

**Indian Institute of Technology Kharagpur**  
**Department of Mathematics**  
**MA11003 - Advanced Calculus**  
**Problem Sheet - 4**  
**Autumn 2025**

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1. Show that the function

$$f(x, y) = \begin{cases} \frac{x^2 + y^2}{|x| + |y|}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

is continuous at  $(0, 0)$ , but  $f_x(0, 0)$  and  $f_y(0, 0)$  do not exist.

2. Show that the following functions

$$(a) \quad f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

$$(b) \quad f(x, y) = \begin{cases} \frac{x^3 + y^3}{x - y}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

possess partial derivatives at  $(0, 0)$ , though it is not continuous at  $(0, 0)$ .

3. Find  $f_x(x, y)$  and  $f_y(x, y)$  using definition for the followings :

$$(a) \quad f(x, y) = x^2 + y^2,$$

$$(b) \quad f(x, y) = \sin(3x + 4y),$$

$$(c) \quad f(x, y) = ye^{-x} + xy.$$

$$(d) \quad f(x, y) = x^2 + xy + y^3,$$

$$(e) \quad f(x, y) = x \sin y + x^2,$$

$$(f) \quad f(x, y) = e^{xy} + \frac{x}{y}.$$

4. Find  $f_x(0, 0)$ ,  $f_y(0, 0)$ ,  $f_x(0, y)$  and  $f_y(x, 0)$  for the followings :

$$(a) \quad f(x, y) = \begin{cases} \frac{xy}{x + y}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

(b)  $f(x, y) = \log(1 + xy)$ ,

(c)  $f(x, y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 0 \text{ or both } x = 0 \text{ and } y = 0; \\ 0, & \text{Otherwise} \end{cases}$

(d)  $f(x, y) = e^{x-y} - e^{y-x}$ ,

(e)  $f(x, y) = \begin{cases} \frac{x^3 - y^3}{x + y}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

5. Show that the following functions

(a)  $f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x^2 + y^2) \neq 0; \\ 0, & \text{if } (x^2 + y^2) = 0. \end{cases}$

(b)  $f(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & \text{if } (x^2 + y^2) \neq 0; \\ 0, & \text{if } (x^2 + y^2) = 0. \end{cases}$

have first order partial derivatives at  $(0, 0)$ , and discuss the differentiability at  $(0, 0)$ .

6. Show that following function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0) \end{cases}$$

is continuous, possess first order partial derivatives but it is not differentiable at the origin.

7. Prove that the function  $f(x, y) = \sqrt{|xy|}$  is not differentiable at  $(0, 0)$ , but that  $f_x$  and  $f_y$  both exists at origin and have the value 0. Show that  $f_x$  and  $f_y$  are continuous everywhere except at the origin.

8. Test the differentiability of the following functions at  $(0, 0)$

(a)  $f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

(b)  $f(x, y) = \begin{cases} \frac{x^3 - 2y^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

9. Let  $f(x, y) = \begin{cases} y \frac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

Find  $f_{xx}(x, y)$ ,  $f_{xy}(x, y)$ ,  $f_{yx}(x, y)$  and  $f_{yy}(x, y)$  at  $(0, 0)$ . Also check the differentiability of the function  $f(x, y)$  at the origin.

10. For the function  $f(x, y) = \begin{cases} \frac{x^2 y(x - y)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$   
check that  $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ . Also check the differentiability of  $f(x, y)$  at the origin
11. Find  $f_{yxx}(x, y)$  and  $f_{xyx}(x, y)$  for the following functions:
- (a)  $f(x, y) = x^4 \sin 3y + 5x - 6y$ .
  - (b)  $f(x, y) = x^5 y^3 + \log(xy) + 10x$ .
  - (c)  $f(x, y) = e^{xy} \tan x + x^3 y^2$ .
  - (d)  $f(x, y) = x^3 \sin y + y^3 \cos x$
  - (e)  $f(x, y) = e^x \ln y + \cos y \ln x$
  - (f)  $f(x, y) = x^3 y^2 + 2xy^3 + \cos(xy^2)$
12. Find the total differential of the following functions
- (a)  $w = x^2 + xy^2 + xy^2 z^3$
  - (b)  $z = \tan^{-1}(x/y)$ ,
  - (c)  $u = e^{(x^2 + y^2 + z^2)}$ ,
  - (d)  $w = \sin(3x + 4y) + 5e^z$
  - (e)  $w = z \ln y + y \ln z + xyz$ ,
  - (f)  $u = \sqrt{x^2 + y^2 + z^2}$ ,
  - (g)  $w = e^x \sin(y + 2z) - x^2 y^2$ ,
  - (h)  $w = e^{\frac{x}{y}} + e^{\frac{z}{y}}$ .