Tutorial 3: Limits and Continuity

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1. (a) Clearly with $y = mx$ this limit does not exist

   (b) Same as above

   (c) Put $x = my^2$, to obtain
   $$L = |m|e^{-|m|}$$
   which depends on m, and thus the limit does not exist

   (d) Put $y = mx$

   (e) Put $y = mx^2$, this gives us the expression
   $$L = \frac{mx^4}{x^4 + mx^4} = \frac{m}{1 + m}$$
   which again depends on m, and thus the limit does not exist

   (f) Put $r = mx$

   (g) $y = mx$

   (h) Limit does exist and is equal to 1

   (i) Consider $u = xy(x + y)$ then
   $$f(x) = \frac{\sin u}{u}(x + y)$$
   Clearly this tends to 0

   (j) Here we have
   $$f(x, y) = (x - y)\left[1 + \frac{xy}{x^2 + y^2}\right]$$
   which reduces to
   $$f(x, y) = \frac{x - y}{\frac{x}{y} + \frac{y}{x}}$$
   The denominator is bounded $|D| \geq 2$ and thus this tends to 0

   (k) Put $z^2 = ky = hx$

2. (a) I will prove this proposition in more detail to outline the method of proof for the others

   We have the function
   $$f(x, y) = \frac{4xy^2}{x^2 + y^2}$$

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and we want to prove that the limit

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

that is to say, the limit exists and is equal to 0, we will use the $\epsilon - \delta$ definition of limit to prove this. We establish a bound on $|f(x,y) - L|$, then we use this bound to find δ such that $|(x,y)| < \delta_\epsilon \implies |f(x,y) - L| < \epsilon$

So in this case, we consider

$$\left|\frac{4xy^2}{x^2 + y^2}\right| \le 4\sqrt{x^2 + y^2}$$

So now if $\epsilon = 4\delta$, $\sqrt{x^2 + y^2} < \delta$ implies

$$\epsilon = 4\delta > 4\sqrt{x^2 + y^2} \ge |f(x,y) - L|$$

which completes our proof

(b) Here we have to make a change of variables to use our template, Let $u = x + 1 \implies x = u - 1$ and similarly for $v = y + 1 \implies y = v - 1$ As before, we want to prove that

$$\lim_{(u,v)\to 0,0} (u-1)(v-1-2u+2) = -1$$

we can assume $|u| < 1$ and $|v| < 1$, now consider

$$|f(u,v) - L| = |(u-1)(v-2u+1) + 1| = |uv - 2u^2 + u - v + 2u|$$

which is

$$|uv - 2u^2 + 3u - v| \le |uv| + 2|u^2| + 3|u| + |v|$$

now apply $|u| < 1$ and $|u| \le \sqrt{u^2 + v^2}$ and $|v| \le \sqrt{u^2 + v^2}$ giving

$$|f(u,v) - L| < 7\sqrt{u^2 + v^2}$$

If we set $7\delta = \epsilon$ then $\sqrt{u^2 + v^2} < \delta$ implies

$$\epsilon = 7\delta > 7\sqrt{u^2 + v^2} > |f(u,v) - L|$$

(c) define $u = x - 1$ so we want

$$\lim_{(u,y)\to(0,0)} f(u,y) = \frac{u^2 \ln(1+u)}{u^2 + y^2} = 0$$

consider

$$|f(u,y) - L| = \left|\frac{u^2 \ln(1+u)}{u^2 + y^2}\right| = |u^2/(u^2 + y^2)| \ln(1+u) \le \ln(1+u) \le \ln(e^u) = \le \sqrt{u^2 + y^2}$$

using $e^u \ge 1 + u$ from calculus, now if $\delta = \epsilon$, $\delta > \sqrt{u^2 + y^2}$ implies

$$\epsilon = \delta > \sqrt{u^2 + y^2} \ge |f(u,y) - L|$$

which completes our proof

(d) Let $u = x + 2$, $v = y - 2$, now we need to prove that

$$\lim_{(u,v)\to(0,0)} u - v + 4 = 4$$

now consider

$$|f(u,v) - L| = |u - v| \le |u| + |v| \le 2\sqrt{u^2 + v^2}$$

Now if $\epsilon = 2\delta$ then $\delta > \sqrt{u^2 + v^2}$ implies

$$\epsilon = 2\delta > 2\sqrt{u^2 + v^2} > |f(u,v) - L|$$

Completing our proof

(e) Here we want to prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2}$$

Clearly $x^2 - y^2 < x^2 + y^2$ which gives us

$$|f(x,y) - L| = |xy\frac{x^2 - y^2}{x^2 + y^2}| \le |xy|$$

Assume $|x| < 1$ then we have

$$|xy| < |y| < \sqrt{x^2 + y^2}$$

now if $\delta = \epsilon$, $\delta > \sqrt{x^2 + y^2}$ implies

$$\epsilon = \delta > \sqrt{x^2 + y^2} \ge |f(x,y) - L|$$

which completes our proof

(f) We have

$$f(x,y) = x\sin x \cos y$$

We want to prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = L = 0$$

consider

$$|f(x,y) - L| = |x\sin x \cos y| \le |x| \le \sqrt{x^2 + y^2}$$

now if $\delta = \epsilon$, $\delta > \sqrt{x^2 + y^2}$ implies

$$\epsilon = \delta > \sqrt{x^2 + y^2} \ge |f(x,y) - L|$$

which completes our proof

(g) We have

$$f(x,y) = \frac{x^2}{\sqrt{x^2 + y^2}}$$

We want to prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = L = 0$$

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consider

$$|f(x,y) - L| = \frac{x^2}{\sqrt{x^2 + y^2}}| \leq \sqrt{x^2 + y^2}$$

since, $x^2 \leq x^2 + y^2$, now if $\delta = \epsilon$, $\delta > \sqrt{x^2 + y^2}$ implies

$$\epsilon = \delta > \sqrt{x^2 + y^2} \geq |f(x,y) - L|$$

which completes our proof

(h) We have

$$f(x,y) = \frac{x^2 y^2}{x^2 + y^2}$$

We want to prove that

$$\lim_{(x,y) \to (0,0)} f(x,y) = L = 0$$

consider

$$|f(x,y) - L| = \frac{x^2 y^2}{x^2 + y^2} < x^2 + y^2 < \sqrt{x^2 + y^2}$$

if we assume $x^2 + y^2 < 1$ now if $\delta = \epsilon$, $\delta > \sqrt{x^2 + y^2}$ implies

$$\epsilon = \delta > \sqrt{x^2 + y^2} \geq |f(x,y) - L|$$

which completes our proof

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