Tutorial 2: Taylor's Theorem

Arjun Ghosh

September 15, 2025

1. Use Taylor's theorem to prove:

   (a) Consider the function $f(x) = \log(1+x)$, applying Taylor's theorem to the first order gives

   $$\log(1+x) = 0 + x - \frac{1}{(1+c)^2}\frac{x^2}{2} < x$$

   For $0 < c < x$, now to the second order we have

   $$\log(1+x) = 0 + x - \frac{x^2}{2} + \frac{1}{6}\frac{2}{(1+c)^3}x^3 > x - \frac{x^2}{2}$$

   combining the two we have

   $$x - \frac{x^2}{2} < \log(1+x) < x$$

   (b) Consider the function $\cos x$ and its second order Taylor's theorem

   $$\cos x = 1 - \frac{x^2}{2!} + (\cos x)'''|_{x=c}\frac{x^3}{3!}$$

   now $(\cos x)''' = -\cos(x)' = \sin(x)$, Now let us break our problem into two cases, $x \in [0, \pi]$ then we have $c \in (0, x)$ and so $\sin c \times x^3 > 0$ similarly if $x \in [-\pi, 0]$ then we have $c \in (x, 0)$ and similarly $\sin c \times x^3 > 0$ and thus

   $$\cos x \geq 1 - \frac{x^2}{2!}$$

   with equality at $x = 0$

   (c) Consider the function $f(x) = \sqrt{1+x}$ and its Taylor's theorem to order 1 in the interval $(0, x)$

   $$\sqrt{1+x} = 1 + \frac{x}{2} + \frac{x^2}{2!}\left(\frac{1}{2} \times \left(-\frac{1}{2}\right) \times (1+c)^{-3/2}\right) > 1 + \frac{x}{2}$$

   and similarly to the second order

   $$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \left(\frac{x^3}{3!}\right)\left[\frac{1}{2} \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right) \times (1+c)^{-5/2}\right] < 1 + \frac{x}{2} - \frac{x^2}{8}$$

2. Since f'' is continuous around c then $f'(x)$ must exist and be continuous around c which gives

$$f'(c-h) = \lim_{h\to 0} \frac{f(c) - f(c-h)}{h}$$

and

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

Subtract these two equations and divide by h

$$\lim_{h\to 0} \frac{f'(c) - f'(c-h)}{h} = \lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$$

3. Ff we apply Taylor's theorem to $f(x+h)$, *i.e.* in the interval $(x, x+h)$ we find $\theta_2 \in (0,1)$ such that

$$hf'(x+\theta h) = \left(f(x) + hf'(x) + \frac{h^2}{2} f''(x+\theta_2 h)\right) - f(x)$$

or

$$\frac{f'(x+\theta h) - f'(x)}{h} = \frac{1}{2} f''(x+\theta_2 h)$$

Now if we take the limit as $h \to 0$

$$\lim_{h\to 0} \frac{f'(x+\theta h) - f(x)}{h} = \theta f''(x) = \lim_{h\to 0} \frac{1}{2} f''(x+\theta_2 h)$$

which implies

$$\lim_{h\to 0} \theta = \frac{1}{2}$$

as a consequence of Taylor's theorem, this however requires the existence of the second derivative of $f(x)$ at x.

4. Evaulate the following series

   (a) Let $u = \pi$ so we have

   $$u - \frac{u^3}{3!} + \frac{u^5}{5!} + \ldots$$

   which is clearly $\sin u = \sin \pi = 0$

   (b) Let $u = \frac{2}{3}$

   $$u - \frac{u^2}{2} + \frac{u^3}{3} + \ldots$$

   is clearly $\ln(1-u) = \ln\left(\frac{1}{3}\right) = -\ln 3$

   (c) Let $u = \frac{1}{\sqrt{3}}$

   $$u - \frac{1}{3}u^3 + \frac{u^5}{5} + \ldots$$

   which is clearly $\arctan(u) = \frac{\pi}{6}$

5. Evaluate the following limits

   (a) We combine the two fractions and substitute $\sin x = x - \frac{x^3}{3!} + \ldots$ which give

   $$\lim_{x\to 0} \frac{x - \frac{x^3}{3!} - x}{x \sin x} = 0$$

(b) expanding both e^x and $\log(1+x)$ we have

$$\lim_{x\to 0} \frac{x\left(1+x+\frac{x^2}{2!}\right) - \left(x+\frac{x^2}{2}\right)}{x^2} = \frac{1}{2}$$

(c) expanding the series for $\tan x = x + \frac{1}{3}x^3 + \dots$

$$\lim_{x\to 0} \frac{x + \frac{1}{3}x^3 - x}{x^2 \tan x} = \frac{1}{3}$$

(d) Here first we find the expansion for $\cosh x = \frac{e^x + e^{-x}}{2}$

$$2\cosh x = 1 + x + \frac{x^2}{2} + \dots + 1 - x + \frac{x^2}{2}$$

or

$$\cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

Substituting in the limit

$$\lim_{x\to 0} \frac{\cosh x - \cos x}{x \sin x} = \lim_{x\to 0} \frac{1 + \frac{x^2}{2} - 1 + \frac{x^2}{2}}{x^2} = 1$$

6. We have

$$\lim_{x\to x_0} \frac{f(x) - a_0 - a_1(x - x_0)}{x - x_0} = 0$$

We know this limit exists and is equal to zero, further we know that a_1 is finite so we can can conclude that the limit

$$\lim_{x\to x_0} \frac{f(x) - a_0}{x - x_0} = a_1$$

exists and is equal to the finite value a_1 Now if we make the manipulation

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{f(x_0) - a_0}{x - x_0} = a_1$$

We can split this limit because we know that the function $f(x)$ is differentiable at x_0 and thus the left limit exists, therefore the right limit must also exist, also note that the numerator is independent of x, thus the only way for this limit to exist is if

$$f(x_0) = a_0$$

which leaves us with

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) = a_1$$

7. Let $f(x) = \sin(m \arcsin x)$ Clearly differentiating this function several times will be quite tedious, also note that the function is only defined in $x \in [-1, 1]$, Also note that the function is odd, *i.e*

$$\sin(m \arcsin -x) = \sin(-m \arcsin x) = -\sin(m \arcsin x)$$

Thus the coefficients of all the even powers of x in the Maclaurin series will be zero, let us try to form a relation between the derivatives of $f(x)$, let $\arcsin x = \theta$

$$f'(x) = \cos m\theta \times m \times \frac{d\theta}{dx}$$

but $\sin\theta = x$ so $\cos\theta \frac{d\theta}{dx} = 1$

$$\cos\theta f'(x) = m\cos m\theta$$

Differentiating again

$$\cos\theta f''(x) - \sin\theta f'(x)\frac{d\theta}{dx} = -m^2 \sin m\theta \times \frac{d\theta}{dx}$$

Or

$$(1-x^2)f''(x) - xf'(x) + m^2 f(x) = 0$$

Now differentiate this equation n times

$$D^n(1-x^2)f''(x) = \sum_{k=0}^{n} \binom{n}{k} (1-x^2)^{(k)} f^{(2+n-k)}(x)$$

Clearly this is zero for all $k > 2$, for $k = 0, 1, 2$

$$= (1-x^2)f^{(n+2)}(x) - 2nxf^{(n+1)}(x) - 2\binom{n}{2} f^{(n)}(x)$$

Repeating for $xf(x)$

$$D^{(n)}xf'(x) = \sum_{k=0}^{n} \binom{n}{k} (x)^{(k)} f^{(1+n-k)}(x)$$

which has only two non zero terms

$$= xf^{(n+1)}(x) + nf^{(n)}(x)$$

Collecting like terms we have

$$(1-x^2)f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) + (m^2 - n^2)f^{(n)}(x) = 0$$

now at $x = 0$

$$f^{(n+2)}(0) = (n^2 - m^2)f^{(n)}(0)$$

Now let $f^{(n)}(0) = a_n$, so we have

$$a_{n+2} = (n^2 - m^2)a_n$$

$a_1 = 1$, so

$$\begin{aligned} a_3 &= (1-m^2) \\ a_5 &= (1-m^2)(9-m^2) \\ a_7 &= (1-m^2)(9-m^2)(25-m^2) \end{aligned}$$

$$a_{2n+1} = \prod_{k=1}^{n} ((2n+1)^2 - m^2)$$

Thus we have

$$f(x) = \sum_{r=1}^{\infty} \frac{x^{2r+1}}{(2r+1)!} \prod_{k=1}^{r} ((2k-1)^2 - m^2)$$

8. We want the smallest n such that

$$\frac{e^c}{(n+1)!}x^n \leq C$$

$\forall x, c \in [-1, 1]$, clearly this is maximum at $x = c = 1$ giving us

$$n! \leq \frac{e}{C} \leq (n+1)!$$

here $C = 0.005$ which gives us

$$n! \leq 543 < (n+1)!$$

which is satisfied by $n = 5$

9. (a) We know that the nth term of the Taylor Series expansion is given by

$$T_n = \frac{x^n}{n!}(D^n f)(0)$$

Now, let $f(x) = (1+x)^h$ so we have

$$D^n f(x) = \underbrace{h(h-1)(h-2)\dots(h-n+1)}_{n \text{ terms}}(1+x)^{h-n} = h^{\underline{n}}(1+x)^{h-n}$$

At $x = 0$ we have

$$T_n = \frac{h^{\underline{n}}}{n!}x^n$$

(b) Clearly for $m \in \mathbb{N}$ we have $m^{\underline{n}} = n!\binom{m}{n}$ so we have

$$(1+x)^m = \binom{m}{0} + \binom{m}{1}x + \dots \binom{m}{k}x^k + \dots \binom{m}{n}x^n$$

Noting that for terms with powers greater than m the value of $m^{\underline{n}}$ is zero because they are integers

10. (a) We have $1.5 = 1 + \frac{1}{2} = 1 + x$, thus by Taylor's theorem we have

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}(1+c)^{-2.5}x^3$$

evaluating the binomial coefficients we have $\binom{\frac{1}{2}}{2} = -\frac{1}{8}$ and $\binom{\frac{1}{2}}{3} = \frac{1}{16}$, where $c \in (0, \frac{1}{2})$. The estimate itself is

$$\sqrt{1.5} = 1 + \frac{1}{4} - \frac{1}{32} = \frac{32 + 8 - 1}{32} = 1\frac{7}{32} \approx 1.21875$$

(b) Clearly the error is maximum when $c = 0$ and minimum when $c = \frac{1}{2}$ so we have, let the error be

$$E = (1+x)^{1/2} - 1 - \frac{1}{2}x + \frac{1}{8}x^2 = \frac{1}{16}(1+c)^{-5/2}x^3$$

Using Lagrange's form we have the error bounded as

$$0.00283 \approx \frac{1}{16}\frac{x^3}{(1+x)^{2.5}} < E < \frac{1}{16}x^3 \approx 0.0078125$$

The actual error is $\sqrt{1.5} - \frac{39}{32} \approx 0.005994$ which falls cleanly within our bounds

11. Estimation of an integral

(a) For convenience let us define $T_0 = 2\pi\sqrt{\frac{l}{g}}$ which is what we expect the answer be close to, further define $\Theta = \frac{\pi}{2}$ which is the maximum angle, thus we have

$$T = \frac{T_0}{\Theta}\int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2\theta}}\,d\theta$$

Now expand the binomial series to 1 term

$$T = \frac{T_0}{\Theta}\int_0^{\Theta} d\theta = T_0 = 2\pi\sqrt{\frac{l}{g}}$$

as expected

(b) Now if we use the binomial series to approximate $(1 - k^2\sin^2\theta)^{-1/2}$ we get

$$T = \frac{T_0}{\Theta}\int_0^{\Theta} 1 + \frac{1}{2}k^2\sin^2\theta\,d\theta$$

we know that $\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{\pi}{4}$ since the average value of $\sin^2 x$ is $\frac{1}{2}$ so we have

$$T \approx T_0\left(1 + \frac{k^2}{4}\right)$$

where $k = \sin\frac{\theta_0}{2}$, if $\theta \ll 1$ then

$$T \approx T_0\left(1 + \frac{\theta_0^2}{16}\right)$$

which matches empirical observations