Tutorial 5: Implicit differentiation and Euler's Theorem

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1. Find $\frac{df}{dt}$ at $t = 0$ for the following

   (a) $f(x, y) = x\cos y + e^x \sin y$ where $x(t) = t^2 + 1$ and $y(t) = t^3 + t$
   The derivative can be found using chain rule

   $$\frac{df}{dt} = f_x\, x_t + f_y\, y_t = (2t)(\cos y + e^x \sin y) + (3t^2 + 1)(-x\sin y + e^x \cos y)$$

   where $x(t)$ and $y(t)$ are given as above

   (b) $f(x, y, z) = x^3 + xz^2 + y^3 + xyz$ where $x(t) = e^t, y(t) = \cos t$ and $z(t) = t^3$
   Again, by chain rule

   $$\frac{d}{dt} f(x, y, z) = f_x\, x_t + f_y\, y_t + f_z\, z_t = (e^t)(3x^2 + z^2 + yz) + (-\sin t)(3y^2 + xz) + (3t^2)(2xz + xy)$$

   where the intermediate variables can be expanded as before

   (c) $f(x_1, x_2, x_3) = 2x_1^2 - x_2x_3 + x_1x_3^2$ where $x_1(t) = 2\sin t, x_2(t) = t^2 - t + 1$ and $x_3(t) = 3^{-t}$
   Again by chain rule we have

   $$\frac{df}{dt} = (2\cos t)(4x_1 + x_3^2) + (2t - 1)(-x_3) + (-\ln 3 \times 3^{-t})(-x_2 + 2x_1x_3)$$

   as before

2. (a) Find $\frac{dy}{dx}$ for the following

   i. $x^y + y^x = c$
   Differentiating we have

   $$x^y d(y \ln x) + y^x d(x \ln y) = x^y \left(\ln(x)dy + \frac{ydx}{x} \right) + y^x \left(\ln(y)dx + \frac{xdy}{y} \right) = 0$$

   Collect like terms

   $$dy \left(x^y \ln(x) + \frac{xy^x}{y} \right) + dx \left(y^x \ln(y) + \frac{y}{x} x^y \right) = 0$$

   Divide to find $\frac{dy}{dx}$

ii. $xy^2 + e^x \sin y^2 + \arctan(x+y) = c$
Again, we differentiate

$$y^2dx + 2xydy + e^x \sin y^2 dx + e^x \cos y^2(2ydy) + \frac{1}{1+(x+y)^2}(dx+dy)$$

And collect like terms

$$dy\left(2xy + 2ye^x\cos y^2 + \frac{1}{1+(x+y)^2}\right) + dx\left(y^2 + e^x \sin y^2 + \frac{1}{1+(x+y)^2}\right)$$

divide to find $\frac{dy}{dx}$

iii. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0$
Differentiate

$$\frac{2xdx}{a^2} + \frac{2ydy}{b^2} = 0$$

and divide to find $\frac{dy}{dx}$

iv. $\ln(x^2+y^2) + \arctan\left(\frac{y}{x}\right) = 0$
Differentiate using the identity for $d\arctan\left(\frac{y}{x}\right)$

$$\frac{2xdx + 2ydy}{x^2+y^2} + \frac{ydx - xdy}{x^2+y^2} = 0$$

Collect like terms and divide to find

$$(2x+y)dx + (2y-x)dy = 0 \implies \frac{dy}{dx} = \frac{y+2x}{x-2y}$$

(b) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the following

i.

ii.

iii.

iv.

3. We have Euler's Theorem in three variables
$u = f(r,s,t)$ which implies

$$xu_x = xu_r r_x + xu_t t_x$$

Repeat for each variable and add to get

$$xu_x + yu_y + zu_y = u_r(xr_x + yr_y) + u_s(ys_y + zs_z) + u_t(xt_x + zt_z)$$

But the terms in the parenthesis vanish because of Euler's theorem and hence

$$xu_x + yu_y + zu_y = 0$$

4. Euler's Theorem in 2 variables Consider v given by

$$v = f(u), u = H_n(x,y)$$

where H_n is a homogenous function of degree n. Then we know what

$$xv_x = xu_x\frac{dv}{du}$$

similarly for y, thus

$$\boxed{xv_x + yv_y = \frac{dv}{du}(xu_x + yu_y) = nu\frac{dv}{du}}$$

by Euler's Theorem

5. Determine whether the function is homogenous and determine its degree
   (a) Not homogenous
   (b) Not homogenous
   (c) Homogenous, degree 1
   (d) Not homogenous
   (e) Not homogenous
   (f) Homogenous, degree $\frac{1}{20}$
   (g) Homogenous, degree 4
   (h) Homogenous, degree -1

6. Clearly $f(x, y)$ is homogenous and of degree 0 hence by Euler's Theorem

$$xf_x + yf_y = x\left(-\frac{y}{x^2} + \frac{1}{y}\right) + y\left(\frac{1}{x} - \frac{x}{y^2}\right) = 0$$

7. Clearly the degree of $f(x, y)$ is 1, hence $k = 1$

8. Linear Transformations If $y = f(x+ct)+\phi(x-ct)$ prove that $y_{tt} = c^2 y_{xx}$, in other words, show that a disturbance travelling at speed $\pm c$ satisfies the wave equation

   Let $u = x + ct$ and $v = x - ct$ then $y = f(u) + \phi(v)$

$$y_{xx} = (y_u u_x + y_v v_x)_x = (y_u + y_v)_x = (f'(u) + \phi'(v))_x = f''(u)u_x + \phi''(v)v_x = f''(u) + \phi''(v)$$

   using $u_x = v_x = 1$

$$\boxed{y_{tt} = (y_u u_t + y_v v_t)_t = c(f'(u) - \phi'(v))_t = c^2(f''(u) + \phi''(v)) = c^2 y_{xx}}$$

   using $u_t = -v_t = c$

9. Heat Diffusion Equation If $u = e^{-mx} sin(nt - mx)$ then prove that $2m^2 u_t = nu_{xx}$ Note that for a given t, that is to say, at a point in time, u undergoes damped oscillations with decay constant m and frequency m, thus m is to be viewed as a wavenumber or spatial frequency, thus as far as space is concerned, as such, let $\phi = nt - mx$ then we have

$$e^{nt}u = e^{\phi} \sin \phi$$

   We have

$$v_{\phi\phi} - 2v_{\phi} +$$

10. Say we have
$$x^x y^y z^z = k \implies x \ln x + y \ln y + z \ln z = \ln k$$
Now take the total differential of $\ln k$
$$0 = (\ln ex)dx + (\ln ey)dy + (\ln ez)dz$$
Re interpret this as the total differential of z which gives
$$dz = \frac{\ln(ex)}{\ln ez}dx + \frac{\ln(ey)}{\ln(ez)}dy$$
comparing this with the standard total differential we have
$$z_x = \frac{\ln(ex)}{\ln ez}; z_y = \frac{\ln(ey)}{\ln ez}$$
thus z_{yx} is given by
$$z_{yx} = \left(\frac{\ln(ey)}{\ln(ez)}\right)_x = -\frac{\ln(ey)}{\ln^2(ez)} \times \frac{1}{ez} \times e \times z_x$$
which is
$$z_{yx} = -\frac{\ln(ey)\ln(ez)}{z \ln^3(ez)}$$
now if $x = y = z$ then this simplifies to
$$\boxed{z_{yx} = -\frac{1}{x \ln ex}}$$

11. We have
$$u = x^2 \arctan\frac{y}{x} - y^2 \arctan\frac{x}{y} = (x^2+y^2)\arctan\frac{y}{x} - \frac{\pi}{2}y^2$$
Now
$$u_y = 2y \arctan\left(\frac{y}{x}\right) - x - \pi y$$
And so
$$\boxed{u_{yx} = \frac{2y}{x^2+y^2}(y) - 1 = \frac{y^2-x^2}{x^2+y^2}}$$

12. Recall from Q4 that
$$xu_x + yu_y = nz\frac{du}{dz}$$
where $n = 2$ and $u(z) = \arctan(z)$ and $z = \frac{x^3+y^3}{x-y}$, so we have
$$xu_x + yu_z = \frac{2z}{1+z^2}$$
but we have $z = \tan u$ so $\sin 2u = \frac{2z}{1+^2}$, thus
$$xu_x + yu_y = \sin 2u$$

13. Generalised Euler's Theorem
Let us define the Eulerian Operator $\mathbf{L} = x\partial_x + y\partial_y$, let u be a function of z which is homogenous in x and y of order n, then

$$v = \mathbf{L}[u] = xu_x + yu_y = (xz_y + yz_y)\frac{du}{dz} = nz\frac{du}{dz}$$

That is to say, that if $a(z)$ is a function of a homogenous function of x and y then

$$\boxed{\mathbf{L}[a] = nza_z}$$

Now consider

$$\begin{aligned}\mathbf{L}[\mathbf{L}[u]] = \mathbf{L}[v] &= xv_x + yv_y \\ &= x(xu_{xx} + u_x + yu_{yx}) + y(yu_{yy} + u_y + xu_{xy}) \\ &= x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} + (xu_x + yu_y) \\ &= x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} + nzu_z\end{aligned}$$

but from the above equation we have

$$\mathbf{L}[\mathbf{L}[u]] = nL[zu_z] = n^2z(zu_z)_z = n^2z(u_z + zu_{zz}) = n^2(zu_z + z^2u_{zz})$$

Equating both evaluations we have

$$x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = n(n-1)\,[zu_z] + n^2z^2\,[u_{zz}]$$

Apply this to the function $u = \arcsin z$ and $z = \sqrt{\ldots}$, $n = -\frac{1}{12}$, we have

14. Now if we define $u = vw$ where $v = x$ and $w = \ln(z)$ and $z = \frac{y}{x}$ and $n = 0$, then by the result of the previous question we have

$$x^2v_{xx} + 2xy\,v_{xy} + y^2v_{yy} = 0$$

and we know

$$u_{xx} = (u_x)_x = (v + xv_x)_x = 2v_x + xv_{xx}$$

and

$$u_{xy} = (v + xv_x)_y = v_y + xv_{xy}$$

and

$$u_{yy} = (xv_x)_x = xv_{yy}$$

Substituting

$$\boxed{x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = 0}$$

15.

16.

17.

18.

19. (a)

(b)

20.

21.

22. Invariance of Laplacian If we have $x = \xi\cos\alpha - \eta\sin\alpha$ and $y = \xi\sin\alpha + \eta\cos\alpha$, which is a rotated set of coordinates, and gives $x_\xi = y_\eta = \cos\alpha$ and $-x_\eta = y_\xi = \sin\alpha$

$$\begin{aligned}
u_{\xi\xi} &= (u_x x_\xi + u_y y_\xi)_\xi \\
&= (u_x \cos\alpha + u_y \sin\alpha)_\xi \\
&= (u_{xx} x_\xi \cos\alpha + u_{yy} y_\xi \sin\alpha) \\
&= (u_{xx} \cos^2\alpha + u_{yy} \sin^2\alpha)
\end{aligned}$$

Similarly

$$u_{\eta\eta} = (u_{xx} \sin^2\alpha + u_{yy} \cos^2\alpha)$$

adding the two we have

$$u_{\eta\eta} + u_{\xi\xi} = u_{xx} + u_{yy}$$