

Tutorial 2: Taylor's Theorem

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1. Use Taylor's theorem to prove:

- (a) Consider the function $f(x) = \log(1+x)$, applying Taylor's theorem to the first order gives

$$\log(1+x) = 0 + x - \frac{1}{(1+c)^2} \frac{x^2}{2} < x$$

For $0 < c < x$, now to the second order we have

$$\log(1+x) = 0 + x - \frac{x^2}{2} + \frac{1}{6} \frac{2}{(1+c)^3} x^3 > x - \frac{x^2}{2}$$

combining the two we have

$$x - \frac{x^2}{2} < \log(1+x) < x$$

- (b) Consider the function $\cos x$ and its second order Taylor's theorem

$$\cos x = 1 - \frac{x^2}{2!} + (\cos x)'''|_{x=c} \frac{x^3}{3!}$$

now $(\cos x)''' = -\cos(x)' = \sin(x)$, Now let us break our problem into two cases, $x \in [0, \pi]$ then we have $c \in (0, x)$ and so $\sin c \times x^3 > 0$ similarly if $x \in [-\pi, 0]$ then we have $c \in (x, 0)$ and similarly $\sin c \times x^3 > 0$ and thus

$$\cos x \geq 1 - \frac{x^2}{2!}$$

with equality at $x = 0$

- (c) Consider the function $f(x) = \sqrt{1+x}$ and its Taylor's theorem to order 1 in the interval $(0, x)$

$$\sqrt{1+x} = 1 + \frac{x}{2} + \frac{x^2}{2!} \left(\frac{1}{2} \times \left(-\frac{1}{2} \right) \times (1+c)^{-3/2} \right) > 1 + \frac{x}{2}$$

and similarly to the second order

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \left(\frac{x^3}{3!} \right) \left[\frac{1}{2} \times \left(-\frac{1}{2} \right) \times \left(-\frac{3}{2} \right) \times (1+c)^{-5/2} \right] < 1 + \frac{x}{2} - \frac{x^2}{8}$$

2. Since f'' is continuous around c then $f'(x)$ must exist and be continuous around c which gives

$$f'(c-h) = \lim_{h \rightarrow 0} \frac{f(c) - f(c-h)}{h}$$

and

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

Subtract these two equations and divide by h

$$\lim_{h \rightarrow 0} \frac{f'(c) - f'(c-h)}{h} = \lim_{h \rightarrow 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$$

3. If we apply Taylor's theorem to $f(x+h)$, *i.e.* in the interval $(x, x+h)$ we find $\theta_2 \in (0, 1)$ such that

$$hf'(x+\theta h) = \left(f(x) + hf'(x) + \frac{h^2}{2}f''(x+\theta_2 h) \right) - f(x)$$

or

$$\frac{f'(x+\theta h) - f'(x)}{h} = \frac{1}{2}f''(x+\theta_2 h)$$

Now if we take the limit as $h \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{f'(x+\theta h) - f'(x)}{h} = \theta f''(x) = \lim_{h \rightarrow 0} \frac{1}{2}f''(x+\theta_2 h)$$

which implies

$$\lim_{h \rightarrow 0} \theta = \frac{1}{2}$$

as a consequence of Taylor's theorem, this however requires the existence of the second derivative of $f(x)$ at x .

4. Evaluate the following series

- (a) Let $u = \pi$ so we have

$$u - \frac{u^3}{3!} + \frac{u^5}{5!} + \dots$$

which is clearly $\sin u = \sin \pi = 0$

- (b) Let $u = \frac{2}{3}$

$$u - \frac{u^2}{2} + \frac{u^3}{3} + \dots$$

is clearly $\ln(1-u) = \ln\left(\frac{1}{3}\right) = -\ln 3$

- (c) Let $u = \frac{1}{\sqrt{3}}$

$$u - \frac{1}{3}u^3 + \frac{u^5}{5} + \dots$$

which is clearly $\arctan(u) = \frac{\pi}{6}$

5. Evaluate the following limits

- (a) We combine the two fractions and substitute $\sin x = x - \frac{x^3}{3!} + \dots$ which give

$$\lim_{x \rightarrow 0} \frac{x - \frac{x^3}{3!} - x}{x \sin x} = 0$$

(b) expanding both e^x and $\log(1+x)$ we have

$$\lim_{x \rightarrow 0} \frac{x \left(1 + x + \frac{x^2}{2!}\right) - \left(x + \frac{x^2}{2}\right)}{x^2} = \frac{1}{2}$$

(c) expanding the series for $\tan x = x + \frac{1}{3}x^3 + \dots$

$$\lim_{x \rightarrow 0} \frac{x + \frac{1}{3}x^3 - x}{x^2 \tan x} = \frac{1}{3}$$

(d) Here first we find the expansion for $\cosh x = \frac{e^x + e^{-x}}{2}$

$$2 \cosh x = 1 + x + \frac{x^2}{2} + \dots + 1 - x + \frac{x^2}{2}$$

or

$$\cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

Substituting in the limit

$$\lim_{x \rightarrow 0} \frac{\cosh x - \cos x}{x \sin x} = \lim_{x \rightarrow 0} \frac{1 + \frac{x^2}{2} - 1 + \frac{x^2}{2}}{x^2} = 1$$

6. We have

$$\lim_{x \rightarrow x_0} \frac{f(x) - a_0 - a_1(x - x_0)}{x - x_0} = 0$$

We know this limit exists and is equal to zero, further we know that a_1 is finite so we can conclude that the limit

$$\lim_{x \rightarrow x_0} \frac{f(x) - a_0}{x - x_0} = a_1$$

exists and is equal to the finite value a_1 Now if we make the manipulation

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \rightarrow x_0} \frac{f(x_0) - a_0}{x - x_0} = a_1$$

We can split this limit because we know that the function $f(x)$ is differentiable at x_0 and thus the left limit exists, therefore the right limit must also exist, also note that the numerator is independent of x , thus the only way for this limit to exist is if

$$f(x_0) = a_0$$

which leaves us with

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) = a_1$$

7. Let $f(x) = \sin(m \arcsin x)$ Clearly differentiating this function several times will be quite tedious, also note that the function is only defined in $x \in [-1, 1]$, Also note that the function is odd, *i.e*

$$\sin(m \arcsin -x) = \sin(-m \arcsin x) = -\sin(m \arcsin x)$$

Thus the coefficients of all the even powers of x in the Maclaurin series will be zero, let us try to form a relation between the derivatives of $f(x)$, let $\arcsin x = \theta$

$$f'(x) = \cos m\theta \times m \times \frac{d\theta}{dx}$$

but $\sin \theta = x$ so $\cos \theta \frac{d\theta}{dx} = 1$

$$\cos \theta f'(x) = m \cos m\theta$$

Differentiating again

$$\cos \theta f''(x) - \sin \theta f'(x) \frac{d\theta}{dx} = -m^2 \sin m\theta \times \frac{d\theta}{dx}$$

Or

$$(1 - x^2)f''(x) - xf'(x) + m^2f(x) = 0$$

Now differentiate this equation n times

$$D^n(1 - x^2)f''(x) = \sum_{k=0}^n \binom{n}{k} (1 - x^2)^{(k)} f^{(2+n-k)}(x)$$

Clearly this is zero for all $k > 2$, for $k = 0, 1, 2$

$$= (1 - x^2)f^{(n+2)}(x) - 2nx f^{(n+1)}(x) - 2 \binom{n}{2} f^{(n)}(x)$$

Repeating for $xf(x)$

$$D^{(n)}xf'(x) = \sum_{k=0}^n \binom{n}{k} (x)^{(k)} f^{(1+n-k)}(x)$$

which has only two non zero terms

$$= xf^{(n+1)}(x) + nf^{(n)}(x)$$

Collecting like terms we have

$$(1 - x^2)f^{(n+2)}(x) - (2n + 1)xf^{(n+1)}(x) + (m^2 - n^2)f^{(n)}(x) = 0$$

now at $x = 0$

$$f^{(n+2)}(0) = (n^2 - m^2)f^{(n)}(0)$$

Now let $f^{(n)}(0) = a_n$, so we have

$$a_{n+2} = (n^2 - m^2)a_n$$

$a_1 = 1$, so

$$a_3 = (1 - m^2)$$

$$a_5 = (1 - m^2)(9 - m^2)$$

$$a_7 = (1 - m^2)(9 - m^2)(25 - m^2)$$

$$a_{2n+1} = \prod_{k=1}^n ((2k+1)^2 - m^2)$$

Thus we have

$$f(x) = \sum_{r=1}^{\infty} \frac{x^{2r+1}}{(2r+1)!} \prod_{k=1}^r ((2k+1)^2 - m^2)$$

8. We want the smallest n such that

$$\frac{e^c}{(n+1)!} x^n \leq C$$

$\forall x, c \in [-1, 1]$, clearly this is maximum at $x = c = 1$ giving us

$$n! \leq \frac{e}{C} \leq (n+1)!$$

here $C = 0.005$ which gives us

$$n! \leq 543 < (n+1)!$$

which is satisfied by $n = 5$

9. (a) We know that the n th term of the Taylor Series expansion is given by

$$T_n = \frac{x^n}{n!} (D^n f)(0)$$

Now, let $f(x) = (1+x)^h$ so we have

$$D^n f(x) = \underbrace{h(h-1)(h-2) \dots (h-n+1)}_{n \text{ terms}} (1+x)^{h-n} = h^n (1+x)^{h-n}$$

At $x = 0$ we have

$$T_n = \frac{h^n}{n!} x^n$$

(b) Clearly for $m \in \mathbb{N}$ we have $m^n = n! \binom{m}{n}$ so we have

$$(1+x)^m = \binom{m}{0} + \binom{m}{1}x + \dots + \binom{m}{k}x^k + \dots + \binom{m}{n}x^n$$

Noting that for terms with powers greater than m the value of m^n is zero because they are integers

10. (a) We have $1.5 = 1 + \frac{1}{2} = 1 + x$, thus by Taylor's theorem we have

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \left(\frac{\frac{1}{2}}{2}\right)x^2 + \left(\frac{\frac{1}{2}}{3}\right)(1+c)^{-2.5}x^3$$

evaluating the binomial coefficients we have $\left(\frac{1}{2}\right) = -\frac{1}{8}$ and $\left(\frac{1}{2}\right) = \frac{1}{16}$, where $c \in (0, \frac{1}{2})$. The estimate itself is

$$\sqrt{1.5} = 1 + \frac{1}{4} - \frac{1}{32} = \frac{32+8-1}{32} = 1\frac{7}{32} \approx 1.21875$$

(b) Clearly the error is maximum when $c = 0$ and minimum when $c = \frac{1}{2}$ so we have, let the error be

$$E = (1+x)^{1/2} - 1 - \frac{1}{2}x + \frac{1}{8}x^2 = \frac{1}{16}(1+c)^{-5/2}x^3$$

Using Lagrange's form we have the error bounded as

$$0.00283 \approx \frac{1}{16} \frac{x^3}{(1+x)^{2.5}} < E < \frac{1}{16} x^3 \approx 0.0078125$$

The actual error is $\sqrt{1.5} - \frac{39}{32} \approx 0.005994$ which falls cleanly within our bounds

11. Estimation of an integral

- (a) For convenience let us define $T_0 = 2\pi\sqrt{\frac{l}{g}}$ which is what we expect the answer be close to, further define $\Theta = \frac{\pi}{2}$ which is the maximum angle, thus we have

$$T = \frac{T_0}{\Theta} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta$$

Now expand the binomial series to 1 term

$$T = \frac{T_0}{\Theta} \int_0^{\Theta} d\theta = T_0 = 2\pi\sqrt{\frac{l}{g}}$$

as expected

- (b) Now if we use the binomial series to approximate $(1 - k^2 \sin^2 \theta)^{-1/2}$ we get

$$T = \frac{T_0}{\Theta} \int_0^{\Theta} 1 + \frac{1}{2}k^2 \sin^2 \theta d\theta$$

we know that $\int_0^{\pi/2} \sin^2 \theta d\theta = \frac{\pi}{4}$ since the average value of $\sin^2 x$ is $\frac{1}{2}$ so we have

$$T \approx T_0 \left(1 + \frac{k^2}{4}\right)$$

where $k = \sin \frac{\theta_0}{2}$, if $\theta \ll 1$ then

$$T \approx T_0 \left(1 + \frac{\theta_0^2}{16}\right)$$

which matches empirical observations