FLUID KINEMATICS

5.1. INTRODUCTION

Fluid kinematics may be defined as follows:

Fluid kinematics *is a branch of 'Fluid mechanics' which deals with the study of velocity and acceleration of the particles of fluids in motion and their distribution in space without considering any force or energy involved.* The motion of fluid can be described fully by an expression describing the location of a fluid particle in space at different times thus enabling determination of the magnitude and direction of velocity and acceleration in the flow field at any instant of time.

In the chapter we shall deal with the conception of fluid flow in general.

5.2. DESCRIPTION OF FLUID MOTION

The motion of fluid particles may be described by the following methods:

1. Langrangian method.
2. Eulerian method.

5.2.1. Langrangian Method

In this method, the observer *concentrates on the movement of a single particle*. The path taken by the particle and the changes in its velocity and acceleration are studied.

In the Cartesian system, the position of the fluid particle in space (x, y, z) at any time t from its position (a, b, c) at time $t = 0$ shall be given as:

$$x = f_1(a, b, c, t)$$
$$y = f_2(a, b, c, t) \qquad \text{...(5.1)}$$
$$z = f_3(a, b, c, t)$$

The velocity and acceleration components (obtained by taking derivatives with respect to time) are given by:

Velocity components:

$$\left.\begin{aligned} u &= \frac{\partial x}{\partial t} \\ v &= \frac{\partial y}{\partial t} \\ w &= \frac{\partial z}{\partial t} \end{aligned}\right\} \quad ...(5.2)$$

Acceleration components :

$$\left.\begin{aligned} a_x &= \frac{\partial^2 x}{\partial t^2} \\ a_y &= \frac{\partial^2 y}{\partial t^2} \\ a_z &= \frac{\partial^2 z}{\partial t^2} \end{aligned}\right\} \quad ...(5.3)$$

At any point, the resultant velocity or acceleration shall be the *resultant* of three components of the respective quantity at that point.

$$\therefore \quad \text{Resultant velocity, } V = \sqrt{u^2 + v^2 + w^2} \quad ...(5.4)$$

$$\text{Acceleration, } a = \sqrt{a_x^2 + a_y^2 + a_z^2} \quad ...(5.5)$$

Similarly, other quantities like pressure, density, etc. can be found.

This method entails the following *shortcomings:*

1. Cumbersome and complex.
2. The equations of motion are very difficult to solve and the motion is hard to understand.

5.2.2. Eulerian Method

In Eulerian method, the observer *concentrates on a point in the fluid system*. Velocity, acceleration and other characteristics of the fluid at that particular point are studied.

This method is almost *exclusively used* in fluid mechanics, especially because of its *mathematical simplicity*. In fluid mechanics, we are not concerned with the motion of each particle, but we study the general state of motion at various points in the fluid system.

The velocities at any point (x, y, z) can be written as:

$$\left.\begin{aligned} u &= f_1(x, y, z, t) \\ v &= f_2(x, y, z, t) \\ w &= f_3(x, y, z, t) \end{aligned}\right\} \quad ...(5.6)$$

The components of acceleration of the fluid particle can be worked out by partial differentiation as follows:

$$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy + \frac{\partial u}{\partial z}.dz + \frac{\partial u}{\partial t}.dt$$

$$= \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \text{ in Eqn. (5.7)}$$

Tangential and normal acceleration: Refer to Fig. 5.1.

When the motion is curvilinear eqn. 5.16 gives the *tangential acceleration*. A particle moving in a curved path will always have a normal acceleration $a_n = \frac{V^2}{r}$ towards the centre of the curved path (r being the radius of the path), though its tangential acceleration (a_s) may be zero as in the case of uniform circular motion.

For motion along a curved path, in general,

$$a = a_s + a_n$$

$$= \left(V\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t} + \frac{V^2}{r} \right) \quad \text{...(5.19)}$$

5.3. TYPES OF FLUID FLOW

Fluids may be *classified* as follows:

1. Steady and unsteady flows
2. Uniform and non-uniform flows
3. One, two and three dimensional flows
4. Rotational and irrotational flows
5. Laminar and turbulent flows
6. Compressible and incompressible flows.

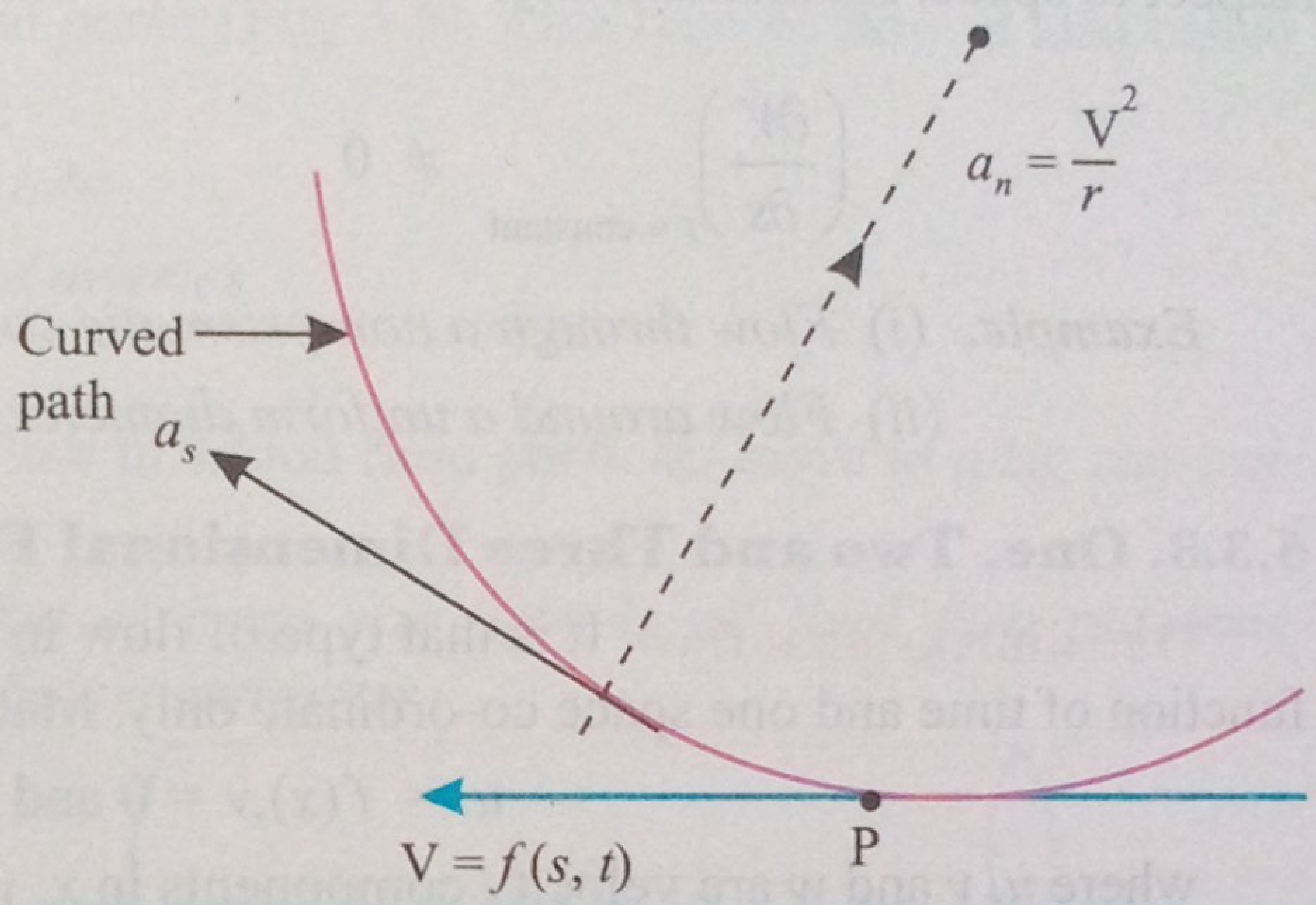

Fig. 5.1. Tangential and normal acceleration.

5.3.1. Steady and Unsteady Flows

Steady flow. The type of flow in which the fluid characteristics like velocity, pressure, density, etc. at a point *do not change* with time is called *steady flow*. Mathematically, we have:

$$\left(\frac{\partial u}{\partial t}\right)_{x_0, y_0, z_0} = 0; \left(\frac{\partial v}{\partial t}\right)_{x_0, y_0, z_0} = 0; \left(\frac{\partial w}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

$$\left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0; \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0; \text{ and so on}$$

where (x_0, y_0, z_0) is a fixed point in a fluid field where these variables are being measured *w.r.t.* time.

Example. *Flow through a prismatic or non-prismatic conduit at a constant flow rate Q m³/s is steady.*

(A prismatic conduit has a constant size shape and has a velocity equation in the form $u = ax^2 + bx + c$, which is independent of time t).

Unsteady flow. It is that type of flow in which the velocity, pressure or density at a point *change* w.r.t. time. Mathematically, we have:

$$\left(\frac{\partial u}{\partial t}\right)_{x_0, y_0, z_0} \neq 0; \left(\frac{\partial v}{\partial t}\right)_{x_0, y_0, z_0} \neq 0; \left(\frac{\partial w}{\partial t}\right)_{x_0, y_0, z_0} \neq 0$$

$$\left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0; \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} \neq 0; \text{ and so on}$$

Example. *The flow in a pipe whose valve is being opened or closed gradually (velocity equation is in the form* $u = ax^2 + bxt$ *).*

5.3.2. Uniform and Non-uniform Flows

Uniform flow. The type of flow, in which the velocity at any given time *does not change* with respect to space is called *uniform flow*. Mathematically, we have:

$$\left(\frac{\partial V}{\partial s}\right)_{t\,=\,\text{constant}} = 0$$

where, ∂V = Change in velocity, and

∂s = Displacement in any direction.

Example. *Flow through a straight prismatic conduit (i.e. flow through a straight pipe of constant diameter).*

Non-uniform flow. It is that type of flow in which the velocity at any given time *changes* with respect to space. Mathematically,

$$\left(\frac{\partial V}{\partial s}\right)_{t\,=\,\text{constant}} \neq 0$$

Example. (*i*) *Flow through a non-prismatic conduit.*

(*ii*) *Flow around a uniform diameter pipe-bend or a canal bend.*

5.3.3. One, Two and Three Dimensional Flows

One dimensional flow. It is that type of flow in which the flow parameter such as velocity is a function of time and one space co-ordinate only. Mathematically:

$$u = f(x), v = 0 \text{ and } w = 0$$

where u, v and w are velocity components in x, y and z directions respectively.

Example. *Flow in a pipe where average flow parameters are considered for analysis.*

Two dimensional flow. The flow in which the velocity is a function of time and two rectangular space coordinates is called *two dimensional flow*. Mathematically:

$$u = f_1(x, y)$$
$$v = f_2(x, y)$$
$$w = 0$$

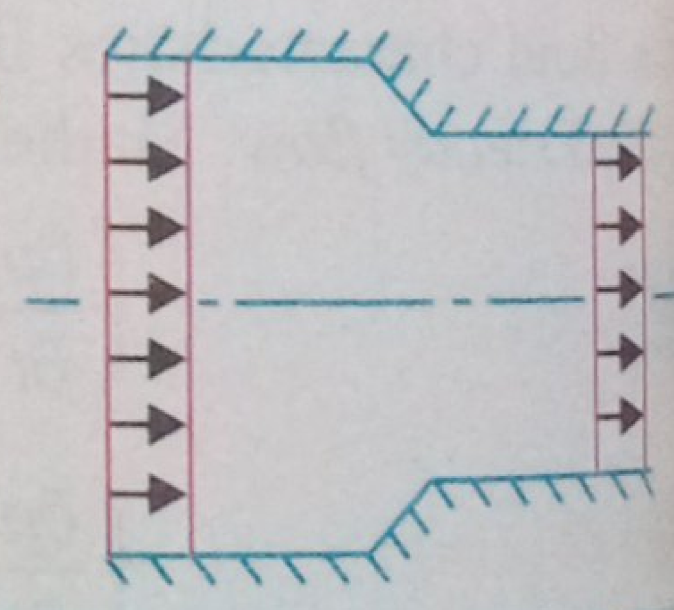

Fig. 5.2. One dimensional flow.

Examples. (*i*) *Flow between parallel plates of infinite extent.*

(*ii*) *Flow in the main stream of a wide river.*

Three dimensional flow. It is that type of flow in which the velocity is a function of time and three mutually perpendicular directions. Mathematically:

$$u = f_1(x, y, z)$$
$$v = f_2(x, y, z)$$
$$w = f_3(x, y, z)$$

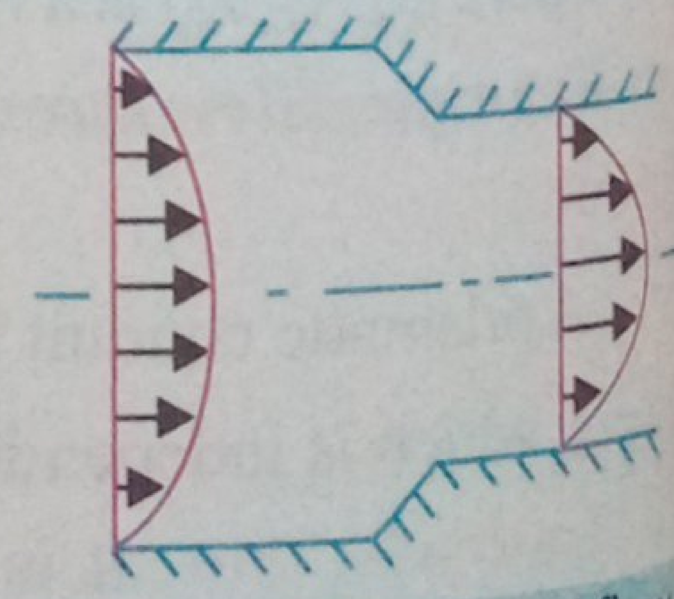

Fig. 5.3. Two dimensional flow.

Examples. (*i*) *Flow in a converging or diverging pipe or channel.*

(*ii*) *Flow in a prismatic open channel in which the width and the water depth are of the same order of magnitude.*

5.3.4. Rotational and Irrotational Flows

Rotational flow. A flow is said to be *rotational* if the fluid particles while moving in the direction of flow *rotate* about their mass centres. *Flow near the solid boundaries is rotational.*

Example. *Motion of liquid in a rotating tank.*

Irrotational flow. A flow is said to be *irrotational* if the fluid particles while moving in the direction of flow *do not rotate* about their mass centres. Flow outside the boundary layer is generally considered irrotational.

Example. *Flow above a drain hole of a stationary tank or a wash basin.*

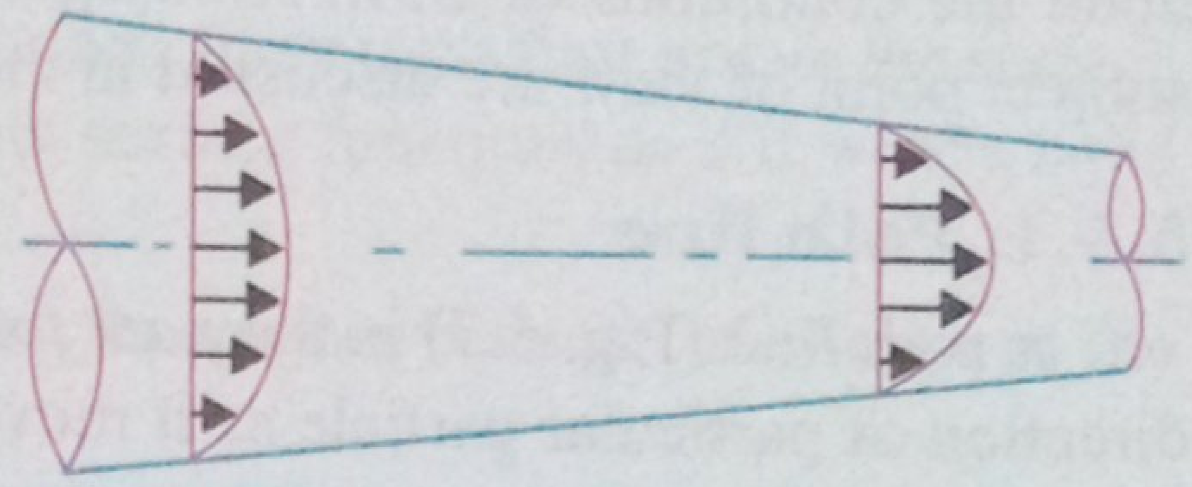

Fig. 5.4. Thee dimensional flow.

Note. If the flow is irrotational as well as steady, it is known as ***Potential flow***.

5.3.5. Laminar and Turbulent Flows

Laminar flow. A laminar flow is one in which *paths taken by the individual particles do not cross one another and move along well defined paths* (Fig. 5.5), This type of flow is also called *stream-line flow or viscous flow*.

Examples. (*i*) *Flow through a capillary tube.*

(*ii*) *Flow of blood in veins and arteries.*

(*iii*) *Ground water flow.*

Turbulent flow. A turbulent flow is that flow in which fluid *particles move in a zig zag way* (Fig. 5.6).

Example. *High velocity flow in a conduit of large size. Nearly all fluid flow problems encountered in engineering practice have a turbulent character.*

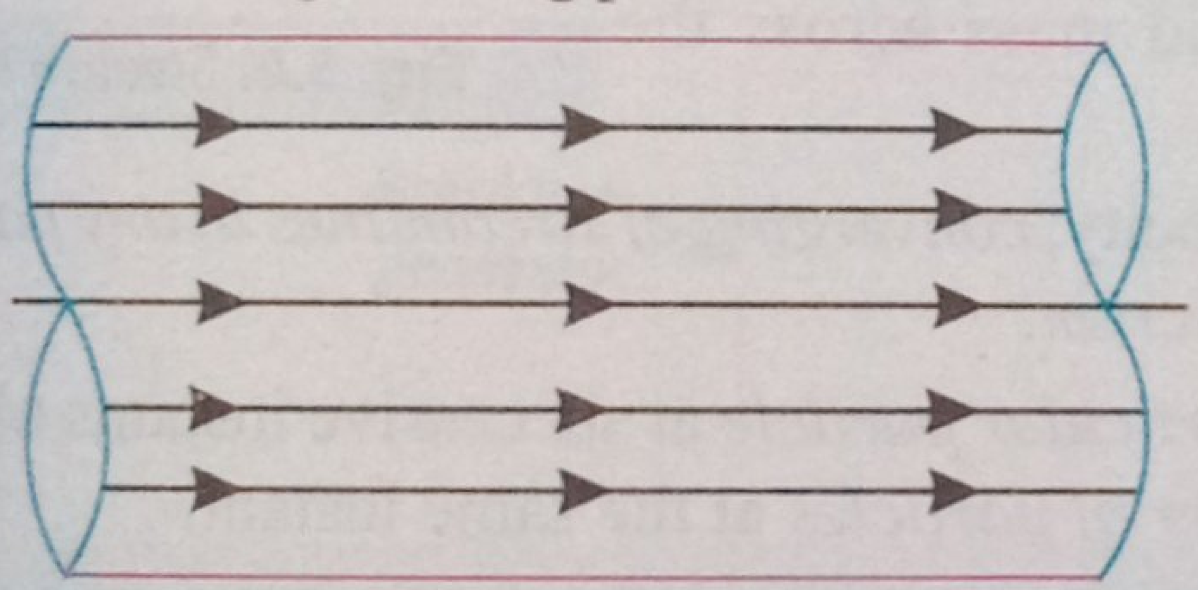

Fig. 5.5. Laminar flow.

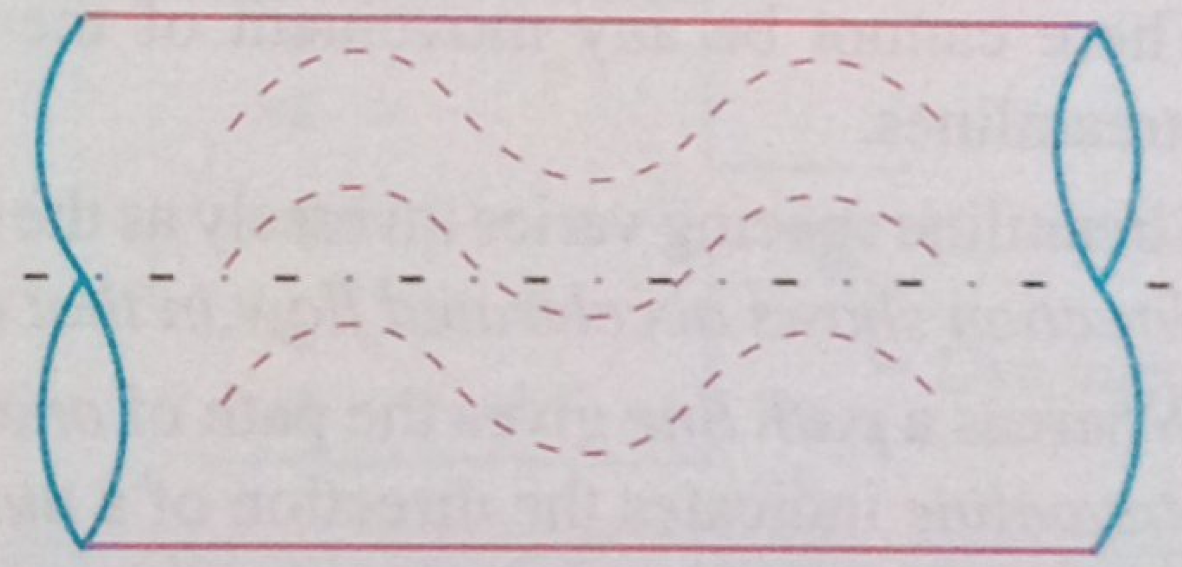

Fig. 5.6. Turbulent flow.

Laminar and turbulent flows are characterised on the basis of Reynolds number (refer to chapter 10).

For Reynolds number (Re) < 2000 ... flow in pipes is *laminar*.

For Reynolds number (Re) > 4000 ... flow in pipes is *turbulent*

For Re between 2000 and 4000 ... flow in pipes *may be laminar or turbulent*.

5.3.6. Compressible and Incompressible Flows

Compressible flow. It is that type of flow in which the *density* (ρ) *of the fluid changes* from point to point (or in other words *density is not constant for this flow*).

Mathematically: $\rho \neq$ constant.

Example. *Flow of gases through orifices, nozzles, gas turbines, etc.*

Incompressible flow. *It is that type of flow in which density is constant for the fluid flow. Liquids are generally considered flowing incompressibly.*

Mathematically: $\rho =$ constant.

Example. *Subsonic aerodynamics.*

5.4. TYPES OF FLOW LINES

Whenever a fluid is in motion, its innumerable particles move along certain lines depending upon the conditions of flow. Although flow lines are of several types, yet some important from subject point of view are discussed in the following subarticles.

5.4.1. Path line

A path line (Fig. 5.7) is the *path followed by a fluid particle in motion*. A path line shows the direction of particular particle as it moves ahead. In general, this is the curve in three-dimensional space. However, if the conditions are such that the flow is two-dimensional the curve becomes two-dimensional.

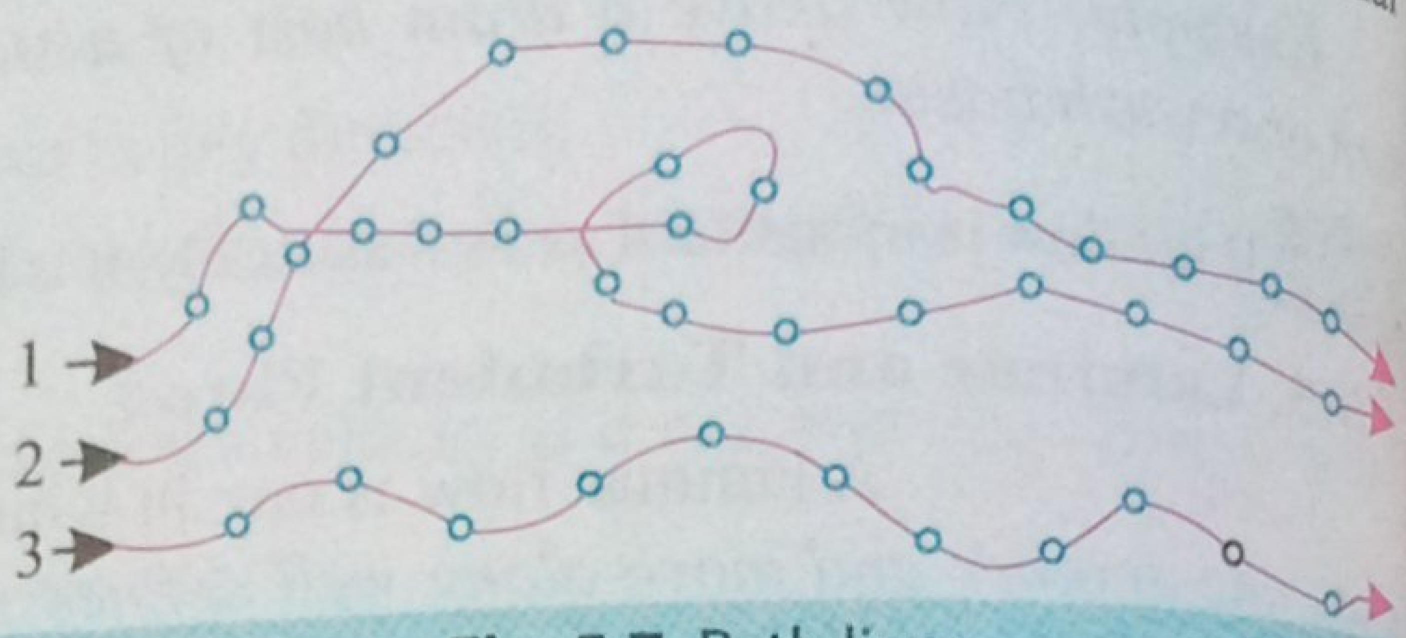

Fig. 5.7. Path lines.

5.4.2. Stream line

A ***stream line*** way be defined as *an imaginary line within the flow so that the tangent at any point on it indicates the velocity at that point.*

Equation of a stream line in a three-dimensional flow is given as:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad ...(5.20)$$

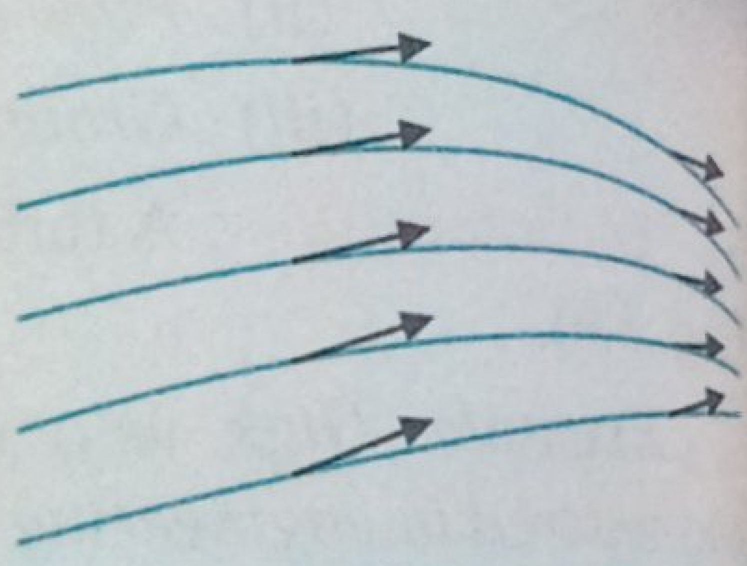

Fig. 5.8. Stream line.

Following *points* about streamlines are *worth noting*:

1. A streamline cannot intersect itself, nor two streamlines can cross.
2. There cannot be any movement of the fluid mass across the streamlines.
3. Streamline spacing varies inversely as the velocity; *converging of streamlines in any particular direction shows accelerated flow in that direction.*
4. Whereas a *path line* gives the path of *one particular particle* at successive instants of time, a *streamline* indicates the direction of a *number of particles* at the same instant.
5. The series of streamlines represent the flow pattern at an instant.

- In *steady flow*, the pattern of streamlines remains invariant with time. The path lines and streamlines will then be identical.
- In *unsteady flow*, the pattern of streamlines may or may not remain the same at the next instant.

5.4.3. Stream Tube

A ***stream tube*** is a *fluid mass bounded by a group of streamlines*. The contents of a stream tube are known as '*current filament*'.

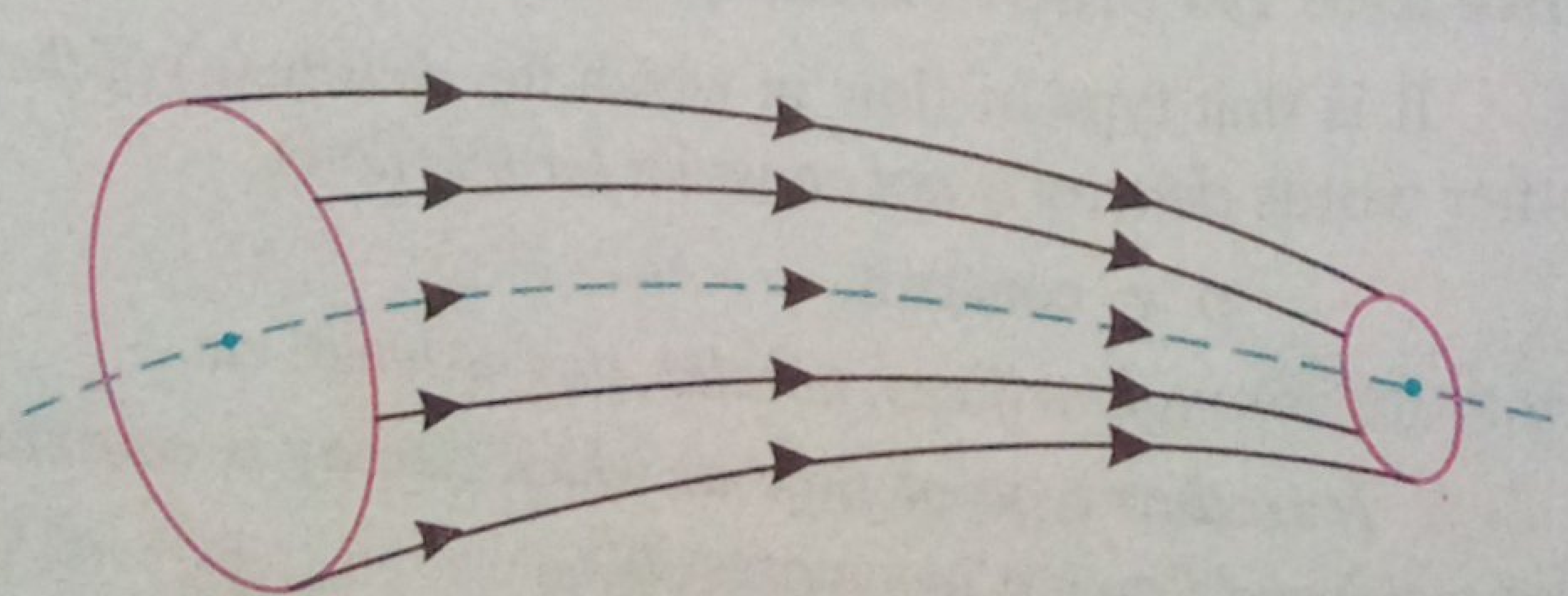

Fig. 5.9. Stream tube.

Examples of stream tube: *Pipes* and *nozzles*.

Following *points* about stream tube are *worth noting*:

1. The stream tube has finite dimensions.
2. As there is no flow perpendicular to stream lines, therefore, there is no flow across the surface (called *stream surface*) of the stream tube. The stream surface functions as if it were a solid wall.
3. The shape of a stream tube changes from one instant to another because of change in the position of streamlines.

5.4.4. Streak Line

The ***streak line*** *is a curve which gives an instantaneous picture of the location of the fluid particles, which have passed through a given point.*

Examples. (*i*) *The path taken by smoke coming out of chimney* (Fig. 5.10).

(*ii*) *In an experimental work to trace the motion of fluid particles, a coloured dye may be injected into the flowing fluid and the resulting coloured filament lines at a given location give the streak lines* (Fig 5.11).

Note. In case of a *steady flow* there is no geometrical distinction between the streamlines, path lines and streak lines; they are coincident if they originate at the same point. For an *unsteady flow* (*e.g.* a person giving out whiff of smoke from a cigarette), the path, streak and stream lines are all different.

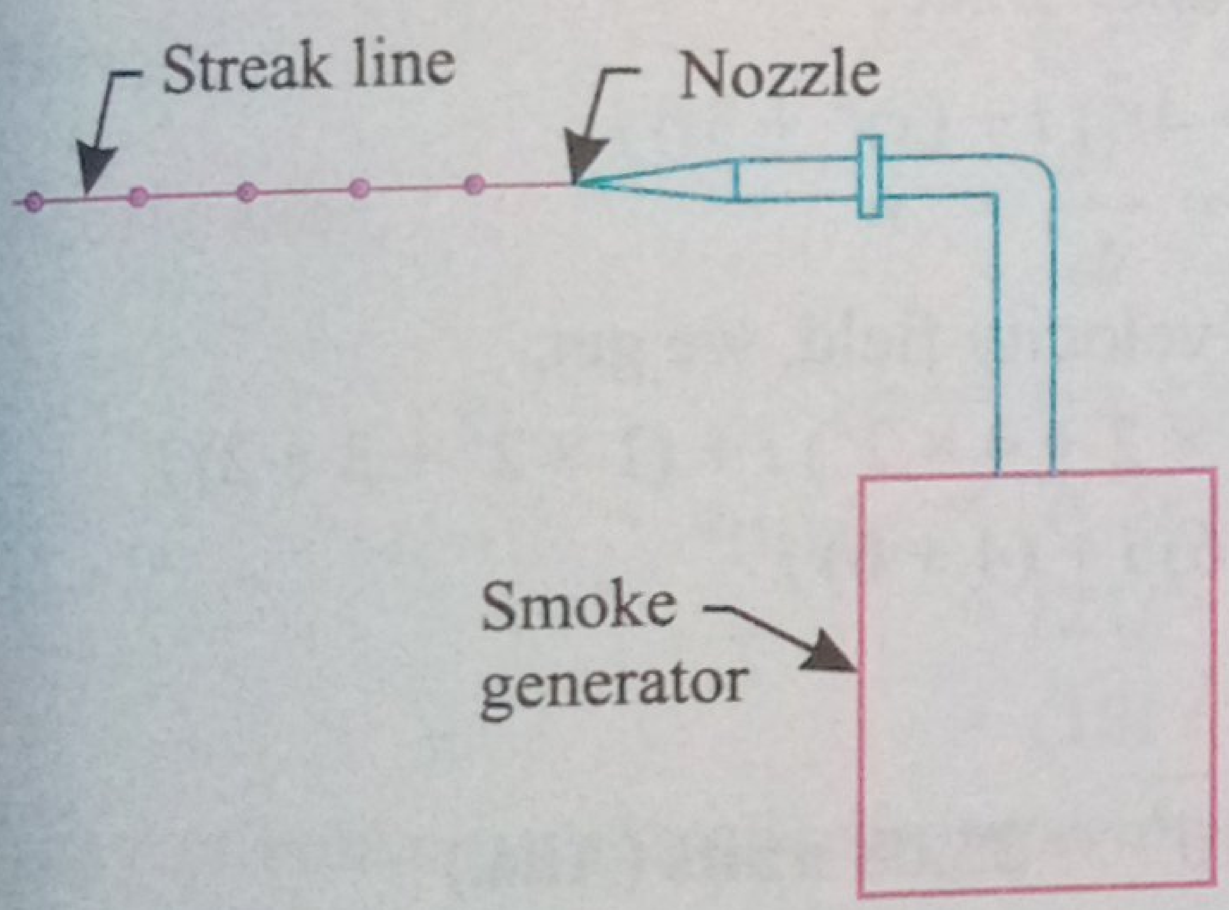

Fig. 5.10. Streak line of smoke issuing from a nozzle.

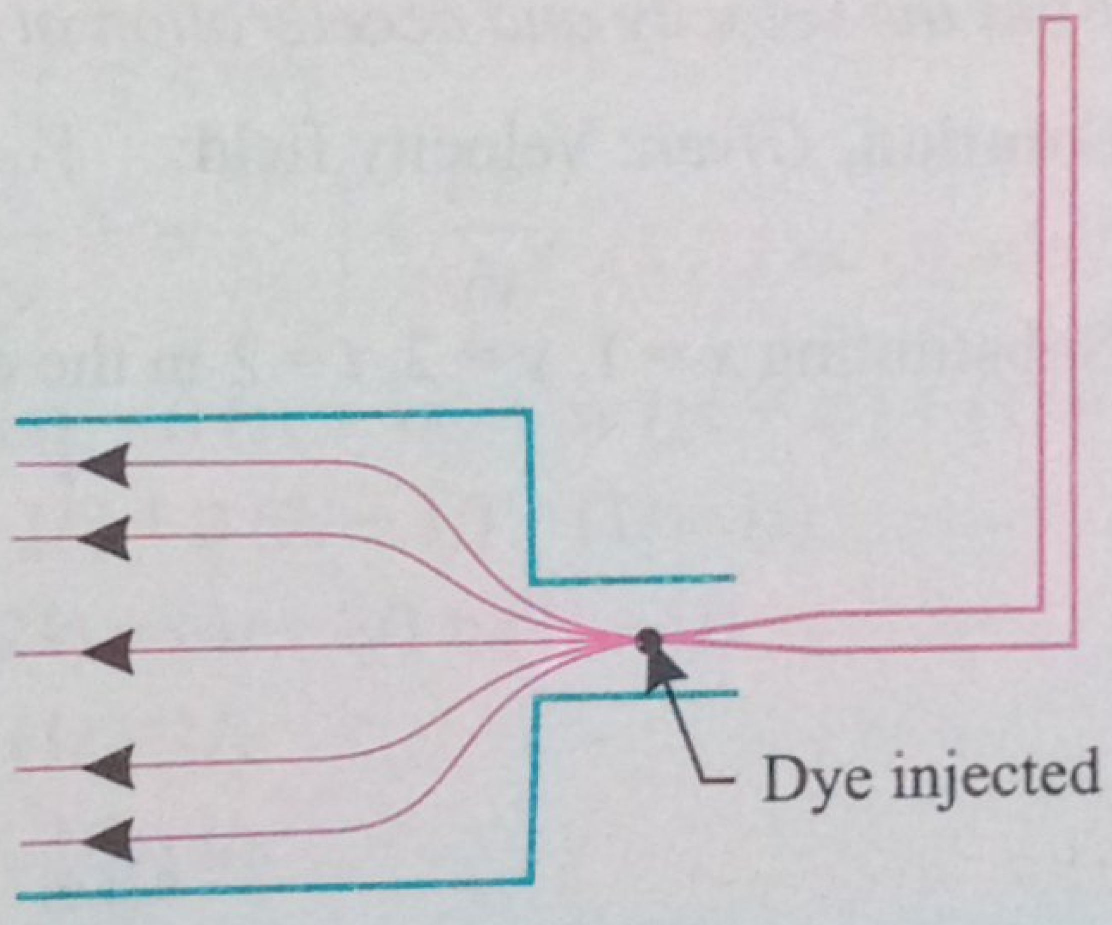

Fig. 5.11. Streak lines at $t = t_1$.

Example 5.1. *In a fluid, the velocity field is given by*

$$V = (3x + 2y)\, i + (2z + 3x^2)\, j + (2t - 3z)\, k$$

Determine:

(*i*) *The velocity components u, v, w at any point in the flow field;*

(*ii*) *The speed at point (1, 1, 1);*

(*iii*) *The speed at time t = 2s at point (0, 0, 2).*

Also classify the velocity field as steady, or unsteady, uniform or non-uniform and one, two or three dimensional.

Solution. *Given*: Velocity field, $V = (3x + 2y)\, i + (2z + 3x^2)\, j + (2t - 3z)\, k$

(*i*) Velocity components:

The velocity components are:

$$\mathbf{u = 3x + 2y, v = (2z + 3x^2), w = (2t - 3z)} \quad \textbf{(Ans.)}$$

(*ii*) Speed at point (1, 1, 1), $V_{(1,1,1)}$:

Substituting $x = 1, y = 1, z = 1$ in the expressions for u, v and w, we have:

$$u = (3 + 2) = 5, v = (2 + 3) = 5, w = (2t - 3)$$

$$\therefore \quad V^2 = u^2 + v^2 + w^2$$

$$= 5^2 + 5^2 + (2t-3)^2$$
$$= 25 + 25 + 4t^2 - 12t + 9$$
$$= 4t^2 - 12t + 59$$

$\therefore$ $$\mathbf{V}_{(1,1,1)} = \sqrt{4t^2 - 12t + 59} \quad \textbf{(Ans.)}$$

(*iii*) Speed at $t = 2s$ at point (0, 0, 2):

Substituting $t = 2, x = 0, y = 0, z = 2$ in the expressions for u, v and w, we

$$u = 0, v = (2 \times 2 + 0) = 4, w = (2 \times 2 - 3 \times 2) = -2$$

$\therefore$ $$V^2 = u^2 + v^2 + w^2 = 0 + 4^2 + (-2)^2 = 20$$

or, $$\mathbf{V}_{(0,0,2)} = \sqrt{20} = \mathbf{4.472 \text{ units}} \textbf{ (Ans.)}$$

Velocity field, type:

(*i*) Since V at given (x, y, z) *depends on t* it is **unsteady flow, (Ans.)**

(*ii*) Since at given t velocity changes in the X direction it is **non-uniform flow. (Ans.)**

(*iii*) Since V depends on x, y, z it is three **dimensional flow. (Ans.)**

Example 5.2 *Velocity for a two dimensional flow field is given by*

$$V = (3 + 2xy + 4t^2)\, i + (xy^2 + 3t)\, j$$

Find the velocity and acceleration at a point (1,2) *after* 2 *sec.*

Solution. *Given*: Velocity field: $V = (3 + 2xy + 4t^2)\, i + (xy^2 + 3t)\, j$

Velocity at (1, 2), $\mathbf{V}_{(1,2)}$:

Substituting $x = 1, y = 2, t = 2$ in the expression of velocity field, we get:

$$V = (3 + 2 \times 1 \times 2 + 4 \times 2^2)\, i + (1 \times 2^2 + 3 \times 2) j$$
$$= (3 + 4 + 16)\, i + (4 + 6)\, j$$
$$= 23i + 10j$$

$$V_{(1,2)} = \sqrt{23^2 + 10^2} = \mathbf{25.08 \text{ units}} \textbf{ (Ans.)}$$

Acceleration at point (1, 2), $\mathbf{a}_{(1,2)}$:

We know that: $$a = \frac{dV}{dt} = \left(u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y}\right) + \frac{\partial V}{\partial t}$$

so, $$V = (3 + 2xy + 4t^2)i + (xy^2 + 3t)\, j$$

$$\frac{\partial V}{\partial x} = 2yi + y^2 j,$$

$$\frac{\partial V}{\partial y} = 2xi + 2xyj, \text{ and}$$

$$\frac{\partial V}{\partial t} = 8\, ti + 3\, j$$

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$$= 23(4i + 4j) + 10\,(2i + 4j) + (16i + 3j)$$
$$= 92i + 92j + 20i + 40j + 16i + 3j$$
$$= 128\,i + 135\,j$$
$$\therefore \quad a_{(1,2)} = \sqrt{128^2 + 135^2} = \mathbf{186.03\ units\ (Ans.)}$$

Example 5.3. *Find the velocity and acceleration at a point (1, 2, 3) after 1 sec. for a three-dimensional flow given by $u = yz + t$, $v = xz - t$, $w = xy$ m/s.*

Solution. *Given*: Three-dimensional flow field is given as:

$$u = yz + t,\ v = xz - t,\ w = xy \text{ m/s}$$

Velocity at a point 1, 2, 3 $V_{(1,2,3)}$:

Velocity at a point (1, 2, 3) after 1s is calculated as follows:

$$u = yz + t = 2 \times 3 + 1 = 7 \text{ m/s},\ v = xz - t = 1 \times 3 - 1 = 2 \text{ m/s and}$$
$$w = xy = 1 \times 2 = 2 \text{ m/s.}$$
$$\therefore \quad V_{(1,2,3)} = 7i + 2j + 2k$$
$$= \sqrt{7^2 + 2^2 + 2^2} = 7.55 \text{ m/s}$$

Hence, $V_{(1,2,3)} = \mathbf{7.55\ m/s\ (Ans.)}$

Acceleration, $a_{(1,2,3)}$:

Now, $V = (yz + t)i + (xz - t)\,j + xy\,k$ m/s

Acceleration, $$a = \frac{dV}{dt} = \left(u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}\right) + \frac{\partial V}{\partial t}$$

$$a = (yz + t)\,(zi + yk) + (xz - t)\,(zi + xk) + xy\,(yi + xj) + (1i - 1j)$$
$$\therefore \quad a_{(1,2,3)} = 7(\,3j + 2k) + 2(3i + 1k) + 2\,(2i + 1j) + (1i - 1j)$$
$$= (21j + 14k) + (6i + 2k) + (4i + 2j) + (1i - 1j)$$
$$= (10i + 23j + 16k) + (1i - 1j)$$

The *convective acceleration components are*: (10, 23, 16) m/s^2

The *local acceleration components are*: (1, –1) m/s^2 along x and y directions.

The total acceleration of fluid particles at the points (1, 2, 3) is given by:

$$a_{(1,2,3)} = \sqrt{(10 + 1)^2 + [23 + (-1)]^2 + 16^2}$$
$$= \sqrt{11^2 + 22^2 + 16^2} = 29.34 \text{ m/s}^2$$

Hence, $a_{(1,2,3)} = \mathbf{29.34\ m/s^2\ (Ans.)}$

Example 5.4. *The velocity along the centreline of a nozzle of length l is given by*

$$V = 2t\left(1 - \frac{x}{2l}\right)^2$$

where V = velocity in m/s, t = time in seconds from commencement of flow, x = distance from inlet to nozzle. Calculate the local acceleration, convective acceleration and the total acceleration when t = 6s, x = 1m and l = 1.6 m.

Solution. The velocity along the centreline of a nozzle, $V = 2t\left(1 - \frac{x}{2l}\right)^2$ ...*(Given)*

Local acceleration:

$$\text{Local acceleration} = \frac{\partial V}{\partial t} = 2\left(1 - \frac{x}{2l}\right)^2$$

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At $t = 6\,s$ and $x = 1m$,

$$\frac{\partial V}{\partial t} = 2\left(1 - \frac{1}{2 \times 1.6}\right)^2 = \mathbf{0.945\ m/s^2}\ \textbf{(Ans.)}$$

Convective acceleration:

$$\text{Convective acceleration} = V\frac{\partial V}{\partial x}$$

$$= 2t\left(1 - \frac{x}{2l}\right)^2 \times 2t \times 2\left(1 - \frac{x}{2l}\right)\left(-\frac{1}{2l}\right)$$

$$= -\frac{4t^2}{l}\left(1 - \frac{x}{2l}\right)^3$$

At $t = 6s$ and $x = 1$ m,

$$\text{Convective acceleration} = -\frac{4 \times 6^2}{1.6}\left(1 - \frac{1}{2 \times 1.6}\right)^3$$

$$= \mathbf{-29.24\ m/s^2}\ \textbf{(Ans.)}$$

Total acceleration:

$$\text{Total acceleration} = \text{Local acceleration} + \text{convective acceleration}$$

$$= 0.945 + (-29.24) = \mathbf{-28.295\ m/s^2}\ \textbf{(Ans.)}$$

Example 5.5. *A conical pipe diverges uniformly from 100 mm to 200 m diameter over a length of 1 m. Determine the local and convective acceleration at the mid-section assuming*

(i) Rate of flow is 0.12 m^3/s and it remains constant;

(ii) Rate of flow varies uniformly from 0.12 m^3/s to 0.24 m^3/s in 5 sec., at t = 2 sec.

Solution. *Given:*

Diameter at the inlet, $D_1 = 0.1m$.

Diameter at the outlet, $D_2 = 0.2\ m$

Length, $l = 1\ m$

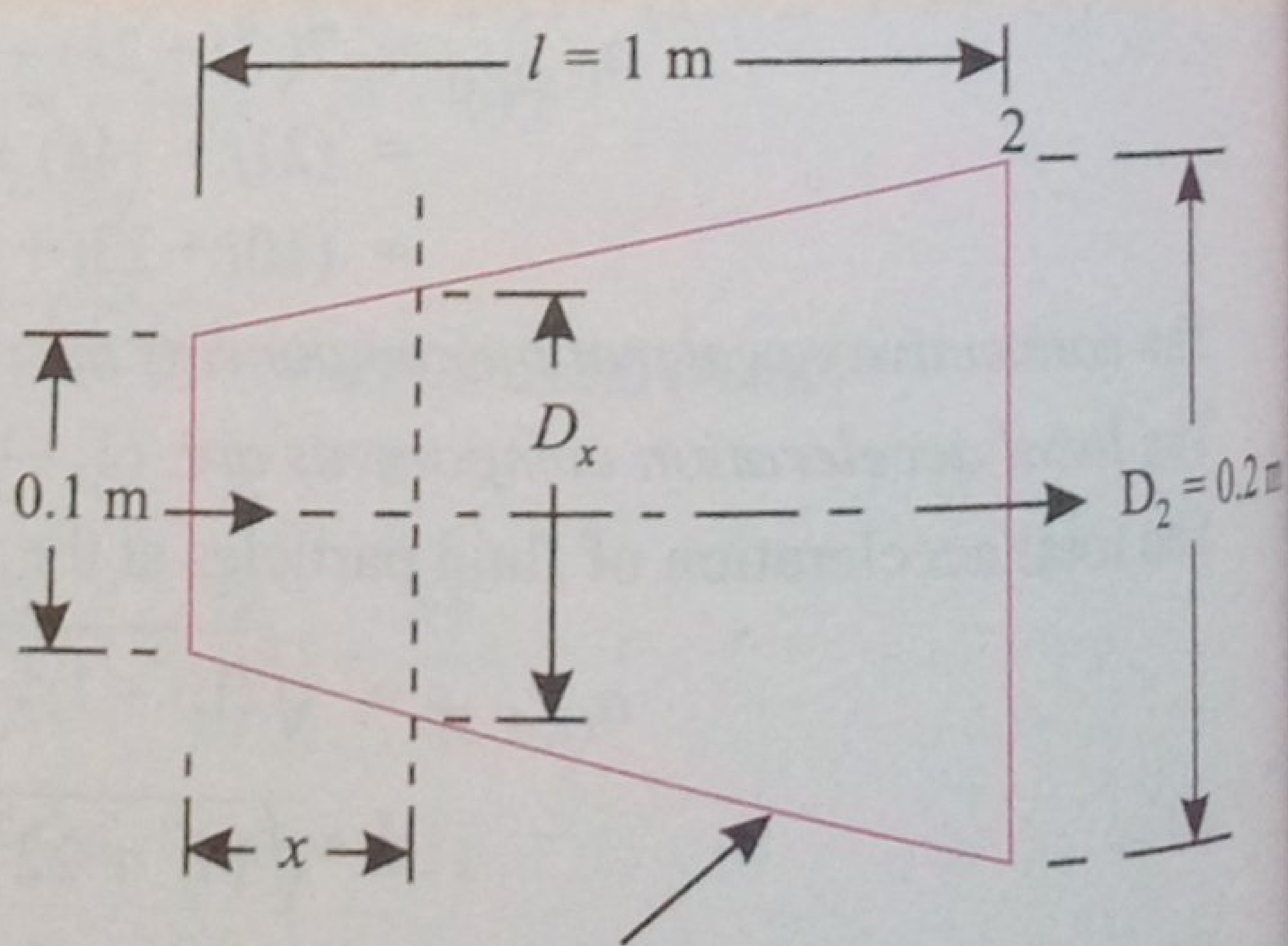

Fig. 5.12

Diameter at any distance x metres from the inlet,

$$D = D_1 + \left(\frac{D_2 - D_1}{l}\right) \times x$$

$$= 0.1 + \left(\frac{0.2 - 0.1}{1}\right) \times x$$

$$= 0.1 + 0.1x = 0.1\,(1 + x)$$

$\therefore$ Cross-sectional area,

$$A_x = \frac{\pi}{4} \times D_x^2 = \frac{\pi}{4}\{0.1(1 + x)\}^2$$

$$= 0.00785\,(1 + x)^2$$

$$\text{Velocity of flow, } u_x\,(= u) = \frac{Q}{A_x} = \frac{Q}{0.00785(1 + x)^2}$$

$$\text{Velocity gradient, } \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{Q}{0.00785\,(1 + x)^2}\right] = \frac{-2Q}{0.00785(1 + x)^3}$$

(i) Discharge Q = 0.12 m^3/s = constant (at any section)

$$\text{Acceleration} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$$

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(*a*) **The local acceleration:**

The local acceleration at *mid-section*

$$= \frac{\partial u}{\partial t} = 0, \text{ since the flow is } steady \textbf{ (Ans.)}$$

(*b*) **The convective acceleration:**

The convective acceleration,

$$a_x = u\frac{\partial u}{\partial x} = \frac{Q}{0.00785\,(1+x)^2} \times \frac{-2Q}{0.00785\,(1+x)^3}$$

$$= \frac{-2Q^2}{(0.00785)^2\,(1+x)^5}$$

$\therefore$ The convective acceleration at *mid-section*,

$$(a_x)_{x=0.5\text{m}} = \frac{-2\times(0.12)^2}{(0.00785)^2\;(1+0.5)^5}$$

$$= -61{\cdot}5 \text{ m/s}^2 \textbf{ (Ans.)}$$

The $-ve$ sign indicates *decrease in velocity* along the direction of flow (this is so as the cross-sectional area is increasing)

(*ii*) **Discharge Q varies w.r.t. time:**

The discharge Q varies from 0.12 m^3/s to 0.24 m^3/s in 5 s.

At t = 2 s, the discharge is,

$$Q = 0.12 + \frac{0.24-0.12}{5}\times 2 = 0.168 \text{ m}^3/\text{s}$$

(*a*) **The local acceleration:**

The local acceleration at *mid-section*,

$$= \frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[\frac{Q}{0.00785\,(1+x)^2}\right] = \frac{1}{0.00785\,(1+x)^2}\times\frac{\partial Q}{\partial t}$$

$$= \frac{1}{0.00785\,(1+0.5)^2}\times\left(\frac{0.168-0.12}{2}\right)$$

[since discharge changes 0.12 m^3/s to 0.168 m^3/s in 2*s*]

$$= 1.36 \text{ m/s}^2 \textbf{ (Ans.)}$$

(*b*) **The convective acceleration:**

The convective acceleration at *mid-section*,

$$(a_x)_{x=0.5} = \frac{-2Q^2}{(0.00785)^2\,(1+x)^5} = \frac{-2\times 0.168^2}{(0.00785)^2\;(1+0.5)^5}$$

$$= -120{\cdot}6 \text{ m/s}^2 \textbf{ (Ans.)}$$

Total acceleration along the main flow is,

$$(a)_{\text{total}} = (a)_{\text{local}} + (a)_{\text{conv.}}$$
$$= 1.36 - 120.6 = -119.24 \text{ m/s}^2 \textbf{ (Ans.)}$$

Example 5.6. *At entry to the pump intake the velocity is found to vary inversely as the square of radial distance from inlet to suction pipe. The velocity was found to be 0.6 m/s at a radial distance of 1.5 m.*

Substituting for u, v and w, we get:

$$\underset{(i)}{\frac{dx}{-x}} = \underset{(ii)}{\frac{dy}{3-y}} = \underset{(iii)}{\frac{dz}{3-z}}$$

Considering the expressions (i) and (ii) and integrating, we get:

$$\int \frac{dx}{-x} = \int \frac{dz}{(3-y)}$$

$$= -\log_e x = -\log_e (3-y) + C_1$$

(where, C_1 = constant of integration).

Since the streamline passes through $x = 1, y = 2$ $\quad \therefore C_1 = 0$

$\therefore$ $\quad (x)^{-1} = (3-y)^{-1}$ $\quad$ or $\quad x = (3-y)$ ...(1)

Considering the expressions (i) and (iii), and integrating, we get:

$$\int \frac{dx}{-x} = \int \frac{dy}{3-z}$$

or, $\quad -\log_e x = -\log (3-z) + C_2$

(where C_2 = constant of integration)

Since the streamline passes through $x = 1, z = 2$ $\quad \therefore C_2 = 0$

$\therefore$ $\quad x^{-1} = (3-z)^{-1}$

or, $\quad x = (3-z)$ ...(2)

From (1) and (2), the equation of the streamline passing through (1, 2, 2) is given as:

$$\boldsymbol{x = (3-y) = (3-z)} \textbf{ (Ans.)}$$

Example 5.8. *Obtain the equation to the streamlines for the velocity field given as:*

$$V = 2x^3 i - 6x^2 yj$$

Solution. *Given*: Velocity field, $V = 2x^3 i - 6x^2 yj$

Here, $\quad u = 2x^3, v = 6x^2 y$

The streamlines in two dimensions are defined by:

$$\frac{dx}{u} = \frac{dy}{v}$$

or, $$\frac{dy}{dx} = \frac{v}{u} = \frac{-6x^2 y}{2x^3} = \frac{-3y}{x}$$

Separating the variables, we have:

$$\frac{dy}{y} = \frac{-3dx}{x}$$

Integrating, we get:

$$ln (y) = -3ln (x) + C_1$$

or, $\quad ln (y) + 3\, ln (x) = C_1$

or, $\quad yx^3 = \mathbf{C}$ **(Ans.)**

Note. The streamlines in the first quadrant can be sketched by giving different values for the constant C$\left(y = \frac{C}{x^3}\right)$.

Solution. The equation of a streamline is given as:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Here, $u = 3x, v = 4y,$ and $w = -7z$

$$\therefore \quad \frac{dx}{3x} = \frac{dy}{4y} = -\frac{dz}{7z}$$

Considering equations involving x and y, on integration we get:

$$\frac{1}{3} ln(x) = \frac{1}{4} ln(y) + ln(C_1') \text{ where } C'_1 = \text{a constant}$$

or, $$y = C_1 x^{4/3} \qquad ...(i)$$

where, C_1 is another constant:

Similarly, by considering equations with x and z and on integration, we have:

$$\frac{1}{3} ln(x) = -\frac{1}{7} ln(z) + ln(C_2'), \text{ where } C_2' = \text{ a constant}$$

or, $$z = \frac{C_2}{x^{7/3}} \qquad ...(ii)$$

where, C_2 is another constant.

Inserting the coordinates of the point L(1, 2, 3), we get:

From eqn. (i) $$C_1 = \frac{y}{(x)^{4/3}} = \frac{2}{(1)^{4/3}} = 2$$

From eqn. (ii) $$C_2 = zx^{7/3} = 3 \times (1)^{7/3} = 3$$

Hence, the streamline passing through L is given by:

$$\mathbf{y = 2x^{4/3} \text{ and } z = \frac{3}{x^{7/3}}} \quad \textbf{(Ans.)}$$

5.5. RATE OF FLOW OR DISCHARGE

Rate of flow (or **discharge**) *is defined as* the *quantity of a liquid flowing per second through a section of pipe or a channel.* It is generally denoted by Q. Let us consider a liquid flowing through a pipe.

Let, A = Area of cross-section of the pipe, and

V = Average velocity of the liquid.

$\therefore$ Discharge, Q = Area × average velocity *i.e.*, $Q = A.V$ ...(5.21)

If area is in m^2 and velocity is in m/s, then the discharge,

$$Q = m^2 \times m/s = m^3/s = \text{cumecs}.$$

5.6. CONTINUITY EQUATION

The **continuity equation** *is based on the principle of conservation of mass.* It states as follows:

"If no fluid is added or removed from the pipe in any length then the mass passing across different sections shall be same."

Consider two cross-sections of a pipe as shown in Fig 5.13

Let, A_1 = Area of the pipe at section 1–1,

V_1 = Velocity of the fluid at section 1–1,

ρ_1 = Density of the fluid at section 1–1,

and A_2, V_2, ρ_2 are corresponding values at sections 2–2.

The total quantity of fluid passing through section 1–1 = $\rho_1 A_1 V_1$

and, the total quantity of fluid passing through section 2–2 = $\rho_2 A_2 V_2$

From the law of conservation of mass (theorem of continuity), we have:

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad ...(5.22)$$

Eqn. (5.22) is applicable to the compressible as well as incompressible fluids and is called ***Continuity Equation.*** In case of *incompressible fluids*, $\rho_1 = \rho_2$ and the continuity eqn. (5.21) reduces to:

$$A_1 V_1 = A_2 V_2 \qquad ...(5.23)$$

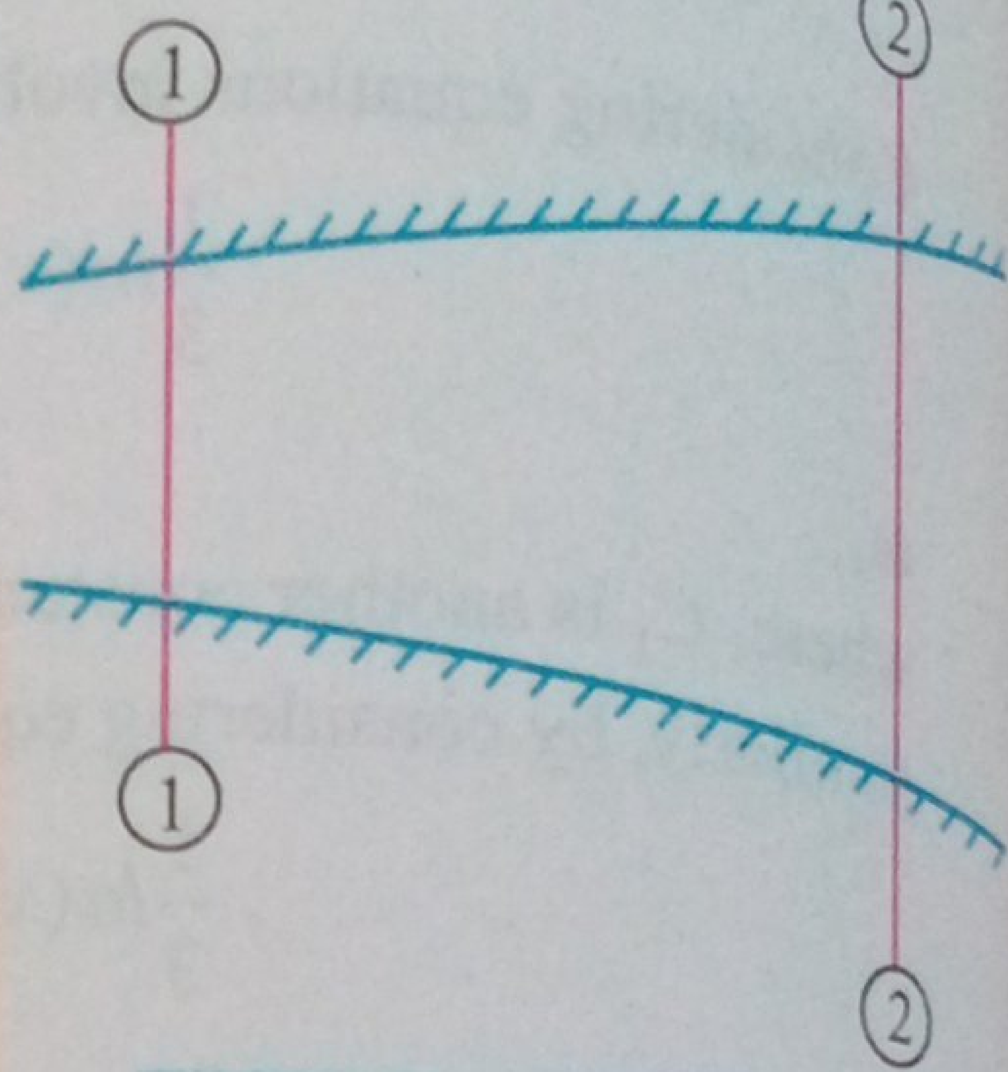


Fig. 5.13. Fluid flow through a pipe.

Example 5.11. *The diameters of a pipe at the sections 1-1 and 2-2 are 200 mm and 300 mm respectively. If the velocity of water flowing through the pipe at section 1-1 is 4m/s, find:*

(i) Discharge through the pipe, and

(ii) Velocity of water at section 2-2

Solution. Diameter of the pipe at *section* 1-1,

$$D_1 = 200 \text{ mm} = 0.2 \text{ m}$$

$$\therefore \quad \text{Area, } A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.2^2 = 0.0314 \text{ m}^2$$

$$\text{Velocity, } V_1 = 4 \text{ m/s}$$

Diameter of the pipe at *section* 2-2,

$$D_2 = 300 \text{ mm}$$

$$\therefore \quad \text{Area, } A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} \times 0.3^2 = 0.0707 \text{ m}^2$$

(i) Discharge through the pipe, Q:

Using the relation,

$Q = A_1 V_1$, we have:

$Q = 0.0314 \times 4 = \mathbf{0.1256\ m^3/s}$ **(Ans.)**

(ii) Velocity of water at section 2-2, V_2:

Using the relation,

$A_1 V_1 = A_2 V_2$, we have:

$$V_2 = \frac{A_1 V_1}{A_2} = \frac{0.0314 \times 4}{0.0707}$$

$$= \mathbf{1.77\ m/s}\ \textbf{(Ans.)}$$

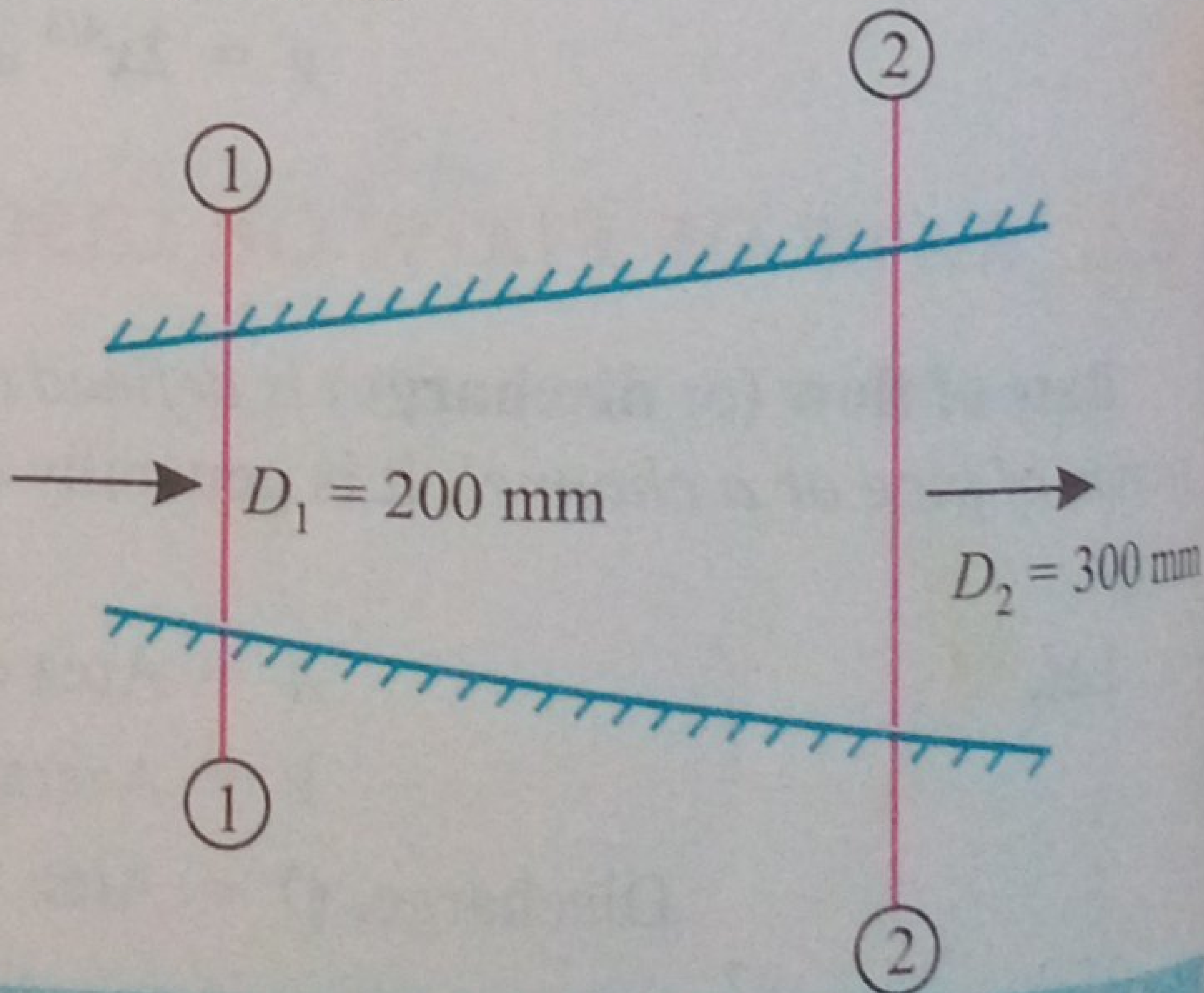


Fig. 5.14

Example 5.12. *A pipe (1) 450 mm in diameter branches into two pipes (2 and 3) of diameters 300 mm and 200 mm respectively as shown in Fig. 5.15. If the average velocity in 450 mm diameter pipe is 3 m/s find:*

(i) Discharge through 450 mm diameter pipe;

(ii) Velocity in 200 mm diameter pipe if the average veloc...

Velocity, $V_1 = 3$ m/s

Diameter, $D_2 = 300$ mm $= 0.3$ m

$\therefore$ Area, $A_2 = \dfrac{\pi}{4} \times 0.3^2 = 0.0707$ m^2

Velocity, $V_2 = 2.5$ m/s

Diameter, $D_3 = 200$ mm $= 0.2$ m

Area, $A_3 = \dfrac{\pi}{4} \times 0.2^2 = 0.0314$ m^2

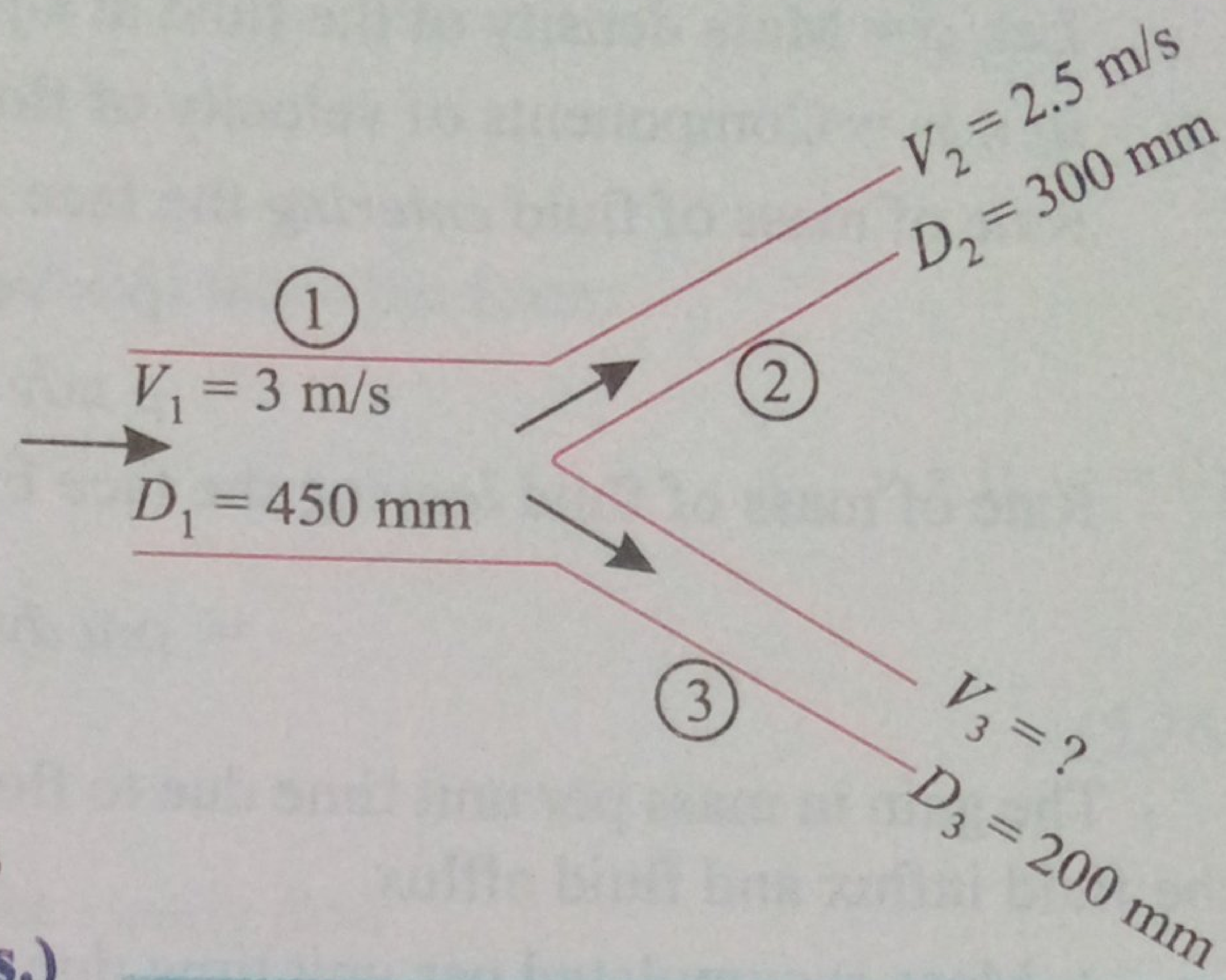


Fig. 5.15

(i) Discharge through pipe (1) Q_1:

Using the relation, $Q_1 = A_1 V_1 = 0.159 \times 3$

$= \mathbf{0.477\ m^3/s}$ **(Ans.)**

(ii) Velocity in pipe of diameter 200 mm i.e. V_3:

Let Q_1, Q_2 and Q_3 be the discharge in pipes 1, 2 and 3 respectively.

Then, according to continuity equation:

$$Q_1 = Q_2 + Q_3 \quad \text{...(i)}$$

where, $Q_1 = 0.477$ m^3/s (calculated earlier)

and, $Q_2 = A_2 V_2 = 0.0707 \times 2.5 = 0.1767$ m^3/s

$\therefore$ $0.477 = 0.1767 + Q_2$ [from eq. (i)]

or, $Q_3 = 0.477 - 0.1767 \simeq 0.3$ m^3/s

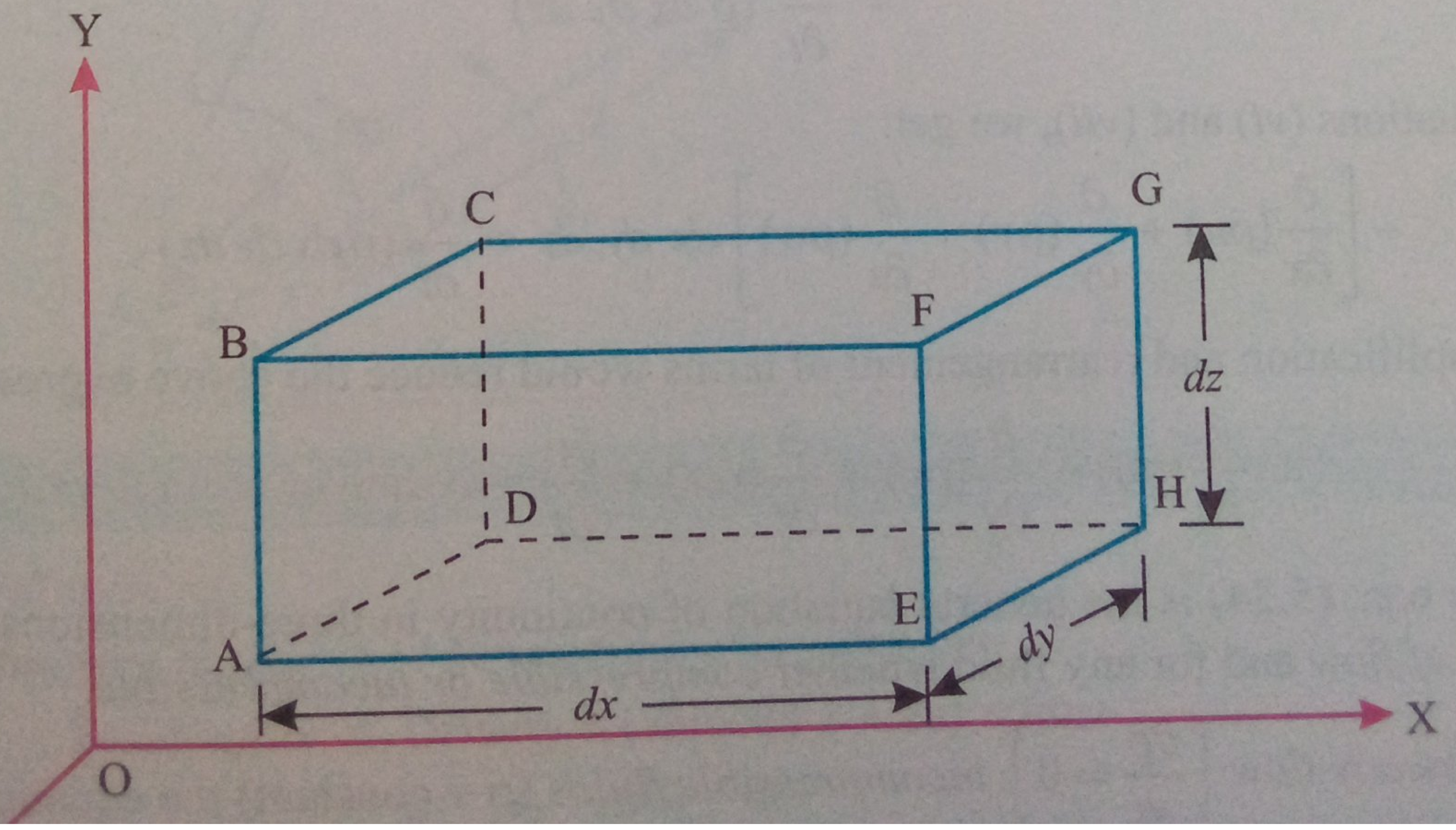
But, $Q_3 = A_3 V_3$

$\therefore$ $V_3 = \dfrac{Q_3}{A_3} = \dfrac{0.3}{0.0314} = 9.55$ m/s

i.e. $V_3 = \mathbf{9.55\ m/s}$ **(Ans.)**

5.7. CONTINUITY EQUATION IN CARTESIAN CO-ORDINATES

Consider a fluid element (control volume) – parallelopiped with sides dx, dy and dz as shown in Fig. 5.16.



7.1. DIMENSIONAL ANALYSIS—INTRODUCTION

Dimensional analysis *is a mathematical technique which makes use of the study of the dimensions for solving several engineering problems.* Each physical phenomenon can be expressed by an equation giving relationship between different quantities, such quantities are dimensional and non-dimensional. Dimensional analysis helps in determining a systematic arrangement of the variables in the physical relationship, combining dimensional variables to form non-dimensional parameters. It is based on the *principle of dimensional homogeneity* and uses the dimensions of relevant variables affecting the phenomenon.

Dimensional analysis has become an important tool for analysing fluid flow problems. It is *specially useful in presenting experimental results in a concise form.*

Uses of dimensional analysis:

The uses of dimensional analysis may be summarised as follows:

1. To test the dimensional homogeneity of any equation of fluid motion.
2. To derive rational formulae for a flow phenomenon.
3. To derive equations expressed in terms of non-dimensional parameters to show the relative significance of each parameter.
4. To plan model tests and present experimental results in a systematic manner, thus making it possible to analyse the complex fluid flow phenomenon.

Advantages of dimensional analysis:

Dimensional analysis entails the following *advantages*:

1. It expresses the functional relationship between the variables in dimensionless terms.
2. In hydraulic model studies it reduces the number of variables involved in a physical phenomenon, generally by *three*.
3. By the proper selection of variables, the dimensionless parameters can be used to make certain logical deductions about the problem.
4. Design curves, by the use of dimensional analysis, can be developed from experimental data or direct solution of the problem.
5. It enables getting up a theoretical equation in a simplified dimensional form.
6. Dimensional analysis provides partial solutions to the problems that are too complex to be dealt with mathematically.
7. The conversion of units of quantities from one system to another is facilitated.

7.2. DIMENSIONS

The various physical quantities used in fluid phenomenon can be expressed in terms of **fundamental quantities** or *primary quantities*. The fundamental quantities are *mass, length, time* and *temperature*, designated by the letters, M, L, T, θ respectively. Temperature is specially useful in *compressible flow*. The *quantities which are expressed in terms of the fundamental or primary quantities are called* **derived or secondary quantities,** (*e.g.*, velocity, area, acceleration etc.). *The expression for a derived quantity in terms of the primary quantities is called the* **dimension** *of the physical quantity.*

A quantity may either be expressed dimensionally in *M-L-T* or *F-L-T* system (some engineers prefer to use force instead of mass as fundamental quantity because the force is easy to measure). Table 7.1 gives the dimensions of various quantities used in both the systems.

Example. 7.1. *Determine the dimensions of the following quantities:*

(*i*) *Discharge,* (*ii*) *Kinematic viscosity,*

(*ii*) *Force, and* (*iv*) *Specific weight.*

Solution. (*i*) *Discharge* = Area × velocity

$$= L^2 \times \frac{L}{T} = \frac{L^3}{T} = \mathbf{L^3T^{-1}}\ \textbf{(Ans.)}$$

(*ii*) *Kinematic viscosity* $(\nu) = \frac{\mu}{\rho}$

where μ is given by: $\tau = \mu \frac{du}{dy}$

$$\therefore \quad \mu = \frac{\tau}{\frac{du}{dy}} = \frac{\text{Shear stress}}{\frac{L}{T} \times \frac{1}{L}} = \frac{\text{Force/area}}{1/T}$$

$$= \frac{\text{Mass} \times \text{acceleration}}{\text{Area} \times 1/T} = \frac{M \times \frac{L}{T^2}}{L^2 \times \frac{1}{T}} = \frac{ML}{L^2T^2 \times \frac{1}{T}}$$

$$= \frac{M}{LT} = ML^{-1}T^{-1} \text{ and } \rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{L^3} = ML^{-3}$$

$\therefore$ Kinematic viscosity (ν) $= \dfrac{\mu}{\rho} = \dfrac{ML^{-1}T^{-1}}{ML^{-3}} = \mathbf{L^2T^{-1}}$ **(Ans.)**

(*iii*) *Force* = mass × acceleration

$$= M \times \frac{\text{length}}{\text{time}^2} = \frac{ML}{T^2} = \mathbf{MLT^{-2}} \textbf{ (Ans.)}$$

(*iv*) *Specific weight* $= \dfrac{\text{Weight}}{\text{Volume}} = \dfrac{\text{Force}}{\text{Volume}} = \dfrac{MLT^{-2}}{L^3} = \mathbf{ML^{-2}T^{-2}}$ **(Ans.)**

Table 7.1. Quantities used in Fluid Mechanics and Heat Transfer and their Dimensions

S.No.	Quantity	Dimensions: M-L-T System	Dimensions: F-L-T System
	(a) Fundamental Quantities		
1.	Mass, M	M	$FL^{-1}T^2$
2.	Length, L	L	L
3.	Time, T	T	T
	(b) Geometric Quantities		
4.	Area, A	L^2	L^2
5.	Volume, $\forall$	L^3	L^3
6.	Moment of inertia	L^4	L^4
	(c) Kinematic Quantities		
7.	Linear velocity, u, V, U	LT^{-1}	LT^{-1}
8.	Angular velocity, ω; rotational speed, N	T^{-1}	T^{-1}
9.	Acceleration, a	LT^{-2}	LT^{-2}
10.	Angular acceleration, α	T^{-2}	T^{-2}
11.	Discharge, Q	L^3T^{-1}	L^3T^{-1}
12.	Gravity, g	LT^{-2}	LT^{-2}
13.	Kinematic viscosity, ν	L^2T^{-1}	L^2T^{-1}
14.	Stream function, ψ, circulation, Γ	L^2T^{-1}	L^2T^{-1}
15.	Vorticity, Ω	T^{-1}	T^{-1}
	(d) Dynamic Quantities		
16.	Force, F	MLT^{-2}	F
17.	Density, ρ	ML^{-3}	$FL^{-4}T^2$
18.	Specific weight, w	$ML^{-2}T^{-2}$	FL^{-3}
19.	Dynamic viscosity, μ	$ML^{-1}T^{-1}$	$FL^{-2}T$
20.	Pressure, p; shear stress, τ	$ML^{-1}T^{-2}$	FL^{-2}
21.	Modulus of elasticity, E, K	$ML^{-1}T^{-2}$	FL^{-2}
22.	Momentum	MLT^{-1}	FT
23.	Angular momentum or moment of momentum	ML^2T^{-1}	FLT
24.	Work, W; energy, E	ML^2T^{-2}	FL
25.	Torque, T	ML^2T^{-2}	FL
26.	Power, P	ML^2T^{-3}	FLT^{-1}
	(e) Thermodynamic Quantities		
27.	Temperature	θ	θ
28.	Thermal conductivity	$MLT^{-3}\theta^{-1}$	$FT^{-1}\theta^{-1}$
29.	Enthalpy per unit mass	L^2T^{-2}	L^2T^{-2}

S.No.	Quantity	Dimensions	
		M-L-T System	F-L-T System
30.	Gas constant	$L^2T^{-2}\theta^{-1}$	$L^2T^{-2}\theta^{-1}$
31.	Entropy	$ML^2T^{-2}\theta^{-1}$	$FL\theta^{-1}$
32.	Internal energy per unit mass	L^2T^{-2}	L^2T^{-2}
33.	Heat transfer	ML^2T^{-3}	FLT^{-1}

7.3. DIMENSIONAL HOMOGENEITY

A physical equation is the relationship between two or more physical quantities. Any *correct equation* expressing a physical relationship between quantities *must be dimensionally homogeneous (according to Fourier's principle of dimensional homogeneity) and numerically equivalent. Dimensional homogeneity* states that *every term in an equation when reduced to fundamental dimensions must contain identical powers of each dimension.* A dimensionally homogeneous equation is applicable to all systems of units. In a dimensionally homogeneous equation, only quantities having the same dimensions can be added, subtracted or equated. Let us consider the equation:

$$p = wh$$

$$\text{Dimensions of L.H.S.} = ML^{-1}T^{-2}$$

$$\text{Dimensions of R.H.S.} = ML^{-2}T^{-2} \times L = ML^{-1}T^{-2}$$

$$\text{Dimensions of L.H.S.} = \text{Dimensions of R.H.S.}$$

$\therefore$ Equation $p = wh$ is dimensionally homogeneous; so it can be used in any system of units.

Applications of Dimensional Homogeneity:

The *principle of homogeneity* proves useful in the following ways:

1. It facilitates to determine the dimensions of a physical quantity.
2. It helps to check whether an equation of any physical phenomenon is dimensionally homogeneous or not.
3. It facilitates conversion of units from one system to another.
4. It provides a step towards dimensional analysis which is fruitfully employed to plan experiments and to present the results meaningfully.

Example 7.2. *Determine the dimensions of E in the dimensionally homogeneous Einstein's equation*

$$E = mc^2\left[\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} - 1\right]$$

where v is the velocity and m is the mass.

Solution. Since the expression is dimensionally homogeneous, the term

$$\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \text{ should be dimensionless}$$

i.e.,

$$[c] = [v] = \frac{L}{T}$$

$$[E] = m[c]^2 = M\left[\frac{L^2}{T^2}\right] = ML^2T^{-2}$$

$\therefore$

i.e. E has the dimensions of **energy. (Ans.)**

7.4. METHODS OF DIMENSIONAL ANALYSIS

With the help of dimensional analysis the equation of a physical phenomenon can be developed in terms of dimensionless groups or parameters and thus reducing the number of variables. *The methods of dimensional analysis are based on the Fourier's principle of homogeneity.* The methods of dimensional analysis are:

1. Rayleigh's method
2. Buckingham's π-method
3. Bridgman's method
4. Matrix-tensor method
5. By visual inspection of the variables involved
6. Rearrangement of differential equations.

Here only first two methods will be dealt with.

7.4.1. Rayleigh's Method

This method gives a special form of relationship among the dimensionless groups, and has the *inherent drawback* that it *does not provide any information regarding the number of dimensionless groups to be obtained as a result of dimensional analysis.* Due to this reason this method has become *obsolete and is not favoured for use.*

Rayleigh's method is used for determining the expression for a variable which depends upon *maximum three or four variables only.* In case the number of independent variables becomes more than four, then it is very difficult to find the expression for the dependent variable.

In this method a functional relationship of some variables is expressed in the form of an exponential equation which must be dimensionally homogeneous. Thus if X is a variable which depends on $X_1, X_2, X_3, ...X_n$; the functional equation can be written as:

$$X = f(X_1, X_2, X_3, ... X_n) \quad ...(7.1)$$

In the above equation X is a *dependent variable,* while $X_1, X_2, X_3, ...X_n$ are *independent variables. A dependent variable is the one about which information is required while independent variables are those which govern the variation of dependent variable.*

Eqn. (7.1) can also be written as:

$$X = C\left(X_1^a, X_2^b, X_3^c, ...X_n^n\right) \quad ...(7.2)$$

where, C is a constant and $a, b, c,...$ are the arbitrary powers. The values of $a, b, c,... n$ are obtained by comparing the powers of the fundamental dimensions on both sides. Thus the expression is obtained for dependent variable.

Example 7.3. *Find an expression for the drag force on smooth sphere of diameter D, moving with a uniform velocity V in a fluid density ρ and dynamic viscosity μ.* (PT)

Solution. The force drag F is a function of

(*i*) Diameter D, (*ii*) Velocity V,

(*iii*) Fluid density ρ, and (*iv*) Dynamic viscosity μ.

Mathematically,

$$F = f(D, V, \rho, \mu) \quad \text{or} \quad F = C\,(D^a \cdot V^b \cdot \rho^c \cdot \mu^d)$$

where, C is a non-dimensional constant.