

## **2 Plates and Shells**

## 2.1 Euler - Bernoulli Beam Theory

### Euler - Bernoulli Beam Theory

a standard structural element is the beam, which is a long, slender solid that carries transverse forces by bending



concentrate on statically determinate beams: cantilevers and simply supported beams

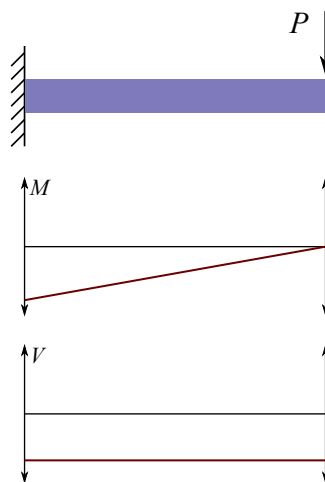
fundamental modelling simplifications are:

1. the beam is a one-dimensional object;
2. plane sections remain plane; and
3. there is no axial extension or contraction of the beam.

note that the principal variable is  $u$ , the vertical deflection of the midplane of the beam

### Euler - Bernoulli Beam Theory

there are several graphical tools that are useful in the analysis of beams



### Euler - Bernoulli Beam Theory

what is the relationship between the shear diagram and the moment diagram?

all of the derivatives of the displacement  $u$  have a physical meaning

$$\frac{du}{dx} = \theta \text{ slope of the beam}$$

$$\frac{d^2u}{dx^2} = \frac{d\theta}{dx} = \kappa \text{ curvature of the beam}$$

$$EI\kappa = M \text{ moment in the beam}$$

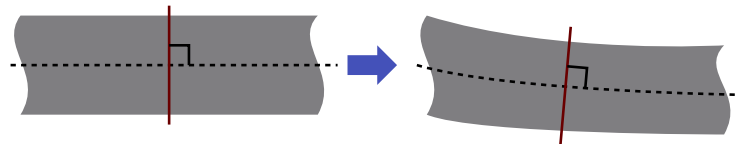
$$-\frac{d}{dx}EI\frac{d^2u}{dx^2} = -\frac{dM}{dx} = V \text{ shear force in the beam}$$

$$\frac{d^2}{dx^2}EI\frac{d^2u}{dx^2} = -\frac{dV}{dx} = w \text{ distributed force on the beam}$$

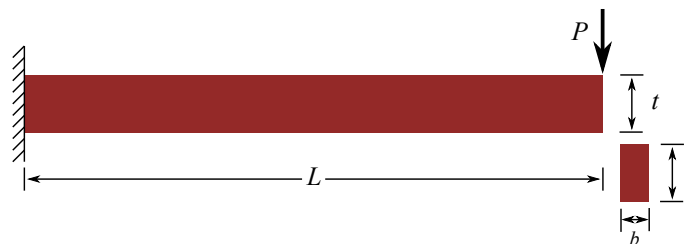
### Euler - Bernoulli Beam Theory

there are several key assumptions associated with Euler-Bernoulli beam theory:

1. the beam is slender: the bending deformations are much larger than the shear deformations
2. there is no through-thickness deformation: the thickness of the beam remains constant
3. plane cross-sections of the beam initially normal to the neutral axis remain plane and normal to the axis of bending after deformation



### Euler - Bernoulli Beam Theory



Euler-Bernoulli beam theory enables the calculation of deflected shapes of beams of various sorts: the cantilever shown above is an example

a complete set of boundary conditions is necessary to use Euler-Bernoulli beam theory: these must be four conditions on the deflection  $u$  or its derivatives

in general, start with the distributed load  $w(x)$  and integrate it to get the shear  $V(x)$ : in this case, the distributed load is zero

$$V(x) = - \int w \, dx = - \int 0 \, dx = C_1$$

because the shear force at any cross-section is  $-P$ , we find that  $C_1 = -P$

#### **Euler - Bernoulli Beam Theory**

second, integrate the shear force to find the moment distribution  $M(x)$ :

$$M(x) = - \int V \, dx = - \int (-P) \, dx = Px + C_2$$

the moment at the free end of the beam (where  $x = L$ ) is zero, so:

$$M(L) = 0 = PL + C_2$$

and therefore:

$$C_2 = -PL$$

third, convert the moment to a curvature,  $\kappa(x)$ :

$$\kappa(x) = \frac{M(x)}{EI} = \frac{P(x - L)}{EI}$$

#### **Euler - Bernoulli Beam Theory**

fourth, integrate the curvature to get the slope,  $\theta(x)$ :

$$\theta(x) = \int \kappa \, dx = \frac{1}{EI} \int (Px - PL) \, dx = \frac{1}{EI} \left( \frac{Px^2}{2} - PLx + C_3 \right)$$

the slope at the wall is zero because it is rigidly fixed and cannot rotate, so evaluating  $\theta(0)$  gives:

$$\theta(0) = C_3 = 0$$

and hence  $C_3 = 0$

#### **Euler - Bernoulli Beam Theory**

finally, integrate the slope to get the deflected shape,  $u(x)$ :

$$u(x) = \int \theta \, dx = \frac{1}{EI} \int \left( \frac{Px^2}{2} - PLx \right) dx = \frac{1}{EI} \left( \frac{Px^3}{6} - \frac{PLx^2}{2} + C_4 \right)$$

the displacement at the wall,  $u(0) = 0$ , so solve for the final constant of integration:

$$C_4 = 0$$

thus, the complete deflected shape of the end-loaded cantilever beam is:

$$u = \frac{1}{EI} \left( \frac{Px^3}{6} - \frac{PLx^2}{2} \right)$$

the maximum deflection, at  $x = L$ , is:

$$u(L) = -\frac{PL^3}{3EI}$$

### Algorithm: Euler-Bernoulli Beam Theory

1. ensure that the beam you are analyzing is statically determinate: if it is not, you will have to use a *superposition* method
2. identify four boundary conditions on the displacement, slope, moment or shear force to solve for the constants of integration
3. starting with the distributed load  $w$  (often this is zero), integrate four times to get the deformed shape  $u$
4. in general solve for the constants of integration as early as possible rather than carrying multiple constants

## 2.2 Plate Theory

### Plates

laminar plate theory enables the calculation of stiffness matrices for laminates; the question now becomes how to use the stiffnesses to solve for displacements for cases that are not simple

beams have one dimension that is much larger than the other two: beams can be treated as effectively one-dimensional

plates, shells and membranes have two dimensions that are much larger than the third dimension, and hence can be treated as two-dimensional objects

the key difference between plates and membranes is that plates can support loads both in-plane and in bending, while membranes can only support in-plane loads

### Plate Midplane

much like in Euler - Bernoulli beam theory, plate theory deals primarily with the midplane of a plate; it is assumed that the forces applied to a plate are acting on the midplane

provided the plate material is isotropic and homogeneous, and the plate has a constant thickness, the midplane is at the middle of the plate thickness

similar to laminate plate theory, once there is material and geometric complexity, such as non-symmetric layers in a plate, the midplane must be found from additional considerations

### Modulus Weighted Plate Midplane

if the stiffness of the plate changes through its thickness (as is the case for composite laminates) the modulus-weighted plate thickness,  $h^*$ , is:

$$h^* = \int_t \frac{E}{E_0} dz_0$$

where  $t$  is the thickness of the plate and  $z_0$  is the transverse coordinate with a conveniently chosen origin

$E$  is the modulus at location  $z$ , while  $E_0$  is a conveniently chosen reference modulus

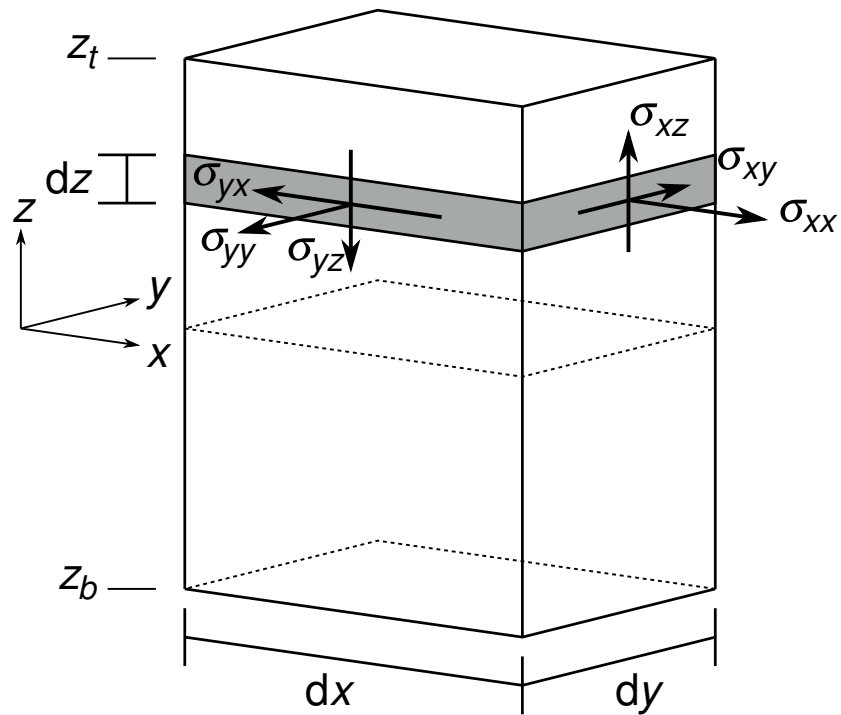
the modulus-weighted midplane is the origin of the  $z$  coordinate; this will be true provided:

$$0 = \int_t z \frac{E}{E_0} dz$$

the two coordinates,  $z$  and  $z_0$  are related by:  $z = z_0 - z_0^*$ ;  $z_0^*$  is the distance from the arbitrary system origin to the modulus weighted midplane, and can be found by:

$$z_0^* h^* = \int_t z_0 \frac{E}{E_0} dz_0 = \sum_i (\bar{z}_0)_i \left( \frac{E h}{E_0} \right)_i$$

### Plate Stresses

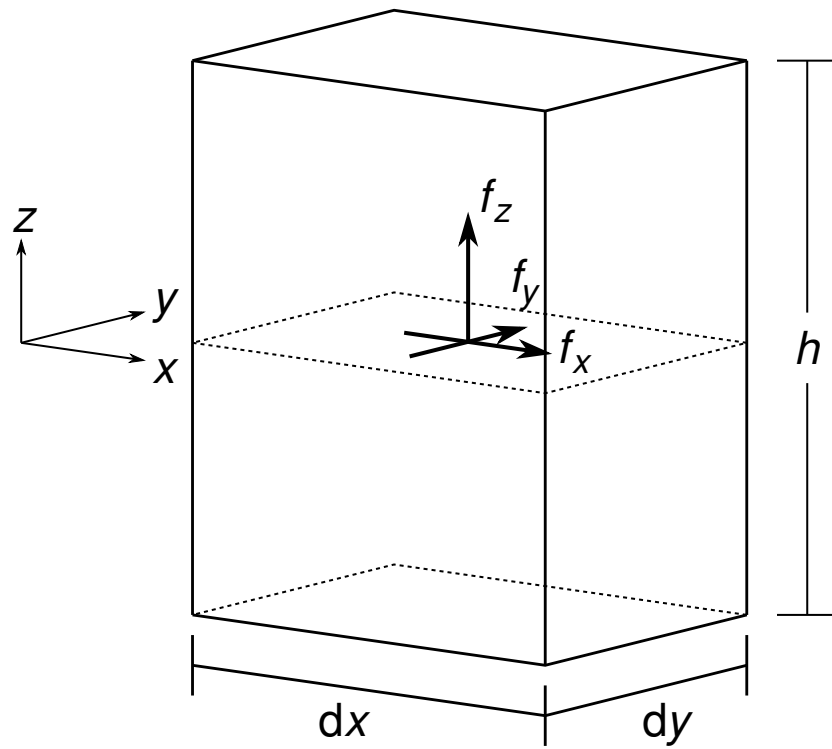


the notation at left to identifies the stresses acting on the faces of a representative element of the plate

this is consistent with standard notation for stresses

note that the stresses may vary in  $x$ ,  $y$  and  $z$ ; for beams, stresses varied only in  $x$  and  $z$

### Applied Forces



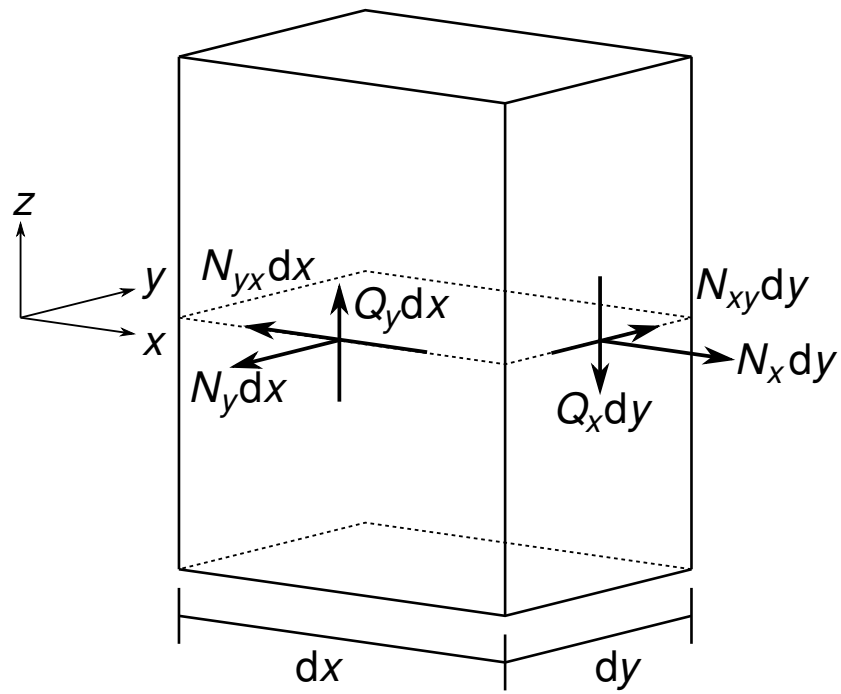
the applied forces are decomposed into three components, aligned with the coordinate axes

the applied forces are assumed to act at the midplane of the plate

in all cases, the applied forces are tractions and have units of force per unit area

#### **Stress Resultants - Normal and Shear**





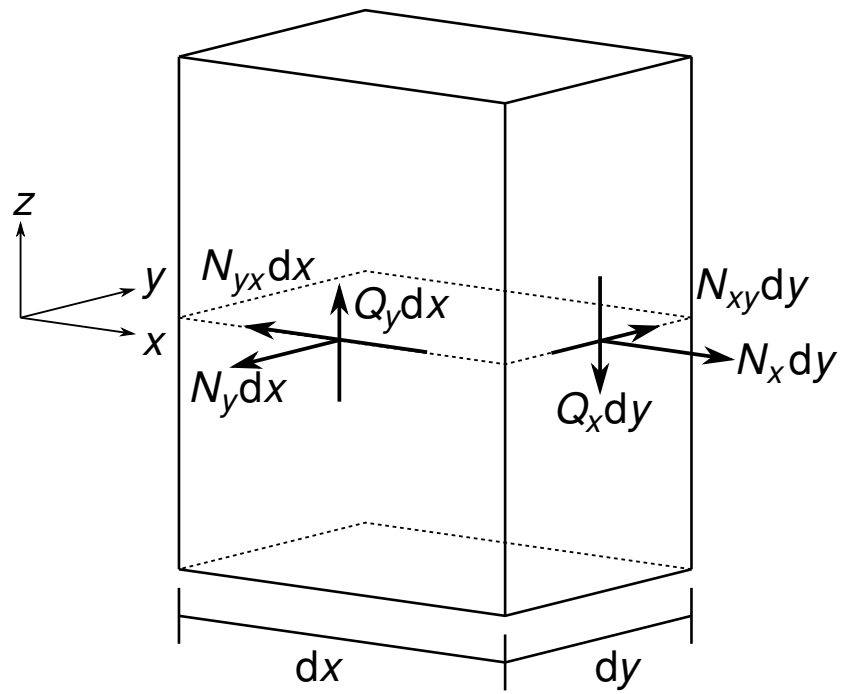
integrating the stresses gives the resultants (per unit width):

$$N_x = \int_{z_b}^{z_t} \sigma_{xx} dz$$

$$N_y = \int_{z_b}^{z_t} \sigma_{yy} dz$$

$$N_{xy} = N_{yx} = \int_{z_b}^{z_t} \sigma_{xy} dz$$

**Stress Resultants - Normal and Shear**

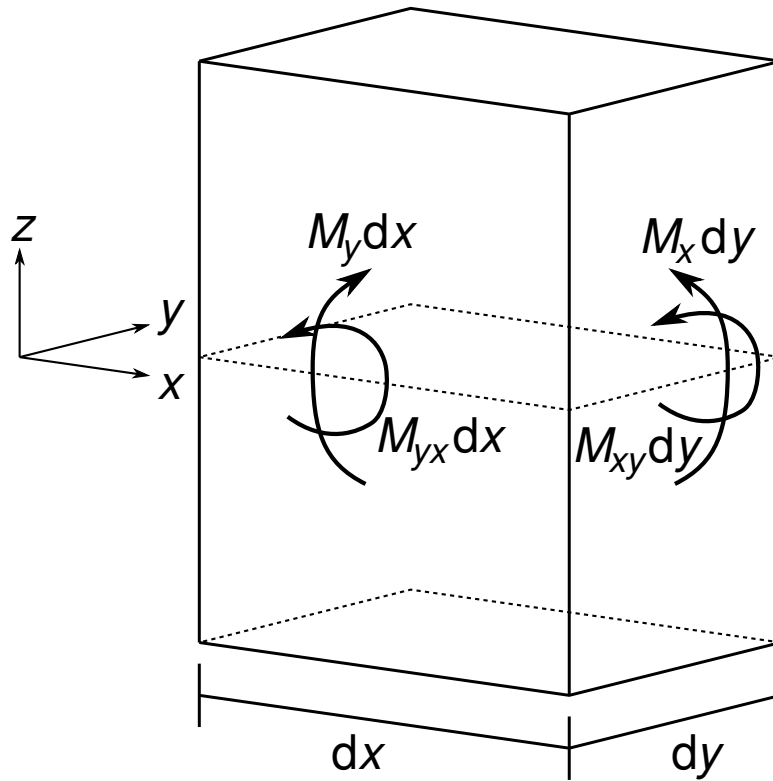


the transverse shear stress resultants are:

$$Q_x = - \int_{z_b}^{z_t} \sigma_{xz} dz$$

$$Q_y = - \int_{z_b}^{z_t} \sigma_{yz} dz$$

### Stress Resultants - Moments



finally, the resultant moments are:

$$M_x = - \int_{z_b}^{z_t} z \sigma_{xx} dz$$

$$M_y = - \int_{z_b}^{z_t} z \sigma_{yy} dz$$

$$M_{xy} = -M_{yx} = - \int_{z_b}^{z_t} z \sigma_{xy} dz$$

### Plate Displacements

the approximations for plate theory are very similar to those for Euler - Bernoulli beam theory and composite laminate theory:

the most important are:

1. straight lines perpendicular to the modulus weighted midplane before bending remain straight and perpendicular after bending
2. the perpendiculars to the modulus weighted midplane remain unchanged in length after deflection

the consequence is that it is necessary only to consider the modulus weighted mid-plane deflections; deflections at other locations through the thickness can be derived:

$$u(x, y, z) = u^{\circ}(x, y, 0) - z\theta_x(x, y)$$

$$v(x, y, z) = v^{\circ}(x, y, 0) - z\theta_y(x, y)$$

$$w(x, y, z) = w^{\circ}(x, y, 0)$$

like in laminate plate theory, the superscripted ‘o’ indicates a midplane quantity

### Strains

there are only three non-zero strains, which are found by differentiating the modulus weighted midplane displacements and adding terms to account for the curvature of the plate:

$$\begin{aligned}\epsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial u^{\circ}}{\partial x} - z \frac{\partial^2 w^{\circ}}{\partial x^2} \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{\partial v^{\circ}}{\partial y} - z \frac{\partial^2 w^{\circ}}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u^{\circ}}{\partial y} + \frac{\partial v^{\circ}}{\partial x} - 2z \frac{\partial^2 w^{\circ}}{\partial x \partial y}\end{aligned}$$

### Stress - Strain Relations

assume a linear elastic material model and hence apply Hooke’s Law to relate the stresses and the strains, in this case in a plane strain manner:

$$\begin{aligned}\sigma_{xx} &= \frac{E}{1 - \nu^2} (\epsilon_{xx} + \nu \epsilon_{yy}) \\ \sigma_{yy} &= \frac{E}{1 - \nu^2} (\epsilon_{yy} + \nu \epsilon_{xx}) \\ \sigma_{xy} &= G \gamma_{xy} = \frac{E \gamma_{xy}}{2(1 + \nu)}\end{aligned}$$

### Stress - Displacement Relations

express the stresses in terms of displacements; to do so, substitute the definitions of strains into the expressions of Hooke’s Law, using commas in the subscripts to indicate differentiation in order to shorten the expressions:

$$\begin{aligned}\sigma_{xx} &= \frac{E}{1 - \nu^2} (u_{,x}^{\circ} + \nu v_{,y}^{\circ} - z(w_{,xx}^{\circ} + \nu w_{,yy}^{\circ})) \\ \sigma_{yy} &= \frac{E}{1 - \nu^2} (v_{,y}^{\circ} + \nu u_{,x}^{\circ} - z(w_{,yy}^{\circ} + \nu w_{,xx}^{\circ})) \\ \sigma_{xy} &= \frac{E}{2(1 + \nu)} (u_{,y}^{\circ} + v_{,x}^{\circ} - 2z w_{,xy}^{\circ})\end{aligned}$$

### Stress Resultant - Displacement Relations

finally this produces the relationship between the stress resultants and the displacements:

$$\begin{aligned}N_x &= K^* (u_{,x}^\circ + \nu v_{,y}^\circ) \\N_y &= K^* (v_{,y}^\circ + \nu u_{,x}^\circ) \\N_{xy} &= \frac{1}{2} (1 - \nu) K^* (u_{,y}^\circ + v_{,x}^\circ)\end{aligned}$$

where:

$$K^* = \frac{E_0 h^*}{1 - \nu^2}$$

### Stress Resultant - Displacement Relations

similarly:

$$\begin{aligned}M_x &= D^* (w_{,xx}^\circ + \nu w_{,yy}^\circ) \\M_y &= D^* (w_{,yy}^\circ + \nu w_{,xx}^\circ) \\M_{xy} &= (1 - \nu^2) D^* w_{,xy}^\circ\end{aligned}$$

where:

$$D^* = E_0 I^*$$

and

$$I^* = \int_t \frac{z^2}{1 - \nu^2} \frac{E}{E_0} dz$$

### Homogeneous Plates

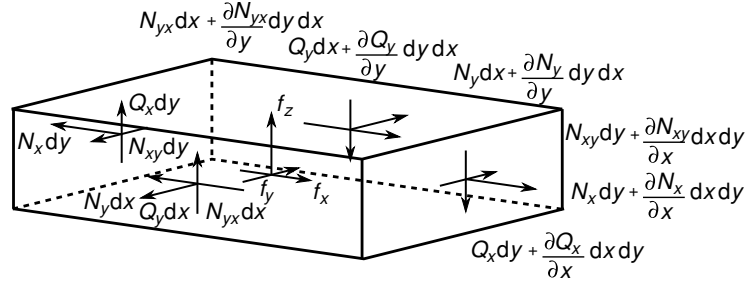
note that in the equations above used  $K^*$ ,  $D^*$ ,  $I^*$  and  $h^*$

this is to provide for the case with a layered material where the layers are not homogeneous, as found in composites

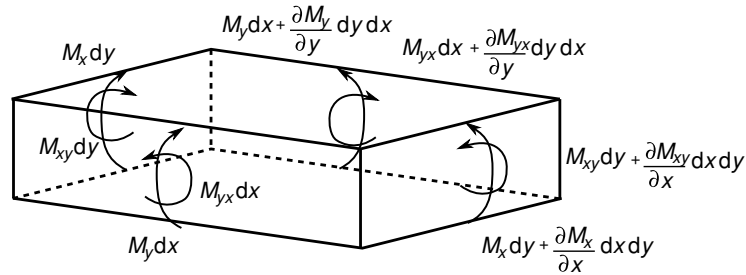
if the plate is homogeneous, then:

$$\begin{aligned}K &= \frac{Eh}{1 - \nu^2} \\I &= \frac{h^3}{12(1 - \nu^2)} \\D &= \frac{Eh^3}{12(1 - \nu^2)}\end{aligned}$$

### Equilibrium of a Plate Element



### Equilibrium of a Plate Element



### Equilibrium of a Plate Element

apply equilibrium equations to the free body diagrams on the previous two slides;  
the key results are, for forces:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = -f_x$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = -f_y$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -f_z$$

and for moments:

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0$$

### Equilibrium of a Plate Element

substitute the stress resultant - displacement equations into the equilibrium equations (again using comma differentiation notation for compactness):

$$K^* \left( u_{,xx}^\circ + \nu v_{,yx}^\circ \right) + \frac{1}{2} (1 - \nu) K^* \left( v_{,xy}^\circ + u_{,yy}^\circ \right) = -f_x$$

$$K^* \left( v_{,yy}^\circ + \nu u_{,xy}^\circ \right) + \frac{1}{2} (1 - \nu) K^* \left( v_{,xx}^\circ + u_{,yx}^\circ \right) = -f_y$$

and moment equilibrium gives:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} = f_z$$

using this and the resultant moment - displacement relation produces, after some simplification:

$$D^* \left( w_{,xxxx}^\circ + 2w_{,xxyy}^\circ + w_{,yyyy}^\circ \right) = f_z$$

this is the constant-thickness, small deflection, isotropic thin plate bending equation; it incorporates equilibrium, continuity of deflection and a linear elastic material model