:

Aux eqn: $m^2 + 2m + 2 = 0 \Rightarrow m = -1 \pm i$

$y_c = e^{ax}(k_1cosbx + k_2sinbx) = e^{-x}(k_1cosx + k_2sinx)$

$(D^2 + 2D + 2)y = \cos 2x \Rightarrow y_p = \frac{1}{D^2+2D+2}\cos 2x$ Applying theorem 7 to D^2, we obtain

$y_p = \frac{1}{-2^2+2D+2}\cos 2x = \frac{1}{2D-2}\cos 2x$

Multiplying top & bottom of the function by $2D + 2$ gives:

$y_p = \frac{(2D+2)\cos 2x}{(2D+2)(2D-2)} = \frac{(2D+2)\cos 2x}{4D^2-4}$ Applying theorem 7 to D^2 again, we have

$y_p = \frac{(2D+2)\cos 2x}{4(-4)-4} = -\frac{1}{20}(2D+2)\cos 2x = -\frac{1}{20}[2D(\cos 2x) + 2\cos 2x]$

$y_p = -\frac{1}{20}(-4\sin 2x + 2\cos 2x) = \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$

$\therefore y_T = e^{-x}(k_1cosx + k_2sinx) + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$ **ans.**

Example 30:

Solve the differential equation by D-operator method: $y'' + 9y = 5\sin 3x$

Solution:

Aux eqn: $m^2 + 9 = 0 \Rightarrow m = \pm j3$

$y_c = k_1cos3x + k_2sin3x$

$y_p = \frac{1}{D^2+9}5\sin 3x$. Applying theorem 7, we obtain:

$y_p = \frac{1}{-3^2+9}5\sin 3x = \frac{5\sin 3x}{0}$. Theorem 7 fails, we apply theorem 8. That is

$\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax$

$y_p = -\frac{x}{2a}5\cos ax = -\frac{x}{2(3)}5\cos 3x = -\frac{5}{6}x\cos 3x$

$\therefore y_T = k_1cos3x + k_2sin3x - \frac{5}{6}x\cos 3x$ **ans.**

5.5 FURTHER PROBLEMS ON D-OPERATOR METHOD

Example 31: Solve the differential equation: $y'' + y' - 2y = 14 + 2x - 2x^2$

Solution:

Aux eqn: $m^2 + m - 2 = 0 \Rightarrow m = 1\ or - 2$

$y_c = k_1e^{x} + k_2e^{-2x}$

$y_p = \frac{1}{D^2+D-2}(14 + 2x - 2x^2) = (D^2 + D - 2)^{-1}(14 + 2x - 2x^2)$

$= 2^{-1}\left[\frac{D^2}{2} + \frac{D}{2} - 1\right]^{-1}(14 + 2x - 2x^2) = -2^{-1}\left[-\frac{D^2}{2} - \frac{D}{2} + 1\right]^{-1}(14 + 2x - 2x^2)$

$y_p = -\frac{1}{2}\left[1 - \left(\frac{D}{2} + \frac{D^2}{2}\right)\right]^{-1}(14 + 2x - 2x^2)$

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Expanding binomially, we have:

$$y_p = -\frac{1}{2}\left[1 + \frac{D}{2} + \frac{D^2}{2} + \frac{-1(-2)}{2!}\left(\frac{D}{2} + \frac{D^2}{2}\right)^2 + \cdots\right](14 + 2x - 2x^2)$$

$$y_p = -\frac{1}{2}\left[1 + \frac{D}{2} + \frac{D^2}{2} + \frac{D^2}{4} + \frac{2D^3}{4} + \frac{D^4}{4}\right](14 + 2x - 2x^2)$$

Since the highest power of x is 2, expansion should stop at D^2 to correspond with x^2

$$y_p = -\frac{1}{2}\left(1 + \frac{D}{2} + \frac{3D^2}{4}\right)(14 + 2x - 2x^2) = -\frac{1}{2}\left[14 + 2x - 2x^2 + \frac{1}{2}(2 - 4x) + \frac{3}{4}(-4)\right]$$

$$y_p = -\frac{1}{2}(14 + 2x - 2x^2 + 1 - 2x - 3) = x^2 - 6$$

$\therefore y_T = k_1e^x + k_2e^{-2x} + x^2 - 6$ **ans.**

Example 32: Solve the differential equation: $(D-2)^2y = 6xe^{2x}$

Aux eqn: $(m-2)^2 = 0 \Rightarrow m = 2(twice)$

$y_c = e^{2x}(k_1 + k_2x)$

$$y_p = \frac{1}{(D-2)^2}6xe^{2x} = \frac{1}{D^2-4D+4}6xe^{2x}$$

Applying theorem 5(shift theorem), we have

$$y_p = e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}6x = e^{2x}\frac{1}{D^2}6x = e^{2x}\iint 6xdx = e^{2x}x^3$$

$\therefore y_T = e^{2x}(k_1 + k_2x) + e^{2x}x^3$ **ans.**

Example 33: Solve the differential equation: $D^2y + 4Dy + 4y = x^3e^{2x}$

Solution:

Aux eqn: $m^2 + 4m + 4 = 0 \Rightarrow m = -2(twice)$

$y_c = e^{-2x}(k_1 + k_2x)$

$y_p = \frac{1}{D^2+4D+4}[x^3e^{2x}]$. Applying the shift theorem, we have

$$y_p = e^{2x}\frac{1}{(D+2)^2+4(D+2)+4}[x^3] = e^{2x}\frac{1}{D^2+8D+16}[x^3] = e^{2x}\frac{1}{(D+4)^2}[x^3]$$

Expanding binomially, we obtain:

$$y_p = e^{2x}\left[4(\frac{D}{4} + \frac{4}{4})\right]^{-2}(x^3) = e^{2x}4^{-2}\left[1 + \frac{D}{4}\right]^{-2}(x^3) = \frac{e^{2x}}{16}\left[1 + \frac{D}{4}\right]^{-2}(x^3)$$

$$y_p = \frac{e^{2x}}{16}\left[1 + -2\left(\frac{D}{4}\right) + \frac{-2(-3)\left(\frac{D}{4}\right)^2}{2!} + \frac{-2(-3)(-4)\left(\frac{D}{4}\right)^3}{3!}\right](x^3)$$

$$y_p = \frac{e^{2x}}{16}\left[1 - \frac{D}{2} + \frac{3}{16}D^2 - \frac{1}{16}D^3\right](x^3)$$

$$y_p = \frac{e^{2x}}{16}\left[x^3 - \frac{1}{2}D(x^3) + \frac{3}{16}D^2(x^3) - \frac{1}{16}D^3(x^3)\right] = \frac{e^{2x}}{16}\left(x^3 - \frac{3}{2}x^2 + \frac{9}{8}x - \frac{3}{8}\right)$$

$\therefore y_T = e^{-2x}(k_1 + k_2x) + \frac{e^{2x}}{16}\left(x^3 - \frac{3}{2}x^2 + \frac{9}{8}x - \frac{3}{8}\right)$ **ans.**

EQUATIONS OF THE FORM: $y'' + a^2y = \sin ax$, $y'' + a^2y = \cos ax$

Consider equations of the form: $y'' + ay = \sin ax$ or $y'' + ay = \cos ax$

Let $y'' + ay = \sin bx$ .....(i)

$y'' + ay = \cos bx$ .....(ii)

From eqn (i), $y_p = \frac{1}{D^2+a^2}[\sin bx] = \frac{1}{-b^2+a^2}[\sin bx] = \frac{\sin bx}{a^2-b^2}$; where $b \neq a$

Similarly, from eqn (ii), $y_p = \frac{1}{D^2+a^2}[\cos bx] = \frac{1}{-b^2+a^2}[\cos bx] = \frac{\cos bx}{a^2-b^2}$; where $b \neq a$

However, when $a = b$, we can no longer apply the above cases. And this occurs when we have differential equations of the form $y'' + a^2y = \sin ax$, $y'' + a^2y = \cos ax$

Let $y'' + a^2y = \sin ax$ .....(iii)

$y'' + a^2y = \cos ax$ .....(iv)

From the eqns. above, we have

$$y_p = \frac{1}{D^2+a^2}[\sin ax], \quad y_p = \frac{1}{D^2+a^2}[\cos ax]$$

Let us device a technique for evaluating $y_p = \frac{1}{D^2+a^2}[\sin ax]$ and $y_p = \frac{1}{D^2+a^2}[\cos ax]$

Recall that $e^{jax} = \cos ax + j\sin ax$ .....(v)

If $h(x) = \cos ax$, the inverse operator method will be applied on the **real** part of e^{jax} and if $h(x) = \sin ax$, it will be applied on the **imaginary** part of e^{jax}. So we have:

$\cos ax = R[e^{jax}]$ .....(vi)

Hence, $y_p = \frac{1}{D^2+a^2}[\cos ax] = \frac{1}{D^2+a^2}R[e^{jax}] = R\frac{1}{(ja)^2+a^2}e^{jax} = R\frac{1}{-a^2+a^2}e^{jax}$ (fails)

We introduce a factor **I** and apply the shift theorem.

$$y_p = R\frac{1}{D^2+a^2}e^{jax}I = Re^{jax}\frac{1}{(D+ja)^2+a^2}I = Re^{jax}\frac{1}{D^2+j2aD-a^2+a^2}I \text{ .....}(vi)$$

If we put $I = e^{0x}$ in eqn (vi), we obtain:

$$y_p = Re^{jax}\frac{1}{D^2+j2aD}e^{0x} = Re^{jax}\frac{1}{D}\cdot\frac{1}{D+j2a}e^{0x} = Re^{jax}\frac{1}{D}\cdot\left[\frac{1}{j2a}\right] = Re^{jax}\int\frac{1}{j2a}dx = Re^{jax}\frac{x}{j2a}$$

$y_p = Re^{jax}\frac{x}{j2a}$ .....(vii)

Substituting eqn (v) in eqn (vii), we obtain:

$$y_p = R\frac{x}{j2a}[\cos ax + j\sin ax] = \frac{R}{j2a}[x\cos ax + jx\sin ax] = \frac{R}{2a}\left[\frac{x\cos ax}{j} + x\sin ax\right]$$

Since $j = \sqrt{-1}$, $\frac{1}{j} = \frac{1}{\sqrt{-1}} = \frac{1}{\sqrt{-1}} \times \frac{\sqrt{-1}}{\sqrt{-1}} = -\sqrt{-1} = -j$

$y_p = \frac{R}{2a}[x\sin ax - jx\cos ax]$ .....$(viii)$

Since, R represents the **real** part of e^{jax}, we shall finally have:

$y_p = \frac{x\sin ax}{2a}$ .....(ix)

Hence, eqn (ix) is the P.I when $h(x) = \cos ax$

Similarly, if $h(x) = \sin ax$, the imaginary part I_m, will be considered. And we have:

$$y_p = \frac{1}{D^2+a^2}[\sin ax]$$

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So from eqn $(viii)$, we deduce that:

$y_p = \frac{I_m}{2a}[x\sin ax - jx\cos ax]$ .....(x)

$y_p = -\frac{x\cos ax}{2a}$ .....(xi)

Hence, eqn (xi) is the P.I when $h(x) = \sin ax$

Example 34: Solve the differential equation: $D^2y + 16y = 4\cos 4x$

Solution:

Aux eqn: $m^2 + 16 = 0 \Rightarrow m = \pm 4i$

$y_c = k_1\cos 4x + k_2\sin 4x$

$y_p = \frac{1}{D^2 + 4^2}[4\cos 4x]$

$y_p = \frac{x\sin ax}{2a} = \frac{4x\sin 4x}{2\times 4} = \frac{x\sin 4x}{2}$

$\therefore y_T = k_1\cos 4x + k_2\sin 4x + \frac{x\sin 4x}{2}$

Example 35: Solve the differential equation: $D^2y + 25y = \sin 5x$

Solution:

Aux eqn: $m^2 + 25 = 0 \Rightarrow m = \pm 5i$

$y_c = k_1\cos 5x + k_2\sin 5x$

$y_p = \frac{1}{D^2 + 5^2}[\sin 5x]$

$y_p = x.\frac{1}{2D}[\sin 5x] = \frac{x}{2}\frac{1}{D}[\sin 5x] = \frac{x}{2}\int \sin 5x dx = \frac{x}{2}\left(-\frac{\cos 5x}{5}\right) = -\frac{x\cos 5x}{10}$

Or simply use: eqn (xi), that is

$y_p = -\frac{x\cos ax}{2a} = -\frac{x\cos 5x}{2\times 5} = -\frac{x\cos 5x}{10}$

$\therefore y_T = k_1\cos 5x + k_2\sin 5x - \frac{x\cos 5x}{10}$

5.6 SUMMARY OF THE RULES (THEOREMS) TO FINDING THE P.I

1. $F(D)e^{ax} = F(a)e^{ax}$
2. $\frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}$, if $F(a) \neq 0$
3. Where $F(a) = 0$, then $\frac{1}{F(D)}e^{ax} = x\frac{1}{F^1(a)}e^{ax}$, where $F'(a) \neq 0$
4. Where $F'(a) = 0$, *then* $\frac{1}{F(D)}e^{ax} = x^2\frac{1}{F''(a)}e^{ax}$, where $F''(a) \neq 0$ *and so on!*
5. $\frac{1}{F(D)}x^n = [F(D)]^{-1}x^n$ and expand $[F(D)]^{-1}$ binomially and then operate.
6. $F(D)e^{ax}F(x) = e^{ax}F(D + a)F(x)$
7. $\frac{1}{F(D)}e^{ax}F(x) = e^{ax}\frac{1}{F(D+a)}F(x)$
8. $\frac{1}{D+a}F(x) = e^{-ax}\int e^{ax}F(x)dx$
9. $F(D)^2\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} = F(-a^2)\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix}$

10. $\frac{1}{f(D^2)}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} = \frac{1}{f(-a^2)}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix}$ *if* $f(-a^2) \neq 0$
11. *If* $f(-a^2) = 0$, *then* $\frac{1}{f(D^2)}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix} = x\frac{1}{f'(-a^2)}\begin{Bmatrix}\sin ax\\ \cos ax\end{Bmatrix}$ *where* $f'(-a^2) \neq 0$
12. $\frac{1}{D^2 + a^2}[\sin ax] = -\frac{x}{2a}\cos ax$
13. $\frac{1}{D^2 + a^2}[\cos ax] = \frac{x}{2a}\sin ax$
14. $\frac{1}{(D^2 + a^2)^n}[\sin ax] = \frac{x^n}{(2a)^n n!}\sin(ax - \frac{1}{2}n\pi)$
15. $\frac{1}{(D^2 + a^2)^n}[\cos ax] = \frac{x^n}{(2a)^n n!}\cos(ax - \frac{1}{2}n\pi)$

EXERCISE 5

Obtain the solutions of the following differential equations by D-operator method:

1. $y'' + 5y' + 6y = 2\cos 5x$ — **Ans:** $y = k_1e^{-3x} + k_2e^{-2x} + \frac{1}{986}(50\sin 5x - 38\cos 5x)$
2. $y'' + y = x^2 + x$ — **Ans:** $y = k_1\cos x + k_2\sin x + x^2 + x - 2$
3. $y'' - y = \cos x$ — **Ans:** $y = k_1e^x + k_2e^{-x} - \frac{1}{2}\cos x$
4. $y'' + 6y = x^3 + 2x$ — **Ans:** $y = k_1\cos\sqrt{6x} + k_2\sin\sqrt{6x} + \frac{1}{6}(x^3 + 5x)$
5. $3y'' + 4y' + 10y = 8e^{4x}$ — **Ans:** $e^{\frac{-2}{3}x}\left(k_1\cos\frac{\sqrt{114}}{6}x + k_2\sin\frac{\sqrt{114}}{6}x\right) + \frac{8}{74}e^{4x}$
6. $y'' + y' = e^x$ — **Ans:** $y = k_1\cos x + k_2\sin x + \frac{1}{5}e^x$
7. $y'' - 3y^1 - 4y = \cos 2x$ — **Ans:** $y = k_1e^{4x} + k_2e^{-x} - \frac{2}{25}\cos 2x - \frac{3}{50}\sin 2x$
8. $y''' - 4y = 2\cos 3x$ — **Ans:** $y = k_1e^{2x} + k_2e^{-2x} - \frac{2}{13}\cos 3x$
9. $y'' + 4y = e^{-x}\sin x$ — **Ans:** $y = k_1\cos 2x + k_2\sin 2x + \frac{e^{-x}}{5}(\sin x + \frac{1}{2}\cos x)$
10. $y'' + 9y = e^x\cos 3x$
11. $y'' + 8y = e^{-2x}\cos 3x$
12. $y'' + 6y = \sin 4x$ — **Ans:** $k_1\cos\sqrt{6}x + k_2\sin\sqrt{6}x - \frac{1}{10}\sin 4x$
13. $(D^2 + 1)y = 2\cos^2 x$

    Ans: $y = k_1e^{-x} + e^{\frac{1}{2}x}(k_2\cos\frac{\sqrt{3}}{2}x + k_3\sin\frac{\sqrt{3}}{2}x) + 1 + \frac{1}{65}(-8\sin 2x + \cos 2x)$
14. $y'' + y = \sin 3x\cos 2x$ — **Ans:** $y = k_1\cos x + k_2\sin x + \frac{1}{48}(-\sin 5x - 12x\cos x)$
15. $y'' + 4y = x^2 + \sin 2x$ — **Ans:** $y = k_1\cos 2x + k_2\sin 2x + \frac{1}{4}(x^2 - x\cos 2x - \frac{1}{2})$
16. $y'' + 4y = e^x + \sin 2x$ — **Ans:** $y = k_1\cos 2x + k_2\sin 2x + \frac{1}{5}e^x - \frac{x}{4}\cos 2x$
17. $y'' - 2y' - 3y = 2e^{2x} + 10\sin 3x$; $y(0) = 2, y'(0) = 4$

    Ans: $y = \frac{29}{12}e^{3x} - \frac{1}{12}e^{-x} - \frac{2}{3}e^{2x} + \frac{1}{3}(\cos 3x - 2\sin 3x)$
18. $y'' - 4y' + 3y = e^{2x} + \cos x$ — **Ans:** $y = k_1e^x + k_2e^{3x} - e^{2x} + \frac{1}{10}(-2\sin x + \cos x)$
19. $(D^2 - 4D + 4)y = 8(e^{2x} + x^2 + \sin 2x)$ **Ans:** $y = (k_1 + k_2x)e^{2x} + 4x^2e^{2x} + 2x^2 + 4x + 3 + \cos 2x$
20. $y'' + 3y' + 2y = \sin 2x$
21. $(D^2 - 5D + 6)y = e^x\sin x$ — **Ans:** $y = k_1e^{2x} + k_2e^{3x} + \frac{e^x}{10}(3\cos x + \sin x)$
22. $(D^2 - 7D + 10)y = e^{2x}\sin x$ — **Ans:** $y = k_1e^{2x} + k_2e^{5x} + \frac{e^{2x}}{10}(3\cos x - \sin x)$

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23. $(D^2 - 4D + 13)y = e^{2x}\cos 3x + (x^2 + x + 9)$

Ans: $y = e^{2x}(k_1\cos 3x + k_2\sin 3x) + \frac{xe^{2x}\sin 3x}{6} + \frac{1}{13}(x^2 + 21 + \frac{x}{13} + \frac{1579}{169})$

24. $(D^2 - 4D + 3)y = 2xe^{3x} + 3e^{3x}\cos 2x$

$y = k_1e^{x} + k_2e^{3x} + \frac{1}{2}e^{3x}(x^2 - x) + \frac{3}{8}e^{3x}(\sin 2x - \cos 2x)$

25. $(D^2 - 4)y = x^2e^{3x}$ Ans: $y = k_1e^{2x} + k_2e^{-2x} + \frac{e^{3x}}{5}(x^2 - \frac{12x}{5} + \frac{62}{25})$

26. $(D^2 - 4D + 1)y = e^{2x}\sin 2x$

27. $(D^2 - 3D + 2)y = 2x^2e^{4x} + 5e^{3x}$ Ans: $y = k_1e^{x} + k_2e^{2x} + \frac{e^{4x}}{54}(18x^2 - 30x + 19) + \frac{5}{2}e^{3x}$

28. $(D^2 + 3D + 2)y = e^{4x}$ Ans: $y = k_1e^{-x} + k_2e^{-2x} + \frac{e^{4x}}{4}$

29. $y'' + 4y' + 3y = \cos 3x$ Ans: $y = k_1e^{-x} + k_2e^{-3x} - \frac{1}{30}(\cos 3x - 2\sin 3x)$

30. $y'' + y = 3e^{x} + 5e^{2x}$ Ans: $y = k_1\cos x + k_2\sin x + \frac{3}{2}e^{x} + e^{2x}$

31. $y'' + 25y = \sin 5x$ Ans: $y = k_1\cos 5x + k_2\sin 5x - \frac{x\cos 5x}{10}$

32. $y'' + 4y' + 4y = 5e^{2x}$ Ans: $y = e^{-2x}(k_1 + k_2x) + 5e^{-3x}$

33. $y'' - 4y' + 5y = \sin 4x$ Ans: $y = e^{2x}(k_1\cos x + k_2\sin x) + \frac{1}{377}(16\cos 4x - 11\sin 4x)$

34. $y'' + 4y' + 5y = 2e^{-2x}; y(0) = 1,\ y'(0) = -2$ Ans: $y = e^{-2x}(2 - \cos x)$

35. $3y'' - 2y' - y = 2x - 3$ Ans: $y = k_1e^{x} + k_2e^{-\frac{x}{3}} - 2x + 7$

36. $D^2y + 3Dy + 2y = xe^{-x}$ Ans: $y = k_1e^{-x} + k_2e^{-2x} - e^{-x}(\frac{x^2}{2} - x)$

37. $D^2y + 2Dy - 3y = 4e^{-3x}$ Ans: $y = k_1e^{x} + k_2e^{-3x} - xe^{-3x}$

38. $D^2y + 4Dy + 4y = \cosh 2x$ Ans: $y = e^{-2x}(k_1 + k_2x) + \frac{1}{32}e^{2x} + \frac{x^2e^{-2x}}{4}$

39. $D^2y + 6Dy + 9y = e^{-3x}\cosh 3x$ Ans: $y = e^{-3x}(k_1 + k_2x) + \frac{1}{18}(1 + e^{-6x})$

40. $D^2y - Dy - 6y = xe^{3x}$ Ans: $y = k_1e^{3x} + k_2e^{-2x} + \frac{e^{3x}}{50}(5x^2 - 2x)$

41. $D^2y + 4Dy + 5y = 8\cos^2 x$ Ans: $y = e^{-2x}(k_1\cos x + k_2\sin x) + \frac{4}{5} + \frac{4}{65}(8\sin 2x + \cos 2x)$

42. $D^2y + 2Dy + 5y = 34\sin x\cos x$ Ans: $y = e^{-x}(k_1\cos 2x + k_2\sin 2x + \sin 2x - 4\cos 2x$

43. $2D^2y + Dy - y = e^{x}\sin 2x$ Ans: $y = k_1e^{\frac{x}{2}} + k_2e^{-x} - \frac{e^{x}}{68}(3\sin 2x\ 5\cos 2x)$

44. $D^2y + 2Dy + 5y = x + e^{-x}\cos 3x$ Ans: $y = e^{-x}(k_1\cos 2x + k_2\sin 2x) + \frac{x}{5} - \frac{2}{25} - \frac{e^{-x}\cos 3x}{5}$

45. $D^2y - 2Dy + 4y = e^{x}\sin 3x$ Ans: $y = e^{x}(k_1\cos\sqrt{3}x + k_2\sin\sqrt{3}x) - \frac{e^{x}\sin 3x}{6}$

46. $D^2y - 9y = \cosh 3x + x^2$ Ans: $y = e^{3x}\left(k_1 + \frac{x}{12}\right) + e^{-3x}\left(k_2 - \frac{x}{12}\right) - \frac{1}{81}(9x^2 + 2)$

47. $D^2y - 4Dy + 4y = e^{2x}$ Ans: $y = e^{2x}(k_1 + k_2x + \frac{x^2}{2})$

48. $D^2y + 3Dy + 2y = e^{-x}\cos x$ Ans: $y = e^{-x}\left(k_1 + \frac{1}{2}\sin x - \frac{1}{2}\cos x\right) + k_2e^{-2x}$

49. $D^2y + 2Dy + 2y = x^2e^{-x}$ Ans: $y = e^{-x}(k_1\cos x + k_2\sin x + x^2 - 2)$

50. $y'' + y' - 6y = 6x + 3x^2 - 6x^3$ Ans: $y = k_1e^{-3x} + k_2e^{2x} + x^3$

51. $y'' - 2y' + 2y = 2x^2$ Ans: $y = e^{x}(k_1\cos x + k_2\sin x) + x^2 + 2x + 1$

52. $16y'' - 8y' + 4y = 8x - 4$ Ans: $y = e^{\frac{x}{4}}(k_1\cos\frac{\sqrt{3}}{4}x + k_2\sin\frac{\sqrt{3}}{4}x) + 2x + 3$

53. $2y'' - 6y' + 9y = e^{-3x}$ Ans: $y = e^{\frac{3x}{2}}(k_1\cos\frac{3}{2}x + k_2\sin\frac{3}{2}x) + \frac{1}{45}e^{-3x}$

54. $y'' + y = 15\sin x$ Ans: $y = \frac{15}{4}\sin x - \frac{15}{2}x\cos x + k_1\cos x + k_2\sin x$

55. $y'' + 4y' + 3y = \sin x + 2\cos x$ Ans: $y = k_1e^{-3x} + k_2e^{x} + \frac{1}{2}\sin x$

56. $y'' + 4y' + 3y = 4\cos x + 2\sin x$ Ans: $y = k_1e^{-3x} + k_2e^{x} + \sin x$

57. $y'' - 2y' + y = e^{x}\sin x$ Ans: $y = e^{x}(k_1 + k_2x - 2x\cos x - \sin x)$

58. $(D^2 + 4)y = 3x\sin x$ Ans: $y = k_1\cos 2x + k_2\sin 2x + x\sin x - \frac{2}{3}\cos x$

59. $y'' - y = x\sin 3x + \cos x$ Ans: $y = k_1e^{x} + k_2e^{-x} - \frac{1}{10}(\frac{3}{5}\cos 3x + x\sin 3x + 5\cos x)$

60. $y'' - y = x\sin x + e^{x} + x^2e^{x}$ Ans: $y = k_1e^{x} + k_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{x}{12}e^{x}(2x^2 - 3x + 9)$

61. $(D^2 - 4D + 4)y = 8x^2e^{2x}\sin 2x$ Ans: $y = e^{2x}(k_1 + k_2x + 3\sin 2x - 4x\cos 2x)$

$(D^2 - 2)^2 - (4D - 2)$

$$y = \frac{e^{2x}\sin 2x}{D^2 - 4D + 1}$$

$$= \frac{e^{(2+2i)x}}{D^2 - 4D + 1} \quad \text{Therem (2)}$$

$$= \frac{e^{(2+2i)x}}{-4i - 7}$$

$f(D) = D^2 - 4D + 1$

$f(2+2i) = (2+2i)(2+2i) - 4(2+2i) + 1$

$= \cancel{4} + 4i - \cancel{4} - \cancel{8} - 8i + 1$

$= (-4i - 7)$

$$= \frac{e^{(2+2i)x}}{-4i - 7} \times (-4i + 7)$$

$$= \frac{e^{(2+2i)x}(-4i + 7)}{-16 - \cancel{27}\ 49}$$

$$= \frac{e^{(2+2i)x}(-4i + 7)}{33}$$

$$= e^{2x}e^{2ix}\frac{(-4i + 7)}{33}$$

$$= e^{2x}\left[\frac{(\cos 2x + i\sin 2x)(-4i + 7)}{33}\right]$$

$$= e^{2x}\left[\frac{-4i\cos 2x + 7\cos 2x + 4\sin 2x - 7i\sin 2x}{33}\right]$$

$$= e^{2x}\left(\frac{-4\cos 2x - 7\sin 2x}{33}\right)$$

$$= e^{2x}\left(-\frac{(4\cos 2x + 7\sin 2x)}{33}\right)$$

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Chapter 6: Series Solution of O.D.E

6.1 INTRODUCTION: In chapter three, we discussed Power Series expansion of functions, such as $sin^2x, \cos x, sin^{-1}x,\ sec^2x,\ etc.$, where both Maclaurin's and Taylor's theorems were employed and demonstrated extensively. The reader may want to revisit that particular chapter for thorough "refreshment" and acquaintance, even though this chapter ultimately stands on its own for understanding!

We recall that if a given function can be represented by a power series of x, that series must be of the form of Maclaurin's series. Thus, we have:

$$F(x) = F(0) + \frac{F'(0)x}{1!} + \frac{F''(0)x^2}{2!} + \frac{F'''(0)x^3}{3!} + \frac{F^{iv}(0)x^4}{4!} + \cdots + \frac{F^{(n-1)}x^{(n-1)}}{(n-1)!} + \cdots$$

The above is a mathematical statement for Maclaurin's theorem or simply Maclaurin's series.

Similarly, if a given function can be represented by a power series in $(x-m)$, that series must be of the form of Taylor's series. Thus, we have:

$$F(x) = F(m) + \frac{F'(m)(x-m)}{1!} + \frac{F''(m)(x-m)^2}{2!} + \frac{F'''(m)(x-m)^3}{3!} + \cdots$$

We present below some standard series of some common functions:

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
3. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$
4. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$
5. $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$
6. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
7. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$
8. $tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$
9. $\ln x = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4 + \cdots$

We note that the Maclaurin's series is the Taylor's expansion of F(x) at $x = m = 0$. Most power series representation of certain elementary functions are from the Maclaurin's series.

Example 1: Find the series representation of the following functions:

$(i)\ y = e^{x^2}$ $(ii)\ y = \cos 3x$ $(iii)\ y = \sin x^3$ $(iv)\ y = tan^{-1}x$

Solution:

$(i)\ y = e^{x^2}$

But $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Replacing x by x^2, we have:

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots$$

$$\therefore e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + \frac{x^{2n}}{n!} \quad \textbf{ans.}$$

$(ii)\ y = \cos 3x$

But $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Replacing x with 3x, i.e. $x \equiv 3x$, we have:

$$\cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots$$

$$= 1 - \frac{9x^2}{2!} + \frac{81x^4}{4!} - \frac{729x^6}{6!} + \cdots$$

$$\therefore \cos 3x = 1 - \frac{9x^2}{2} + \frac{27x^4}{8} - \frac{81x^6}{80} + \cdots \quad \textbf{ans.}$$

$(iii)\ y = \sin x^3$

But $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

Replacing x with x^3, *we have:*

$$\sin x^3 = x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} - \frac{(x^3)^7}{7!} + \frac{(x^3)^9}{9!} -$$

$$\therefore \sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \frac{x^{27}}{9!} \quad \textbf{ans.}$$

$(iv)\ y = tan^{-1}x$

$$\tan^{-1}x = \int \frac{1}{1+x^2}dx$$

Now if we take $f(x) = \frac{1}{1+x^2} \rightarrow f(0) = 1$

$$f'(x) = \frac{-2x}{(1+x^2)^2} \rightarrow f'(0) = 0$$

$$f''(x) = \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} \rightarrow f''(0) = -2$$

$$f'''(x) = \frac{24x(1+x^2) - 48x^3}{(1+x^2)^4} \rightarrow f'''(0) = 0$$

$$f^{iv}(x) = \frac{24(1+x^2)^2 - 288x^2(1+x^2) + 384x^4}{(1+x^2)^5} \rightarrow f^{iv}(0) = 24$$

Substituting in the Maclaurin's series, we obtain:

$$F(x) = F(0) + \frac{F'(0)x}{1!} + \frac{F''(0)x^2}{2!} + \frac{F'''(0)x^3}{3!} + \frac{F^{iv}(0)x^4}{4!}$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

Hence, $\tan^{-1}x$ can be evaluated by term by term integration of the series:

$$\therefore \tan^{-1}x = \int(1 - x^2 + x^4 - x^6 + \cdots) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \textbf{ans.}$$

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EXERCISE 6.1

1. Find the Taylor's series expansion of the following functions:

(i) $\sin 4x$ (ii) e^{-x^2} (iii) $\cos 2x^4$ (iv) $\sin x^2$ (v) $\sinh x^2$ (vi) $\ln x^2$ (vii) $\tanh 4x$

2. Find the Maclaurin's series expansion of the following functions:

(i) $\ln(1+x)$ (ii) $\tan x$ (iii) $\sin x^2$ (iv) $\frac{\cos x}{1+x}$ (v) $\frac{x}{e^x+1}$ (vi) $\frac{x}{e^x-1}$ (vii) $\sin^{-1}x$

6. 2 POWER SERIES SOLUTION OF FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

The major task at hand is to show that solutions of differential equations obtained by power series method yield similar results with those obtained by algebraic means or through calculus, as shown in our previous chapters.

Definition:

A power series in powers of $(x - m)$ is an infinite series of the form:

$$y = \sum_{n=0}^{\infty} a_n (x-m)^n$$

Expanding, we write:

$$y = a_0 + a_1(x-m) + a_2(x-m)^2 + a_3(x-m)^3 + \cdots + a_n(x-m)^n + \cdots$$

Where $a_0, a_1, a_2 \ldots a_n$ are constants, m is another constant known as the center and x is the independent variable. If $m = 0$, then the series is centered at the origin, and we have:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

Example 2: Solve the differential equation by power series method: $y' - y = 0$

Solution:

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$ .....(i)

$y^1 = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$ .....(ii)

Putting these in the original eqn, we obtain:

$$y' - y = a_1 + 2a_2 x + 3a_3 x^2 + \cdots - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = 0$$

Collecting like terms:

$$a_1 - a_0 = 0 \rightarrow \therefore a_1 = a_0$$

For terms in x, we have:

$$2a_2 x - a_1 x = 0 \rightarrow 2a_2 x = a_1 x \Rightarrow \therefore a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$

For terms in x^2, we have:

$$3a_3 x^2 - a_2 x^2 = 0 \rightarrow 3a_3 x^2 = a_2 x^2 \Rightarrow \therefore a_3 = \frac{a_2}{3} = \frac{a_0}{3 \times 2}$$

For terms in x^3, we have:

$$4a_4 x^3 - a_3 x^3 = 0 \rightarrow 4a_4 x^3 = a_3 x^3 \Rightarrow \therefore a_4 = \frac{a_3}{4} = \frac{a_0}{4 \times 3 \times 2}$$

Substituting the values of $a_1, a_2, a_3,$ etc in eqn (i), we obtain:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3 \times 2}x^3 + \frac{a_0}{4 \times 3 \times 2}x^4 + \cdots$$

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) \quad \textbf{ans:}$$

The above expression is easily recognizable as the exponential series of e^x. So we write:

$\therefore y = a_0 e^x$ **ans.**

Check By Calculus Method:

As a check, we can use the calculus method to verify the answer obtained from the power series method.

$$y' - y = 0$$

$$\int \frac{dy}{y} = \int dx \Rightarrow \ln y = x + c \Rightarrow y = e^{x+c}$$

$$y = e^x . e^c \Rightarrow let\ a_0 = e^c$$

$\therefore y = a_0 e^x$ where a_0 is a constant. **ans.**

Example 3: Solve the differential equation: $(1+x)y' + xy = 0$

Solution:

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$ .....(i)

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$ .....(ii)

Putting these in the original equation, we have:

$$(1+x)y' + xy = 0$$

$$= (1+x)[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots] + x[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots] = 0$$

$$a_1 + (2a_2 + a_1 + a_0)x + (3a_3 + 2a_2 + a_1)x^2 + (4a_4 + 3a_3 + a_2)x^3 + (5a_5 + 4a_4 + a_3)x^4 + 6a_6 + 5a_5 + a_4)x^5 = 0$$

Comparing coefficients of terms of x, we obtain:

$$a_1 = 0$$

$$2a_2 + a_1 + a_0 = 0 \rightarrow a_2 = -\frac{a_0}{2}$$

$$3a_3 + 2a_2 + a_1 = 0 \rightarrow a_3 = \frac{a_0}{3}$$

$$4a_4 + 3a_3 + a_2 = 0 \rightarrow a_4 = -\frac{a_0}{8}$$

$$5a_5 + 4a_4 + a_3 = 0 \rightarrow a_5 = \frac{a_0}{30}$$

Substituting these in eqn (i), we obtain:

$$y = a_0 - \frac{a_0}{2}x^2 + \frac{a_0}{3}x^3 - \frac{a_0}{8}x^4 + \frac{a_0}{30}x^5 - \cdots$$

$$\therefore y = a_0\left(1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{8} + \frac{x^5}{30} - \cdots\right) \quad \textbf{ans.}$$

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Check by Calculus Method:

$(1+x)\frac{dy}{dx} = -xy \rightarrow (1+x)dy = -xydx \Rightarrow \frac{dy}{y} = \frac{-xdx}{1+x}$

Integrating both sides, we obtain:

$\ln y = \int \frac{-xdx}{1+x} = \int \left[\frac{1}{1+x} - 1\right] dx = \ln(1+x) - x + c$

$\ln \frac{y}{1+x} = c - x \Rightarrow \frac{y}{1+x} = e^{c-x} = \frac{e^c}{e^x} = e^c e^{-x}$

Let $e^c = a_0$, so that we have: $\frac{y}{1+x} = a_0 e^{-x} \Rightarrow y = a_0 e^{-x}(1+x)$

Recall that $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots$

$y = a_0 e^{-x}(1+x) = a_0\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right)(1+x)$

$y = a_0\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots + x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \frac{x^5}{24} - \cdots\right)$

Collecting like terms and simplifying, we obtain:

$y = a_0\left(1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{8} + \frac{x^5}{30} - \cdots\right)$ **ans.**

What we have done so far is to show that power series offers an alternative method of solving linear differential equations. With this knowledge in mind, the student may extend use of power series method to find solutions of differential equations whose solutions may be difficult with the use of calculus, algebraic or analytical method.

6. 3 HIGHER ORDER DERIVATIVES

If $y = \sin ax$, then $y' = a\cos ax = a\sin\left(ax + \frac{\pi}{2}\right)$, $y'' = -a^2 \sin ax$

$= a^2 \sin(ax + \pi) = a^2 \sin\left(ax + \frac{2\pi}{2}\right)$, $y''' = -a^3 \cos ax = a^3 \sin\left(ax + \frac{3\pi}{2}\right)$, *etc.* In general, $y^n = a^n \sin(ax + \frac{n\pi}{2})$.

Example 4: Find the value of y^6, given that $y = \sin 4x$

Solution:

$y^6 = 4^6 \sin\left(4x + \frac{6\pi}{2}\right) = 4^6 \sin(4x + 3\pi) = 4096(\sin 4x \cos 3\pi + \cos 4x \sin 3\pi)$

$\therefore y^6 = -4096 \sin 4x$

Example 5: Determine the value of y^5 given that $y = \sin 2x$

Solution:

$y^5 = 2^5 \sin\left(2x + \frac{5\pi}{2}\right) = 32(\sin 2x \cos\frac{5\pi}{2} + \cos 2x \sin\frac{5\pi}{2}) = 32\cos 2x$

$\therefore y^5 = 32\cos 2x$ **ans.**

1. If $y = \cos ax$, then $y' = -a\sin ax = a\cos(ax + \frac{\pi}{2})$, $y'' = -a^2 \cos ax$ $y'' = a^2 \cos(ax + \frac{2\pi}{2})$ $, y''' = a^3 \sin ax = a^3 \cos\left(ax + \frac{3\pi}{2}\right)$, *etc.* In general, $y^n = a^n \cos(ax + \frac{n\pi}{2})$

Example 6: If $y = 2\cos 3x$, determine the value of y^5

Solution:

$y^5 = 2 \times 3^5 \cos(3x + \frac{5\pi}{2}) = 486(\cos 3x \cos\frac{5\pi}{2} - \sin 3x \sin\frac{5\pi}{2})$

$\therefore y^5 = -486 \sin 3x$ **ans.**

2. If $y = e^{ax}$, then $y' = ae^{ax}, y'' = a^2 e^{ax}, y''' = a^3 e^{ax}$, *etc.* In general, $y^n = a^n e^{ax}$

Example 7: If $y = 6e^{4x}$, determine the value of y^4

Solution:

$y^4 = 6 \times 4^4 e^{4x} = 1536e^{4x}$

$\therefore y^4 = 1536e^{4x}$ **ans.**

3. If $y = x^a$, then $y' = ax^{a-1}, y'' = a(a-1)x^{a-2}, y''' = a(a-1)(a-2)x^{a-3}$.

In general, $y^n = \frac{a!}{(a-n)!}x^{a-n}$, where "a" is a positive integer.

Example 8: Determine the derivative y^6, if $y = 2x^8$

Solution:

$y^6 = 2 \times \frac{8!}{(8-6)!}x^{8-6} = 2 \times \frac{8!}{2!}x^2 = 40320x^2$ **ans.**

4. If $y = \ln ax$, then $y' = \frac{1}{x}, y'' = -\frac{1}{x^2}, y''' = \frac{2}{x^3}$, *etc.*

In general, $y^n = (-1)^{n-1}\frac{(n-1)!}{x^n}$

Example 9: Determine the value of y^4, if $y = 2\ln 3x$

Solution:

$y^4 = (-1)^{4-1}\frac{(4-1)!}{x^4} = -\frac{3!}{x^4} = -\frac{6}{x^4}$

$\therefore 2\ln 3x = 2 \times -\frac{6}{x^4} = -\frac{12}{x^4}$

6.4 SOLUTION OF SECOND-ORDER DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Our previous discussions on second-order differential equations centered on those with constant coefficients of the form:

$ay'' + by' + cy = 0$ .....(i)

Where the coefficients $a, b,$ and c were independent of x, but were constants. In solving such equations, we employed various techniques like the analytical methods involving the auxiliary equation, algebraic methods, D-operator method, and so on. However, when the coefficients are variables and dependent on x, we must device further methods of solutions.

Consider a second-order differential equation with variable coefficients:

$a(x)y'' + b(x)y' + c(x)y = 0$ .....(ii)

$y'' + p(x)y' + q(x)y = 0$ .....(iii)

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In general, equations in the forms of eqns (ii) & (iii) cannot be solved by the analytic methods already discussed, but in the form of an infinite series of powers of x. Hence, we have presented in this chapter some of methods that can make this possible!

6.5 CAUCHY (OR EULER) METHOD

The general solution to the Cauchy equation is the sum of the homogeneous part and the particular integral. The complementary function is obtained by assuming a solution of the form: $y = x^m$. The solution, however, depends on the nature of the roots. When the roots are **real & distinct**, that is $m_1 \neq m_2$, the general solution is $y = k_1x^{m_1} + k_2x^{m_2} + \cdots + k_nx^{m_n}$. When the roots are **equal**, that is $m_1 = m_2 = m$, the general solution is $y = x^m(k_1 + k_2 \ln x)$. And when the roots are **complex & conjugate**, that is $m = a \pm jb$, the general solution is $y = x^a[k_1 \cos(b \ln x) + k_2 \sin(b \ln x)]$.

Example 10: Solve the differential equation: $x^2y'' - 3xy' + 3y = 0$

Solution:

Let $y = x^m \Rightarrow y' = mx^{m-1}, \Rightarrow y'' = m(m-1)x^{m-2}$

Substituting the values of y, y' & y'' in the original equation, we have:

$x^2\{m(m-1)\}x^{m-2} - 3x(mx^{m-1}) + 3x^m = 0$

$x^2\{m(m-1)\}\frac{x^m}{x^2} - 3xm\frac{x^m}{x} + 3x^m = 0$

$x^m\{m(m-1) - 3m + 3\} = 0$

$x^m \neq 0$ and $m(m-1) - 3m + 3 = 0$

$m^2 - 4m + 3 = 0 \Rightarrow m = 3$ or 1. Hence, the general solution is:

$y = k_1x^{m_1} + k_2x^{m_2} \Rightarrow y = k_1x^3 + k_2x^1$

$\therefore\ y = k_1x^3 + k_2x$ or $y = x(k_1x^2 + k_2)$ **ans.**

Example 11: Solve the Euler equation: $x^2y'' - xy' + y = 0$

Solution:

Let $y = x^m \Rightarrow y' = mx^{m-1}, \Rightarrow y'' = m(m-1)x^{m-2}$

Substituting the values of y, y^1 & y^{11} in the Euler equation, we have:

$x^2\{m(m-1)\}x^{m-2} - x(mx^{m-1}) + x^m = 0$

$m^2 - m - m + 1 = 0 \Rightarrow m^2 - 2m + 1 = 0 \Rightarrow \therefore m = 1(twice)$

Hence, the general solution is:

$y = x^m(k_1 + k_2 \ln x) = x(k_1 + k_2 \ln x)$ **ans.**

Example 12: Solve the differential equation: $x^4y'' - 4x^3y' + 6x^2y = 0$

Solution:

$x^4y'' - 4x^3y' + 6x^2y = 0$ .....(i)

Dividing through by x^2, we obtain the Cauchy equation:

$x^2y'' - 4xy' + 6y = 0$ .....(ii)

Let $y = x^m \Rightarrow y' = mx^{m-1}, \Rightarrow y'' = m(m-1)x^{m-2}$

Substituting the values of y, y' & y'' in the Euler equation, we have:

$x^2\{m(m-1)\}x^{m-2} - 4x(mx^{m-1}) + 6x^m = 0 \Rightarrow m^2 - 5m + 6 = 0 \Rightarrow m = 2$ or 3

$\therefore y = k_1x^{m_1} + k_2x^{m_2} = y = k_1x^2 + k_2x^3$ **ans.**

Example 13: Solve the non-homogeneous Cauchy equation: $x^2y'' + 2xy' - 4y = x^2$

Solution:

Let $y = x^m \Rightarrow y' = mx^{m-1}, \Rightarrow y'' = m(m-1)x^{m-2}$

Substituting the values of y, y' & y'' in the Euler equation, we have:

$x^2\{m(m-1)\}x^{m-2} + 2x(mx^{m-1}) - 4x^m = 0 \Rightarrow m(m-1) + 2m - 4 = 0$

$m^2 + m - 4 = 0 \Rightarrow m = 1.56$ or -2.56

$y_c = k_1x^{1.56} + k_2x^{-2.56}$

Let $y_p = ax^2 + bx + c \Rightarrow y^1 = 2ax + b, \Rightarrow y^{11} = 2a$

Substituting the values in the Cauchy equation, we have:

$x^2(2a) + 2x(2ax + b) - 4(ax^2 + bx + c) = x^2$

$2ax^2 - 2bx - 4c \equiv x^2$

Comparing coefficients and identities, we obtain:

$2a = 1 \Rightarrow a = \frac{1}{2},\ b = 0,\ and\ c = 0$

$\therefore\ y_p = ax^2 + bx + c = \frac{x^2}{2}$

Hence, the general solution is $y_T = y_c + y_p$

$\therefore\ y_T = k_1x^{1.56} + k_2x^{-2.56} + \frac{x^2}{2}$ **ans.**

EXERCISE 6.2

Solve the following Euler (or Cauchy) equations:

3. $x^2y'' + 2xy' - 2y = \ln x^4$ — **Ans:** $y = k_1x + k_2x^{-2} + 4\ln x - 4$
4. $(x+2)^2y'' - (x+2)y' = 2x + 4$ — **Ans:** $y = k_1 + k_2(x-2)^2 - 2(x-2)$
5. $x^2y''' + 2xy' - 2y = x^2 \ln x + 3x$
   Ans: $y = k_1x + x(k_2 \cos \ln x + k_3 \sin \ln x) + 3x \ln x - x^2 + \frac{1}{2}x^2 \ln x$
6. $x^2y'' - 4xy' + 5y = \cos \ln x$
   Ans: $y = k_1x^{1.38} + k_2x^{3.62} + 0.0975 \cos \ln x - 0.122 \sin \ln x$
7. $xy''' + 3y'' + \frac{1}{x}y' = 2$ — **Ans:** $y = k_1 + k_2 \ln x + k_3(\ln x)^2 + \frac{x^2}{4}$
8. $x^3y''' + 3x^2y'' - 2xy' + 2y = 0$ — **Ans:** $y = k_1x + k_2x \ln x + k_3x^{-2}$
9. $xy'' + y' = 0$ — **Ans:** $y = k_1 + k_2 \ln x$
10. $x^2y'' + xy' - y = 0;\ y(1) = 2,\ y'(1) = 0$
11. $x^2y'' = 4xy'$ — **Ans:** $y = k_1 + k_2x^5$

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6.6 POWER SERIES SOLUTION TO SECOND-ORDER O.D.E

Power series solution offers a convenient method of solving second-order differential equations with variable coefficients. That is, equations of the form: $y'' + p(x)y' + q(x)y = 0$, where p and q are functions of x. We shall first consider a number of theorems that will be of assistance.

Leibnitz Theorem: -the nth derivative of a product of two functions.

Let $y = uv$ .....(i), where u and v are functions of x, then by the product rule,

$y' = uv' + vu'$ where $v' = \frac{dv}{dx}$ and $u' = \frac{du}{dx}$ .....(ii)

$y'' = uv'' + v'u' + vu'' + u'v' = u''v + 2u'v' + uv''$ .....(iii)

$y''' = u''v' + vu''' + 2u'v'' + 2v'u'' + uv''' + v''u'$

$= u'''v + 3u''v' + 3u'v'' + uv'''$ .....(iv)

A further stage of differentiation would give:

$y^{(4)} = u^{(4)}v^{(0)} + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + u^{(0)}v^{(4)}$

where $u^{(0)} = u$ and $v^{(0)} = v$, *we have:*

$y^{(4)} = u^{(4)}v + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)}$ .....(v)

From the above equations (i-v), we observe that:

a. The superscript of u decreases regularly by 1 from left to right
b. The superscript of v increases regularly by 1 from left to right
c. The coefficients of 1, 4, 6, 1 are the normal binomial coefficients. Indeed, $(uv)^n$ can be obtained by expanding $(u+v)^n$ using the binomial theorem, where the "powers" are interpreted as derivatives.

So, the expression for the nth derivative can be written as:

$$y^{(n)} = (uv)^n = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \cdots \quad .....(vi)$$

Equation (vi) is a statement of the Leibnitz theorem, which can be used to differentiate a product n times.

> **Note:** For the product y = uv, the function taken as "u" is the one whose n^{th} derivative can readily be obtained, whereas the function taken as "v" is the one whose derivatives steadily reduce to zero during the process of differentiation.

Example 14: Given that $y = x^3e^{4x}$, use the Leibnitz theorem to determine the values of:

(i) y^n (ii) $y^{(5)}$ (iii) $y^{(8)}$

Solution:

$u = e^{4x} \Rightarrow u^1 = 4e^{4x} \Rightarrow u^{11} = 4^2e^{4x} \Rightarrow u^{111} = 4^3e^{4x} \Rightarrow u^n = 4^ne^{4x} \Rightarrow u^{n-1} = 4^{n-1}e^{4x}$

$v = x^3 \Rightarrow v^1 = 3x^2 \Rightarrow v^{11} = 6x \Rightarrow v^{111} = 6 \Rightarrow v^{(4)} = 0$

From the Leibnitz theorem,

$$y^{(n)} = (uv)^n = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \cdots$$

(i) $y^n = 4^ne^{4x}x^3 + n.4^{n-1}e^{4x}.3x^2 + \frac{n(n-1)}{2!}4^{n-2}e^{4x}.6x + \frac{n(n-1)(n-2)}{3!}4^{n-3}e^{4x}.6 + \frac{n(n-1)(n-2)(n-3)}{4!}4^{n-3}e^{4x}.0$

$y^n = 4^ne^{4x}x^3 + \frac{3n4^ne^{4x}x^2}{4} + \frac{3n(n-1)4^ne^{4x}x}{16} + \frac{n(n-1)(n-2)4^ne^{4x}}{64} + 0$

Multiplying through by 64, we obtain:

$64y^n = 4^ne^{4x}\{64x^3 + 48nx^2 + 12n(n-1)x + n(n-1)(n-2)\}$

$\therefore y^n = 4^{n-3}e^{4x}\{64x^3 + 48nx^2 + 12n(n-1)x + n(n-1)(n-2)\}$ **ans.**

(ii) $y^5 = 4^{5-3}e^{4x}\{64x^3 + 48 \times 5x^2 + 12 \times 5(5-1)x + 5(5-1)(5-2)\}$

$y^5 = 16e^{4x}(64x^3 + 240x^2 + 240x + 60)$

$\therefore y^5 = 64e^{4x}(16x^3 + 60x^2 + 60x + 15)$ **ans.**

(iii) $y^8 = 4^{8-3}e^{4x}\{64x^3 + 48 \times 8x^2 + 12 \times 8(8-1)x + 8(8-1)(8-2)\}$

$y^8 = 1024e^{4x}(64x^3 + 384x^2 + 672x + 336)$

$\therefore y^8 = 16384e^{4x}(4x^3 + 24x^2 + 42x + 21)$ **ans.**

Example 15: Use the Leibnitz theorem to determine the value of $y^{(3)}$ if $y = x^4 \ln x$

Solution:

$u = \ln x \Rightarrow u^{(1)} = \frac{1}{x} \Rightarrow u^{(2)} = -\frac{1}{x^2} \Rightarrow u^{(3)} = \frac{2}{x^3} \Rightarrow u^{(4)} = -\frac{6}{x^4}, etc$

$v = x^4 \Rightarrow v^{(1)} = 4x^3 \Rightarrow v^{(2)} = 12x^2 \Rightarrow v^{(3)} = 24x \Rightarrow v^{(4)} = 24 \Rightarrow v^{(5)} = 0$

From the Leibnitz theorem,

$$y^{(n)} = (uv)^n = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \cdots$$

$y^{(3)} = u^{(3)}v + 3u^{(2)}v^{(1)} + \frac{3(2)}{2!}u^{(1)}v^{(2)} + \frac{3(2)(1)}{3!}u^{(0)}v^{(3)} + \cdots$

$y^{(3)} = \frac{2}{x^3}(x^4) + 3\left(-\frac{1}{x^2}\right)(4x^3) + 3\left(\frac{1}{x}\right)(12x^2) + (\ln x)(24x)$

$\therefore y^{(3)} = 26x + 24x\ln x = 2x(13 + 12\ln x)$ **ans.**

Example 16: Use the Leibnitz theorem to obtain the 5th derivative of the function $y = x^4\cos 2x$

Solution:

$u = \cos 2x$, $u^{(1)} = -2\sin 2x \Rightarrow u^{(2)} = -4\cos 2x \Rightarrow u^{(3)} = 8\sin 2x \Rightarrow u^{(4)} = 16\cos 2x$

$u^{(5)} = -32\sin 2x \Rightarrow u^{(6)} = -64\cos 2x$

$v = x^4 \Rightarrow v^{(1)} = 4x^3 \Rightarrow v^{(2)} = 12x^2 \Rightarrow v^{(3)} = 24x \Rightarrow v^{(4)} = 24 \Rightarrow v^{(5)} = 0$

From Leibnitz theorem,

$$y^{(n)} = (uv)^n = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \cdots$$

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$y^5 = u^{(5)}v + 5u^{(4)}v^{(1)} + \frac{5(4)u^{(3)}v^{(2)}}{2!} + \frac{5(4)(3)u^{(2)}v^{(3)}}{3!} + \frac{5(4)(3)(2)u^{(1)}v^{(4)}}{4!} + \frac{5(4)(3)(2)(1)u^{(0)}v^{(5)}}{5!}$

$y^5 = (-32\sin 2x)(x^4) + 5(16\cos 2x)(4x^3) + 10(8\sin 2x)(12x^2) + 10(-4\cos 2x)(24x) + 5(-2\sin 2x)(24) + 0$

$= -32x^4\sin 2x + 320x^3\cos 2x + 960x^2\sin 2x - 960x\cos 2x - 240\sin 2x$

$\therefore y^5 = 32\sin 2x(-x^4 + 30x^2 - 7.5) + 320\cos 2x(x^3 - 3x)$ **ans.**

Example 17: If $2x^2y'' + xy' + 3y = 0$, show that

$2x^2y^{(n+2)} + (4n+1)xy^{(n+1)} + (2n^2 - n + 3)y^n = 0$

Solution:

From Leibnitz theorem,

$y^{(n)} = (uv)^n = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \cdots$

$2x^2y'' = 2x^2y^{(n+2)} + 4xny^{(n+1)} + 2n(n-1)y^n + 0.y^{(n-1)}$ .....(i)

$xy' = xy^{(n+1)} + ny^n + 0.y^{(n-1)}$ .....(ii)

$3y = 3y^n + 0.y^{(n-1)}$ .....(iii)

Combining the three eqns, we have:

$2x^2y^{(n+2)} + 4xny^{(n+1)} + 2n(n-1)y^n + xy^{(n+1)} + ny^n + 3y^n = 0$

$2x^2y^{(n+2)} + (4n+1)xy^{(n+1)} + [2n(n-1) + n + 3]y^n = 0$

$2x^2y^{(n+2)} + (4n+1)xy^{(n+1)} + (2n^2 - n + 3)y^n = 0$ **ans.** Which is what was required to show.

In order to solve a second-order differential equation by power series method, the following steps may be of importance!

1. Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots + a_nx^n$ to be the solution of the differential equation
2. Find y', y'', y''' etc
3. Substitute your y', y'', y''' etc in the given differential equation
4. Calculate a_0, a_1, a_2, etc coefficients of various powers of x by equating coefficients to zero.
5. Substitute the values of a_0, a_1, a_2, etc in the differential equation to obtain the required series solution.

Example 18: Use the power series method to solve the differential equation:

$(x^2 + 2)y'' + xy' - (1+x)y = 0$

Solution:

$(x^2 + 2)y'' + xy' - (1 + x)y = 0$ .....(i)

Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots +$ .....(ii)

$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$ .....(iii)

$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots$ .....(iv)

Substituting eqns (i), (ii), & (iii) in eqn (i), we obtain:

$(x^2 + 2)(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots) +$
$x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots) -$
$(1 + x)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots) = 0$

Expanding each of the expressions in the bracket, we obtain:

$2a_2x^2 + 4a_2 + 6a_3x^3 + 12a_3x + 12a_4x^4 + 24a_4x^2 + 20a_5x^5 + 40a_5x^3 + 30a_6x^6 + 60a_6x^4 + \ldots +$
$a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 + 5a_5x^5 + 6a_6x^6 + \cdots - a_0 - a_0x - a_1x - a_1x^2 - a_2x^2 - a_2x^3 -$
$a_3x^3 - a_3x^4 - a_4x^4 - a_4x^5 - a_5x^5 - a_5x^6 = 0$

Grouping like terms together, we obtain:

$(4a_2 - a_0) + (12a_3 - a_0)x + (24a_4 + 3a_2 - a_1)x^2 + (40a_5 + 8a_3 - a_2)x^3 +$
$(6a_6 + 15a_4 - a_3)x^4 + \cdots \equiv 0$

Comparing coefficients of powers of x, we have:

$4a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{4}$

$12a_3 - a_0 = 0 \Rightarrow a_3 = \frac{a_0}{12}$

$24a_4 + 3a_2 - a_1 = 0 \Rightarrow a_4 = \frac{4a_1 - 3a_0}{96}$

$40a_5 + 8a_3 - a_2 = 0 \Rightarrow a_5 = -\frac{a_0}{96}$

$6a_6 + 15a_4 - a_3 = 0 \Rightarrow a_6 = \frac{53a_0 - 60a_1}{5760}$, etc

Next, we substitute the values obtained above in eqn (ii), we obtain:

$y = a_0 + a_1x + \frac{a_0}{4}x^2 + \frac{a_0}{12}x^3 + \frac{1}{96}(4a_1 - 3a_0)x^4 - \frac{a_0}{96}x^5 + \frac{1}{5760}(53a_0 - 60a_1)x^6 + \ldots$

$y = a_0\left[1 + \frac{x^2}{4} + \frac{x^3}{12} - \frac{3x^4}{96} - \frac{x^5}{96} + \frac{53x^6}{5760} - \cdots\right] + a_1\left[x + \frac{4x^4}{96} - \frac{60x^6}{5760} + \cdots\right]$

$\therefore y = a_0\left[1 + \frac{x^2}{4} + \frac{x^3}{12} - \frac{x^4}{32} - \frac{x^5}{96} + \frac{53x^6}{5760} - \cdots\right] + a_1\left[x + \frac{x^4}{24} - \frac{x^6}{96} + \cdots\right]$ **ans.**

EXERCISE 6.4

Solve the following differential equations by power series method:

10. $y'' + xy = 0$
11. $y'' + xy' + x^2y = 0$
12. $(x^2 + 1)y'' + xy' - xy = 0$
13. $y'' - 2x^2y' + 4xy = x^2 + 2x + 4$
14. $(1 - x^2)y'' - xy' + 4y = 0$
15. $(1 - x^2)y'' - 2xy' + 2y = 0$

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ANSWERS

10. $y = a_0\left(1 - \frac{x^3}{3!} + \frac{4x^6}{6!} + \frac{28x^9}{9!} + \dots\right) + a_1\left(x - \frac{2x^4}{4!} + \frac{10x^7}{7!} + \cdots\right)$

11. $y = a_0\left(1 - \frac{1}{12}x^4 - \cdots\right) + a_1\left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \cdots\right)$

12. $y = a_0\left(1 + \frac{x^3}{6} - \frac{3x^5}{40} + \dots\right) + a_1\left(x - \frac{x^3}{6} + \frac{x^4}{12} + \frac{3}{40}x^5 + \cdots\right)$

13. $y = a_0\left(1 - \frac{2}{3}x^2 - \frac{2}{45}x^6 - \frac{2}{405}x^9 - \cdots\right) + a_1(x - \frac{1}{6}x^4 - \frac{1}{63}x^7 - \frac{1}{567}x^{10} - \cdots)$
$+ 2x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{1}{126}x^7 + \frac{1}{405}x^9 + \frac{1}{1134}x^{10} + \cdots$

14. $y = a_0(1 - 2x^2) + a_1\left(x - \frac{x^5}{8} - \frac{x^7}{16} + \cdots\right)$

15. $y = a_0\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} \dots\right) + a_1 x$

6.7 LEIBNITZ-MACLAURIN METHOD

At this point, it is believed that the reader is already familiar with the Leibnitz theorem and the Maclaurin's theorem. Having said this, we present the following simple 5-step procedures for the Leibnitz-Maclaurin method:

a. Differentiate the given equation n times, using the Leibnitz theorem
b. Rearrange the result to obtain the recurrence relation at $x = 0$
c. Determine the values of the derivatives at $x = 0$, that find y_0 and $y^1{}_0$
d. Substitute in the Maclaurin expansion for y = f(x)
e. Simplify the result where possible and apply boundary values, if given.

Leibnitz Theorem: States that if y = uv, where u and v are functions of x, then

$$y^{(n)} = (uv)^n = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \cdots$$

Maclaurin's Theorem: States

$$y = y_{(0)} + xy'_{(0)} + \frac{x^2}{2!}y''_{(0)} + \frac{x^3}{3!}y'''_{(0)} + \frac{x^4}{4!}y^{(4)}{}_{(0)} + \frac{x^5}{5!}y^{(5)}{}_{(0)} + \cdots$$

Example 19: Use the Leibnitz-Maclaurin method to determine the power series solution of the differential equation: $(1 + x^2)y'' + xy' - 9y = 0$, at $y(0) = 1$ and $y'(0) = 4$.

Solution:

Step 1: We differentiate using the Leibnitz theorem:

$$(1 + x^2)y^{(n+2)} + 2nxy^{(n+1)} + n(n-1)y^n + xy^{(n+1)} + ny^n - 9y^n = 0$$

$$(1 + x^2)y^{(n+2)} + (2nx + x)y^{(n+1)} + (n^2 - n + n - 9)y^n = 0$$

$$(1 + x^2)y^{(n+2)} + (2nx + x)y^{(n+1)} + (n^2 - 9)y^n = 0$$

Step 2: We find the recurrence relation at $x = 0$

$$y^{(n+2)} = -(n^2 - 9)y^n$$

Step 3: Substitute $n = 0, 1, 2, 3, \dots$ to determine the values of the derivatives:

For $n = 0$, $y''_{(0)} = 9y^{(0)}{}_{(0)} = 9y_{(0)}$

$n = 1$, $y'''_{(0)} = 8y'_{(0)}$

$n = 2$, $y^{(4)}{}_{(0)} = 5y''_{(0)} = 45y_{(0)}$

$n = 3$, $y^{(5)}{}_{(0)} = 0$

$n = 4$, $y^{(6)}{}_{(0)} = -7\,y^{(4)}{}_{(0)} = -315y_{(0)}$

$n = 5$, $y^{(7)}{}_{(0)} = -16y^{(5)}{}_{(0)} = 0$

$n = 6$, $y^{(8)}{}_{(0)} = -27y^{(6)}{}_{(0)} = 8505y_{(0)}$

$n = 7$, $y^{(9)}{}_{(0)} = -40y^{(7)}{}_{(0)} = 0$

Step 4: Substitute the values of n in the Maclaurin's theorem:

$$y = y_{(0)} + xy'_{(0)} + \frac{x^2}{2!}y''_{(0)} + \frac{x^3}{3!}y'''_{(0)} + \frac{x^4}{4!}y^{(4)}{}_{(0)} + \frac{x^5}{5!}y^{(5)}{}_{(0)} + \cdots$$

$$y = y_{(0)} + xy'_{(0)} + \frac{x^2}{2!}(9y_{(0)}) + \frac{x^3}{3!}(8y'_{(0)}) + \frac{x^4}{4!}(45y_{(0)}) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-315y_{(0)}) +$$
$$\frac{x^7}{7!}(0) + \frac{x^8}{8!}(8505y_{(0)}) + \frac{x^9}{9!}(0) + \cdots$$

Step 5: Collecting like terms and simplifying, we have:

$$y = y_{(0)}\left[1 + \frac{9x^2}{2!} + \frac{45x^4}{4!} - \frac{315x^6}{6!} + \frac{8505x^8}{8!} + \cdots\right] + y'_{(0)}\left[x + \frac{8x^3}{3!}\right]$$

$$y = y_{(0)}\left[1 + \frac{9x^2}{2} + \frac{15x^4}{8} - \frac{7x^6}{16} + \frac{27x^8}{128} + \cdots\right] + y'_{(0)}\left[x + \frac{4x^3}{3}\right]$$

Step 6: Applying the boundary values:

When $x = 0, y = 1$

$$1 = y_{(0)}[1 + 0 + 0] + y^1{}_{(0)}[0] \Rightarrow y_{(0)} = 1$$

$$y^1 = y_{(0)}\left[0 + 9x + \frac{15x^3}{2} - \cdots\right] + y'_{(0)}[1 + 4x^2]$$

When $x = 0, y' = 4$

$$4 = y_{(0)}[0] + y'_{(0)}[1 + 0] \Rightarrow y'_{(0)} = 4$$

Hence,

$$y = \left[1 + \frac{9x^2}{2} + \frac{15x^4}{8} - \frac{7x^6}{16} + \frac{27x^8}{128} + \cdots\right] + 4\left[x + \frac{4x^3}{3}\right]$$

$$\therefore y = 1 + 4x + \frac{9x^2}{2} + \frac{16x^3}{3} + \frac{15x^4}{8} - \frac{7x^6}{16} + \frac{27x^8}{128} + \cdots \quad \textbf{ans.}$$

Example 20: Use the Leibnitz-Maclaurin method to find the particular solution of the differential equation: $(1 - x^2)y'' - 2xy' + 2y = 0$, given that $y(0) = 1$ and $y'(0) = 2$

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Solution:

Step 1: we differentiate using the Leibnitz theorem:

$$y^{(n)} = (uv)^n = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \cdots$$

$$(1-x^2)y^{(n+2)} - 2nxy^{(n+1)} - n(n-1)y^n - 2xy^{(n+1)} - 2ny^n + 2y^n = 0$$

$$(1-x^2)y^{(n+2)} - (2nx+2x)y^{(n+1)} - (n^2+n-2)y^n = 0$$

Step 2: We find the recurrence relation at $x = 0$

$$y^{(n+2)} = (n^2+n-2)y^n$$

Step 3: Substitute $n = 0, 1, 2, 3, \ldots$ to determine the values of the derivatives:

For $n = 0,\ y''(0) = -2y_{(0)}, \Rightarrow n = 1,\ y'''(0) = 0, \Rightarrow n = 2,\ y^{(4)}(0) = 4y''(0) = -8y(0)$

For $n = 3, y^{(5)}(0) = 10\,y'''(0) = 0 \Rightarrow n = 4,\ y^{(6)}(0) = 18\ y^{(4)}(0) = -144y(0)$

$n = 5,\ y^{(7)}(0) = 28y^{(5)}(0) = 0 \Rightarrow n = 6,\ y^{(8)}(0) = 40\,y^{(6)}(0) = -5760y(0)$

Step 4: Substitute the values of n in the Maclaurin's theorem:

$$y = y_{(0)} + xy'_{(0)} + \frac{x^2}{2!}y''_{(0)} + \frac{x^3}{3!}y'''_{(0)} + \frac{x^4}{4!}y^{(4)}_{(0)} + \frac{x^5}{5!}y^{(5)}_{(0)} + \cdots$$

$$y = y_{(0)} + xy'_{(0)} - x^2y_{(0)} - \frac{x^4}{3}y_{(0)} - \frac{x^6}{5}y_{(0)} - \frac{x^8}{7}y_{(0)} - \cdots$$

$$y = y_{(0)}\left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - \cdots\right] + xy'_{(0)}$$

Step 5: Applying the boundary values:

When $x = 0, y = 1 = y_{(0)}$ and at $x = 0,\ y' = 2 = y'_{(0)}$

$$\therefore\ y = 1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - \cdots + 2x$$

$$\therefore\ y = 1 + 2x - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - \cdots \quad \textbf{ans.}$$

EXERCISE 6.3

16. Determine $y^{(8)}$, if $y = \cos 2x$
17. If $y = 3\ln 2x$, determine $y^{(5)}$
18. Determine $y^{(3)}$, when $y = x^4 \ln x$
19. If $(1+x^2)y'' + 2xy' - 3y = 0$, show that
    $(1+x^2)y^{(n+2)} + 2(n+1)xy^{(n+1)} + (n^2+n-3)y^n = 0$
20. Determine the 5th derivative of $y = x^4 \sin x$
21. If $y = (2x-\pi)^4 \sin\left(\frac{\pi}{2}\right)$, *evaluate* $y^{(6)}$ when $x = \frac{\pi}{2}$
22. If $y = e^{-x}\cos x$, show that $y^{(4)} + 4y = 0$
23. If $y = (x^3 + 3x^2)e^{2x}$, determine an expression for $y^{(6)}$
24. If $y = x^3e^{2x}$, determine $y^{(n)}$ and hence $y^{(3)}$
25. If $y = x^3\cos x$, determine the 5th derivative.
26. If $y = x^5 \ln 2x$, find $y^{(3)}$
27. Determine an expansion for $y^{(5)}$, given that $y = (x^3 + 2x^2)e^{2x}$

28. Use the Leibnitz-Maclaurin method to determine the power series solution of the differential equation: $(1-x^2)y'' - 4xy' - (1+x^2)y = 0$
29. Use the Leibnitz-Maclaurin method to determine the power series solution of the differential equation: $(1-x^2)y'' - 7xy' - 9y = 0$
30. Determine the power series solution of the differential equation: $y'' + 2xy' + y = 0$ using the Leibnitz-Maclaurin method, given that $y(0) = 1,\ y'(0) = 2$
31. Determine the particular solution of the differential equation:
    $(x^2+1)y'' + xy' - 4y = 0$, using the Leibnitz-Maclaurin theorem at $y(0) = y' = 1$
32. Use the Leibnitz-Maclaurin method to determine the power series solution of the differential equation: $xy'' + y' + xy = 1$, given that $y(0) = 1;\ y'(0) = 2$
33. Use the Leibnitz-Maclaurin method to determine the power series solution of the differential equation: $x(x+1)^2y'' + (1-x^2)y' + (x-1)y = 0;\ y(0) = y'(0) = 1$

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Chapter 7: Applications of Second-Order Differential Equations

Second-order linear differential equations can be applied in science and engineering. The areas which it can be applied include: electric circuits and vibration of springs as in simple harmonic motion, damped vibration and forced vibration.

In this chapter we provide an introduction to the concept of modeling of engineering systems that leads to second order differential equations.

7.1 MODELING FREE MECHANICAL OSCILLATIONS

Consider the following schematic diagram of a shock absorber:

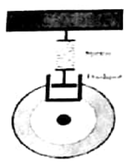

Figure 7.1: Diagram of shock absorber

The equation of motion can be described in terms of the vertical displacement x of the mass. Let m be the mass, $K\frac{dx}{dt}$ the damping force resulting from the dashpot and nx the restoring force resulting from the spring. Here, k and n are constants. Then the equation of motion is:

$$m\frac{d^2x}{dt^2} = -k\frac{dx}{dt} - nx \quad \text{.....7.1}$$

Suppose that the mass is displaced a distance $x0$ initially and released from rest. Then at $t = 0$, $x = x_0$ and $\frac{dx}{dt} = 0$. Writing the differential equation in standard form we have:

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + nx = 0 \quad \text{---------- 7.2}$$

The value of k controls the amount of damping in the system. We explore the solution for two particular values of k.

No damping

If $k = 0$ then there is no damping. We expect, in this case, that once motion has started it will continue forever. The motion that ensues is called simple harmonic motion. In this case we have:

$\lambda = \frac{\pm\sqrt{-4mn}}{2m}$, That is, $\lambda = \sqrt{\frac{n}{m}}\,i$, Where $i = \sqrt{-1}$

In this case the solution for the displacement x is:

$$x = A\cos\left(\sqrt{\frac{n}{m}}\,t\right) + B\sin\left(\sqrt{\frac{n}{m}}\,t\right) \quad \text{---------- 7.3}$$

Where A, B are arbitrary constants.

Light damping

If $k^2 - 4mn < 0$, i.e. $k^2 < 4mn$ then the roots of the auxiliary equation are complex:

$\lambda_1 = \frac{-k+i\sqrt{4mn-k^2}}{2m}$, $\lambda_2 = \frac{-k-i\sqrt{4mn-k^2}}{2m}$, Then after some rearrangement we have:

$$x = e^{-kt/2m}[A\cos pt + B\sin pt] \quad \text{----------7.4}$$

In which $p = \sqrt{4mn - k^2/2m}$

Heavy damping

If $k^2 - 4mn > 0$, i.e. $k^2 > 4mn$ then there are two real roots of the auxiliary equation, λ_1 and λ_2:

$\lambda_1 = \frac{-k+\sqrt{k^2-4mn}}{2m}$, $\lambda_2 = \frac{-k-\sqrt{k^2-4mn}}{2m}$, Then, $x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

7.2 MODELING FORCED MECHANICAL OSCILLATIONS

Suppose now that the mass is subject to a force $f(t)$ after the initial disturbance. Then the equation of motion is:

$$m\frac{d^2x}{dt^2} + k\frac{dx}{dt} + nx = f(t) \quad \text{---------- 7.5}$$

Consider the case $f(t) = F\cos\omega t$, that is, an oscillatory force of magnitude F and angular frequency ω. Choosing specific values for the constants in the model: $m = n = 1$, $k = 0$, and $\omega = 2$ we find;

$$\frac{d^2x}{dt^2} + x = F\cos t \quad \text{---------- 7.6}$$

Vibrating Springs:

Let us consider the motion of an object with mass m at the end of a spring that is either vertical as shown below:

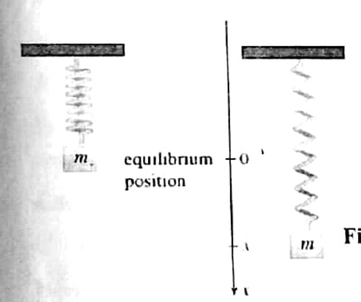

Figure 7.2: Vertical Spring

Applying Hooke's Law, which says that if the spring is stretched (or compressed) units from its natural length, then it exerts a force that is proportional to x: then the restoring force = - kx, where is a positive constant (called the **spring constant**). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have:

$$\frac{d^2x}{dt^2} = kx; \text{ » } \frac{d^2x}{dt^2} + kx = 0 \qquad 7.7$$

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This is a second-order linear differential equation. Its auxiliary equation is $mr^2 + k = 0$ with roots $r = \pm i\omega$,

where $\omega = \sqrt{k/m}$. Thus, the general solution is, $X(t) = C_1\cos\omega t + C_2\cos\omega t$. Which can also be written as

$X(t) = A\cos(\omega t + \delta)$, Where $\omega = \sqrt{k/m}$ *(frequency)*, $A = \sqrt{C_1^2 + C_2^2}$ *(Amplitude)*

$\cos\delta = \frac{C_1}{A}$, $\sin\delta = \frac{C_2}{A}$ *(δ = the phase angle)*

EXAMPLE 1: A spring with a mass of 2 kg has natural length 0.5m. A force of 25.6N is required to maintain it stretched to a length of m. If the spring is stretched to a length of 0.7m and then released with initial velocity 0, find the position of the mass at any time t.

SOLUTION: From Hooke's Law, the force required to stretch the spring is

$k(0.2) = 25.6$, So, $k = \frac{25.6}{0.2} = 128$

Using the value of the spring constant k, together with m = 2 in equation 7.7 we have:

$2\frac{d^2x}{dt^2} + 128x = 0$. As in the earlier discussion, the solution of this equation is

$x(t) = C_1\cos 8t + C_2\sin 8t$

But we are given the initial condition that $x(0) = 0.2$, therefore from the above solution $x(0) = C_1$.

So, $C_1 = 0.2$. And if we differentiate,

$x(t) = C_1\cos 8t + C_2\sin 8t$. We have $x'(t) = -8C_1\sin 8t + 8C_2\cos 8t$

Since the initial velocity is given as $x'(0) = 0$ we have $C_2 = 0$

So, the solution is $x(t) = {}^1/_5\cos 8t$

Let us consider the motion of a spring that is subject to a frictional force (in the case of the horizontal spring) or a damping force (in the case where a vertical spring moves through a fluid as shown in the figure below). An example is the damping force supplied by a shock absorber in a car or a bicycle.

Figure 7.3: Vertical Spring moving through a Fluid

We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion.

$$Damping\ force = -C\frac{dx}{dt}$$

Where C is a positive constant, called the **damping constant**. Thus, in this case, Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = restoring\ force + damping\ force = -kc - c\frac{dx}{dt}$$

» $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} - kc = 0$ (compare with equation 5.2)

$mx'' + cx' + kx = F(t)$ ---------- 7.8

that governs the motion of the mass. If there is no dashpot (and we ignore all frictional forces), then we set c = 0 in Eq. (7.8) and call the motion undamped; it is damped motion if c > 0. If there is no external force, we replace F (t) with 0 in Eq. (7.8). We refer to the motion as free in this case and forced in the case F (t) ≠ 0. Thus the homogeneous equation describes free motion of a mass on a spring with dashpot but with no external forces applied.

$mx'' + cx' + kx = 0$

Equation 5.8 is a second-order linear differential equation and its auxiliary equation is

$Mr^2 + cr + k = 0$. The roots are:

$r_1 = \frac{-c+\sqrt{c^2-4mk}}{2m}$, $r_1 = \frac{-c+\sqrt{c^2-4mk}}{2m}$ ---------- 7.9

As we have seen earlier in this discussion, three cases are involved:

Case I

$c^2 - 4mk > 0$ **(Overdamping)**

In this case r_1 and r_2 are distinct real roots and

$x = c_1e^{r_1t} + c_2e^{r_2t}$

Since c, m, and k are all positive, we have $\sqrt{c^2 - 4mk} < c$, so the roots r_1 and r_2 given by Equations 7.9 must both be negative. This shows that $x \to 0$ as $t \to \infty$. Typical graphs of x as a function of are shown in Figure 7.4. Notice that oscillations do not occur. (It's possible for the mass to pass through the equilibrium position once, but only once.) This is because $c^2 > 4mk$ means that there is a strong damping force (high-viscosity oil or grease) compared with a weak spring or small mass.

Case II

$c^2 - 4mk = 0$ ***(Critical Damping)*** This case corresponds to equal roots $r_1 = r_2 = -\frac{c}{2m}$ and the solution is given by

$X = (c_1 + c_2t)e^{-(c/2m)t}$ ---------- 7.10

In this case, the damping is just sufficient to suppress vibrations. Any decrease in the viscosity of the fluid leads to the vibrations of the following case.

Case III

$c^2 - 4mk < 0$

Here the roots are complex:

$r_1 = -\frac{c}{2m} + \omega i$, $r_2 = -\frac{c}{2m} - \omega i$ ---------- 7.11

Where $\omega = \frac{\sqrt{4mk-c^2}}{2m}$ The solution is given by

$x = e^{-(c/2m)t}[c_1\cos\omega t + c_2\sin\omega t]$ ---------- 7.12

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We see that there are oscillations that are damped by the factor $e^{-(c/2m)t}$. since c > 0 and m>0, we have $-(c/2m)<0$ so $e^{-(c/2m)t} \to 0$ as $t \to \infty$. This implies that $x \to 0$ as $t \to \infty$; i.e the motion decays as time increases.

EXAMPLE 2: Suppose that the spring of Example 1 is immersed in a fluid with damping constant c = 40. Find the position of the mass at any time if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6m/s.

SOLUTION: From Example 1 the mass is m= 2 and the spring constant is k = 128, so the differential equation (3) becomes

$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0 = \frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 64x = 0$ The auxiliary equation is

$r^2 + 20r + 64 = (r+4)(r+16) = 0$ with roots -4 and -16, so the motion is overdamped and the solution is

$c_1e^{-4t} + c_2e^{-16t}$ But we are given that $x(0) = 0$, so $c_1 + c_2 = 0$

Differentiating we get: $x'(t) = -4c_1e^{-4t} - 16c_2e^{-16t}$ and $x'(0) = -4c_1 - 16c_2 = 0.6$

Since $c_2 = -c_1$, this gives $12c_1 = 0.6$ or $c_1 = 0.05$ Therefore, $x = 0.05(e^{-4t} - e^{-16t})$

The simple pendulum:

A simple pendulum consists of a mass m swinging back and forth on the end of a string (or better, a massless rod) of length L, as shown in figure below. We may specify the position of the mass at time t by giving the counterclockwise angle $\theta = \theta$ (t) that the string or rod makes with the vertical at time t. To analyze the motion of the mass m, we will apply the law of the conservation of mechanical energy, according to which the sum of the kinetic energy and the potential energy of m remains constant.

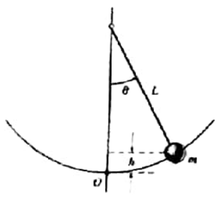

Figure 7.4: Simple Pendulum

The distance along the circular arc from 0 to m is $s = L\theta$, so the velocity of the mass is $v = v = \frac{ds}{dt} = L(\frac{d\theta}{dt})$ and therefore its kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\frac{ds}{dt})^2 = \frac{1}{2}mL^2(\frac{d\theta}{dt})^2$$

We next choose as reference point the lowest point O reached by the mass. Then its potential energy V is the product of its weight mg and its vertical height $h = L(1 - \cos\theta)$ above O , so

$V = mgL(1 - \cos\theta)$

The fact that the sum of T and V is a constant C therefore gives

$$\frac{1}{2}mL^2(\frac{d\theta}{dt})^2 + mL(1 - \cos\theta) = C$$

We differentiate both sides of this`identity with respect to t to obtain

$$mL^2\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) + mgl(\sin\theta)\frac{d\theta}{dt} = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \quad \text{---------- 7.13}$$

After removal of the common factor $mL^2\left(\frac{d\theta}{dt}\right)$ This differential equation can be derived in a seemingly more elementary manner using the familiar second law $F = ma$ of Newton (applied to tangential components of the acceleration of the mass and the force acting on it). However, derivations of differential equations based on conservation of energy are often seen in more complex situations where Newton's law is not so directly applicable, and it may be instructive to see the energy method in a simpler application like the pendulum.

Recall that if θ is small, then $\sin\theta \approx \theta$ (this approximation obtained by retaining just the first term in the Taylor series for $\sin\theta$). In fact, $\sin\theta$ and θ agree to two decimal places when $|\theta|$ is at most $\frac{\pi}{12}$ (that is, 1 5 °). In a typical pendulum clock, for example, θ would never exceed 1 5 °. It therefore seems reasonable to simplify our mathematical model of the simple pendulum by replacing $\sin\theta$ with θ in Eq. (7.13). If we also insert a term $c\theta'$ to account for the frictional resistance of the surrounding medium, we have:

$$\theta'' + C\theta' + k\theta = 0 \quad \text{(Compare with equation 7.8) ---------- 7.14}$$

Where $K = g/L$. Note that this equation is independent of the mass m on the end of the rod. We might, however, expect the effects of the discrepancy between θ and $\sin\theta$ to accumulate over a period of time, so that Eq. (5.14) will probably not describe accurately the actual motion of the pendulum over a long period of time. Next, we will analyze free undamped motion and then free damped motion.

Free Undamped Motion

If we have only a mass on a spring, with neither damping nor external force, then Eq . (7.5) takes the simpler form.

7.3 MODELING FORCED MECHANICAL OSCILLATIONS AND RESONANCE

In Section 5.1 we derived the differential equation

$$mx'' + cx' + kx = F(t) \quad \text{---------- 7.15}$$

The above equation governs the one-dimensional motion o f a mass m that is attached to a spring (with constant k) and a dashpot (with constant c) and is also acted on by an external force F (t). Machines with rotating components commonly involve mass-spring systems (or their equivalents) in which the external force is simple harmonic: $F(t) = F_0\cos\omega t$ or $F(t) = F_0\sin\omega t$

Where the constant F_0 is the amplitude of the periodic force and ω is its circular frequency.

For example, the figure below shows a rotating machine component can provide a simple harmonic force. Consider the cart with a rotating vertical flywheel as shown:

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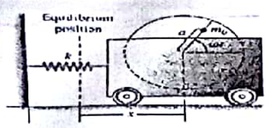

Figure 7.5: The Cart with Flywheel system

The cart has mass $m - m_0$, not including the flywheel of mass mo. The centroid of the flywheel is off center at a distance a from its center, and its angular speed is ω radians per second. The cart is attached to a spring (with constant k) as shown. Assume that the centroid of the cart itself is directly beneath the center of the flywheel, and denote by $x(t)$ its displacement from its equilibrium position (where the spring is unstretched).

It can be seen from the figure above that the displacement $\bar{x}$ of the centroid of the combined cart plus flywheel is given by

$$\bar{x} = \frac{(m-m_0)x+m_0(x+a\cos\omega t)}{m} = x + \frac{m_0 a}{m}\cos\omega t \quad \text{---------- } 7.16$$

Let us ignore friction and apply Newton's second law $mx'' = -kx$ because the force exerted by the spring is $-kx$. We substitute for x in the last equation to obtain

$mx'' - m_0 a\omega^2 \cos\omega t = -kx$; That is,

$$mx'' + kx = m_0 a\omega^2 \cos\omega t \quad \text{---------- } 7.17$$

Thus the cart with its rotating flywheel acts like a mass on a spring under the influence of a simple harmonic external force with amplitude $F_0 = m_0 a\omega^2$. Such a system is a reasonable model of a front-loading washing machine with the clothes being washed loaded off center. This illustrates the practical importance of analyzing solutions of Eq. (7.15) with external forces as in (7.16).

Un-damped Forced Oscillations

To study un-damped oscillations under the influence of the external force $(t) = F_0 \cos\omega t$, we set $c = 0$ in Eq. (5.15), and thereby begin with the equation

$$mx'' + kx = F_0 \cos\omega t \quad \text{---------- } 7.18$$

Whose complementary function is $x_c = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$. Here $\omega_0 = \sqrt{\frac{k}{m}}$.

Solved Examples

Example 3: A body with mass m = $\frac{1}{2}$ kiiogram (kg) is attached to the end of a spring that is stretched 2 meters (m) by a force of 100 newtons (N). It is set in motion with initial position x_0 = 1 (m) and initial velocity v_0= -5 (m/s). (Note that these initial conditions indicate that the body is displaced to the right and is moving to the left at time t = 0.) Find the position function of the body as well as the amplitude, frequency, period of oscillation, and time lag of its motion.

Solution:

The spring constant is k = $\frac{(100N)}{2m}$ = 50 (N/m), so Eq. 5.5 yields, $\frac{1}{2}$x" + 50x = 0; that is, x" + 1 00x = 0; that is.

Consequently, the circular frequency of the resulting simple harmonic motion of the body will be $\omega_0 = \sqrt{100}$ = 10 (rad/s). Hence it will oscillate with period. $T = \frac{2\pi}{\omega_0} = \frac{\pi}{10} \approx 0.6283S$. And with frequency

$$V = \frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{10}{2\pi} = 1.5915Hz$$

We now impose the initial conditions x (0) = I and x' (0) = -5 on the position function

x (t) = Acos 10t + B sin 10t with x'(t) = - 10A sin 10t + 10B cos 10t .

It follows readily that A = 1 and B = -$\frac{1}{2}$, so the position function of the body is

x(t) = cos10t - $\frac{1}{2}$sin 10t . Hence its amplitude of motion is $C = \sqrt{(1)^2 + (-\frac{1}{2})^2} = \frac{1}{2}\sqrt{5}m$

To find the time lag, we write $X(t) = \frac{\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}\cos 10t - \frac{1}{\sqrt{5}}\sin 10t\right) = \frac{\sqrt{5}}{2}\cos(10t - \alpha)$

Where the phase angle α satisfies cos $\alpha = \frac{2}{\sqrt{5}} > 0$ and sin $\alpha = \frac{1}{\sqrt{5}} < 0$

Hence α is the Fourth quadrant angle $\alpha = 2\pi + \tan^{-1}\left(\frac{1}{\sqrt{5}} \Big/ \frac{2}{\sqrt{5}}\right) = 2\pi - \tan^{-1}(\frac{1}{2}) \approx 5.8195$

And the time lag of the motion is $\delta = \frac{\delta}{\omega_0} \approx 0.5820$

With the amplitude and approximate phase angle shown explicitly, the position function of the body takes the form $X(t) = \frac{1}{2}\sqrt{5}\cos(10t - 5.8195)$

Example 4: The mass and spring of Example 1 are now attached also to a dashpot that provides 1 N of resistance for each meter per second of velocity. The mass is set in motion with the same initial position x (0) = 1 and initial velocity x' (0) = -5 as in Example 1. Now find the position function of the mass, its new frequency and pseudoperiod of motion, its new time lag, and the times of its first four passages through the initial position x = 0.

Solution

Rather than memorizing the various formulas given in the preceding discussion, it is better practice in a particular case to set up the differential equation and then solve it directly. Recall that m = 1 and k = 50; we are now given c = 1 in mks units. Hence Eq. (5.5) is given as $\frac{1}{2}x'' + x' + 50x = 0$; that is, $x'' + 2x' + 1\ 00x = 0$.

The characteristic equation $r^2 + 2r + 100 = (r+1)2 + 99 = 0$ has roots $r_1, r_2 = -1 \pm \sqrt{99}i$, so the general solution is $e^{-t}(A\cos\sqrt{99}t + B\sin\sqrt{99})$

Consequently, the new circular (pseudo)frequency is $\omega_1 = \sqrt{99} \approx 9.9499$ (as compared with ω_0= 10 in Example 1). The new (pseudo)period and frequency are

$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{99}} \approx 0.6315s$. And $V_1 = \frac{1}{T_1} = \frac{\omega_1}{2\pi} = \frac{\sqrt{99}}{2\pi} \approx 1.5836$

(as compared with T ≈ 0.6283 < T_1 and V ≈ 1 . 5 9 1 5 > V_1in Example 1) . We now impose the initial conditions x (0) = 1 and x' (0) = -5 on the position function and the resulting velocity function

$x^1(t) = -e^{-t}(A\cos\sqrt{99}t + B\sin\sqrt{99}t) + \sqrt{99}te^{-t}(-A\sin\sqrt{99}t + B\cos\sqrt{99}t)$ it follows that

$X(0)$ = A = 1 and $x^1(0) - A + \sqrt{99} - 5$. Whence we find that A = 1 and B = $-4/\sqrt{99}$

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Thus the new position function of the body is $x(t) = e^{-t}(\cos\sqrt{99}t - {}^{4}/_{\sqrt{99}}\sin\sqrt{99}t)$

Hence its time-varying amplitude of motion is $C_1e^{-1} = \sqrt{(1)^2 + ({}^{-4}/_{\sqrt{99}})^2e^{-t}} = \sqrt{\frac{115}{99}}e^{-t}$

We therefore write $x(t) = \sqrt{\frac{115}{99}}e^{-t}\left(\frac{\sqrt{99}}{\sqrt{115}}\cos\sqrt{99}t - \frac{4}{\sqrt{115}}\sin\sqrt{99}t\right) = \sqrt{\frac{115}{99}}e^{-t}\cos(\sqrt{99}t - \propto_1)$

Where the phase angle $\propto_1$ satisfies $\text{Cos}\propto_1 = \frac{\sqrt{99}}{\sqrt{115}} > 0$ and $\sin\propto_1 = -\frac{4}{\sqrt{115}} < 0$

Hence $\propto_1$ is the fourth quadrant angle $\propto_1 = 2\pi - \tan^{-1}\left(\frac{{}^{-4}/_{\sqrt{115}}}{{}^{\sqrt{99}}/_{\sqrt{115}}}\right) = 2\pi - \tan^{-1}\frac{4}{\sqrt{99}} \approx 5.999$

And the time lag of the motion is $\delta_1 = \frac{\propto_1}{\omega_1} = 0.5931s$

(as compared with $\delta \approx 0.5820 < \delta_1$ in Example 1). With the time-varying amplitude and approximate phase angle shown explicitly, the position function of the mass takes the form $x(t) = \sqrt{\frac{115}{99}}e^{-1}\cos(\sqrt{99}t - 5.9009)$

7.4 MODELING ELECTRICAL CIRCUITS

Second order differential equation can be applied in the analysis of electric circuits. Here we examine the RLC circuit that is a basic building block in more complicated electrical circuits and networks.

Consider the figure shown below:

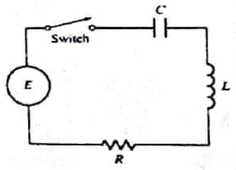

Figure 7.6: Electric Circuit

A **resistor** with a resistance of R ohms, an **inductor** with an inductance of L henries, and a **capacitor** with a capacitance of C farads

If the switch is closed, this results in a current of I (t) amperes in the circuit and a charge of Q (t) coulombs on the capacitor at time t. The relation between the functions Q and I is

$\frac{dQ}{dt} = I(t)$ 7.19

It is known from physics that the voltage drops across the resistor, inductor, and capacitor are $RI, L\frac{dI}{dt}, \frac{Q}{C}$ respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage: $L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$

Since $\frac{dQ}{dt} = I$, this equation becomes $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$ ---------- 7.20

Which is a second-order linear differential equation with constant coefficients. If the charge Q_0 and the current I_0 are known at time 0, then we have the initial conditions. $Q(0) = Q_0$ $Q'(0) = I(0) = I_0$ and the initial-value problem can be solved by the methods of *Nonhomogeneous Linear Equations*. A

differential equation for the current can be obtained by differentiating Equation 5.20 with respect to and remembering that $\frac{dQ}{dt} = I$

$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)$

$$\boxed{L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)}$$

EXAMPLE 5 Find the charge and current at time in the circuit of Figure 5.6 if R = 40Ω, L = 1H, $16 \times 10^{-4}F$, E(t) = 100cos10t and the initial charge and current are both 0.

SOLUTION With the given values of L, R, C, and E(t) Equation 5.6 becomes

$\frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + 625Q = 100\cos 10t$ The auxiliary equation is $r^2 + 40r + 625 = 0$

With roots $\frac{-40\pm\sqrt{-900}}{2} = 20 \pm 15i$ so the solution of the complimentary equation is

$Q_c(t) = e^{-20t}(C_1\cos 15t + C_2\sin 15t)$

From the method of undetermined coefficient, we try the particular solution

$Q_p(t) = A\cos 10t + 10\sin 10t.\ Q_p'(t) = -10A\sin 10t + 10B\cos 10t,$

$Q_p''(t) = 100A\cos 10t - 100B\sin 10t$

Substituting into equation 5.21 we have

$(-100A\cos 10t - 100B\sin 10t) + 40(-10A\sin 10t + 10B\cos 10t) + 625(A\cos 10t + B\sin 10t) =$

$(625A + 400B)\cos 10t + (-400A + 525B)\sin 10t = 100\cos 10t$

Equating the coefficients, we have

$525A + 400B = 100$ or $21A + 16B = 4$

$-400A + 525B = 0$ or $-16A + 21B = 0$

The solution of the system is $A = \frac{84}{697}$ and $B = \frac{64}{697}$, so a particular solution is

$Q_p(t) = \frac{1}{697}(84\cos 10t + 64\sin 10t)$ and the general solution is

$Q(t) = Q_c(t) + Q_p(t) = e^{-20t}(C_1\cos 15t + C_2\sin 15t) + \frac{4}{697}(21\cos 10t + 16\sin 10t)$

Imposing the initial condition $Q(0) = 0$, we get . $Q(0) = C_1 + \frac{84}{697} = 0$

$C_1 = \frac{84}{697}$. To impose the other initial condition we first differentiate to find the current

$$I = \frac{dQ}{dt} = e^{-20t}[(20C_1 + 15C_2)\cos 15t + (-15C_1 - 20C_2)\sin 15t] + \frac{40}{697}(-21\sin 10t + 16\cos 10t$$

$$I(0) = -20C_1 + 15C_2 + \frac{640}{697} = 0$$

$C_2 + \frac{464}{2091}$. Thus the formula for the charge is

$$Q(t) = \frac{4}{697}\left[\frac{e^{-20t}}{3}(-63\cos 15t - 116\sin 15t) + (21\cos 10t + 16\sin 10t)\right]$$

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And the expression for the current is

$$I(t) = \frac{1}{2091}[e^{-20t}(-1920\cos 15t + 13060\sin 15t) + 120(-21\sin 10t + 16\cos 10t)]$$

EXERCISE 7

1. A spring with a 3-kg mass is held stretched 0.6m beyond its natural length by a force of 20 N. If the spring begins at its equilibrium position but a push gives it an initial velocity of 1.2m/s, find the position of the mass after t seconds.
2. A spring with a 4-kg mass has natural length 1 m and is maintained stretched to a length of 1.3m by a force of 24.3N. If the spring is compressed to a length of 0.8m and then released with zero velocity, find the position of the mass at any time t.
3. A spring with a mass of 2 kg has damping constant 14, and a force of 6 N is required to keep the spring stretched 0.5m beyond its natural length. The spring is stretched 1 m beyond its natural length and then released with zero velocity. Find the position of the mass at any time t.
4. A spring with a mass of 3 kg has damping constant 30 and spring constant 123.
   a. Find the position of the mass at time t if it starts at the equilibrium position with a velocity of 2 m/s.
   b. Draw a graph of the position function of the mass.
5. A spring has a mass of 1 kg and its spring constant is k=100. The spring is released at a point 0.1 m above its equilibrium position. Graph the position function for the following values of the damping constant c: 10, 15, 20, 25, 30. What type of damping occurs in each case?
6. A spring has a mass of 1 kg and its damping constant is C=100. The spring starts from its equilibrium position with a velocity of 1 m/s. Graph the position function for the following values of the spring constant k: 10, 20, 25, 30, 40. What type of damping occurs in each case?
7. A series circuit consists of a resistor with 20 Ohms, an inductor with 1H, a capacitor with 0.002F, and a 12-V battery. If the initial charge and current are both 0, find the charge and current at time t.
8. A series circuit contains a resistor with 24 ohms, an inductor with2 H, a capacitor with 0.005 F, and a 12-V bat- tery. The initial charge is Q=0.001C and the initial current is 0. (a) Find the charge and current at time t. (b) Graph the charge and current functions.
9. Suppose a spring has mass m and spring constant K and let $\omega_0 = \sqrt{\frac{k}{m}}$ suppose that the damping constant is so small that the damping force is negligible. If an external force $F(t) = F_0 \cos\omega t$ is applied, where $\omega_0 \neq \omega$, use the method of undetermined coefficients to show that the motion of the mass is described by $x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{F_0}{m(\omega^2 - \omega_0^2)}\cos\omega t$
10. A 16 lb object stretches a spring $\frac{8}{9}$ft by itself. There is no damping and no external forces acting on the system. The spring is initially displaced 6 inches upwards from its equilibrium position and given an initial velocity of 1 ft/sec downward. Find the displacement at any time t, $x(t)$.
11. A series circuit consists of a resistor with R = 20Ω, an inductor with L = 1H, a capacitor with C = 0.002F, and a 12-V battery. If the initial charge and current are both 0, find the charge and current at time t.

7. Find the transpose of $\begin{pmatrix} 4 & 1 & -6 \\ -2 & 3 & -3 \\ 5 & 7 & -4 \end{pmatrix}$
8. Given that $A = \begin{pmatrix} -1 & 3 \\ 6 & 0 \\ -1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 2 \\ -8 & 1 & 9 \end{pmatrix}$, determine:
   (i) $(AB)^T$
   (ii) $(AB)^{-1}$

8.2 SOLUTION OF A SET OF LINEAR EQUATIONS

8.2.1 By Matrix Method (Inverse Method)

For Equations in two variables:

Consider a set of simultaneous equations in two variables

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

We form the matrix equation corresponding to these equations:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Next, we determine the inverse matrix of $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, that is: $\frac{1}{a_1b_2 - b_1a_2}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}$

Finally, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a_1b_2 - b_1a_2}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and we solve for x and y by equating corresponding elements.

Example 2: Use matrix method to find solution of the simultaneous equations:

$3x + 5y = 11$

$4x - 3y = -24$

Solution:

$$\begin{pmatrix} 3 & 5 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -24 \end{pmatrix}$$

Let $A = \begin{pmatrix} 3 & 5 \\ 4 & -3 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\begin{pmatrix} -3 & -5 \\ -4 & 3 \end{pmatrix} = \frac{1}{-29}\begin{pmatrix} -3 & -5 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{29} & \frac{5}{29} \\ \frac{4}{29} & -\frac{3}{29} \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{29} & \frac{5}{29} \\ \frac{4}{29} & -\frac{3}{29} \end{pmatrix}\begin{pmatrix} 11 \\ -24 \end{pmatrix} = \begin{pmatrix} \frac{33}{29} - \frac{120}{29} \\ \frac{44}{29} + \frac{72}{29} \end{pmatrix} = \begin{pmatrix} -\frac{87}{29} \\ \frac{116}{29} \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

$\therefore x = -3,\ y = 4$

For Equations in three variables:

Consider simultaneous linear equations in three unknowns

$$a_1x + b_1y + c_1z = d_1$$

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$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3 = d_3$$

We write the matrix equation corresponding to the above equations:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Next, we find the inverse matrix of A, where $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

Finally, we have $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$, where the values of x, y, z can be found by comparing corresponding elements.

Also consider a set of linear equations:

$$a_1x_1 + a_2x_2 + a_3x_3 = s_1$$
$$b_1x_1 + b_2x_2 + b_3x_3 = s_2$$
$$c_1x_1 + c_2x_2 + c_3x_3 = s_3$$

The matrix equation corresponding to the above equations is:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

So we have the equation: $A.x = S$, where $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $S = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$

$\therefore x = A^{-1}.\,S$; that is $x = A^{-1}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$, where A^{-1} is the inverse of the matrix A. We then solve for x by equating the corresponding elements.

Example 3: Solve the equations by matrix method:

$3x_1 + 2x_2 + x_3 = 1$

$x_1 - x_2 + 3x_3 = 5$

$2x_1 + 5x_2 - 2x_3 = 0$

Solution:

We first write the matrix equation, which is

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 2 & 5 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$$

$Ax = B$

$x = A^{-1}B$

$\frac{C^T}{|A|} \times B$

If we let $A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 2 & 5 & -2 \end{pmatrix}$, then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$, where A^{-1} is the inverse of A.

So our major concern will be to find A^{-1}. To do this, we:

i. Form the determinant of A, that is $|A| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 2 & 5 & -2 \end{vmatrix} = -16$

ii. Form a new matrix C consisting of the cofactors of the elements in A, that is

$C = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where $a_{11} = +\begin{vmatrix} -1 & 3 \\ 5 & -2 \end{vmatrix} = -13$; $a_{12} = -\begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} = 8$; $a_{13} = +\begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} = 7$; $a_{21} = -\begin{vmatrix} 2 & 1 \\ 5 & -2 \end{vmatrix} = 9$; $a_{22} = +\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} = -8$;

$a_{23} = -\begin{vmatrix} 3 & 2 \\ 2 & 5 \end{vmatrix} = -11$; $a_{31} = +\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = 7$; $a_{32} = -\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = -8$;

$a_{33} = +\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = -5$

Now, $C = \begin{pmatrix} -13 & 8 & 7 \\ 9 & -8 & -11 \\ 7 & -8 & -5 \end{pmatrix}$ and $C^T = \begin{pmatrix} -13 & 9 & 7 \\ 8 & -8 & -8 \\ 7 & -11 & -5 \end{pmatrix}$

$$A^{-1} = \frac{C^T}{|A|} = \frac{1}{-16}\begin{pmatrix} -13 & 9 & 7 \\ 8 & -8 & -8 \\ 7 & -11 & -5 \end{pmatrix}$$

Therefore, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} = \frac{1}{-16}\begin{pmatrix} -13 & 9 & 7 \\ 8 & -8 & -8 \\ 7 & -11 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} = \frac{1}{-16}\begin{pmatrix} -13+45+0 \\ 8-40+0 \\ 7-55+0 \end{pmatrix}$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{-16}\begin{pmatrix} 32 \\ -32 \\ -48 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$$

$\therefore x_1 = -2,\ x_2 = 2,\ x_3 = 3$

Example 4: If $Ax = b$ where $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & -3 \\ 5 & 2 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ -8 \\ 9 \end{pmatrix}$, determine A^{-1} and hence solve the set of equations.

Solution:

Since $Ax = b \Rightarrow x = A^{-1}.b$

To find A^{-1}, we follow the usual method.

$$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & -3 \\ 5 & 2 & 1 \end{vmatrix} = -28$$

$C = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where $a_{11} = +\begin{vmatrix} -1 & -3 \\ 2 & 1 \end{vmatrix} = 5$; $a_{12} = -\begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} = -17$; $a_{13} = +\begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} = 9$; $a_{21} = -\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 1$; $a_{22} = +\begin{vmatrix} 1 & 2 \\ 5 & 1 \end{vmatrix} = -9$; $a_{23} = -\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 13$;

$a_{31} = +\begin{vmatrix} 3 & 2 \\ -1 & -3 \end{vmatrix} = -7$; $a_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = 7$; $a_{33} = +\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} = -7$

$$\therefore C = \begin{pmatrix} 5 & -17 & 9 \\ 1 & -9 & 13 \\ -7 & 7 & -7 \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} 5 & 1 & -7 \\ -17 & -9 & 7 \\ 9 & 13 & -7 \end{pmatrix}$$

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$$A^{-1} = \frac{C^T}{|A|} = \frac{1}{-28}\begin{pmatrix} 5 & 1 & -7 \\ -17 & -9 & 7 \\ 9 & 13 & -7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\begin{pmatrix} 3 \\ -8 \\ 9 \end{pmatrix} = \frac{1}{-28}\begin{pmatrix} 5 & 1 & -7 \\ -17 & -9 & 7 \\ 9 & 13 & -7 \end{pmatrix}\begin{pmatrix} 3 \\ -8 \\ 9 \end{pmatrix} = \frac{1}{-28}\begin{pmatrix} 15-8-63 \\ -51+72+63 \\ 27-104-63 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{-28}\begin{pmatrix} -56 \\ 84 \\ -140 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

$\therefore\ x_1 = 2,\ x_2 = -3,\ x_3 = 5$

Summary:

To solve $A\,.\,x = b$, where $x = A^{-1}\,.\,b$, we evaluate $|A|$. If $|A| \neq 0$, then form C and form C^T, the transpose of C and then find A^{-1}, where $A^{-1} = \frac{C^T}{|A|}$. But if $|A| = 0$, then there's no inverse and hence no unique solution.

8.2.2 By Determinant Method

The notation of determinants arises from the process of elimination of the unknowns of simultaneous linear equations. Consider a pair of simultaneous equations:

$$a_1x + b_1y + d_1 = 0$$
$$a_2x + b_2y + d_2 = 0$$

We obtain that $x = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}$ and $y = -\frac{a_1d_2 - a_2d_1}{a_1b_2 - a_2b_1}$

Each of these numerators & denominators can be expressed as a determinant. Thus, we have:

$$x = \frac{\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \Rightarrow \frac{x}{\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } \frac{y}{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Combining these results, we obtain:

$$\frac{x}{\begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

If we denote $\Delta_x = \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$ and $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, then we shall have:

$$\frac{x}{\Delta_x} = \frac{-y}{\Delta_y} = \frac{1}{\Delta}$$

Note: To form Δ_x, omit the x-terms and write down the coefficients and constant terms in the order in which they stand. Similarly, to form Δ_y from the given equations, omit the y-terms and write down the coefficients and constant terms and finally to form Δ, omit the constant terms and write down the coefficients in the order in which they stand.

Example 5: Solve the pair of simultaneous equations by determinant method:

$$2x + y - 16 = 0$$
$$3x - 2y - 3 = 0$$

Solution:

$\frac{x}{\Delta_x} = \frac{-y}{\Delta_y} = \frac{1}{\Delta}$, where $\Delta_x = \begin{vmatrix} 1 & -16 \\ -2 & -3 \end{vmatrix} = -35$; $\Delta_y = \begin{vmatrix} 2 & -16 \\ 3 & -3 \end{vmatrix} = 42$; $\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = -7$;

$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-35}{-7} = 5,\ y = -\frac{\Delta_y}{\Delta} = -\frac{42}{-7} = 6$

$\therefore x = 5,\ y = 6$

Example 6: Solve the equations by determinants:

$$5x - 6y + 52 = 0$$
$$7x + 3y - 7 = 0$$

Solution:

$\frac{x}{\Delta_x} = \frac{-y}{\Delta_y} = \frac{1}{\Delta}$, where $\Delta_x = \begin{vmatrix} -6 & 52 \\ 3 & -7 \end{vmatrix} = -114$; $\Delta_y = \begin{vmatrix} 5 & 52 \\ 7 & -7 \end{vmatrix} = -399$; $\Delta = \begin{vmatrix} 5 & -6 \\ 7 & 3 \end{vmatrix} = 57$;

$\therefore x = \frac{\Delta_x}{\Delta} = -\frac{114}{57} = -2,\ y = -\frac{\Delta_y}{\Delta} = -\frac{399}{57} = -7$

$\therefore x = -2,\ y = 7$

Equations in Three Variables – By Determinant method

Consider the equations:

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$
$$a_3x + b_3y + c_3z + d_3 = 0$$

If we find x, y, and z by elimination method, we shall obtain the following results:

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

We can further reduce the above equation to the following:

$\frac{x}{\Delta_x} = \frac{-y}{\Delta_y} = \frac{z}{\Delta_z} = \frac{-1}{\Delta}$ where

$\Delta_x = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$ = the determinant of the coefficients omitting the x – terms;

$\Delta_y = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$ = the determinant of the coefficients omitting the y – terms;

$\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$ = the determinant of the coefficients omitting the z – terms;

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ = the determinant of the coefficients omitting the constant terms.

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Example 7: Solve the equations by determinant method:

$$x - 3y + 4z - 5 = 0$$
$$2x + y + z - 3 = 0$$
$$4x + 3y + 5z - 1 = 0$$

Solution:

$$\Delta_x = \begin{vmatrix} -3 & 4 & -5 \\ 1 & 1 & -3 \\ 3 & 5 & -1 \end{vmatrix} = -84; \quad \Delta_y = \begin{vmatrix} 1 & 4 & -5 \\ 2 & 1 & -3 \\ 4 & 5 & -1 \end{vmatrix} = -56; \quad \Delta_z = \begin{vmatrix} 1 & -3 & -5 \\ 2 & 1 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 28;$$

$$\Delta = \begin{vmatrix} 1 & -3 & 4 \\ 2 & 1 & 1 \\ 4 & 3 & 5 \end{vmatrix} = 28$$

But $\frac{x}{\Delta_x} = \frac{-y}{\Delta_y} = \frac{z}{\Delta_z} = \frac{-1}{\Delta}$

$x = \frac{-\Delta_x}{\Delta} = \frac{-(-84)}{28} = 3; \quad y = \frac{\Delta_y}{\Delta} = \frac{-56}{28} = -2; \quad z = \frac{-\Delta_z}{\Delta} = \frac{-28}{28} = -1$

$\therefore x = 3, \ y = -2, \ z = -1$

Example 8: Solve the linear equations by determinant method:

$$4x - 5y + 7z = -14$$
$$9x + 2y + 3z = 47$$
$$x - y - 5z = 11$$

Solution:

$$\frac{x}{\Delta_x} = \frac{-y}{\Delta_y} = \frac{z}{\Delta_z} = \frac{-1}{\Delta}$$

$$\Delta_x = \begin{vmatrix} -5 & 7 & 14 \\ 2 & 3 & -47 \\ -1 & -5 & -11 \end{vmatrix} = 1725; \quad \Delta_y = \begin{vmatrix} 4 & 7 & 14 \\ 9 & 3 & -47 \\ 1 & -5 & -11 \end{vmatrix} = -1380; \quad \Delta_z = \begin{vmatrix} 4 & -5 & 14 \\ 9 & 2 & -47 \\ 1 & -1 & -11 \end{vmatrix} = -690; \quad \Delta =$$

$$\begin{vmatrix} 4 & -5 & 7 \\ 9 & 2 & 3 \\ 1 & -1 & -5 \end{vmatrix} = -345$$

$x = \frac{-\Delta_x}{\Delta} = \frac{-1725}{-345} = 5; \quad y = \frac{\Delta_y}{\Delta} = \frac{-1380}{-345} = 4; \quad z = \frac{-\Delta_z}{\Delta} = \frac{-(-690)}{-345} = -2$

$\therefore x = 5, \ y = 4, \ z = -2$

8.2.3 By Cramer's Rule

Cramer's rule states that if we consider the set of equations:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$x = \frac{\Delta_x}{\Delta}; \quad y = \frac{\Delta_y}{\Delta}; \quad z = \frac{\Delta_z}{\Delta}$$

Example 9: Solve the set of linear equations by Cramer's rule

$$x - 3y + 4z - 5 = 0$$
$$2x + y + z - 3 = 0$$
$$4x + 3y + 5z - 1 = 0$$

Solution:

$$x - 3y + 4z = 5$$
$$2x + y + z = 3$$
$$4x + 3y + 5z = 1$$

$$\Delta = \begin{vmatrix} 1 & -3 & 4 \\ 2 & 1 & 5 \\ 4 & 3 & 5 \end{vmatrix} = 28; \quad \Delta_x = \begin{vmatrix} 5 & -3 & 4 \\ 3 & 1 & 1 \\ 1 & 3 & 5 \end{vmatrix} = 84; \quad \Delta_y = \begin{vmatrix} 1 & 5 & 4 \\ 2 & 3 & 1 \\ 4 & 1 & 5 \end{vmatrix} = -56;$$

$$\Delta_z = \begin{vmatrix} 1 & -3 & 5 \\ 2 & 1 & 3 \\ 4 & 3 & 1 \end{vmatrix} = -28$$

By Cramer's rule, $x = \frac{\Delta_x}{\Delta} = \frac{84}{28} = 3; \quad y = \frac{\Delta_y}{\Delta} = \frac{-56}{28} = -2; \quad z = \frac{\Delta_z}{\Delta} = \frac{-28}{28} = -1$

$\therefore x = 3, \ y = -2, \ z = -1$

Example 10: Solve the equations by Cramer's rule:

$$x_1 + 3x_2 + 2x_3 = 3$$
$$2x_1 - x_2 - 3x_3 = -8$$
$$5x_1 + 2x_2 + x_3 = 9$$

Solution:

$$\Delta = \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & -3 \\ 5 & 2 & 1 \end{vmatrix} = -28; \quad \Delta_{x_1} = \begin{vmatrix} 3 & 3 & 2 \\ -8 & -1 & -3 \\ 9 & 2 & 1 \end{vmatrix} = -56; \quad \Delta_{x_2} = \begin{vmatrix} 1 & 3 & 2 \\ 2 & -8 & -3 \\ 5 & 9 & 1 \end{vmatrix} = 84;$$

$$\Delta_{x_3} = \begin{vmatrix} 1 & 3 & 3 \\ 2 & -2 & -8 \\ 5 & 2 & 9 \end{vmatrix} = -140$$

By Cramer's rule, $x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{-56}{-28} = 2, \quad x_2 = \frac{\Delta_{x_2}}{\Delta} = \frac{84}{-28} = -3, \quad x_3 = \frac{\Delta_{x_3}}{\Delta} = \frac{-140}{-28} = 5$

$\therefore x_1 = 2, \ x_2 = -3, \ x_3 = 5$

8.2.4 By Gaussian Elimination Method

Example 11: Solve the set of linear equations by Gaussian elimination method:

$$x_1 - 3x_2 - 2x_3 = 8$$

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$$2x_1 + 2x_2 + x_3 = 4$$
$$3x_1 - 4x_2 + 2x_3 = -3$$

Solution:

We begin by forming the matrix equation:

$$\begin{pmatrix} 1 & -3 & -2 \\ 2 & 2 & 1 \\ 3 & -4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ -3 \end{pmatrix}$$

We then form the augmented matrix by including the constants as an extra column on the right-hand side of the matrix

$$\left(\begin{array}{ccc|c} 1 & -3 & -2 & 8 \\ 2 & 2 & 1 & 4 \\ 3 & -4 & 2 & -3 \end{array}\right)$$

Subtracting $\frac{2}{1}$ times the first row from the 2nd row, and subtracting $\frac{3}{1}$ times the first row from the 3rd row, we have:

$$\left(\begin{array}{ccc|c} 1 & -3 & -2 & 8 \\ 0 & 8 & 5 & -12 \\ 0 & 5 & 8 & -27 \end{array}\right)$$

Subtracting $\frac{5}{8}$ times the 2nd row from the 3rd row,

$$\left(\begin{array}{ccc|c} 1 & -3 & -2 & 8 \\ 0 & 8 & 5 & -12 \\ 0 & 0 & \frac{39}{8} & -\frac{39}{2} \end{array}\right)$$

Finally, we detach the right-hand column back to its original position

$$\begin{pmatrix} 1 & -3 & -2 \\ 0 & 8 & 5 \\ 0 & 0 & \frac{39}{8} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -12 \\ -\frac{39}{2} \end{pmatrix}$$

Expanding from the bottom row, working upwards, we obtain:

$$\frac{39}{8}x_3 = -\frac{39}{2} \Rightarrow x_3 = -4$$

$$8x_2 + 5x_3 = -12 \Rightarrow 8x_2 + 5(-4) = -12 \Rightarrow x_2 = 1$$

$$x_1 - 3x_2 - 2x_3 = 8 \Rightarrow x_1 - 3(1) - 2(-4) = 8 \Rightarrow x_1 = 3$$

$$\therefore x_1 = 3, \quad x_2 = 1, \quad x_3 = -4$$

Example 12: Use Gaussian elimination method to solve the equation A . x = b, where

$$A = \begin{pmatrix} 1 & -2 & -4 \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{pmatrix} \text{ and } b = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 1 & -2 & -4 \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$$

We form the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & -4 & -3 \\ 2 & 1 & -3 & 4 \\ 1 & 3 & 2 & 5 \end{array}\right)$$

Subtracting $\frac{2}{1}$ times the first row from the 2nd row and $\frac{1}{1} = 1$ times the first row from the 3rd row, we have:

$$\left(\begin{array}{ccc|c} 1 & -2 & -4 & -3 \\ 0 & 5 & 5 & 10 \\ 0 & 5 & 6 & 8 \end{array}\right)$$

Subtracting $\frac{5}{5} = 1$ times the 2nd row from the 3rd row, we obtain:

$$\left(\begin{array}{ccc|c} 1 & -2 & -4 & -3 \\ 0 & 5 & 5 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Detaching the right-hand column back to its former position, we obtain:

$$\begin{pmatrix} 1 & -2 & -4 \\ 0 & 5 & 5 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ -2 \end{pmatrix}$$

Finally, expanding from the bottom row upwards, we have:

$$x_3 = -2; \quad 5x_2 + 5x_3 = 10 \Rightarrow 5x_2 = 20 \therefore x_2 = 4$$

$$x_1 - 2x_2 - 4x_3 = -2 \Rightarrow x_1 - 2(4) - 4(-2) = -2 \therefore x_1 = -3$$

$$\therefore x_1 = -3, \quad x_2 = 4, \quad x_3 = -2$$

Example 13: Use Gaussian elimination method to find the values of x, y & z in the following set of equations:

$$x + 3y + 2z = 3$$
$$2x - y - 3z = -8$$
$$5x + 2y + z = 9$$

Solution:

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & -3 \\ 5 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ 9 \end{pmatrix}$$

We form the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 2 & -1 & -3 & -8 \\ 5 & 2 & 1 & 9 \end{array}\right)$$

Subtracting $\frac{2}{1}$ times the 1st row from the 2nd row & $\frac{5}{1}$ times the 1st row from the 3rd row, we have:

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & -7 & -7 & -14 \\ 0 & -13 & -9 & -6 \end{array}\right)$$

Subtracting $\frac{-13}{-7}$ times the 2nd row from the 3rd row, we have:

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$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & -7 & -7 & -14 \\ 0 & 0 & 4 & 20 \end{array}\right)$$

Detaching the right-hand column back to its former position, we have:

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & -7 & -7 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -14 \\ 20 \end{pmatrix}$$

Finally, expanding from the bottom row upwards, we obtain:

$4z = 20 \therefore z = 5$

$-7y - 7z = -14 \Rightarrow -7y - 7(5) = -14 \Rightarrow -7y = 21 \therefore y = -3$

$x + 3y + 2z = 3 \Rightarrow x + 3(-3) + 2(5) = 3 \Rightarrow \therefore x = 2$

$\therefore x = 2,\ y = -3,\ z = 5$

The above example is similar to example 10 which we solved by Cramer's rule.

EXERCISE 12.2

9. Solve the set of linear equations by matrix method
   $5x_1 + 2x_2 + 3x_3 = 6$
   $3x_1 - 2x_2 - 2x_3 = 5$
   $4x_1 + 3x_2 + x_3 = -5$ **Ans: $x_1 = 1,\ x_2 = -4,\ x_3 = 3$**
10. If $A.x = b$, where $A = \begin{pmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 2 & -3 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 10 \\ 9 \end{pmatrix}$, determine A^{-1} and hence solve the set of equations. **Ans: $x_1 = -4,\ x_2 = 2,\ x_3 = -3$**
11. Use matrix method to find solution of the set of linear equations:
    $x + 2y + 3z = 5$
    $2x - 3y - z = 3$
    $-3x + 4y + 5z = 3$ **Ans: $x = 1,\ y = -1,\ z = 2$**
12. Repeat the above using determinant method
13. Repeat the above using Cramer's rule
14. Solve the following set of equations by Gaussian elimination method:
    $x_1 + 2x_2 + 3x_3 = 5$
    $3x_1 - x_2 + 2x_3 = 8$
    $4x_1 - 6x_2 - 4x_3 = -2$ **Ans: $x_1 = -1,\ x_2 = -3,\ x_3 = 4$**
15. Use Gaussian elimination method to solve the equations:
    $x - 3y + 2z = 8$
    $2x - y + z = 9$
    $3x + 2y + 3z = 5$ **Ans: $x = 4,\ y = -2,\ z = -1$**
16. Use Gaussian elimination method to solve the equation given that $Ax = b$, where $A = \begin{pmatrix} 5 & 2 & 3 \\ 3 & -2 & -2 \\ 4 & 3 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ 5 \\ -5 \end{pmatrix}$ **Ans: $x_1 = 1,\ x_2 = -4,\ x_3 = 3$**

17. Solve by Gaussian elimination method:
    $x_1 - 2x_2 + x_3 = 7$
    $3x_1 + x_2 - 2x_3 = -3$
    $5x_1 + 3x_2 + 3x_3 = 5$ **Ans: $x_1 = 1,\ x_2 = -2,\ x_3 = 2$**
18. Solve the set of linear equations by Cramer's rule:
    $4x - 3y + 2z = -7$
    $6x + 2y - 3z = 33$
    $2x - 4y - z = -3$
19. Solve the set of equations by matrix method:
    $3x_1 + 2x_2 - 2x_3 = 16$
    $4x_1 + 3x_2 + 3x_3 = 2$
    $-2x_1 + x_2 - x_3 = 1$

8.3 EIGENVALUES AND EIGENVECTORS

Definitions

Let A be a complex square matrix. Then, if λ is a complex number and X a non-zero complex column vector satisfying the equation $AX = \lambda X$, we call X an eigenvector of A, while λ is called an eigenvalue of A. We also say that X is an eigenvector corresponding to the eigenvalue λ.

Consider a set of equations:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{.....(i)}$$

For the set of homogeneous linear equations (right-hand side constant terms all zero) to have non-trivial solutions, **$|A - \lambda I|$ must be zero.** This simplifies into the following:

$$\begin{pmatrix} (a_{11} - \lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{22} - \lambda) & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & (a_{nn} - \lambda) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{.....(ii)}$$

From eqns (i) and (ii), we can easily deduce that **$Ax = \lambda x$ or $Ax - \lambda x = 0$ or**

$$(A - \lambda I)x = 0$$

Where **I** is the unit matrix being introduced since we can subtract only a matrix from another matrix. This now brings us to our earlier assertion that for non-trivial solutions,

$|A - \lambda I|$ must be zero.

This is known as the *characteristic determinant* of the matrix **A** and **$|A - \lambda I| = 0$** is the *characteristic equation*, the values of λ are known as the *eigenvalues, characteristic values, or latent roots* of the matrix **A** and the corresponding solutions of the given equations **$Ax = \lambda x$** are called the *eigenvectors, or characteristic vectors* of **A.**

Example 14: If $Ax = \lambda x$, determine the eigenvalues & eigenvectors of the matrix A where $A = \begin{pmatrix} 2 & -5 \\ 1 & -4 \end{pmatrix}$

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Solution:

The characteristic equation is $|\mathbf{A} - \lambda\mathbf{I}| = \mathbf{0}$. That is

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -4-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(-4-\lambda) + 5 = 0 \Rightarrow \lambda^2 + 2\lambda - 3 = 0 \Rightarrow \lambda = 1 \text{ or } -3$

$\therefore \lambda_1 = 1$ and $\lambda_2 = -3$

For $\lambda_1 = 1$, $\begin{bmatrix} (2-\lambda) & -5 \\ 1 & (-4-\lambda) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -5 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - 5x_2 = 0 \Rightarrow x_1 = 5x_2 \therefore x_1 = 5$ and $x_2 = 1$

Hence, for $\lambda_1 = 1$, $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

For $\lambda_2 = -3$, $\begin{bmatrix} (2-\lambda) & -5 \\ 1 & (-4-\lambda) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & -5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$5x_1 - 5x_2 = 0 \Rightarrow x_1 = x_2 \therefore x_1 = 1$ and $x_2 = 1$

Hence, for $\lambda_2 = -3$, $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore X_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 1$

$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = -3$

Example 15: Determine the eigenvalues and corresponding eigenvectors of the matrix A, where $A = \begin{pmatrix} -6 & 5 \\ 4 & 2 \end{pmatrix}$

Solution:

The characteristic equation is $|\mathbf{A} - \lambda\mathbf{I}| = \mathbf{0}$. That is

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -6-\lambda & 5 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$(-6-\lambda)(2-\lambda) - 20 = 0 \Rightarrow \lambda^2 + 4\lambda - 32 = 0 \Rightarrow \lambda = -8 \text{ or } 4$

$\therefore \lambda_1 = -8$ and $\lambda_2 = 4$

For $\lambda_1 = -8$, we substitute in the characteristic equation

$\begin{pmatrix} 2 & 5 \\ 4 & 10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2x_1 + 5x_2 = 0 \Rightarrow 5x_2 = -2x_1 \therefore x_2 = -2,\ x_1 = 5$

$\therefore X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ for $\lambda_1 = -8$

For $\lambda_2 = 4$, we substitute in the characteristic equation

$\begin{pmatrix} -10 & 5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-10x_1 + 5x_2 = 0 \Rightarrow 5x_2 = 10x_1 \Rightarrow x_2 = 2x_1 \therefore x_2 = 2,\ x_1 = 1$

$\therefore X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for $\lambda_2 = 4$

Hence, the required eigenvectors are:

$X_1 = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ Corresponding to $\lambda_1 = -8$

$X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ Corresponding to $\lambda_2 = 4$

Example 16: If $A.x = \lambda x$, where $A = \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, determine the eigenvalues and corresponding eigenvectors of the matrix A.

Solution:

As usual, we obtain the characteristic equation

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} (2-\lambda) & 2, & -2 \\ 1 & (3-\lambda) & 1 \\ 1 & 2 & (2-\lambda) \end{vmatrix} = 0$$

$(2-\lambda)\{(3-\lambda)(2-\lambda) - 2\} - 2\{(2-\lambda) - 1\} - 2\{2 - (3-\lambda)\} = 0$

$(2-\lambda)(3-\lambda)(2-\lambda) - 2(2-\lambda) - 2(2-\lambda) + 2 - 4 + 2(3-\lambda) = 0$

$(2-\lambda)(\lambda^2 - 5\lambda + 6) - 2(2-\lambda) = 0$

$(2-\lambda)(\lambda^2 - 5\lambda + 4) = 0 \Rightarrow \lambda = 2,\ 4,\ \text{or } 1$

$\therefore \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 4$

For $\lambda_1 = 1$,

$\begin{pmatrix} (2-\lambda) & 2 & -2 \\ 1 & (3-\lambda) & 1 \\ 1 & 2 & (2-\lambda) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ Putting $\lambda_1 = 1$, we obtain

$\begin{pmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x_1 + 2x_2 - 2x_3 = 0$ .....(i)

$x_1 + 2x_2 + x_3 = 0$ .....(ii)

From eqns (i) & (ii), we have: $x_1 = 2x_3 - 2x_2 = -x_3 - 2x_2$

$3x_3 = 0 \therefore x_3 = 0$ and $x_1 = -2x_2 \Rightarrow \therefore x_1 = -2,\ x_2 = 1$

$\therefore x_1 = -2,\ x_2 = 1,\ x_3 = 0$

$\therefore X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ for $\lambda_1 = 1$

For $\lambda_2 = 2$,

$\begin{pmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$2x_2 - 2x_3 = 0 \Rightarrow x_2 = x_3 \therefore x_2 = 1,\ x_3 = 1$

$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2 \therefore x_1 = -2,\ x_2 = 1$

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$\therefore X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ for $\lambda_2 = 2$

For $\lambda_3 = 4$,

$$\begin{pmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ 1 & 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2x_1 + 2x_2 - 2x_3 = 0$ .....(i)

$x_1 - x_2 + x_3 = 0$ .....(ii)

$x_1 + 2x_2 - 2x_3 = 0$ .....(iii)

From eqns (ii) & (iii), we obtain: $x_1 = x_2 - x_3 = 2x_3 - 2x_2 \Rightarrow 3x_2 = 3x_3 \therefore x_2 = 1, x_3 = 1$

$\therefore x_1 = 0$

$\therefore X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ for $\lambda_3 = 4$

Hence, the required eigenvectors are:

$X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ Corresponding to $\lambda_1 = 1$

$X_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ Corresponding to $\lambda_2 = 2$

$X_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ Corresponding to $\lambda_3 = 4$

Example 17: Given that $Ax = \lambda x$, where $A = \begin{bmatrix} 4 & 10 & -8 \\ 1 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$, determine the eigenvalues and the corresponding eigenvectors of the matrix A.

Solution:

$$|A - \lambda I| = \begin{vmatrix} (4-\lambda) & 10 & -8 \\ 1 & (2-\lambda) & 1 \\ -1 & 2 & (3-\lambda) \end{vmatrix} = 0$$

$(4-\lambda)\{(2-\lambda)(3-\lambda) - 2\} - 10(3 - \lambda + 1) - 8(2 + 2 - \lambda) = 0$

$(4-\lambda)(\lambda^2 - 5\lambda + 4) - 18(4 - \lambda) = 0$

$(4-\lambda)(\lambda - 7)(\lambda + 2) = 0 \Rightarrow \lambda = -2, 4, 7$

$\therefore \lambda_1 = -2, \lambda_2 = 4, \lambda_3 = 7$

For $\lambda_1 = -2$,

$$\begin{bmatrix} 6 & 10 & -8 \\ 1 & 4 & 1 \\ -1 & 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$6x_1 + 10x_2 - 8x_3 = 0$ .....(i)

$x_1 + 4x_2 + x_3 = 0$ .....(ii)

$-x_1 + 2x_2 + 5x_3 = 0$ .....(iii)

From eqns (ii) & (iii), we have: $x_1 = -4x_2 - x_3 = 2x_2 + 5x_3$

$2x_2 + 5x_3 = -4x_2 - x_3 \Rightarrow x_2 = -x_3 \Rightarrow x_2 = -1, x_3 = 1$

But $x_1 = -4x_2 - x_3 \Rightarrow x_1 = -4(-1) - 1 = 3$

$\therefore x_1 = 3, x_2 = -1, x_3 = 1$

$-4x_2 - x_3 = 2x_2 + 5x_3$

$-4x_2 - 2x_2 = 6x_3$

$-6x_2 = 6x_3$

$x_2 = 1$

$x_3 = -1$

$\therefore X_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ Corresponding to the eigenvalue $\lambda_1 = -2$

For $\lambda_2 = 4$,

$$\begin{bmatrix} 0 & 10 & -8 \\ 1 & -2 & 1 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$10x_2 - 8x_3 = 0$ .....(i)

$x_1 - 2x_2 + x_3 = 0$ .....(ii)

$-x_1 + 2x_2 - x_3 = 0$ .....(iii)

From eqn (i), $5x_2 = 4x_3 \therefore x_2 = 4, x_3 = 5$

Subst. in eqn (ii), $x_1 = 2x_2 - x_3 = 2(4) - 5 = 3$

$\therefore x_1 = 3, x_2 = 4, x_3 = 5$

$\therefore X_2 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ Corresponding to the eigenvalue $\lambda_2 = 4$

For $\lambda_3 = 7$,

$$\begin{bmatrix} -3 & 10 & -8 \\ 1 & -5 & 1 \\ -1 & 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 10x_2 - 8x_3 = 0$ .....(i)

$x_1 - 5x_2 + x_3 = 0$ .....(ii)

$-x_1 + 2x_2 - 4x_3 = 0$ .....(iii)

From eqns (iii) & (ii), we have: $x_1 = 2x_2 - 4x_3 = 5x_2 - x_3 \Rightarrow -3x_2 = 3x_3$

$\therefore x_2 = 1, x_3 = -1$

Subst. in eqn (iii), $x_1 = 2x_2 - 4x_3 = 2(1) - 4(-1) = 6$

$\therefore x_1 = 6, x_2 = 1, x_3 = -1$

$\therefore X_3 = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$ Corresponding to the eigenvalue $\lambda_3 = 7$

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Hence, the required eigenvectors are:

$$X_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \text{ for } \lambda_1 = -2; \quad X_2 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \text{ for } \lambda_2 = 4; \quad X_3 = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix} \text{ for } \lambda_3 = 7$$

EXERCISE 8.3

Given that $Ax = \lambda x$, determine the eigenvalues and the corresponding eigenvectors of the following matrices:

20. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$

21. $A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

22. $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 10 & 4 \\ 1 & 3 & -3 \end{bmatrix}$

23. $A = \begin{pmatrix} -3 & 0 & 6 \\ 4 & 5 & 3 \\ 1 & 2 & 1 \end{pmatrix}$

24. $A = \begin{bmatrix} 1 & -3 & 5 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

25. $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$

26. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

Compute the eigenvalues and eigenvectors of A and A^2

27. $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$

28. $A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$

29. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

30. $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Compute the eigenvalues and eigenvectors of A and B and A + B

31. $A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ and $A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

Chapter 9: Fourier Analysis

9.1 INTRODUCTION:

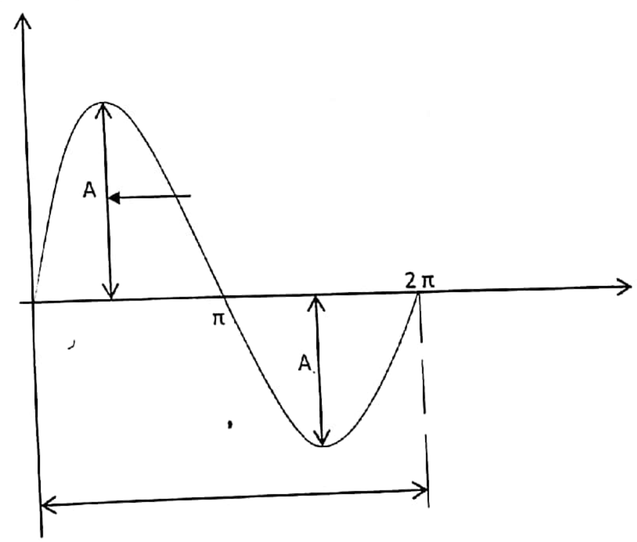
A signal whether continuous-time or discrete-time may be analyzed with the help of Fourier series and Fourier Transform. While Fourier series is used for periodic signals, Fourier Transform is used for aperiodic signals. In general, Fourier Transform may be used for both periodic & aperiodic signals. Fourier series represents a periodic waveform in the form of sum of infinite number of sine and cosine terms. The series provides a method of analyzing periodic functions into their constituent components. It is a representation of a signal in a time domain series form. One of the advantages of Fourier series is that it can be used to represent both continuous and discontinuous functions. Fourier series is a tool for the analysis of any periodic signal. This is quite unlike Maclaurin's and Taylor's series or Power series, which require the function to be continuous throughout. Hence, Fourier series can effectively represent a function with discontinuities. This is actually a great advantage!

9.2 PERIODIC FUNCTION:

A function f(x) is said to be periodic, if its function values repeat at regular intervals of the independent variable. The regular interval between repetitions is said to be the period of oscillation. Hence, a function $y = f(x)$ is a periodic function if and only if there is real number p such that $f(x + p) = f(x)$ where p is the period of the function f(x).

For example, $f(x + 2p)$ is a periodic function, since $f(x + 2p) = f(x + p + p) = f(x + p) = f(x)$.

A typical example of a periodic function is $y = \sin x$ for $0 \le x \le 2\pi$



In general, if $y = \sin \omega t$, then the period of the waveform is $p = \frac{2\pi}{\omega}$. For example, the function shown below has period of 2π and is defined by:

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

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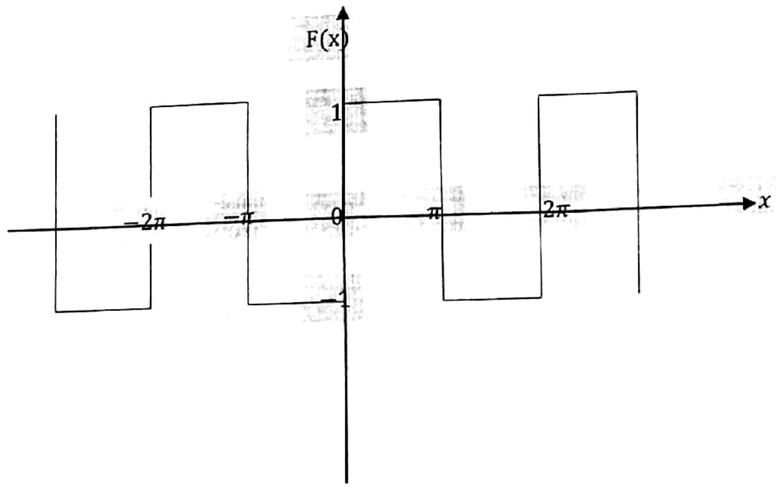


Fig. 9.2: A Square Wave Graph

If a graph of a function has no sudden jumps or breaks, it is called a continuous function. An example is the graphs of sine and cosine functions. However, some other graphs make finite jumps at a point or points in the interval. For example, the square wave of figure 2 has finite discontinuities at $x = \pi,\ 2\pi,\ 3\pi$, etc. That is, typical periodic functions in Engineering Mathematics are sine and cosine functions and have continuous graphs and are easily differentiable. Other examples of periodic functions are the square wave, saw-tooth and rectified half-wave used to represent voltages or currents.

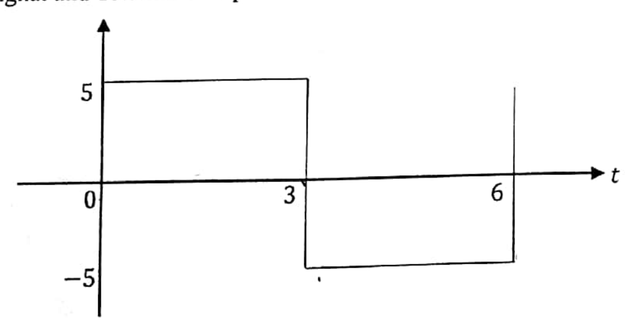
9.3 ANALYTIC DESCRIPTION OF A PERIODIC FUNCTION

A periodic function can be described analytically in many ways. Let us consider a few examples.

Example 1: A signal $f(t)$ is defined by the expression $f(t) = \begin{cases} 5 & 0 \le t \le 3 \\ -5 & 3 \le t \le 6 \end{cases}$

Sketch the signal and determine its period.

Solution:

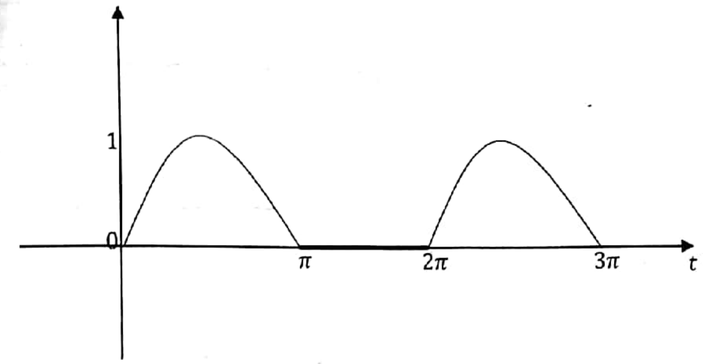


From the diagram above, we can see that the period is 6.

Example 2: A signal $f(t)$ is defined by the expression: $f(t) = \begin{cases} \sin t & 0 \le t \le \pi \\ 0 & \pi \le t \le 2\pi \\ \sin t & 2\pi \le t \le 3\pi \end{cases}$

Sketch the signal and determine its period.

Solution:

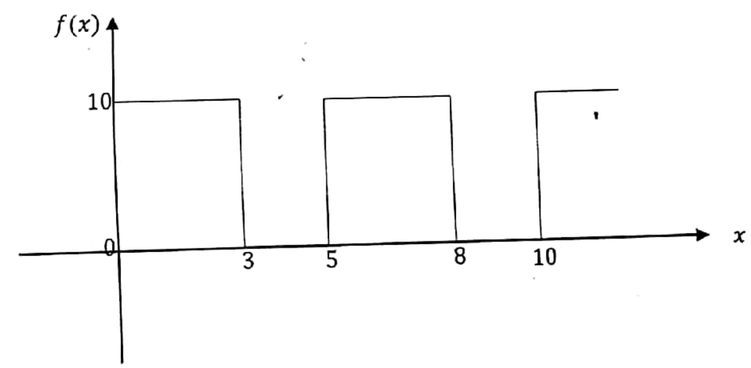


From the graph above, we can see that the period is 2π.

Example 3: A periodic function is defined by the expression: $f(x) = \begin{cases} 10 & 0 < x < 3 \\ 0 & 3 < x < 5 \end{cases}$

Determine the period of the function using the analytic method.

Solution:



Between $x = 0$ and $= 3, y = 10$; and between $x = 3$ and $x = 5, y = 0$. Hence, the period of the function is 5 units. That is, $f(x) = f(x + 5)$.

Example 4: Given that $f(x) = \begin{cases} x & 0 \le x \le 2 \\ 3 - \frac{x}{2} & 2 \le x \le 6 \end{cases}$

Sketch a graph of the function using the analytic method, and determine its period.

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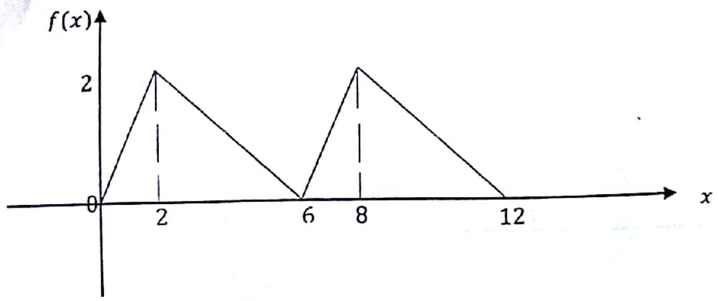
Solution:

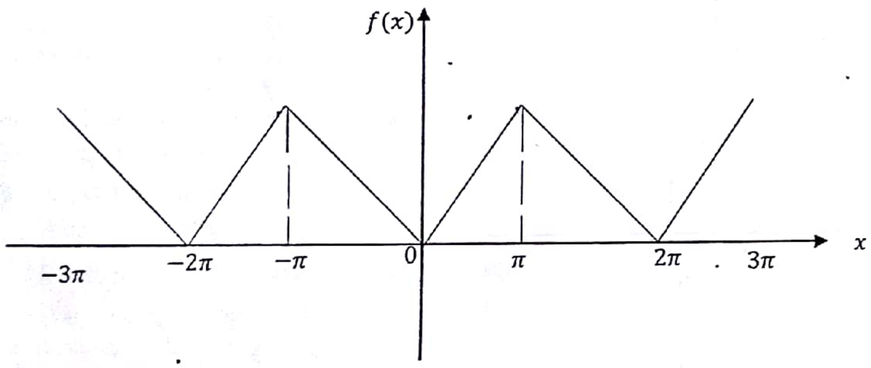


From the graph above, we can easily deduce that the period is 6 units.

Example 5: Given that $f(x) = \begin{cases} -x, & if\ -\pi < x < 0 \\ 0, & if\ x = 0 \\ x, & if\ 0 < x < \pi \end{cases}$

Sketch a graph of the function using the analytic method, and determine its period.

Solution:



EXERCISE 9.1

For each of the following functions, use the analytic method to graph the functions. Hence, determine the period.

1. $f(x) = \begin{cases} 3 & 0 < x < 5 \\ -1 & 5 < x < 8 \end{cases}$

2. $f(x) = \begin{cases} \frac{x}{2} & 0 < x < \pi \\ \pi - \frac{x}{2} & \pi < x < 2\pi \end{cases}$

3. $f(x) = \begin{cases} -\cos x & -\pi < x < 0 \\ 0 & x = 0 \\ \cos x & 0 < x < \pi \end{cases}$

Determine the Fourier series up to the 5th element.

Solution:

Let us begin by determining the values of the constants.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\mathrm{d}x + \int_{0}^{\pi} x\mathrm{d}x\right] = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0} -x\cos nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\mathrm{d}x$$

$$= -\frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right] + \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right] = -\frac{1}{n^2\pi}[1 - \cos n\pi] + \frac{1}{n^2\pi}[\cos n\pi - 1]$$

$\therefore\ a_n = -\frac{4}{n^2\pi}$ for odd values of n (i.e. $1, 3, 5, \ldots$)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{0} -x\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x = \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} = 0$$

Having obtained the Fourier coefficients, we then substitute in the Fourier expression as:

$$\mathbf{f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx}$$

$f(x) = \frac{\pi}{2} + -\frac{4}{n^2\pi}\cos nx$ (for odd values of n = $1, 3, 5, \ldots$)

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{3^2\pi}\cos 3x - \frac{4}{5^2\pi}\cos 5x - \cdots = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right]$$

Alternatively,

Since the graph of the function is symmetrical about the y-axis, the function $f(x)$ is even. Hence, $\mathbf{b_n = 0}$.

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\mathrm{d}x + \int_{0}^{\pi} x\mathrm{d}x\right] = \frac{\pi}{2}$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -x\cos nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\mathrm{d}x = -\frac{4}{n^2\pi}$ (for odd values of n)

Hence, the Fourier series $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right]$

Example 8: Given that $f(x) = \begin{cases} \pi + x, & if\ -\pi < x < 0 \\ 0, & if\ x = 0, \\ \pi - x, & if\ 0 < x < \pi \end{cases}$

Determine the Fourier series of the function.

Solution:

Since the function is symmetrical about the y-axis, it is even. Hence, $\mathbf{b_n = 0}$.

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(\pi + x)\mathrm{d}x + \int_{0}^{\pi}(\pi - x)\mathrm{d}x\right] = \frac{\pi}{2}$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi + x)\cos nx\mathrm{d}x + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nx\mathrm{d}x = \frac{4}{n^2\pi}$ (for odd values of n only)

Hence, the Fourier series $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$f(x) = \frac{\pi}{2} + \frac{4}{n^2\pi}\cos nx$ (for odd values of n = $1, 3, 5, \ldots$)

$$\therefore f(x) = \frac{\pi}{2} + \frac{4}{\pi}\cos x + \frac{4}{3^2\pi}\cos 3x + \frac{4}{5^2\pi}\cos 5x + \cdots = \frac{\pi}{2} + \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right]$$

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Example 9: A periodic signal $f(t)$ is given by the expression: $f(t) = \begin{cases} -c, & if -\pi < t < 0 \\ 0, & if\ t = 0, \\ c, & if\ 0 < t < \pi \end{cases}$

Determine the Fourier series of the signal.

Solution:

Since the graph of the signal is symmetrical about the origin, it is odd. Hence, $a_n = a_0 = 0$. We are left with only the sine terms.

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{dx} = \frac{1}{\pi}\int_{-\pi}^{0} -c\sin nx\,\mathrm{dx} + \int_{0}^{\pi} c\sin nx\,\mathrm{dx}$

$b_n = \frac{c}{n\pi}[1-\cos n\pi] - \frac{c}{n\pi}[\cos n\pi - 1] = \frac{4c}{n\pi}$ (for odd values of n)

Hence, the Fourier series $f(x) = \sum_{n=1}^{\infty} \mathrm{b_n}\sin \mathrm{nx} = \frac{4c}{n\pi}\sin nx$ (for odd values of n.)

$\therefore f(x) = \frac{4c}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$

Example 10: Determine the Fourier series for the signal defined by: $f(t) = \begin{cases} -t, & if -\pi < t < 0 \\ 0, & if\ 0 < t < \pi \end{cases}$

Solution:

As usual, we begin by finding the values of the Fourier coefficients.

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} f(t)\mathrm{dt} = \frac{1}{2\pi}\int_{-\pi}^{0} -t\mathrm{dt} + \frac{1}{2\pi}\int_{0}^{\pi} 0\mathrm{dt} = \frac{\pi}{4}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\mathrm{dt} = \frac{1}{\pi}\int_{0}^{\pi} -t\cos nt\mathrm{dt} = -\frac{2}{n^2\pi}$ (for odd values of n) [Using integration by parts]

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,\mathrm{dt} = \frac{1}{\pi}\int_{0}^{\pi} -t\sin nt\,\mathrm{dt} = -\frac{\cos n\pi}{n} = \begin{cases} -\frac{1}{n}, & \text{if n is even} \\ \frac{1}{n}, & \text{if n is odd} \end{cases}$

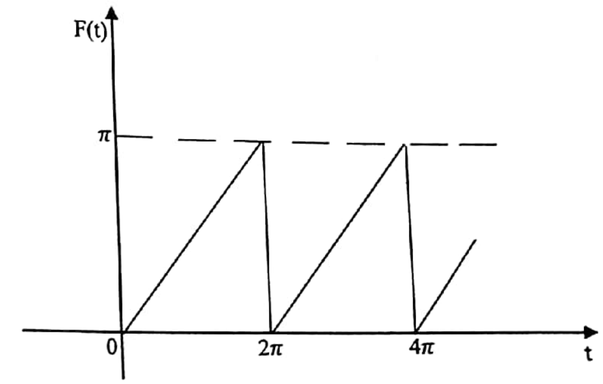
Next, we substitute the values so obtained in the Fourier equation, as shown below.

$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[-\frac{2}{n^2\pi}\cos nt + \frac{1}{n}\sin nt - \frac{1}{n}\sin nt\right]$

$f(t) = \frac{\pi}{4} - \frac{2}{\pi}\cos t - \frac{2}{9\pi}\cos 3t - \frac{2}{25\pi}\cos 5t - \cdots + \sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots - \frac{1}{2}\sin 2t - \frac{1}{4}\sin 4t - \ldots$

$f(t) = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos t + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \cdots\right] + \left[\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \frac{1}{5}\sin 5t - \cdots\right]$

Example 11: Determine the Fourier series for the periodic signal shown below:



Solution:

$y = \mathrm{mx} + \mathrm{c}$, where $\mathrm{m} = \frac{\Delta y}{\Delta x} = \frac{\pi}{2\pi} = \frac{1}{2}$, and $\mathrm{c} = 0$.

Hence, $\mathrm{y} = \frac{1}{2}x + 0$. In the case of the graph above, we obtain $\mathrm{y} = \frac{1}{2}t + 0 \rightarrow f(t) = \frac{1}{2}t$

Therefore, the equation of the graph above is $f(t) = \frac{1}{2}t$ at $0 < t < 2\pi$

Next, we determine the values of the Fourier coefficients.

$a_0 = \frac{1}{2\pi}\int_{0}^{2\pi} f(t)\mathrm{dt} = \frac{1}{2\pi}\int_{0}^{2\pi}\frac{1}{2}t\mathrm{dt} = \frac{1}{4\pi}\int_{0}^{2\pi} t\mathrm{dt} = \frac{\pi}{2}$

$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(t)\cos nt\mathrm{dt} = \frac{1}{\pi}\int_{0}^{2\pi}\frac{1}{2}t\cos nt\mathrm{dt} = \frac{1}{2\pi}\int_{0}^{2\pi} t\cos nt\mathrm{dt} = 0$

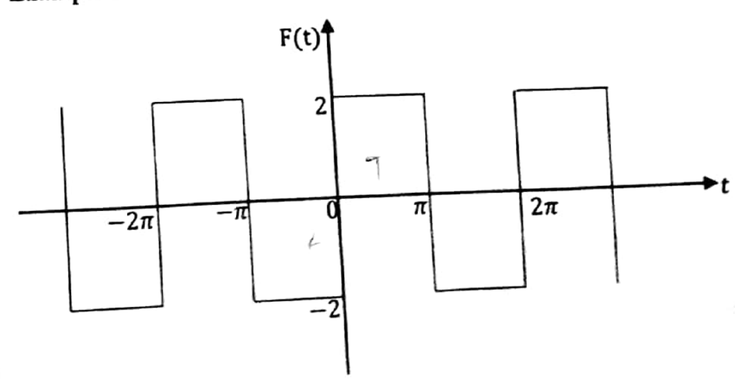
$b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(t)\sin nt\,\mathrm{dt} = \frac{1}{\pi}\int_{0}^{2\pi}\frac{1}{2}t\sin nt\,\mathrm{dt} = \frac{1}{2\pi}\int_{0}^{2\pi} t\sin nt\,\mathrm{dt} = -\frac{1}{n}$

Having obtained the values of the Fourier coefficients, we substitute in the Fourier expression to form the series.

$f(t) = a_0 + \sum_{n=1}^{\infty}\mathrm{b_n}\sin \mathrm{nt} = \frac{\pi}{2} + \sum_{n=1}^{\infty} -\frac{1}{n}\sin \mathrm{nt} = \frac{\pi}{2} + \left[-\sin t - \frac{1}{2}\sin 2t - \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t - \cdots\right]$

$\therefore f(t) = \frac{\pi}{2} - \left[\sin t + \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t + \frac{1}{4}\sin 4t + \cdots\right]$

Example 12: Determine the Fourier series for the signal shown below:



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Solution:

The student is expected to try this. When you are done, check whether your solution corresponds with the one in the box.

$$\boxed{f(t) = \frac{8}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right]}$$

Since the signal is symmetrical about the origin, it is odd. Hence, $a_0 = 0$ and $a_n = 0$. We are then left with b_n.

$\therefore f(t) = \sum_{n=1}^{\infty} b_n \sin nt$

From the diagram above, we note that the function is defined by $f(t) = \begin{cases} -2, & -\pi < t < 0 \\ 2, & 0 < t < \pi \end{cases}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -2\sin nt dt + \frac{1}{\pi}\int_{0}^{\pi} 2\sin nt dt = \frac{8}{n\pi}$ (if n is odd, i.e. 1, 3, 5, ...)

$f(t) = \sum_{n=1}^{\infty} b_n \sin nt = \sum_{n=1}^{\infty} \frac{8}{n\pi}\sin nt = \frac{8}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right]$

EXERCISE 9.2

11. Given that $f(x) = \begin{cases} -\cos x, & if -\pi < x < 0 \\ 0, & if\ x = 0 \\ \cos x, & if\ 0 < x < \pi \end{cases}$
    (a) Sketch a graph of the function
    (b) Determine the Fourier series of $f(x)$

12. A signal is defined by the expression $f(t) = \begin{cases} t, & if\ 0 \le t \le 1 \\ 1, & if\ 1 < t \le 3 \\ (4-t), & if\ 3 < t \le 4 \end{cases}$

    Sketch a graph of the signal using the analytic method, and find its Fourier series.

13. If $f(x) = \begin{cases} x, & if\ 0 < x < \frac{\pi}{2} \\ \pi - x, & if\ \frac{\pi}{2} < x < \pi \end{cases}$

    Sketch a graph of the function, and hence determine the Fourier series of $f(x)$.

14. Given that $f(x) = \begin{cases} 1-x, & if -\pi < x < 0 \\ 0, & if\ x = 0, \\ 1+x, & if\ \ 0 < x < \pi \end{cases}$ $f(x+2\pi) = f(x)$
    (a) Sketch a graph of $f(x)$ up to 6π
    (b) Find the Fourier series of $f(x)$ up to the 5th element.

Chapter 10: Laplace Analysis

10.1 INTRODUCTION

The Laplace transform is an integral transform named after mathematician and astronomer Pierre-Simon Laplace, who happens to be its discoverer. It is an operation that transforms a function of t (i.e. a function of time domain), defined on [0, ∞], to a function of s (i.e. of frequency domain). F(s) is the Laplace transform, or simply the transform, of $f(t)$. Hence, the Laplace transform is a frequency-domain approach for continuous time signals irrespective of whether the system is stable or unstable. The Laplace transform is used to solve linear differential equations (both ordinary and partial). It can be used to find the solution of first order differential equations, homogeneous first order differential equations, linear first order differential equations, numerical methods for first order differential equations and second order differential equations. Laplace transforms provide a much more systematic way of solving such equations by converting the differential equation into an algebraic equation and also have the added advantage of considering or incorporating the boundary conditions from the beginning.

10.2 DEFINITIONS OF LAPLACE TRANSFORMS

Let $f(t)$ be defined for $t \ge 0$. The Laplace transform of a function $f(t)$ is denoted by F(s) or $Lf\{(t)\}$ and is defined by the integral:

$$\boxed{L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt}$$

The Laplace transform of $f(t)$ is a function of s. From the above equation, the parameter s is assumed to be positive and a real number, and large enough to ensure that the integral converges. The Laplace transform is said to exist, if the integral exists. The symbol L is called the transform operator. The following are common notations for the Laplace transform: $L\{f(t)\}, or\ L(f), or\ Lf(t), or\ Lf\ or\ \bar{f}(s)$. In this book, we will adopt any of the notations especially the first notation.

10.3 APPLICATIONS OF LAPLACE TRANSFORMS

The Laplace transform is applied frequently in engineering and physics, much extensively in electrical engineering to solve differential equations. It reduces a linear differential equation to an algebraic equation, which can then be solved by the formal rules of algebra. The Laplace transform is a system that relies on algebra rather than calculus-based methods to solve linear differential equations. While it might seemingly appear somewhat cumbersome method at times, it is however a very powerful tool that enables us to readily deal with linear differential equations with discontinuities.

One interesting aspect of Laplace transform is that it finds the particular solution of an initial-value problem directly, without solving for the general solution first. Indeed, it usually takes more effort to find the general solution of an equation than it takes to find a particular solution! With Laplace transform method, we can solve differential equations of any order including first-order, second-

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order, etc. The method will also solve non-homogeneous linear differential equations directly without having to separately solving for the complementary & particular solutions, which is the case with the method of undetermined coefficients and D-Operator method.

10.4 LAPLACE TRANSFORM OF BASIC FUNCTIONS

In this section, we shall be looking at the applications of Laplace transform to Engineering problems. We shall begin by considering some of the elementary or basic functions such as exponential, trigonometric and hyperbolic functions.

So, from the definition of Laplace transform, if $f(t) = 3, for\ t \geq 0$, $L\{3\} = \int_0^\infty 3e^{-st}dt = 3\left[\frac{e^{-st}}{-s}\right] = \frac{3}{-s}[e^{-\infty} - e^0] = \frac{3}{-s}\left[\frac{1}{\infty} - 1\right] = \frac{3}{s}$

$$\therefore L\{3\} = \frac{3}{s}, \qquad provided\ s > 0$$

By the same token, given that 'a' is some constant, then

$$\boxed{L\{a\} = \frac{a}{s}, \qquad provided\ s > 0}$$

Let us consider a few examples:

Example 1: Determine the Laplace Transform of e^{at}

Solution:

Now, let $f(t) = e^{at}$, then from the definition, we have

$$L\{e^{at}\} = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_{0\to\infty} = -\frac{1}{s-a}[e^{-\infty} - e^0] = \frac{1}{s-a}$$

$$\boxed{\therefore\ L\{e^{at}\} = \frac{1}{s-a} \quad \text{provided } (s-a) > 0, or\ s > a}$$

From the above solution, we can deduce the following:

(a) When $a = 0$, $L\{e^{at}\} = L\{e^0\} = L\{1\} = \frac{1}{s}$

(b) When $a = 3, 6, 9, etc.$ $L\{e^{3t}\} = \frac{1}{s-3}$, $L\{e^{6t}\} = \frac{1}{s-6}$, $L\{e^{9t}\} = \frac{1}{s-9}, etc.$

Example 2: Determine the Laplace Transform of t

Solution:

Let $f(t) = t$

$$L\{t\} = \int_0^\infty e^{-st}tdt = \left[t\frac{e^{-st}}{-s} - \int\frac{e^{-st}}{-s}\right](0 \to \infty) = \left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right](0 \to \infty) = \frac{1}{s^2}$$

$$\boxed{\therefore L\{t\} = \frac{1}{s^2} \quad (provided\ s > 0)}$$

Example 3: Determine the Laplace Transform of t^n

Solution:

$$L\{t^n\} = \int_0^\infty e^{-st}t^n dt$$

Let $st = z$, $\therefore t = \frac{z}{s}$, and $dt = \frac{dz}{s}$

$$L\{t^n\} = \int_0^\infty e^{-st}\left(\frac{z}{s}\right)^n\left(\frac{dz}{s}\right) = \int_0^\infty e^{-st}\frac{z^n}{s^{n+1}}dz = \frac{1}{s^{n+1}}\int_0^\infty e^{-z}z^n dz$$

By definition, $\int_0^\infty e^{-z}z^n dz = n!$ and is known as the gamma function.

$$\boxed{\therefore\ L\{t^n\} = \frac{1}{s^{n+1}}n! = \frac{n!}{s^{n+1}}}$$

By the same token, we can find the Laplace Transform the following t, t^5, t^{10}, as

$$L\{t\} \equiv L\{t^n\} = \frac{n!}{s^{n+1}}$$

Here $n = 1$, so we obtain $L\{t\} = \frac{1!}{s^{(1+1)}} = \frac{1}{s^2}$

$$L\{t^5\} = \frac{5!}{s^{(5+1)}} = \frac{120}{s^6}$$

$$L\{t^{10}\} = \frac{10!}{s^{(10+1)}} = \frac{10!}{s^{11}}$$

Example 4: Determine the Laplace Transforms of $\sin at$ and $\cos at$

Solution:

$$L\{\sin at\} = \int_0^\infty e^{-st}\sin at\, dt \text{ and } L\{\cos at\} = \int_0^\infty e^{-st}\cos at\, dt$$

We can use integration by parts or Euler's formula. However, it is much shorter with Euler's formula.

That is, $e^{j\theta} = \cos\theta + j\sin\theta \to e^{jat} = \cos at + j\sin at$

So that $\sin\theta$ is the imaginary part while $\cos\theta$ is the real part.

The function $\sin at$ can be written as $1e^{jat}$ so that

$$L\{\sin at\} = L\{1e^{jat}\} = 1\int_0^\infty e^{-st}e^{jat}dt = 1\int_0^\infty e^{-(s-ja)t}dt = 1\left\{\left[\frac{e^{-(s-ja)t}}{-(s-ja)}\right](0 \to \infty)\right\}$$

$$L\{\sin at\} = 1\left\{\frac{1}{-(s-ja)}(0-1)\right\} = 1\left[\frac{1}{s-ja}\right]$$

Rationalizing the denominator, we have:

$$L\{\sin at\} = 1\left[\frac{s+ja}{(s-ja)(s+ja)}\right] = 1\left[\frac{s+ja}{s^2-j^2a^2}\right] = 1\left[\frac{s+ja}{s^2+a^2}\right] \text{ Since } j^2 = -1$$

Hence, $L\{\sin at\} = 1\left[\frac{s+ja}{s^2+a^2}\right]$

Since we are interested in the imaginary part, we will have

$$\boxed{\therefore\ L\{\sin at\} = \frac{a}{s^2 + a^2}}$$

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Similarly, $\cos at$ represents the real part, and so we shall have

$L\{\cos at\} = \mathrm{R}\left[\frac{s+ja}{s^2+a^2}\right]$

$$\therefore L\{\cos at\} = \frac{s}{s^2+a^2}$$

By the same token, we can find $L\{\sin 3t\}$ & $L\{\cos 4t\}$. Hence,

$L\{\sin 3t\} = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$ and $L\{\cos 4t\} = \frac{s}{s^2+4^2} = \frac{s}{s^2+16}$

Example 5: Determine the Laplace Transform of $\sinh at$

Solution:

Let $f(t) = \sinh at$

$L\{f(t)\} = L\{\sinh at\} = \int_0^\infty e^{-st}\sinh at\, dt$

But $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$ and $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$. These are the exponential definitions of $\sinh at$ *and* $\cosh at$

Now, $L\{\sinh at\} = \int_0^\infty e^{-st}\sinh at\, dt = \frac{1}{2}\int_0^\infty e^{-st}(e^{at} - e^{-at})dt = \frac{1}{2}\int_0^\infty [e^{-(s-a)t} - e^{-(s+a)t}]$

$L\{\sinh at\} = \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right](0 \to \infty) = \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2-a^2}$

$$\therefore L\{\sinh at\} = \frac{a}{s^2-a^2}$$

We can use alternative method to find the $L\{\sinh at\}$ and $L\{\cosh at\}$

We recall that $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$

$L\{\cosh at\} = L\left\{\frac{1}{2}(e^{at} + e^{-at})\right\} = \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\} = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$

$$L\{\cosh at\} = \frac{s}{s^2-a^2}$$

This appears very simpler and easier than the first approach!

Similarly, $L\{\sinh at\} = L\left\{\frac{1}{2}(e^{at} - e^{-at})\right\} = \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\} = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$

$$\therefore L\{\sinh at\} = \frac{a}{s^2-a^2}$$

Just as before!

10.5 LINEARITY OF THE LAPLACE TRANSFORM

The Laplace Transform is a linear transform. The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is,

$$L\{f(t) \pm h(t)\} = L\{f(t)\} \pm L\{h(t)\}$$

10.7 TABLE OF STANDARD TRANSFORM

This sections looks at a list of table of transform applications.

S/N	$f\{t\}$	$L\{f(t)\} = F(s)$	S/N	$f\{t\}$	$L\{f(t)\} = F(s)$
1	a	$\frac{a}{s}$	20	$t\cosh wt$	$\frac{s^2+w^2}{(s^2-w^2)^2}$
2	3	$\frac{3}{s}$	21	$e^{at}\sinh wt$	$\frac{w}{(s-a)^2-w^2}$
3	e^{at}	$\frac{1}{s-a}$	22	$e^{at}\cosh wt$	$\frac{s-a}{(s-a)^2-w^2}$
4	e^{-at}	$\frac{1}{s+a}$	23	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
5	t	$\frac{1}{s^2}$	24	$\sin wt - wt\cos wt$	$\frac{2w^3}{(s^2+w^2)^2}$
6	t^n	$\frac{n!}{s^{n+1}}$	25	$\sin(wt+k)$	$\frac{s\sin k + w\cos k}{s^2+w^2}$
7	$\sin wt$	$\frac{w}{s^2+w^2}$	26	$\cos wt - wt\sin wt$	$\frac{s(s^2-w^2)}{(s^2+w^2)^2}$
8	$\cos wt$	$\frac{s}{s^2+w^2}$	27	$\sin wt + wt\cos wt$	$\frac{2ws^2}{(s^2+w^2)^2}$
9	$e^{at}\sin wt$	$\frac{w}{(s-a)^2+w^2}$	28	$\cos wt + wt\sin wt$	$\frac{s(s^2+3w^2)}{(s^2+w^2)^2}$
10	$e^{-at}\sin wt$	$\frac{w}{(s+a)^2+w^2}$	29	$\cos(wt+k)$	$\frac{\cos k - w\sin k}{s^2+w^2}$
11	$t\sin wt$	$\frac{2ws}{(s^2+w^2)^2}$	30	$t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
12	$e^{at}\cos wt$	$\frac{s-a}{(s-a)^2+w^2}$	31	t^{n-1}	$\frac{\Gamma(n)}{s^2} = \frac{(n-1)!}{s^n}$
13	$e^{-at}\cos wt$	$\frac{s+a}{(s+a)^2+w^2}$	32	$\frac{1}{\sqrt{(\pi t)}}$	$s^{-1/2}$
14	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	33	$1-\cos wt$	$\frac{w^2}{s(s^2+w^2)^2}$
15	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	34	$wt - \sin wt$	$\frac{w^3}{s^2(s^2+w^2)^2}$
16	$t\cos wt$	$\frac{s^2-w^2}{(s^2+w^2)^2}$	35	$\frac{1-\cos wt}{t}$	$\frac{1}{2}\ln\left(\frac{s^2+w^2}{s^2}\right)$
17	$\sinh wt$	$\frac{w}{\ldots}$	36	$\frac{\sin wt}{\ldots}$	$\tan^{-1}\left(\frac{w}{\ldots}\right)$

TABLE 2.1: Table of Standard Transform

Sinh at
$\frac{a}{s^2-a^2}$
Cosh
$\frac{s}{s^2-a^2}$

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10.8 THE INVERSE LAPLACE TRANSFORMS

This is the reverse process. If $L\{f(t)\} = F(s)$, then $L^{-1}F(s) = f(t)$, where L^{-1} is called the inverse Laplace transform operator. The above statement indicates that given a Laplace transform, we can find the function t to which it belongs. For instance, we know that $L\{\cos wt\} = \frac{s}{s^2+w^2}$. So, we can write $L^{-1}\left\{\frac{s}{s^2+w^2}\right\} = \cos wt$. Hence,

(a). $L^{-1}\left\{\frac{2}{s+6}\right\} = 2e^{-6t}$ (b). $L^{-1}\left\{\frac{10}{s}\right\} = 10$ (c). $L^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t$

(d). $L^{-1}\left\{\frac{s}{s^2-9}\right\} = \cosh 3t$ (e). $L^{-1}\left\{\frac{20}{s^2-16}\right\} = 5\sinh 4t$ (f). $L^{-1}\left\{\frac{15}{s^2+25}\right\} = 3\sin 5t$

So far, we have come to agree that provided that a Laplace transform can be recognized from our table of transforms, its corresponding expression can easily be written down.

It is however very instructive to note that operations with Laplace inverse are not very clear-cut. We shall anticipate the function, which we intend to obtain. In general, simplification of the function using fundamental principles in algebra such as **factorization method, perfect square method, partial fraction method**, etc. shall be used.

For instance, let us consider the following example.

Example 11: Evaluate $L^{-1}\left\{\frac{4s+9}{s^2+s-12}\right\}$

Solution:

The above problem does not seem to appear in our table of standard transforms. But we can write $\frac{4s+9}{s^2+s-12}$ as the sum of two simpler functions $\frac{4s+9}{s^2+s-12} = \frac{1}{s+4} + \frac{3}{s-3}$. Hence,

$$L^{-1}\left\{\frac{4s+9}{s^2+s-12}\right\} = L^{-1}\left\{\frac{1}{s+4}\right\} + L^{-1}\left\{\frac{3}{s-3}\right\} = e^{-4t} + 3e^{3t}$$

The two simpler expressions are the partial fractions of $\frac{4s+9}{s^2+s-12}$. It is expected that the student is still very familiar with the elementary techniques used in resolving complicated algebraic expressions in terms of their partial fractions. If confused and where necessary, you may need to consult some text books that treated partial fractions to refresh your knowledge, since the topic in question is far below the standard of this text book. However, we shall try to be as simplistic as possible throughout our discussion!

Here are a few other examples!

Example 12: Determine $L^{-1}\left\{\frac{2s-3}{s^2}\right\}$

Solution:

$$L^{-1}\left\{\frac{2s-3}{s^2}\right\} = L^{-1}\left\{\frac{2s}{s^2}\right\} - L^{-1}\left\{\frac{3}{s^2}\right\} = L^{-1}\left\{\frac{2}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} = 2 - 3t$$

Example 13: Find $L^{-1}\left\{\frac{4s-3}{s^2+4}\right\}$

Solution:

$$L^{-1}\left\{\frac{4s-3}{s^2+4}\right\} = L^{-1}\left\{\frac{4s}{s^2+4}\right\} - L^{-1}\left\{\frac{3}{s^2+4}\right\} = 4L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{3}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\} = 4\cos 2t - \frac{3}{2}\sin 2t$$

Example 14: Evaluate $L^{-1}\left\{\frac{1}{s^2+2s}\right\}$

Solution:

$$\frac{1}{s^2+2s} = \frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} = \frac{1}{2s} - \frac{1}{2(s+2)}$$

$$L^{-1}\left\{\frac{1}{s^2+2s}\right\} = L^{-1}\left\{\frac{1}{2s} - \frac{1}{2(s+2)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s} - \frac{1}{s+2}\right\} = \frac{1}{2}(1 - e^{-2t})$$

Example 15: Find $L^{-1}\left\{\frac{5s^2-4s-7}{(s-3)(s^2+4)}\right\}$

Solution:

$$\frac{5s^2-4s-7}{(s-3)(s^2+4)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4} = \frac{2}{s-3} + \frac{3s+5}{s^2+4} = \frac{2}{s-3} + \frac{3s}{s^2+4} + \frac{5}{s^2+4}$$

$2e^{3t} + 3\cos 2t + \frac{5}{2}\sin 2t$

$$L^{-1}\left\{\frac{5s^2-4s-7}{(s-3)(s^2+4)}\right\} = L^{-1}\left\{\frac{2}{s-3} + \frac{3s}{s^2+4} + \frac{5}{s^2+4}\right\} = 2e^{3t} + 3\cos 2t + \frac{5}{2}\sin 2t$$

Example 16: Determine $L^{-1}\left\{\frac{s^2-2s+3}{(s-3)^3}\right\}$

Solution:

$$\frac{s^2-2s+3}{(s-3)^3} = \frac{A}{s-3} + \frac{B}{(s-3)^2} + \frac{C}{(s-3)^3} \equiv \frac{1}{s-3} + \frac{4}{(s-3)^2} + \frac{6}{(s-3)^3}$$

$$L^{-1}\left\{\frac{s^2-2s+3}{(s-3)^3}\right\} = L^{-1}\left\{\frac{1}{s-3} + \frac{4}{(s-3)^2} + \frac{6}{(s-3)^3}\right\} = e^{3t} + 4te^{3t} + 3t^2e^{3t} = e^{3t}(3t^2 + 4t + 1)$$

That was good enough! One thing is very clear in all the above examples. The denominators were easily factorized. However, when the denominator is not easily factorized, we form a perfect square of the denominator. This includes such expressions like:

$L^{-1}\left\{\frac{s}{s^2+bs+c}\right\}$ OR $L^{-1}\left\{\frac{k}{s^2+bs+c}\right\}$. That is, the following steps may be taken to handle such expressions:

(i) Form a perfect square with the quadratic equation:

$$\frac{s}{s^2+bs+c} = \frac{s}{\left(s+\frac{b}{2}\right)^2 + \left(\frac{\sqrt{4c-b^2}}{2}\right)^2}$$

(ii) Add and subtract $\frac{b}{2}$ from the numerator

$$\frac{s}{s^2+bs+c} = \frac{s+\frac{b}{2}-\frac{b}{2}}{\left(s+\frac{b}{2}\right)^2 + \left(\frac{\sqrt{4c-b^2}}{2}\right)^2}$$

(iii) Split the equation into two such that the standard forms are obtained as:

$$\frac{s}{s^2+bs+c} = \frac{\left(s+\frac{b}{2}\right)}{\left(s+\frac{b}{2}\right)^2 + \left(\frac{\sqrt{4c-b^2}}{2}\right)^2} - \frac{\left(\frac{b}{2}\right)}{\left(s+\frac{b}{2}\right)^2 + \left(\frac{\sqrt{4c-b^2}}{2}\right)^2}$$

(iv) Finally, use the Laplace transform table and compute the solution.

Example 17: Determine $L^{-1}\left\{\frac{4}{s^2+4s+13}\right\}$

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Solution:

$$4L^{-1}\left\{\frac{1}{s^2+4s+13}\right\} = 4L^{-1}\left\{\frac{1}{(s+2)^2+3^2}\right\} = \frac{4}{3}L^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\} = \frac{4}{3}e^{-2t}\sin 3t$$

$$\therefore\ L^{-1}\left\{\frac{4}{s^2+4s+13}\right\} = \frac{4}{3}e^{-2t}\sin 3t$$

$e^{-2t}\sin 3t$

Example 18: Determine $L^{-1}\left\{\frac{s}{s^2+4s+13}\right\}$

Solution:

$$L^{-1}\left\{\frac{s}{(s+2)^2+3^2}\right\} = L^{-1}\left\{\frac{(s+2)-2}{(s+2)^2+3^2}\right\} = L^{-1}\left\{\frac{(s+2)}{(s+2)^2+3^2}\right\} - L^{-1}\left\{\frac{2}{(s+2)^2+3^2}\right\}$$

$$= L^{-1}\left\{\frac{(s+2)}{(s+2)^2+3^2}\right\} - \frac{2}{3}L^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\} = e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t$$

$$\therefore\ L^{-1}\left\{\frac{s}{s^2+4s+13}\right\} = e^{-2t}\left(\cos 3t - \frac{2}{3}\sin 3t\right)$$

$\frac{6}{2} - \frac{6}{2}$ $\quad e^{-2t}\cos 3t$

Example 19: Determine $L^{-1}\left\{\frac{3s-4}{s^2-6s+10}\right\}$

$2(3+1)-2-3$

$2\cdot 3+2(-2\cdot 3)-5+2$

$2\cdot 3-3$

Solution:

$$L^{-1}\left\{\frac{3s-4}{s^2-6s+10}\right\} = L^{-1}\left\{\frac{3s-4}{(s-3)^2+1}\right\} = L^{-1}\left\{\frac{3(s-3)+9-4}{(s-3)^2+1}\right\} = L^{-1}\left\{\frac{3(s-3)+5}{(s-3)^2+1}\right\} = 3L^{-1}\left\{\frac{(s-3)}{(s-3)^2+1}\right\} + L^{-1}\left\{\frac{5}{(s-3)^2+1}\right\}$$

$$= 3e^{3t}\cos t + 5e^{3t}\sin t = e^{3t}(3\cos t + 5\sin t)$$

$3s-9+4-4$

$3s-3+3-4$

$3(s-1)+5-4$

$3(s-1)-1$

EXERCISE 10.2

Determine the inverse Laplace transforms of the following functions:

27. $\frac{s-5}{s^2+4s+20}$ 28. $\frac{2s+5}{s^2+4s-5}$ 29. $\frac{2s-1}{s^2-2s+10}$

30. $\frac{26-s^2}{s(s^2+4s+13)}$ 31. $\frac{1}{s^2-3s+3}$ 32. $\frac{1+s}{(s+2)^2}$

33. $\frac{3s+2}{s^2-8s+25}$ 34. $\frac{6-s}{s^2+4s+20}$ 35. $\frac{2-11s}{(s-2)(s^2+2s+2)}$

36. $\frac{s^2-3s-4}{(s-3)(s-1)^2}$ 37. $\frac{s^2+3s-7}{(s-1)(s^2+2)}$ 38. $\frac{5s+10}{s^2+5s+7}$

$4-4(1)(n)$

The results are in the table below.

27. $e^{-2t}(\cos 4t - \frac{7}{4}\sin 4t)$	28. $\frac{5}{6}e^{-5t} + \frac{7}{6}e^{t}$	29. $e^{t}\left(2\cos 3t + \frac{1}{3}\sin 3t\right)$
30. $2 - e^{-2t}\left(3\cos 3t - \frac{2}{3}\sin 3t\right)$	31. $\frac{2}{\sqrt{3}}e^{\frac{3}{2}t}\sin(\frac{\sqrt{3}}{2}t)$	32. $e^{-2t}(1-t)$
33. $e^{2t}\left(3\cos 3t + \frac{4}{3}\sin 3t\right)$	34. $e^{-2t}(2\sin 4t - \cos 4t)$	35. $e^{-t}(2\cos t - 5\sin t) - 2e^{2t}$
36. $e^{t}(3t+2) - e^{3t}$	37. $2\cos\sqrt{2}t + \frac{5}{\sqrt{2}}\sin\sqrt{2}t - e^{t}$	38. $5e^{\frac{5}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + \frac{5}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$

Putting boundary values and collecting like terms, we obtain:

$$(s^2+3s+2)\bar{y} + 0.9s + 0.7 + 2.7 = \frac{3}{s^2+1} \rightarrow (s^2+3s+2)\bar{y} = \frac{3}{s^2+1} - 0.9s - 3.4$$

$$\bar{y} = \frac{3-0.9s(s^2+1)-3.4(s^2+1)}{(s^2+1)(s^2+3s+2)} = \frac{-0.9s^3-3.4s^2-0.9s-0.4}{(s^2+1)(s^2+3s+2)} = \frac{-0.9s^3-3.4s^2-0.9s-0.4}{(s+2)(s+1)(s^2+1)}$$

Resolving into partial fractions, and solving for the constant terms, we obtain:

$$\bar{y} = \frac{-0.9s^3-3.4s^2-0.9s-0.4}{(s+2)(s+1)(s^2+1)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1} = \frac{1}{s+2} - \frac{1}{s+1} - \frac{0.9s}{s^2+1} + \frac{0.3}{s^2+1}$$

Taking the inverse Laplace transform of both sides, we obtain:

$$y = L^{-1}\left(\frac{1}{s+2} - \frac{1}{s+1} - \frac{0.9s}{s^2+1} + \frac{0.3}{s^2+1}\right) = e^{-2t} - e^{-t} - 0.9\cos t + 0.3\sin t$$

$$\therefore y = e^{-2t} - e^{-t} + \frac{3}{10}(\sin t - 3\cos t)$$

Example 25: Solve the differential equation using Laplace transform method:

$y'' - 9y = e^{3t} + \sin 3t;\ \ y(0) = 2,\ \ y'(0) = 0$

Solution:

$$L\{y''\} - 9L\{y\} = L\{e^{3t}\} + L\{\sin 3t\} \rightarrow s^2\bar{y} - sy_0 - y'(0) - 9\bar{y} = \frac{1}{s-3} + \frac{3}{s^2+9}$$

Putting boundary values and collecting like terms, we obtain:

$$(s^2-9)\bar{y} - 2s = \frac{s^2+3s}{(s-3)(s^2+9)} \rightarrow (s^2-9)\bar{y} = \frac{s^2+3s}{(s-3)(s^2+9)} + 2s = \frac{s^2+3s+2s(s^3-3s^2+9s-27)}{(s-3)(s^2+9)}$$

$$\bar{y} = \frac{2s^4-6s^3+19s^2-51s}{(s-3)(s^2+9)(s^2-9)} = \frac{2s^4-6s^3+19s^2-51s}{(s+3)(s-3)^2(s^2+9)} = \frac{A}{s+3} + \frac{B}{s-3} + \frac{C}{(s-3)^2} + \frac{Ds+E}{s^2+9} = \frac{1}{s+3} + \frac{1}{s-3} + \frac{1}{6}\left(\frac{1}{(s-3)^2}\right) - \frac{1}{6}\left(\frac{1}{s^2+9}\right)$$

Taking the inverse Laplace transform of both sides, we obtain:

$$y = L^{-1}\left[\frac{1}{s+3} + \frac{1}{s-3} + \frac{1}{6}\left(\frac{1}{(s-3)^2}\right) - \frac{1}{6}\left(\frac{1}{s^2+9}\right)\right] = e^{-3t} + e^{3t} + \frac{1}{6}te^{3t} - \frac{1}{18}\sin 3t$$

$$\therefore y = e^{-3t} + e^{3t} + \frac{1}{6}te^{3t} - \frac{1}{18}\sin 3t$$

Example 26: Solve the differential equation by Laplace transform method:

$y'' + 4y = e^{-t}\sin t;\ y(0) = 0,\ y'(0) = \frac{1}{10}$

Solution:

$$L\{y''\} + 4L\{y\} = L\{e^{-t}\sin t\} \rightarrow s^2\bar{y} - sy_0 - y'(0) - 4\bar{y} = \frac{1}{(s+1)^2+1}$$

Putting boundary values and collecting like terms, we obtain:

$$s^2\bar{y} - \frac{1}{10} - 4\bar{y} = \frac{1}{s^2+2s+2} \rightarrow (s^2+4)\bar{y} = \frac{1}{s^2+2s+2} + \frac{1}{10} = \frac{s^2+2s+12}{10(s^2+2s+2)}$$

$$10\bar{y} = \frac{s^2+2s+12}{(s^2+4)(s^2+2s+2)} = \frac{As+B}{(s^2+4)} + \frac{Cs+D}{(s^2+2s+2)} = \frac{(As+B)(s^2+2s+2)+(Cs+D)(s^2+4)}{(s^2+4)(s^2+2s+2)}$$

$$s^2+2s+12 \equiv As(s^2+2s+2) + B(s^2+2s+2) + Cs(s^2+4) + D(s^2+4)$$

$$s^2+2s+12 \equiv (A+C)s^3 + (2A+B+D)s^2 + (2A+2B+4C)s + (2B+4D)$$

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Comparing coefficients, we obtain:

$A + C = 0$.....(i)

$2A + B + D$.....(ii)

$2A + 2B + 4C = 2$.....(iii)

$2B + 4D = 12$.....(iv)

Solving the equations gives: $A = -1,\ B = 0,\ C = 1,\ D = 3$. Hence, we obtain:

$10\bar{y} = \frac{s+3}{s^2+2s+2} - \frac{s}{s^2+4} \rightarrow \bar{y} = \frac{1}{10}\left[\frac{s+3}{s^2+2s+2} - \frac{s}{s^2+4}\right]$. Taking the inverse Laplace transform gives:

$y = \frac{1}{10}L^{-1}\left[\frac{s+3}{s^2+2s+2} - \frac{s}{s^2+4}\right] = \frac{1}{10}L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] + \frac{1}{10}L^{-1}\left[\frac{3-1}{(s+1)^2+1}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+4}\right]$

$y = \frac{1}{10}e^{-t}\cos t + \frac{1}{5}e^{-t}\sin t - \frac{1}{10}\cos 2t$

Example 27: Solve the differential equation by Laplace transform method:

$y'' + 6y' + 9y = e^{-3t}\cosh 3t; y(0) = \frac{10}{9}, y'(0) = -\frac{1}{3}$

Solution:

We may begin by transforming $\cosh 3t$. Hence, we recall that $\cosh 3t = \frac{1}{2}(e^{3t} + e^{-3t})$

$y'' + 6y' + 9y = e^{-3t}.\frac{1}{2}(e^{3t} + e^{-3t}) = \frac{1}{2} + \frac{1}{2}e^{-6t}$

$L\{y''\} + 6L\{y'\} + 9L\{y\} = L\left\{\frac{1}{2} + \frac{1}{2}e^{-6t}\right\} \rightarrow s^2\bar{y} - sy_0 - y'(0) + 6(s\bar{y} - y_0) + 9\bar{y} = \frac{1/2}{s} + \frac{1/2}{s+6}$

Putting the boundary values and collecting like terms together gives:

$s^2\bar{y} - \frac{10}{9}s + \frac{1}{3} + 6s\bar{y} - \frac{60}{9} + 9\bar{y} = \frac{1/2}{s} + \frac{1/2}{s+6}$

$(s^2 + 6s + 9)\bar{y} = \frac{1}{2s} + \frac{1}{2(s+6)} + \frac{10}{9}s - \frac{1}{3} + \frac{60}{9} = \frac{20s^3+234s^2+702s+54}{18s(s+6)}$

Resolving into partial fractions, we obtain:

$\bar{y} = \frac{20s^3+234s^2+702s+54}{18s(s+6)(s^2+6s+9)} = \frac{20s^3+234s^2+702s+54}{18s(s+6)(s+3)^2} = \frac{A}{18s} + \frac{B}{s+6} + \frac{C}{s+3} + \frac{D}{(s+3)^2}$

$20s^3 + 234s^2 + 702s + 54 \equiv A(s+6)(s+3)^2 + B(18s)(s+3)^2 + C(18s)(s+6)(s+3) + D(18s)(s+6)$

Expanding each of the terms in the RHS and collecting like terms gives:

$20s^3 + 234s^2 + 702s + 54 \equiv A(s^3 + 12s^2 + 45s + 54) + B(18s^3 + 108s^2 + 162s) + C(18s^3 + 162s^2 + 324s) + D(18s^2 + 108s)$

Grouping the coefficients together, we obtain:

$\equiv (A + 18B + 18C)s^3 + (12A + 108B + 162C + 18D)s^2 + (45A + 162B + 324C + 108D)s + 54A$

Comparing the coefficients of LHS & RHS of the equation, we have:

$A + 18B + 18C = 20$.....(i)

$12A + 108B + 162C + 18D = 234$.....(ii)

$45A + 162B + 324C + 108D = 702$.....(iii)

$54A = 54$.....(iv)

Solving the above equations gives: $A = 1, B = \frac{1}{18},\ C = 1, D = 3$. Hence,

$\bar{y} = \frac{1}{18s} + \frac{1}{18(s+6)} + \frac{1}{s+3} + \frac{3}{(s+3)^2}$

Taking the inverse Laplace transform of both sides gives:

$y = \frac{1}{18} + \frac{1}{18}e^{-6t} + e^{-3t} + 3te^{-3t}$

$\therefore y = \frac{1}{18}(1 + e^{-6t}) + e^{-3t}(1 + 3t)$

Having gone thus far, you should be able to attempt the following exercises.

EXERCISE 10.3

Use Laplace transform to find the solution of the following differential equations:

39. $x' + 3x = e^{-2t}; x(0) = 2$
40. $3x' - 6x = \sin 2t; x(0) = 1$
41. $y' - 4y = 3\sin 2t; y(0) = 3$
42. $y' + 2y = 4\cos t; y(0) = 2$
43. $y' + 3y = \sin 2t; y(0) = 2$
44. $y'' + y = 2e^x;\ y(0) = 4, y'(0) = 2$
45. $y'' - 4y' + 4y = 6te^{2t};\ y(0) = 1, y'(0) = 1$
46. $y'' + 4y = t^2 + \sin 2t;\ y(0) = y'(0) = 0$
47. $y'' - 2y' - 3y = 2e^{2t} + 10\sin 3t;\ y(0) = 1,\ y'(0) = 4$
48. $y'' + y = \sin 3t\cos 2t;\ y(0) = 1,\ y'(0) = -\frac{17}{48}$
49. $y'' + 4y = e^t + \sin 2t;\ y(0) = \frac{11}{5}, y'(0) = \frac{9}{20}$
50. $y'' - 4y' + 3y = e^{2t} + \cos t;\ y(0) = \frac{1}{10}, y'(0) = -\frac{1}{5}$
51. $y'' - y = t\sin t + e^t + t^2e^t;\ y(0) = -\frac{1}{2},\ y'(0) = -\frac{1}{4}$
52. $y'' + 4y' + 4y = \cosh 2t;\ y(0) = \frac{1}{32},\ y'(0) = 0$
53. $y'' + 4y' + 5y = 8\cos^2 t;\ y(0) = \frac{56}{65}, y'(0) = \frac{116}{65}$
54. $y'' + 2y' + 5y = 34\sin t\cos t; y(0) = y'(0) = 0$

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The results are in the table below.

39. $x = e^{-2t} + e^{-3t}$	40. $y = \frac{1}{12}[13e^{2t} - \cos 2t - \sin 2t]$
41. $y = 2.7e^{4t} + 0.3\cos 2t + 1.2\sin 2t$	42. $y = 0.4e^{-2t} + 1.60\cos t + 0.8\sin t$
43. $y = 1.846e^{-3t} + 0.154\cos 2t - 0.23\sin 2t$	44. $y = 2e^{x} + 2(\cos x - \sin x)$
45. $y = e^{2t}(t^3 - t + 1)$	46. $y = \frac{1}{8}[\sin 2t + \cos 2t] + \frac{1}{4}\left[t^2 - t\cos 2t - \frac{1}{2}\right]$
47. $y = \frac{1}{12}[29e^{3t} - e^{-t}] - \frac{2}{3}e^{2t} + \frac{1}{3}(\cos 3t - 2\sin 3t)$	48. $y = \cos t + \frac{1}{48}[-\sin 5t - 12t\cos t]$
49. $y = 2\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{5}e^{t} - \frac{1}{4}t\cos 2t$	50. $y = \frac{1}{2}[e^{t} + e^{3t}] - e^{2t} + \frac{1}{10}(\cos t - 2\sin t)$
51. $y = \frac{1}{2}(e^{-t} - e^{t} - t\sin t - \cos t) + \frac{t}{12}e^{t}(2t^2 - 3t + 9)$	52. $y = \frac{1}{32}(e^{2t} - 2e^{-2t}) + \frac{t^2}{4}e^{-2t}$
53. $y = \frac{4}{5}(1 + e^{-2t}\sin t) + \frac{4}{65}(8\sin 2t + \cos 2t)$	
54. $y = e^{-t}(4\cos 2t - 3\sin 2t) + \sin 2t - 4\cos 2t$	

10.11 SOLUTION TO SIMULTANEOUS DIFFERENTIAL EQUATIONS

We shall consider a few examples of using Laplace transform to solve simultaneous differential equations.

Example 28: Use Laplace transform method to solve the pair of simultaneous differential equations: $\left.\begin{aligned} y' + 3x &= e^{-2t} \\ x' - 3y &= e^{2t} \end{aligned}\right\}$ $y(0) = x(0) = 0$

Solution:

We begin by expressing both equations in Laplace transform.

$[s\bar{y} - y(0)] + 3\bar{x} = \frac{1}{s+2}$ .....(i)

$[s\bar{x} - x(0)] - 3\bar{y} = \frac{1}{s-2}$ .....(ii)

Next, we substitute the initial conditions, $y(0) = x(0) = 0$

$s\bar{y} + 3\bar{x} = \frac{1}{s+2}$ .....(iii)

$s\bar{x} - 3\bar{y} = \frac{1}{s-2}$ .....(iv)

Applying normal elimination method, say: eqn. (iii) × s and eqn. (iv) × 3 gives:

$s^2\bar{y} + 3s\bar{x} = \frac{s}{s+2}$ .....(v)

$-9\bar{y} + 3s\bar{x} = \frac{3}{s-2}$ .....(vi)

Eqn. (v) – eqn. (vi) gives:

$s^2\bar{y} + 9\bar{y} = \frac{s}{s+2} - \frac{3}{s-2} \rightarrow (s^2+9)\bar{y} = \frac{s^2-5s-6}{(s+2)(s-2)} \rightarrow \bar{y} = \frac{s^2-5s-6}{(s+2)(s-2)(s^2+9)} = \frac{A}{s+2} + \frac{B}{s-2} + \frac{Cs+D}{s^2+9}$

Solving for the values of A, B, C & D using any standard algebraic methods gives

$A = -\frac{2}{13}, B = -\frac{3}{13}, C = \frac{5}{13}, D = \frac{15}{13}$. Hence,

$\bar{y} = -\frac{2}{13(s+2)} - \frac{3}{13(s-2)} + \frac{5s+15}{13(s^2+9)}$

Finally, we take the inverse Laplace transform.

$y = L^{-1}\left\{-\frac{2}{13(s+2)} - \frac{3}{13(s-2)} + \frac{5s+15}{13(s^2+9)}\right\} = -\frac{2}{13}e^{-2t} - \frac{3}{13}e^{2t} + \frac{5}{13}\cos 3t + \frac{5}{13}\sin 3t$

$\therefore y = \frac{1}{13}(5\cos 3t + 5\sin 3t - 3e^{2t} - 2e^{-2t})$

Next, we revert to our earlier eqn. (iii) and eliminate $\bar{y}$ to obtain $\bar{x}$ and hence x, in the same way.

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PAST QUESTIONS

ENGINEERING MATHEMATICS II (ENG 202)

Instruction: Answer *Question No 1* and any other **FOUR** Questions.
Clarity of work shall attract Bonus Marks!! Use of programmable calculators is highly prohibited!
Time Allowed: 3 Hours. *EXAM Date:* 16th September, 2015 *TOTAL Marks:* 60%
Unit Load: 3 *ACADEMIC Year:* 2014/2015

Question 1: [12 marks]

(a) Use Newton-Raphson method to solve the following equations, correct to three significant figures: **[3MKS]**

(i) $3\ln x + 4x = 5$ **[3MKS]**

(ii) $300e^{-2\theta} + \frac{\theta}{2} = 6$

(b) Using Euler's method, obtain a numerical solution of the differential equation: $y' = 3(1+x) - y$ given the initial conditions that $y(1) = 4$, for the range $x = 1.0(0.2)2.0$ **[6MKS]**

Question 2: [12 marks]

(a) An electric circuit consists of an 8 Ω resistor in series with an inductor of 0.5H, and a battery of E volts. At $t = 0$, the current $I = 0$. Using mathematical modelling, determine the current I at any time $t > 0$, and the maximum current if $E = 32e^{-8t}$ **[6MKS]**

(b) Compute the integral of the complex function $f(z) = (x^2 + y) + j(xy)$ from the point $A(z = 0)$ to the point $B(z = 1 + j)$ along:

(i) A straight line, **[3MKS]**

(ii) A parabola $y = x^2$. **[3MKS]**

Question 3: [12 marks]

(a) Use the method of undetermined coefficients to solve the differential equation: $y'' - 5y' + 6y = e^x \sin x$ **[6MKS]**

(b) A closed path C consists of a rectangle formed by joining the points A (z = 2 + j), B (z = 2 + j4), C (z = j4), and D (z = j). Given that $f(z) = z^2$, evaluate $\oint f(z)dz$, and hence show that $\oint f(z)dz = 0$ **[4MKS]**

(c) Using relevant theorems, determine the analyticity of the function: $f(z) = \sin x \cosh y + j \cos x \sinh y$ **[2MKS]**

Question 4: [12 marks]

(a) For the Bernoulli's equation: $y' + y\tan x = y^3 sec^4 x$; $y(0) = 1$, show that the solution can be expressed in the form of $y^2 = \frac{1}{4(1-2\sqrt{3})}$, when $x = 60°$. **[5MKS]**

(b) Obtain the solution of the differential equation: $y'' + y = \sin 3x \cos 2x$, using D-operator method. **[5MKS]**

(c) Use relevant theorems to show that $f'(z)$ exists at every point if $f(z) = z^3$ **[2MKS]**

Question 5: [12 marks]

(a) Use the Leibnitz-Maclaurin's method to find the particular solution of the differential equation: $(1 - x^2)y'' - 2xy' + 2y = 0$, given that $y(0) = 1$ and $y'(0) = 2$. **[5MKS]**

(b) Use the D-operator method to solve the differential equation: $D^2y + 4Dy + 4y = 2cos^2x$; given that $y(0) = \frac{9}{4}$, $y'(0) = -\frac{11}{4}$. **[5MKS]**

(c) Solve the equation: $(x - j2y) + (y - j3x) = 5 - j6$ **[2MKS]**

Question 6: [12 marks]

(a) Use the power series method to solve the differential equation: $(1 - x^2)y'' - 2xy' + 2y = 0$; given that $y(0) = 1$ and $y'(0) = 2$. **[5MKS]**

(b) A triangle ABC in the z-plane with vertices A $(-2 + j3)$, B $(1 + j2)$, C $(-1 + j)$ is mapped onto the w-plane under the transformation equation: $w = (-3 + j)z + (2 + j4)$. Determine the image A′B′C′ of ABC in the w-plane. Hence, calculate the magnification and rotation involved. **[4MKS]**

(c) If $x = \cos 15° + j \sin 15°$, evaluate $x^4 + \frac{1}{x^4}$ **[3MKS]**

Question 7: [12 marks]

(a) A closed path consists of three straight lines joining the points P $(z = 1 + j)$, Q (z = 3 + j3) and R (z = −1 + j3). If $f(z) = z - 1 + j$, evaluate $\oint f(z)dz$, and hence show that $\oint f(z)dz = 0$. **[5MKS]**

(b) Use the method of undetermined coefficients to solve the differential equation: $y'' - 6y' + 9y = e^{3x} + e^{-3x}$ **[5MKS]**

(c) Given that $z = x + jy$, show that $Re\left[\frac{z-j}{1-\bar{z}j}\right] = Im\left[\frac{z-j}{1-\bar{z}j}\right] \times B$, where $B = \frac{2x-2xy}{x^2+2y-y^2-1}$ **[3MKS]**

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ENGINEERING MATHEMATICS II –ENG 202

Instructions: *Answer **Only** Five (5) Questions in all.*

Time Allowed: *3 hours*

Marks: *60 Marks*

Unit: *3* ***Exam Date:*** 18th September, 2014

1. **(i)** Using Euler's modified formula, find the approximate value of y when $x = 0.8$, given that $\frac{dy}{dx} = x^2 + y$, and $y(0) = 1$, taking the interval 0.2 .....10mks
   (ii) Show that the vertices represented by the complex numbers: $(1+3j), (2+3j), (2+j), (1+j)$ form a rectangle. 10mks
   (iii) If an arc of a parabola is represented as $y = x^2$, compute the integral of the function $f(z) = \bar{z}^2 = (x^2 - y^2) - j(2xy)$ along a parabola from point $z = 0$ to $z = 1 + j$. .10mks

2. **(i)** Solve the non-linear differential equation: $y' \cos x + y \sin x = y^3 \cos^2 x$ 10mks
   (ii) Use the method of undetermined coefficients to solve the differential equation: $y'' + 4y' + 4y = 2\cos^2 x$; given that $y(0) = \frac{9}{4}$, and $y'(0) = -\frac{11}{4}$ 12mks
   (iii) Evaluate $e^{2x} \sin 3x$ using power series expansion up to four terms. 8mks

3. **(i)** Obtain the solution of the differential equation using D-operator method: $D^2y + 4Dy + 4y = 2\cos^2 x$; given that $y(0) = \frac{9}{4}$, $y'(0) = -\frac{11}{4}$. 10mks
   (ii) Determine the power series solution of the differential equation: $(x^2+2)y'' + xy' - (1+x)y = 0$; given that $y(0) = 1$, and $y'(0) = 1$ 12mks
   (iii) Use Power series expansion to evaluate $\lim_{x \to 0} \frac{\sin^2 x}{x^2}$ 10mks

4. **(i)** Using the 2nd order Runge-Kutta algorithm, with a step size of 0.2, solve the initial value problem $\frac{dy}{dx} + 2y = \sin 3x$ with $y(0) = 1$ in the interval $0 \le x \le 0.6$ 12mks
   (ii) Show that $f'(z)$ does not exist at any point if $f(z) = 2x + jy^2$ 8mks
   (iii) If $x = \cos\theta + j\sin\theta$, find $x^m + \frac{1}{x^m}$ 10mks

5. **(i)** Using D-operator method, obtain the solution of the differential equation $y'' + 6y' + 9y = 2e^{-3x} \cosh 3x$ 10mks
   (ii) Use the Leibnitz-Maclaurin method to obtain the particular solution of the differential equation: $(1 - x^2)y'' - 2xy' + 2y = 0$; $y(0) = 1$ and $y'(0) = 2$ 12mks
   (iii) Use the Binomial theorem to evaluate $(4.0446)^6$ correct to one decimal place. .8mks

6. **(i)** Prove that $\frac{x}{e^x + 1} = \frac{x}{2} - \frac{x^2}{4} + \frac{x^4}{48}$ 10mks
   (ii) Solve the Bernoulli's equation: $y' + y \tan x = y^3 \sec^4 x$; given that $y\left(\frac{\pi}{2}\right) = 1$ .10mks
   (iii) Solve the differential equation: $D^2y + 4y = e^{-x} \sin x$; $y(0) = \frac{21}{10}$ and $y'(0) = \frac{11}{10}$ 10mks

7. **(i)** Use the method of undetermined coefficients to obtain the solution of the differential equation: $y'' - 9y = e^{3x} + \sin 3x$; given that $y(0) = 0$, and $y'(0) = 6$ 12mks
   (ii) Obtain the solution of the differential equation using D-operator method: $D^2y + 4y = e^x + \sin 2x$ 8mks
   (iii) Evaluate $\frac{9}{(3+x)^3}$ using binomial theorem in ascending powers of x, and determine the limits of x for which the expansion is valid. 10mks

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