Tutorial 1: 2D and 3D Equilibrium

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1. **Vector Form of a force**
   Clearly the force points along the vector $\vec{MP}$, now

$$\vec{MP} = \vec{r} = \vec{P} - \vec{M} = \begin{bmatrix} R\cos\theta\sin\phi \\ R\sin\theta\sin\phi \\ R\cos\phi \end{bmatrix} - \begin{bmatrix} \frac{R}{2}\cos\theta \\ \frac{R}{2}\sin\theta \\ 0 \end{bmatrix} = \frac{R}{2}\begin{bmatrix} \cos\theta\,(2\sin\phi - 1) \\ \sin\theta\,(2\sin\phi - 1) \\ R\cos\phi \end{bmatrix}$$

   Thus,

$$\vec{F} = F\hat{r} = \frac{F}{r}\vec{r}$$

   where

$$r = \frac{R}{2}\sqrt{(2\sin\phi - 1)^2 + \cos^2\phi}$$

2. **Force Triangle**
   Consider the force triangle so formed

$$F_a = \frac{F}{\sin 60}; F_b = \frac{F}{\tan 60}$$

   where $F = 600$ N , which gives

$$F_a = 692.8 \text{ N}; F_b = 346.4 \text{ N}$$

3. **2D Torque**
   In 2D, a cross product

$$\vec{A} \times \vec{B} = \mathrm{Im}[A^* B]\hat{k}$$

   where A and B are the complex representations of the vectors $\vec{A}$ and $\vec{B}$ The torque is given by

$$\vec{\tau} = \vec{r} \times \vec{F} = (-100, -200)\text{mm} \times 300\angle 25 \text{ N} = -30(1+2i) \times (\angle\bar{25}) = 41.699 \text{ Nm}$$

   The torque is maximum when $\sin\theta = 1$, and do

$$F_{\min} = \frac{\tau}{r_{DB}} = \mathrm{Im}\left(\frac{30(1-2i)\angle 25}{|200 + 200i|}\right) = 147.43 \text{ N}$$

4. **3D Torque**
   We want to calculate the torque

$$\vec{\tau} = \vec{r} \times \vec{F} = 24 \cdot (0, 18, 30) \times \mathrm{uvec}(-6, -5, -30) = (-301.93, -139.35, 83.61)$$

   where uvec is the unit vector function

5. **Splitting the problem**
   Divide the rod into three parts, along z axis, along y axis and along x axis, their weights are
   $$w_x = 137.34 \text{ N}, w_y = 75.537 \text{ N}$$
   and their moment arms are
   $$y_y = 0.55m; y_x = 1.1m; x_x = 0.2m$$
   The moment about the x axis
   $$M_{Ox} = -w_x y_x - w_y y_y = -(137.34 \times 1.1 + 75.537 \times 0.55) = 192.61935 \text{ Nm}$$
   while about the
   $$M_{Oy} = -w_x x_x = -137.34 \times 0.2 = -27.428$$
   And hence
   $$M_O = 194.55938289 \text{ Nm}$$

6. **Moment along an axis**
   Project everything down the z axis to obtain a 2D figure, now
   $$\tau_z = \vec{r_{xy}} \times \vec{F_{xy}} = \text{Im}(2 \times (1.2)\overline{\text{uvec(-1.2+0.5i)}}) = 0.923 \text{ kNm}$$

7. **Moments in a plane**
   The torque about each axis is just the vector torque about O, for which $\vec{r} = (0, 0, 2.25)$ hence there can not be any force along the z axis, otherwise the torque is
   $$\vec{\tau} = \vec{r} \times \vec{F} = F(\vec{r} \times \hat{F}) = (1349 \text{ N})(0, 0, 2.25) \times \text{uvec}(0.9, 1.5, -2.25) = (-1597.5, 958.5, 0) \text{ Nm}$$

8. **Decomposing Moments**
   The angle ϕ is between P and the z axis, so we can find the cartesian vector for $\vec{P}$
   $$\vec{P} = P(0, -\sin\phi, \cos\phi)$$
   Using the two force components M_y and M_z
   $$\begin{aligned} M_y = xF_z - zF_x &= Px\cos\phi \\ M_z = r_z F_y - yF_x &= Px\sin\phi \end{aligned}$$
   We obtain ϕ by dividing the two
   $$\tan\phi = \frac{M_z}{M_y} = 3.43 \implies \phi = 73.74°$$
   and P by squaring and adding them
   $$M_y^2 + M_z^2 = P^2x^2 \implies P = \frac{1}{x}\sqrt{M_y^2 + M_z^2} \implies P = 125 \text{ N}$$
   Now expand the magnitude of the moment about x to get $\theta + \phi$
   $$M_x = P \times (200 \text{ mm}) \times \sin(\theta + \phi)$$
   and thus obtain θ
   $$\theta = \arcsin\left(\frac{x}{l}\frac{M_x}{\sqrt{M_z^2 + M_y^2}}\right) - \arctan\left(\frac{M_z}{M_y}\right) = 53.13°$$

9. **Re-Scaled Force Triangle**
Consider the force triangle, and the two shown triangles, a triangle similar to the fore triangle can be constructed by scaling down AC by $\frac{BO}{CO}$

$$\frac{T_{AB}}{T_{AC}} = \frac{AB}{AC} \times \frac{CO}{BO} \implies T_{AB} = (8 \text{ kN})\frac{|(40,50)|}{|(40,60)|} \times \frac{40}{50} = 5.682 \text{ kN}$$

Further

$$R = T_{AC}\left(\frac{y_C}{x_C} + \frac{y_B}{x_B}\right)\frac{x_C}{AC} = 10.206 \text{ kN}$$

10. **Equivalent Force**
Then the torque about O must be zero, hence

$$M = 400 \times 150\cos 30 + 320 \times 300 = 147 \text{ N mm}$$

and it must be anti clock wise

11. **Equivalent Moment Arm in 2D**
Since this is in 2D, we can use complex numbers

$$F_{\text{equiv}} = T + T\angle 15^\circ = T(1 + \angle 15^\circ)$$

While

$$\tau = \text{Im}(T\overline{(-10+3i)} + T\angle 15^\circ\overline{(-10-3i)}) = T\,\text{Im}([-10(2+\angle 15^\circ) + 3i(\angle 15 - 1)])$$

If F passes through some point on the x axis then its torque is

$$\tau = xF_y \implies x = 10.3949 \text{ m}$$

12. **Equivalent Moment Arm in 3D**
Clearly

$$F_{\text{equiv}} = 3 \times 90 = 270 \text{ kN}$$

also, since all four forces produced no torque, the torque they produce now, must the negative of what engine 3 produced

$$\vec{\tau} = -\vec{\tau_3} = -\vec{r_3} \times \vec{F_3} = \vec{r} \times (3\vec{F_3}) \implies \vec{r} = -\frac{1}{3}\vec{r_3}$$

thus $y = -4$m and $z = 1$m