

# Problem Set 6

Arjun Ghosh

September 12, 2025

## 1 Question 1

Consider two  $x$  values,  $x = 0$  and  $x = L$  where  $L \rightarrow \infty$ , clearly

$$f(x) = 2x^3 + 5x - 9 = -9$$

at  $x = 0$  and

$$f(L) \rightarrow \infty \text{ as } L \rightarrow \infty$$

thus  $\exists x \in [0, \infty) : f(x) = 0$  hence,  $f(x)$  has at least one root, further, for the sake of contradiction let us assume there exist two or more roots of  $f(x)$ , let any two of them be  $x_1$  and  $x_2$ , then by Rolle's theorem

$$f'(x) = 6x^2 + 5 = 0$$

for some  $c \in (x_1, x_2)$  which is clearly absurd, thus our assumption must be false, and thus there is only one root of  $f(x)$

## 2 Question 2

### 2.1 Part (a)

Consider  $f(x) = |x - 1|$  in the interval  $[0, 2]$ , clearly  $f(x)$  is not differentiable at  $x = 1$ , since  $f'(1^-) = -1$  and  $f'(1^+) = 1$ , thus it does not satisfy the conditions for Rolle's theorem and it doesn't satisfy its consequences either since  $\nexists x \in (0, 2) : f'(x) = 0$

## 2.2 Part (b)

$$f(x) = 1 + x^{2/3}$$

in the interval  $[-8, 8]$ , simply differentiating we have

$$f'(x) = \frac{2}{3}x^{-1/3}$$

which is not defined for  $x = 0$  and clearly the derivative is not continuous at  $x = 0$  anyways, thus it does not satisfy the conditions for Rolle's theorem and clearly there does not exist  $c \in [-8, 8] : f'(c) = 0$

## 2.3 Part (c)

$$f(x) = x \sin \frac{1}{x}; f(0) = 0$$

Consider

$$f'(0) = \lim_{h \rightarrow 0} \frac{(h \sin(\frac{1}{h}))}{h} = \lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right)$$

does not exist, hence it does not satisfy the conditions for Rolle's theorem, differentiating on the differentiable intervals

$$f'(x) = \sin\left(\frac{1}{x}\right) - \frac{x \cos\left(\frac{1}{x}\right)}{x^2} = \sin\left(\frac{1}{x}\right) - \frac{1}{x} \cos\left(\frac{1}{x}\right)$$

Setting it to 0

$$\tan\left(\frac{1}{u}\right) = \frac{1}{u}$$

There are infinitely many solutions to this equation in the interval  $\frac{1}{u} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  or  $u \in \left(\infty, -\frac{2}{\pi}\right) \cup \left(\frac{2}{\pi}, u\right)$ , so there do exist value of  $c$  in the interval which satisfy the consequences of Rolle's theorem even though its conditions are not satisfied.

## 3 Question 3

### 3.1 Part (a)

In the interval  $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ , Cauchy's MVT gives us

$$\frac{f'(\xi)}{g'(\xi)} = \frac{1 - 1}{1 - (-1)} = 0 = -\frac{\cos(\xi)}{\sin(\xi)}$$

Clearly  $\xi = \frac{\pi}{2}$  satisfies this condition

### 3.2 Part (b)

In the interval  $[0, \frac{1}{2}]$  Cauchy's MVT gives us

$$\frac{f'(\xi)}{g'(\xi)} = \frac{\frac{3}{2}(1+\xi)^{1/2}}{\frac{1}{2}(1+\xi)^{-1/2}} = 3(1+\xi) = \frac{f(b)-f(a)}{g(b)-g(a)}$$

which gives us

$$3(1+\xi) = \frac{(u)^{3/2}-1}{\sqrt{u}-1} \implies \xi = \frac{1}{3}(u + \sqrt{u} + 1) - 1$$

where  $u = \frac{3}{2}$

## 4 Question 4

We write LMVT as: for an interval  $[a, b]$  if  $f(x)$  is continuous and differentiable in the corresponding open interval  $(a, b)$ , then  $\exists c \in (a, b) : f'(c) = \frac{f(b)-f(a)}{b-a}$

Now define  $\theta = \frac{c-a}{b-a}$ , since

$$c \in (a, b) \implies c - a \in (0, b - a) \implies \theta \in (0, 1)$$

Thus we have

$$\boxed{f'(c) = f'(a + c - a) = f'(a + \theta(b - a)) = \frac{f(b) - f(a)}{b - a}}$$

Now we set  $b - a = h$  and  $a = x$  so we have

$$f'(x + \theta h) = \frac{f(x + h) - f(x)}{h}$$

Now, before we continue, let us examine this equation, if we apply Taylor's theorem to  $f(x + h)$ , *i.e.* in the interval  $(x, x + h)$  we find  $\theta_2 \in (0, 1)$  such that

$$hf'(x + \theta h) = \left( f(x) + hf'(x) + \frac{h^2}{2}f''(x + \theta_2 h) \right) - f(x)$$

or

$$\frac{f'(x + \theta h) - f'(x)}{h} = \frac{1}{2} f''(x + \theta_2 h)$$

Now if we take the limit as  $h \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{f'(x + \theta h) - f'(x)}{h} = \theta f''(x) = \lim_{h \rightarrow 0} \frac{1}{2} f''(x + \theta_2 h)$$

which implies

$$\lim_{h \rightarrow 0} \theta = \frac{1}{2}$$

as a consequence of Taylor's theorem, this however requires the existence of the second derivative of  $f(x)$  at  $x$ .

Clearly,  $\theta = g(x, h)$ , *i.e.*  $\theta$  is a function of  $x$  and  $h$ , thus

#### 4.1 Part (a)

for  $f(x) = x^2$  We have

$$f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$$

or

$$\frac{f(x + h) - f(x)}{h} = 2x + h = f'(x + \theta h) = 2(x + \theta h) = 2x + 2\theta h$$

Clearly

$$\theta = \frac{1}{2} \forall x, h$$

which matches what we had earlier

#### 4.2 Part (b)

$$\frac{f(x + h) - f(x)}{h} = e^x \left( \frac{e^h - 1}{h} \right)$$

While

$$f'(x + \theta h) = e^x e^{\theta h}$$

Thus we have

$$\theta = \frac{1}{h} \ln \frac{e^h - 1}{h}$$

In this case we have  $\theta$  not dependent on  $x$ , a graph of this function is given, taking the limit as

$$\theta_0 = \lim_{h \rightarrow 0} \frac{1}{h} \ln \frac{e^h - 1}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \ln \frac{h + \frac{h^2}{2} + O(h^3)}{h}$$

which is

$$\theta_0 = \lim_{h \rightarrow 0} \frac{1}{h} \ln \left( 1 + \frac{h}{2} + O(h^2) \right) = \lim_{h \rightarrow 0} \frac{1}{2} + O(h) = \frac{1}{2}$$

which matches what we had earlier.

### 4.3 Part (c)

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h} \ln \left( 1 + \frac{h}{x} \right) = f'(x + \theta h) = \frac{1}{x + \theta h}$$

or we have

$$\theta = \frac{1}{h} \left( \frac{h}{\ln \left( 1 + \frac{x}{h} \right)} - x \right)$$

This time,  $\theta$  is a function of  $x$  and  $h$ , our claim is that the dependence on  $x$  dies out as  $h \rightarrow 0$  we have

$$\theta = \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{h}{\left( \frac{x}{h} \right) - \frac{1}{2} \left( \frac{x}{h} \right)^2 + O\left( \frac{h^3}{x^3} \right)} - x \right)$$

which is

$$\lim_{h \rightarrow 0} \frac{h}{x} \left( \frac{1}{1 - \frac{1}{2} \left( \frac{x}{h} \right) + O\left( \left( \frac{x}{h} \right)^2 \right)} - 1 \right) = \frac{1}{2}$$

which confirms our claim

## 5 Question 5

### 5.1 Part (a)

By LMVT, since  $f$  is continuous and differentiable, for  $c \in [1, 2]$

$$f(2) = f(1) + f'(c)(1)$$

so

$$f(1) = f(2) - f'(c) = -5 - f'(c)$$

but this lies between

$$f(1) \in [-7, -3]$$

since  $f'(c) \in [-2, 2]$  and this maximum value is achieved if  $f(x)$  is linear in the interval  $(1, 2)$

## 5.2 Part (b)

We know that

$$28^{1/3} = (27 + 1)^{1/3} = 3 \left( 1 + \frac{1}{27} \right)^{1/3}$$

applying LMVT we have

$$(1 + u)^{1/3} = 1 + \frac{u}{3}(1 + v)^{-2/3}$$

where  $u = \frac{1}{27}$  and  $v \in (0, u)$ , let us assume without proof for now that  $(1 + v)^{-2/3} = 1 + O(u)$ , which is a consequence of Taylor's theorem. So we have

$$(28)^{1/3} = 3 + \frac{1}{27} + O\left(\left(\frac{1}{27}\right)^2\right) \approx \frac{82}{27}$$

## 5.3 Part (c)

We have  $f''(x) \geq 0$  for  $x \in [a, b]$ , Let us assume for the sake of contradiction that

$$f\left(\frac{x_1 + x_2}{2}\right) > \frac{1}{2}[f(x_1) + f(x_2)]$$

for \*some\*  $x_1, x_2 \in [a, b]$ . Assume WLOG  $x_2 > x_1$ , now consider the point  $\bar{x} = \frac{1}{2}(x_1 + x_2)$ , also define  $\Delta x = \bar{x} - x_1 = x_2 - \bar{x}$ , thus  $2\Delta x = x_2 - x_1 > 0$  hence  $\Delta x > 0$

then by LMVT on the intervals  $(x_1, \bar{x})$  and  $(\bar{x}, x_2)$  (whose conditions are satisfied by the assumption that  $f''(x)$  exists and is positive) yield  $\xi_1 \in (x_1, \bar{x})$  and  $\xi_2 \in (\bar{x}, x_2)$  and thus  $\xi_2 > \xi_1$  such that

$$f'(\xi_1) = \frac{f(\bar{x}) - f(x_1)}{x - x_1}; f'(\xi_2) = \frac{f(x_2) - f(\bar{x})}{x_2 - \bar{x}}$$

which we rewrite as

$$f(\bar{x}) = f(x_1) + f'(\xi_1)\Delta x = f(x_2) - f'(\xi_2)\Delta x$$

Adding both RHS sides and doubling the LHS

$$2f(\bar{x}) = f(x_1) + f(x_2) + \Delta x[f'(\xi_1) - f'(\xi_2)]$$

Or

$$f(\bar{x}) = \frac{1}{2}(f(x_1) + f(x_2)) + \frac{\Delta x}{2}(f'(\xi_1) - f'(\xi_2))$$

But our assumption now implies

$$\frac{\Delta x}{2}(f'(\xi_1) - f'(\xi_2)) > 0$$

but since  $\Delta x > 0$  and  $\xi_2 > \xi_1$  this contradicts the statement that  $f''(x) > 0$  for  $[a, b]$ , thus our assumption must be false and thus

$$\boxed{f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}[f(x_1) + f(x_2)]}$$

Completing our proof

## 6 Question 6

### 6.1 Part (a)

To prove the right hand side of the inequality we apply CMVT to  $\sin x$  and  $x$  on the interval  $[0, x]$  to get

$$\cos \xi = \frac{\sin x}{x} \implies \frac{\sin x}{x} < 1 \implies \sin x < x; \forall x \in \left(0, \frac{\pi}{2}\right)$$

For the LHS, consider the function

$$\zeta(x) = \frac{x}{\sin x} \implies \zeta'(x) = \frac{\sin x - x \cos x}{(\sin x)^2} > 0$$

since  $\tan x > x$  in  $x \in \left(0, \frac{\pi}{2}\right)$ , therefore

$$\zeta(x) < \zeta\left(\frac{\pi}{2}\right) = \frac{\pi}{2}; \forall x \in \left(0, \frac{\pi}{2}\right)$$

which gives

$$\boxed{\frac{2x}{\pi} < \sin x < x}$$

(b) Consider the functions  $x^n$  and  $x$  on the interval  $[a, b]$ , clearly CMVT gives

$$n\xi^{n-1} = \frac{b^n - a^n}{b - a}$$

Now if  $n > 1$  then  $n - 1 > 0$  which implies that  $x^{n-1}$  is an increasing function which implies

$$na^{n-1} < n\xi^{n-1} < nb^{n-1}$$

which gives us the inequality

$$\boxed{na^{n-1} < \frac{b^n - a^n}{b - a} < nb^{n-1}}$$

(c) For the right hand side consider the functions  $\log(1 + x)$  and  $x$  on the interval  $[0, x]$  CMVT gives

$$1 > \frac{1}{1 + \xi} = \frac{\log(1 + x)}{x}$$

which gives

$$\log(1 + x) < x$$

apply LMVT to the function  $\log(1 + x)$  on the interval  $[0, x]$  gives

$$\log(1 + x) = 0 + \frac{x}{1 + \xi} > \frac{x}{1 + x}$$

since  $\xi < x$  or

$$\boxed{\frac{x}{1 + x} < \log(1 + x) < x}$$

$\forall x > 0$

7(a) Let the intersection happen at some point  $\eta \in (a, b)$ , now apply LMVT on  $(a, \eta)$  and  $(\eta, b)$  to give  $\xi_1, \xi_2 : \xi_2 > \xi_1$  such that

$$f'(\xi_1) = f'(\xi_2) = m = \frac{f(b) - f(a)}{b - a}$$



now apply Rolle's theorem on  $f'(x)$  on the interval  $(\xi_1, \xi_2)$

$$f''(\zeta) = 0$$

for some  $\zeta \in (\xi_1, \xi_2) \subset (a, b)$

(b) Apply CMVT to  $f(x)$  and  $x^2$  on the interval  $[0, 1]$  to give

$$\frac{f'(\xi)}{2\xi} = f(1) - f(0)$$

for  $\xi \in (0, 1)$

(c) Consider  $g(x) = f(x) - x$  then  $g'(x) = f'(x) - 1$ , so we have

$$g(a) = g(b) = 0 \implies g'(\xi) = 0$$

for some  $\xi \in (a, b)$ , Now apply LMVT to  $(a, \xi)$  and  $(\xi, b)$  to find  $\zeta_1, \zeta_2$  such that

$$g'(\zeta_1) = \frac{g(\xi)}{\xi - a}, g'(\zeta_2) = \frac{-g(\xi)}{b - \xi}$$

which must have opposite signs, thus applying IVT between  $(\zeta_1, \xi)$  and  $(\xi, \zeta_2)$  we can find two numbers  $c_1, c_2$  such that

$$\boxed{g'(c_1) + g'(c_2) = 0 \implies f'(c_1) + f'(c_2) = 2}$$

8 (a) We want to prove that there exists  $c \in (a, b)$

$$\begin{vmatrix} f(a) + (b-a)f'(c) & f(b) \\ \phi(a) + (b-a)\phi'(c) & \phi(b) \end{vmatrix} = 0$$

Consider CMVT on  $f(x)$  and  $\phi(x)$  on the interval  $(a, b)$  we have

$$f(b) = f(a) + \frac{f'(c)}{\phi'(c)}(\phi(b) - \phi(a))$$

(b) Apply CMVT on the functions  $f(x)$ ,  $x$ ,  $x^2$ ,  $x^3$  to get

$$f(b) - f(a) = f'(x_1)(b - a) = \frac{f'(x)}{2x_2}(b^2 - a^2) = \frac{f'(x_3)}{3x_3^2}(b^3 - a^3)$$

or

$$f'(x_1) = \frac{f'(x_2)}{2x_1} = \frac{f'(x_3)}{3x_3^2}$$

9 (a) Consider  $1 - \cos x = 2 \sin^2 \frac{x}{2}$ , setting  $u = x/2$  we need to prove

$$\sin x < x$$

which we can do by applying CMVT to these functions

$$\cos \xi = \frac{\sin x}{x}$$

which completes our proof

(b) consider  $\frac{f(x)}{x^2}$  and  $\frac{1}{x^2}$  applying CMVT we have

$$\frac{\frac{f(b)}{b^2} - \frac{f(a)}{a^2}}{\frac{1}{b^2} - \frac{1}{a^2}} = \frac{\frac{c^2 f'(c) - 2cf(c)}{c^4}}{-\frac{2}{c^3}}$$

$$\frac{a^2 f(b) - b^2 f(a)}{a^2 - b^2} = \frac{1}{2}(2f(c) - cf'(c))$$

as required (c) Apply CMVT on the functions  $\ln x$  and  $\arcsin x$  on the interval  $[x, 1]$  to get

$$\frac{-\ln x}{\frac{\pi}{2} - \arcsin x} = \frac{2 \ln x}{2 \arcsin x - \pi} = \sqrt{1 - \frac{1}{\xi^2}}$$

which is an increasing function for  $\xi$  so we have

$$\frac{2 \ln x}{2 \arcsin x - \pi} < \frac{\sqrt{1 - x^2}}{x}$$

for  $x \in (0, 1)$

10 Apply LMVT between  $(a, x_0)$  and  $(x_0, b)$  to find  $\xi_1 < \xi_2$  such that

$$f'(\xi_1) = \frac{f(x_0) - f(a)}{x_0 - a}; f'(\xi_2) = \frac{f(b) - f(x_0)}{b - x_0}$$

Now assume for the sake of contradiction that there does not exist  $c \in (a, b)$  such that  $f''(c) < 0$  which would imply  $f'(x)$  is an increasing function which implies

$$f'(\xi_2) > f'(\xi_1) \text{ since } \xi_2 > \xi_1$$

but this contradicts what we found and thus our assumption, and therefore our assumption must be wrong and there exists  $c \in (a, b)$  such that  $f''(c) < 0$