Calculus

Practise Exam 4

Radboud University *2019-10*

Exercise 1

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1, \quad \text{then } f(x) \sim g(x)$$

Exercise 2

Note that some tasks have 2 solutions. You are free to shoose the one you understand better.

2.1

With limits tending to 0, divide each term by the largest polynomial in the equation.

2.1.1

$$\begin{aligned} \lim_{n \to \infty} \frac{n^2 + 3n - 1}{8n^2 + 3n - 1/n^3} &= \lim_{n \to \infty} \frac{\frac{n^2}{n^2} + \frac{3n}{n^2} - \frac{1}{n^2}}{\frac{8n^2}{n^2} + \frac{3n}{n^2} - \frac{1/n^3}{n^2}} \\ &= \lim_{n \to \infty} \frac{1 + \frac{3}{n} - \frac{1}{n^2}}{8 + \frac{3}{n} - \frac{1}{n^5}} \\ &= \frac{1 + 0 - 0}{8 + 0 - 0} \\ &= \frac{1}{8} \end{aligned}$$

Alternatively:

$$\begin{gathered} \lim_{n\to\infty} \tfrac{n^2+3n-1}{8n^2+3n-\frac{1}{n^3}} = \lim_{n\to\infty} \tfrac{n^2}{8n^2+3n-\frac{1}{n^3}} + \lim_{n\to\infty} \tfrac{3n-1}{8n^2+3n-\frac{1}{n^3}} = \\ = \lim_{n\to\infty} \tfrac{\frac{8}{8}n^2+\frac{3}{8}n-\frac{1}{8}\frac{1}{n^3}-\frac{3}{8}n+\frac{1}{8}\frac{1}{n^3}}{8n^2+3n-\frac{1}{n^3}} + \lim_{n\to\infty} \tfrac{3n-1}{8n^2+3n-\frac{1}{n^3}} = \\ = \tfrac{1}{8} \lim_{n\to\infty} \tfrac{8n^2+3n-\frac{1}{n^3}}{8n^2+3n-\frac{1}{n^3}} + \lim_{n\to\infty} \tfrac{-\frac{3}{8}n+\frac{1}{8}\frac{1}{n^3}}{8n^2+3n-\frac{1}{n^3}} + \lim_{n\to\infty} \tfrac{3n-1}{8n^2+3n-\frac{1}{n^3}} = \\ \tfrac{1}{8} \lim_{n\to\infty} 1 + 0 + 0 = \tfrac{1}{8} \end{gathered} \tag{1}$$

2.1.2

$$\begin{aligned} \lim_{n \to \infty} \frac{n^2 + 4n}{3n^3 + n^2} &= \lim_{n \to \infty} \frac{\frac{n^2}{n^3} + \frac{4n}{n^3}}{\frac{3n^3}{n^3} + \frac{n^2}{n^3}} \\ &= \lim_{n \to \infty} \frac{\frac{1}{n} + \frac{4}{n^2}}{3 + \frac{1}{n}} \\ &= \frac{0 + 0}{3 + 0} \\ &= 0 \end{aligned}$$

Alternatively:

$$\begin{aligned}\lim_{n\to\infty}\tfrac{n^2+4n}{3n^3+n^2} = \lim_{n\to\infty}\tfrac{n(n+4)}{n^2(3n+1)} = \lim_{n\to\infty}\tfrac{n+4}{n(3n+1)} = \\ = \tfrac{n}{n(3n+1)} + \tfrac{4}{n(3n+1)} = \tfrac{1}{3n+1} + \tfrac{4}{n(3n+1)} = 0+0=0\end{aligned} \tag{2}$$

2.1.3

$$\begin{aligned}\lim_{n\to\infty}\frac{n^6+9n^2+n}{7n^6-100n^4} &= \lim_{n\to\infty}\frac{\frac{n^6}{n^6}+\frac{9n^2}{n^6}+\frac{n}{n^6}}{\frac{7n^6}{n^6}-\frac{100n^4}{n^6}} \\ &= \lim_{n\to\infty}\frac{1+\frac{9}{n^4}+n}{7-\frac{100}{n^2}} \\ &= \frac{1+0+0}{7-0} \\ &= \frac{1}{7}\end{aligned}$$

Alternatively:

$$\lim_{n\to\infty}\frac{n^6+9n^2+n}{7n^6-100n^4} = \lim_{n\to\infty}\frac{n^6}{7n^6} = \frac{1}{7} \tag{3}$$

Because $\quad n^6+9n^2+n \sim n^6$ for $n\to\infty$
and $\quad 7n^6-100n^4 \sim 7n^6$ for $n\to\infty$

2.2

With limits tending to 0, divide each term with the lowest polynomial possible in the equation.

2.2.1

$$\begin{aligned}\lim_{x\to 0}\frac{x^5+3x^3}{3x^5+x^3} &= \lim_{x\to 0}\frac{\frac{x^5}{x^3}+\frac{3x^3}{x^3}}{\frac{3x^5}{x^3}+\frac{x^3}{x^3}} \\ &= \lim_{x\to 0}\frac{x^2+3}{3x^2+1} \\ &= \frac{0+3}{3\cdot 0+1} \\ &= 3\end{aligned}$$

Alternatively:

$$\lim_{x\to 0}\frac{x^5+3x^3}{3x^5+x^3} = \lim_{x\to 0}\frac{x^3(x^2+3)}{x^3(3x^2+1)} = \lim_{x\to 0}\frac{x^2+3}{3x^2+1} = \frac{0^2+3}{3\cdot 0^2+1} = \frac{3}{1} = 3 \tag{4}$$

2.2.2

$$\begin{aligned}\lim_{x\to 0}\frac{x\left(\mathrm{e}^x-1\right)}{8x^2+4x^4} &= \lim_{x\to 0}\frac{x\cdot x}{8x^2+4x^4} \text{ using the fact that } \mathrm{e}^x-1\sim x\\ &= \lim_{x\to 0}\frac{\frac{x^2}{x^2}}{\frac{8x^2}{x^2}+\frac{4x^4}{x^2}}\\ &= \lim_{x\to 0}\frac{1}{8+4x^2}\\ &= \frac{1}{8}\end{aligned}$$

Alternatively:

$$\begin{aligned}&\lim_{x\to 0}\tfrac{x(e^x-1)}{8x^2+4x^4} = \lim_{x\to 0}\tfrac{x(e^x-1)}{x^2(8+4x^2)} = \lim_{x\to 0}\tfrac{e^x-1}{x(8+4x^2)} =\\ &= \lim_{x\to 0}\tfrac{e^x-1}{x} * \tfrac{1}{(8+4x^2)} = \lim_{x\to 0}\tfrac{e^x-1}{x} * \lim_{x\to 0}\tfrac{1}{(8+4x^2)} = 1 * \tfrac{1}{8+4*0^2} = \tfrac{1}{8}\end{aligned} \tag{5}$$

2.2.3

$$\begin{aligned}\lim_{x\to 0}\frac{x\left(\mathrm{e}^x-1\right)}{3x\cdot\sin(x)} &= \lim_{x\to 0}\frac{x^2}{3x^2} \text{ using the fact that } \mathrm{e}^x-1\sim x\\ &= \frac{\frac{x^2}{x^2}}{\frac{3x^2}{x^2}}\\ &= \frac{1}{3}\end{aligned}$$

Alternatively:

$$\begin{aligned}&\lim_{x\to 0}\tfrac{x(e^x-1)}{3x(x)} = \lim_{x\to 0}\tfrac{x(e^x-1)}{3x(x)} = \tfrac{1}{3}\lim_{x\to 0}\tfrac{e^x-1}{x}\cdot\tfrac{x}{sin(x)} =\\ &= \tfrac{1}{3}\lim_{x\to 0}\tfrac{e^x-1}{x}\cdot\lim_{x\to 0}\tfrac{x}{sin(x)} = \tfrac{1}{3}\cdot 1\cdot 1 = \tfrac{1}{3}\end{aligned} \tag{6}$$

The second to last step is made based on the hints.

Exercise 3

3.1

Here the sum rule is used

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{8}x^8+\frac{1}{4}x^4+6x^2+x\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{8}x^8\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4}x^4\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(6x^2\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(x\right)\\ &= x^7+x^3+12x+1\end{aligned}$$

3.1.1

Here the product rule is used

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(6x^3\mathrm{e}^x\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(6x^3\right)\cdot\mathrm{e}^x+6x^3\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x\right)\\ &= 18x^2\mathrm{e}^x+6x^3\mathrm{e}^x\end{aligned}$$

3.1.2

Here the product rule and chain rule are used

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 \sin(\mathrm{e}^x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)\sin(\mathrm{e}^x) + x^3 \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(\mathrm{e}^x)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)\sin(\mathrm{e}^x) + x^3 \frac{\mathrm{d}\sin(\mathrm{e}^x)}{\mathrm{d}\mathrm{e}^x}\cdot \mathrm{d}\mathrm{e}^x \\
&= 3x^2 \sin(\mathrm{e}^x) + x^3 \cos(\mathrm{e}^x)\cdot \mathrm{e}^x
\end{aligned}$$

3.1.3

Here the quotient rule and chain rule is used

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin(x)}{(\cos(x))^2}\right) &= \frac{\cos^2(x)\cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(x)\right) - \sin(x)\cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^2(x)\right)}{(\cos(x))^2} \\
&= \frac{\cos^2(x)\cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(x)\right) - \sin(x)\cdot \frac{\mathrm{d}\cos^2(x)}{\mathrm{d}\cos(x)}\cdot \mathrm{d}\cos(x)}{(\cos(x))^2} \\
&= \frac{(\cos(x))^2\cdot\cos(x) - (\sin(x))(2\cos(x)\sin(x))}{(\cos(x))^4} \\
&= \frac{\cos^3(x) - 2\cos(x)\sin^2(x)}{\cos^4(x)}
\end{aligned}$$

3.2

Here we use the fact that $\mathrm{e}^x - 1 \sim x$, and in the same way $\mathrm{e}^{\Delta x} - 1 \sim \Delta x$

$$\begin{aligned}
\lim_{\Delta x\to 0}\frac{\mathrm{e}^{x+\Delta x+1} - \mathrm{e}^{x+1}}{\Delta x} &= \lim_{\Delta x\to 0}\frac{\mathrm{e}^{x+1}\mathrm{e}^{\Delta x} - \mathrm{e}^{x+1}}{\Delta x} \\
&= \mathrm{e}^{x+1}\cdot \lim_{\Delta x\to 0}\frac{\mathrm{e}^{\Delta x} - 1}{\Delta x} \\
&= \mathrm{e}^{x+1}\cdot \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x} \\
&= \mathrm{e}^{x+1}\cdot 1 \\
&= \mathrm{e}^{x+1}
\end{aligned}$$

3.3

Calculate the first and second derivative of the function. Find the roots of the first derivative, and insert those solutions into the second derivative. If the answer is larger than 0, then it is a local minimum, if it is smaller than 0, then it is a local maximum, and if it is equal to 0, then it is a saddle point.

$$\begin{aligned} f(x) &= x^2 \mathrm{e}^x \\ \frac{\mathrm{d}}{\mathrm{d}x} f(x) &= 2x\mathrm{e}^x + x^2\mathrm{e}^x \\ &= \mathrm{e}^x(x^2 + 2x) \end{aligned}$$

$$\begin{aligned} \mathrm{e}^x(x^2 + 2x) &= 0 \\ \mathrm{e}^x = 0 \vee x^2 + 2x &= 0 \\ \text{impossible} \vee x(x+2) &= 0 \\ x = 0 \vee x &= -2 \end{aligned}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = (2x+2)\mathrm{e}^x + (x^2 + 2x)\mathrm{e}^x$$

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(0) &= (2 \cdot 0 + 2)\mathrm{e}^0 + (0^2 + 2 \cdot 0)\mathrm{e}^0 \\ &= 2\mathrm{e}^0 + 0 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(-2) &= (2 \cdot -2 + 2)\mathrm{e}^{-2} + ((-2)^2 + 2 \cdot -2)\mathrm{e}^{-2} \\ &= -2 \cdot \mathrm{e}^{-2} + 0 \\ &= \frac{-2}{\mathrm{e}^2} \end{aligned}$$

$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(0) > 0$, thus this is a local minimum

$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(-2) < 0$, thus this is a local maximum

Exercise 4

4.1

4.1.1

Here the sum rule is used

$$\begin{aligned} \int_0^3 3x^3 + 2x^2 + 2x\mathrm{d}x &= \frac{3}{4}x^4 + \frac{2}{3}x^3 + x^2 \Big|_0^3 \\ &= \left(\frac{3}{4} \cdot (3)^4 + \frac{2}{3} \cdot (3)^3 + (3)^2\right) - \left(\frac{3}{4} \cdot (0)^4 + \frac{2}{3} \cdot (0)^3 + (0)^2\right) \\ &= \frac{243}{4} + \frac{54}{3} + 9 \end{aligned}$$

4.1.2

Here partial integration is used

$$
\begin{aligned}
\int_{-2}^{1} x\mathrm{e}^x \mathrm{d}x &= x\mathrm{e}^x - \int_{-2}^{1} \mathrm{e}^x \mathrm{d}x \\
&= x\mathrm{e}^x - \mathrm{e}^x \Big|_{-2}^{1} \\
&= \left(1 \cdot \mathrm{e}^1 - \mathrm{e}^1\right) - \left(-2 \cdot \mathrm{e}^{-2} - \mathrm{e}^{-2}\right) \\
&= 0 + 2 \cdot \mathrm{e}^{-2} + \mathrm{e}^{-2} \\
&= \frac{3}{\mathrm{e}^2}
\end{aligned}
$$

4.1.3

Here the sum rule is used, as well as partial integration on $x\mathrm{e}^x$ (the steps of this can be seen in the previous equation)

$$
\begin{aligned}
\int_{-1}^{1} x\mathrm{e}^x + x^3 \mathrm{d}x &= \int_{-1}^{1} x\mathrm{e}^x \mathrm{d}x + \int_{-1}^{1} x^3 \mathrm{d}x \\
&= x\mathrm{e}^x - \mathrm{e}^x + \frac{1}{4}x^4 \Big|_{-1}^{1} \\
&= \left(1 \cdot \mathrm{e}^1 - \mathrm{e}^1 + \frac{1}{4} \cdot 1\right) - \left(-1 \cdot \mathrm{e}^{-1} - \mathrm{e}^{-1} + \frac{1}{4} \cdot -1\right) \\
&= \frac{1}{2} + \frac{2}{\mathrm{e}}
\end{aligned}
$$

4.2

$$\int_0^b x^2 \mathrm{d}x = \lim_{N\to\infty} \sum_{n=0}^{N-1} x_n^2 \cdot \Delta x$$

$$\Delta x = \frac{b-0}{N} = \frac{b}{N}$$

$$\begin{aligned} x_n &= n \cdot \Delta x \\ &= \frac{n \cdot b}{N} \end{aligned}$$

$$\begin{aligned} \lim_{N\to\infty} \sum_{n=0}^{N-1} x_n^2 \cdot \Delta x &= \lim_{N\to\infty} \sum_{n=0}^{N-1} \left(\frac{n \cdot b}{N}\right)^2 \cdot \frac{b}{N} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N-1} \frac{n^2 \cdot b^3}{N^3} \\ &= \lim_{N\to\infty} \frac{b^3}{N^3} \sum_{n=0}^{N-1} n^2 \\ &= \lim_{N\to\infty} \frac{b^3}{N^3} \cdot \frac{(N-1)N(2N-1)}{6} \\ &= \lim_{N\to\infty} b^3 \cdot \frac{2N^3 - 3N^2 + N}{6N^3} \\ &= \lim_{N\to\infty} b^3 \cdot \frac{2 - {}^3/_N + {}^1/_{N^2}}{6} \\ &= b^3 \cdot \frac{2-0+0}{6} \\ &= \frac{1}{3} b^3 \end{aligned}$$

Exercise 5

If f(x) is asymptotic to 0, then $\lim_{x\to 0} \frac{0}{f(x)} = 0 \cdot \lim_{x\to 0} \frac{1}{f(x)}$ needs to be equal to 1, by definition of asymptotic relations. This cannot happen as everything multiplied by 0 is 0.