Principal Equations of AER 1403

rule of mixtures:

$$\rho = v_f \rho_f + \left(1 - v_f\right)\rho_m$$

Reuss bound:

$$\frac{1}{E} = \frac{v_f}{E_f} + \frac{\left(1 - v_f\right)}{E_m}$$

Voigt Bound:

$$E = v_f E_f + \left(1 - v_f\right) E_m$$

compliance matrix, orthogonal laminae:

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}$$

compliance matrix, unidirectional lamina:

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}$$

where:

$$S_{11} = \frac{1}{E_1}$$

$$S_{22} = \frac{1}{E_2}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}$$

$$S_{66} = \frac{1}{G_{12}}$$

stiffness matrix, unidirectional lamina:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

where:

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{E_1}{1 - \nu_{12}\nu_{21}}$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{-S_{12}}{S_{11}S_{22} - S_{12}^2} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{66} = \frac{1}{S_{66}} = G_{12}$$

rotation of axes:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \left[\overline{Q}\right] \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

where

$$\left[\overline{Q}\right] = [T]^{-1} [Q] \left[T^T\right]^{-1}$$

and:

$$\begin{bmatrix} \cos^2\theta & \sin^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} = \left[T^T\right]^{-1}$$

the components of $\left[\overline{Q}\right]$ are given explicitly by:

$$\overline{Q}_{11} = Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta$$

$$\overline{Q}_{12} = \left(Q_{11} + Q_{22} - 4Q_{66}\right)\cos^2\theta\sin^2\theta + Q_{12}\left(\cos^4\theta + \sin^4\theta\right)$$

$$\overline{Q}_{22} = Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta$$

$$\overline{Q}_{16} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)\cos^3\theta\sin\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\cos\theta\sin^3\theta$$

$$\overline{Q}_{26} = \left(Q_{11} - Q_{12} - 2Q_{66}\right)\cos\theta\sin^3\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\cos^3\theta\sin\theta$$

$$\overline{Q}_{66} = \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)\cos^2\theta\sin^2\theta + Q_{66}\left(\cos^4\theta + \sin^4\theta\right)$$

laminate plate theory:
the total strains are:

$$\epsilon_x = \frac{\partial u}{\partial x} = \epsilon_x^\circ + z\kappa_x$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \epsilon_y^\circ + z\kappa_y$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy}^\circ + z\kappa_{xy}$$

the strains on the mid-plane are:

$$\epsilon_x^\circ = \frac{\partial u^\circ}{\partial x} \quad \epsilon_y^\circ = \frac{\partial v^\circ}{\partial y} \quad \gamma_{xy}^\circ = \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x}$$

and the curvatures of the mid-plane are:

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2} \quad \kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y}$$

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \epsilon_x^\circ \\ \epsilon_y^\circ \\ \gamma_{xy}^\circ \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

the extensional stiffnesses are:

$$A_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\overline{Q}_{ij}\right)_k \, \mathrm{d}z = \sum_{k=1}^{N} \left(\left(\overline{Q}_{ij}\right)_k (z_k - z_{k-1})\right)$$

and the coupling stiffnesses are:

$$B_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\overline{Q}_{ij}\right)_k z \, \mathrm{d}z = \frac{1}{2} \sum_{k=1}^{N} \left(\left(\overline{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right)\right)$$

and the bending stiffnesses are:

$$D_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\overline{Q}_{ij}\right)_k z^2 \, \mathrm{d}z = \frac{1}{3} \sum_{k=1}^{N} \left(\left(\overline{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right)\right)$$

Tsai-Hill:

$$\frac{\sigma_1^2}{\sigma_L^2} - \frac{\sigma_1 \sigma_2}{\sigma_L^2} + \frac{\sigma_2^2}{\sigma_T^2} + \frac{\tau_{12}^2}{\tau_{LT}^2} \geq 1$$

Tsai-Wu:

$$F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 \geq 1$$

where:

$$F_{11} = \frac{1}{\sigma_L^+ \sigma_L^-} \quad F_{22} = \frac{1}{\sigma_T^+ \sigma_T^-}$$

$$F_1 = \frac{1}{\sigma_L^+} - \frac{1}{\sigma_L^-} \quad F_2 = \frac{1}{\sigma_T^+} - \frac{1}{\sigma_T^-} \quad F_{66} = \frac{1}{\tau_{LT}^2}$$

sandwich equivalent flexural rigidity:

$$(EI)_{eq} = \frac{E_f b t d^2}{2} + \frac{E_f b t^3}{6} + \frac{E_c b c^3}{12} \approx \frac{E_f b t d^2}{2}$$

equivalent shear rigidity:

$$(AG)_{eq} = \frac{bd^2 G_c}{c} \approx bdG_c$$

microbuckling:

$$P = \frac{4bdt\sigma_f}{L}$$

core shear:

$$P = 2bd\tau_c$$

wrinkling:

$$P = \frac{2btd}{L}\sqrt[3]{E_f E_c G_c}$$

indentation:

$$P = bt\left(\frac{\pi^2 dE_f \sigma_c^2}{3L}\right)^{\frac{1}{3}} \qquad P = 2bt\left(\sigma_c \sigma_f\right)^{\frac{1}{2}}$$

Euler's equation of variational calculus:

$$F_y\left(x, y, y'\right) - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\left(x, y, y'\right) = 0$$

the first variation of the potential energy is:

$$\delta V = \int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V - \int_V X_i \delta u_i\,\mathrm{d}V - \int_S T_i \delta u_i\,\mathrm{d}S$$

strong form:

$$\frac{1}{2}\left(C_{ijkl}\left(u_{k,l} + u_{l,k}\right)\right)_{,j} + X_i = 0$$

weak form:

$$\frac{1}{2}\int_V \delta w_{i,j} C_{ijkl}\left(u_{k,l} + u_{l,k}\right)\,\mathrm{d}V = \int_V \delta w_i X_i\,\mathrm{d}V + \frac{1}{2}\int_S \left(\delta w_i C_{ijkl}\left(u_{k,l} + u_{l,k}\right)\right)\nu_j\,\mathrm{d}S$$

strong form in one dimension:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + B = 0$$

weak form in one dimension:

$$\int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int_0^L \delta w B\,\mathrm{d}x + \left(\delta w A\bar{T}\right)_{x=0}$$

approximation in one dimension:

$$u(x) \approx \begin{bmatrix} \frac{\ell - x}{\ell} & \frac{x}{\ell} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \mathbf{Nd}$$

$$u'(x) \approx \begin{bmatrix} -\frac{1}{\ell} & \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \mathbf{Hd}$$

$$\delta w \approx \begin{bmatrix} \frac{\ell-x}{\ell} & \frac{x}{\ell} \end{bmatrix} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} = \mathbf{N}\mathbf{w} \qquad \delta w' \approx \begin{bmatrix} -\frac{1}{\ell} & \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} = \mathbf{H}\mathbf{w}$$

$$\sum^{\text{elems}} \mathbf{w} \left(\int_{x_1}^{x_2} \mathbf{H} A E \mathbf{H} \, \mathrm{d}x \mathbf{d} - \int_{x_1}^{x_2} \mathbf{N} B \, \mathrm{d}x - \left(\mathbf{N} A \bar{T} \right)_{x=0} \right) = 0$$

plane stress:

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

plane strain:

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

triangular element shape functions:

$$N_a = \frac{1}{2A} \left((x_{1b} x_{2c} - x_{1c} x_{2b}) + (x_{2b} - x_{2c}) x_1 + (x_{1c} - x_{1b}) x_2 \right)$$

$$N_b = \frac{1}{2A} \left((x_{1c} x_{2a} - x_{1a} x_{2c}) + (x_{2c} - x_{2a}) x_1 + (x_{1a} - x_{1c}) x_2 \right)$$

$$N_c = \frac{1}{2A} \left((x_{1a} x_{2b} - x_{1b} x_{2a}) + (x_{2a} - x_{2b}) x_1 + (x_{1b} - x_{1a}) x_2 \right)$$

and derivative:

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial N_a}{\partial x_1} & 0 & \dfrac{\partial N_b}{\partial x_1} & 0 & \dfrac{\partial N_c}{\partial x_1} & 0 \\ 0 & \dfrac{\partial N_a}{\partial x_2} & 0 & \dfrac{\partial N_b}{\partial x_2} & 0 & \dfrac{\partial N_c}{\partial x_2} \\ \dfrac{\partial N_a}{\partial x_2} & \dfrac{\partial N_a}{\partial x_1} & \dfrac{\partial N_b}{\partial x_2} & \dfrac{\partial N_b}{\partial x_1} & \dfrac{\partial N_c}{\partial x_2} & \dfrac{\partial N_c}{\partial x_1} \end{bmatrix}$$

or, with only the non-zero, non-repeating terms:

$$\mathbf{H} = \frac{1}{2A} \begin{bmatrix} (x_{2b} - x_{2c}) & (x_{2c} - x_{2a}) & (x_{2a} - x_{2b}) \\ (x_{1c} - x_{1b}) & (x_{1a} - x_{1c}) & (x_{1b} - x_{1a}) \end{bmatrix}$$

rectangular element shape functions:

$$N_a = \frac{x_1 - x_{1b}}{x_{1a} - x_{1b}} \frac{x_2 - x_{2d}}{x_{2a} - x_{2d}} = \frac{1}{A} (x_1 - x_{1b})(x_2 - x_{2d})$$

$$N_b = \frac{x_1 - x_{1a}}{x_{1b} - x_{1a}} \frac{x_2 - x_{2d}}{x_{2a} - x_{2d}} = -\frac{1}{A} (x_1 - x_{1a})(x_2 - x_{2d})$$

$$N_c = \frac{x_1 - x_{1a}}{x_{1b} - x_{1a}} \frac{x_2 - x_{2a}}{x_{2d} - x_{2a}} = \frac{1}{A}(x_1 - x_{1a})(x_2 - x_{2a})$$

$$N_d = \frac{x_1 - x_{1b}}{x_{1a} - x_{1b}} \frac{x_2 - x_{2a}}{x_{2d} - x_{2a}} = -\frac{1}{A}(x_1 - x_{1b})(x_2 - x_{2a})$$

isoparametric shape functions:

$$N_a^{4Q}(\xi_1, \xi_2) = \frac{1}{4}(1 + \xi_{1a}\xi_1)(1 + \xi_{2a}\xi_2)$$

$$N_b^{4Q}(\xi_1, \xi_2) = \frac{1}{4}(1 + \xi_{1b}\xi_1)(1 + \xi_{2b}\xi_2)$$

$$N_c^{4Q}(\xi_1, \xi_2) = \frac{1}{4}(1 + \xi_{1c}\xi_1)(1 + \xi_{2c}\xi_2)$$

$$N_d^{4Q}(\xi_1, \xi_2) = \frac{1}{4}(1 + \xi_{1d}\xi_1)(1 + \xi_{2d}\xi_2)$$

strains:

$$\nabla \mathbf{u} \approx \mathbf{Hd}$$

where:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial N_a^{4Q}}{\partial x_1} & \frac{\partial N_b^{4Q}}{\partial x_1} & \frac{\partial N_c^{4Q}}{\partial x_1} & \frac{\partial N_d^{4Q}}{\partial x_1} \\ \frac{\partial N_a^{4Q}}{\partial x_2} & \frac{\partial N_b^{4Q}}{\partial x_2} & \frac{\partial N_c^{4Q}}{\partial x_2} & \frac{\partial N_d^{4Q}}{\partial x_2} \end{bmatrix}$$

Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_a^{4Q}}{\partial \xi_1} & \frac{\partial N_b^{4Q}}{\partial \xi_1} & \frac{\partial N_c^{4Q}}{\partial \xi_1} & \frac{\partial N_d^{4Q}}{\partial \xi_1} \\ \frac{\partial N_a^{4Q}}{\partial \xi_2} & \frac{\partial N_b^{4Q}}{\partial \xi_2} & \frac{\partial N_c^{4Q}}{\partial \xi_2} & \frac{\partial N_d^{4Q}}{\partial \xi_2} \end{bmatrix} \begin{bmatrix} x_{1a} & x_{2a} \\ x_{1b} & x_{2b} \\ x_{1c} & x_{2c} \\ x_{1d} & x_{2d} \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \frac{\partial}{\partial \xi_1} \\ \frac{\partial}{\partial \xi_2} \end{bmatrix}$$

$$\mathbf{H} = \mathbf{J}^{-1}\mathbf{G}\mathbf{N}^{4Q}$$

Gauss quadrature in two dimensions:

$$I = \sum_{j=1}^{n_{gp}} \left(\sum_{i=1}^{n_{gp}} W_j W_i \left| \mathbf{J}(\xi_i, \xi_j) \right| f(\xi_i, \xi_j) \right)$$

global system:

$$\mathbf{K}^g \mathbf{d}^g = \mathbf{f}^g + \mathbf{r}^g$$

$$\mathbf{K}^g = \sum_{elements} \mathbf{L}^T \mathbf{K} \mathbf{L}$$