Tutorial 5: Implicit differentiation and Euler's Theorem

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1. Find f_t at $t = 0$ for the following

   (a) $f(x, y) = x \cos y + e^x \sin y$ where $x(t) = t^2 + 1$ and $y(t) = t^3 + t$
   The derivative can be found using chain rule

   $$f_t = f_x \, x_t + f_y \, y_t = (2t)(\cos y + e^x \sin y) + (3t^2 + 1)(-x \sin y + e^x \cos y)$$

   where $x(t)$ and $y(t)$ are given as above

   (b) $f(x, y, z) = x^3 + xz^2 + y^3 + xyz$ where $x(t) = e^t, y(t) = \cos t$ and $z(t) = t^3$
   Again, by chain rule

   $$f_t = f_x \, x_t + f_y \, y_t + f_z \, z_t = (e^t)(3x^2 + z^2 + yz) + (-\sin t)(3y^2 + xz) + (3t^2)(2xz + xy)$$

   where the intermediate variables can be expanded as before

   (c) $f(x_1, x_2, x_3) = 2x_1^2 - x_2 x_3 + x_1 x_3^2$ where $x_1(t) = 2\sin t, x_2(t) = t^2 - t + 1$ and $x_3(t) = 3^{-t}$
   Again by chain rule we have

   $$f_t = (2\cos t)(4x_1 + x_3^2) + (2t - 1)(-x_3) + (-\ln 3 \times 3^{-t})(-x_2 + 2x_1 x_3)$$

   as before

2. (a) Find y_x for the following

   i. $x^y + y^x = c$
      Differentiating we have

      $$x^y d(y \ln x) + y^x d(x \ln y) = x^y \left(\ln(x)dy + \frac{ydx}{x} \right) + y^x \left(\ln(y)dx + \frac{xdy}{y} \right) = 0$$

      Collect like terms

      $$dy \left(x^y \ln(x) + \frac{xy^x}{y} \right) + dx \left(y^x \ln(y) + \frac{y}{x}x^y \right) = 0$$

      Divide to find $\frac{dy}{dx}$

   ii. $xy^2 + e^x \sin y^2 + \arctan(x + y) = c$
      Again, we differentiate

      $$y^2 dx + 2xydy + e^x \sin y^2 dx + e^x \cos y^2 (2ydy) + \frac{1}{1 + (x + y)^2}(dx + dy)$$

And collect like terms

$$dy\left(2xy + 2ye^x\cos y^2 + \frac{1}{1+(x+y)^2}\right) + dx\left(y^2 + e^x\sin y^2 + \frac{1}{1+(x+y)^2}\right)$$

divide to find $\frac{dy}{dx}$

iii. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1 = 0$

Differentiate

$$\frac{2xdx}{a^2} + \frac{2ydy}{b^2} = 0$$

and divide to find $\frac{dy}{dx}$

iv. $\ln(x^2 + y^2) + \arctan\left(\frac{y}{x}\right) = 0$

Differentiate using the identity for $d\arctan\left(\frac{y}{x}\right)$

$$\frac{2xdx + 2ydy}{x^2+y^2} + \frac{ydx - xdy}{x^2+y^2} = 0$$

Collect like terms and divide to find

$$(2x+y)dx + (2y-x)dy = 0 \implies \frac{dy}{dx} = \frac{y+2x}{x-2y}$$

3. We have Euler's Theorem in three variables
   $u = f(r,s,t)$ which implies

$$xu_x = xu_r r_x + xu_t t_x$$

Repeat for each variable and add to get

$$xu_x + yu_y + zu_y = u_r(xr_x + yr_y) + u_s(ys_y + zs_z) + u_t(xt_x + zt_z)$$

But the terms in the parenthesis vanish because of Euler's theorem and hence

$$xu_x + yu_y + zu_y = 0$$

4. Euler's Theorem in 2 variables
   Consider v given by

$$v = f(u), u = H_n(x,y)$$

where H_n is a homogenous function of degree n. Then we know what

$$xv_x = xu_x\frac{dv}{du}$$

similarly for y, thus

$$\boxed{xv_x + yv_y = \frac{dv}{du}(xu_x + yu_y) = nu\frac{dv}{du}}$$

by Euler's Theorem

5. Determine whether the function is homogenous and determine its degree

   (a) Homogenous, degree 0

   (b) Homogenous degree 0

(c) Homogenous, degree 1

(d) Homogenous, degree 0

(e) Homogenous, degree 2

(f) Homogenous, degree $\frac{1}{20}$

(g) Not homogenous

(h) Homogenous, degree -1

6. Clearly $f(x, y)$ is homogenous and of degree 0 hence by Euler's Theorem

$$xf_x + yf_y = x\left(-\frac{y}{x^2} + \frac{1}{y}\right) + y\left(\frac{1}{x} - \frac{x}{y^2}\right) = 0$$

7. Clearly the degree of $f(x, y)$ is 1, hence $k = 1$

8. Linear Transformations If $y = f(x+ct) + \phi(x-ct)$ prove that $y_{tt} = c^2 y_{xx}$, in other words, show that a disturbance travelling at speed $\pm c$ satisfies the wave equation

   Let $u = x + ct$ and $v = x - ct$ then $y = f(u) + \phi(v)$

   $$y_{xx} = (y_u u_x + y_v v_x)_x = (y_u + y_v)_x = (f'(u) + \phi'(v))_x = f''(u)u_x + \phi''(v)v_x = f''(u) + \phi''(v)$$

   using $u_x = v_x = 1$

   $$\boxed{y_{tt} = (y_u u_t + y_v v_t)_t = c(f'(u) - \phi'(v))_t = c^2(f''(u) + \phi''(v)) = c^2 y_{xx}}$$

   using $u_t = -v_t = c$

9. Advection Equation
   If $u = e^{-mx}\sin(nt - mx)$ then prove that $2m^2 u_t = nu_{xx}$
   Note that for a given t, that is to say, at a point in time, u undergoes damped oscillations with decay constant m and frequency m, thus m is to be viewed as a wavenumber or spatial frequency, thus as far as space is concerned, as such, let $\phi = nt - mx$ then we have

   $$e^{nt}u = e^\phi \sin\phi$$

   We have

   $$v_{\phi\phi} - 2v_\phi +$$

   ???

10. Say we have
    $$x^x y^y z^z = k \implies x\ln x + y\ln y + z\ln z = \ln k$$
    Now take the total differential of $\ln k$

    $$0 = (\ln ex)dx + (\ln ey)dy + (\ln ez)dz$$

    Re interpret this as the total differential of z which gives

    $$dz = \frac{\ln(ex)}{\ln ez}dx + \frac{\ln(ey)}{\ln(ez)}dy$$

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comparing this with the standard total differential we have

$$z_x = \frac{\ln(ex)}{\ln ez}; z_y = \frac{\ln(ey)}{\ln ez}$$

thus z_{yx} is given by

$$z_{yx} = \left(\frac{\ln(ey)}{\ln(ez)}\right)_x = -\frac{\ln(ey)}{\ln^2(ez)} \times \frac{1}{ez} \times e \times z_x$$

which is

$$z_{yx} = -\frac{\ln(ey)\ln(ez)}{z\ln^3(ez)}$$

now if $x = y = z$ then this simplifies to

$$\boxed{z_{yx} = -\frac{1}{x\ln ex}}$$

11. We have
$$u = x^2 \arctan\frac{y}{x} - y^2 \arctan\frac{x}{y} = (x^2 + y^2)\arctan\frac{y}{x} - \frac{\pi}{2}y^2$$

Now
$$u_y = 2y \arctan\left(\frac{y}{x}\right) - x - \pi y$$

And so
$$\boxed{u_{yx} = \frac{2y}{x^2 + y^2}(y) - 1 = \frac{y^2 - x^2}{x^2 + y^2}}$$

12. Recall from Q4 that
$$\mathbf{L}[u] = xu_x + yu_y = nzu_z$$

where $n = 2$ and $u(z) = \arctan(z)$ and $z = \frac{x^3 + y^3}{x - y}$, so we have

$$xu_x + yu_z = \frac{2z}{1 + z^2}$$

but we have $z = \tan u$ so $\sin 2u = \frac{2z}{1 + 2}$, thus

$$xu_x + yu_y = \sin 2u$$

13. Generalised Euler's Theorem
Let us define the Eulerian Operator $\mathbf{L} = x\partial_x + y\partial_y$, let u be a function of z which is homogenous in x and y of order n, then

$$v = \mathbf{L}[u] = xu_x + yu_y = (xz_y + yz_y)\frac{du}{dz} = nz\frac{du}{dz}$$

That is to say, that if $a(z)$ is a function of a homogenous function of x and y then

$$\boxed{\mathbf{L}[a] = nza_z}$$

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Now consider

$$\mathbf{L}[\mathbf{L}[u]] = \mathbf{L}[v] = xv_x + yv_y$$
$$= x(xu_{xx} + u_x + yu_{yx}) + y(yu_{yy} + u_y + xu_{xy})$$
$$= x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} + (xu_x + yu_y)$$
$$= x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} + nzu_z$$

but from the above equation we have

$$\mathbf{L}[\mathbf{L}[u]] = nL[zu_z] = n^2 z(zu_z)_z = n^2 z(u_z + zu_{zz}) = n^2(zu_z + z^2 u_{zz})$$

Equating both evaluations we have

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} = n(n-1)\left[zu_z\right] + n^2 z^2\left[u_{zz}\right]$$

Apply this to the function $u = \arcsin z$ and $z = \sqrt{\ldots}$, $n = -\frac{1}{12}$, we have ???

14. From the previous part, if $u = z$, we have $u_z = 1$ and $u_{zz} = 0$ and so

$$x^2 u_{xx} + +2xy\,u_{xy} + y^2 u_{yy} = n(n-1)u = 0$$

since $n = 1$

15. ???

16. Let $z = z_1 + z_2$ then

$$xz_x + yz_y = mz_1 + nz_2$$

and

$$x^2 z_{xx} + 2xy\,z_{xy} + y^2 z_{yy} = m(m-1)z_1 + n(n-1)z_2$$

adding the two we have

$$x^2 z_{xx} + 2xy\,z_{xy} + y^2 z_{yy} = m^2 z_1 + n^2 z_2 - (xz_z + yz_y)$$

add $mnz = mnz_1 = mnz_2$ to both sides we have

$$x^2 z_{xx} + 2xy\,z_{xy} + y^2 z_{yy} + mnz = (m+n-1)(xz_z + yz_y)$$

17. Using the linearity of the Eulerian operator

$$\mathbf{L}[h] = L[f] + L[g] = mf + ng = 0 \implies f = -\frac{n}{m}g = \alpha g$$

18. ???

19. **Polar Coordinates**
The transformation

$$x = r\cos\theta; y = r\sin\theta$$

can be inverted to

$$r^2 = x^2 + y^2; \theta = \arctan\left(\frac{y}{x}\right)$$

which can be used to find

$$r_x = \cos\theta; r_y = \sin\theta$$

and

$$\theta_x = \frac{\sin\theta}{r}; \theta_y = -\frac{\cos\theta}{r}$$

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(a) $(z_x)^2 + (z_y)^2 = z_r^2 + \frac{1}{r^2}z_\theta^2$

Evaluate

$$z_x = z_r r_x + z_\theta \theta_x = z_r \cos\theta + \frac{z_\theta \sin\theta}{r}$$

and

$$z_y = z_r r_y + + z_\theta \theta_y = z_r \sin\theta - \frac{z_\theta \cos\theta}{r}$$

Square and add to get

$$z_x^2 + z_y^2 = z_r^2 + \frac{1}{r^2}z_\theta^2$$

(b) This question is much longer

$$\Delta z = z_{xx} + z_{yy} = z_{rr} = \frac{1}{r}z_r + \frac{1}{r^2}z_{\theta\theta}$$

that is to say, the Laplacian has no mixed derivatives and no single θ derivatives. On the LHS we have

$$z_{xx} = (z_x)_x = \left(z_r \cos\theta + \frac{z_\theta \sin\theta}{r} \right)_x = (A+B)_x$$

and

$$z_{yy} = (z_y)_y = \left(z_r \sin\theta - \frac{z_\theta \cos\theta}{r} \right)_y = (C+D)_y$$

Evaluate any one,

$$A_x = \frac{z_r(-\sin\theta)(\sin\theta)}{r} + \cos\theta\left[z_{rr}\cos\theta + \frac{z_{r\theta}\sin\theta}{r} \right] = z_{rr}\cos^2\theta - \frac{z_r\sin^2\theta}{r} + z_{r\theta}\frac{\sin\theta\cos\theta}{r}$$

Clearly what's going to happen is that the squared terms will add and the cross terms will cancel

$$B_x = \frac{\sin\theta}{r}\left[z_{\theta r}\cos\theta + \frac{z_{\theta\theta}\sin\theta}{r} \right] + z_\theta Q$$

Clearly the Q term will contribute zero to the final answer

$$z_{xx} + z_{yy} = z_{rr}(\sin^2\theta + \cos^2\theta) + \frac{1}{r}z_r(\sin^2\theta + \cos^2\theta) + \frac{1}{r^2}z_{\theta\theta}(\sin^2\theta + \cos^2\theta)$$

while all the other terms cancel out

20. We need to find the value of k such that

$$\nabla^2 r^{2k} = 0; \ \forall v \in \mathbb{R}^n$$

where $r = \sqrt{\sum x_i^2}$ is the L_2 norm of the vector v, consider the expression

$$r^2 = \sum x_i^2 \implies r_{x_i} = \frac{x_i}{r}$$

which implies that

$$(r_{2k})_{x_i x_i} = \left(2k(r^{2k-1})\frac{x_i}{r} \right)_{x_i} = 2k(r^{2k-2}x_i)_{x_i} = 2k \times A$$

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where A is given by

$$A = r^{2k-2} + x_i(2k-2)r^{2k-3}\frac{x_i}{r} = r^{2k-4}[r^2 + (2k-2)x_i^2]$$

Thus

$$\nabla^2 r^{2k} = 2kr^{2k-4} \times \sum r^2 + (2k-2)x_i^2 = 2kr^{2k-2}[n + 2 - 2k]$$

Thus $k = 0$ or $k = 1 + \frac{n}{2}$

21. ???

22. **Invariance of Laplacian** If we have $x = \xi\cos\alpha - \eta\sin\alpha$ and $y = \xi\sin\alpha + \eta\cos\alpha$, which is a rotated set of coordinates, and gives $x_\xi = y_\eta = \cos\alpha$ and $-x_\eta = y_\xi = \sin\alpha$

$$\begin{aligned}
u_{\xi\xi} &= (u_x x_\xi + u_y y_\xi)_\xi \\
&= (u_x \cos\alpha + u_y \sin\alpha)_\xi \\
&= (u_{xx} x_\xi \cos\alpha + u_{yy} y_\xi \sin\alpha) \\
&= (u_{xx} \cos^2\alpha + u_{yy} \sin^2\alpha)
\end{aligned}$$

Similarly

$$u_{\eta\eta} = (u_{xx}\sin^2\alpha + u_{yy}\cos^2\alpha)$$

adding the two we have

$$u_{\eta\eta} + u_{\xi\xi} = u_{xx} + u_{yy}$$