10.3 Figure P10.3 shows a general closed-loop control system. The plant transfer function is

$$G_P(s) = \frac{1}{s^2 + 6s + 8}$$

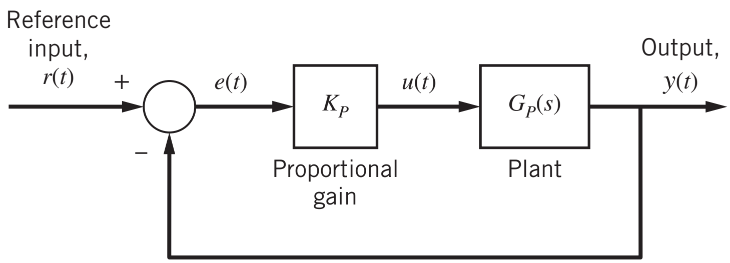

Figure P10.3

a. Determine whether the closed-loop system is stable for control gain $K_P = 2$.
b. Compute the controller gain K_P so that step response shows 25% overshoot.
c. Estimate the settling time for a step reference input if the control gain is $K_P = 0.5$.

10.3 The closed-loop transfer function (CLTF) is

$$T(s) = \frac{K_P G_P(s)}{1 + K_P G_P(s) H(s)} = \frac{K_P \dfrac{1}{s^2 + 6s + 8}}{1 + K_P \dfrac{1}{s^2 + 6s + 8}} = \frac{K_P}{s^2 + 6s + 8 + K_P}$$

a) When gain $K_P = 2$, the denominator of the closed-loop transfer function is

Characteristic equation: $s^2 + 6s + 10 = 0$ Closed-loop roots: $s = -3 \pm j$

Since the closed-loop roots have negative real parts the closed-loop system is **stable** for $K_P = 2$.

b) The CLTF is second order. Maximum overshoot is solely a function of damping ratio ζ

$$M_{OS} = e^{-\zeta\pi / \sqrt{1-\zeta^2}} = 0.25$$

Hence, for $M_{OS} = 0.25$ we require $\zeta = 0.4037$. The first-order denominator term of the CLTF is

$$6 = 2\zeta\omega_n = 2(0.4037)\omega_n$$

where the undamped natural frequency is $\omega_n = \sqrt{8 + K_P} = 7.4313$ rad/s. Therefore $\boldsymbol{K_P = 47.224}$**.**

c) When gain $K_P = 0.5$, the denominator of the closed-loop transfer function is $s^2 + 6s + 8.5$. The closed-loop roots are $s_1 = -3.7071$ and $s_2 = -2.2929$. The "slowest" closed-loop root is s_2 and its associated time constant is $\tau_2 = 1/2.2929 = 0.4361$ s. The settling time is $t_S = 4\tau_2$ **= 1.7445 s.**

10.4 Figure P10.4 shows a closed-loop control system.

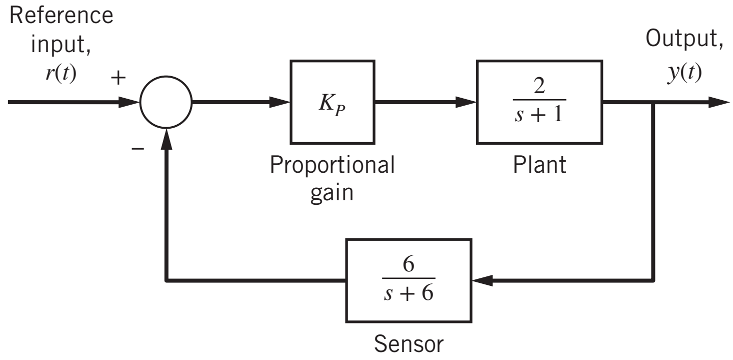

Figure P10.4

a. Compute the controller gain K_P so that the undamped natural frequency of the closed-loop system is $\omega_n = 4$ rad/s.

b. Compute the controller gain K_P so that the damping ratio of the closed-loop system is $\zeta = 0.7$.

c. Compute the steady-state output for a step reference input $r(t) = 4U(t)$ and controller gain $K_P = 2$.

10.4 The closed-loop transfer function (CLTF) is

$$T(s) = \frac{K_P G_P(s)}{1 + K_P G_P(s) H(s)} = \frac{K_P \dfrac{2}{s+1}}{1 + K_P \dfrac{2}{s+1}\dfrac{6}{s+6}} = \frac{2K_P(s+6)}{s^2 + 7s + 6 + 12K_P}$$

a) The (closed-loop) undamped natural frequency is obtained from the zeroth-order term, or $\omega_n = \sqrt{6 + 12K_P} = 4$ rad/s. Hence the gain setting must be ***K*$_P$ = 0.8333.**

b) The (closed-loop) damping ratio is obtained from the first-order term $7 = 2\zeta\omega_n$, where the undamped natural frequency is $\omega_n = \sqrt{6 + 12K_P}$. Setting $\zeta = 0.7$ and solving for the gain yields the solution ***K*$_P$ = 1.5833.**

c) With gain setting $K_P = 2$, the CLTF is

$$T(s) = \frac{4(s+6)}{s^2 + 7s + 30}$$

Because the input is a constant (magnitude of 4) we can use the CLTF DC gain: $T(s = 0) = 24/30 = 0.8$ and hence the steady-state output is **y_{ss} = (4)(0.8) = 3.2.**

10.5 A closed-loop control system is shown in Fig. P10.5.

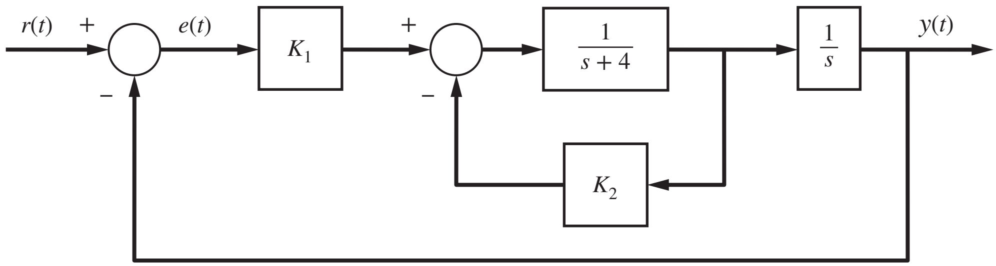

Figure P10.5

a. Compute the closed-loop transfer function for the overall system, $T(s) = Y(s)/R(s)$.

b. Two gain pairs are considered: Option 1 ($K_1 = 15$, $K_2 = 2$) and Option 2 ($K_1 = 30$, $K_2 = 3$). Which gain pair provides the greatest closed-loop damping ratio? Justify your answer.

10.5 a) The complete closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_1 G(s)\frac{1}{s}}{1 + K_1 G(s)\frac{1}{s}}$$

where $G(s)$ is the closed-loop transfer function of the "inner" closed-loop feedback system:

$$G(s) = \frac{\frac{1}{s+4}}{1+\frac{K_2}{s+4}} = \frac{1}{s+4+K_2}$$

Substituting this result for $G(s)$ yields the overall CLTF:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K_1 \frac{1}{s+4+K_2}\frac{1}{s}}{1+K_1\frac{1}{s+4+K_2}\frac{1}{s}} = \boxed{\frac{K_1}{s^2+(4+K_2)s+K_1}}$$

b) The (closed-loop) damping ratio is obtained from the first-order term $4 + K_2 = 2\zeta\omega_n$, where the undamped natural frequency is $\omega_n = \sqrt{K_1}$. Substituting the Option 1 gains ($K_1 = 15$, $K_2 = 2$) yields $\omega_n = 3.873$ rad/s and damping ratio $\zeta = 0.7746$. Substituting the Option 2 gains ($K_1 = 30$, $K_2 = 3$) yields $\omega_n = 5.4772$ rad/s and damping ratio $\zeta = 0.6390$. Therefore **Option 1** gains provide the largest closed-loop damping ratio.

10.6 Figure P10.6 shows a unity-feedback closed-loop system. The reference input is a ramp, $r(t) = 0.2t$.

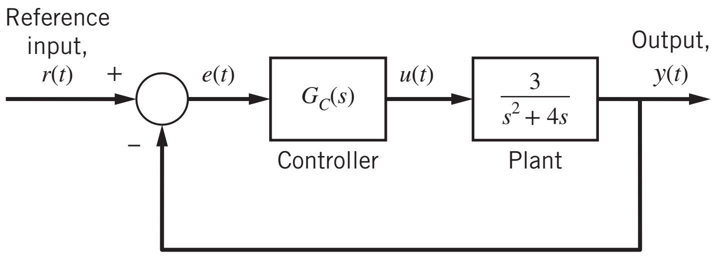

Figure P10.6

a. Compute the steady-state tracking error if the controller $G_C(s)$ is a simple proportional gain $K_P = 2$.
b. Compute the steady-state tracking error if we use a PI controller with gains $K_P = 3$ and $K_I = 1.5$.

10.6 a) We may use Table 10-3 to compute the steady-state tracking error. Note that the plant is a type 1 system (it has one "pure integrator"). For a ramp input (with slope 0.2) the steady-state error for a type-1 system is

$$\text{Type-1 system:} \quad e_{ss} = \frac{0.2}{K_{sv}}$$

where the "static velocity error constant" is $K_{sv} = \lim_{s \to 0} sG(s)$. The forward transfer function is

$$G(s) = \frac{3K_P}{s(s+4)}$$

When the control gain is $K_P = 2$, we get $K_{sv} = 6/4 = 1.5$, and the steady-state error $\boldsymbol{e_{ss}} = \mathbf{0.1333.}$

b) If we use a PI controller the forward transfer function becomes

$$G(s) = G_C(s)G_P(s) = \frac{K_P s + K_I}{s} \frac{3}{s(s+4)} = \frac{3(K_P s + K_I)}{s^2(s+4)}$$

Because the forward transfer function is type 2 the steady-state error for a ramp input is **zero**.

10.7 A simple closed-loop PI control system is shown in Fig. P10.7.

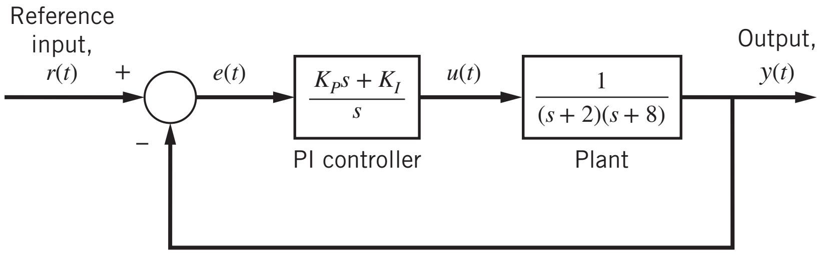

Figure P10.7

a. Show that the closed-loop system is stable for PI gains $K_P = 5$ and $K_I = 25$.

b. Determine "by hand" the closed-loop output $y(t)$ at time $t = 8$ s if the reference input is a ramp function $r(t) = 1.4t$. Use the PI gains from part (a).

10.7 The closed-loop transfer function (CLTF) is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\frac{K_P s + K_I}{s} \frac{1}{(s+2)(s+8)}}{1 + \frac{K_P s + K_I}{s} \frac{1}{(s+2)(s+8)}} = \frac{K_P s + K_I}{s^3 + 10s^2 + (16 + K_P)s + K_I}$$

a) Using gains $K_P = 5$ and $K_I = 25$ the CLTF becomes

$$T(s) = \frac{5s + 25}{s^3 + 10s^2 + 21s + 25}$$

The three roots of the closed-loop transfer function are $s_1 = -7.6926$ and $s_{2,3} = -1.1537 \pm j1.3852$. Since all closed-loop roots have <u>negative</u> real parts the closed-loop system is **stable** for this gain setting.

b) We want the output at time $t = 8$ s for a ramp input $r(t) = 1.4t$. The complex closed-loop root at $s_{2,3} = -1.1537 \pm j1.3852$ is the "slowest" root and its settling time is approximately 3.5 s. Hence, the closed-loop transient response has died out by $t = 8$ s. Using Table 10-1 we see that the for a type 1 system the steady-state error for a ramp input (with slope 1.4) is

$$\text{Type-1 system:} \quad e_{ss} = \frac{1.4}{K_{sv}}$$

The system is clearly type 1 due to the PI controller in the forward path. The "static velocity error constant" is $K_{sv} = \lim_{s \to 0} sG(s)$. The forward transfer function is

$$G(s) = G_C(s)G_P(s) = \frac{K_P s + K_I}{s} \frac{1}{(s+2)(s+8)} = \frac{5s + 25}{s(s+2)(s+8)} \quad \text{(using the PI gains)}$$

Hence $K_{sv} = 25/16 = 1.5625$, and the steady-state error for a ramp input is $e_{ss} = 0.8960$. The output at time $t = 8$ s can be computed using the tracking error since $e = r - y$. Therefore

$$y(8) = r(8) - e_{ss} = (1.4)(8) - 0.8960 = \mathbf{10.3040}$$