Tutorial 1: Mean Value Theorems

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1. Consider two x values, $x = 0$ and $x = L$ where $L \to \infty$, clearly

$$f(x) = 2x^3 + 5x - 9 = -9$$

at $x = 0$ and

$$f(L) \to \infty \text{ as } L \to \infty$$

thus $\exists\, x \in [0, \infty) : f(x) = 0$ hence, $f(x)$ has at least one root, further, for the sake of contradiction let us assume there exist two or more roots of $f(x)$, let any two of them be x_1 and x_2, then by Rolle's theorem

$$f'(x) = 6x^2 + 5 = 0$$

for some $c \in (x_1, x_2)$ which is clearly absurd, thus our assumption must be false, and thus there is only one root of $f(x)$

2. (a) Consider $f(x) = |x-1|$ in the interval $[0, 2]$, clearly $f(x)$ is not differentiable at $x = 1$, since $f'(1^-) = -1$ and $f'(1^+) = 1$, thus it does not satisfy the conditions for Rolle's theorem and it doesn't satisfy its consequences either since $\nexists\, x \in (0, 2) : f'(x) = 0$

(b)

$$f(x) = 1 + x^{2/3}$$

in the interval $[-8, 8]$, simply differentiating we have

$$f'(x) = \frac{2}{3}x^{-1/3}$$

which is not defined for $x = 0$ and clearly the derivative is not continuous at $x = 0$ anyways, thus it does not satisfy the conditions for Rolle's theorem and clearly there does not exist $c \in [-8, 8] : f'(c) = 0$

(c)

$$f(x) = x \sin \frac{1}{x}; f(0) = 0$$

Consider

$$f'(0) = \lim_{h\to 0} \frac{\left(h \sin\left(\frac{1}{h}\right)\right)}{h} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

does not exist, hence it does not satisfy the conditions for Rolle's theorem, differentiating on the differentiable intervals

$$f'(x) = \sin\left(\frac{1}{x}\right) - \frac{x \cos\left(\frac{1}{x}\right)}{x^2} = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right)$$

Setting it to 0

$$\tan\left(\frac{1}{u}\right) = \frac{1}{u}$$

There are infinitely many solutions to this equation in the interval $\frac{1}{u} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ or $u \in \left(\infty, -\frac{2}{\pi}\right) \cup \left(\frac{2}{\pi}, u\right)$, so there do exist value of c in the interval which satisfy the consequences of Rolle's theorem even though its conditions are not satisfied.

3. (a) In the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$, Cauchy's MVT gives us

$$\frac{f'(\xi)}{g'(\xi)} = \frac{1-1}{1-(-1)} = 0 = -\frac{\cos(\xi)}{\sin(\xi)}$$

Clearly $\xi = \frac{\pi}{2}$ satisfies this condition

(b) In the interval $\left[0, \frac{1}{2}\right]$ Cauchy's MVT gives us

$$\frac{f'(\xi)}{g'(\xi)} = \frac{\frac{3}{2}(1+\xi)^{1/2}}{\frac{1}{2}(1+\xi)^{-1/2}} = 3(1+\xi) = \frac{f(b)-f(a)}{g(b)-g(a)}$$

which gives us

$$3(1+\xi) = \frac{(u)^{3/2}-1}{\sqrt{u}-1} \implies \xi = \frac{1}{3}(u+\sqrt{u}+1) - 1$$

where $u = \frac{3}{2}$

4. We write LMVT as: for an interval $[a, b]$ if $f(x)$ is continuous and differentiable in the corresponding open interval (a, b), then $\exists\, c \in (a, b) : f'(c) = \frac{f(b)-f(a)}{b-a}$

Now define $\theta = \frac{c-a}{b-a}$, since

$$c \in (a, b) \implies c - a \in (0, b-a) \implies \theta \in (0, 1)$$

Thus we have

$$\boxed{f'(c) = f'(a + c - a) = f'(a + \theta(b-a)) = \frac{f(b)-f(a)}{b-a}}$$

Now we set $b - a = h$ and $a = x$ so we have

$$f'(x + \theta h) = \frac{f(x+h) - f(x)}{h}$$

Now, before we continue, let us examine this equation, if we apply Taylor's theorem to $f(x+h)$, *i.e.* in the interval $(x, x+h)$ we find $\theta_2 \in (0, 1)$ such that

$$hf'(x+\theta h) = \left(f(x) + hf'(x) + \frac{h^2}{2} f''(x + \theta_2 h)\right) - f(x)$$

or

$$\frac{f'(x+\theta h) - f'(x)}{h} = \frac{1}{2} f''(x + \theta_2 h)$$

Now if we take the limit as $h \to 0$

$$\lim_{h\to 0} \frac{f'(x+\theta h) - f(x)}{h} = \theta f''(x) = \lim_{h\to 0} \frac{1}{2} f''(x+\theta_2 h)$$

which implies

$$\lim_{h\to 0} \theta = \frac{1}{2}$$

as a consequence of Taylor's theorem, this however requires the existence of the second derivative of $f(x)$ at x.

Clearly, $\theta = g(x,h)$, *i.e.* θ is a function of x and h, thus

(a) for $f(x) = x^2$ We have

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$$

or

$$\frac{f(x+h) - f(x)}{h} = 2x + h = f'(x+\theta h) = 2(x+\theta h) = 2x + 2\theta h$$

Clearly

$$\theta = \frac{1}{2} \,\forall\, x, h$$

which matches what we had earlier

(b)

$$\frac{f(x+h) - f(x)}{h} = e^x \left(\frac{e^h - 1}{h} \right)$$

While

$$f'(x+\theta h) = e^x e^{\theta h}$$

Thus we have

$$\theta = \frac{1}{h} \ln \frac{e^h - 1}{h}$$

In this case we have θ not dependent on x, a graph of this function is given, taking the limit as

$$\theta_0 = \lim_{h\to 0} \frac{1}{h} \ln \frac{e^h - 1}{h} = \lim_{h\to 0} \frac{1}{h} \ln \frac{h + \frac{h^2}{2} + O(h^3)}{h}$$

which is

$$\theta_0 = \lim_{h\to 0} \frac{1}{h} \ln \left(1 + \frac{h}{2} + O(h^2) \right) = \lim_{h\to 0} \frac{1}{2} + O(h) = \frac{1}{2}$$

which matches what we had earlier.

(c)

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h} \ln \left(1 + \frac{h}{x} \right) = f'(x+\theta h) = \frac{1}{x+\theta h}$$

or we have

$$\theta = \frac{1}{h} \left(\frac{h}{\ln\left(1 + \frac{x}{h}\right)} - x \right)$$

This time, θ is a function of x and h, our claim is that the dependence on x dies out as $h \to 0$ we have

$$\theta = \lim_{h\to 0} \frac{1}{h}\left(\frac{h}{\left(\frac{x}{h}\right) - \frac{1}{2}\left(\frac{x}{h}\right)^2 + O\left(\frac{h^3}{x^3}\right)} - x\right)$$

which is

$$\lim_{h\to 0} \frac{h}{x}\left(\frac{1}{1 - \frac{1}{2}\left(\frac{x}{h}\right) + O\left(\left(\frac{x}{h}\right)^2\right)} - 1\right) = \frac{1}{2}$$

which confirms our claim

5. (a) By LMVT, since f is continuous and differentiable, for $c \in [1, 2]$

$$f(2) = f(1) + f'(c)(1)$$

so

$$f(1) = f(2) - f'(c) = -5 - f'(c)$$

but this lies between

$$f(1) \in [-7, -3]$$

since $f'(c) \in [-2, 2]$ and this maximum value is achieved if $f(x)$ is linear in the interval $(1, 2)$

(b) We know that

$$28^{1/3} = (27+1)^{1/3} = 3\left(1 + \frac{1}{27}\right)^{1/3}$$

applying LMVT we have

$$(1+u)^{1/3} = 1 + \frac{u}{3}(1+v)^{-2/3}$$

where $u = \frac{1}{27}$ and $v \in (0, u)$, let us assume without proof for now that $(1+v)^{-2/3} = 1 + O(u)$, which is a consequence of Taylor's theorem. So we have

$$(28)^{1/3} = 3 + \frac{1}{27} + O\left(\left(\frac{1}{27}\right)^2\right) \approx \frac{82}{27}$$

(c) We have $f''(x) \geq 0$ for $x \in [a, b]$, Let us assume for the sake of contradiction that

$$f\left(\frac{x_1 + x_2}{2}\right) > \frac{1}{2}[f(x_1) + f(x_2)]$$

for *some* $x_1, x_2 \in [a, b]$. Assume WLOG $x_2 > x_1$, now consider the point $\bar{x} = \frac{1}{2}(x_1 + x_2)$, also define $\Delta x = \bar{x} - x_1 = x_2 - \bar{x}$, thus $2\Delta x = x_2 - x_1 > 0$ hence $\Delta x > 0$ then by LMVT on the intervals $(x_1, \bar{x})$ and $(\bar{x}, x_2)$ (whose conditions are satisfied by the assumption that $f''(x)$ exists and is positive) yield $\xi_1 \in (x_1, \bar{x})$ and $\xi_2 \in (\bar{x}, x_2)$ and thus $\xi_2 > \xi_1$ such that

$$f'(\xi_1) = \frac{f(\bar{x}) - f(x_1)}{x - x_1}; f'(\xi_2) = \frac{f(x_2) - f(\bar{x})}{x_2 - \bar{x}}$$

which we rewrite as

$$f(\bar{x}) = f(x_1) + f'(\xi_1)\Delta x = f(x_2) - f'(\xi_2)\Delta x$$

Adding both RHS sides and doubling the LHS

$$2f(\bar{x}) = f(x_1) + f(x_2) + \Delta x[f'(\xi_1) - f(\xi_2)]$$

Or

$$f(\bar{x}) = \frac{1}{2}(f(x_1) + f(x_2)) + \frac{\Delta x}{2}(f'(\xi_1) - f'(\xi_2))$$

But our assumption now implies

$$\frac{\Delta x}{2}(f'(\xi_1) - f'(\xi_2)) > 0$$

but since $\Delta x > 0$ and $\xi_2 > \xi_1$ this contradicts the statement that $f''(x) > 0$ for $[a, b]$, thus our assumption must be false and thus

$$\boxed{f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}[f(x_1) + f(x_2)]}$$

Completing our proof

6. (a) To prove the right hand side of the inequality we apply CMVT to $\sin x$ and x on the interval $[0, x]$ to get

$$\cos\xi = \frac{\sin x}{x} \implies \frac{\sin x}{x} < 1 \implies \sin x < x; \forall x \in \left(0, \frac{\pi}{2}\right)$$

For the LHS, consider the function

$$\zeta(x) = \frac{x}{\sin x} \implies \zeta'(x) = \frac{\sin x - x\cos x}{(\sin x)^2} > 0$$

since $\tan x > x$ in $x \in \left(0, \frac{\pi}{2}\right)$, therefore

$$\zeta(x) < \zeta\left(\frac{\pi}{2}\right) = \frac{\pi}{2}; \forall x \in \left(0, \frac{\pi}{2}\right)$$

which gives

$$\boxed{\frac{2x}{\pi} < \sin x < x}$$

(b) Consider the functions x^n and x on the interval $[a, b]$, clearly CMVT gives

$$n\xi^{n-1} = \frac{b^n - a^n}{b - a}$$

Now if $n > 1$ then $n - 1 > 0$ which implies that x^{n-1} is an increasing function which implies

$$na^{n-1} < n\xi^{n-1} < nb^{n-1}$$

which gives us the inequality

$$\boxed{na^{n-1} < \frac{b^n - a^n}{b - a} < nb^{n-1}}$$

(c) For the right hand side consider the functions $\log(1+x)$ and x on the interval $[0, x]$ CMVT gives

$$1 > \frac{1}{1+\xi} = \frac{\log(1+x)}{x}$$

which gives

$$\log(1+x) < x$$

apply LMVT to the function $\log(1+x)$ on the interval $[0, x]$ gives

$$\log(1+x) = 0 + \frac{x}{1+\xi} > \frac{x}{1+x}$$

since $\xi < x$ or

$$\boxed{\frac{x}{1+x} < \log(1+x) < x}$$

$\forall x > 0$

7. (a) Let the intersection happen at some point $\eta \in (a, b)$, now apply LMVT on (a, η) and (η, b) to give $\xi_1, \xi_2 : \xi_2 > \xi_1$ such that

$$f'(\xi_1) = f'(\xi_2) = m = \frac{f(b) - f(a)}{b - a}$$

now apply Rolle's theorem on $f'(x)$ on the interval (ξ_1, ξ_2)

$$f''(\zeta) = 0$$

for some $\zeta \in (\xi_1, \xi_2) \subset (a, b)$

(b) Apply CMVT to $f(x)$ and x^2 on the interval $[0, 1]$ to give

$$\frac{f'(\xi)}{2\xi} = f(1) - f(0)$$

for $\xi \in (0, 1)$

(c) Consider $g(x) = f(x) - x$ then $g'(x) = f'(x) - 1$, so we have

$$g(a) = g(b) = 0 \implies g'(\xi) = 0$$

for some $\xi \in (a, b)$, Now apply LMVT to (a, ξ) and (ξ, b) to find ζ_1, ζ_2 such that

$$g'(\zeta_1) = \frac{g(\xi)}{\xi - a}, g'(\zeta_2) = \frac{-g(\xi)}{b - \xi}$$

which must have opposite signs, thus applying IVT between (ζ_1, ξ) and (ξ, ζ_2) we can find two numbers c_1, c_2 such that

$$\boxed{g'(c_1) + g'(c_2) = 0 \implies f'(c_1) + f'(c_2) = 2}$$

8. (a) We want to prove that there exists $c \in (a, b)$

$$\begin{vmatrix} f(a) + (b-a)f'(c) & f(b) \\ \phi(a) + (b-a)\phi'(c) & \phi(b) \end{vmatrix} = 0$$

Consider CMVT on $f(x)$ and $\phi(x)$ on the interval (a, b) we have

$$f(b) = f(a) + \frac{f'(c)}{\phi'(c)}(\phi(b) - \phi(a))$$

(b) Apply CMVT on the functions $f(x)$, x, x^2, x^3 to get

$$f(b) - f(a) = f'(x_1)(b-a) = \frac{f'(x)}{2x_2}(b^2 - a^2) = \frac{f'(x_3)}{3x_3^2}(b^3 - a^3)$$

or

$$f'(x_1) = \frac{f'(x_2)}{2x_1} = \frac{f'(x_3)}{3x_3^2}$$

9. (a) Consider $1 - \cos x = 2\sin^2\frac{x}{2}$, setting $u = x/2$ we need to prove

$$\sin x < x$$

which we can do by applying CMVT to these functions

$$\cos\xi = \frac{\sin x}{x}$$

which completes our proof

(b) consider $\frac{f(x)}{x^2}$ and $\frac{1}{x^2}$ applying CMVT we have

$$\frac{\frac{f(b)}{b^2} - \frac{f(a)}{a^2}}{\frac{1}{b^2} - \frac{1}{a^2}} = \frac{\frac{c^2 f'(c) - 2cf(c)}{c^4}}{-\frac{2}{c^3}}$$

$$\frac{a^2 f(b) - b^2 f(a)}{a^2 - b^2} = \frac{1}{2}(2f(c) - cf'(c))$$

as required

(c) Apply CMVT on the functions $\ln x$ and $\arcsin x$ on the interval $[x, 1]$ to get

$$\frac{-\ln x}{\frac{\pi}{2} - \arcsin x} = \frac{2\ln x}{2\arcsin x - \pi} = \sqrt{1 - \frac{1}{\xi^2}}$$

which is an increasing function for ξ so we have

$$\frac{2\ln x}{2\arcsin x - \pi} < \frac{\sqrt{1 - x^2}}{x}$$

for $x \in (0, 1)$

10. Apply LMVT between (a, x_0) and (x_0, b) to find $\xi_1 < \xi_2$ such that

$$f'(\xi_1) = \frac{f(x_0)}{x_0 - a}; f(\xi_2) = \frac{-f(x_0)}{b - x_0}$$

Now assume for the sake of contradiction that there does not exist $c \in (a, b)$ such that $f''(c) < 0$ which would imply $f'(x)$ is an increasing function which implies

$$f'(\xi_2) > f'(\xi_1) \text{ since } \xi_2 > \xi_1$$

but this contradicts what we found and thus our assumption, and therefore our assumption must be wrong and there exists $c \in (a, b)$ such that $f''(c) < 0$