WP 1.1 Mathematical Foundations

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1 Contents

This works package details the mathematical foundations for the thesis work. This will be split up into several sections:

- Coordinate systems and transformations

- Three-body orbital mechanics (summary)

- Definitions for the numerical methods

- Definitions for the Kalman filter methods

- Definitions for the neural networks

2 Coordinate systems and transformations

This section will detail the necessary coordinate systems and corresponding transformations.

Often, an orbit is expressed in terms of the Keplerian orbital elements:

- Eccentricity e

- Semi-major axis a

- Inclination i

- Right Ascension of the Ascending Node Ω

- Argument of Periapsis ω

- True anomaly at Epoch θ

These elements provide a full specification of the orbit in 3D, but they are not that useful for our application. Therefore, some additional coordinate frames are defined, along with transformations. The most important frames are the heliocentric and the geocentric reference frames. Additionally, the camera reference frame of the spacecraft needs to be specified.

2.1 Heliocentric reference frame

The heliocentric, or *non-rotating heliocentric ecliptic* reference frame is a non-rotating, quasi-inertial reference frame. As XY-plane, the plane of the ecliptic is used. Therefore, the Z+ direction is perpendicular to the plane of the ecliptic, pointing North. The X+ direction points towards the vernal equinox (first point of Aries). The Y+ direction completes the right-handed coordinate system. The coordinate system is centered on the center of mass of the Sun.

2.2 Geocentric reference frame

The geocentric reference frame that will be used is the *non-rotating geocentric ecliptic* reference frame. The geocentric reference frame is defined similar to the heliocentric frame, but centered on the center of mass of the Earth. Note that all axes are parallel: celestial north and the first point of Aries are located at an infinite distance from the origin, and therefore the directions from the Sun and from Earth are parallel.

2.3 Spacecraft camera reference frame

To specify the observations of the target in the spacecraft's camera, the spacecraft camera reference frame is defined. In order to specify this coordinate frame, first the spacecraft (quasi-)inertial reference frame is defined. The spacecraft inertial frame is defined similar to the heliocentric and geocentric reference frames, but with the coordinate system centered on the spacecraft center of mass. From this, a simplifying assumption is made: it is assumed the spacecraft focal plane and optical axis coincide with its center of mass. As the dimensions of the spacecraft are negligible compared to the distance to the targets, this assumption is justified. From here, the spacecraft camera reference frame can be defined as the reference frame centered on the intersection of the optical axis of the camera and the camera focal plane, with axis L+ pointing in the direction of the optical axis of the camera ("out" from the focal plane), M+ pointing to the left and N+ completing the right-handed system by pointing up, both as seen from the focal plane. This is seen in Figure 1.
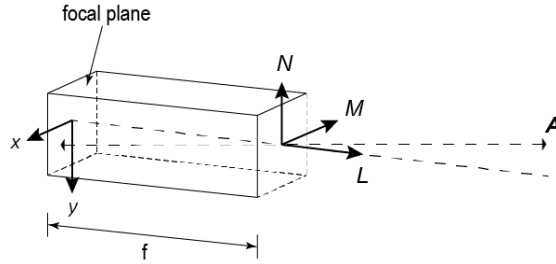

Figure 1: Camera reference frame

TRANSFORMATIONS

3 Alternative coordinate systems

A problem that will be encountered later in the process is the "conversion" from a preliminary orbit determination to an impact threat assessment. For this, it might be necessary to tell the system where Earth is. (alternatively, other metrics such as MOID could be looked at, leading to a system where the target is analysed in more detail from Earth). Instead of giving the system a measure of Earth's location, it might be convenient to define a coordinate system in which both Earth and the Sun (as it is the main gravitational attractor) are fixed. We shall define the Sun-Earth ecliptic reference frame as the right-handed frame with the origin at the center of the Sun, the Z+ axis pointing North perpendicular to the ecliptic, the X+ axis pointing through the center of Earth, and the Y+ axis completing the right-handed system.

The advantage of this system is obvious: if we assume Earth's orbit to be circular, the position of Earth will always be $[1, 0, 0]^T$ AU, and therefore the problem simplifies to determining the hazard to that point. However, as the reference frame is no longer non-rotating, the dynamics of the motion vastly complicate.

4 Transformations

Most coordinate transformations that are required are quite straightfoward. The transformation between heliocentric frame h, geocentric frame e and spacecraft centered frame s can be performed through vector addition. Example for vector $\vec{r}$:

$$\vec{r}^s = \vec{r}^h - \vec{r_s}^h \tag{1}$$

With $\vec{r_s}^h$ denoting the position of the spacecraft in the heliocentric frame. In words: the position of the target in the spacecraft-centered frame is the difference of the position of the target and the position of the spacecraft in any of the other reference frames.

Thus, when the position of Sun, Earth and spacecraft are known in each of the respective frames, the coordinate transformation is straightforward. The other coordinate frames require a more elaborate transformation to achieve. The Sun-Earth ecliptic frame r (for rotating) can be achieved by a rotation of the heliocentric frame around the Z-axis. If we define the true anomaly of the Earth θ_e as 0 at the vernal equinox, we find that:

$$\vec{r}^r = \mathbf{R_3}(\theta_e)\vec{r}^h \tag{2}$$

With $\mathbf{R_3}$ denoting a rotation around the third axis. Similarly, the camera coordinate frame can be achieved by a rotation matrix $\mathbf{C}$ which can be given in terms of twist angle ϕ, right ascension α and declination δ:

$$\mathbf{C} = \mathbf{R_3}(\phi)\mathbf{R_1}(90° - \delta)\mathbf{R_3}(\alpha + 90°) \tag{3}$$

or:

$$\mathbf{R_3}(\phi + 90°)\mathbf{R_2}(90° - \delta)\mathbf{R_3}(\alpha) \tag{4}$$

Then:
$$\vec{r}^c = \mathbf{C}(\alpha, \delta, \phi)\vec{r}^s \tag{5}$$

The rotation matrices are given by:

$$\mathbf{R_1}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{6}$$

$$\mathbf{R_2}(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{7}$$

$$\mathbf{R_3}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

5 Two- and Three-body Orbital Mechanics

The equations of motion need to be known in order to accurately model the asteroids and to e.g. apply the kalman filter and parts of the numerical methods. The motion of the spacecraft around the Sun is governed soleley by gravity. By assuming the Sun to be at rest (a fair assumption, as $m_{S/C} \ll m_\odot$), only gravitational force is to be considered:

$$\frac{d^2\vec{r}}{dt^2} = \frac{\mu_\odot}{||r||^3}\vec{r} \tag{9}$$

Consequently, velocity and position are obtained by integration of the acceleration. Or, in equations, the position of the spacecraft at time $t = T$ can be determined from its position and velocity at time $t = 0$:

$$\vec{r}(t = T) = \vec{r}(t = 0) + \int_0^T \frac{d\vec{r}}{dt}dt \tag{10}$$

$$\frac{d\vec{r}}{dt}(t = T) = \frac{d\vec{r}}{dt}(t = 0) + \int_0^T \frac{d^2\vec{r}}{dt^2}dt \tag{11}$$

$$\frac{d^2\vec{r}}{dt^2}(t = T) = \frac{\mu_\odot}{||r(t = T)||^3}\vec{r}(t = T) \tag{12}$$

If a discrete-time solution is required, the following result is obtained (with velocity v and acceleration a:

$$\vec{r}(T + \Delta t) = \vec{r}(T) + \vec{v}(T)\Delta t \tag{13}$$
$$\vec{v}(T + \Delta t) = \vec{v}(T) + \vec{a}(T)\Delta t \tag{14}$$
$$\vec{a}(T + \Delta t) = \frac{\mu_\odot}{||r(T + \Delta t)||^3}\vec{r}(T + \Delta t) \tag{15}$$

The resulting equations of motion for two-body problem are simple:

$$r(\theta) = \frac{a(1 - e^2)}{1 + e\cos(\theta)} \tag{16}$$

As the motion is in a single plane, it is only necessary to know the distance to the barycenter (which can be assumed to coincide with the Sun), and the

true anomaly θ. The orbital elements Ω, i and ω can be seen as Euler angles rotating the coordinate system. If we define the *orbital reference frame o* with X+ towards the periapsis, Z+ perpendicular to the plane of the orbit and Y+ completing the coordinate system right-handed, we can define the following transformations:

$$
\begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_h \\ y_h \\ z_h \end{bmatrix}
\tag{17}
$$

Finding the inverse is straightforward: firstly, one should note all matrices are diagonal and thus easily invertible. However, perhaps more intuitively, the rotation sequence can be "reversed" by simply reversing the order of the matrices and inverting the sign of the angles (essentially "rotating" the system "back" into its original orientation):

$$
\begin{bmatrix} x_h \\ y_h \\ z_h \end{bmatrix} = \begin{bmatrix} \cos-\Omega & \sin-\Omega & 0 \\ -\sin-\Omega & \cos-\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos-i & \sin-i \\ 0 & -\sin-i & \cos-i \end{bmatrix} \begin{bmatrix} \cos-\omega & \sin-\omega & 0 \\ -\sin-\omega & \cos-\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}
\tag{18}
$$

Lastly, conversion from r, θ to x_o, y_o, z_o is simple (note that z_o is always zero in the two-body solution as the orbit is planar):

$$
\begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} r
\tag{19}
$$

6 Numerical Methods

As mentioned in the literature study, the numerical methods based on the method of very short arcs require knowing the parameter space of the targets. As determined by the original authors, there are some initial requirements:

- If the object is within the SoI of Earth, it can't be orbitting Earth.

- The object should orbit the sun

- The object should have a reasonable magnitude

These can be further defined when examining whether an asteroid is dangerous:

- The object should be a NEA, i.e. $q \leq 1.017$AU and $Q \geq 0.983$AU

- The object should be large enough to cause damage, i.e. $D \geq 30$m, corresponding to $H \leq 25$

- The object should be small enough that there's a chance it hasn't been found, i.e. $H \geq 15$

This leaves the question of how to efficiently estimate the parameter space. Essentially, we can accomplish this by getting an estimate of $\vec{r}$ and $\dot{\vec{r}}$. Imagine

extending an imaginary line from the camera in the direction of the target during a measurement. It is known the target is somewhere along this line. Thus, this constrains the possibilities for $\vec{r}$ to a small subspace. When taking a second measurement, we can fit a straight line to the trajectory (which should be accurate for a short timestep; for higher timesteps the method of short arcs is unneccessary). This allows computation of $\dot{\vec{r}}$. With both of these vectors known, the orbit is essentially determined. If more than 2 measurements are available, the straight line can be fit more accurately using e.g. a least-squares estimation.

The process of determining the admissible values of $\vec{r}$ is as follows:

1. Use the apparent magnitude of the object to determine the range of $||r||$

2. Perform coordinate transforms to get $\vec{r}$ in the heliocentric reference frame

3. Repeat step 1 for the second measurement, using the constraints on orbital energy to determine $\dot{\vec{r}}$

4. Calculate risk to Earth using monte-carlo methods.

The first step assume phase integral is:

$$\phi(\alpha) = |\cos\theta| \tag{20}$$

With absolute magnitude H and apparent magnitude m, we can find for body B orbiting Sun S as seen from observer O, d in AU:

$$H = m - 5\log_{10}(d_{BS}d_{BO}) + 2.5\log_{10}\phi(\alpha) \tag{21}$$

Initially, I think it's best to just do this iteratively. There will be a large range of initial $\vec{r}$ values. However, the second step will have a much smaller solution space because of the other restrictions. Then, using e.g. scikit-learn.model_selection.RandomSearchCV (grid search is too slow) the worst-case approach near Earth can be determined. Care should be taken to allow the system to run feasibly on a limited processor inside a spacecraft.

IMPORTANT: As the spacecraft is also orbitting the sun, all calculations should be performed in the heliocentric reference frame.

7 Kalman Filters

8 Artificial Neural Networks