Simulation Costs for Quantum Dynamical Semigroups Generated by Lindbladians

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Abstract

Given a quantum system, the simulation problem involves determining the cost at which the system can be approximated on a quantum computer. This project investigates the simulation costs of a quantum dynamical semigroup of channels generated by Lindbladians — operators that solve the master equation for modeling Markovian open quantum systems. In particular, instead of relying on universal gates, this project explores the cost of simulation with customized resource sets based on implementability and availability of the gates on the given quantum computing hardware.

We provide a recipe for estimating both the upper and lower bounds of simulation costs, based on the quantum gates permitted within the resource set. Lower estimates are usually challenging to obtain. In our framework, they are acquired thanks to the Lipschitz complexity of random quantum circuits, defined as a seminorm on the semigroup of channels, introduced in [ACJW23]. On the other hand, the upper bound follows from standard tools such as the Suzuki-Trotter formula [Suz91, CST+21] or by determining the specific unitaries used in particular examples. This report focuses on the exploration of two models — the LCU model and the noisy model — and offers examples where these bounds can be explicitly calculated. As the generators of the GNS symmetric semigroups, understanding the above two models provides hints as to the simulation costs for any GNS symmetric semigroups.

Contents

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1 Introduction

1.1 Historical Motivation

Hamiltonian simulation is first proposed by Richard Feynman in [Fey18]. In his keynote speech, he explored a range of topics as of simulating physical problems on different computational devices, classical and quantum, both deterministically and probabilistically. In particular, Feynman was interested in a general quantum mechanical simulator, that is capable of simulating any Hamiltonian on the given device. He was optimistic about the potential of this approach and remarked:

> "The question is, if we wrote a Hamiltonian which involved only these operators [Pauli matrices], locally coupled to corresponding operators on the other space-time points, could we imitate every quantum mechanical system which is discrete and has a finite number of degrees of freedom? I know, almost certainly, that we could do that for any quantum mechanical system which involves Bose particles. I'm not sure whether Fermi particles could be described by such a system. So I leave that open."

Since Feynman's initial speech, the field of quantum simulation has expanded significantly. Notably, Lloyd's work on universal quantum simulator provides a formal framework for quantum simulation [Llo96]. The development of algorithms like the Trotter-Suzuki decomposition further allowed for the approximation of exponential operators, which are crucial for simulating time evolution under various Hamiltonians. These methods have opened doors to understanding complex quantum phenomena such as superconductivity and quantum phase transitions.

Moreover, the realization of quantum computing technologies, from trapped ions to superconducting qubits, has transitioned quantum simulation from theoretical construct to experimental feasibility. The implementation of quantum error correction and fault-tolerant quantum computing are essential for the accuracy and scalability of these simulations. As we move into an era of noisy intermediate-scale quantum (NISQ) devices, researchers are now focusing on hybrid quantum-classical algorithms, like the variational quantum eigensolver (VQE) and quantum approximate optimization algorithm (QAOA), to perform simulations that are resilient to errors.

1.2 Hamiltonian Simulation

Hamiltonian simulation is key for probing complex quantum systems that overwhelm classical computers. To realize it, quantum simulators reproduce these systems, crucial for studying phenomena like superposition and entanglement, on quantum computation devices. They are especially valuable in physics research in low temperatures and for many-particle systems. In addition, these simulations have facilitated breakthroughs such as the discovery of time crystals and quantum spin liquids. By simulating quantum behaviors of many particles in controlled settings, they offer insights into material characteristics and the principles of quantum mechanics beyond traditional experimental mise-en-scène.

From the perspective of computational complexity, Hamiltonian simulation is the quantum analogue of constraint satisfaction problem (CSP) in classical computation theory, where the Hamiltonian's energy term correspond to the number of constraints satisfied and the optimal solution, i.e. the most possible constraints satisfied, corresponds to Hamiltonian's ground state. 3-SAT is a well-studied CSP of which can be encoded into local

Hamiltonian where the ground energy of the Hamiltonian is zero if and only if the 3-SAT formula is satisfiable [GHLS15]. By selecting specific parameters, it is known that accurate estimation of a Hamiltonian's ground state energy is possible if the Hamiltonian can be simulated. Consequently, all findings regarding the difficulty of determining ground state energy are also applicable to simulation tasks.

A special case of Hamiltonian simulation is to simulate k-local Hamiltonian. It means that the Hamiltonian can be written in terms of summation of local Hamiltonian acting on no more than k-qubits. This is important in the theoritical computer science perspective as the k-local Hamiltonian problem is known to be a complete problem in the complexity class **QMA**, standing for Quantum Merline Arthur. In relation to other complexity classes,

$$\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{MA} \subseteq \mathbf{QCMA} \subseteq \mathbf{QMA} \subseteq \mathbf{PP}.$$

QCMA falls within **QMA** because the verifier can require the prover to provide a classical proof by immediately measuring any proofs upon receipt. The inclusion of **QMA** within **PP** is established in [MW05].

The complexity class **QMA** is defined in terms of circuit depth, indicating the quantum resources required to complete a given task. Similarly, Hamiltonian simulation investigates the number of quantum gates needed to simulate a given Hamiltonian. Specifically, these problems examine the increasing difficulty of achieving such tasks as more qubits are simulated. Therefore, the escalation in task difficulty corresponds to the growth in circuit depth required for the simulation.

1.3 Quantum Computing Devices

Quantum computing hardware has seen significant advancements over the past decade, with several methodologies emerging, each offering unique advantages. Here are a few prominent models that have been instrumental in building the first quantum computers:

- Photonic quantum computers exploit the properties of photons such as superposition and entanglement, employing an array of optical devices to control light paths and perform quantum computations, offering advantages in quantum communication and specialized algorithms.

- Trapped ion systems, known for their high fidelity and long coherence times, use electromagnetically confined charged atoms with laser-induced quantum state manipulations, providing versatility across diverse quantum algorithms.

- Neutral atom quantum computers leverage optical tweezers to trap neutral atoms, encoding qubits in their electronic states and using laser-induced interactions for scalable array-based quantum operations.

- Superconducting qubits, utilizing low-temperature circuits with Josephson junctions, facilitate rapid gate operations and integrate easily with classical technologies, leading the field in developing scalable quantum systems.

- Additionally, emerging technologies such as quantum dots and topological qubits are exploring new frontiers in quantum stability and error resistance, aiming to enhance computational capabilities and reduce operational errors in quantum computing environments.

For any quantum computing device, computation is realized by applying quantum gates, which are represented as unitary matrices regardless of their physical implementation. Consequently, the depth of circuits correlates directly with the number of physical gates required to complete the given task.

It is worth noting that different devices have their own advantages and disadvantages when implementing quantum gates. For example, trapped ion devices can perform single-qubit gates with high precision, while their performance for two-qubit gates is suboptimal. Consequently, no single type of physical device has a definitive advantage over all others, making the study of quantum hardware and their respective strengths an ongoing area of research.

1.4 Simulation Cost Approach to Semigroups

The inherent nature of quantum systems is characterized by noise. As a result, no quantum system is entirely immune to disturbances. In practical terms, physicists often focus on calibrating these systems, which requires a thorough understanding of the system's behavior under various conditions. This understanding is typically enhanced by fitting theoretical models to parameters derived from observational data. In this process, experimentalists hypothesize about potential models of error — using Hamiltonian or Lindbladian models — based on preliminary observations.

To evaluate the accuracy of these hypotheses, simulations are necessary. These simulations help in visualizing how the theoretical model behaves under different conditions, as of computing what is actually going on for the proposed guesses. Given the complexity of quantum systems, classical computers often fall short, necessitating the use of quantum devices.

Quantum channels are essential for modeling how information is inevitably lost to the surrounding environment. This loss of information requires the consideration of both the environment and the primary system of interest. The mathematical representation of this interaction is given by:

$$\Phi(\rho) = \mathrm{tr}_E(V\rho V^*)$$

where $V : \mathbb{B}(A) \to \mathbb{B}(EA)$ represents a partial isometry, encapsulating the interaction between the system and its environment.

A quantum dynamical semigroup describes the evolution of quantum states over time through a collection of channels expressed as $(e^{-tL})_{t\geq 0}$, where L is a Lindbladian. This representation provides a framework for analyzing the continuous-time behavior of quantum systems, crucial for understanding and controlling quantum noise.

The project operates under the assumption that classical resources are inexpensive. Techniques such as qDRIFT demonstrate that a classical framework can reduce circuit depth, leading us to adopt a convexified version of simulation cost. We treat classical averaging as a freely available resource and define the effective channel complexity $l_\delta^{\mathcal{U}}(T_t)$ as follows:

$$l_\delta^{\mathcal{U}}(T_t) = \inf\left\{m : \exists \mu(\mathcal{U}^m) = 1 \text{ and } \|T_t - \mathbb{E}_\mu \,\mathrm{ad}_{u_1} \dots \mathrm{ad}_{u_m}\|_\diamond\right\}$$

where the resource set S determines the allowable unitary gates $\mathcal{U}$.

Calculating $l_\delta^{\mathcal{U}}$ is highly challenging even for simple semigroup models. Our goal is to approximate this cost by establishing both upper and lower bounds. Typically, deriving an upper estimate is more straightforward, while providing a rigorous lower estimate is significantly more challenging. For this project, the lower bound is derived from a well-defined channel complexity concept outlined in [ACJW23]. Upper estimates are obtained

using either the Suzuki-Trotter formula, which provides an estimate albeit with certain limitations on error degrees, or by explicitly writing out the gates, allowing for zero-error simulations subject to scalability compression.

Our findings conclude that adjusting resources and unitaries for noise models is crucial in selecting the appropriate gates and optimizing the simulation algorithm, thereby enhancing benchmarking efforts.

1.5 Overview of the Paper

In this paper, we consider two primary models with four examples. We consider the LCU model, where the Lindbladian is generated by self-adjoint operators. Our examples focuses on the case of Pauli matrices. In particular, we considered 1-local Lindbladians given by

- $L = \sum_{j=1}^{n} L_{Y_j}$
- $L = \sum_{j=1}^{n} L_{X_j} + L_{Y_j}$

where X and Y are Pauli matrices (see section 3.5) and $A_j = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes A \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$.

The other model we considered is the noisy one, where the generator for Lindbladian is non-hermitian. We investigated

- $L = \sum_{j=1}^{n} L_{a_j}$ where $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
- $L = \sum_{j=1}^{n} L_{b_j}$ where $b = \begin{bmatrix} 0 & e^{-\frac{i\theta}{2}} \\ e^{\frac{i\theta}{2}} & 0 \end{bmatrix}$.

This paper begins by surveying the relative concepts from Hilbert space and operator space theory (Section 2) and quantum mechanics (Section 3). A brief discussion on open quantum systems and Lindbladians (Section 4) follow.

Then the problem statement and the proposed framework will be presented. In Section 5, the main framework of the paper is introduced. The central question of concern is the upper and lower estimate of the simulation cost. To realize the lower bound on the cost, we rely on the notion of Lipschitz complexity (Section 6). The Suzuki-Trotter formula is a well-known tool for achieving upper estimates (Section 7).

Finally, this paper explores application of the method in the main models of quantum dynamical semigroups: the LCU model (Section 8) and the noisy model (Section 9). Examples in which estimates can be explicitly calculated will be provided for both cases.

2 Hilbert Spaces and Normed Operator Spaces

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2.1 Hilbert Spaces

The primary object of interest in quantum information theory is Hilbert spaces and operators on them. Let us begin by defining what a Hilbert space is. Throughout the paper, all vector spaces will be over $\mathbb{C}$.

Definition 2.1.1. *Given a vector space V, an **inner product** on V is a sesquilinear form $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{C}$ such that*

- $\langle x|x\rangle \geq 0$ *and* $\langle x|x\rangle = 0$ *if and only if* $x = 0$
- $\langle x|\alpha y + \beta z\rangle = \alpha \langle x|y\rangle + \beta \langle x|z\rangle$ *for all* $x, y, z \in V$ *and* $\alpha, \beta \in \mathbb{C}$.
- $\langle x|y\rangle = \overline{\langle y|x\rangle}$ *for all* $x, y \in V$.

*The pair $(V, \langle\cdot|\cdot\rangle)$ is called an **inner product space**.*

An inner product induces a norm $\|x\|_2 = \sqrt{\langle x|x\rangle}$. Any norm can define a metric as $d(x, y) = \|x - y\|$. Hence, any inner product space can be viewed as a metric space.

Definition 2.1.2. *A **Hilbert space** is an inner product space that is complete with respect to the induced metric.*

The followings are a few common examples for Hilbert spaces.

Example. The n-dimensional complex vector space $\mathbb{C}^n$ is an inner product space with respect to the inner product $\langle x|y\rangle = \sum_{i=1}^{n} \overline{x_i} y_i$.

Example. The sequence space

$$\ell_2(\mathbb{N}) = \left\{ (a_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$$

with the inner product $\langle a|b\rangle = \sum_{n=1}^{\infty} \overline{a_n} b_n$, where $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$, forms a Hilbert space.

Example. Given $([0, 1], \mathcal{R}, \mu)$ with the Borel σ-algebra $\mathcal{R}$ on $[0, 1]$ and the corresponding Lebesgue measure, the function space $L_2[0, 1]$, where

$$L_2[0, 1] = \left\{ f : \mathbb{R} \to \mathbb{R} : \int_0^1 |f|^2 \, d\mu < \infty \right\}.$$

The inner product is given by $\langle f|g\rangle = \int_0^1 \overline{f} g \, d\mu$.

Remark. In fact, any Hilbert spaces is isometrically isomorphic to $\ell_2(I)$ for some index set I. The dimension of H equals the cardinality of the index set I.

Remark. Throughout the paper, we will only consider finite-dimensional Hilbert spaces and finite von Neumann algebras. Hence, the only relevant Hilbert space will be $\mathbb{C}^n$.

In this project, we work with a nice set of operators, namely the set of bounded operators $\mathbb{B}(H)$ on the Hilbert space H. We begin by defining bounded operators.

Definition 2.1.3. *Given a Hilbert space H and its induced norm $\|\cdot\|$, a linear map $T : H \to H$ is **bounded** if there exists $M > 0$ such that for all $x \in H$,*

$$\|Tx\| \leq M \|x\| .$$

*We define the **operator norm** $\|T\|$ as the smallest of such number*

$$\|T\| = \inf \{M : \|Tx\| \leq M \|x\|\} .$$

Remark. Equivalently, the operator norm can be defined as

$$\begin{aligned} \|T\| &= \inf \{M : \|Tx\| \leq M \text{ where } \|x\| \leq 1\} \\ &= \inf \{M : \|Tx\| \leq M \text{ where } \|x\| = 1\} \\ &= \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}. \end{aligned}$$

Remark. For finite-dimensional Hilbert space, every operator is bounded. So $\mathbb{B}(H)$ is exactly the collection of all linear maps from H to itself. When $H = \mathbb{C}^d$, the set of bounded linear operators $\mathbb{B}(H)$ is exactly the complex matrix algebra $\mathbb{M}_d(\mathbb{C})$.

2.2 Operators on Hilbert Spaces

Here we will collect the definition for the operators we will use in this paper.

- Given an operator $A : H \to H$, its **adjoint** is defined as

  $$\langle A^* x | y \rangle = \langle x | A y \rangle$$

  for all $x, y \in H$.

- If $A^* = A$, then we call it **self-adjoint** or **Hermitian**.

- An operator T is called **positive (semi-definite)** if $\langle v | Tv \rangle \geq 0$ for all $v \in H$, denoted as $T \geq 0$.

- A map V is called an **isometry** if $V^* V = I$.

- U is an **unitary** operator if

  $$U^* U = U U^* = I$$

  if and only if U is a surjective isometry.

Example. $S : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ where $S(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$ is an isometry that is not surjective, hence not unitary.

- P is a **projection** if $P^2 = P = P^*$.

- A map V is a **partial isometry** if $V^* V = P$ for some projection P.

2.3 Normed Spaces

Inner product resembles the Euclidean distance in $\mathbb{C}^n$. However, there are more general measurement of distance on vector spaces.

Definition 2.3.1. *Given a vector space V, a **semi-norm** $\|\cdot\| : V \to \mathbb{R}$ is a positive-valued function such that*

- $\|x\| \geq 0$ *for all* $x \in V$,
- $\|\alpha x\| = |\alpha| \, \|x\|$ *for all* $\alpha \in \mathbb{R}$ *and for all* $x \in V$, *and*
- $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in V$ *(triangular inequality).*

If, in addition, it is non-degenerate, meaning that

- $\|x\| = 0$ *if and only if* $x = 0$,

*then the function $\|\cdot\|$ is called a **norm** and the pair $(V, \|\cdot\|)$ is called a normed space.*

Example. Given a Hilbert space H, its inner product induces a norm. Namely

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

If $H = \mathbb{C}^n$, then this norm is exactly the ℓ_2-norm, defined as

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

Example. This is a special case of a class of norm on $\mathbb{C}^n$ called ℓ_p norm. For $1 \leq p < \infty$,

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Example. It is worth emphasizing the 1-norm and the ∞-norm.

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \quad \text{and} \quad \|x\|_\infty = \sup |x_i|$$

We will see the operator analogous of these two norms.

For sequence spaces, there are many sequences in which their norm diverges. For example, consider the constant sequence $c = (1, 1, 1, \dots)$. Its norm $\|c\|_1 = \infty$. Hence, when we talk about ℓ_p spaces, we do not consider these cases. Therefore, we define

$$\ell_p = \left\{ x = (x_j)_{j=1}^{\infty} : \|x\|_p < \infty \right\}.$$

Remark. A semi-norm can be modified into a norm by quotienting the kernel out. We will see that trick later in the paper.

2.4 Norms on Operators

Since $\mathbb{B}(H)$ forms a vector space, we will explore the norms one can define on the space of operators. We have seen the operator norm in the discussion before. Here we will introduce Schatten p-norm and completely bounded norm.

Definition 2.4.1. *Given $T \in \mathbb{B}(H)$, the trace norm, also known as **Schatten 1-norm**, is defined as*

$$\|T\|_1 = \mathrm{tr}(|T|),$$

where $|T| = \sqrt{T^*T}$.

Note that the Schatten 1-norm is the sum of all singular values of the operator.

Definition 2.4.2. *More generally, we can define **Schatten p-norm** for $1 \leq p < \infty$.*

$$\|T\|_p = \left(\mathrm{tr}(|T|^p)\right)^{1/p}.$$

Remark. It is a convention to write the operator norm $\|T\| = \|T\|_\infty$ as the Schatten ∞-norm. Note that the operator norm is precisely the largest singular value. For positive operators, it is also the largest eigenvalue.

It is worth emphasizing **Schatten 2-norm** gives rise to an inner product. By definition,

$$\|T\|_2 = \mathrm{tr}(T^*T)^{1/2} := \sqrt{\langle T, T\rangle}.$$

Generally, the inner product is defined as

$$\langle T, S\rangle = \mathrm{tr}(T^*S).$$

Similar to the cases of sequence spaces, we are only interested in those operators with finite norm. Thus, we define **Schatten p-class** as

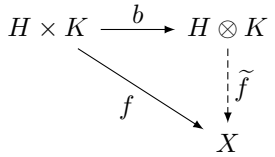
$$S_p(H) = \left\{T \in \mathbb{B}(H) : \|T\|_p < \infty\right\}.$$

The "Schatten ∞-class" is the Schatten class with respect to the operator norm, but this is exactly $\mathbb{B}(H)$.

Remark. Note that when $\dim H < \infty$, all operators have finite norm. Hence $S_1(H)$ and $\mathbb{B}(H)$ are the same as sets. However, we will use such notation to emphasize the different norm we are considering.

2.5 Tensor Product and Completely Bounded Norms

Let H, K be two Hilbert space. We are usually interested in the system that combines both spaces. That is modeled using tensor product. The tensor product $H \otimes K$ is a vector space together with a bilinear map $b : H \times K \to H \otimes K$ such that the universal property is satisfied. That is, For any other Hilbert space X and a linear map $f : H \times K \to X$, there exists a unique map $\widetilde{f} : H \otimes K \to X$ such that $f = \widetilde{f} \circ b$. Equivalently, the following diagram commutes:



A easy way to realize the structure of the tensor product for finite-dimensional spaces is to consider its basis elements. Let $\{h_i\}_{i=1}^m$ and $\{k_j\}_{j=1}^n$. Then $\{h_i \otimes k_j\}_{i,j=1}^{m,n}$ will be the basis for $H \otimes K$. It is immediate that if $\dim H = m$ and $\dim K = n$, then $\dim H \otimes K = mn$. In particular, if $H = \mathbb{C}^m$ and $K = \mathbb{C}^n$, then $H \otimes K = \mathbb{C}^m \otimes \mathbb{C}^n \cong \mathbb{C}^{mn}$.

When working with matrices, it is possible write the tensor explicitly via Kronecker product. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and B be any matrix, then

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}.$$

Definition 2.5.1. *Let H be the Hilbert space and $\Phi \in \mathbb{B}(H)$. The **completely bounded (cb) norm** of Φ is defined as*

$$\|\Phi\|_{\mathrm{cb}} = \sup_{n \in \mathbb{N}} \|\Phi \otimes \mathbb{1}_n\|_\infty .$$

Here $\|\cdot\|_\infty$ is the operator norm.

Definition 2.5.2. *The **diamond norm** is the cb norm with respect to the Schatten 1-norm. That means, the diamond norm of a Φ is defined as*

$$\|\Phi\|_\diamond = \sup_{n \in \mathbb{N}} \|\Phi \otimes \mathbb{1}_n\|_1$$

where $\|\cdot\|_1$ is the Schatten 1-norm.

Remark. Throughout the paper, cb-norm refers exclusively to the cb-norm with respect to the operator norm. We will use diamond norm for cb-norm with respect to the Schatten 1-norm.

The usefulness of diamond norm in our context is illustrated as the proposition below:

Proposition 2.5.3. *Let Φ, Φ', Ψ, Ψ' be channels. Suppose that $\|\Phi - \Phi'\|_\diamond < \varepsilon$ and $\|\Psi - \Psi'\|_\diamond < \sigma$, then*

$$\|\Phi \otimes \Psi - \Phi' \otimes \Psi'\|_\diamond < \varepsilon + \sigma.$$

Proof. This is a easy consequence from a few properties of the diamond norm.

Note that

$$\|\Phi \otimes \mathbb{1}\|_\diamond = \|\Phi\|_\diamond .$$

Thus

$$\|\Phi \otimes \Psi - \Phi' \otimes \Psi'\|_\diamond = \|(\Phi \otimes \mathbb{1})(\mathbb{1} \otimes \Psi) - (\Phi' \otimes \mathbb{1})(\mathbb{1} \otimes \Psi')\|_\diamond .$$

Also notice

$$\|AB - A'B'\|_\diamond \leq \|A - A'\|_\diamond + \|B - B'\|_\diamond .$$

Thus

$$\begin{aligned}\|(\Phi\otimes\mathbb{1})(\mathbb{1}\otimes\Psi)-(\Phi'\otimes\mathbb{1})(\mathbb{1}\otimes\Psi')\|_\diamond &\le \|\Phi\otimes\mathbb{1}-\Phi'\otimes\mathbb{1}\|_\diamond+\|\mathbb{1}\otimes\Psi-\mathbb{1}\otimes\Psi'\|_\diamond\\ &= \|\Phi-\Phi'\|_\diamond+\|\Psi-\Psi'\|_\diamond\\ &< \varepsilon+\sigma.\end{aligned}$$

□

2.6 Properties of Diamond Norm and cb-Norm

We used two key properties of the diamond norm in the proof of approximating the tensor of two channels. Here we will give a list of properties that have various uses. We will state the properties without proof. For detailed accounts, see [Wat18].

- If $\Phi\in\mathbb{B}(H)$, then
$$\|\Phi\otimes\mathbb{1}\|_\diamond=\|\Phi\|_\diamond.$$
- For all $\Phi,\Psi\in\mathbb{B}(H)$,
$$\|\Phi\Psi\|_\diamond\le\|\Phi\|_\diamond\|\Psi\|_\diamond.$$
- For $\Phi,\Phi',\Psi,\Psi'\in\mathbb{B}(H)$,
$$\|\Phi\Psi-\Phi'\Psi'\|_\diamond\le\|\Phi-\Phi'\|_\diamond+\|\Psi-\Psi'\|_\diamond$$
- Let $\Phi,\Psi\in\mathbb{B}(H)$, then
$$\|\Phi\otimes\Psi\|_\diamond=\|\Phi\|_\diamond\|\Psi\|_\diamond.$$

Remark. The same properties hold for *cb*-norm as well due to duality.

3 Fundamentals of Quantum Mechanics

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3.1 Quantum States

Quantum systems are modeled as Hilbert spaces. Once again, vectors in H is denoted as $|\varphi\rangle$ and its dual vector is written as $\langle\varphi| = (|\varphi\rangle)^* = \overline{|\varphi\rangle}^T$.

Recall the computational basis

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Definition 3.1.1. *A **quantum state** can be written as*

$$|\varphi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. This represents a probability where $|\varphi\rangle$ is in the superposition of different quantum states.

Definition 3.1.2. *A **pure state** $|\varphi\rangle$ is a normalized vector in H where $\langle\varphi|\varphi\rangle = 1$.*

Pure states represent complete knowledge of the quantum state, which is not a realistic assumption in quantum mechanics. Therefore to understand more general quantum states, we need to introduce a better representation of quantum states, called the density matrix.

Definition 3.1.3. *A **mixed state** is a positive semi-definite operator $\rho \in \mathbb{B}(H)^+$ with $\operatorname{tr}\rho = 1$.*

Notice that pure states are exactly the quantum states of rank 1. In particular, they can be written as $\psi = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in H$.

We can always write a mixed state ρ as a probabilistic mixture of pure states by spectral decomposition

$$\rho = \sum_{i=1}^{k} p_i\,|\psi_i\rangle\langle\psi_i|\,.$$

To describe how the quantum system interacts with another quantum system, such as how atoms interact with each other, we need to consider composite quantum systems. It is modeled by the tensor product of the individual quantum systems, i.e. Hilbert spaces.

Definition 3.1.4. *Let H and K be two Hilbert spaces associated with the respective quantum systems, the **composite system** of HK is given by*

$$HK = H \otimes K.$$

To isolate the information of individual systems from the composite system, i.e. recovering the marginal states, we need something called partial trace.

Definition 3.1.5. *Given a bipartite system $\mathbb{B}(HK) = \mathbb{B}(H) \otimes \mathbb{B}(K)$, **partial trace** over H is defined as $\operatorname{tr}_H : \mathbb{B}(HK) \to \mathbb{B}(H)$ where*

$$\operatorname{tr}_H = \operatorname{tr} \otimes \mathrm{id}\,.$$

Partial trace over K is defined similarly where $\operatorname{tr}_K : \mathbb{B}(HK) \to \mathbb{B}(K)$ by

$$\operatorname{tr}_K = \mathrm{id} \otimes \operatorname{tr}\,.$$

If given two matrices A and B with the Kronecker product written as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix},$$

its partial trace over the B system is

$$\operatorname{tr}_B(A \otimes B) = \begin{bmatrix} a_{11}\operatorname{tr}B & a_{12}\operatorname{tr}B & \dots & a_{1n}\operatorname{tr}B \\ a_{21}\operatorname{tr}B & a_{22}\operatorname{tr}B & \dots & a_{2n}\operatorname{tr}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\operatorname{tr}B & a_{m2}\operatorname{tr}B & \dots & a_{mn}\operatorname{tr}B \end{bmatrix} = \operatorname{tr}B \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Definition 3.1.6. *Given any $\rho_{HK} \geq 0$, where $\rho_{HK} \in \mathbb{B}(HK)$,*

$$\rho_H := \operatorname{tr}_K \rho_{HK} \in \mathbb{B}(H) \quad \textit{and} \quad \rho_K := \operatorname{tr}_H \rho_{HK} \in \mathbb{B}(K)$$

*We call ρ_H and ρ_K **marginal states** of ρ_{HK} on H and K respectively.*

3.2 Quantum Observables and Duality

At the early stages of developing quantum theory, there are two schools of scholars studying different objects. One school studies quantum states, as described above, and another school investigate the observables on quantum systems. For example, the two most well-known observables are momentum and position. In general, one can have a chain of observables, and due to their non-commutative nature, will define a new observable.

The school that studies quantum states is called the Schrödinger picture and the school that studies quantum observables Heisenberg picture. It is soon realized that these two are mathematically equivalent. In the language of operator algebra, quantum observables are the dual maps of quantum states. This mathematical relationship establishes a link between the behavior of quantum observables and quantum states, embodying a duality that is reflected in the outcomes.

Mathematically, quantum observables are Hermitian matrices. For simplicity, we will assume every Hermitian matrix represent some observables, though it is unclear what the correspondence between arbitrary observable and Hermitian is.

Recall that given a vector space V, its **dual space**

$$V^* = \{v^* : V \to \mathbb{C}\}$$

is the collection of all linear functionals on V. A space V_* is called the **predual** of V if the dual of V_* is V. In our context, $\mathbb{B}(H)$ is the (post)-dual for $S_1(H)$. That means, $S_1(H)^* \simeq \mathbb{B}(H)$.

3.3 Quantum Channel

In quantum information, we want to consider noise models for quantum processes. That means, we want to model the process in which the quantum systems are distorted. Such process is given by quantum channels.

Definition 3.3.1. *A **quantum channel** $\Phi : \mathbb{B}(H) \to \mathbb{B}(K)$ is a super-operator that is completely positive (CP) and trace preserving (TP).*

Definition 3.3.2. *A linear operator Φ is **trace-preserving** if given any $\rho_H \in \mathbb{B}(H)$,*

$$\mathrm{tr}(\rho_H) = \mathrm{tr}(\Phi\rho_H).$$

*Φ is called **unital** if*

$$\Phi(\mathbb{1}) = \mathbb{1}.$$

Definition 3.3.3. *A linear operator Φ is **completely positive** if $\Phi \otimes \mathrm{id}_n \geq 0$ for all $n \in \mathbb{N}$. Such definition ensures that we can sending a probability distribution to a probability distribution even in the composite systems or arbitrary dimension.*

Here are some examples of CPTP maps:

Example. Conjugation by unitary operators are CPTP.

Example. Conjugation by (partial) isometries are CPTP.

Example. Trace operators are CPTP.

Example. Partial trace is CPTP

Not all positive maps are completely positive. That is, there are examples of positive map that is not completely positive. Here is one:

Example. Let A be any operator, the map that sends $A \mapsto A^T$ to its transpose is a positive map, but not completely positive.

Since we are working with both pictures, it is important to understand the channel described in the Heisenberg picture. For that, we need a theorem for the dual of CPTP maps.

Let $\Phi : S_1(H) \to S_1(H)$ denote the channel in Schrödinger picture, i.e. a CPTP map. Channels in Heisenberg picture will be written as $\Phi^* : \mathbb{B}(H) \to \mathbb{B}(H)$ to emphasize the duality.

Theorem 3.3.4. *Φ is CP iff Φ^* is CP.*

Proof.

$$\langle(\Phi^* \otimes \mathrm{id}_n)x|\rho\rangle = \langle x|(\Phi \otimes \mathrm{id}_n)\rho\rangle \geq 0.$$

□

Theorem 3.3.5. *Φ is TP iff Φ^* is unital.*

Proof.

$$\langle\Phi^*\mathbb{1}|\rho\rangle = \langle\mathbb{1}|\Phi\rho\rangle = \mathrm{tr}(\Phi(\rho)) = \langle\mathbb{1}|\rho\rangle$$

□

Remark. Recall that Φ is TP if for any $\rho \in S_1(H)$, $\mathrm{tr}\,\Phi(\rho) = \mathrm{tr}\,\rho$ and Φ^* is unital if $\Phi^*(\mathbb{1}) = \mathbb{1}$.

Thus, channels in Heisenberg picture are exactly unital and completely positive (UCP) maps.

3.4 Kraus Representation

The requirement of channels being CP arises from physical concern, which naturally give rise to operator sum representation of quantum channels.

Starting with an state ρ, prepare it with respect to some pure state $|0\rangle\langle 0|$ (otherwise, apply purification to make it into a pure state), it goes through some unitary evolution $\mathrm{ad}_U(\cdot) = U(\cdot)U^*$. Since we are most concerned about the state in our principal system, we should then take partial trace tr_E over the environment. The above data can be formulated in terms of the Stinespring dilation as

$$\Phi(\rho) = \mathrm{tr}_E\left[U(\rho \otimes |0\rangle\langle 0|)U^*\right].$$

Since these are all the allowed transformations of a state, all channels can be modeled this way. Requiring such process to be CP assures the terminal state after such transformation is still a legit state, being positive in particular.

Theorem 3.4.1. *Given a channel (CPTP map) Φ, there exists partial isometries $\{E_k\}$, also known as the **Kraus operators**, such that for any $\rho \in S_1(H)$*

$$\Phi(\rho) = \sum_k E_k \rho E_k^*$$

Proof. Stinespring dilation exists for any CP map. We will first shown how the dilation give rise to the operator sum representation and then give a criteria for TP.

Without lost of generality, assume the initial environment is given by a pure state $\rho_E = |0\rangle\langle 0|$. Otherwise, we apply purification to bring the state back to a pure state. The operator sum representation of a channel is derived from

$$\begin{aligned}\Phi(\rho) &= \mathrm{tr}_E\left[U(\rho \otimes |0\rangle\langle 0|)U^*\right] \\ &= \sum_k \langle e_k| U(\rho \otimes |0\rangle\langle 0|)U^* |e_k\rangle \\ &= \sum_k \langle e_k| U |0\rangle \rho \langle 0| U^* |e_k\rangle .\end{aligned}$$

Set $E_k \equiv \langle e_k|U|0\rangle$, and we get that

$$\Phi(\rho) = \sum_k E_k \rho E_k^*.$$

For TP, note that $\mathrm{tr}\,\rho = \mathbb{1}$ for all $\rho \in S_1(H)$. Then

$$\begin{aligned}\mathbb{1} &= \mathrm{tr}\,\Phi(\rho) \\ &= \mathrm{tr}\left(\sum_k E_k \rho E_k^*\right) \\ &= \mathrm{tr}\left(\sum_k E_k^* E_k \rho\right).\end{aligned}$$

Since this equation holds for all ρ, it holds for $\rho \sim \mathbb{1}$ in particular. Hence $\sum_k E_k^* E_k = \mathbb{1}$. □

Remark. Dually, in Heisenberg picture, given any observables $x \in \mathbb{B}(H)$, it has that

$$\Phi^*(x) = \sum E_k^* x E_k$$

and Φ^* is unital iff $\sum_k E_k E_k^* = \mathbb{1}$.

gates

3.5 Elementary Quantum Gates

Here are some useful gates that is used in quantum computation:
Pauli Gates:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hadamard Gate:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

SWAP Gate:

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Controlled Gates:

$$C{-}NOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad C{-}Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad C{-}e^{itZ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{it} & 0 \\ 0 & 0 & 0 & e^{-it} \end{bmatrix}$$

4 Open Quantum System

open_qu_sys

4.1 Closed and Open Quantum Systems

The initial studies of quantum mechanics begins by considering ideal models of quantum systems that is isolated from the affect from the environment. For examples, examples of closed quantum system includes infinite potential well, quantum harmonic oscillator, hydrogen aton, and electron spin systems.

The dynamics of such systems are described by the Schrödinger equation, particularly when external potentials are absent given by

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

where

$$\hbar = \frac{h}{2\pi} = 1.054573 \times 10^{-34} Js.$$

is the Planck constant. The general solution for time-dependent Schrödinger equation without potential is

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$$

where H is the Hamiltonian, a self-adjoint operator. Since e^{itH} is unitary for self-adjoint matrices H, all dynamics in closed system are described by unitary operators.

Closed quantum systems are idealized models where no energy or information is exchanged with the environment. However, this assumption is often impractical and unrealistic. To study open quantum systems effectively, physicists have devised two general approaches.

One approach is to consider the system along with the environment affecting it, thereby treating the larger combined system as closed. In the most extreme cases, one could consider the system in conjunction with the entire universe, which by definition is closed. However, dealing with such an expansive system is usually not feasible.

Hence, alternative methods for studying open quantum systems, which accept their inherent openness, have been developed. This necessitates a modified version of the Schrödinger equation that accounts for the system's behavior in light of its interactions with the environment.

One of the models for studying Markovian open quantum systems is known as the master equation, also referred to as the Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation. It is expressed as

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho] + \sum_i \gamma_i \left(L_i\rho L_i^* - \frac{1}{2}L_i^*L_i\rho - \frac{1}{2}\rho L_i^*L_i\right).$$

Here, ρ represents the density matrix. L_i, also known as the jump operator or Lindbladian operator, models the open dynamics of the quantum system.

Throughout this project, we shall assume that the Lindbladian operator is devoid of closed dynamics. This means our Lindbladian is given by

$$L_a(\rho) = a^*a\rho + \rho a^*a - 2a\rho a^*$$

where a is called the generator of the Lindbladian and ρ is a density matrix within the quantum dynamical semigroup $(e^{-tL})_{t\geq 0}$. We will first explore some properties of the semigroup.

4.2 Quantum Dynamical Semigroup

Since quantum evolution is continuous, parameterized by time, it is common to consider the behavior of the evolution itself. That means, we need to consider a family of quantum channels $(\Phi_t)_{t\geq 0}$ and the result of the evolution is a snapshot given by $\Phi_t(\rho)$. This family is commonly known as quantum dynamical semigroup. When studying evolution in closed systems, our semigroup will be given by $(e^{tH})_{t\geq 0} = (U_t)_{t\geq 0}$ for some H that solves the Schrödinger equation for the specific case. More generally, when we consider the evolution generated by Lindbladians, we will have a Markovian quantum dynamical semigroup $(e^{-tL})_{t\geq 0}$.

We explore some key properties of the semigroup. Let us consider a one-parameter semigroup of channels $(\Phi_t : S_1(H) \to S_1(H))_{t\geq 0}$ with the properties that

- $\Phi_0(\rho) = \mathrm{id}\,\rho = \rho$.
- $\Phi_{s+t} = \Phi_s \circ \Phi_t$
- Φ_t is normal and CPTP (or UCP) for all $t \geq 0$.
- for all $x \in S_1(H)$, $T_t x \to x$ in ultra-weak topology as $t \to 0$.

It is immediate that $\Phi_t = \Phi_{t/n}^n$. This corollary plays an important role in our calculation.

simu_cost

5 Simulation Cost

5.1 Scope of Inquiry

In this paper, we examine the 1-local Lindbladian generated by an operator a_k. This is expressed as:

$$L = \sum_{j=1}^{n} L_{a_{1_j}} + L_{a_{2_j}} + \cdots + L_{a_{K_j}}$$

where

$$a_{k_j} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underbrace{a_k}_{j\text{-th component}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

and

$$L_{a_k}(\rho) = a_k^* a_k \rho + \rho a_k^* a_k - 2 a_k \rho a_k^*.$$

Given that these operators act on distinct qubits, the $L_{a_{k_j}}$ commute with each other. Therefore, we will focus on the behavior of a single qubit, treating other aspects as a matter of scaling.

5.2 Resource Dependent Simulation

The goal for this project is to determine the simulation cost of a given one continuous parameter quantum dynamical semigroup $(e^{-tL})_{t\geq 0}$ of channels generated by Lindbladians. In classical computation, such tasks are typically carried out assuming universal gates, which are capable of implementing any Boolean function without the need for other types of gates. For instance, both NAND and NOR gates qualify as universal gates due to their ability to execute any logical operation on their own. Therefore, it is natural to inquire the same in the context of quantum computation given some quantum universal sets. Some examples of quantum universal sets includes the collection of C-NOT and all single-qubit gates [Bar95][BBC+95] and C-NOT gate, Hadamard gate, and phase gate [Boy00][BMP+00].

However, from a practical standpoint, not all quantum computing devices can implement universal sets well. In fact, different models of quantum computing devices have their unique characteristics in terms of the gates they can implement well. Hence, we propose a resource dependent version of simulation cost. Namely, given a resource set, how costly is it to simulate the given semigroup.

The choice of approperate resource set will be discussed throughout the paper when we have particular model in mind. For now, assume there is a set of unitaries $\mathcal{U} \subseteq U(A)$, where A is the principal system of interests, that we can use for simulation. Given a probability measure μ on $\mathcal{U}^m$, we acquire a random unitary channel

$$\Psi_\mu = \mathbb{E}_\mu \operatorname{ad}_{u_1} \dots \operatorname{ad}_{u_m}$$

i.e.

$$\Psi_\mu(\rho) = \int_{\mathcal{U}^m} u_1 \dots u_m \rho u_m^* \dots u_1^* \, d\mu(u_1, \dots, u_m)$$

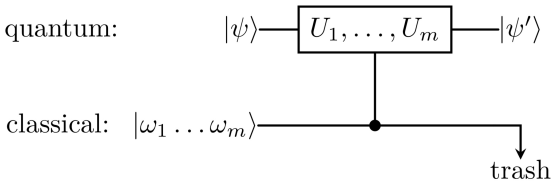
for $u_1, \dots, u_m \in \mathcal{U}$. Note that this is the convexified version of the regular simulation length in Hamiltonian simulation. Physically, it correspond to simulation with the help of classical resources. Here, we perceive classical resources as free, and it helps to reduce the simulation length via methods such as qDRIFT [Cam19][Cam19]. We will use this concept to formally define our query.

5.3 Length of Simulation

We define the simulation cost of the given random unitary channel as

$$\begin{aligned} l_\delta^{\mathcal{U}}(T_t) &= \inf\left\{m : \exists \mu(\mathcal{U}^m) = 1 \left\|T_t - \Psi_\mu\right\|_\diamond\right\} \\ &= \inf\left\{m : \exists \mu(\mathcal{U}^m) = 1 \left\|T_t - \mathbb{E}_\mu \operatorname{ad}_{u_1} \dots \operatorname{ad}_{u_m}\right\|_\diamond\right\} \end{aligned}$$

Notice that, in addition to taking the shortest length, we also allow arbitrary probability measure. This means, we are able to have greater autonomy in terms of the ways in which the classical resources are applied. Intuitively, this measure should rapidly mix the unitaries to minimize the norm. Therefore, it should be a measure that deviates from the Haar measure. We can represent our framework via the following circuit:



Since the behavior of the semigroup is hard to predict, we will consider the maximal simulation cost

$$M_{\alpha,\beta} = \sup_{t \geq 0} l_{\alpha t^\beta}^{\mathcal{U}}(T_t)$$

when needed. Via the following facts, we predict the witness of the maximal simulation cost will be at small t as the system stabilizes quickly.

The calculation of $l_\delta^{\mathcal{U}}$ is notably difficult, as it falls under the category of a **QMP** problem. Consequently, exact computation is often impractical, and we resort to approximations. Notably, standard methodologies like the Suzuki-Trotter formula facilitate deriving a general solution for the upper bound. Conversely, establishing a lower bound is considerably more intricate. This is because confirming a lower bound implies that the optimal solution cannot be worse than this established minimum, a type of assurance that is less commonly available even in classical complexity theory.

Fortunately, the semigroup approach combined with complexity theory offers a systematic method for computing lower bounds. For upper bounds, the Suzuki-Trotter formula provides a reliable tool, and the accuracy can sometimes be improved if the unitaries can be explicitly calculated.

Theorem 5.3.1. *Let $u, u^* \in \mathcal{U}$,*

$$l_\delta^{\mathcal{U}}(\mathrm{id}) \leq 2$$

Proof. Let $\Psi_\mu = \mathbb{E}_\mu \operatorname{ad}_{u_1} \operatorname{ad}_{u_2} = \operatorname{ad}_u \operatorname{ad}_{u^*} = \mathrm{id}$ with the point mass measure at $\operatorname{ad}_u \operatorname{ad}_{u^*}$. Hence $l_\delta \leq l_0 \leq 2$. □

Theorem 5.3.2. *Let $u, u^* \in \mathcal{U}$,*

$$l_2^{\mathcal{U}}(T_t) \leq 2$$

Proof. Same as before, let $\Psi_\mu = \mathbb{E}_\mu \operatorname{ad}_{u_1} \operatorname{ad}_{u_2} = \operatorname{ad}_u \operatorname{ad}_{u^*} = \mathrm{id}$ with the point mass measure at $\operatorname{ad}_u \operatorname{ad}_{u^*}$.

$$\left\|T_t - \mathbb{E}_\mu \operatorname{ad}_u \operatorname{ad}_{u^*}\right\|_\diamond = \left\|T_t - \mathrm{id}\right\|_\diamond \leq \left\|T_t\right\|_\diamond + \left\|\mathrm{id}\right\|_\diamond \leq 2.$$

□

Theorem 5.3.3.

$$l_{\delta}^{\mathcal{U}}(T_t) \leq m l_{\frac{\delta}{m}}^{\mathcal{U}}(T_{\frac{t}{m}})$$

Proof. Let $\left\|T_{\frac{t}{m}} - \Psi\right\|_{\diamond} \leq \frac{\delta}{m}$, where $\Psi = \mathbb{E}_{\mu} \operatorname{ad}_{u_1} \dots \operatorname{ad}_{u_k}$. Then

$$\|T_t - \Psi^m\|_{\diamond} \leq \left\|T_{\frac{t}{m}}^m - \Psi^m\right\|_{\diamond} \leq m \left\|T_{\frac{t}{m}} - \Psi\right\|_{\diamond} \leq \delta.$$

Whence,

$$\left\{km : \left\|T_{\frac{t}{m}} - \Psi\right\|_{\diamond} \leq \frac{\delta}{m}\right\} \subseteq \left\{k : \|T_t - \Psi\|_{\diamond} \leq \delta\right\}.$$

Take the infimum over n on both sides and the claim follows. □

It is worth mentioning that with a fix β, bound of different choices of α can be obtained.

Corollary 5.3.4.

$$M_{\alpha,\beta} \leq \frac{\alpha'}{\alpha} M_{\alpha',\beta}$$

Hence, if we can calculate $M_{\alpha,\beta}$ for some α, we also have $M_{1,\beta}$.

Theorem 5.3.5.

$$\|T_t - \mathrm{id}\|_{\diamond} \leq t \|L\|_{\diamond} e^{t\|L\|_{\diamond}}.$$

Proof.

$$\begin{aligned}
\|T_t - \mathrm{id}\|_{\diamond} &= \left\|e^{-tL} - \mathrm{id}\right\|_{\diamond} \\
&= \left\|\sum_{k=1}^{\infty} \frac{(-tL)^k}{k!}\right\|_{\diamond} \\
&= \left\|(-tL) \sum_{k=1}^{\infty} \frac{(-tL)^{k-1}}{k!}\right\|_{\diamond} \\
&\leq \|(-tL)\|_{\diamond} \left\|\sum_{k=0}^{\infty} \frac{(-tL)^k}{(k+1)!}\right\|_{\diamond} \\
&\leq t \|L\|_{\diamond} \left\|\sum_{k=0}^{\infty} \frac{(-tL)^k}{(k+1)!}\right\|_{\diamond} \\
&\leq t \|L\|_{\diamond} \left\|\sum_{k=0}^{\infty} \frac{(-tL)^k}{k!}\right\|_{\diamond} \\
&\leq t \|L\|_{\diamond} \sum_{k=0}^{\infty} \frac{(t\|-L\|_{\diamond})^k}{k!} \\
&= t \|L\|_{\diamond} \sum_{k=0}^{\infty} \frac{(t\|L\|_{\diamond})^k}{k!} \\
&\leq t \|L\|_{\diamond} e^{t\|L\|_{\diamond}}
\end{aligned}$$

□

Lip_comp

6 Lipschitz Complexity

6.1 Axioms of Complexity

LBK22
Inspired by [LBK+22], following the initial works done in ACJW23 [ACJW23], we take an axiomatic approach to defining the complexity.

Axiom. *Let $\Phi : S_1(H) \to S_1(H)$ be a channel on the respective space, and C is a positive-valued cost function.*

- $C(\mathrm{id}) = 0$
- $C(\Phi\Psi) \leq C(\Phi)C(\Psi)$
- $C(\sum_i p_i\Phi_i) \leq \sum_i p_i C(\Phi_i)$
- $C(\Phi \otimes \Psi) = C(\Phi) + C(\Psi)$
- $|C(\Phi) - C(\Psi)| \leq C(E) \|\Phi - \Psi\|_\diamond$, *given* $E\Phi = E = E\Psi$ *where E is the corresponding conditional expectation.*
- $C(\mathrm{ad}_u) \leq D$ *for some constant D.*

Definition 6.1.1. *Let $E : \mathbb{B}(H) \to \mathcal{N}$ be a UCP map. It is called* ***conditional expectation*** *if*

- $E(x) = x$ *for all* $x \in \mathcal{N}$ *and*
- $E(xay) = aE(x)b$ *for all* $a, b \in \mathcal{N}$ *and* $x \in \mathbb{B}(H)$

These axioms are necessary and sufficient for establishing the existence of a lower bound, according to our derivation. This list exclusively comprises all the properties utilized in the proof. While selecting axioms retrospectively from the proof might appear cyclical, this approach becomes clear when considering noisy models. In such cases, a new complexity measure must be defined. This measure will appropriately account for the notion of complexity in light of the surrounding environment, as discussed later in the paper.

6.2 Complete Lipschitz Complexity

Our complexity measure is derived from an norm on channel. We define the concept of resource-dependent complexity using Lipschitz semi-norm. The choice of resource set, which is central to our discussion, will be further explored when we examine specific models. For now, we will develop the theory using a general resource set.

Definition 6.2.1. *Given a resource set $S \subseteq \mathbb{B}(H)$, the induced* ***Lipschitz semi-norm*** *is defined as*

$$|||f|||_S = \sup_{s \in S} \|[s, f]\|_\infty$$

for any $f \in \mathbb{B}(H)$, where $[s, f] = sf - fs$ is the commutator.

It is clear that the norm is dependent on the resource set. Different choices of resource set will induce different Lipschitz norms.

We will take some liberty in the notation. We sometimes write $\|\!|\cdot|\!\|_{\mathrm{Lip}}$ to emphasize the norm associated with the operator space and sometimes write $\|\!|\cdot|\!\|_S$ to emphasize the resource set. This notation will come in handy when we discuss the difference choice of resource sets.

Note that if $f \in S'$, then $[s, f] = 0$ for all $s \in S$ and hence $\|\!|f|\!\|_S = 0$. That means,

$$S' = \{f \in \mathbb{B}(H) : \|\!|f|\!\|_S = 0\}.$$

This shows that anything in S' has norm 0, which does not satisfy the faithfulness property of norms.

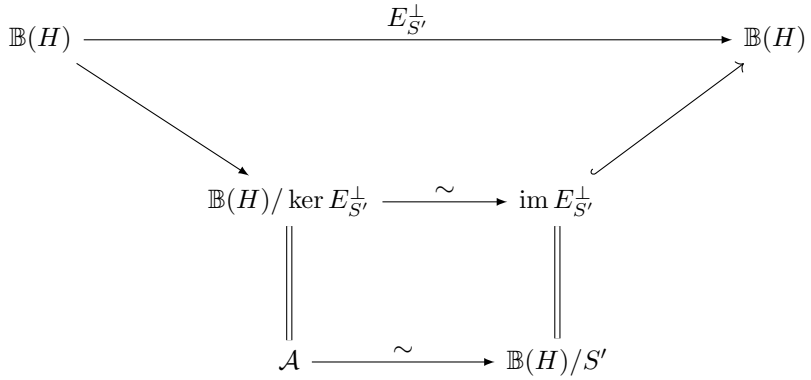
To convert it to an actual norm, we quotient out the kernel from $\mathbb{B}(H)$. Define

$$\mathcal{A} = \mathbb{B}(H)/\{f \in \mathbb{B}(H) : \|\!|f|\!\|_S = 0\} = \mathbb{B}(H)/S'$$

be the desired quotient space. It is easy to check that $\|\!|\cdot|\!\|_S$ is an actual norm on $\mathcal{A}$ (by force).

Such procedure can be realized via an (orthogonal) projection. Let $E_{S'} : \mathbb{B}(H) \to S'$ be a conditional expectation (orthogonal projection) onto S'. That means, $E_{S'}|_{S'} = \mathrm{id}$, and $E_{S'}^{\perp} = \mathrm{id} - E_{S'}|_{S'} = 0$. Equivalently, $E_{S'}(s) = s$ for all $s \in S'$ and $E_{S'}(f) = 0$ for all $f \in \mathbb{B}(H)\backslash S'$.

Thus, $\operatorname{im} E_{S'} = S'$ and by the first isomorphism theorem, $\operatorname{im} E_{S'} \simeq \mathbb{B}(H)/\ker E_{S'} = \mathcal{A}$.



Hence, we conclude that

$$\mathcal{A} \cong \{f \in \mathbb{B}(H) : E_{S'}(f) = 0\}.$$

Now we are ready to define the complete Lipschitz complexity.

Definition 6.2.2. *Let $\Phi : S_1(A) \to S_1(A)$ be a channel, **complete complexity** is defined as*

$$C_{S_A}^{\mathrm{cb}}(\Phi) = \left\| \Phi^* - \mathrm{id} : (\mathbb{B}(H), \|\!|\cdot|\!\|_{\mathrm{Lip}}) \to (\mathbb{B}(H), \|\cdot\|_{\infty}) \right\|_{\mathrm{cb}}.$$

Remark. We will be abusive on the notion here and write $\|\cdot\|_{\mathrm{cb}}$ for $\|\cdot\|_{\mathrm{Lip}\to\infty,\mathrm{cb}}$ when the context is clear. We write A here to emphasize this is the principal system of interests.

In the axioms, we ask for a notion of conditional expectation. In our context, we define

$$E_A = \lim_{t\to\infty} \frac{1}{t} \int_0^t T_s \, ds.$$

Theorem 6.2.3. *$E_A = E_{S'}$ if and only if $\operatorname{im} E_A = \operatorname{im} E_{S'} = S'$, where $E_{S'}$ is the conditional expectation onto S'.*

Proof. It is clear that $\operatorname{im} E_{S'} = S'$. Since both E_A and $E_{S'}$ are conditional expectations (projections), if $\operatorname{im} E_A = S'$, then they project onto the same space, hence the map has to agree. □

Corollary 6.2.4.

$$E_A = E_{S'}$$

Remark. It is always true that $S' \subseteq \operatorname{im} E_A$. Hence, to show the two projections agree, it is sufficient to show that $\operatorname{im} E_A \subseteq S'$.

Remark. We will see later in the case where dilation theorem is needed, E_A and $E_{S'}$ ease to coincide.

The mixing time of a given semigroup will play a big role in establishing concrete lower bound estimate.

mixing_time **Definition 6.2.5.** *Fix a semigroup $(T_t)_{t\geq 0}$, we define the **mixing time** as*

$$t_{\text{mix}}(\varepsilon) = \inf\left\{t : \|T_t - E_{S'}\|_\diamond < \varepsilon\right\}.$$

Remark. It is commonly assumed $\varepsilon = \frac{1}{2}$. Hence, for the rest of the note, unless otherwise stated, $t_{\text{mix}} = t_{\text{mix}}(\frac{1}{2})$. One can also replace it by a different number and a similar calculation holds.

Before proceeding to showing the lower bound, we will first check that the complete Lipschitz complexity indeed satisfies the axiom.

Theorem 6.2.6.

$$C^{\text{cb}}_{S_A}(\text{id}) = 0$$

Proof. By definition,

$$C^{\text{cb}}_{S_A}(\text{id}) = \|\text{id} - \text{id}\|_{\text{cb}} = \|0\|_{\text{cb}} = 0$$

□

Theorem 6.2.7.

$$C^{\text{cb}}_{S_A}(\Phi\Psi) \leq C^{\text{cb}}_{S_A}(\Phi) + C^{\text{cb}}_{S_A}(\Psi)$$

Proof.

$$\begin{aligned}
C^{\text{cb}}_{S_A}(\Phi\Psi) &= \|\Psi^*\Phi^* - \text{id}\|_{\text{cb}} \\
&= \|\Psi^*\Phi^* - \Psi^* + \Psi^* - \text{id}\|_{\text{cb}} \\
&= \|\Psi^*(\Phi^* - \text{id}) + \Psi^* - \text{id}\|_{\text{cb}} \\
&= \|\Psi^*(\Phi^* - \text{id})\|_{\text{cb}} + \|\Psi^* - \text{id}\|_{\text{cb}} \\
&= \|\Psi^*\|_{\text{cb}} \|\Phi^* - \text{id}\|_{\text{cb}} + \|\Psi^* - \text{id}\|_{\text{cb}} \\
&\leq \|\Phi^* - \text{id}\|_{\text{cb}} + \|\Psi^* - \text{id}\|_{\text{cb}} \\
&= C^{\text{cb}}_{S_A}(\Phi) + C^{\text{cb}}_{S_A}(\Psi)
\end{aligned}$$

□

Theorem 6.2.8.

$$C_{S_A}^{\mathrm{cb}}(\sum_i p_i\Phi_i) \leq \sum_i p_i C_{S_A}^{\mathrm{cb}}(\Phi_i)$$

Proof.

$$\begin{aligned}
C_{S_A}^{\mathrm{cb}}(\sum_i p_i\Phi_i) &= \left\|\sum_i p_i\Phi_i^* - \mathrm{id}\right\|_{\mathrm{cb}} \\
&= \left\|\sum_i p_i\Phi_i^* - \sum_i p_i\,\mathrm{id}\right\|_{\mathrm{cb}} \\
&= \left\|\sum_i p_i(\Phi_i^* - \mathrm{id})\right\|_{\mathrm{cb}} \\
&= \sum_i p_i \left\|\Phi_i^* - \mathrm{id}\right\|_{\mathrm{cb}} \\
&= \sum_i p_i C_{S_A}^{\mathrm{cb}}(\Phi_i)
\end{aligned}$$

□

Theorem 6.2.9.

$$C_{S_A}^{\mathrm{cb}}(\Phi\otimes\Psi) \leq C_{S_A}^{\mathrm{cb}}(\Phi) + C_{S_A}^{\mathrm{cb}}(\Psi)$$

Proof.

$$\begin{aligned}
C_{S_A}^{\mathrm{cb}}(\Phi\otimes\Psi) &= \left\|\Phi^*\otimes\Psi^* - \mathrm{id}\otimes\mathrm{id}\right\|_{\mathrm{cb}} \\
&= \left\|\Phi^*\otimes\Psi^* - \mathrm{id}\otimes\Psi^* + \mathrm{id}\otimes\Psi^* - \mathrm{id}\otimes\mathrm{id}\right\|_{\mathrm{cb}} \\
&\leq \left\|\Phi^*\otimes\Psi^* - \mathrm{id}\otimes\Psi^*\right\|_{\mathrm{cb}} + \left\|\mathrm{id}\otimes\Psi^* - \mathrm{id}\otimes\mathrm{id}\right\|_{\mathrm{cb}} \\
&= \left\|(\Phi^*\otimes\mathrm{id} - \mathrm{id}\otimes\mathrm{id})(\mathrm{id}\otimes\Psi^*)\right\|_{\mathrm{cb}} + \left\|\mathrm{id}\otimes\Psi^* - \mathrm{id}\otimes\mathrm{id}\right\|_{\mathrm{cb}} \\
&\leq \left\|(\Phi^*\otimes\mathrm{id} - \mathrm{id}\otimes\mathrm{id})\right\|_{\mathrm{cb}} \left\|\mathrm{id}\otimes\Psi^*\right\|_{\mathrm{cb}} + \left\|\mathrm{id}\otimes\Psi^* - \mathrm{id}\otimes\mathrm{id}\right\|_{\mathrm{cb}} \\
&\leq \left\|(\Phi^*\otimes\mathrm{id} - \mathrm{id}\otimes\mathrm{id})\right\|_{\mathrm{cb}} + \left\|\mathrm{id}\otimes\Psi^* - \mathrm{id}\otimes\mathrm{id}\right\|_{\mathrm{cb}} \\
&= \left\|\Phi^* - \mathrm{id}\right\|_{\mathrm{cb}} + \left\|\Psi^* - \mathrm{id}\right\|_{\mathrm{cb}} \\
&\leq C_{S_A}^{\mathrm{cb}}(\Phi) + C_{S_A}^{\mathrm{cb}}(\Psi)
\end{aligned}$$

□

cont_bound **Theorem 6.2.10.** *Suppose* $E_{S'}\Phi = E_{S'} = E_{S'}\Psi$,

$$|C_{S_A}^{\mathrm{cb}}(\Phi) - C_{S_A}^{\mathrm{cb}}(\Psi)| \leq \kappa(S)\left\|\Phi - \Psi\right\|_\diamond$$

Proof.

$$\begin{aligned}
C_{S_A}^{\mathrm{cb}}(\Phi) &= \|\Phi^* - \mathrm{id}\|_{\mathrm{cb}} \\
&= \|\Phi^* - \Psi^* + \Psi^* - \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|\Phi^* - \Psi^*\|_{\mathrm{cb}} + \|\Psi^* - \mathrm{id}\|_{\mathrm{cb}} \\
&= \|\Phi^* - \Psi^* - E_{S'}^* + E_{S'}^*\|_{\mathrm{cb}} + \|\Psi^* - \mathrm{id}\|_{\mathrm{cb}} \\
&= \|\Phi^* - \Psi^* - \Phi^* E_{S'}^* + \Psi^* E_{S'}^*\|_{\mathrm{cb}} + \|\Psi^* - \mathrm{id}\|_{\mathrm{cb}} \\
&= \|(\Phi^* - \Psi^*)(\mathrm{id} - E_{S'}^*)\|_{\mathrm{cb}} + \|\Psi^* - \mathrm{id}\|_{\mathrm{cb}} \\
&= \|\Phi^* - \Psi^*\| \, \|\mathrm{id} - E_{S'}^*\|_{\mathrm{cb}} + \|\Psi^* - \mathrm{id}\|_{\mathrm{cb}} \\
&= C_{S_A}^{\mathrm{cb}}(E_{S'}) \, \|\Phi^* - \Psi^*\|_{\mathrm{cb}} + C_{S_A}^{\mathrm{cb}}(\Psi) \\
&= C_{S_A}^{\mathrm{cb}}(E_{S'}) \, \|\Phi - \Psi\|_{\diamond} + C_{S_A}^{\mathrm{cb}}(\Psi)
\end{aligned}$$

□

Theorem 6.2.11.

$$C_{S_A}^{\mathrm{cb}}(\mathrm{ad}_u) < D$$

Proof. We can write $u = e^{-itH}$ for some Hamiltonian H. Let $f_t(x) = e^{-itH} x e^{itH}$. Its derivative is

$$\begin{aligned}
\frac{d}{dt} f_t(x) &= \frac{d}{dt} e^{-itH} x e^{itH} \\
&= e^{-itH}(-iH) x e^{itH} + e^{-itH} x (iH) e^{itH} \\
&= i e^{-itH} [H, x] e^{itH}
\end{aligned}$$

We have

$$\int_0^t i e^{-isH} [H, x] e^{isH} \, ds = f_t(x) - x$$

by the fundamental theorem of calculus. Taking cb-norm on both sides,

$$\|f_t(x) - x\|_{\mathrm{cb}} \leq |t - 0| \left\| e^{-isH} [H, x] e^{isH} \right\| \leq t \, \|[H, x]\|_{\infty} \leq t |||[H, x]|||_{\mathrm{Lip}}.$$

Hence,

$$\frac{\|f_t(x) - x\|_{\mathrm{cb}}}{|||[H, x]|||_{\mathrm{Lip}}} \leq t \leq 2\pi.$$

Whence,

$$C_{S_A}^{\mathrm{cb}}(\mathrm{ad}_u) < 2\pi.$$

□

6.3 Lower Estimate for Maximal Simulation Cost

To realize the lower bound for $M_{\alpha,\beta}$, we want to connect the complexity of the conditional expectation $E_{S'}$ to $M_{\alpha,\beta}$. Then, by approximating $E_{S'}$, we will be able to obtain a lower bound for $M_{\alpha,\beta}$. We first establish a key lemma.

key_ineq **Lemma 6.3.1.**

$$C_{S_A}^{\mathrm{cb}}(\Phi) \leq D l_{\delta}(\Phi) + \kappa(S)\delta$$

Proof. Let $\Psi = \mathbb{E}_\mu \operatorname{ad}_{u_1} \dots \operatorname{ad}_{u_m}$. Apply 6.2.10, we have

$$\begin{aligned}
C^{\mathrm{cb}}_{S_A}(\Phi) &\le C^{\mathrm{cb}}_{S_A}(\Psi) + \kappa(S) \left\|\Phi - \Psi\right\|_\diamond \\
&\le C^{\mathrm{cb}}_{S_A}(\mathbb{E}_\mu \operatorname{ad}_{u_1} \dots \operatorname{ad}_{u_m}) + \kappa(S) \left\|\Phi - \mathbb{E}_\mu \operatorname{ad}_{u_1} \dots \operatorname{ad}_{u_m}\right\|_\diamond \\
&\le C^{\mathrm{cb}}_{S_A}(\mathbb{E}_\mu \operatorname{ad}_{u_1} \dots \operatorname{ad}_{u_m}) + \kappa(S)\delta \\
&\le \mathbb{E}_\mu C^{\mathrm{cb}}_{S_A}(\operatorname{ad}_{u_1} \dots \operatorname{ad}_{u_m}) + \kappa(S)\delta \\
&\le \mathbb{E}_\mu \left(C^{\mathrm{cb}}_{S_A}(\operatorname{ad}_{u_1}) + \dots + C^{\mathrm{cb}}_{S_A}(\operatorname{ad}_{u_m})\right) + \kappa(S)\delta \\
&\le \mathbb{E}_\mu (D + \dots + D) + \kappa(S)\delta \\
&\le Dm + \kappa(S)\delta
\end{aligned}$$

Now take infimum over all m, the claim follows. □

To relate $\kappa(S)$ with $M_{\alpha,\beta}$, we need the following:

kappa_tmix

Lemma 6.3.2.

$$\frac{1}{2}\kappa(S) \le C^{\mathrm{cb}}_{S_A}(T_{t_{\mathrm{mix}}})$$

Proof. Recall that $\left\|T_{t_{\mathrm{mix}}} - E_{S'}\right\|_\diamond \le \frac{1}{2}$

$$\begin{aligned}
C^{\mathrm{cb}}_{S_A}(E_{S'}) &\le C^{\mathrm{cb}}_{S_A}(T_{t_{\mathrm{mix}}}) + \kappa(S) \left\|T_{t_{\mathrm{mix}}} - E_{S'}\right\|_\diamond \\
&\le C^{\mathrm{cb}}_{S_A}(T_{t_{\mathrm{mix}}}) + \frac{1}{2}\kappa(S)
\end{aligned}$$

□

The following theorem gives the general lower estimate for the maximal simulation length.

low_bound_cha

Theorem 6.3.3.

$$C^{\mathrm{cb}}_{S_A}(T_t) \le 2(DM_{\alpha,\beta})^{\frac{\beta-1}{\beta}} \max\left\{(DM_{\alpha,\beta})^{\frac{1}{\beta}}, 2(\alpha\kappa(S))^{\frac{1}{\beta}} t\right\}$$

Proof. By the property of semigroup and subadditivity of Lipschitz norm,

$$C^{\mathrm{cb}}_{S_A}(T_t) = C^{\mathrm{cb}}_{S_A}(T^n_{\frac{t}{n}}) \le nC^{\mathrm{cb}}_{S_A}(T_{\frac{t}{n}}).$$

Apply 6.3.1 with $\delta = \alpha t^\beta$

$$\begin{aligned}
nC^{\mathrm{cb}}_{S_A}(T_{\frac{t}{n}}) &\le n\left(Dl_\delta(T_{\frac{t}{n}}) + \kappa(S)\alpha\left(\frac{t}{n}\right)^\beta\right) \\
&\le nDl_\delta(T_{\frac{t}{n}}) + n\kappa(S)\alpha\left(\frac{t}{n}\right)^\beta \\
&= nDl_\delta(T_{\frac{t}{n}}) + n^{1-\beta}\kappa(S)\alpha t^\beta \\
&\le nDM_{\alpha,\beta} + n^{1-\beta}\alpha t^\beta.
\end{aligned}$$

Notice that the first term is increasing and the second one is decreasing. The two terms are equal when

$$\begin{aligned}
n_0 DM_{\alpha,\beta} &= n_0^{1-\beta}\kappa(S)\alpha t^\beta \\
n_0^\beta &= \left(\frac{\alpha\kappa(S)}{DM_{\alpha,\beta}}\right) t^\beta.
\end{aligned}$$

Substitute $n_0 = \left(\frac{\alpha\kappa(S)}{DM_{\alpha,\beta}}\right)^{\frac{1}{\beta}} t$,

$$\begin{aligned} C^{\mathrm{cb}}_{S_A}(T_t) &\leq 2\left(\frac{\alpha\kappa(S)}{DM_{\alpha,\beta}}\right)^{\frac{1}{\beta}} t \cdot DM_{\alpha,\beta} \\ &= 2(DM_{\alpha,\beta})^{1-\frac{1}{\beta}} \cdot (\alpha\kappa(S))^{\frac{1}{\beta}} t \end{aligned}$$

This gives the second term in the maximum and we are done. □

In most context, the growth of $\kappa(S)$ is easier to calculate. Hence, we have the following corollary.

lower_bound **Corollary 6.3.4.**

$$\kappa(S) \leq DM_{\alpha,\beta} \max\left\{4, 8^{\frac{\beta}{\beta-1}} \alpha^{\frac{1}{\beta-1}} t_{\mathrm{mix}}^{\frac{\beta}{\beta-1}}\right\}$$

Proof. Combine 6.3.2 and 6.3.3, we have that

$$\frac{1}{2}\kappa(S) \leq C^{\mathrm{cb}}_{S_A}(T_{t_{\mathrm{mix}}}) \leq 2(DM_{\alpha,\beta})^{\frac{\beta-1}{\beta}} \max\left\{(DM_{\alpha,\beta})^{\frac{1}{\beta}}, 2(\alpha\kappa(S))^{\frac{1}{\beta}} t\right\}$$

That means, either

$$\kappa(S) \leq 4DM_{\alpha,\beta}$$

or

$$\begin{aligned} \kappa(S) &\leq 8(DM_{\alpha,\beta})^{\frac{\beta-1}{\beta}} (\alpha\kappa(S))^{\frac{1}{\beta}} t_{\mathrm{mix}} \\ \kappa(S)^{\frac{\beta-1}{\beta}} &\leq 8(DM_{\alpha,\beta})^{\frac{\beta-1}{\beta}} \alpha^{\frac{1}{\beta}} t_{\mathrm{mix}} \\ \kappa(S) &\leq 8^{\frac{\beta}{\beta-1}} \alpha^{\frac{1}{\beta-1}} t_{\mathrm{mix}}^{\frac{\beta}{\beta-1}} \cdot DM_{\alpha,\beta}. \end{aligned}$$

□

In general, it is challenging to calculate $\kappa(S)$ explicitly. However, in many examples, it is still possible to bound it. In particular, several examples shows $\kappa(S) \sim n$ grows as the number of qubits. We will show examples in which such growth is realized.

ST_formula

7 Suzuki-Trotter Formula

7.1 Matrix Exponentials

Recall that exponential maps are formally defined in terms of the power series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Most of the calculation in regard to the elementary exponential relies on the following fact:

Theorem 7.1.1. *For $x, y \in \mathbb{R}$, we have the identity that*

$$e^{x+y} = e^x e^y.$$

Proof.

$$e^{x+y} = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{1}{k!} \frac{k!}{m!(k-m)!} x^m y^{k-m} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{y^n}{n!} = e^x e^y.$$

□

However, if we were to do exponential with matrices, the above identity fails since matrix multiplication is not commutative in general. That means, the difference is:

$$(x+y)^2 = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2,$$

but

$$(A+B)^2 = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2.$$

In fact, it is easy to see there is no term in $e^A e^B$ that produces BA.

Hence, it is not true that $e^{A+B} - e^A e^B = 0$. However, even in the case of Hamiltonian simulation, we would like to ask what is the error of approximating H by smaller pieces. Consider the simplest case where $H = X + Z$ we want to determine how large is

$$e^{itH} - e^{itX} e^{itY}.$$

Furthermore, we would also like to answer are there better approximation of e^{itH} up to some error of degree β, or $\mathcal{O}(t^\beta)$.

Let us begin by reviewing the tools we have for computing exponential of nice matrices. Recall that if Λ is a diagonal matrix with $\lambda_1, \ldots, \lambda_n$ as the entries, then

$$e^{\Lambda} = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix}.$$

If we are given a Hermitian matrix A, by spectral theorem, $A = U\Lambda U^*$ and thus

$$e^A = e^{U\Lambda U^*} = U e^{\Lambda} U^*.$$

Further, for any matrices where $AB = BA$, then

$$e^{A+B} = e^A e^B.$$

More generally, if $A_1, \ldots, A_n$ all commutes, i.e. $A_i A_j = A_j A_i$ for all i, j, then

$$e^{\sum_{j=1}^n A_j} = \prod_{j=1}^n e^{A_j}.$$

Example. Let $A_j = X_j$, performing Pauli-X gate on the j-th component of a string of tensors products. Then $X_i X_j = X_j X_i$ for all i, j. Thus

$$e^{\sum_{j=1}^n X_j} = \prod_{j=1}^n e^{X_j}.$$

Similar result holds if $A_j = Y_j$ or $A_j = Z_j$.

We will introduce two more general formula one can use for matrix exponential: Lie product formula and Baker–Campbell–Hausdorff formula [Hall15].

Theorem 7.1.2 (Lie product formula)**.** *For (not necessarily commuting) matrices A and B,*

$$e^{A+B} = \lim_{k\to\infty} \left(e^{\frac{1}{k}A} e^{\frac{1}{k}B}\right)^k.$$

Using a large finite k to approximate the above is basis of the Suzuki-Trotter expansion, which we will discuss in detail.

There is another useful equation for sufficiently small matrices.

Theorem 7.1.3 (Baker-Campbell-Hausdorff Formula)**.** *Let A and B be matrices.*

$$e^A e^B = e^C,$$

where

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] + \ldots,$$

where the remaining terms are all iterated commutators with A and B.

7.2 Suzuki-Trotter Product Formula

Let $H = \sum_{i=1}^\Gamma H_i$ be the sum of time independent operators such that the evolution generated by H is e^{tH}. Note that there is no requirement on H_i, i.e. not necessarily self-adjoint. [Suz91] provides a concrete method to approximate e^{tH} by products of e^{tH_i} up to an arbitrary degree. In general, given a degree d approximation,

$$e^{tH} = C_p(t) + \mathcal{O}(t^{p+1}).$$

The first and second order approximation is given explicitly.

$$C_1(t) = e^{tH_\Gamma} \cdots e^{tH_1}$$
$$C_2(t) = e^{\frac{t}{2}H_1} \cdots e^{\frac{t}{2}H_\Gamma} e^{\frac{t}{2}H_\Gamma} \cdots e^{\frac{t}{2}H_1}.$$

Higher-order Suzuki formulae are defined recursively by

$$C_{2k}(t) = C_{2k-2}^2(u_k t) C_{2k-2}((1-4u_k)t) C_{2k-2}^2(u_k t).$$

where $u_k = \frac{1}{4-4^{\frac{1}{2k-1}}}$, and $k \geq 2$. Note that when $k = 2$, $u_k \approx 0.4144$. Furthermore as $k \to \infty$, $u_k \to \frac{1}{3}$. This means $1 - 4u_k < 0$ for all k.

Having negative coefficient is troublesome for us since their norm is unbounded. Thus, a natural direction of investigation is to look for other formula that do have a positive decomposition. However, Suzuki proved something much stronger:

non_positive_approx

Theorem 7.2.1 ([Suz91], Theorem 3)**.** *For non-commutating operator A and B, there exists no decomposition of the form*

$$e^{t(A+B)} = e^{t_1 A} e^{t_2 B} e^{t_3 A} e^{t_4 B} \dots e^{t_M A} + \mathcal{O}(t^{p+1}),$$

with all t_j positive for finite M when $p > 3$.

_non_positive_approx

Corollary 7.2.2. *For $\Gamma \geq 2$, there exists no real positive decomposition*

$$\exp\left(t \sum_{\gamma=1}^{\Gamma} H_\gamma\right) = e^{t_1 H_1} e^{t_2 H_2} \dots e^{t_\Gamma H_\Gamma} \cdots + \mathcal{O}(t^{p+1}),$$

for $p \geq 3$ and for a finite number of products.

The theorem says it is impossible to have a positive decomposition regardless of the formula. That is, the above recursive formula, or any other product formula, will necessarily have negative coefficient somewhere for any higher order approximation.

Remark. Having no positive decomposition means some of the terms in the product is nor contractive with respect to the diamond norm. It is questionable if the negative term is a valid channel, i.e. being CP.

7.3 Childs Norm Estimate

Let

$$L = \sum_{j=1}^{n} L_{A_{1_j}} + L_{A_{2_j}} + \cdots + L_{A_{\Gamma_j}} = \sum_{j=1}^{n} L_j$$

be the Lindbladian, where

$$A_{\gamma_j} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes A_\gamma \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

for any operator A_{k_j}.

Theorem 7.3.1. *Let C_p be the degree p approximation, i.e. $e^{-tL} = C_p + \mathcal{O}(t^{p+1})$,*

$$\left\| e^{-tL_j} - C_p(t) \right\|_\diamond \leq \frac{2\left(\frac{2}{5}\sqrt{5}^p\right)^{p+1} \left(\sum_{\gamma=1}^{\Gamma} \left\| L_{A_{\gamma_j}} \right\|_\diamond\right)^{p+1}}{(p+1)!} \cdot t^{p+1}$$

Proof. To obtain the upper bound for the diamond norm estimate, we apply Lemma 6 in [CST21] [CST+21].

$$
\begin{aligned}
&\left\|e^{-tL_j} - C_p(t)\right\|_\diamond \\
&\leq \frac{t^{p+1}}{(p+1)!}\left[\left(\Upsilon_p \sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^{p+1} e^{-t\Upsilon_p\sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond} + \left(\sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^{p+1} e^{-t\sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond}\right] \\
&\leq \frac{t^{p+1}}{(p+1)!}\left[\left(\Upsilon_p \sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^{p+1} e^{-t\Upsilon_p\sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond} + \left(\Upsilon_p \sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^{p+1} e^{-t\sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond}\right] \\
&\leq \frac{t^{p+1}}{(p+1)!}\left[2\left(\Upsilon_p \sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^{p+1} e^{-t\sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond}\right] \\
&\leq \frac{t^{p+1}}{(p+1)!}\left[2\left(\Upsilon_p \sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^{p+1} \left\|e^{-t\sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond}\right\|_\diamond\right] \\
&\leq \frac{t^{p+1}}{(p+1)!}\left[2\left(\Upsilon_p \sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^{p+1}\right] \\
&\leq \frac{2\left(\Upsilon_p \sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^{p+1}}{(p+1)!} t^{p+1}
\end{aligned}
$$

Since $\Upsilon_p = \frac{2}{5}\sqrt{5}^p$,

$$
\left\|e^{-tL_j} - C_p(t)\right\|_\diamond \leq \frac{2\left(\frac{2}{5}\sqrt{5}^p\right)^{p+1}\left(\sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^{p+1}}{(p+1)!} \cdot t^{p+1}
$$

□

However, recall that 7.2.2 says that there is no positive decomposition for $p \geq 3$. Hence, the above bound only works in the case where $p = 1$ or $p = 2$, since, though $e^{-tL_{A_{k_j}}}$ is contractive, $e^{tL_{A_{k_j}}}$ is not, and thus, there is no way to bound such terms. Whence, we can write the only two cases explicitly:

beta2 **Corollary 7.3.2.** *When $p = 1$, i.e. $\beta = 2$,*

$$
\left\|e^{-tL} - C_1(t)\right\|_\diamond \leq \frac{4}{5}\left(\sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^2 t^2.
$$

beta3 **Corollary 7.3.3.** *When $p = 2$, i.e. $\beta = 3$,*

$$
\left\|e^{-tL} - C_2(t)\right\|_\diamond \leq \frac{8\sqrt{5}}{75}\left(\sum_{\gamma=1}^{\Gamma}\left\|L_{A_{\gamma_j}}\right\|_\diamond\right)^3 t^3.
$$

LCU_model

8 Linear Combination of Unitaries Model

8.1 The Magical Formula for Lindbladian Generated by Single Hamiltonian

Recall that the model for Lindbladian in this paper is given by

$$L_a(\rho) = a^*a\rho + \rho a^*a - 2a\rho a^*.$$

We begin by considering the simplest case where the Lindbladian is generated by a self-adjoint operator $H = H^*$. Thus,

$$L_H(\rho) = H^2\rho + \rho H^2 - 2H\rho H.$$

It is possible to write out what the channel induced by L_H is explicitly.

Theorem 8.1.1. *Let $H = H^* \in \mathbb{B}(H)$. Then*

$$e^{-tL_H}(x) = \mathbb{E}_\mu \operatorname{ad}_{e^{i\sqrt{2t}gH}}(x))$$

for some standard Gaussian variable $g = N(0,1)$.

Proof. Since H is self-adjoint, we can write $H = \sum_{j=1}^d \lambda_j e_j$ for eigenvalues $\{\lambda_1, \ldots, \lambda_d\}$. Consider $x \in \mathbb{B}(H)$, written as $x = \sum_{r,s=1}^d e_r x e_s$, and $I = \sum_{j=1}^d e_j$. Now, compute

$$\begin{aligned}
L_H(x) &= H^2x + xH^2 - 2HxH \\
&= \left(\sum_{j=1}^d \lambda_j^2 e_j\right)\left(\sum_{r,s=1}^d e_r x e_s\right) + \left(\sum_{r,s=1}^d e_r x e_s\right)\left(\sum_{j=1}^d \lambda_j^2 e_j\right) - 2\left(\sum_{j=1}^d \lambda_j e_j\right)\left(\sum_{r,s=1}^d e_r x e_s\right)\left(\sum_{j=1}^d \lambda_j e_j\right) \\
&= \left(\sum_{r=1}^d \lambda_r^2 e_r x\right) + \left(\sum_{s=1}^d \lambda_s^2 x e_s\right) - 2\left(\sum_{r,s=1}^d \lambda_r\lambda_s e_r x e_s\right) \\
&= \left(\sum_{r=1}^d \lambda_r^2 e_r x\right) I + I\left(\sum_{s=1}^d \lambda_s^2 x e_s\right) - 2\left(\sum_{r,s=1}^d \lambda_r\lambda_s e_r x e_s\right) \\
&= \left(\sum_{r=1}^d \lambda_r^2 e_r x\right)\left(\sum_{s=1}^d e_s\right) + \left(\sum_{r=1}^d e_r\right)\left(\sum_{s=1}^d \lambda_s^2 x e_s\right) - 2\left(\sum_{r,s=1}^d \lambda_r\lambda_s e_r x e_s\right) \\
&= \left(\sum_{r,s=1}^d \lambda_r^2 e_r x e_s\right) + \left(\sum_{r,s=1}^d \lambda_s^2 e_r x e_s\right) - 2\left(\sum_{r,s=1}^d \lambda_r\lambda_s e_r x e_s\right) \\
&= \lambda_r^2 e_r x e_s + \lambda_s^2 e_r x e_s - \sum_{r,s=1}^d 2\lambda_r\lambda_s e_r x e_s \\
&= \sum_{r,s=1}^d \lambda_r^2 e_r x e_s + \lambda_s^2 e_r x e_s - 2\lambda_r\lambda_s e_r x e_s \\
&= \sum_{r,s=1}^d -(\lambda_r - \lambda_s)^2 e_r x e_s.
\end{aligned}$$

Now we see clearly that L_H acts as a Schur multiplier. We need to exponentiate this Schur multiplier, doing so, we see that we just need to exponentiate the individual components of the Schur multiplier, i.e.

$$e^{-tL_H}(x) = \sum_{r,s=1}^{d} e^{-t(\lambda_r-\lambda_s)^2} e_r x e_s.$$

Recall for the $g = N(0,1)$ standard Gaussian variable $g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ that

$$\mathbb{E}(e^{iag}) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-iay} e^{-\frac{y^2}{2}}\, dy = e^{-\frac{a^2}{2}}.$$

Let $a_{rs}(t) = \sqrt{2t}(\lambda_r - \lambda_s)$. Now, we see

$$\mathbb{E}(e^{ia_{rs}(t)g}) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-i\sqrt{2t}(\lambda_r-\lambda_s)y} e^{-\frac{y^2}{2}}\, dy = e^{-t(\lambda_r-\lambda_s)^2}.$$

Extending the above to matrices and Schur multiplication, the left hand side becomes

$$\mathbb{E}(e^{ia_{rs}(t)g} e_r x e_s) = \mathbb{E}(e^{i\sqrt{2t}(\lambda_r-\lambda_s)g} e_r x e_s) = \mathbb{E}(e^{i\sqrt{2t}\lambda_r g} e_r x e_s e^{-i\sqrt{2t}\lambda_r g}).$$

Summing everything up yields

$$\sum_{r,s=1}^{d} \mathbb{E}(e^{i\sqrt{2t}\lambda_r g} e_r x e_s e^{-i\sqrt{2t}\lambda_r g}) = \mathbb{E}(\mathrm{ad}_{e^{i\sqrt{2t}gH}}(x))$$

where the ad map is acting as a Schur multiplier inherited from e^{-tL_H}. □

Hence, if our Lindbladian is represented by a single local Lindbladian generated by a self-adjoint operator, the preceding equation provides a general method to express it as a gate. We will now extend our discussion to consider channels that are somewhat more complicated.

8.2 General Setting for Linear Combination of Unitary

In the LCU model, we consider Lindbladians as linear combinations of local Lindbladians, each generated by specific Hamiltonians. Let Γ be a finite index set,

$$L = \sum_{j=1}^{n} L_{H_{1_j}} + \cdots + L_{H_{\Gamma_j}}$$

where

$$H_{\gamma_j} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underbrace{H_\gamma}_{j\text{-th component}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}.$$

Since our framework is resource-dependent, we need to define what the resource set is. Usually, we consider

$$S = \{H_1, \ldots, H_\Gamma\} \quad \text{and} \quad \mathcal{U} = \left\{e^{itH_\gamma} : H_\gamma \in S, 0 \le t \le 2\pi\right\}.$$

We can also consider a bigger resource set given by

$$\widehat{S} = \mathrm{Alg}\left\{H_1, \ldots, H_\Gamma\right\} \quad \text{and} \quad \mathcal{U} = \left\{e^{itH_\gamma} : H_\gamma \in \widehat{S}, 0 \le t \le 2\pi\right\}.$$

In the LCU model, our upper estimate is derived using the Suzuki-Trotter formula. Therefore, employing $\widehat{S}$ does not alter the results. However, we will later demonstrate that such a method can provide advantages for more general error models.

8.3 Lindbladian Generated by Pauli Matrices

Consider the semigroup $(e^{-tL})_{t\ge 0}$ where the Lindbladian given by

$$L = \sum_{j=1}^{n} L_{X_j} + L_{Y_j} = \sum_{j=1}^{n} L_j.$$

Our resource set will be

$$S = \{X_j, Y_j\} \quad \text{and} \quad \mathcal{U} = \left\{e^{itX_j} : 0 \le t \le 2\pi\right\} \cup \left\{e^{itY_j} : 0 \le t \le 2\pi\right\}.$$

We may also consider

$$\widehat{S} = \mathrm{Alg}\left\{X_j, Y_j\right\} = \{\mathbb{1}_j, X_j, Y_j, Z_j\} \quad \text{and} \quad \mathcal{U} = \left\{e^{itH_j} : H = \mathbb{1}, X, Y, Z \ 0 \le t \le 2\pi\right\},$$

but as we discussed, it yields no difference in our calculation.

Since the Lindbladian only acts on a single qubit, L_j's commutes. Thus, it is sufficient to consider the simulation for L_j.

To obtain the lower bound, we will use 6.3.4. However, that means we need to approximate $\kappa(S)$ and calculate the mixing time t_{mix}.

$$T_t \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \frac{1+e^{-8t}}{2}\rho_{11} + \frac{1-e^{-8t}}{2}\rho_{22} & e^{-2t}\rho_{12} \\ e^{-2t}\rho_{21} & \frac{1-e^{-8t}}{2}\rho_{11} + \frac{1+e^{-8t}}{2}\rho_{22} \end{bmatrix}.$$

The conditional expectation $E_{S'}$ is obtained by taking $t \to \infty$ and hence

$$E_{S'} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \rho_{11}+\rho_{22} & 0 \\ 0 & \rho_{11}+\rho_{22} \end{bmatrix}.$$

Recall that mixing time is defined in 6.2.5 as

$$t_{\mathrm{mix}} = \inf\left\{t : \|T_t - E_{S'}\|_\diamond < \frac{1}{2}\right\}.$$

To estimate $\|T_t - E_{S'}\|_\diamond$,

$$\begin{aligned}
\|T_t - E_{S'}\|_\diamond &= \left\| \begin{bmatrix} \frac{e^{-8t}}{2}\rho_{11} + \frac{-e^{-8t}}{2}\rho_{22} & e^{-2t}\rho_{12} \\ e^{-2t}\rho_{21} & \frac{-e^{-8t}}{2}\rho_{11} + \frac{e^{-8t}}{2}\rho_{22} \end{bmatrix} \right\|_\diamond \\
&= \left\| e^{-2t} \begin{bmatrix} \frac{e^{-6t}}{2}\rho_{11} + \frac{-e^{-6t}}{2}\rho_{22} & \rho_{12} \\ \rho_{21} & \frac{-e^{-6t}}{2}\rho_{11} + \frac{e^{-6t}}{2}\rho_{22} \end{bmatrix} \right\|_\diamond \\
&\le \left\| e^{-2t} \begin{bmatrix} \frac{1}{2}\rho_{11} + \frac{1}{2}\rho_{22} & \rho_{12} \\ \rho_{21} & \frac{1}{2}\rho_{11} + \frac{1}{2}\rho_{22} \end{bmatrix} \right\|_\diamond \\
&\le 4ne^{-2t} \\
&\le \frac{1}{2}
\end{aligned}$$

Taking the infimum over the time t of which the above condition is satisfied, the mixing time is

$$t_{\text{mix}} \sim \frac{\ln 4n + \ln 2}{2}.$$

Applying 6.3.4 with the knowledge that $n \leq \kappa(S)$,

$$\frac{n}{8^{\frac{\beta}{\beta-1}} D\alpha^{\frac{1}{\beta-1}} t_{\text{mix}}^{\frac{\beta}{\beta-1}}} \leq M_{\alpha,\beta}.$$

To obtain the upper bound, since we can only bound the norm of the Sukuzi-Trotter formula for $\beta = 2$ and $\beta = 3$, we will apply 7.3.2 and 7.3.3. Since

$$\|L_X(\rho)\|_\diamond = \|2\rho - 2X\rho X\|_\diamond \leq 2\|\rho\|_\diamond + 2\|X\rho X\|_\diamond \leq 4$$

and

$$\|L_Y(\rho)\|_\diamond = \|2\rho - 2Y\rho Y\|_\diamond \leq 2\|\rho\|_\diamond + 2\|Y\rho Y\|_\diamond \leq 4$$

for any $\|\rho\|_\diamond \leq 1$. We get

$$\left\|e^{-tL_j} - C_1(t)\right\|_\diamond \leq \frac{256}{5}t^2$$

and

$$\left\|e^{-tL_j} - C_2(t)\right\|_\diamond \leq \frac{4096\sqrt{5}}{75}t^3.$$

That means

$$M_{\frac{256}{5}n,2} \leq 2n$$

and

$$M_{\frac{4096\sqrt{5}}{75}n,3} \leq 3n.$$

noisy_model

9 Noisy Model

9.1 Setup and Issues

In the LCU model, the generators for the Lindbladians are self-adjoint. In particular, we studied the example

$$L = \sum_{j=1}^{n} L_{X_j} + L_{Y_j}.$$

Since the generators X_j and Y_j are both self-adjoint, we can put them directly in the resource set and their exponential e^{itX_j} and e^{itY_j} are unitaries.

However, the naïve approach seize to work for more general generators. We have seen the example

$$L = \sum_{j=1}^{n} L_{a_j} \text{ where } a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since a here is not self-adjoint, its exponential will not be unitary, meaning it is not a valid gate that quantum computer can use.

9.2 Stinespring dilation

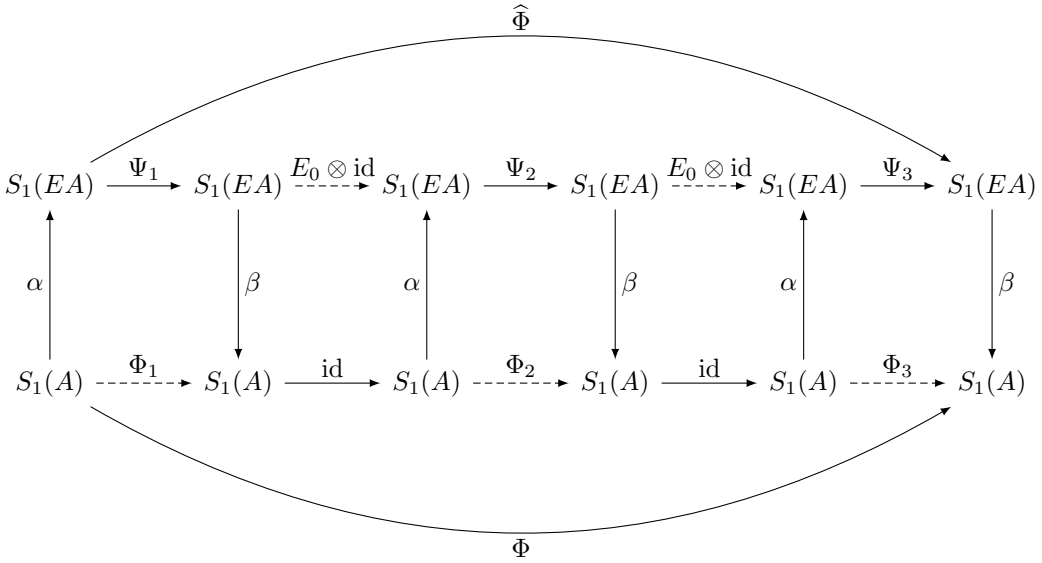
Thankfully, when we consider our system in a bigger space composite with some environment, it is possible to realize $e^{ita'}$ as a unitary operator. Hence, we can modify our approximation methods so that the quantum computer will run on bigger space $S_1(EA)$ and use the standard tools of state preparation and partial trace connect the two levels and to acquire the simulation cost on the principal system of interest. Via diagram the information of Stinespring dilation can be written as

$$\begin{array}{ccc} S_1(EA) & \xrightarrow{\Psi = \mathbb{E}_\mu \operatorname{ad}_{u_1} \dots \operatorname{ad}_{u_m}} & S_1(EA) \\ \Big\uparrow \alpha = |0\rangle\langle 0| \otimes - & & \Big\downarrow \beta = \operatorname{tr} \otimes \operatorname{id} \\ S_1(A) & \xrightarrow{\Phi} & S_1(A) \end{array}$$

Via this method, our new approximation takes the form of $\Psi = \beta\Psi\alpha$.

There are more effective ways of using the environment. Instead of doing the approximation all at once, we can divide it into stages $\Phi_1, \dots, \Phi_m$. This allows us to reset the system after several steps of approximation so that we do not need to carry the noise inserted in the environment as simulation goes. Such reset mechanism have the potential of reducing the cost of the simulation. For each stage, $\Phi_i = \beta\Psi_i\alpha$ where the length for Ψ_i is m_i, and the system gets reset and proceed to Φ_{i+1}.

Diagrammatically,



Here $E_0 \otimes \mathrm{id} = \alpha \cdot \mathrm{id} \cdot \beta = \alpha\beta$. More generally, we define $E_0 \otimes - = \alpha(-)\beta$.

We wanted to study the simulation cost at both levels. Looking only at the top row, the simulation cost is defined as

$$l_{S_{EA}}((E_0 \otimes \mathrm{id})\Psi_1(E_0 \otimes \mathrm{id})\Psi_2(E_0 \otimes \mathrm{id}) \dots (E_0 \otimes \mathrm{id})\Psi_m(E_0 \otimes \mathrm{id})) \leq \sum_{i=1}^{m} m_i + (m+1)\widetilde{C}^{\mathrm{cb}}_{S_{EA}}(E_0 \otimes \mathrm{id})$$

We will artificially define $l_{S_{EA}}(E_0) = \widetilde{C}^{\mathrm{cb}}_{S_{EA}}(E_0 \otimes \mathrm{id})$ here, but it can be placed with the real cost of applying $\alpha\beta$. Since our question investigate the approximation length at $S_1(A)$, we define the simulation cost at the lower level as

$$l_{S_A}(\beta\Psi_1\alpha\beta\Psi_2\alpha \dots \beta\Psi_m\alpha) = l_{S_A}(\Phi_1\Phi_2 \dots \Phi_m).$$

It is clear that the approximation $l_{S_{EA}}$ and l_{S_A} coincide up to a fixed number of initial β and terminal α.

9.3 New Complexity Measure

We rely on the appropriate notion of complexity on the semigroup to obtain the lower estimate. In particular, we need the complexity of the conditional expectation to lower bound $M_{\alpha,\beta}$.

To generalize the method we used to obtain lower bound for the LCU model to the noisy model, via the introduction before, we will need to consider the dilation of the principal system. Hence, we will need a complexity on $S_1(EA)$.

We will first collect the critical axioms that we used in the prior case. First recall that we define the conditional expectation as

$$E_A = \lim_t \frac{1}{t} \int_0^t T_s \, ds.$$

Note that it is not necessarily onto S', so hence $E_A \neq E_{S'}$ in general (it might be, but we do not know yet). We do have, however,

$$E_A|_{S'} = \mathrm{id}\,|_{S'}.$$

Recall that Lipschitz seminorm is defined as

$$|||f|||_S = \sup_{s\in S} \|[s,f]\|_\infty$$

for $f \in \mathbb{B}(H)$. With this, the complete complexity of a channel is defined as

$$C^{\mathrm{cb}}_{S_A}(\Phi) = \left\| \Phi^* - \mathrm{id} : (\mathbb{B}(A), |||\cdot|||_{\mathrm{Lip}}) \to (\mathbb{B}(A), \|\cdot\|_\infty) \right\|_{\mathrm{cb}}.$$

Since we need to use dilation theorem for the noisy model, we also need a complexity on the bigger space at $\mathbb{B}(EA)$ level:

$$\widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Phi) = \left\| E_0^* \otimes \Phi^* - E_0^* \otimes \mathrm{id} : (\mathbb{B}(EA), |||\cdot|||_{\mathrm{Lip}}) \to (\mathbb{B}(EA), \|\cdot\|_\infty) \right\|_{\mathrm{cb}}.$$

If we can verify the above axioms for $\widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Phi)$, then the lower bound estimate for LCU model will follow for the noisy model.

It is easy to verify $\widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\mathrm{id}) = 0$, additivity, and convexity. Thus, we will show the continuity bound and boundedness of the complexity for $\mathbb{E}_\mu \,\mathrm{ad}_{u_1} \dots \mathrm{ad}_{u_m}$.

To check that 6.3.3 and 6.3.4 holds for $\widetilde{C}^{\mathrm{cb}}_{S_{EA}}$, we first need to check the axioms of complexity. subadditivity, tensor subadditivity, and convexity follows easily from the definition and the same proof for $C^{\mathrm{cb}}_{S_A}$ holds. Hence, it left to show the continuity bound and the key inequality.

Theorem 9.3.1. *Suppose $E_A\Phi = E_A = E_A\Psi$,*

$$|\widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Phi) - \widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Psi)| \leq \widetilde{C}^{\mathrm{cb}}_{S_{EA}}(E_A)\,\|\Phi - \Psi\|_\diamond$$

Proof.

$$\begin{aligned}
\widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Phi) &= \|E_0^* \otimes \Phi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|E_0^* \otimes \Phi^* - E_0^* \otimes \Psi^* + E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|E_0^* \otimes (\Phi^* - \Psi^*) + E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|E_0^* \otimes (\Phi^* - \Psi^*)\|_{\mathrm{cb}} + \|E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|E_0^* \otimes (\Phi^* - \Psi^*) - E_0 \otimes (E_A^* - E_A^*)\|_{\mathrm{cb}} + \|E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|E_0^* \otimes (\Phi^* - \Psi^*) - E_0 \otimes (\Phi^* E_A^* - \Psi^* E_A^*)\|_{\mathrm{cb}} + \|E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|E_0^* \otimes (\Phi^* - \Psi^*) - E_0 \otimes (\Phi^* E_A^* - \Psi^* E_A^*)\|_{\mathrm{cb}} + \|E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|E_0^* \otimes (\Phi^* - \Psi^* - \Phi^* E_A^* + \Psi^* E_A^*)\|_{\mathrm{cb}} + \|E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|E_0^* \otimes (\Phi^* - \Psi^*)(\mathrm{id} - E_A^*)\|_{\mathrm{cb}} + \|E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|(\mathrm{id} \otimes (\Phi^* - \Psi^*))(E_0^* \otimes (\mathrm{id} - E_A^*))\|_{\mathrm{cb}} + \|E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|\mathrm{id} \otimes (\Phi^* - \Psi^*)\|_{\mathrm{cb}}\, \|E_0^* \otimes (\mathrm{id} - E_A^*)\|_{\mathrm{cb}} + \|E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|\Phi^* - \Psi^*\|_{\mathrm{cb}}\, \|E_0^* \otimes (\mathrm{id} - E_A^*)\|_{\mathrm{cb}} + \|E_0^* \otimes \Psi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \widetilde{C}^{\mathrm{cb}}_{S_{EA}}(E_A)\,\|\Phi^* - \Psi^*\|_{\mathrm{cb}} + \widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Psi) \\
&= \widetilde{C}^{\mathrm{cb}}_{S_{EA}}(E_A)\,\|\Phi - \Psi\|_\diamond + \widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Psi)
\end{aligned}$$

□

Next, we show the boundedness of commplexity for ad_u

Theorem 9.3.2.

$$\widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Phi) \leq K$$

Proof.

$$\begin{aligned}
\widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Phi) &= \|(E_0 \otimes \Phi)^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|(E_0 \otimes (\beta \,\mathrm{ad}_u\, \alpha))^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|(\alpha\beta \,\mathrm{ad}_u\, \alpha\beta)^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|(E_0^* \otimes \mathrm{id})\,\mathrm{ad}_{u^*}(E_0^* \otimes \mathrm{id}) - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|(E_0^* \otimes \mathrm{id})(\mathrm{ad}_{u^*}(E_0^* \otimes \mathrm{id}) - \mathrm{id} \otimes \mathrm{id})\|_{\mathrm{cb}} \\
&\leq \|E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \|\mathrm{ad}_{u^*}(E_0^* \otimes \mathrm{id}) - \mathrm{id} \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|\mathrm{ad}_{u^*}(E_0^* \otimes \mathrm{id}) - \mathrm{id} \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|\mathrm{ad}_{u^*}(E_0^* \otimes \mathrm{id}) - \mathrm{ad}_{u^*} + \mathrm{ad}_{u^*} - \mathrm{id} \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|\mathrm{ad}_{u^*}(E_0^* \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id}) + \mathrm{ad}_{u^*} - \mathrm{id} \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|\mathrm{ad}_{u^*}(E_0^* \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id})\|_{\mathrm{cb}} + \|\mathrm{ad}_{u^*} - \mathrm{id} \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|\mathrm{ad}_{u^*}\|_{\mathrm{cb}} \|(E_0^* \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id})\|_{\mathrm{cb}} + \|\mathrm{ad}_{u^*} - \mathrm{id} \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|(E_0^* \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id})\|_{\mathrm{cb}} + \|\mathrm{ad}_{u^*} - \mathrm{id} \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= C^{\mathrm{cb}}_{S_A}(E_0 \otimes \mathrm{id}) + C^{\mathrm{cb}}_{S_A}(\mathrm{ad}_u) \\
&\leq C^{\mathrm{cb}}_{S_A}(E_0 \otimes \mathrm{id}) + D
\end{aligned}$$

□

We wanted the approximation to be on $S_1(A)$, and such complexity is defined to be inclusive of $S_1(EA)$. Fortunately, we have a lemma stating that the complexity on both levels coincide.

Lemma 9.3.3. *Let resource set $S = S_E \otimes \mathbb{1} \cup \mathbb{1} \otimes S_A$*

$$C^{\mathrm{cb}}_{S_A}(\Phi) = \widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Phi)$$

Proof.

$$\begin{aligned}
\widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Phi) &= \|E_0^* \otimes \Phi^* - E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|(E_0^* \otimes \mathrm{id})(\mathrm{id} \otimes \Phi^* - \mathrm{id} \otimes \mathrm{id})\|_{\mathrm{cb}} \\
&\leq \|E_0^* \otimes \mathrm{id}\|_{\mathrm{cb}} \|\mathrm{id} \otimes \Phi^* - \mathrm{id} \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&\leq \|\mathrm{id} \otimes \Phi^* - \mathrm{id} \otimes \mathrm{id}\|_{\mathrm{cb}} \\
&= \|\Phi^* - \mathrm{id}\|_{\mathrm{cb}} \\
&= C^{\mathrm{cb}}_{S_A}(\Phi)
\end{aligned}$$

To show the converse, note that elements in S is either of the norm $\mathrm{id} \otimes t$ or $s \otimes \mathrm{id}$. Note

that

$$\begin{aligned}
\|\!|x|\!\|_{S_A} &= \|\!|\mathbb{1}\otimes x|\!\|_S \\
&= \max\left\{\sup_{\mathbb{1}\otimes t}\|[\mathbb{1}\otimes t, \mathbb{1}\otimes x]\|_\infty, \sup_{s\otimes\mathbb{1}}\|[s\otimes\mathbb{1}, \mathbb{1}\otimes x]\|_\infty\right\} \\
&= \max\left\{\sup_{\mathbb{1}\otimes t}\|[\mathbb{1}\otimes t, \mathbb{1}\otimes x]\|_\infty, 0\right\} \\
&= \sup_{\mathbb{1}\otimes t}\|[\mathbb{1}\otimes t, \mathbb{1}\otimes x]\|_\infty \\
&= \sup_{t\in S_A}\|[t,x]\|_\infty
\end{aligned}$$

□

In practice, this choice is resource set might not be the most convenient object to work with. For example, it is hard to implement the gate $e^{itX\otimes Y}$ with this resource set. The following theorem solve this problem by showing the resilience of the the lower bound with respect to more convenient choices of resource set.

diff_resource

Theorem 9.3.4. *Given $S_{EA} = S_E\otimes\mathbb{1}\cup\mathbb{1}\otimes S_A$ and $\widehat{S} = S_E\otimes S_A$,*

$$\widetilde{C}^{\mathrm{cb}}_{S_{EA}}(\Phi) = \widetilde{C}^{\mathrm{cb}}_{\widehat{S}}(\Phi)$$

Proof. It suffice to show that $\|\!|\cdot|\!\|_{S_{EA}}$ and $\|\!|\cdot|\!\|_{\widehat{S}}$ are equivalent as norms. It is clear that

$$\|\!|[1\otimes s_A, f]|\!\| \leq \|\!|[s_E\otimes s_A, f]|\!\|.$$

Recall that

$$[ab, c] = a[b,c] + [a,c]b$$

where $[a,b] = ab - ba$. Hence

$$\begin{aligned}
\|\!|[s_E\otimes s_A, f]|\!\| &= \|\!|[(s_E\otimes\mathbb{1})(\mathbb{1}\otimes s_A), f]|\!\| \\
&= \|\!|(s_E\otimes\mathbb{1})[(\mathbb{1}\otimes s_A), f] + [s_E\otimes\mathbb{1}, f](\mathbb{1}\otimes s_A)|\!\| \\
&\leq \|\!|(s_E\otimes\mathbb{1})[(\mathbb{1}\otimes s_A), f]|\!\| + \|\!|[s_E\otimes\mathbb{1}, f](\mathbb{1}\otimes s_A)|\!\| \\
&\leq \|\!|s_E\otimes\mathbb{1}|\!\|\,\|\!|[\mathbb{1}\otimes s_A, f]|\!\| + \|\!|[s_E\otimes\mathbb{1}, f]|\!\|\,\|\!|\mathbb{1}\otimes s_A|\!\| \\
&\leq B\left(\|\!|[\mathbb{1}\otimes s_A, f]|\!\| + \|\!|[s_E\otimes\mathbb{1}, f]|\!\|\right)
\end{aligned}$$

where $B = \max\left\{\|\!|s_E\otimes\mathbb{1}|\!\|, \|\!|\mathbb{1}\otimes s_A|\!\|\right\}$. □

This gives us some flexibility of choosing the resource set without altering the lower bound. Usually, $\widehat{S}$ will afford a smaller upper bound, and in fact, we could also replace S_{EA} by $\widehat{S} = \mathrm{Alg}\left\{S_E\otimes\mathbb{1}, \mathbb{1}\otimes S_A\right\}$ and the same argument will show the norms are equivalent.

We will take advantage of this fact and work with $\widehat{S}$ in most of the calculation for the aim of getting a better upper estimate. We will further discuss the potential of replacing the unitary set $\mathcal{U}$ with something more elementary in specific examples, without sacrificing the cost too much.

9.4 Photon Emission Model

We consider the example where

$$L = \sum_{j=1}^{n} L_{a_j}$$

where

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$a_j = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underbrace{a}_{j\text{-th component}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}.$$

We begin by calculating the upper bound. It is possible to apply the Suzuki-Trotter formula, but it this particular example, by calculating the unitary in the dilation, we are able to realize the gate directly. Hence, we are able to achieve zero-error approximation via this method. First, we may calculate the channel explicitly

Theorem 9.4.1.

$$e^{-tL_a} \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \rho_{11} + (1 - e^{-2t})\rho_{22} & e^{-t}\rho_{12} \\ e^{-t}\rho_{21} & e^{-2t}\rho_{22} \end{bmatrix}.$$

Proof. Let us first evaluate the behavior of L_a on matrix units. Notice that

$$L_a(\rho) = |2\rangle\langle 2|\rho + \rho|2\rangle\langle 2| - 2|1\rangle\langle 2|\rho|2\rangle\langle 1|.$$

Hence,

$$\begin{aligned} L_a|1\rangle\langle 1| &= 0 \\ L_a|1\rangle\langle 2| &= |1\rangle\langle 2| \\ L_a|2\rangle\langle 1| &= |2\rangle\langle 1| \\ L_a|2\rangle\langle 2| &= 2|2\rangle\langle 2| - 2|1\rangle\langle 1| \end{aligned}$$

By rearranging the entries, we get two blocks as

$$L_a \begin{bmatrix} |1\rangle\langle 1| \\ |2\rangle\langle 2| \\ |1\rangle\langle 2| \\ |2\rangle\langle 1| \end{bmatrix} = \begin{bmatrix} 0 & 0 & & \\ -2 & 2 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{bmatrix} \begin{bmatrix} |1\rangle\langle 1| \\ |2\rangle\langle 2| \\ |1\rangle\langle 2| \\ |2\rangle\langle 1| \end{bmatrix}$$

Now, we should take the exponential of this and obtain that

$$\exp(-tL_a) \begin{bmatrix} |1\rangle\langle 1| \\ |2\rangle\langle 2| \\ |1\rangle\langle 2| \\ |2\rangle\langle 1| \end{bmatrix} = \begin{bmatrix} 1 & 0 & & \\ 1 - e^{-2t} & e^{-2t} & & \\ & & e^{-t} & 0 \\ & & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} |1\rangle\langle 1| \\ |2\rangle\langle 2| \\ |1\rangle\langle 2| \\ |2\rangle\langle 1| \end{bmatrix}$$

Hence, finding the correspondence with the channel by reading columns, we get that

$$\exp -tL_a) \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \rho_{11} + (1 - e^{-2t})\rho_{22} & e^{-t}\rho_{12} \\ e^{-t}\rho_{21} & e^{-2t}\rho_{22} \end{bmatrix}.$$

□

Theorem 9.4.2.

$$M_{\alpha,\beta} \leq 2n$$

Proof. Since the channel is given by

$$e^{-tL_a}\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \rho_{11} + (1-e^{-2t})\rho_{22} & e^{-t}\rho_{12} \\ e^{-t}\rho_{21} & e^{-2t}\rho_{22} \end{bmatrix},$$

we shall then choose the partial isometries

$$V_t\left|1^E 1\right\rangle = \left|1^E 1\right\rangle$$
$$V_t\left|1^E 2\right\rangle = e^{-t}\left|1^E 2\right\rangle + \sqrt{1-e^{-2t}}\left|2^E 1\right\rangle$$

In matrix form,

$$V_t = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \\ 0 & \sqrt{1-e^{-2t}} \\ 0 & 0 \end{bmatrix}$$

In general, we can use any unitary completion of the partial isometry, but the choice

$$U_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-t} & -\sqrt{1-e^{-2t}} & 0 \\ 0 & \sqrt{1-e^{-2t}} & e^{-t} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

makes the corresponding gates easy to implement as of Pauli matrices. To verify this, we need to find the self-adjoint matrix H in which $e^{itH} = U_t$. By taking the logarithm of the unitary above, we get that

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{Y\otimes X - X\otimes Y}{2}.$$

Since $Y \otimes X$ and $X \otimes Y$ commutes, it is true that

$$e^{\frac{it}{2}(Y\otimes X - X\otimes Y)} = e^{\frac{it}{2}Y\otimes X}e^{\frac{-it}{2}X\otimes Y}.$$

□

Hence, to pick a natural resource set, one may propose to include $Y \otimes X - X \otimes Y$ in S. However, this might be hard to implement physically. Hence, if we require $S_{EA} = S_E \otimes \mathbb{1} \cup \mathbb{1} \otimes S_A$ then we should include $Y \otimes \mathbb{1}$, $X \otimes \mathbb{1}$, $\mathbb{1} \otimes X$, and $\mathbb{1} \otimes Y$. However, note that $\mathbb{1} \otimes X$ and $\mathbb{1} \otimes Y$ do not commute, hence we will need to Trotterize the approximation.

An alternate proposal is to include the algebra generated by the above gates. Namely,

$$\widehat{S} = \mathrm{Alg}\left\{Y\otimes\mathbb{1}, X\otimes\mathbb{1}, \mathbb{1}\otimes X, \mathbb{1}\otimes Y\right\} = \left\{I,X,Y,Z\right\}\otimes\left\{I,X,Y,Z\right\}.$$

The corresponding gates will be of the form

$$\mathcal{U} = \left\{e^{its} : s \in \widehat{S}, 0 \leq t \leq 2\pi\right\}.$$

Recall that 9.3.4 shows taking the algebra generated by the initial resource set S_{EA} does not alter the lower estimate for the approximation, but does yield a much better upper estimate as in this case, we only need to implement two gates since $Y \otimes X$ and $X \otimes Y$ commutes.

Remark. It is also possible to have less gates in the resource set that does not change the estimate by more than a constant. Consider the resource set

$$\widetilde{U} = \left\{C - e^{itZ}, H, SWAP, CNOT\right\}.$$

Now, we calculate the lower bound. To do so, we will use 6.3.4. For that, we need the mixing time and an estimate on $\kappa(S)$.

Theorem 9.4.3.

$$t_{\text{mix}} \sim \ln 4n + \ln 2$$

Proof. Recall that

$$e^{-tL_a} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \rho_{11} + (1 - e^{-2t})\rho_{22} & e^{-t}\rho_{12} \\ e^{-t}\rho_{21} & e^{-2t}\rho_{22} \end{bmatrix}.$$

The conditional expectation E_A is obtained by taking $t \to \infty$

$$E_{S'} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \rho_{11} + \rho_{22} & 0 \\ 0 & 0 \end{bmatrix}.$$

By the calculation before,

$$e^{-tL_a} - E_{S'} = \begin{bmatrix} -e^{-2t}\rho_{22} & e^{-t}\rho_{12} \\ e^{-t}\rho_{21} & e^{-2t}\rho_{22} \end{bmatrix}$$

Hence,

$$\begin{aligned} \left\|e^{-tL_a} - E_{S'}\right\|_\diamond &= \left\| \begin{bmatrix} -e^{-2t}\rho_{22} & e^{-t}\rho_{12} \\ e^{-t}\rho_{21} & e^{-2t}\rho_{22} \end{bmatrix} \right\|_\diamond \\ &= \left\| e^{-t} \begin{bmatrix} -e^{-t}\rho_{22} & \rho_{12} \\ \rho_{21} & e^{-t}\rho_{22} \end{bmatrix} \right\|_\diamond \\ &\leq \left\| e^{-t} \begin{bmatrix} \rho_{22} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \right\|_\diamond \\ &\leq 4e^{-t}. \end{aligned}$$

Thus,

$$\left\|e^{-tL} - E_A\right\|_\diamond = \left\|(e^{-tL_a})^{\otimes n} - E_A^{\otimes n}\right\|_\diamond \leq n \left\|e^{-tL_a - E_A}\right\|_\diamond \leq 4ne^{-t} \leq \frac{1}{2}.$$

By taking the infimum over the time with above conditions satisfied, the mixing time is

$$t_{\text{mix}} \sim \ln 4n + \ln 2.$$

□

Since we know that the unitaries that is used is of the form of Pauli matrices tensor Pauli matrices, the estimates for $\kappa(S)$ in the LCU case still holds. In particular, $n \leq \kappa(S)$. Apply 6.3.4, we have that

$$\frac{n}{8^{\frac{\beta}{\beta-1}} D\alpha^{\frac{1}{\beta-1}} t_{\text{mix}}^{\frac{\beta}{\beta-1}}} \leq M_{\alpha,\beta} \leq 2n$$

where

$$t_{\text{mix}} \sim \ln 4n + \ln 2.$$

Since both sides of the bound are linear, we can conclude that $M_{\alpha,\beta}$ grows linearly.

9.5 An Almost Unitary Example

The last example we will show is a modified version of the photon emission model. Consider the Lindbladian

$$L = \sum_{j=1}^{n} L_{b_j}$$

where

$$b = \begin{bmatrix} 0 & e^{-\frac{i\theta}{2}} \\ e^{\frac{i\theta}{2}} & 0 \end{bmatrix}.$$

The channel is given by

$$e^{-tL_{b_j}} \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \frac{1+e^{-4t}}{2}\rho_{11} + \frac{1-e^{-4t}}{2}\rho_{22} & \frac{1+e^{-4t}}{2}\rho_{12} + \frac{(1-e^{-4t})e^{i\theta}}{2}\rho_{21} \\ \frac{1-e^{-4t}}{2}\rho_{21} + \frac{(1+e^{-4t})e^{-i\theta}}{2}\rho_{12} & \frac{1+e^{-4t}}{2}\rho_{11} + \frac{1-e^{-4t}}{2}\rho_{22} \end{bmatrix}$$

To show the upper bound, we will, again, find the unitary of the dilation. For notational purposes, let

$$a = \frac{1+e^{-4t}}{2} \quad \text{and} \quad b = \frac{1-e^{-4t}}{2}.$$

The partial isometry is given by

$$V_t \left|1^E 1\right\rangle = \sqrt{a}\left|1^E 1\right\rangle + e^{-\frac{i\theta}{2}}\sqrt{b}\left|2^E 2\right\rangle$$
$$V_t \left|1^E 2\right\rangle = e^{\frac{i\theta}{2}}\sqrt{b}\left|2^E 1\right\rangle + \sqrt{a}\left|1^E 2\right\rangle.$$

In matrix form, we have that

$$V_t = \begin{bmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a} \\ 0 & e^{\frac{i\theta}{2}}\sqrt{b} \\ e^{-\frac{i\theta}{2}}\sqrt{b} & 0 \end{bmatrix}$$

Again, we have some autonomy in terms of the choice of unitary, here I will show two choices and discuss why one is better than the other. The first unitary we tried is

$$W = \begin{bmatrix} \sqrt{a} & & & e^{\frac{i\theta}{2}}\sqrt{b} \\ & \sqrt{a} & e^{-\frac{i\theta}{2}}\sqrt{b} & \\ & e^{\frac{i\theta}{2}}\sqrt{b} & -\sqrt{a} & \\ e^{-\frac{i\theta}{2}}\sqrt{b} & & & -\sqrt{a} \end{bmatrix}$$

The corresponding self-adjoint matrix is

$$K = \begin{bmatrix} -\sin\phi + 1 & & & -\phi - e^{\frac{i\theta}{2}}\cos\phi \\ & -\sin\phi + 1 & -e^{-\frac{i\theta}{2}}\cos\phi & \\ & -e^{\frac{i\theta}{2}}\cos\phi & \sin\phi + 1 & \\ -e^{-\frac{i\theta}{2}}\cos\phi & & & \sin\phi + 1 \end{bmatrix}$$
$$= I \otimes I - \sin\phi\ I \otimes Z - \cos\phi\ b \otimes X.$$

where $\sin\phi = \sqrt{a}$ and $\cos\phi = \sqrt{b}$.

The non-commutation of $b \otimes X$ and $I \otimes Z$ makes direct implementation unfeasible. Instead, one could use Trotterized approximation of the unitary, which only yields approximations of order $\mathcal{O}(t^2)$ and $\mathcal{O}(t^3)$. However, it is possible to achieve a more accurate result by selecting a different unitary. Notably, the unitary completion of a partial isometry is not uniquely determined, and neither is Stinespring dilation. This opens up considerable flexibility for our approach, although calculating the specific partial isometry and unitary for a given case may pose greater challenges.

For this particular example, consider the completion of $V_t\,|1^E 1\rangle$ and $V_t\,|1^E 2\rangle$ as

$$U = \begin{bmatrix} \sqrt{a} & & & -e^{-\frac{i\theta}{2}}\sqrt{b} \\ & \sqrt{a} & -e^{-\frac{i\theta}{2}}\sqrt{b} & \\ & e^{\frac{i\theta}{2}}\sqrt{b} & \sqrt{a} & \\ e^{\frac{i\theta}{2}}\sqrt{b} & & & \sqrt{a} \end{bmatrix}$$

The corresponding self-adjoint matrix is

$$H = \begin{bmatrix} & & & -\phi e^{-\frac{i\theta}{2}} \\ & & -\phi e^{-\frac{i\theta}{2}} & \\ & \phi e^{\frac{i\theta}{2}} & & \\ \phi e^{\frac{i\theta}{2}} & & & \end{bmatrix}$$

where $\cos(\phi) = \sqrt{a}$. It is not hard to see that $H = i\phi Y \otimes b$. Hence, if we add b into our resource set, the upper bound for this model is n.

To show the lower bound, since we have seen this is still under the regime of Pauli matrix tensor Pauli matrix, so $n \leq \kappa(S)$ still holds. Hence, it is left to calculate the mixing time given by

$$t_{\text{mix}} = \frac{\ln 4n + \ln 2}{4}.$$

Together with 6.3.4, we get that

$$\frac{n}{8^{\frac{\beta}{\beta-1}} D \alpha^{\frac{1}{\beta-1}} t_{\text{mix}}^{\frac{\beta}{\beta-1}}} \leq M_{\alpha,\beta} \leq n$$

with the resource set $S = \{b, \mathbb{1}, X, Y, Z\} \otimes \{b, \mathbb{1}, X, Y, Z\}$, where b is the generator of the Lindbladian.

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