Indian Institute of Technology Kharagpur
Department of Mathematics
MA11003 - Advanced Calculus
Problem Sheet - 5
Autumn 2025

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1. Find $\dfrac{df}{dt}$ at $t = 0$ for the following functions:

   (a) $f(x, y) = x\cos y + e^x \sin y$, where $x(t) = t^2 + 1$, $y(t) = t^3 + t$.

   (b) $f(x, y, z) = x^3 + xz^2 + y^3 + xyz$, where $x(t) = e^t$, $y(t) = \cos t$, $z(t) = t^3$.

   (c) $f(x_1, x_2, x_3) = 2x_1^2 - x_2x_3 + x_1x_3^2$, where $x_1(t) = 2\sin(t)$, $x_2(t) = t^2 - t + 1$, $x_3(t) = 3^{-t}$.

2. (a) Using implicit differentiation, find $\dfrac{dy}{dx}$ from the following:

   (i) $x^y + y^x = c$

   (ii) $xy^2 + \exp(x)\sin(y^2) + \tan^{-1}(x + y) = c$

   (iii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + 1 = 0$

   (iv) $\ln(x^2 + y^2) + \tan^{-1}(y/x) = 0$

   (b) Using implicit differentiation, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ from the following:

   (i) $xy^2z^2 + \sin(yz) - \exp(xz^2) = 0$

   (ii) $x\tan^{-1}(\frac{y}{z}) + y\tan^{-1}(\frac{z}{x}) + z\tan^{-1}(\frac{x}{y}) = 0$

   (iii) $xy^2 + z^3 + \sin(xyz) = 0$

   (iv) $x - yz + \cos(xyz) - x^2z^2 = 1$

3. If $u = f(r, s, t)$, where $r = \dfrac{x}{y}$, $s = \dfrac{y}{z}$ and $t = \dfrac{z}{x}$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

4. If $v = f(u)$, where u is a homogeneous function of x and y of degree n, then prove that

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nu\frac{dv}{du}$$

5. Check whether the following functions are homogeneous or not, if so, determine the degree of the function:

   (a) $\tan^{-1}\dfrac{y}{x} + \sin^{-1}\dfrac{x}{y}$

   (b) $\cos^{-1}(\dfrac{y}{\sqrt{x^2 + y^2}})$

   (c) $\dfrac{x^2}{y} + \dfrac{y^2}{x}$

   (d) $\dfrac{x}{y}\sin(\dfrac{y}{x})$

   (e) $x^{2/3}y^{4/3}\tan\dfrac{y}{x}$

   (f) $\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

   (g) $x^2y^2 + xy^3 + x^2y + x^3y$

   (h) $\dfrac{x^2 + y^2}{x^3 + y^3}$

6. If $f(x, y) = \dfrac{y}{x} + \dfrac{x}{y}$, then show that $xf_x + yf_y = 0$.

7. If $u = \dfrac{x^2+y^2}{\sqrt{x+y}}$, $(x,y) \neq (0,0)$, what should be the value of k so that $xu_x + yu_y = ku$?

8. If $y = f(x+ct) + \phi(x-ct)$, then show that $y_{tt} = c^2 y_{xx}$.

9. If $u = e^{-mx}\sin(nt - mx)$, prove that $u_t = \dfrac{n}{2m^2} u_{xx}$.

10. If $x^x\ y^y\ z^z = k$ (constant), then show that at the point (x,y,z), where $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{x \log_e(ex)}.$$

11. If $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$, prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{x^2 - y^2}{x^2+y^2}$.

12. If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, then prove that $xu_x + yu_y = \sin 2u$.

13. If $u = \sin^{-1}\sqrt{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$, then show that

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right).$$

14. If $u(x,y) = x \log\left(\dfrac{y}{x}\right)$, for $xy \neq 0$, then show that

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0.$$

15. If $u = \dfrac{(ax^3+by^3)^n}{3n(3n-1)} + xf\left(\dfrac{y}{x}\right)$, then prove that

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = \left(ax^3+by^3\right)^n$$

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16. If $z = x^m f\left(\dfrac{y}{x}\right) + y^n g\left(\dfrac{x}{y}\right)$, then show that

$$x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} + mnz = (m+n-1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right).$$

17. Let $f(x,y)$ and $g(x,y)$ be two homogeneous functions of degree m and n respectively, where $m \neq 0$. Let $h = f + g$, and $x\dfrac{\partial h}{\partial x} + y\dfrac{\partial h}{\partial y} = 0$. Then show that $f = \alpha g$ for some constant α.

18. By the transformation $\xi = a + \alpha x + \beta y, \eta = b - \beta x + \alpha y$, where α, β, a, b are all constants and $\alpha^2 + \beta^2 = 1$, the function $u(x,y)$ is transfered into $U(\xi,\eta)$. Prove that

$$U_{\xi\xi}U_{\eta\eta} - U_{\xi\eta}^2 = u_{xx}u_{yy} - u_{xy}^2.$$

19. If z be a differentiable function of x and y (rectangular cartesian co-ordinates) and let $x = r\cos\theta, y = r\sin\theta$ (r, θ are polar co-ordinates), then show that

    (a) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$

    (b) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}.$

20. Given $w = (x_1^2 + x_2^2 + \cdots + x_n^2)^k$, for $n \geq 2$. Then, for what values of k, the following relation holds:
$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \cdots + \frac{\partial^2 w}{\partial x_n^2} = 0.$$

21. Let $u(x, y)$ be such that all its second order partial derivatives exist. If $x = r\cos\theta, y = r\sin\theta$, then show that
$$r^2\frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial \theta^2} - r\frac{\partial u}{\partial r} = (x^2 - y^2)\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) + 4xy\frac{\partial^2 u}{\partial x \partial y}.$$

22. Let $u(x, y)$ be such that all its second order partial derivatives exists. If $x = \xi\cos\alpha - \eta\sin\alpha, \quad y = \xi\sin\alpha + \eta\cos\alpha$, then prove that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}.$$