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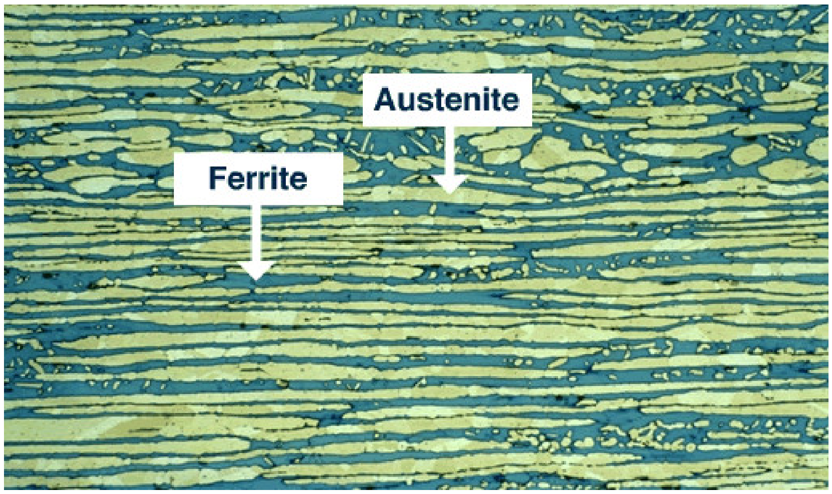
1 Composite Materials

1.1 Introduction to Composites

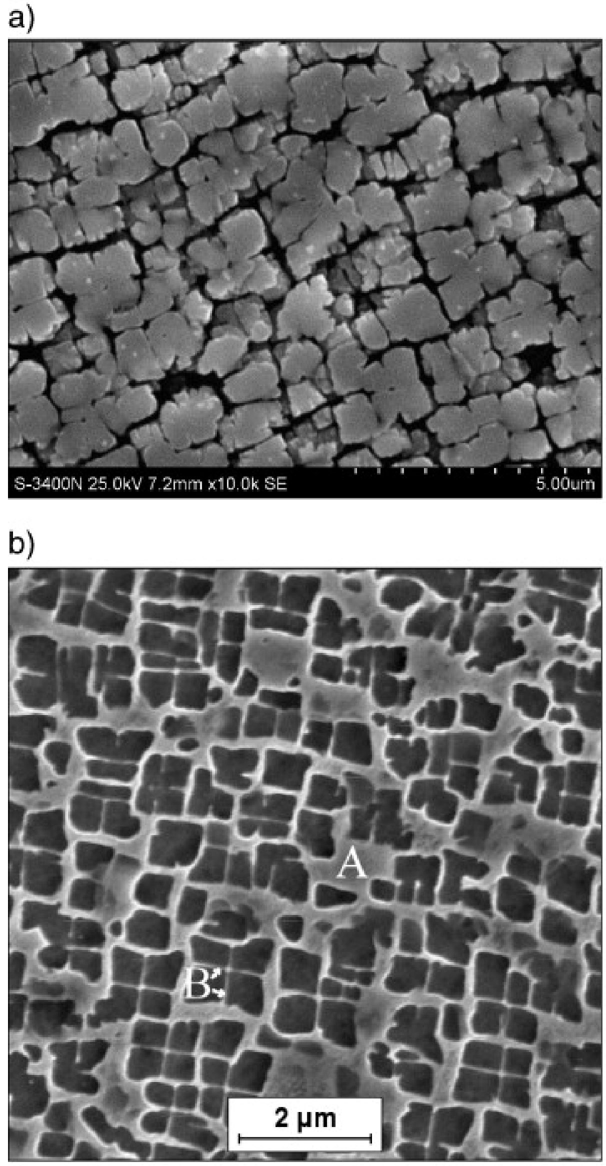
What Are Composites?

materials consisting of two or more phases

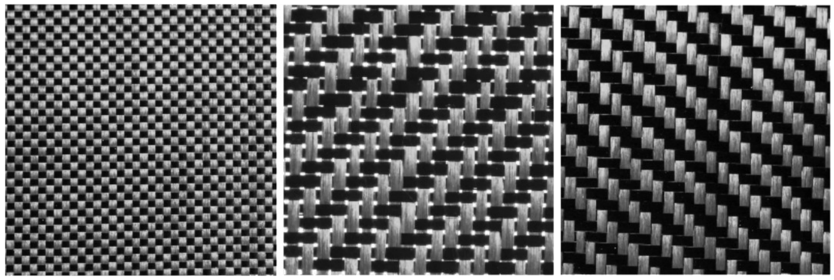


What Are Composites?

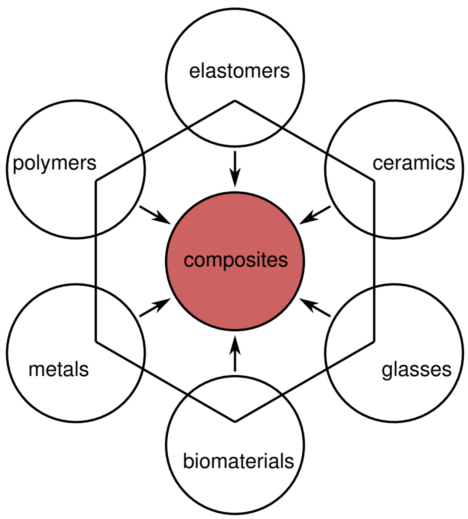
materials with two or more phases that are engineered so that the phases are geometrically related in a beneficial manner



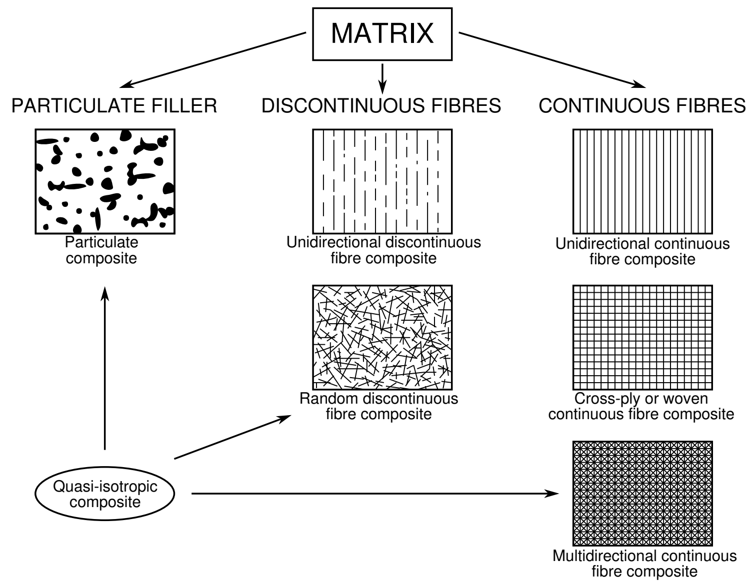
What Are Composites?



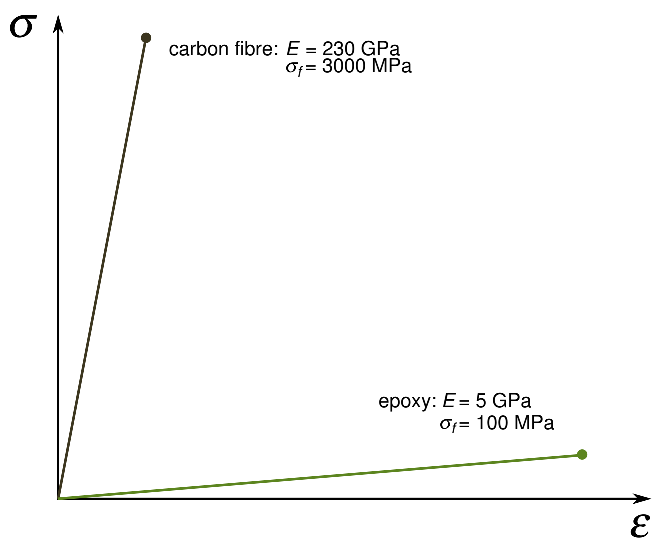
What Are Composites?

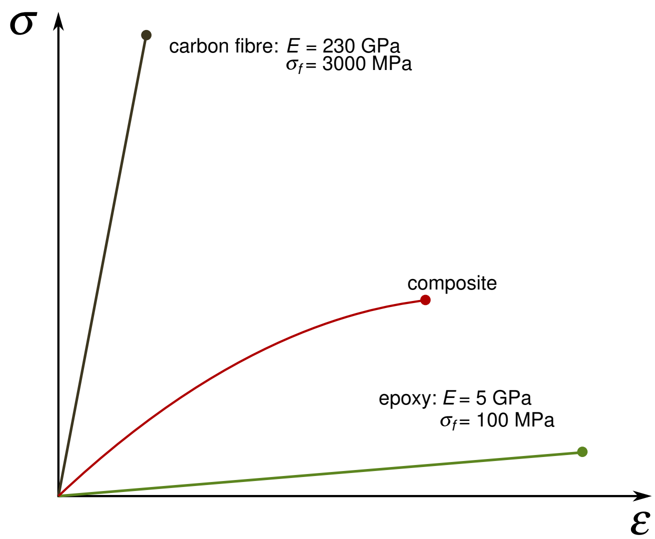


Classification of Composites



Why Composites?





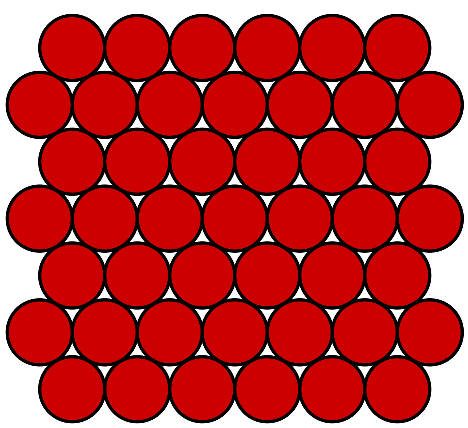
Composite Properties

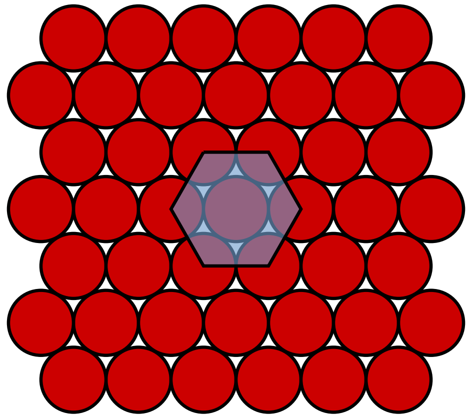
define:

v_f	volume fraction of the reinforcing material
ρ_m	density of the matrix material
ρ_f	density of the reinforcing material
E_m	Young's modulus of the matrix material
E_f	Young's modulus of the reinforcing material

Volume Fraction

looking at a unidirectional composite in cross-section, the cylindrical fibres look like closely packed circles





Composite Properties: Density

if we know the densities of the two components of the composites (ρ_f and ρ_m), as well as the volume fraction of the filler (v_f), we can directly calculate the density of the composite:

$$\rho = v_f \rho_f + \left(1 - v_f\right) \rho_m$$

this is the basic rule of mixtures, and is exact for the density of the composite material

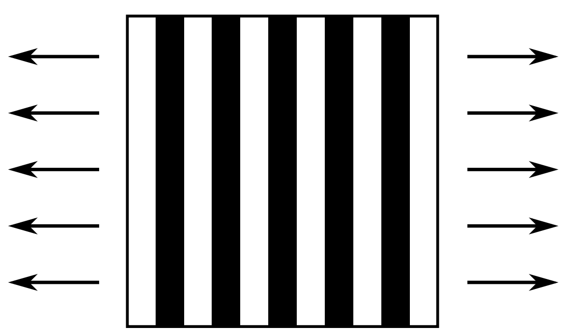
what other composite properties are described exactly by the rule of mixtures?

if we cannot calculate the property exactly, it is generally the case that we can find *bounds* on the possible values of the property

the bounds are the maximum and minimum values a property can have, independent of the arrangement of the reinforcement or the orientation of the composite

most importantly, we can calculate bounds on the elastic modulus of a composite

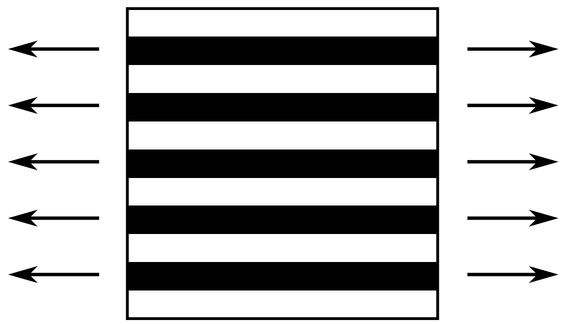
Bounds on Composite Properties: Modulus Lower (Reuss) Bound



the Reuss bound assumes that the *stress* is constant through all the layers of the composite and the strain is calculated:

$$\frac{1}{E} = \frac{v_f}{E_f} + \frac{\left(1 - v_f\right)}{E_m}$$

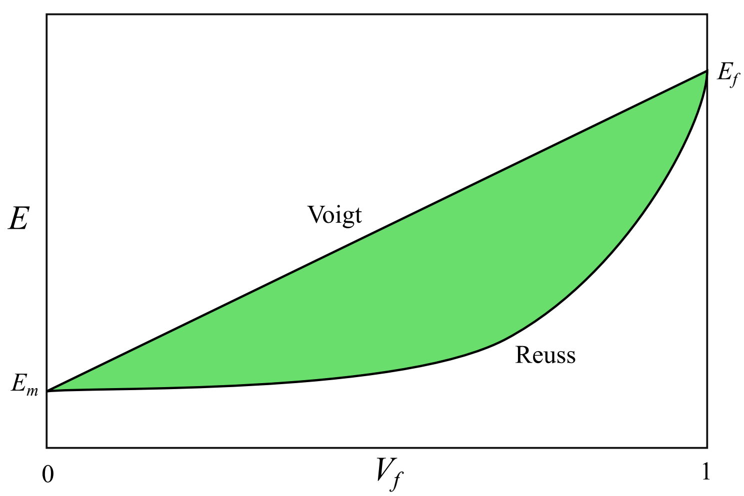
Bounds on Composite Properties: Modulus Upper (Voigt) Bound



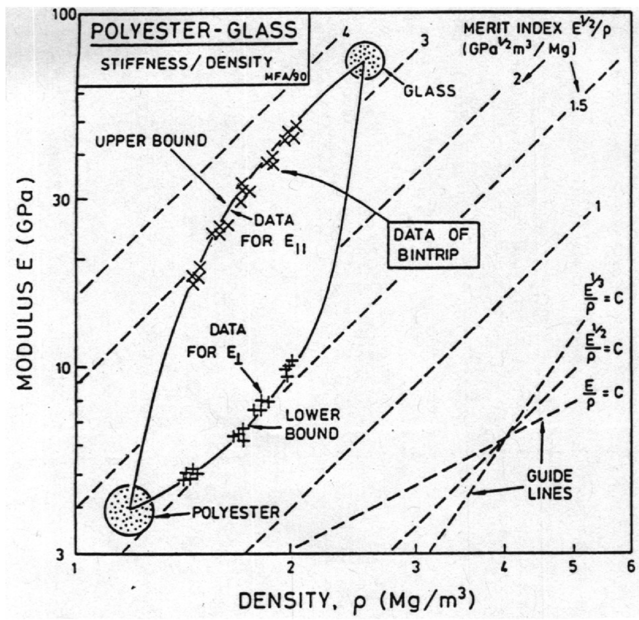
the Voigt bound assumes that the *strain* is constant through all the layers of the composite and the stress is calculated:

$$E = v_f E_f + \left(1 - v_f\right) E_m$$

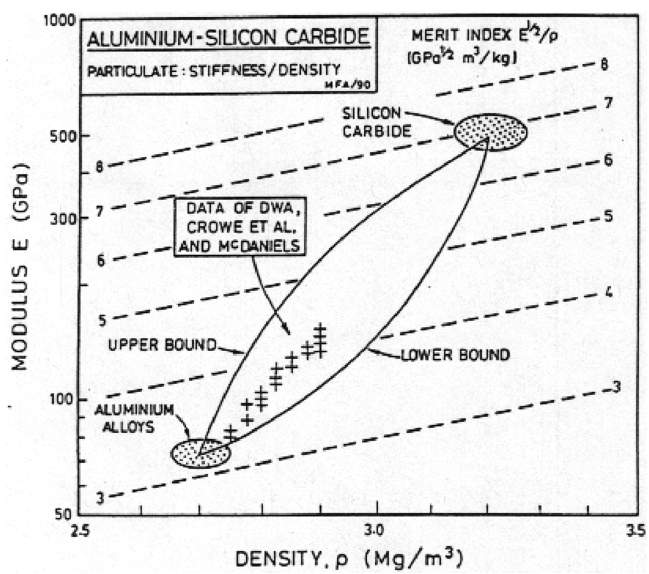
Modulus Bounds



Modulus Bounds: Polyester / Glass



Modulus Bounds: Aluminum / Silicon Carbide



Algorithm: Modulus Bounds

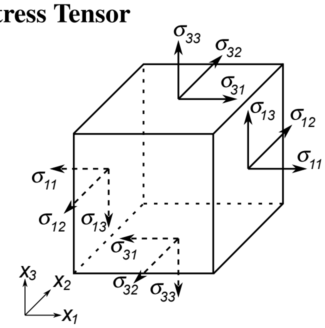
1. identify the materials of interest

2. determine the Young's modulus and density of the materials
3. for a given volume fraction, calculate the density and the Reuss and Voigt bounds for the composite
4. repeat for as many volume fractions are needed
5. plot the two bounds of composite modulus against volume fraction or density, as required

1.2 Elastic Properties of Composites

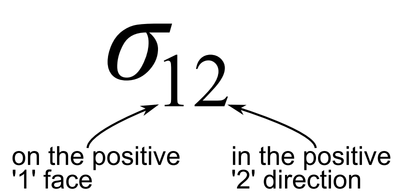
1.2.1 Linear Elasticity

Stress Tensor



this is a very small cube that surrounds a material point in a stressed medium in an $x_1 - x_2 - x_3$ axis system

what is the stress at this material point?
identify the various stresses on the cube using the following notation:



note that the stresses on the '2' faces have been omitted for clarity

this is the Cauchy stress tensor: when expressed in a particular coordinate system, it has the components:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Strain Tensor; Elastic Relations

similarly, the strain tensor is given by:

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

where

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

if the material behaves in a linear elastic manner, the stress and strain are related by a fourth order tensor:

$$\sigma_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl}\epsilon_{kl} \equiv C_{ijkl}\epsilon_{kl}$$

where C_{ijkl} is the tensor of elastic constants

for a general material, this requires 81 constants; for an isotropic material, two constants are sufficient

composites are somewhere in between these two extremes

Vector Notation for Stresses and Strains

in the composites world, the second-order stress and strain tensors are contracted into six-component vectors, and the tensor of elastic constants into a two-dimensional matrix:

$$\begin{aligned} \sigma &= [\sigma_{11}\ \ \sigma_{22}\ \ \sigma_{33}\ \ \sigma_{23}\ \ \sigma_{13}\ \ \sigma_{12}]^T \\ &= [\sigma_1\ \ \sigma_2\ \ \sigma_3\ \ \sigma_4\ \ \sigma_5\ \ \sigma_6]^T \end{aligned}$$

and

$$\begin{aligned} \epsilon &= [\epsilon_{11}\ \ \epsilon_{22}\ \ \epsilon_{33}\ \ 2\epsilon_{23}\ \ 2\epsilon_{13}\ \ 2\epsilon_{12}]^T \\ &= [\epsilon_{11}\ \ \epsilon_{22}\ \ \epsilon_{33}\ \ \gamma_{23}\ \ \gamma_{13}\ \ \gamma_{12}]^T \\ &= [\epsilon_1\ \ \epsilon_2\ \ \epsilon_3\ \ \epsilon_4\ \ \epsilon_5\ \ \epsilon_6]^T \end{aligned}$$

stress is related to strain by:

$$\sigma_i = Q_{ij}\epsilon_j$$

in the modern composites literature it is conventional to make this modification even though it discards all of the useful mathematics associated with tensors

Why Multiply the Shear Strains by Two?

the shear strains are multiplied by two in order to maintain an accurate work increment per unit volume:

$$\begin{aligned} \mathrm{d}W &= \sigma_{11}\,\mathrm{d}\epsilon_{11} + \sigma_{22}\,\mathrm{d}\epsilon_{22} + \sigma_{33}\,\mathrm{d}\epsilon_{33} + \sigma_{12}\,\mathrm{d}\epsilon_{12} + \sigma_{21}\,\mathrm{d}\epsilon_{21} \\ &+ \sigma_{13}\,\mathrm{d}\epsilon_{13} + \sigma_{31}\,\mathrm{d}\epsilon_{31} + \sigma_{23}\,\mathrm{d}\epsilon_{23} + \sigma_{32}\,\mathrm{d}\epsilon_{32} \end{aligned}$$

the corresponding expression for $\mathrm{d}W$ in the composites notation is:

$$\mathrm{d}W = \sigma_1\,\mathrm{d}\epsilon_1 + \sigma_2\,\mathrm{d}\epsilon_2 + \sigma_3\,\mathrm{d}\epsilon_3 + \sigma_4\,\mathrm{d}\epsilon_4 + \sigma_5\,\mathrm{d}\epsilon_5 + \sigma_6\,\mathrm{d}\epsilon_6$$

and hence $\epsilon_4 = 2\epsilon_{23}$

it is necessary to be very careful handling the shear stresses, shear strains and the elastic constants relating them

Strain Energy

for a linearly elastic solid, the strain energy density is given by:

$$W = \frac{1}{2}\sigma_i\epsilon_i = \frac{1}{2}Q_{ij}\epsilon_i\epsilon_j = \frac{1}{2}S_{ij}\sigma_i\sigma_j$$

where S_{ij} is the compliance matrix, and $S_{ij} = Q_{ij}^{-1}$; note that W is scalar

to get the total strain energy, the strain energy density must be integrated over the volume of interest

now:

$$\frac{\partial W}{\partial \epsilon_i} = Q_{ij}\epsilon_j$$

and

$$\frac{\partial^2 W}{\partial \epsilon_i \partial \epsilon_j} = Q_{ij}$$

reversing the order of differentiation shows that the stiffness matrix must be symmetric; the same can be said for the compliance matrix

hence, if elastic strain energy exists, the stiffness and compliance matrices must be symmetric

Compliance for an Isotropic Solid

the **elastic compliance matrix** S connects the stresses and strains:

$$\epsilon = S\sigma$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

for an isotropic solid, from the theory of elasticity:

$$\epsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}$$

or, in the composites notation:

$$\epsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} - \nu\frac{\sigma_3}{E}$$

because

$$\epsilon_1 = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3 + S_{14}\sigma_4 + S_{15}\sigma_5 + S_{16}\sigma_6$$

by equating corresponding coefficients, it is possible to identify some of the components of the elastic compliance matrix:

$$S_{11} = \frac{1}{E}; \quad S_{12} = -\frac{\nu}{E}; \quad S_{13} = -\frac{\nu}{E} \quad S_{14} = S_{15} = S_{16} = 0$$

Compliance for an Isotropic Solid

also:

$$\epsilon_4 = \frac{\sigma_4}{G}$$

this means that another component of the compliance matrix is:

$$S_{44} = \frac{1}{G}$$

while

$$S_{41} = S_{42} = S_{43} = S_{45} = S_{46} = 0$$

for an isotropic solid, there are only two elastic constants, so Young's modulus, Poisson's ratio and the shear modulus must be related:

$$G = \frac{E}{2\,(1+\nu)}$$

Compliance for an Isotropic Solid

the overall form of the compliance matrix for an **isotropic solid** in this notation is:

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}$$

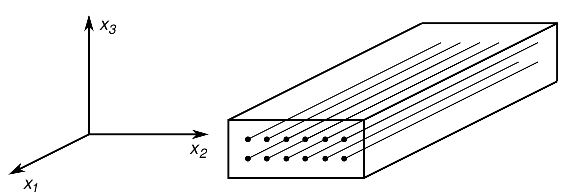
filling in the material constants:

$$S_{ij} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix}$$

1.2.2 Orthotropic Solids

Compliance of Orthotropic Solids

however, the previous expressions were for **isotropic solids**; composites are **not** isotropic



typically, a composite lamina is a solid with three orthogonal axes of material symmetry, here shown oriented with the coordinate axes

this is what is called an **orthotropic** solid

the **forms** of the compliance and stiffness matrices are the same as for the isotropic case; how many elastic constants are needed?

Compliance Matrix for Orthogonal Laminae

the compliance matrix for a general orthotropic solid has the same non-zero components as the compliance matrix for an isotropic solid; however, more material constants are needed

$$
\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}
$$

Unidirectional Laminae

composites are typically configured as stacks of laminae and each lamina is unidirectional, which means it has a constant fibre direction

for a unidirectional lamina, the fibre spacing is uniform in both transverse directions (approximately close-packed), so the response associated with the x_2 direction is the same as the response corresponding to the x_3 direction

the consequences of that are:

$$
\begin{aligned}
E_2 &= E_3 \\
G_{12} &= G_{13} \\
\nu_{12} &= \nu_{13} \\
\nu_{21} &= \nu_{31} \\
\nu_{23} &= \nu_{32} \\
G_{23} &= \frac{E_2}{2(1+\nu_{32})}
\end{aligned}
$$

Laminae in Plane Stress

in laminate plate theory, the through-thickness stresses are neglected and each lamina is assumed to exist in a state of plane stress:

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

by neglecting the corresponding strain components ϵ_{33}, γ_{13} and γ_{23}, this simplifies the compliance relation to:

$$
\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}
$$

where:

$$S_{11} = \frac{1}{E_1}$$

$$S_{22} = \frac{1}{E_2}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}$$

$$S_{66} = \frac{1}{G_{12}}$$

Stiffness Matrix in Plane Stress

invert the compliance matrix to get the stiffness matrix:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

where:

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{E_1}{1 - \nu_{12}\nu_{21}}$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{-S_{12}}{S_{11}S_{22} - S_{12}^2} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{66} = \frac{1}{S_{66}} = G_{12}$$

Example

consider T300/934 carbon-epoxy lamina with the properties: thickness $t = 0.125$ mm; fibre diameter $d = 6$ μm; fibre volume fraction $v_f = 0.65$

measured properties: $E_1 = 131$ GPa; $E_2 = 10.3$ GPa; $G_{12} = 6.0$ GPa; $\nu_{12} = 0.22$

the stiffness matrix is:

$$Q = \begin{bmatrix} 132 & 2.3 & 0 \\ 2.3 & 10.3 & 0 \\ 0 & 0 & 6.0 \end{bmatrix} \text{GPa}$$

what is the stress state if $\epsilon_{11} = 0.001$ and all other strains are zero?

what is the strain state if $\sigma_{11} = 100$ MPa and all other stresses are zero?

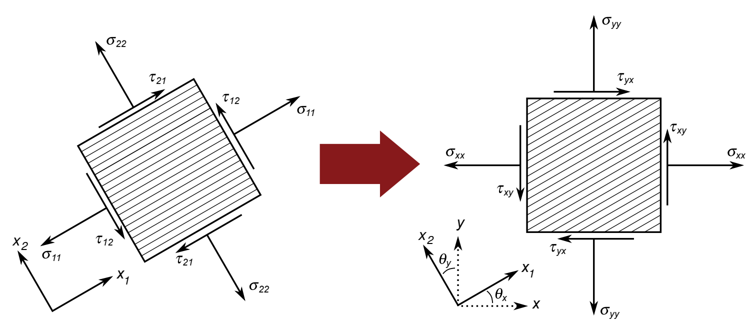
Algorithm: Calculating the Stiffness Matrix for a Unidirectional Lamina

1. collect all necessary material properties: $E_1, E_2, \nu_{12}, \nu_{21}, G_{12}$
2. calculate or estimate any properties that are missing
3. determine the compliance terms $S_{11}, S_{22}, S_{12}, S_{66}$ using the equations on page 29
4. invert the resulting matrix to get the stiffness terms $Q_{11}, Q_{22}, Q_{12}, Q_{66}$ using the equations on page 30

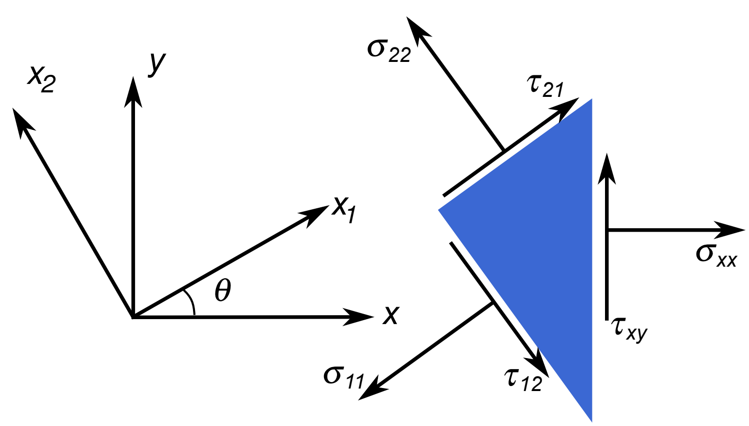
1.2.3 Rotation of Axes

Rotation of Axes

these are the expressions for the elastic properties of a lamina oriented with the principal material axes (x_1, x_2); how are these transformed into the expressions for the elastic properties in global axes (x, y)?

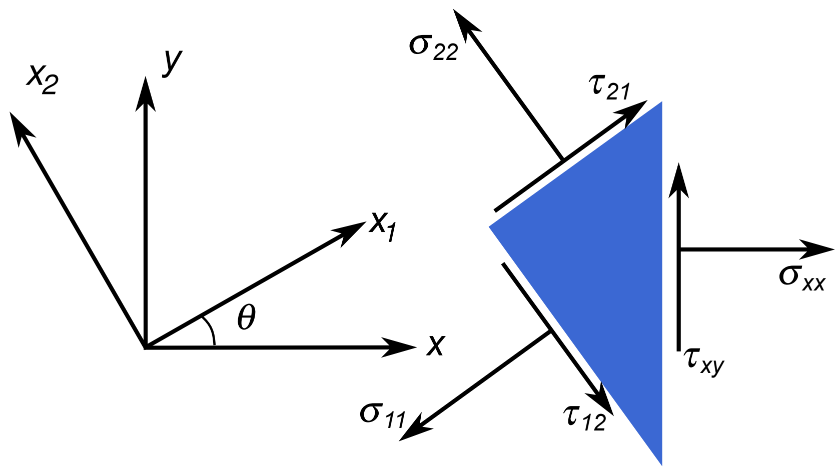


Rotation of Axes



this is an infinitesimal triangle surrounding a point in an elastic continuum

Rotation of Axes



determine the equilibrium of the infinitesimal triangle:

$$
\begin{aligned}
\sigma_{xx}\,\mathrm{d}A &= \sigma_{11}\cos^2\theta\,\mathrm{d}A - \tau_{12}\sin\theta\cos\theta\,\mathrm{d}A \\
&\quad +\sigma_{22}\sin^2\theta\,\mathrm{d}A - \tau_{21}\cos\theta\sin\theta\,\mathrm{d}A
\end{aligned}
$$

$$
\begin{aligned}
\tau_{xy}\,\mathrm{d}A &= \sigma_{11}\sin\theta\cos\theta\,\mathrm{d}A + \tau_{12}\cos^2\theta\,\mathrm{d}A \\
&\quad -\sigma_{22}\cos\theta\sin\theta\,\mathrm{d}A - \tau_{21}\sin^2\theta\,\mathrm{d}A
\end{aligned}
$$

simlarly for σ_{yy} and τ_{yx}

note that $\mathrm{d}A_1 = \mathrm{d}A\cos\theta$ and $\mathrm{d}A_2 = \mathrm{d}A\sin\theta$

Rotation of Axes

collect all of the terms into a matrix equation:

$$
\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 2\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix}
$$

invert this relation to get:

$$
\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}
$$

write this as:

$$
\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = [T] \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}
$$

Rotation of Axes

strains transform in the same manner:

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12}/2 \end{pmatrix} = [T] \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy}/2 \end{pmatrix}$$

transform this:

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 2\cos\theta\sin\theta & 2\left(\cos^2\theta - \sin^2\theta\right) \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy}/2 \end{pmatrix}$$

and another transformation:

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

Rotation of Axes

it turns out that:

$$\begin{bmatrix} \cos^2\theta & \sin^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} = \left[T^T\right]^{-1}$$

hence invert the previous relation to produce:

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \left[T^T\right] \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

in full:

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & \cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

Rotation of Axes

the key relation is the one between $\left(\sigma_{xx} \;\; \sigma_{yy} \;\; \tau_{xy}\right)^T$ and $\left(\epsilon_{xx} \;\; \epsilon_{yy} \;\; \gamma_{xy}\right)^T$; that is: the matrix $\left[\overline{Q}\right]$ where $(\sigma) = \left[\overline{Q}\right](\epsilon)$

recall that:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = [Q] \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

then:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = [T]^{-1} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = [T]^{-1} [Q] \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = [T]^{-1} [Q] \left[T^T\right]^{-1} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

Rotation of Axes

that is the necessary relation; write it as:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \left[\overline{Q}\right] \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

where

$$\left[\overline{Q}\right] = [T]^{-1} [Q] \left[T^T\right]^{-1}$$

$$\left[\overline{Q}\right] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}$$

note that $\overline{Q}_{21} = \overline{Q}_{12}$, $\overline{Q}_{62} = \overline{Q}_{26}$ and $\overline{Q}_{61} = \overline{Q}_{16}$: the matrix is symmetric

Rotation of Axes

the components of $\left[\overline{Q}\right]$ are given explicitly by:

$$\overline{Q}_{11} = Q_{11} \cos^4\theta + Q_{22} \sin^4\theta + 2\,(Q_{12} + 2Q_{66}) \sin^2\theta \cos^2\theta$$

$$\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \cos^2\theta \sin^2\theta + Q_{12} \left(\cos^4\theta + \sin^4\theta\right)$$

$$\overline{Q}_{22} = Q_{11} \sin^4\theta + Q_{22} \cos^4\theta + 2\,(Q_{12} + 2Q_{66}) \sin^2\theta \cos^2\theta$$

$$\overline{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \cos^3\theta \sin\theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta$$

$$\overline{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \cos\theta \sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos^3\theta \sin\theta$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \cos^2\theta \sin^2\theta + Q_{66} \left(\cos^4\theta + \sin^4\theta\right)$$

note that this is a full matrix: this means that there is coupling between normal strains and shear stresses (and shear strains and normal stresses); uncoupling of direct and shear straining occurs only when the loading is aligned with the principal material directions

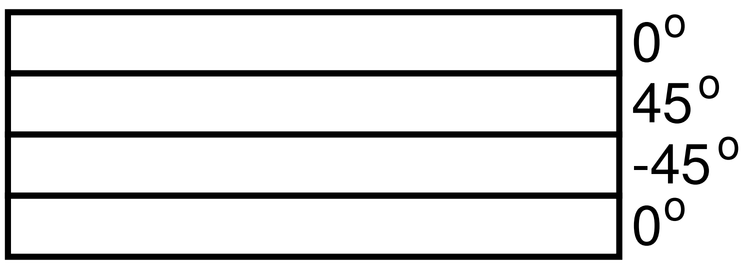
Algorithm: Calculating the Rotated Stiffness Matrix for a Unidirectional Lamina

1. find the stiffness matrix for the lamina in the material coordinate system
2. determine the angle θ by which the global coordinate system is rotated from the material system
3. use the equations on page 41 to determine the $\bar{Q}_{ij}$ components of the stiffness matrix

1.2.4 Laminate Plate Theory

Laminates

at present, composites are usually produced by stacking layers of fabric / prepreg to produce laminates



the laminate to the left is produced by stacking four layers of unidirectional prepreg in the order 0°, -45°, 45°, 0°

we use the conventional notation [0/-45/45/0] to identify this composite layup

note that this is from bottom to top: the first ply is the one closest to the tooling

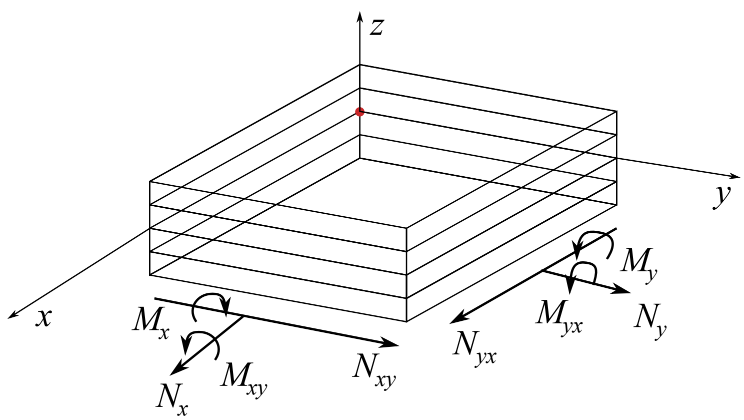
Laminate Notation

there is a standard convention for the identification of composite laminate configurations

1. a laminate description is enclosed by square brackets
2. the angles of the plies are separated by forward slashes
3. symmetry is indicated by a subscripted 's'
4. a bilayer consisting of a +45° ply and a -45° ply is often identified as [±45°]
5. the central ply of a symmetric laminate with an odd number of plies is indicated by an overbar
6. angles are always between +90 and -90

two more common terms: an 'angle-ply' laminate has $\pm\theta$ plies ([30 / -30] or [45 / -45]) and a 'cross-ply' laminate has 0° and 90° plies ($[0 / 90]_s$ or [0 / 90 / 0 / 90])

Laminate Plate Theory



the laminate is assumed to have its middle surface on the (x, y) plane

the displacements at a point (x, y, z) are (u, v, w)

Laminate Plate Theory

there are several important assumptions:

1. the plate consists of orthotropic laminae bonded together, with the principal material directions of the laminae oriented arbitrarily with respect to the (x, y) axes
2. each ply is linearly elastic
3. the thickness of the plate, t, is small compared to the other dimensions of the plate
4. the plate thickness t is constant
5. the displacements u and v are small compared to the plate thickness
6. tangential displacements u and v are are linear functions of the through-thickness z coordinate
7. the in-plane strains ϵ_x, ϵ_y and γ_{xy} are small
8. the transverse normal strain ϵ_z is neglected
9. transverse shear strains γ_{xz} and γ_{yz} are zero
10. the transverse shear stresses τ_{xz} and τ_{yz} vanish on the plate surfaces defined by $z = \pm\frac{t}{2}$

Laminate Plate Theory

identify the displacements and strains on the mid-plane of the laminate with superscripts: $u^\circ, v^\circ, w^\circ, \epsilon_x^\circ, \epsilon_y^\circ, \gamma_{xy}^\circ$

as a consequence, the displacements at points away from the mid-plane can be expressed as:

$$u(x,y,z) = u^\circ(x,y) + zF_1(x,y)$$

$$v(x,y,z) = v^\circ(x,y) + zF_2(x,y)$$

$$w(x,y,z) = w^\circ(x,y)$$

these are consequences of assumptions 4, 6 and 8

Laminate Plate Theory

differentiate the expressions for u and v with respect to z to get $\partial u/\partial z = F_1$ and $\partial v/\partial z = F_2$

substitute these into the definitions of transverse shear strain, recalling that they have been defined by assumption to be zero:

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = F_1(x,y) + \frac{\partial w}{\partial x} = 0$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = F_2(x,y) + \frac{\partial w}{\partial y} = 0$$

consequently it must be the case that:

$$F_1(x,y) = -\frac{\partial w}{\partial x} \qquad \text{and} \qquad F_2(x,y) = -\frac{\partial w}{\partial y}$$

and

$$u = u^\circ - z\frac{\partial w}{\partial x} \qquad \text{and} \qquad v = v^\circ - z\frac{\partial w}{\partial y}$$

Laminate Plate Theory

substituting these results into the definitions of in-plane strains and differentiating gives:

$$\epsilon_x = \frac{\partial u}{\partial x} = \epsilon_x^\circ + z\kappa_x$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \epsilon_y^\circ + z\kappa_y$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy}^\circ + z\kappa_{xy}$$

the strains on the mid-plane are:

$$\epsilon_x^\circ = \frac{\partial u^\circ}{\partial x} \quad \epsilon_y^\circ = \frac{\partial v^\circ}{\partial y} \quad \gamma_{xy}^\circ = \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x}$$

and the curvatures of the mid-plane are:

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2} \quad \kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

Laminate Plate Theory

these give the strains for any arbitrary distance z from the mid-plane; find the stresses in the kth lamina by substituting into the lamina stress-strain relations:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}_k \begin{pmatrix} \epsilon_x^\circ + z\kappa_x \\ \epsilon_y^\circ + z\kappa_y \\ \gamma_{xy}^\circ + z\kappa_{xy} \end{pmatrix}$$

it is convenient to use forces and moments per unit width instead of stresses, particularly when the component has relatively simple geometry

force per unit width is:

$$N_i = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_i \, \mathrm{d}z = \sum_{k=1}^{N} \left(\int_{z_{k-1}}^{z_k} (\sigma_i)_k \, \mathrm{d}z \right)$$

and moment per unit width is:

$$M_i = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_i z \, \mathrm{d}z = \sum_{k=1}^{N} \left(\int_{z_{k-1}}^{z_k} (\sigma_i)_k z \, \mathrm{d}z \right)$$

with z_k the z-coordinate of the top surface of lamina k and z_{k-1} the z-coordinate of the bottom surface of lamina k

Laminate Plate Theory

substitute the lamina stress-strain relations into the integrals to get:

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \epsilon_x^\circ \\ \epsilon_y^\circ \\ \gamma_{xy}^\circ \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

the extensional stiffnesses are:

$$A_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\overline{Q}_{ij}\right)_k \, \mathrm{d}z = \sum_{k=1}^{N} \left(\left(\overline{Q}_{ij}\right)_k (z_k - z_{k-1}) \right)$$

the coupling stiffnesses are:

$$B_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\overline{Q}_{ij}\right)_k z \, \mathrm{d}z = \frac{1}{2} \sum_{k=1}^{N} \left(\left(\overline{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right) \right)$$

and the bending stiffnesses are:

$$D_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\overline{Q}_{ij}\right)_k z^2 \, \mathrm{d}z = \frac{1}{3} \sum_{k=1}^{N} \left(\left(\overline{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right) \right)$$

Laminate Plate Theory

this equation is often written in abbreviated, partitioned form:

$$\begin{pmatrix} N \\ M \end{pmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{pmatrix} \epsilon^{\circ} \\ \kappa \end{pmatrix}$$

the extensional terms A relate the in-plane forces to the in-plane strains; the coupling terms B relate the the in-plane forces to bending and twisting, and the moments to in-plane strains; the bending terms D relate the moments and torques to the curvatures and twists

NB: if a laminate is symmetric around the mid-plane, the B terms are all zero - there is no coupling between bending and in-plane forces

Algorithm: Calculating the Stiffness Matrix for a Laminate

1. calculate the $\bar{Q}_{ij}$ matrix for all N lamina in the laminate in a single global coordinate system

2. determine the locations z of the top and bottom of each lamina; z_k is at the top of the lamina and z_{k-1} is at the bottom (distances measured from the mid-plane)

3. for N layers, calculate the sum of the $\bar{Q}_{ij}$ terms times the layer thickness for all layers; this provides A_{ij} as shown on page 51

4. for N layers, calculate the sum of the $\bar{Q}_{ij}$ terms times the difference between the square of the distances to the top and bottom surfaces of the lamina; this provides B_{ij} (if the laminate is symmetric, the B_{ij} terms are all zero)

5. for N layers, calculate the sum of the $\bar{Q}_{ij}$ terms times the difference between the cube of the distances to the top and bottom surfaces of the lamina; this provides D_{ij}

Laminate Plate Theory - Calculating Stresses

the equation that to be solved for the laminate is:

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \epsilon_x^\circ \\ \epsilon_y^\circ \\ \gamma_{xy}^\circ \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

typically what is known is the vector of applied forces:

$$\begin{pmatrix} N_x & N_y & N_{xy} & M_x & M_y & M_{xy} \end{pmatrix}^T$$

and the matrix equation is solved for the strains:

$$\begin{pmatrix} \epsilon_x^\circ & \epsilon_y^\circ & \gamma_{xy}^\circ & \kappa_x & \kappa_y & \kappa_{xy} \end{pmatrix}^T$$

Laminate Plate Theory - Calculating Stresses

solving for the vector of laminate mid-plane strains allows assembly of the vector of strains in each lamina, knowing the distance z of the individual laminae from the plate mid-plane:

$$\begin{pmatrix} \epsilon_x^\circ + z\kappa_x & \epsilon_y^\circ + z\kappa_y & \gamma_{xy}^\circ + z\kappa_{xy} \end{pmatrix}^T$$

here, z is the position of the mid-plane of the individual lamina

this vector is used with the stress-strain relation for lamina k:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}_k \begin{pmatrix} \epsilon_x^\circ + z\kappa_x \\ \epsilon_y^\circ + z\kappa_y \\ \gamma_{xy}^\circ + z\kappa_{xy} \end{pmatrix}$$

this gives the vector of stresses in each lamina, which is used to get the lamina stresses in the material coordinate system:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = [T] \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}$$

failure analysis compares this stress state to some failure criterion

Algorithm: Calculating Stresses

1. using the laminate stiffness matrix and the vector of applied forces and moments, determine the vector of mid-plane strains and curvatures

2. for each lamina in the laminate, calculate the vector of strains for that lamina given by the equation on page 55

3. determine the vector of stresses in each lamina using the $\bar{Q}_{ij}$ matrix for that lamina and the vector of lamina strains; this generates stresses in the global coordinate system

4. for each lamina, convert the stresses in the global coordinate system to stresses in the material coordinate system using the vector of global stresses and the $[T]$ matrix for the appropriate lamina

1.3 Strength of Composites

Models of Composite Strength

there are two general types of models for composite strength:

Engineering models: robust and simple enough to be used in everyday engineering calculations, but can be inaccurate if used in the wrong situation and therefore have to be supported by careful testing, both numerically and experimentally

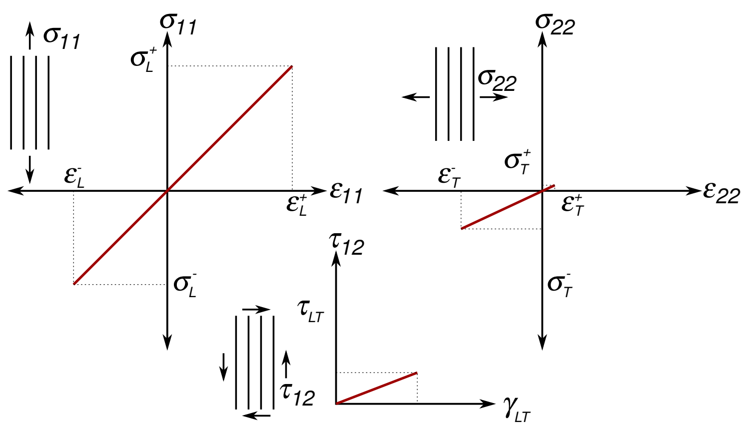
Micromechanical models: capture the essential physical mechanism for a particular failure mode and should be accurate for a range of materials and loading conditions, but are too complex for everyday use

for the most part, this course concentrates on engineering models

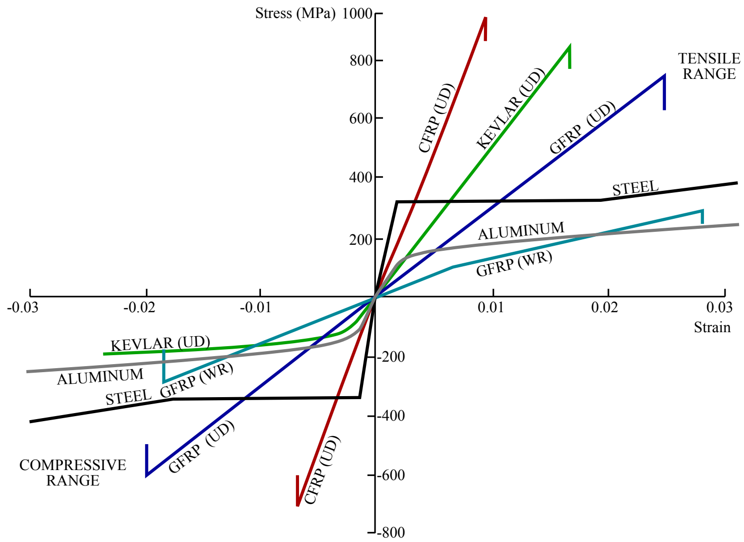
Stress-based Lamina Failure

consider the in-plane loading of a unidirectional composite lamina; assume that the resolved stresses and strains in terms of the material axes of the lamina are known

this may be the result of simple calculations or of a large finite element simulation



Typical Stress-Strain Curves



Typical Failure Stress Data

Material	Longitudinal tension σ_L^+ (MPa)	Longitudinal compression σ_L^- (MPa)	Transverse tension σ_T^+ (MPa)	Transverse compression σ_T^- (MPa)	In-plane shear τ_{LT} (MPa)
Al 7075-T6	450	450	450	450	225
Glass - polyester	650-950	600-900	20-25	90-120	45-60
Carbon - epoxy	850-1500	700-1200	35-40	130-190	60-75
Kevlar - epoxy	1100-1250	240-290	20-30	110-140	40-60

note that while strengths are normally given as positive quantities, compressive strengths refers to negative stresses

for example, for Al 7075-T6, provided -450 MPa $\leq \sigma_{11} \leq 450$ MPa, the material will not yield in tension or compression

Typical Failure Strain Data

Material	Longitudinal tension ϵ_L^+	Longitudinal compression ϵ_L^-	Transverse tension ϵ_T^+	Transverse compression ϵ_T^-	In-plane shear γ_{LT}
Aluminum alloy	15%		15%		100%
Glass - polyester	2.5%	2%	0.5%	3%	10%
Carbon - epoxy	1-2%	1-2%	0.5%	2-3%	10%
Kevlar - epoxy	1.5%	2.5%	0.5%	1%	10%

1.3.1 Engineering Models

Standard Criteria for Composite Failure

the most common approach is to use laminate plate theory or finite element calculations to determine the stress state in a laminate, and from that analysis determine the stress state in all of the constituent laminae

empirical failure criteria are used to decide when laminate failure occurs; typically this is after failure of the first ply

this means that the stresses or strains in each ply are compared to **measured** material properties to determine whether failure has occurred

the are three main types of failure criteria:

1. maximum stress
2. maximum strain
3. quadratic / interaction

Maximum Stress Failure Criterion

the stresses in the lamina are compared to maximum allowable stresses

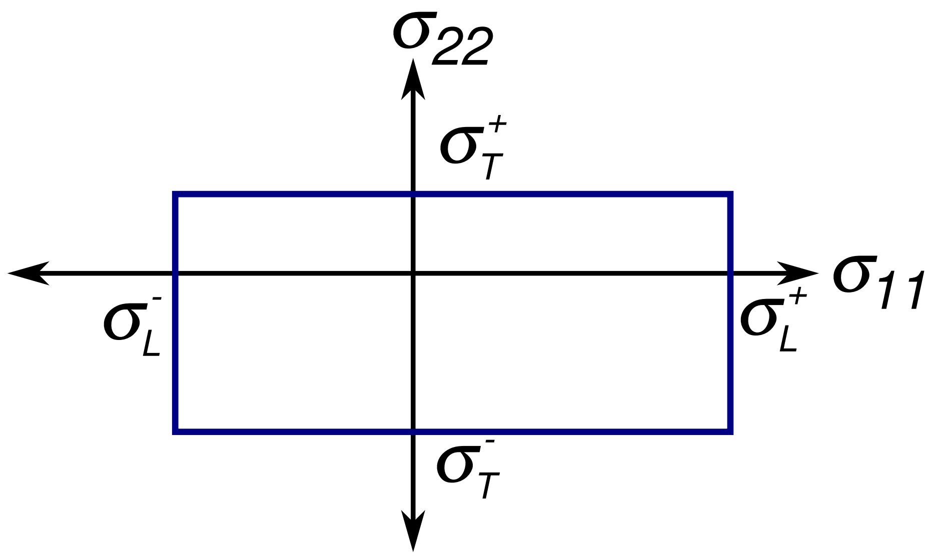
the stresses are resolved into the material coordinate system so that σ_{11} aligns with the fibre direction

failure will not occur provided the following conditions are met:

$$\sigma_L^- < \sigma_{11} < \sigma_L^+$$
$$\sigma_T^- < \sigma_{22} < \sigma_T^+$$
$$|\tau_{12}| < \tau_{LT}$$

this requires that all of the stresses are between the limits set by the strengths of the material

graphically, consider a **stress space** with axes σ_{11} and σ_{22}



this is a section of the failure surface for a maximum stress failure criterion; τ_{12} would lie in a third dimension

the stress point $(\sigma_{11}, \sigma_{22})$ must lie within the region set by σ_L and σ_T for the material to survive

Maximum Strain Failure Criterion

next consider the maximum allowable strain in the lamina

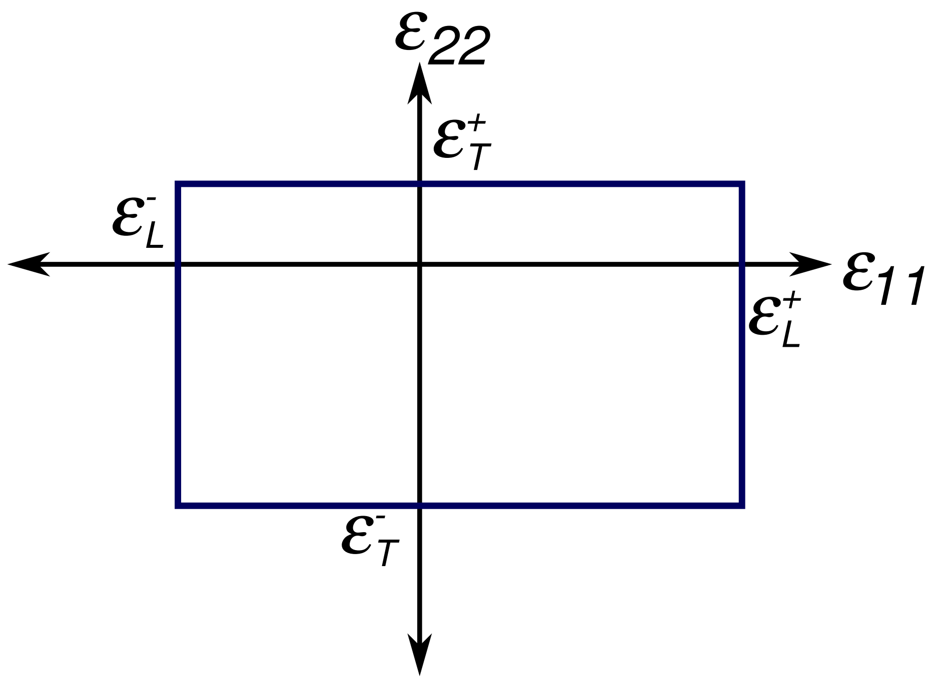
the strains are resolved such that ϵ_{11} aligns with the fibre direction

failure will not occur provided the following conditions are met:

$$\epsilon_L^- < \epsilon_{11} < \epsilon_L^+$$

$$\epsilon_T^- < \epsilon_{22} < \epsilon_T^+$$

$$|\gamma_{12}| < \gamma_{LT}$$



this is a section of the failure surface for a maximum strain failure criterion

Quadratic Failure Criterion - Tsai-Hill

failure criteria that are quadratic in the stresses allow for interaction between the stress components

consider plane stress with $\sigma_{33} = \tau_{13} = \tau_{23} = 0$

for homogeneous isotropic ductile solids, the von Mises criterion predicts yielding when:

$$\sigma_{11}^2 + \sigma_{22}^2 + (\sigma_{22} - \sigma_{11})^2 + 6\tau_{12}^2 \geq 2Y^2$$

where Y is the uniaxial yield strength

for **anisotropic** solids, the equivalent is the Tsai-Hill criterion:

$$\frac{\sigma_{11}^2}{\sigma_L^2} - \frac{\sigma_{11}\sigma_{22}}{\sigma_L^2} + \frac{\sigma_{22}^2}{\sigma_T^2} + \frac{\tau_{12}^2}{\tau_{LT}^2} \geq 1$$

where σ_L and σ_T are the strengths in either tension or compression, as appropriate for the respective signs of σ_{11} and σ_{22}

note that composite materials **do not yield**

Quadratic Failure Criterion - Tsai-Wu

the Tsai-Wu failure criterion uses a more general formulation for interaction between the various components of stresses

the most important terms of the relevant expression are:

$$F_{11}\sigma_{11}^2 + F_{22}\sigma_{22}^2 + F_{66}\tau_{12}^2 + F_1\sigma_{11} + F_2\sigma_{22} + 2F_{12}\sigma_{11}\sigma_{22} \geq 1$$

where:

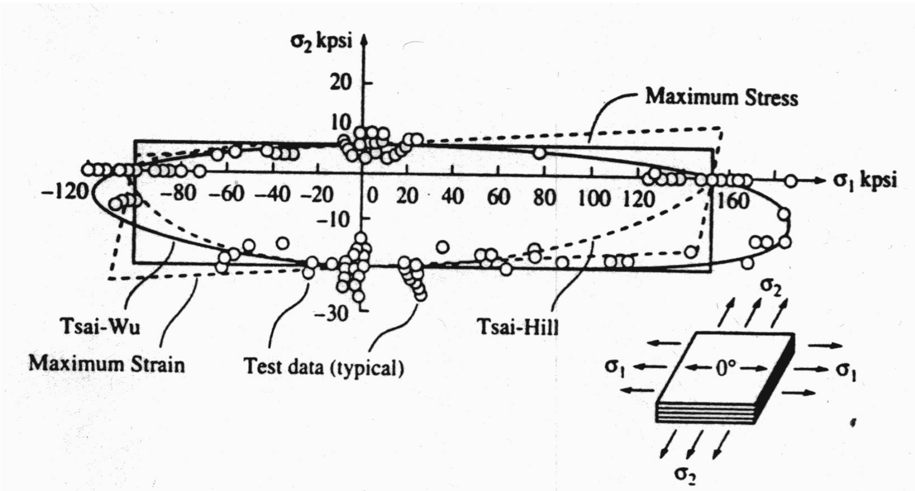
$$F_{11} = \frac{1}{|\sigma_L^+\sigma_L^-|} \qquad F_{22} = \frac{1}{|\sigma_T^+\sigma_T^-|}$$

$$F_1 = \frac{1}{\sigma_L^+} - \frac{1}{|\sigma_L^-|} \qquad F_2 = \frac{1}{\sigma_T^+} - \frac{1}{|\sigma_T^-|} \qquad F_{66} = \frac{1}{\tau_{LT}^2}$$

the interaction term should be optimised using experimental data, but is usually approximated as:

$$F_{12} \approx -\frac{(F_{11}F_{22})^{\frac{1}{2}}}{2}$$

Comparison of Failure Surfaces



Algorithm: Applying Failure Criteria

1. calculate the vector of stresses (or strains if needed) in the material coordinate system for the ply or plies of interest
2. select an appropriate failure criterion; for most composite design, a quadratic criterion is most likely to be used
3. identify the relevant parameters associated with the failure criterion
4. evaluate the equations associated with the chosen criterion

Failure as a Function of Ply Angle

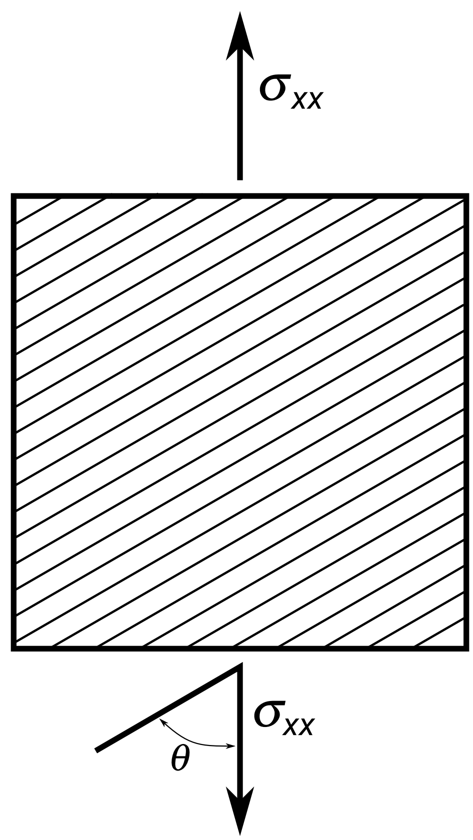
the maximum axial stress that can be applied to a ply is a strong function of the angle of the ply

here, the plies make angle θ with the loading direction

how does the strength of the ply vary with the angle of the ply?

resolve the stresses for each angle from 0° to 90° (after which the result is periodic)

apply a maximum stress or quadratic failure criterion to find the maximum σ_{xx} that can be applied before failure occurs



1.3.2 Micromechanical Models

Micromechanical Models

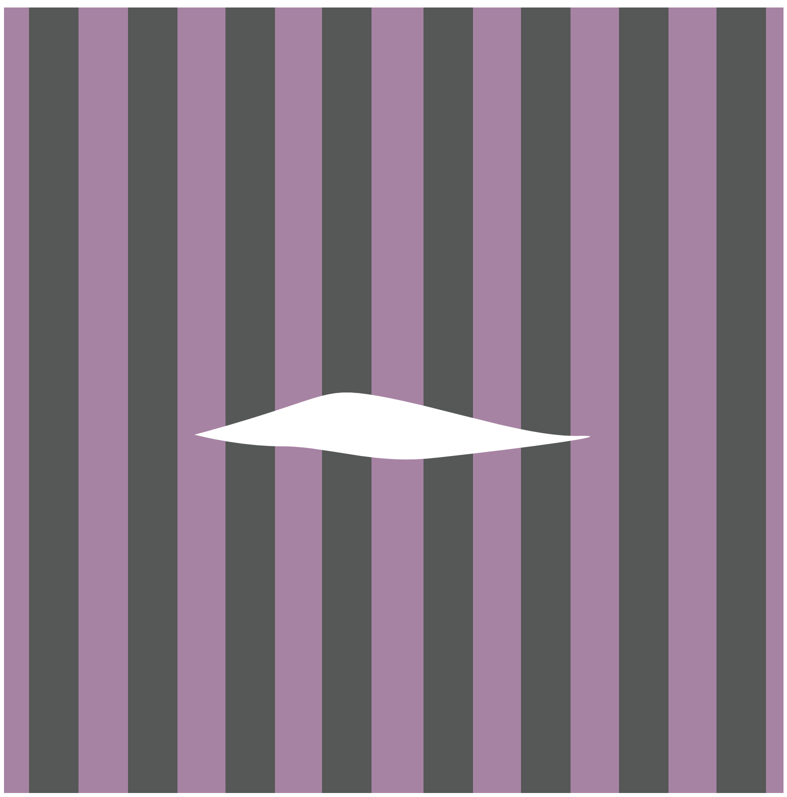
micromechanical models depend upon the accurate description of the material and geometry of a composite structure or specimen typically they involve far more effort than the application of empirical failure models (which tend to be highly phenomenological)

because micromechanical models accurately capture the physics of the behaviour of composites, they provide predictions of the strength of composite when strength is governed by a particular failure mechanism, even outside the range of experimental data

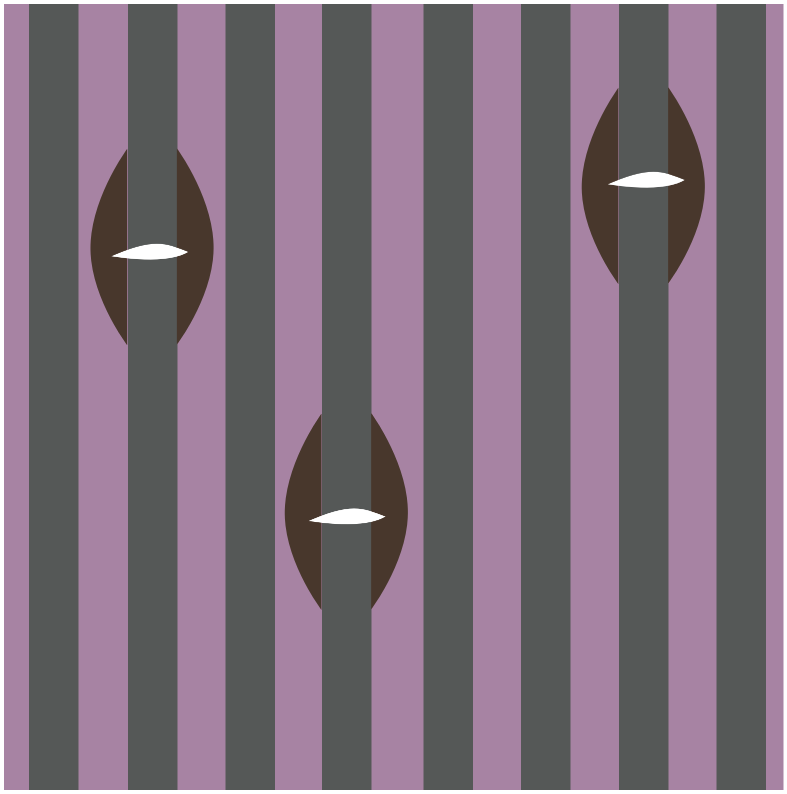
Micromechanisms of Failure - Fibre Failure

composite failure is governed by either fibre or matrix failure

stress concentrations due to fibre failure may lead to a cascading failure, where nearby fibres also break



alternately, shear lag zones enable fibres to pick up axial stress again some distance from the fibre break; the fibre breaks remain isolated

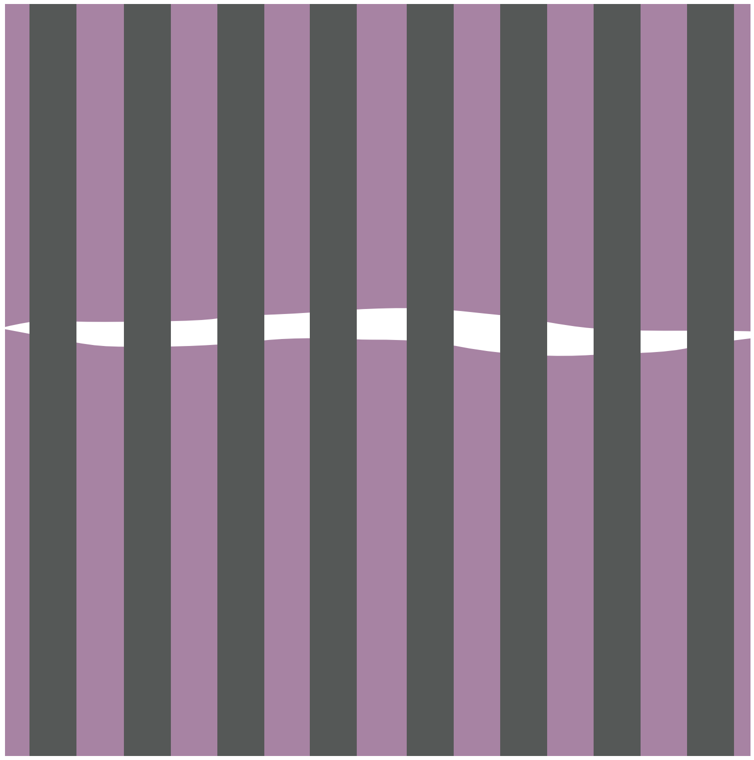


Micromechanisms of Failure - Matrix Failure

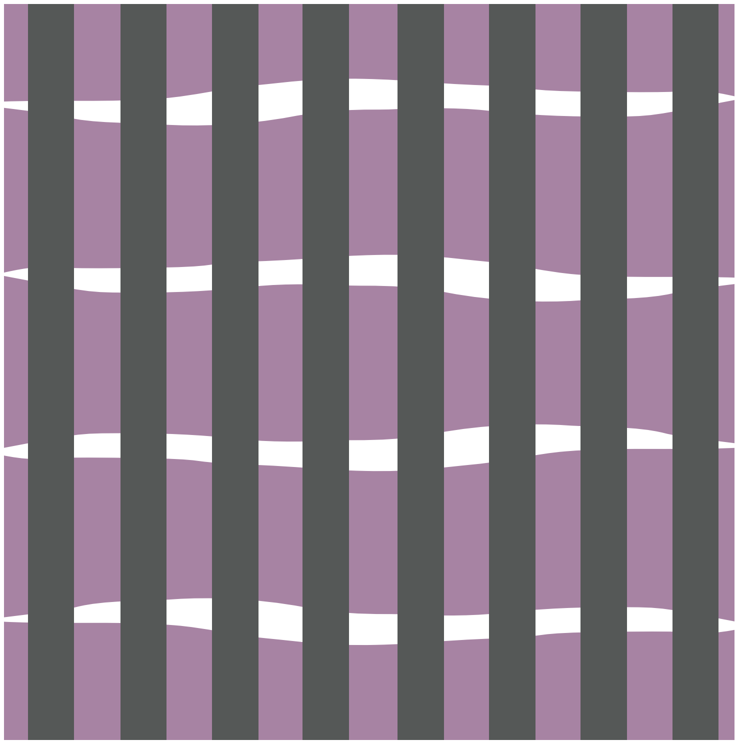
matrix failure occurs when the strain to failure of the matrix is less than half that of the fibres

fibres may bridge the matrix cracks

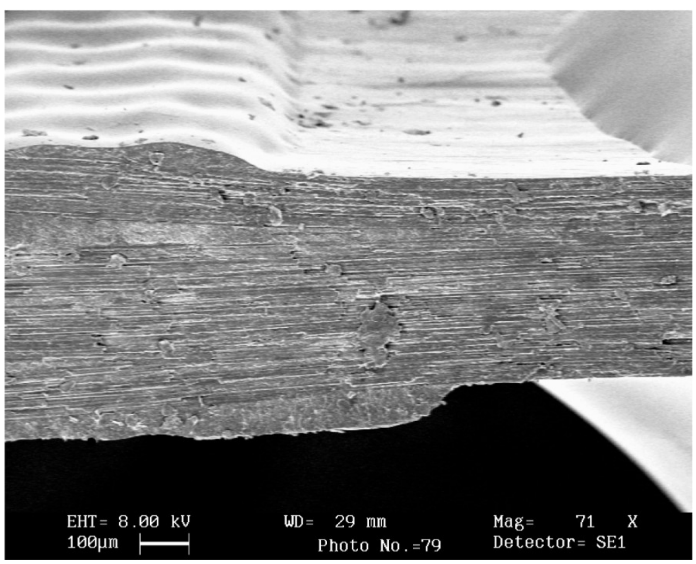
this is usually more common in ceramic matrix composites



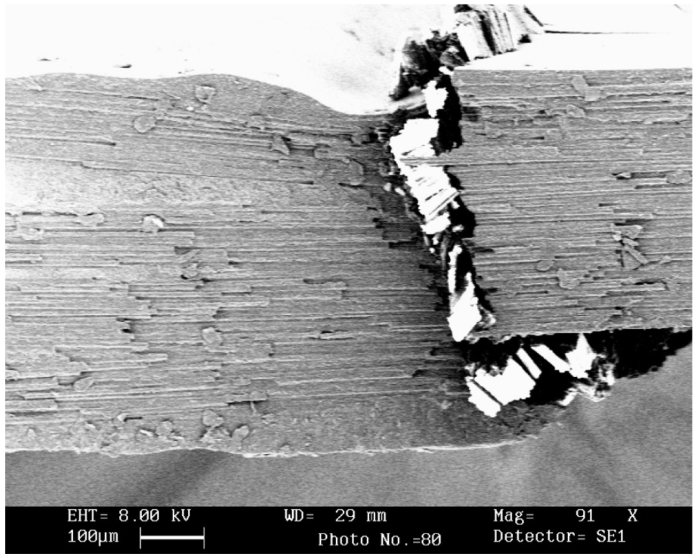
it is possible that multiple matrix fractures will occur; the distance between the fractures has a characteristic length scale depending upon the material properties of the fibre and matrix



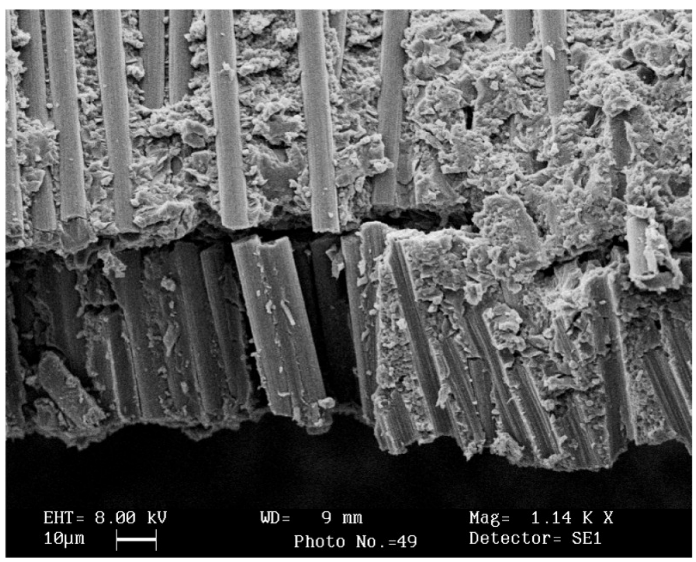
Microbuckling



Microbuckling



Microbuckling



Microbuckling

microbuckling is the standard mechanism of failure for long-fibre composites in compression

microbuckling is driven by an interaction between the misalignment of the primary load-bearing fibres and the shear failure of the matrix material when the matrix material shears plastically, kink bands of a characteristic length form in the fibres

the microbuckling strength of a composite can be estimated by:

$$\sigma_M = \frac{\tau_Y}{\phi_0}$$

see Fleck, *Advances in Applied Mechanics*, 1997, for comprehensive detail

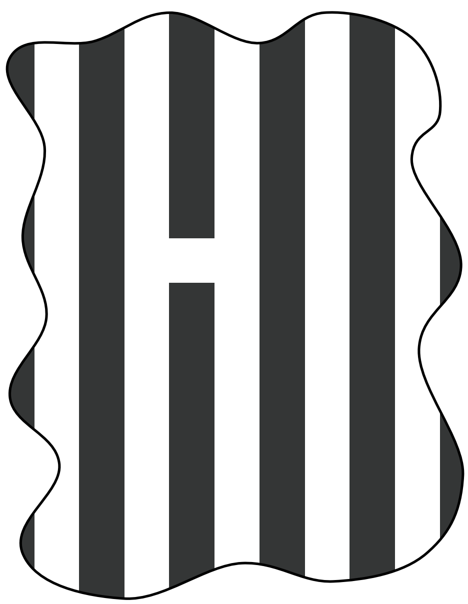
Shear Lag

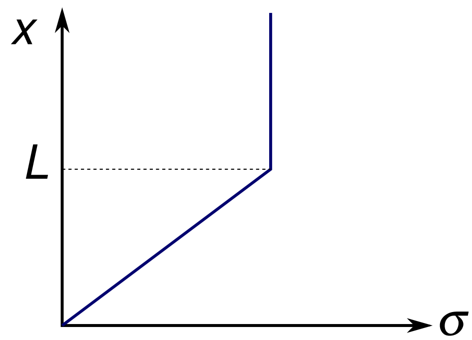
tensile failure of a composite can be associated with sudden, catastrophic large-scale damage, or with progressive fibre failure

in the second case, a micromechanical model of shear lag in fibres is useful

if there is a fibre break within the composite, the tensile stress on the end of the fibre must be zero

the stress in the fibre builds up as shear is transferred from the matrix to the fibre



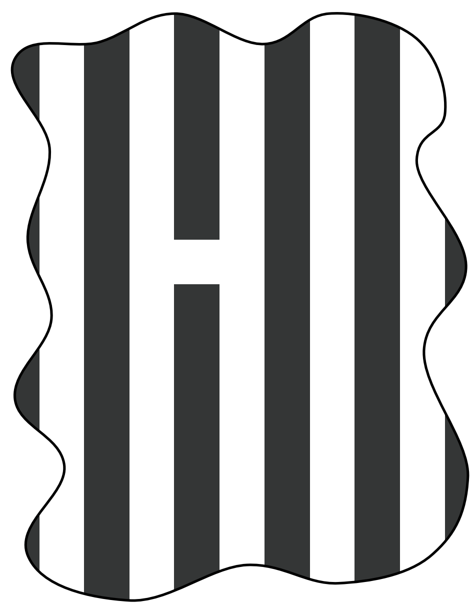


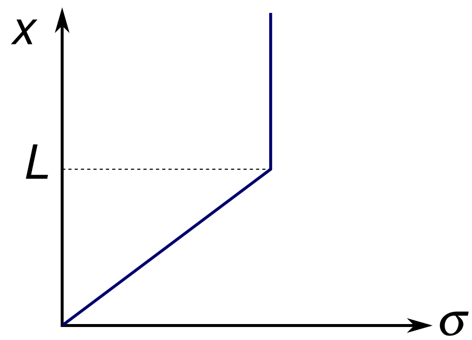
Shear Lag

Aveston, Cooper and Kelly developed a micromechanical model for shear lag: see Aveston and Kelly, *Journal of Material Science*, 1973

they found that the length of the shear lag zone is given by:

$$L = \frac{d\sigma_1 E_f}{4\tau_f E_1}$$





1.4 Sandwich Panels

Sandwich Panels and Beams

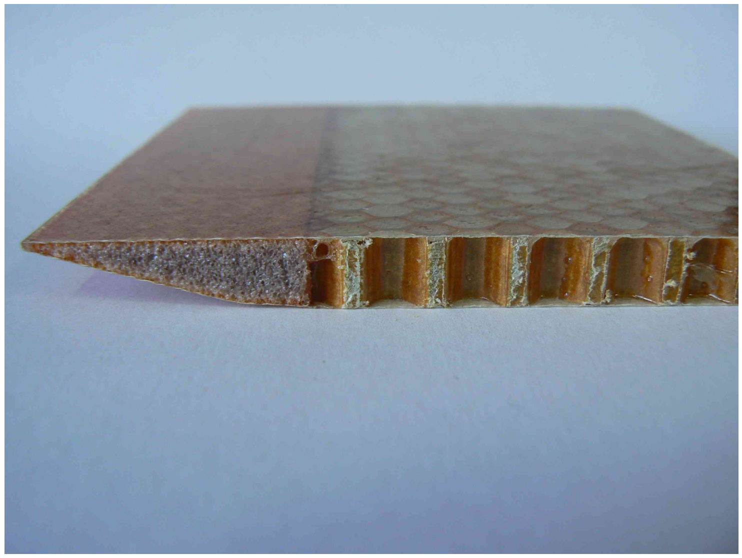
in aerospace engineering (and in many other areas) it is common to use composites in the form of sandwich structures, panels and beams

sandwich structures are composed of two strong, stiff face sheets separated by a lightweight core, which comes in a variety of configurations

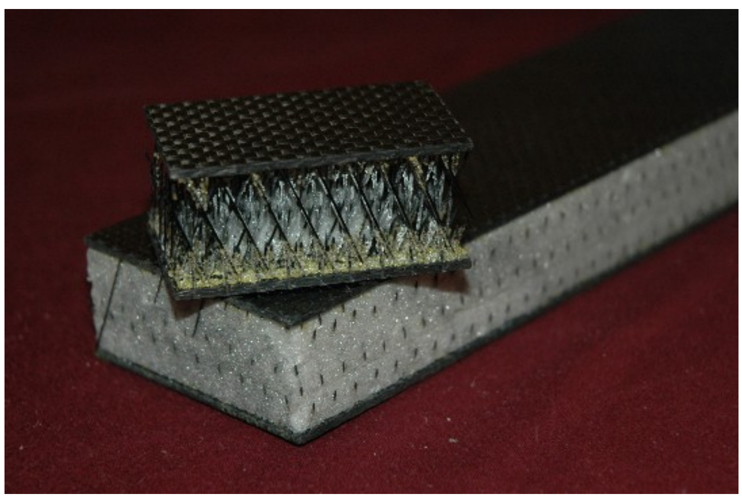
the purpose of this geometry is that the sandwich structure has a much higher ratio of bending stiffness to mass than do monolithic plates and beams; a sandwich is analogous to an I-beam

Polymer Foam Core Sandwiches

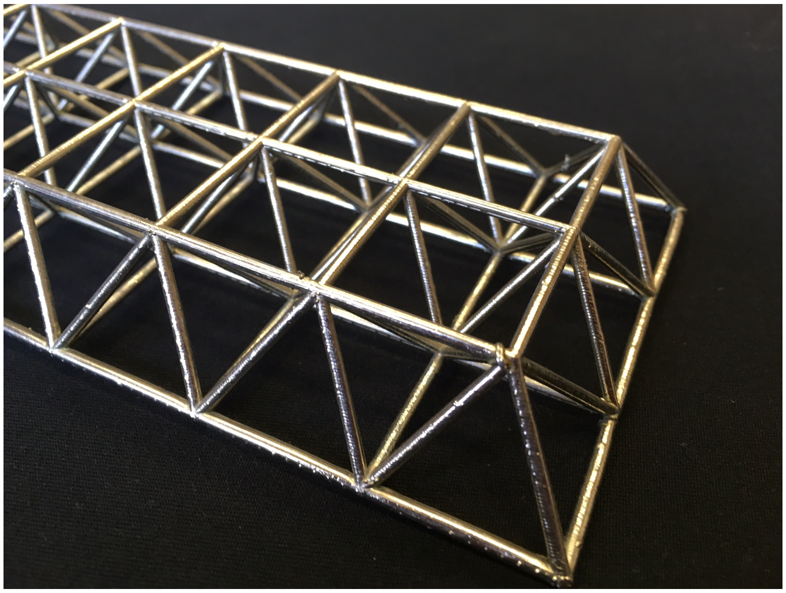
Honeycomb Core Sandwiches



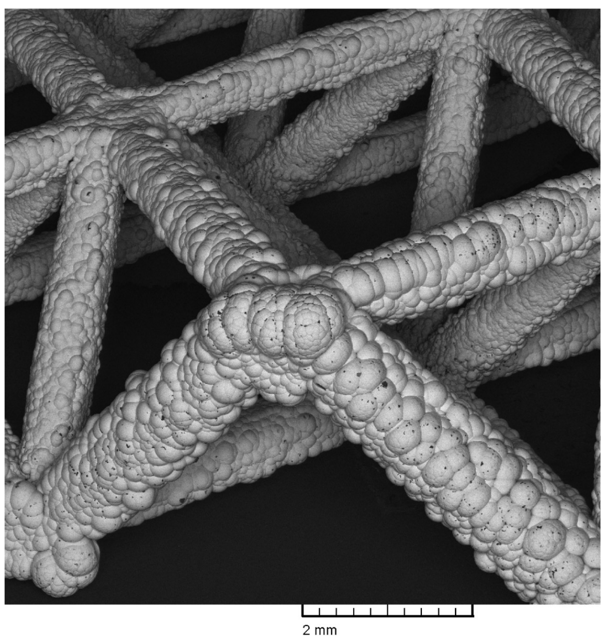
Truss Core Sandwiches



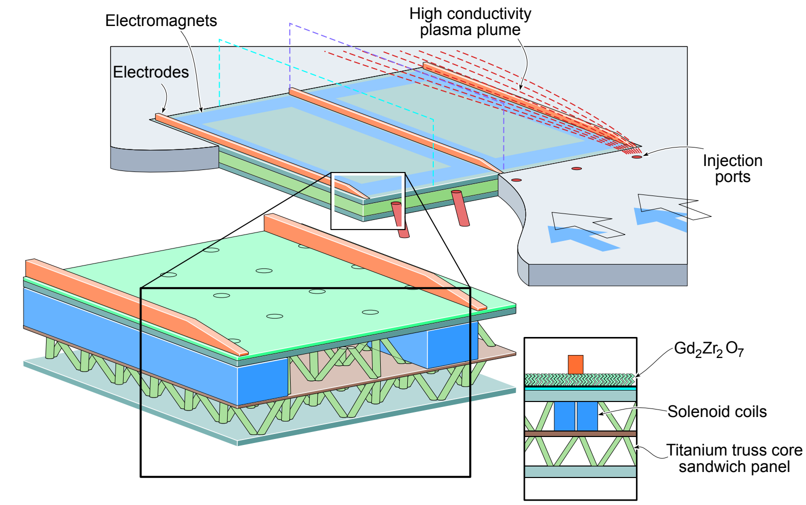
Hybrid Nanocrystalline Sandwiches



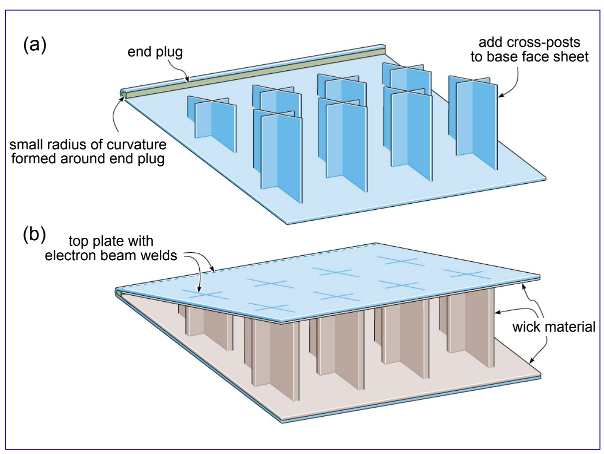
Hybrid Nanocrystalline Sandwiches



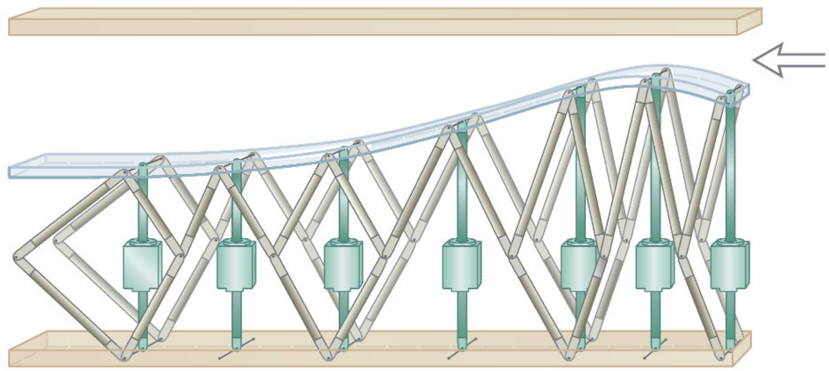
Multifunctionality - MHD Reentry Vehicle



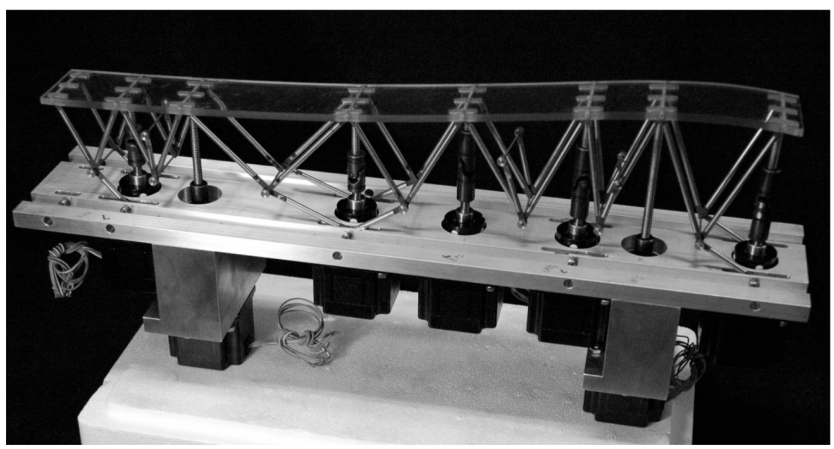
Multifunctionality - Leading Edge Heat Pipe



Multifunctionality - Morphing Wind Tunnel

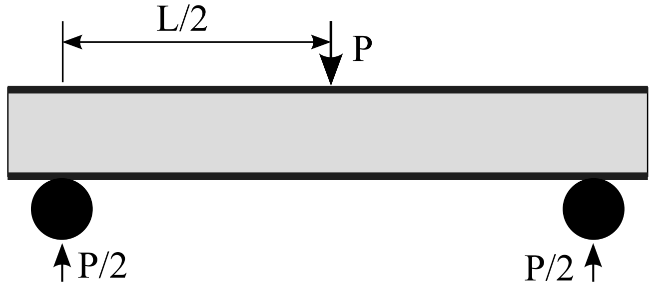


Multifunctionality - Morphing Wind Tunnel



Classical Sandwich Analysis

the prototypical sandwich loading configuration is three point bending:



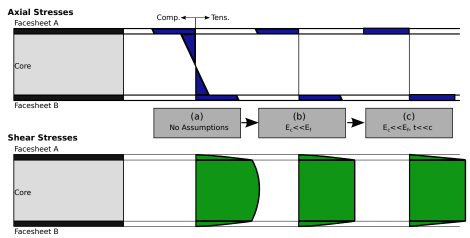
at all locations on the beam there is a bending moment and a transverse shear force

Classical Sandwich Analysis

classical sandwich analysis makes two crucial assumptions:

1. the face sheets carry all of the axial load associated with the bending moment; in the example geometry here, the top face sheet carries compressive load while the bottom face sheet is in tension

2. the core carries only shear load



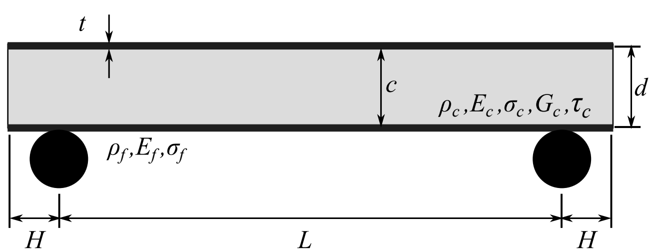
the large differences in the moduli and thicknesses of the face sheets and core enable two additional assumptions:

1. the axial stress in the face sheets is constant through the thickness

2. the shear stress in the core is constant through the thickness

Classical Sandwich Analysis

to analyse sandwich beams, use the following nomenclature:



note that the width of the sandwich beam is denoted b

Classical Sandwich Analysis - Stiffness

the total deflection δ at the mid-point of a sandwich beam loaded in three-point bending is the sum of the deflections due to bending and shear:

$$\delta = \frac{PL^3}{48(EI)_{eq}} + \frac{PL}{4(AG)_{eq}}$$

$(EI)_{eq}$ is the equivalent flexural rigidity:

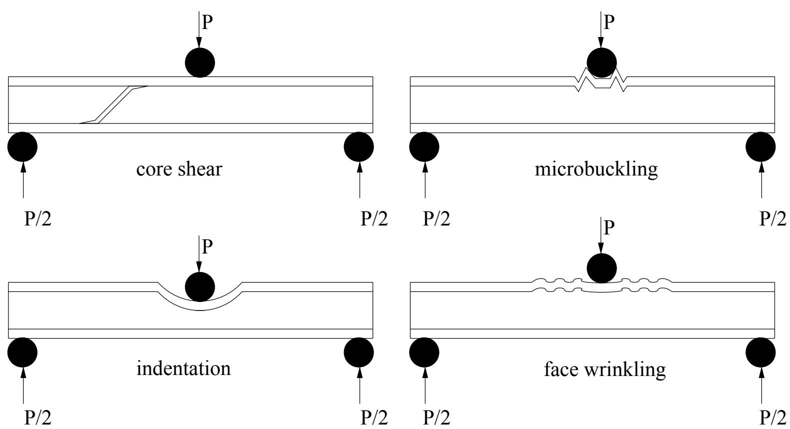
$$(EI)_{eq} = \frac{E_f btd^2}{2} + \frac{E_f bt^3}{6} + \frac{E_c bc^3}{12} \approx \frac{E_f btd^2}{2}$$

and $(AG)_{eq}$ is the equivalent shear rigidity:

$$(AG)_{eq} = \frac{bd^2 G_c}{c} \approx bdG_c$$

generally, the shear compliance of a sandwich beam is neglected because the slenderness of the beam makes it very small compared to the bending compliance

Classical Sandwich Analysis - Failure Mechanisms



Classical Sandwich Analysis - Strength

when analysing sandwich beam strength, the key thing to recognize is that, because they are complex structures, they are subject to complex failure mechanisms

for every combination of materials and geometry, the maximum load that a sandwich beam can carry (in, for example, three point bending) can be calculated for each mechanism

the mechanism that can carry the *least* load is the active failure mechanism for that geometry

the design space can be divided into regions associated with each mechanism of failure

Classical Sandwich Analysis - Failure Mechanisms

there are well-developed analytical models for each of these mechanisms of failure (and several others as well):

microbuckling:

$$P = \frac{4bdt\sigma_f}{L}$$

core shear:

$$P = 2bd\tau_c$$

wrinkling:

$$P = \frac{2btd}{L}\sqrt[3]{E_f E_c G_c}$$

indentation:

$$P = bt\left(\frac{\pi^2 d E_f \sigma_c^2}{3L}\right)^{\frac{1}{3}} \qquad P = 2bt\left(\sigma_c \sigma_f\right)^{\frac{1}{2}}$$

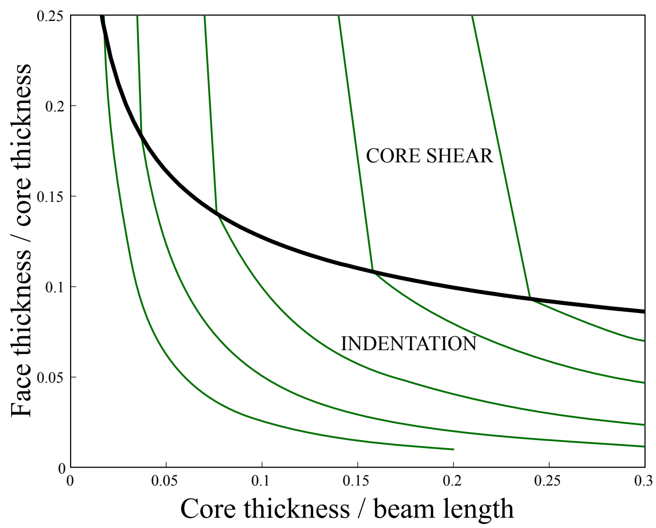
Classical Sandwich Analysis - Failure Mechanism Maps

once the material pair for a sandwich structure has been selected and the length L and width b have been determined by the functional requirements, the only remaining variables are the thickness of the core c and the thickness of the face t

these two variables define a design space

as mentioned earlier, the design space can be divided into regions in which one mechanism of failure is dominant

Classical Sandwich Analysis - Failure Mechanism Maps



Classical Sandwich Analysis - Optimisation

the strength of any sandwich beam loaded in three point bending can be calculated, given the geometry and several material properties; this means it is possible to design beams that fulfill some functional requirement

a better question is: What is the lightest beam that fulfills the requirements?

calculate the mass of the sandwich beam, which consists of two face sheets and one core:

$$M = 2bLt\rho_f + bLc\rho_c$$

Classical Sandwich Analysis - Optimisation

because the mass of the sandwich beam is a function only of face thickness and core thickness once the materials have been selected, this enables the plotting of contours of mass on the failure mechanism map

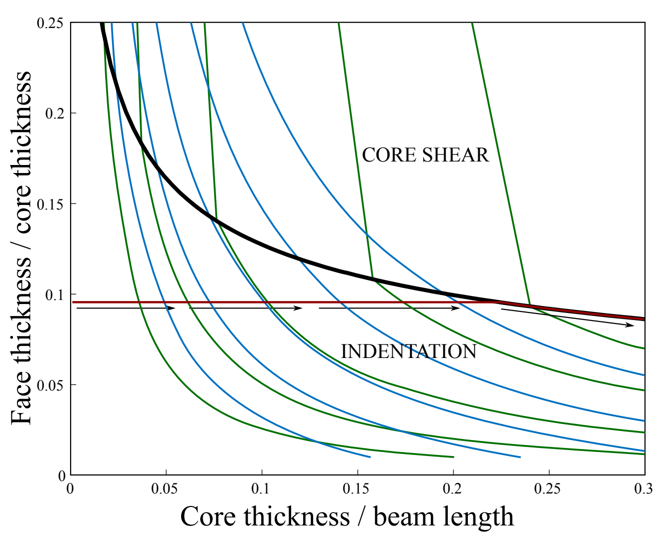
analytical techniques show that mass is minimised for a required strength when the gradient of the mass is parallel to the gradient of the strength; mathematically speaking:

$$\nabla M = \lambda \nabla P$$

where λ is a Lagrange multiplier

these locations can be found analytically or graphically on a failure mechanism map; the locus of all such points determines an optimal trajectory for minimum mass design

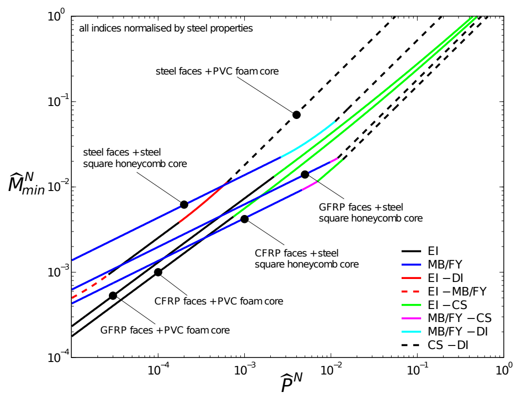
Classical Sandwich Analysis - Optimisation



Classical Sandwich Analysis - Optimisation

each trajectory in design space corresponds to a trajectory in load - mass space; hence plot the minimum mass as a function of the desired applied load

do this for a selection of different material pairs; this reveals which material choice is lightest for a given load



Classical Sandwich Analysis - Optimisation

note that finding the optimal design for minimum mass for a given strength often results in sandwich beams which are exceptionally compliant; in general it is necessary to impose a stiffness constraint as well

in addition, it would be helpful to know what is the ideal density for the core, rather than simply choosing the material *a priori*

if so, for a general sandwich beam, it is probable that the optimally designed beam will have just enough indentation strength and just enough stiffness to satisfy the requirements; the optimal values of the design variables are (in non-dimensional form):

$$\bar{\rho} = \left(\frac{\hat{P}_{req}}{2\gamma}\right)^{\frac{9}{14}} \left(\frac{3}{\tilde{\rho}}\right)^{\frac{5}{14}} \left(\frac{1}{\hat{E}I_{req}}\right)^{\frac{2}{7}}$$

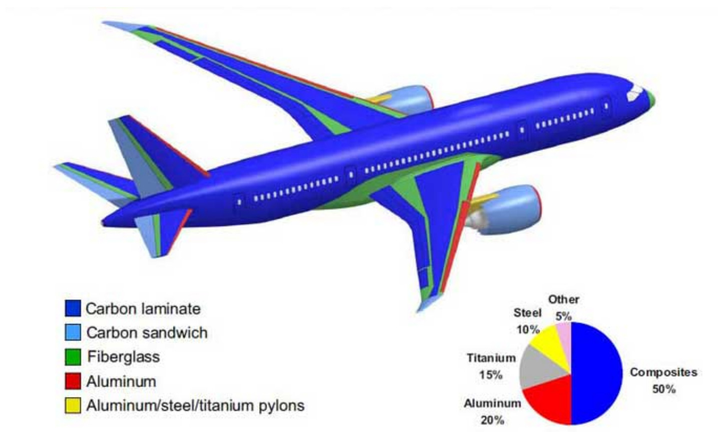
$$\bar{c} = \left(\frac{6\gamma \hat{E}I_{req}^2}{\hat{P}_{req}\tilde{\rho}}\right)^{\frac{3}{14}}$$

and

$$\bar{t} = \frac{2}{\hat{E}I_{req}^{\frac{2}{7}}} \left(\frac{\hat{P}_{req}\tilde{\rho}}{6\gamma}\right)^{\frac{9}{14}}$$

1.5 Fabrication of Composites

Composites in the Boeing 787



Breguet Range Equation

$$R = \frac{V}{C}\frac{C_L}{C_D}\ln\frac{w_i}{w_f}$$

V = cruise velocity C = specific fuel consumption of engines C_L = lift coefficient C_D = drag coefficient w_i = initial weight w_f = final weight

we are interested in how the weights change when we convert from metals to composites

Airbus A320

Aluminum	76.5%
Titanium	4.5%
Steel	13.5%
Composites	5.5%

Operating empty weight	42600 kg
Maximum zero-fuel weight	62500 kg
Maximum take-off weight	78000 kg
Maximum landing weight	64500 kg
Structural weight	22000 kg

what happens to the vehicle range when we convert all of the structural aluminum to composite?

Converting to Composites

assuming that the proper conversion from aluminum to composite represents the replacement of an aluminum tensile part with an equivalently stiff composite tensile part, the masses convert as:

$$\frac{m_c}{m_a} = \frac{\rho_c E_a}{\rho_a E_c}$$

if we replace a standard aerospace aluminum with a carbon fibre epoxy system, the A320 would be reduced in mass by over 11500 kg: this is over 50% of the total mass of the vehicle structure

consequently, the range of the A320 would increase by nearly 20% (or fuel burn would decrease commensurately): this justifies the effort needed to replace metal parts with composites

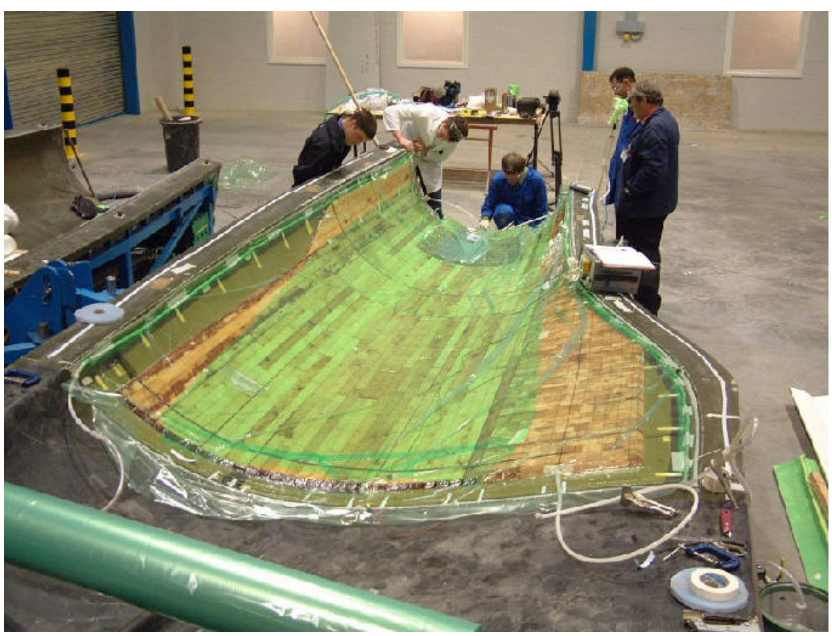
Fabrication Techniques (Polymer Matrix Composites)

OPEN MOULD TECHNIQUES:

1. hand lay-up
2. autoclaving
3. filament winding

CLOSED MOULD TECHNIQUES

1. resin transfer moulding
2. hot press moulding
3. pultrusion

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