Tutorial 3: Advanced Calculus

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September 15, 2025

Solutions to Problems

1. (a) Clearly with $y = mx$ this limit does not exist

   (b) Same as above

   (c) Put $x = my^2$, to obtain
   $$L = |m|e^{-|m|}$$
   which depends on m, and thus the limit does not exist

   (d) Put $y = mx$

   (e) Put $y = mx^2$, this gives us the expression
   $$L = \frac{mx^4}{x^4 + mx^4} = \frac{m}{1+m}$$
   which again depends on m, and thus the limit does not exist

   (f) Put $r = mx$

   (g) $y = mx$

   (h) Limit does exist and is equal to 1

   (i) Consider $u = xy(x + y)$ then
   $$f(x) = \frac{\sin u}{u}(x + y)$$
   Clearly this tends to 0

   (j) Here we have
   $$f(x, y) = (x - y)\left[1 + \frac{xy}{x^2 + y^2}\right]$$
   which reduces to
   $$f(x, y) = \frac{x - y}{\frac{x}{y} + \frac{y}{x}}$$
   The denominator is bounded $|D| \geq 2$ and thus this tends to 0

   (k) Put $z^2 = ky = hx$

2. (a) I will prove this proposition in more detail to outline the method of proof for the others

We have the function
$$f(x,y) = \frac{4xy^2}{x^2+y^2}$$
and we want to prove that the limit
$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$
that is to say, the limit exists and is equal to 0, we will use the $\epsilon - \delta$ definition of limit to prove this. We establish a bound on $|f(x,y) - L|$, then we use this bound to find δ such that $|(x,y)| < \delta_\epsilon \implies |f(x,y) - L| < \epsilon$

So in this case, we consider
$$|\frac{4xy^2}{x^2+y^2}| \leq 4\sqrt{x^2+y^2}$$
So now if $\epsilon = 4\delta$, $\sqrt{x^2+y^2} < \delta$ implies
$$\epsilon = 4\delta > 4\sqrt{x^2+y^2} \geq |f(x,y) - L|$$
which completes our proof

(b) Here we have to make a change of variables to use our template, Let $u = x+1 \implies x = u-1$ and similarly for $v = y+1 \implies y = v-1$ As before, we want to prove that
$$\lim_{(u,v)\to 0,0} (u-1)(v-1-2u+2) = -1$$
we can assume $|u| < 1$ and $|v| < 1$, now consider
$$|f(u,v) - L| = |(u-1)(v-2u+1)+1| = |uv - 2u^2 + u - v + 2u|$$
which is
$$|uv - 2u^2 + 3u - v| \leq |uv| + 2|u^2| + 3|u| + |v|$$
now apply $|u| < 1$ and $|u| \leq \sqrt{u^2+v^2}$ and $|v| \leq \sqrt{u^2+v^2}$ giving
$$|f(u,v) - L| < 7\sqrt{u^2+v^2}$$
If we set $7\delta = \epsilon$ then $\sqrt{u^2+v^2} < \delta$ implies
$$\epsilon = 7\delta > 7\sqrt{u^2+v^2} > |f(u,v) - L|$$

(c) define $u = x-1$ so we want
$$\lim_{(u,y)\to(0,0)} f(u,y) = \frac{u^2\ln(1+u)}{u^2+y^2} = 0$$
consider
$$|f(u,y) - L| = |\frac{u^2\ln(1+u)}{u^2+y^2}| = |u^2/(u^2+y^2)|\ln(1+u) \leq \ln(1+u) \leq \ln(e^u) =\leq \sqrt{u^2+y^2}$$
using $e^u \geq 1+u$ from calculus, now if $\delta = \epsilon$, $\delta > \sqrt{u^2+y^2}$ implies
$$\epsilon = \delta > \sqrt{u^2+y^2} \geq |f(u,y) - L|$$
which completes our proof

(d) Let $u = x + 2$, $v = y - 2$, now we need to prove that

$$\lim_{(u,v)\to(0,0)} u - v + 4 = 4$$

now consider

$$|f(u,v) - L| = |u - v| \leq |u| + |v| \leq 2\sqrt{u^2 + v^2}$$

Now if $\epsilon = 2\delta$ then $\delta > \sqrt{u^2 + v^2}$ implies

$$\epsilon = 2\delta > 2\sqrt{u^2 + v^2} > |f(u,v) - L|$$

Completing our proof

(e) Here we want to prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2}$$

Clearly $x^2 - y^2 < x^2 + y^2$ which gives us

$$|f(x,y) - L| = |xy\frac{x^2 - y^2}{x^2 + y^2}| \leq |xy|$$

Assume $|x| < 1$ then we have

$$|xy| < |y| < \sqrt{x^2 + y^2}$$

now if $\delta = \epsilon$, $\delta > \sqrt{x^2 + y^2}$ implies

$$\epsilon = \delta > \sqrt{x^2 + y^2} \geq |f(x,y) - L|$$

which completes our proof

(f) We have

$$f(x,y) = x \sin x \cos y$$

We want to prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = L = 0$$

consider

$$|f(x,y) - L| = |x \sin x \cos y| \leq |x| \leq \sqrt{x^2 + y^2}$$

now if $\delta = \epsilon$, $\delta > \sqrt{x^2 + y^2}$ implies

$$\epsilon = \delta > \sqrt{x^2 + y^2} \geq |f(x,y) - L|$$

which completes our proof

(g) We have

$$f(x,y) = \frac{x^2}{\sqrt{x^2 + y^2}}$$

We want to prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = L = 0$$

consider

$$|f(x,y) - L| = \frac{x^2}{\sqrt{x^2+y^2}}| \leq \sqrt{x^2+y^2}$$

since, $x^2 \leq x^2 + y^2$, now if $\delta = \epsilon$, $\delta > \sqrt{x^2+y^2}$ implies

$$\epsilon = \delta > \sqrt{x^2+y^2} \geq |f(x,y) - L|$$

which completes our proof

(h) We have

$$f(x,y) = \frac{x^2y^2}{x^2+y^2}$$

We want to prove that

$$\lim_{(x,y)\to(0,0)} f(x,y) = L = 0$$

consider

$$|f(x,y) - L| = \frac{x^2y^2}{x^2+y^2} < x^2 + y^2 < \sqrt{x^2+y^2}$$

if we assume $x^2 + y^2 < 1$ now if $\delta = \epsilon$, $\delta > \sqrt{x^2+y^2}$ implies

$$\epsilon = \delta > \sqrt{x^2+y^2} \geq |f(x,y) - L|$$

which completes our proof