

# Tutorial 3: Advanced Calculus

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## Solutions to Problems

1. (a) Clearly with  $y = mx$  this limit does not exist
- (b) Same as above
- (c) Put  $x = my^2$ , to obtain

$$L = |m|e^{-|m|}$$

which depends on  $m$ , and thus the limit does not exist

- (d) Put  $y = mx$
- (e) Put  $y = mx^2$ , this gives us the expression

$$L = \frac{mx^4}{x^4 + mx^4} = \frac{m}{1 + m}$$

which again depends on  $m$ , and thus the limit does not exist

- (f) Put  $r = mx$
- (g)  $y = mx$
- (h) Limit does exist and is equal to 1
- (i) Consider  $u = xy(x + y)$  then

$$f(x) = \frac{\sin u}{u}(x + y)$$

Clearly this tends to 0

- (j) Here we have

$$f(x, y) = (x - y) \left[ 1 + \frac{xy}{x^2 + y^2} \right]$$

which reduces to

$$f(x, y) = \frac{x - y}{\frac{x}{y} + \frac{y}{x}}$$

The denominator is bounded  $|D| \geq 2$  and thus this tends to 0

- (k) Put  $z^2 = ky = hx$

2. (a) I will prove this proposition in more detail to outline the method of proof for the others

We have the function

$$f(x, y) = \frac{4xy^2}{x^2 + y^2}$$

and we want to prove that the limit

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$$

that is to say, the limit exists and is equal to 0, we will use the  $\epsilon - \delta$  definition of limit to prove this. We establish a bound on  $|f(x, y) - L|$ , then we use this bound to find  $\delta$  such that  $|(x, y)| < \delta_\epsilon \implies |f(x, y) - L| < \epsilon$

So in this case, we consider

$$\left| \frac{4xy^2}{x^2 + y^2} \right| \leq 4\sqrt{x^2 + y^2}$$

So now if  $\epsilon = 4\delta$ ,  $\sqrt{x^2 + y^2} < \delta$  implies

$$\epsilon = 4\delta > 4\sqrt{x^2 + y^2} \geq |f(x, y) - L|$$

which completes our proof

- (b) Here we have to make a change of variables to use our template, Let  $u = x + 1 \implies x = u - 1$  and similarly for  $v = y + 1 \implies y = v - 1$  As before, we want to prove that

$$\lim_{(u,v) \rightarrow 0,0} (u - 1)(v - 1 - 2u + 2) = -1$$

we can assume  $|u| < 1$  and  $|v| < 1$ , now consider

$$|f(u, v) - L| = |(u - 1)(v - 2u + 1) + 1| = |uv - 2u^2 + u - v + 2u|$$

which is

$$|uv - 2u^2 + 3u - v| \leq |uv| + 2|u^2| + 3|u| + |v|$$

now apply  $|u| < 1$  and  $|u| \leq \sqrt{u^2 + v^2}$  and  $|v| \leq \sqrt{u^2 + v^2}$  giving

$$|f(u, v) - L| < 7\sqrt{u^2 + v^2}$$

If we set  $7\delta = \epsilon$  then  $\sqrt{u^2 + v^2} < \delta$  implies

$$\epsilon = 7\delta > 7\sqrt{u^2 + v^2} > |f(u, v) - L|$$

- (c) define  $u = x - 1$  so we want

$$\lim_{(u,y) \rightarrow (0,0)} f(u, y) = \frac{u^2 \ln(1 + u)}{u^2 + y^2} = 0$$

consider

$$|f(u, y) - L| = \left| \frac{u^2 \ln(1 + u)}{u^2 + y^2} \right| = |u^2 / (u^2 + y^2)| \ln(1 + u) \leq \ln(1 + u) \leq \ln(e^u) \leq \sqrt{u^2 + y^2}$$

using  $e^u \geq 1 + u$  from calculus, now if  $\delta = \epsilon$ ,  $\delta > \sqrt{u^2 + y^2}$  implies

$$\epsilon = \delta > \sqrt{u^2 + y^2} \geq |f(u, y) - L|$$

which completes our proof

(d) Let  $u = x + 2$ ,  $v = y - 2$ , now we need to prove that

$$\lim_{(u,v) \rightarrow (0,0)} u - v + 4 = 4$$

now consider

$$|f(u, v) - L| = |u - v| \leq |u| + |v| \leq 2\sqrt{u^2 + v^2}$$

Now if  $\epsilon = 2\delta$  then  $\delta > \sqrt{u^2 + v^2}$  implies

$$\epsilon = 2\delta > 2\sqrt{u^2 + v^2} > |f(u, v) - L|$$

Completing our proof

(e) Here we want to prove that

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2}$$

Clearly  $x^2 - y^2 < x^2 + y^2$  which gives us

$$|f(x, y) - L| = |xy \frac{x^2 - y^2}{x^2 + y^2}| \leq |xy|$$

Assume  $|x| < 1$  then we have

$$|xy| < |y| < \sqrt{x^2 + y^2}$$

now if  $\delta = \epsilon$ ,  $\delta > \sqrt{x^2 + y^2}$  implies

$$\epsilon = \delta > \sqrt{x^2 + y^2} \geq |f(x, y) - L|$$

which completes our proof

(f) We have

$$f(x, y) = x \sin x \cos y$$

We want to prove that

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = L = 0$$

consider

$$|f(x, y) - L| = |x \sin x \cos y| \leq |x| \leq \sqrt{x^2 + y^2}$$

now if  $\delta = \epsilon$ ,  $\delta > \sqrt{x^2 + y^2}$  implies

$$\epsilon = \delta > \sqrt{x^2 + y^2} \geq |f(x, y) - L|$$

which completes our proof

(g) We have

$$f(x, y) = \frac{x^2}{\sqrt{x^2 + y^2}}$$

We want to prove that

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = L = 0$$

consider

$$|f(x, y) - L| = \frac{x^2}{\sqrt{x^2 + y^2}} \leq \sqrt{x^2 + y^2}$$

since,  $x^2 \leq x^2 + y^2$ , now if  $\delta = \epsilon$ ,  $\delta > \sqrt{x^2 + y^2}$  implies

$$\epsilon = \delta > \sqrt{x^2 + y^2} \geq |f(x, y) - L|$$

which completes our proof

(h) We have

$$f(x, y) = \frac{x^2 y^2}{x^2 + y^2}$$

We want to prove that

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = L = 0$$

consider

$$|f(x, y) - L| = \frac{x^2 y^2}{x^2 + y^2} < x^2 + y^2 < \sqrt{x^2 + y^2}$$

if we assume  $x^2 + y^2 < 1$  now if  $\delta = \epsilon$ ,  $\delta > \sqrt{x^2 + y^2}$  implies

$$\epsilon = \delta > \sqrt{x^2 + y^2} \geq |f(x, y) - L|$$

which completes our proof  $x + 3$