Mathematics I

Full Marks: 80 + 20
Pass Marks: 32 + 8
Credit Hrs: 3 *Semester:* I

Course Title: Mathematics I
Course No: MTH112
Nature of the Course: Theory

Course Description:
The course covers the concepts of functions, limits, continuity, differentiation, integration of function of one variable; logarithmic, exponential, applications of derivative and antiderivatives, differential equations, vectors and applications, partial derivatives and Multiple Integrals.

Course Objectives:
The objective of this course is to make students able to
- understand and formulate real world problems into mathematical statements.
- develop solutions to mathematical problems at the level appropriate to the course.
- describe or demonstrate mathematical solutions either numerically or graphically.

Course Contents:

Unit 1: Function of One Variable **5 hrs.**
Four ways of representing a function, Linear mathematical model, Polynomial, Rational, Trigonometric, Exponential and Logarithmic functions, Combination of functions, Range and domain of functions and their Graphs

Unit 2: Limits and Continuity **4 hrs.**
Precise definition of Limit, Limits at infinity, Continuity, Horizontal asymptotes, Vertical and Slant asymptotes

Unit 3: Derivatives **4 hrs.**
Tangents and velocity, Rate of change, Review of derivative, Differentiability of a function, Mean value theorem, Indeterminate forms and L'Hospital rule

Unit 4: Applications of Derivatives **4 hrs.**
Curve sketching, Review of maxima and minima of one variable, Optimization problems, Newton's method

Unit 5: Antiderivatives **5 hrs.**
Review of antiderivatives, Rectilinear motion, Indefinite integrals and Net change, Definite integral, The Fundamental theorem of calculus, Improper integrals

Unit 6: Applications of Antiderivatives **5 hrs.**
Areas between the curves, Volumes of cylindrical cells, Approximate Integrations, Arc length, Area of surface of revolution

Unit 7: Ordinary Differential Equations **6 hrs.**
Introduction, Introduction to first order equations Separable equations, Linear equations, Second order linear differential equations, Non homogeneous linear equations, Method of undetermined coefficients

Unit 8: Infinite Sequence and Series **5 hrs.**

Infinite sequence and series, Convergence tests and power series, Taylor's and Maclaurin's series

Unit 9: Plane and Space Vectors **4 hrs.**

Introduction, Applications, Dot product and cross Product, Equations of lines and Planes, Derivative and integrals of vector functions, Arc length and curvature, Normal and binormal vectors, Motion in space

Unit 10: Partial Derivatives and Multiple Integrals **3 hrs.**

Limit and continuity, Partial derivatives, Tangent planes, Maximum and minimum values, Multiple integrals

Text Book:

1. Calculus Early Transcendentals, James Stewart, 7E, CENGAGE Learning.

Reference Book:

1. Calculus Early Transcendentals, Thomas, 12th Editions, Addision Wesley.

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Chapter 1

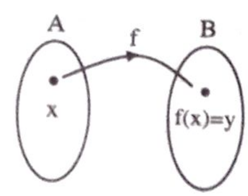
Function of One Variable

The goal of study on calculus is opened after understanding the functions. In this unit functions can be represented by different ways. Here, we also describe the process how the functions act as mathematical model in real world phenomena. Beside this, we explain how the functions are combined and transformed with its graphs.

1.1 Functions and Its Representation

Let A and B be the non-empty sets. Then a rule from A to B is called function if every element of domain A has unique association with elements of codomain B.

Note 1: Let $f: A \to B$ be a function.



(i) Set A is called domain.

(ii) Set B is called co-domain

(iii) y is called image of x under f i.e. $y = f(x)$

(iv) x is called pre-image of y

(v) Range of f is denoted by $f(A)$ and defined by $f(A) = \{f(x) : \forall x \in A\}$.

Note 2: The **domain of function** is set of all possible input values x, which plug in the function formula to produce the output values y (OR $f(x)$). While finding the **domain** note that:

- The denominator of fraction can not be zero.
- The sign of variables under the even roots (square root, fourth root and so on) must be positive.

The **range of function** is set of all possible output values y (or $f(x)$) which result from plugging all the input values x in the function formula.

While finding the **range**:

Transfer the functional formula $y = f(x)$ in to $x = g(y)$ and do as above for domain (here for y).

Example 1: Identify the domain and range of function

(i) $y = x^2$ (ii) $y = \frac{1}{x}$ (iii) $y = \sqrt{4-x}$ (iv) $y = \sqrt{9-x^2}$

Solution:

(i) Given $y = x^2$.

For domain, for all real value of x, y is exist. So, domain is set of all real number i.e domain is $(-\infty, \infty)$.

For range, $y = x^2$ or $x = \sqrt{y}$. For all $y \geq 0$, x is defined. Thus, y is set of all non–negative real number. Hence, range is $y \geq 0$ or $[0, \infty)$.

(ii) Given $y = \frac{1}{x}$.

For domain; when $x = 0$, y doesn't exist, it means y is exist for all real number except 0. The domain is set of all real number except 0 i.e. $(-\infty, 0) \cup (0, \infty)$.

For range, $y = \frac{1}{x}$ or $x = \frac{1}{y}$. Here, x is exist for all real number except $y = 0$. Thus, the range is set of all real number except 0 i.e. $(-\infty, 0) \cup (0, \infty)$.

(iii) Given, $y = \sqrt{4-x}$.

For domain, y is exist only when $4 - x \geq 0$ i.e. $x - 4 \leq 0$ i.e. $x \leq 4$.

Thus, domain is $x \leq 4$ i.e. $(-\infty, 4]$.

For range,
$$y = \sqrt{4-x}$$
Or,
$$x = 4 - y^2$$

Here, x is exist for all real value of y. But given $y = \sqrt{4-x}$ which is non–negative (because for $x \leq 4$, y is non–negative)

Thus, range is set of all non–negative real number i.e. $[0, \infty)$.

(iv) Given, $y = \sqrt{9-x^2}$.

For domain; y is exist for $9 - x^2 \geq 0$.

i.e. $x^2 - 9 \leq 0$.

i.e. $(x-3)(x+3) \leq 0$.

i.e. $-3 \leq x \leq 3$.

Hence, domain is $-3 \leq x \leq 3$ i.e. $[-3, 3]$.

For range,
$$y = \sqrt{9-x^2}.$$
$$\Rightarrow y^2 = 9 - x^2.$$
$$\Rightarrow x = \sqrt{9-y^2}.$$

Here, x is exist for $9 - y^2 \geq 0$.

i.e. $y^2 - 9 \leq 0$

i.e. $-3 \leq y \leq 3$.

But given $y = \sqrt{9 - x^2}$, which is non-negative (because for $-3 \le x \le 3$, y is non-negative) hence range is $0 \le y \le 3$ i.e. $[0, 3]$.

Functions are represented by four ways which are **verbally** (it means by a description in words), **numerically** (it means by table of values), **visually** (it means by a graph) and **algebraically** (it means by an explicit formula).

The following example of a function which is verbal description and we obtain an algebraic formula.

Example 2: A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs $10 per square meter, material for the sides costs $6 per square meter. Express the cost of material as a function of width of the base.

Solution: Let w be the width of the base of container, so its length is 200. Suppose h be its height.

So area of base = $2w.w = 2w^2$

$\therefore$ Cost for base = $10 \times 2w^2 = 20w^2$

Total area of 4 sides = $2 \times wh + 2 \times 2wh = 6wh$

$\therefore$ Cost for sides = $6 \times 6wh = 36wh$

$\therefore$ Total cost (C) = $20w^2 + 36wh$ ....(1)

Since, Volume (V) = $w.2w.h = 2w^2h$

But V = 10 m^3

Hence, $10 = 2w^2h$

$$\therefore \quad h = \frac{5}{w^2}$$

From (1), we get

$$C = 20w^2 + 36wh$$

$$= 20\,w^2 + 36w \times \frac{5}{w^2}$$

$$\therefore \quad C = 20w^2 + \frac{180}{w}$$

Vertical Line Test for Function

Not every curve in xy plane can be the graph of a function, so the vertical line test test that which curves in xy plane are graph of functions. This test state that a curve in the xy plane is the graph of a function of x if and only if no vertical line intersects the more than onces.

Note: In general, the equation where power of y is even then such curve is not function of x.

Example 3: Identify graph of $x^2 + y^2 = 4$ is function or not.

Given equation

$$x^2 + y^2 = 4$$

Put $x = 0$ then

$$y = \pm 2$$

Here vertical line $x = 0$ meet curve at two points $y = \pm 2$.

Thus $x^2 + y^2 = 4$ is not function.

Example 4: **Identify the graph of function $y = x^3 + 2$ is function or not.**

Given equation $y = x^3 + 2$

Here, for any vertical line, say $x = 1$.

$y = 1^3 + 2 = 3$, curve meet single point $y = 3$

Thus, $y = x^3 + 2$ is function.

Even and Odd function: Symmetry

A function $y = f(x)$ is an even function if $f(-x) = f(x)$, and odd function if $f(-x) = -f(x)$ for every value of x.

Note that the graph of even function is symmetrical about y-axis and the graph of odd function is symmetrical about origin.

Example 5: Identify following functions are even or odd or neither.

(i) $f(x) = x^{-4}$ (ii) $f(x) = x^3$ (iii) $f(x) = |x| + 5$ (iv) $f(x) = x + 1$

Solution:

(i) Since, $f(x) = x^{-4} = \frac{1}{x^4}$.

So, $f(-x) = \frac{1}{(-x)^4} = \frac{1}{x^4} = f(x)$.

Therefore, the given $f(x) = x^{-4}$ is even function.

(ii) Since, $f(x) = x^3$.

So, $f(-x) = (-x)^3 = -x^3 = -f(x)$

Therefore, the given function $f(x) = x^3$ is odd function.

(iii) Since, $f(x) = |x| + 5$.

So, $f(-x) = |-x| + 5 = |1|\ |x| + 5 = |x| + 5 = f(x)$.

Therefore, the given $f(x) = |x| + 5$ is even function.

(iv) Since, $f(x) = x + 1$.

So, $f(-x) = -x + 1 = -(x - 1) \neq f(x)$.

and $f(-x) = -x + 1 = -(x - 1) \neq -f(x)$.

Therefore, the given function is neither even nor odd.

Example 6: If f and g are even function, prove that f + g also even function.

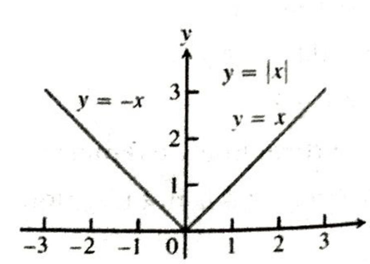
Solution: Let f and g are even so $f(-x) = f(x)$ and $g(-x) = g(x)$. Suppose $f + g = h$,

So, $h(-x) = (f + h)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)x = h(x)$. So h is even function.

Therefore, f + g is a even function.

Piecewise Defined Function

A piecewise function is a function whose formula changes depending on different part of its domain. A simple example is absolute value function, which is



$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Example 7: A cell phone plan has a basic charge of \$35 a month. The plan includes 400 free minutes and charge 10 cents for each additional minute of uses. Write the monthly cost C as function of number x of minutes used. Also find the monthly cost for call when 200 minutes used. What cost for 500 minutes.

Let c be the monthly cost and x be the minutes used. Then

$C = 35$ for $0 \leq x \leq 400 = 35 + 0.1(x - 400)$ for $x > 400$

For 200 minutes i.e. $x = 200$

$c = 35\,\$$

For 500 minutes, i.e. $x = 500$

$c = 35 + 0.1(500 - 400) = 35 + 10 = 45\$$

Exercise 1.1

1. Evaluate the difference quotient for given function

   (i) $f(x) = 4 - 3x; \quad \dfrac{f(3+h) - f(3)}{h}$ (ii) $f(x) = \dfrac{x+3}{x+1}; \quad \dfrac{f(x) - f(1)}{x - 1}$

2. Find the domain of the function

   (i) $f(x) = \dfrac{x+4}{x^2 - 9}$ (ii) $f(x) = \dfrac{2x^3 - 5}{x^2 + x - 6}$

   (iii) $f(t) = \sqrt[3]{2t - 1}$ (iv) $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$

   (v) $f(p) = \sqrt{2 - \sqrt{p}}$ (vi) $G(x) = \dfrac{3x + |x|}{x}$

3. Find domain and range of the function

   (i) $h(x) = \sqrt{4 - x^2}$ (ii) $f(x) = \sqrt{x - 5}$

   (iii) $g(x) = \dfrac{2x + 1}{x - 3}$

4. Identify which one is graph of functions

   (i) $y = x + 2$ (ii) $x = y^2$

   (iii) $y = x^2$ (iv) $y = -\sqrt{x + 2}$

   (v) $x^2 + y = 5$ (vi) $x = y^2 - 2$

5. Determine whether following functions even or odd or neither

   (i) $f(x) = \dfrac{x^2}{x^4 + 1}$ (ii) $g(x) = x|x|$

   (iii) $h(x) = 1 + x^3 - x^5$ (iv) $f(x) = 2|x| + 1$

   (v) $g(x) = 3$

6. If f and g are both even function then prove that f.g is also even function.

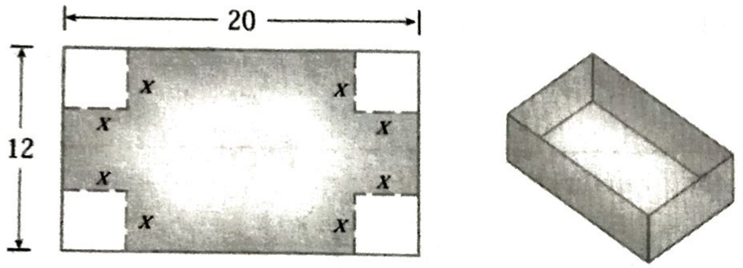
7. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one its side.

8. A rectangle has are 16 m^2. Express the perimeter of the rectangle as a function of the length of one of its sides.

9. An open rectangular box with volume $2m^3$ has a square base. Express the surface area of the box a function of the length of a side of the base.

10. Norman window has the shape of a rectangular surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as function of the width x of window.

11. A box with an open top is to be constructed from a rectangular piece of cardboard with dimension 12 m, by 20 m, by cutting out equal squares of side x at each corner and then folding up the side as in the figure. Express the volume V of the box as function of x.



12. In a certain state the maximum speed permitted on freeways is 65 ml/h and the minimum speed is 40 ml/h. The fine for violating these limits is $15 for every mile per hour above the maximum speed or below the minimum speed. Express the amount of the fine F as function of driving speed x.

13. An electricity company charges its customers a base rate of $10 a month plus 6 cents per kelowatt hour (kwh) for the first 1200 kwh and 7 cents per kwh for all usage over 1200 kwh. Express the monthly cost E as function of the amount x of electricity used.

Answer:

1. (i) - 3 (ii) $\dfrac{-1}{x+1}$
2. (i) $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ (ii) $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$ (iii) $(-\infty, \infty)$ (iv) $[-2, 3]$ (v) $[0, 4]$ (vi) $(-\infty, 0) \cup (0, \infty)$
3. (i) Domain $[-2, 2]$, Range $[0, 2]$
   (ii) Domain $[5, \infty)$, Range $[0, \infty)$
   (iii) Domain $(-\infty, 3) \cup (3, \infty)$, Range $(-\infty, 2) \cup (2, \infty)$
4. (i) Function (ii) not function (iii) Function (iv) Function (v) Function (vi) Not function
5. (i) Even (ii) Odd (iii) Neither (iv) Even (v) Even

7. $A = 10l - l^2$

8. $P = \dfrac{32}{l} + 2l$

9. $A = a^2 + \dfrac{8}{a}$

10. $A = 15x - \dfrac{x^2(\pi + 4)}{8}$

11. $V = 4x^3 - 64x^2 + 240x$

12. $f(x) = \begin{cases} 15(40 - x) \text{ for} & 0 \le x < 40 \\ 0 \quad \text{for} & 40 \le x \le 65 \\ 15(x - 65) \text{ for} & x > 65 \end{cases}$

13. $E = \begin{cases} 10 + 0.06x \text{ for} & 0 \le x \le 1200 \\ 82 + 0.07(x - 1200) \text{ for} & x > 1200 \end{cases}$

1.2 Linear Mathematical Model

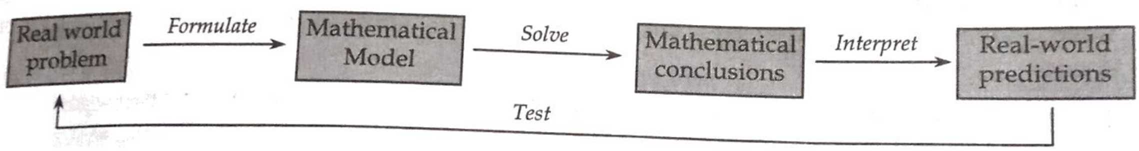
A mathematical model is method of simulating real life situation with mathematical equation to forecast their future behaviour. The mathematical model described in term of four stages.

Stage 1 (Formulation): On give real problem by using the dependent and independent variable establish an equation called mathematical model.

Stage 2 (Solving the Model): We use mathematical methods to solve the mathematical model. Calculus is the primary tool of analysis in this text.

Stage 3 (Interpretation): After the mathematical model has been solved any conclusions that may be obtained from solving are applied to the original real world problem.

Stage 4 (Testing): In this stage, the model is tested by gathering new data to check the accuracy of any predictions inferred from the analysis. If the predictions are not confirmed then the assumptions of model are adjusted and modeling process is repeated.



There are may different type of functions that can be used to model relationship observed in the real world. Here we discuss the linear mathematical model. A linear mathematical model can be described with lines we know that the equation of straight line is $y = mx + c$, where m is its slope and c is y intercept. This straight line's equation is linear mathematical model where y is dependent variable and x is independent variable. In this model m is rate of change and c is initial amount.

Example 1: A dry air moves upward, it expand and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C.

a. Express the temperature T (in°C) as a function of the height h (in km) assuming that a linear model is appropriate.

b. What slope represent?

c. What is the temperature at a height of 2.5 km?

Solution: (a) Since linear model, so

$$T = mh + b \qquad \text{........(i)}$$

Where, T is temperature at height h, m is slope of model and b is T intercept (i.e. temperature at ground level)

Given that $T = 20$ when $h = 0$

From (i), $b = 20$

Again, given $T = 10$ when $h = 1$ so from (i)

$$10 = m.1 + 20$$

$$\therefore \quad m = -10$$

So (i) becomes

$T = -10h + 20$, is required model.

(b) Slope = m = - 10

It means rate of change of temperature with respect to height = $\frac{dT}{dh}$ = –10 i.e. when 1 km height is increased then temperature is decrease by 10°C.

(c) Here, T = ? where h = 2.5 km

Since our model (i) is

$T = -10h + 20$

$T = -10 \times 2.5 + 20$

$T = -25 + 20$

$T = -5$

At height 2.5 km from ground, temperature is –5°C.

Example 2: At the surface of the ocean, the water pressure is the same the air pressure above the water 15 lb/m². Below the surface, the water pressure increased by 4.34 lb/m² for every 10 ft of decent.

a. Express the water pressure as a function of the depth below the ocean surface.

b. At what depth is pressure 100 lb/m².

Solution: Let P be the water pressure and x be the depth below the ocean surface. Here water pressure increased by 4.34 lb/m² for every 10 ft of decent, so model is linear and of the form

$P = mx + c$ ....(1)

Given m = Rate of change of pressure with respect to depth $= \frac{\Delta P}{\Delta x} = \frac{\text{Change in pressure}}{\text{Change in depth}}$

$= \frac{4.34}{10} = 0.434$

Hence, from (1)

$P = 0.434x + c$ ....(2)

Given that at surface of ocean i.e. x = 0, the pressure P = 15 lb/m².

So using (2),

$15 = 0 + c$

i.e. $c = 15$

Thus (2) becomes P = 0.434x + 15, is required model.

(b) Here x = ? when P = 100 lb/m²

Since model is P = 0.434x + 15

or $100 = 0.434x + 15$

or $0.434x = 85$

or $x = \frac{85}{0.434} = 195.86$

Thus, at depth approximately 196 ft the water pressure is 100 lb/m².

Exercise 1.2

1. Find an equation of the family of linear functions such that $f(2) = 1$.
2. Recent studies indicate that the average surface, temperature of the earth has been rising steadily some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in 0°C and t represents years since 1900.
   a. What do slope and T-intercepts represents?
   b. Use the equation to predict the average global surface temperature in 2100.
3. The Manager of a furniture factory finds that it costs $2200 to manufacture 100 chairs in one day and $4800 to produce 300 chairs in one day.
   a. Express to cost as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph.
   b. What is the slope of the graph and what does it represents?
   c. What is y-intercept of the graph and what does it represents?
4. Biologist has noticed that the chirping rate of crickets of certain species is related to temperature and the relationship appears to be very nearly linear. A cricket produces 113 chirps per minute at 70°F and 173 chirps per minute at 80°F.
   a. Find a linear equation that models the temperature T as a function of the number of chirps per minutes N.
   b. What is the slope of graph? What does it represent?
   c. If the crickets are chirping at 150 chirps per minute, estimate the temperature?
5. Since the beginning of the year, the price of a bottle of soda at a local discount supermarket has been rising at a constant rate of 2 cents per month. By November first, the price has reached $1.56 per bottle. Express the price of the soda as function of time and determine the price at the beginning of the year.
6. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her $380 to drive 480 ml and in June it cost her $460 to drive 800 ml.
   a. Express the monthly cost C as a function of the distance driven d, assuming that a linear relation relationship gives a suitable model.
   b. Use part (a) to predict the cost of driving 1500 miles per month.
   c. Draw the graph of the linear function. What does the slope represents?
   d. What does the C-intercept represent?
7. Temperature measured in degree Fahrenheit is linear function of temperature measured in degrees Celsius. Use the fact that 0° Celsius is equal to 32° Fahrenheit and 100° Celsius is equal to 212° Fahrenheit to write an equation for this linear function.
   a. Using this function convert 15° Celsius to Fahrenheit.
   b. Convert 68° Fahrenheit to Celsius.
   c. What temperature is the same in both the Celsius and Fahrenheit scale?

8. In certain part of the world, the number of death N per week have been observed to be linearly related to the average concentration x of sulfur dioxide in the air. Suppose these are 97 death when $x = 100\ mg/m^3$ and 110 death when $x = 500\ mg/m^3$.

   a. What is the functional relationship between N and x?

   b. Using (a) find the number of death per week when $300\ mg/m^3$ sulfur dioxide is in air.

9. The average score of incoming students at an estern liberal arts college in the SAT mathematics examination have been declining at a constant rate in recent years. In 1995, the average SAT score was 575 while in 2000 it was 545.

   a. Express the average score as a function of time.

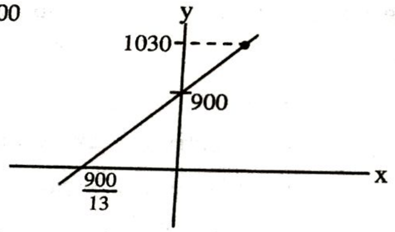
   b. If the trend continue, what will be average SAT score of incoming student be in 2005?

   c. If the trend continue, when will the average SAT score be 527?

Answers:

1. $f(x) = mx + (1 - 2m)$
2. (a) Slope (m) = 0.02, T-intercept (T) = 8.50

   (b) 12.5°C
3. (a) $y = 13x + 900$

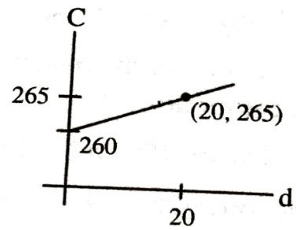


   (b) Slope (m) = 13, Rate of change of cost is 13. (c) y-intercept = 900, Fixed cost = 900$
4. (a) $T = \frac{1}{6}N + \frac{101}{2}$ (b) Slope (m) $= \frac{1}{6}$, Rate of change of temperature is $\frac{1}{6}$ (c) T = 75.5°C
5. $P = 2t + 136$; P = 1.36$
6. (a) $C = 0.25\,d + 260$

   (b) C = 536$

   (c) m = 0.25

   (d) C-intercept = 260



7. $F = \frac{9}{5}C + 32$ (a) 59°C (b) C = 20°C (c) F = C = – 40
8. (a) $N = \frac{13}{400}x + \frac{375}{4}$ (b) N = 104
9. (a) $y = -6t + 575$ (b) y = 515 (c) In 2003.

1.3 Combination of Functions

Arithmetic Combinations of Functions

Like numbers functions can be added, subtracted, multiplied and divided (except where the denominator is zero) to product new function. Let f and g are two functions then f + g, f - g, f.g and $\frac{f}{g}$ are new function which are defined by

Sum: $(f + g)(x) = f(x) + g(x)$;

Difference: $(f - g)(x) = f(x) - g(x)$

Product: $(f.g)(x) = f(x)\, g(x)$ [This is not composite]

Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ as long as $g(x) \neq 0$

The domain of each of these combination is intersection of the domain of f and the domain of g.

Example 1: If functions are defined by the formulas

$$f(x) = \sqrt{x} \text{ and } g(x) = \sqrt{2 - x}.$$

Find the formulas for functions f + g, f − g, f.g, $\frac{f}{g}$ and $\frac{g}{f}$.

Also find their domain.

Here $(f + g)(x) = f(x) + g(x) = \sqrt{x} + \sqrt{2 - x}$

The domain for $f(x) = \sqrt{x}$ is $x \geq 0$ i.e. $A = [0, \infty)$

The domain for $g(x) = \sqrt{2 - x}$;

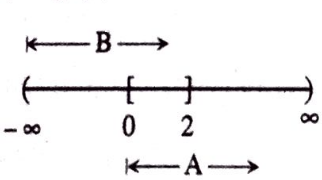
$$2 - x \geq 0$$

$$x - 2 \leq 0$$

i.e. $x \leq 2$ i.e. $B = (-\infty, 2]$

Domain for $(f + g)(x)$ is $A \cap B$.

Here



Here $A \cap B = [0, 2]$

Thus domain for $(f + g)$ is $[0, 2]$.

Again $(f - g)(x) = f(x) - g(x) = \sqrt{2} - \sqrt{2 - x}$.

and domain is $A \cap B = [0, 2]$.

$$(f.g)(x) = f(x) \,.\, g(x) = \sqrt{x}\sqrt{2 - x} = \sqrt{2x - x^2}$$

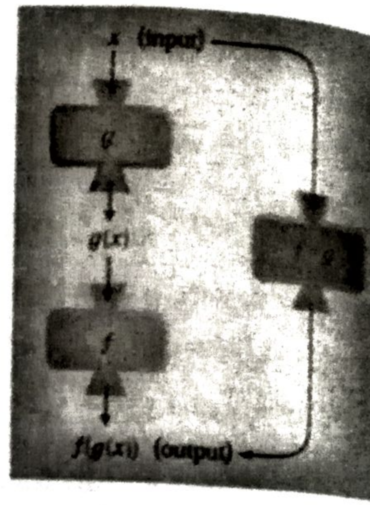
Thus domain for f.g is $[0, 2]$

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{2-x}};\quad x \neq 2$$

Domain for $\frac{f}{g}$ is $[0, 2)$

and $\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{2-x}}{\sqrt{x}}, x \neq 0$

Domain for $\frac{g}{f}$ is $(0, 2]$.



Composite functions

If f and g are functions, then composite function fog is defined by

$$(fog)(x) = f(g(x))$$

Note that the domain of fog(x) is intersection of domain of g(x) and f(g(x)).

Example 2: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$ find the each function and its domain.

(a) fog (b) gof (c) fof (d) gog

Solution:

(a) $fog(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = (2-x)^{1/4}$

To find domain of fog

First we find domain of $g(x) = \sqrt{2-x}$

Domain of g(x) is

$$2 - x \geq 0$$

i.e. $x \leq 2$ i.e. $A = (-\infty, 2]$.

and domain of $f(g(x)) = (2-x)^{1/4}$

$$2 - x \geq 0$$

$x \leq 2$ i.e. $B = (-\infty, 2]$

Here $A \cap B = (\infty, 2]$ is domain of fog.

(b) $gof(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$

For domain of gof

The domain of $f(x) = \sqrt{x}$ is $x \geq 0$ i.e. $A = [0, \infty)$

The domain of $g(f(x)) = \sqrt{2-\sqrt{x}}$ is $2 - \sqrt{x} \geq 0$

i.e. $\sqrt{x} \leq 2$

i.e. $0 \leq x \leq 4$

So, $B = [0, 4]$

$\therefore$ Domain of gof is $A \cap B = [0, 4]$.

(c) $f.f(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{1/4}$

The domain of f(x) is $A = [0, \infty)$

The domain of f(f(x)) is $B = [0, \infty)$

$\therefore$ Domain of fof is $A \cap B = [0, \infty)$

(d) $gog(x) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}$

Domain of g(x) is $A = (-\infty, 2]$

Domain of g(g(x)) is $2 - \sqrt{2-x} \geq 0$

i.e. $\sqrt{2-x} \leq 2$

i.e. $0 \leq 2 - x \leq 4$

i.e. $-2 \leq -x \leq 2$

i.e. $2 \geq x \geq -2$

Domain of g(g(x)) i.e. $B = [-2, 2]$

$\therefore$ Domain of gog is $A \cap B = [-2, 2]$.

Example 3: If $f(x) = \frac{1}{x+3}$ and $g(x) = \frac{x}{x-2}$. Find formula for fog and also find its domain.

Solution: Here, $fog(x) = f(g(x))$

$$= f\left(\frac{x}{x-2}\right) = \frac{1}{\frac{x}{x-2}+3} = \frac{1}{\frac{x+3x-6}{x-2}} = \frac{x-2}{4x-6}$$

$$\therefore \quad fog(x) = \frac{x-2}{2(2x-3)}$$

Here, domain of g(x) is $\mathcal{R} - \{2\} = A$

and domain of f(g(x)) is $\mathcal{R} - \left\{\frac{3}{2}\right\} = B$

Thus domain of fog(x) is $A \cap B = \mathcal{R} - \left\{2, \frac{3}{2}\right\}$

Example 4: Find fogoh if $f(x) = \sqrt{x-3}$, $g(x) = x^2$ and $h(x) = x^3 + 2$

Solution: Since $fogoh(x) = fog(h(x)) = fog(x^3 + 2) = f(g(x^3 + 2))$

$$= f((x^3 + 2)^2)$$

$$= \sqrt{(x^3 + 2)^2 - 3}$$

$$= \sqrt{x^6 + 4x^3 + 4 - 3}$$

$$= \sqrt{x^6 + 4x^3 + 1}$$

$\therefore$ fogoh $(x) = \sqrt{x^6 + 4x^3 + 1}$

Example 5: Given $F(x) = \cos^2(x + 9)$. Find function f, g and h such that F = fogoh.

Solution: Given, $F(x) = [\cos(x + 9)]^2$ so formula of F in x say add 9 then take cosine and that make square. So

$$h(x) = x + 9,\ g(x) = \cos x \text{ and } f(x) = x^2$$

Hence, $fogoh\,(x) = (fog)\,(h(x)) = (fog)\,(x + 9) = f[g(x + 9)]$

$$= f[\cos(x + 9)] = [\cos(x + 9)]^2$$

$$= \cos^2(x + 9)$$

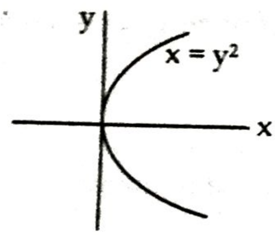
$$= F(x)$$

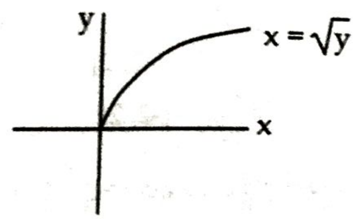
Some Well Known Functions and their Graph

(i) The function $y = mx + c$ is linear function and its graph is straight line.

(ii) The function $y = ax^2 + bx + c$, $a \neq 0$ is quadratic function. Its graph is parabola.

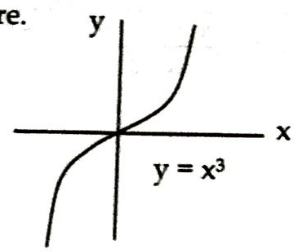
Graph of $x = y^2$ Graph of $x = \sqrt{y}$ are shown in figure.





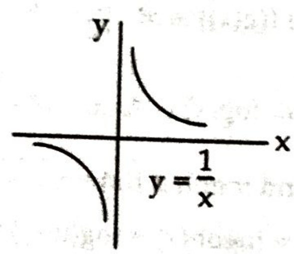
(iii) The function $y = ax^3 + bx^2 + cx + d$ is $a \neq 0$ cubic function

Graph of $y = x^3$ is shown in figure.



(iv) The function $y = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$ is rotational function.

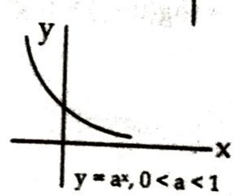
Graph of $y = \dfrac{1}{x}$ is shown in figure.

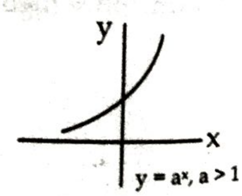


(v) The function $y = a^x$, $a > 0$ is exponential function

Graph of $y = a^x$, $a > 1$ and

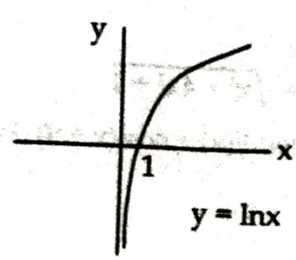
Graph of $y = a^x$, $0 < a < 1$ are shown in figure.





(vi) The function of $y = \log_a x$ is logarithmic function

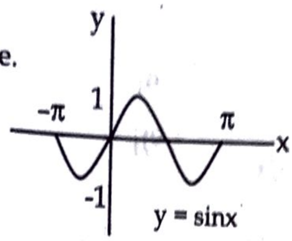
Graph of $y = \ln x$ is shown in figure.

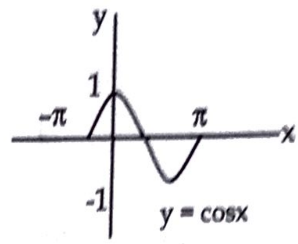


(vii) The function y = sinx, y = cosx are trigonometric functions

Graph of y = sinx and

Graph of y = cosx are shown in figure.





Transformations of Functions

By applying certain transformations to the graph of a given function we can obtain the graph of new function. Here we discuss the two type of transformations one is translations and another is stretching and reflecting transformations.

Translation Transformation

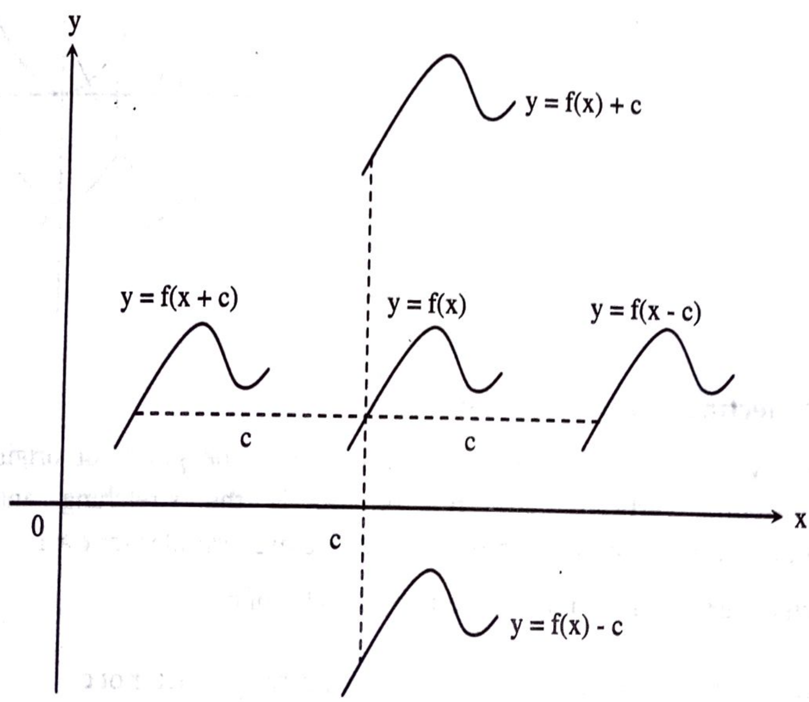
The transformation by which the graph of new function is the graph of original function shifted vertically or horizontally is the translation transformation. Shift formulas: Suppose $c > 0$.

$y = f(x) + c$, shift the graph of $y = f(x)$ at distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ at distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ at distance c units to the right

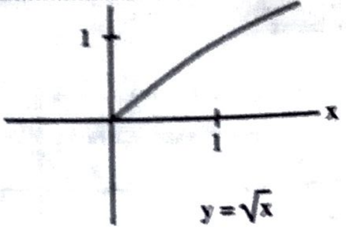
$y = f(x + c)$, shift the graph of $y = f(x)$ at distance c units to the left

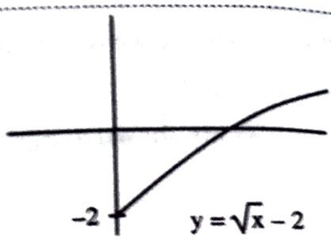


Example 6: Given the graph of $y = \sqrt{x}$, use transformations to graph the following functions by hand, not by plotting points. (i) $y = \sqrt{x} - 2$ (ii) $\sqrt{x - 2}$.

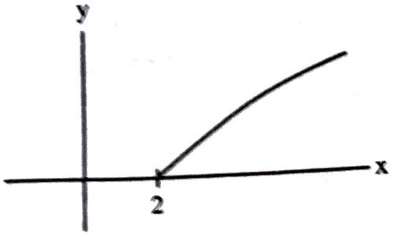
Solution: (i) $y = \sqrt{x} - 2$

Graph of this function is, shifting the graph of $y = \sqrt{x}$ at distance 2 unit downward so, it graph is





(ii) $y = \sqrt{x-2}$, graph of this function is, shifting the graph of $y = \sqrt{x}$ at distance 2 unit to the right; so it's graph is



Example 7: Find the equation of shifted graph using given transformation to given function. Also sketch the original and shifted graphs together. (Sketch by hand not by plotting points).

(i) $y = x^2$ down 3, left 2.

Given $y = x^2$

For down 3, $y = f(x) - 3$

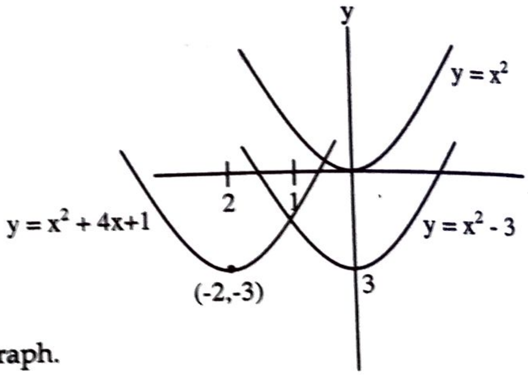
$$y = x^2 - 3$$

For left 2,

$$y = f(x + 2)$$

$$= (x + 2)^2 - 3$$

i.e. $y + 3 = (x + 2)^2$ is required equation of shifted graph.



Stretching and Reflecting Transformation

The transformation by which the graph of new function is the graph of original function stretching and reflecting vertically and horizontally is the stretching and reflecting transformation. Vertical and horizontal stretching and reflecting formulas for $c > 1$:

$y = cf(x)$ stretch the graph of $y = f(x)$ vertically by a factor of c.

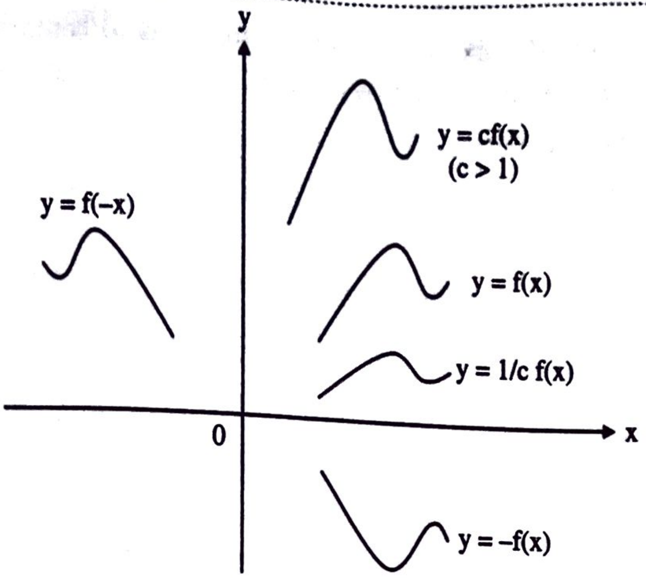
$y = \frac{1}{c} f(x)$ shrink (compress) the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$ shrink (compress) the graph of $y = f(x)$ horizontally by a factor of c

$y = f\left(\frac{x}{c}\right)$ stretch the graph of the $y = f(x)$ horizontally by a factor of c

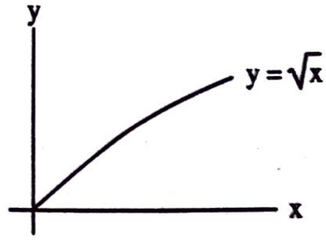
$y = -f(x)$ reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$ reflect the graph of $y = f(x)$ about the y-axis



Example 8: Given the graph of $y = \sqrt{x}$, use transformation to graph the following functions by hand, not by plotting points.

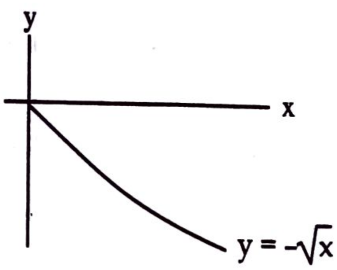
(i) $y = -\sqrt{x}$ (ii) $y = 2\sqrt{x}$ (iii) $y = \sqrt{-x}$



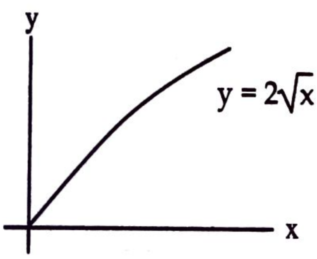
Solution:

(i) $y = -\sqrt{x}$, graph of this function is reflecting of $y = \sqrt{x}$ about x-axis.

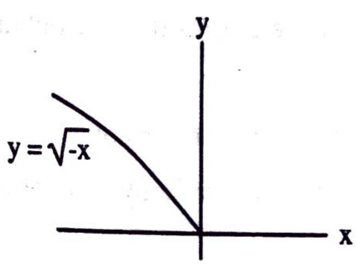
So it graph is



(ii) $y = 2\sqrt{x}$, graph of this function is stretching vertically by factor 2, of $y = \sqrt{x}$. So it graph is



(iii) $y = \sqrt{x}$, graph of this function is reflecting of $y = \sqrt{x}$ about y-axis, so its graph is



Example 9: Given the function $f(x) = x^4 - 4x^3 + 10$, find the formulas to compress the graph horizontally by a factor 2 followed by a reflection across the y-axis.

Solution: Given function is $f(x) = x^4 - 4x^3 + 10$

Compress horizontally by factor 2 so

$$y = f(2x)$$

$$y = (2x)^4 - 4(2x)^3 + 10 = g(x)$$

and followed by reflection about y-axis,

$$y = g(-x) = (-2x)^4 - 4(-2x)^3 + 10$$

$$y = 16x^4 + 32x^3 + 10$$

Exercise 1.3

1. Find $f + g$, $f - g$, $f \cdot g$, and f/g and state their domain

   (i) $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$

   (ii) $f(x) = \sqrt{3 - x}$, $g(x) = \sqrt{x^2 - 1}$

   (iii) $f(x) = \sqrt{x}$; $g(x) = \sqrt{1 - x}$

   (iv) $f(x) = x$, $g(x) = \sqrt{x - 1}$

   (v) $f(x) = \sqrt{x + 1}$, $g(x) = \sqrt{x - 1}$

2. Find fog, gof, fof and gog and state their domain

   (i) $f(x) = \sqrt{x}$; $g(x) = x + 1$

   (ii) $f(x) = x^2 - 1$; $g(x) = 2x + 1$

   (iii) $f(x) = \sqrt{x}$; $g(x) = \sqrt[3]{1 - x}$

   (iv) $f(x) = x + \frac{1}{x}$; $g(x) = \frac{x + 1}{x + 2}$

   (v) $f(x) = \sqrt{x + 1}$, $g(x) = \frac{1}{x}$

   (vi) $f(x) = x^2$, $g(x) = 1 - \sqrt{x}$

3. Find $f_o g_o h$

   (i) $f(x) = 3x - 2$, $g(x) = \sin x$, $h(x) = x^2$

   (ii) $f(x) = |x - 4|$, $g(x) = 2^x$, $h(x) = \sqrt{x}$

4. Express the function in form of fog if

   (i) $f(x) = (2x + x^2)^4$

   (ii) $f(x) = \cos^2 x$

   (iii) $v(t) = \sec(t^2) \tan(t^2)$

5. Express the function in form $f_o g_o h$ if

   (i) $R(x) = \sqrt{\sqrt{x} - 1}$

   (ii) $H(x) = \sqrt[8]{2 + |x|}$

   (iii) $H(x) = \text{sex}^4(\sqrt{x})$

6. Suppose the graph of f is given. Write equation for the graphs that are obtained from the graph of f as follows:

   (i) Shift 3 unit upward

   (ii) Shift 2 unit to the right

   (iii) Reflect about y-axis

   (iv) Stretch vertically by a factor of 3.

   (v) Compressed horizontally by a factor of 1.

7. Explain how each graph is obtained from the graph of $y = f(x)$

(i) $y = f(x) + 8$
(ii) $y = f(x + 8)$
(iii) $y = f(8x)$
(iv) $y = -f(x) - 1$
(v) $y = 8f\left(\frac{1}{8}x\right)$

8. Find the new function by using given transformation on given functions

(i) $f(x) = -\sqrt{x}$ shifted right by 3
(ii) $y = 2x - 7$ shifted up by 7.
(iii) $y = x^2 - 1$ stretched vertically by a factor of 3.
(iv) $y = \sqrt{x+1}$ compressed horizontally by factor 4.
(v) $y = \frac{1}{2}(x + 1) + 5$ shifted down by 5 followed by right 1.
(vi) $f(x) = \frac{1}{x^2}$ shifted left by 2 followed by down 1.
(vii) $f(x) = x^3 - 4x^2 - 10$ compress vertically by 2 followed by reflection about x-axis.

9. Find the appropriate transformation used thus obtained new function as below, and graph the function by hand not by plotting point.

(i) $y = |x| - 2$
(ii) $y = x^2 + 2$
(iii) $y = (x + 1)^2$
(iv) $y = \sqrt{x-2} - 1$
(v) $1 - 2\sqrt{x+3}$

Answers:

1.(i) $(f + g)(x) = x^3 + 5x^2 - 1$ $(f - g)(x) = x^3 - x^2 + 1$

$(fg)(x) = 3x^5 + 6x^4 - x^3 - 2x^2$ $(f/g)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1}$

Domain for $(f + g)$, $(f - g)$ and (fg) is $\mathcal{R}$

Domain of f/g is $\mathcal{R} - \left\{\pm\frac{1}{\sqrt{3}}\right\}$

(ii) $(f + g)(x) = \sqrt{3-x} + \sqrt{x^2-1}$ $(f - g)(x) = \sqrt{3-x} - \sqrt{x^2-1}$

$(fg)(x) = \sqrt{(3-x)(x^2-1)}$ $(f/g)(x) = \sqrt{\dfrac{3-x}{x^2-1}}$; $x \neq \pm 1$

Domain for $(f + g)$, $(f - g)$ and (fg) is $(-\infty, -1]$

Domain for f/g is $(-\infty, -1)$

(iii) $(f + g)(x) = \sqrt{x} + \sqrt{1-x}$ $(f - g)(x) = \sqrt{x} - \sqrt{1-x}$

$(fg)(x) = \sqrt{x(1-x)}$ $(f/g)(x) = \sqrt{\dfrac{x}{x-1}}$ $x \neq 1$

Domain for $(f + g)$, $(f - g)$ and (fg) is $[0, 1]$

Domain for f/g is $[0, 1)$

(iv) $(f+g)(x) = x + \sqrt{x-1}$ $\quad$ $(f-g)(x) = x - \sqrt{x-1}$

$(fg)(x) = x\sqrt{x-1}$ $\quad$ $(f/g)(x) = \dfrac{x}{\sqrt{x-1}}, \; x \neq 1$

Domain for $(f+g)$, $(f-g)$ and (fg) is $[1, \infty)$

Domain for f/g is $(1, \infty)$

(v) $(f+g)(x) = \sqrt{x+1} + \sqrt{x-1}$ $\quad$ $(f-g)(x) = \sqrt{x+1} - \sqrt{x-1}$

$(fg)(x) = \sqrt{x^2-1}$ $\quad$ $(f/g)(x) = \sqrt{\dfrac{x+1}{x-1}}, \; x \neq 1$

Domain for $(f+g)$, $(f-g)$ and (fg) is $[1, \infty)$

Domain for f/g is $(1, \infty)$

2.(i) $(f \circ g)(x) = \sqrt{x+1}$ $\quad$ domain is $[-1, \infty)$

$(g \circ f)(x) = \sqrt{x} + 1$ $\quad$ domain is $[0, \infty)$

$(f \circ f)(x) = x^{1/4}$ $\quad$ domain is $[0, \infty)$

$(g \circ g)(x) = x + 2$ $\quad$ domain is $(-\infty, \infty)$

(ii) $(f \circ g)(x) = 4x^2 + 4x$ $\quad$ domain is $(-\infty, \infty)$

$(g \circ f)(x) = 2x^2 - 1$ $\quad$ domain is $(-\infty, \infty)$

$(f \circ f)(x) = x^4 - 2x^2$ $\quad$ domain is $(-\infty, \infty)$

$(g \circ g)(x) = 4x + 3$ $\quad$ domain is $(-\infty, \infty)$

(iii) $(f \circ g)(x) = \sqrt[6]{1-x}$ $\quad$ domain is $(-\infty, 1]$

$(g \circ f)(x) = (1 - \sqrt{x})^{1/3}$ $\quad$ domain is $[0, \infty)$

$(f \circ f)(x) = x^{1/4}$ $\quad$ domain is $[0, \infty)$

$(g \circ g)(x) = \sqrt[3]{1 - \sqrt[3]{1-x}}$ $\quad$ domain is $(-\infty, \infty)$

(iv) $(f \circ g)(x) = \dfrac{2x^2 + 6x + 5}{(x+2)(x+1)}$ $\quad$ domain is $\mathfrak{R} - \{-1, -2\}$

$(g \circ f)(x) = \dfrac{x^2 + x + 1}{(x+1)^2}$ $\quad$ domain is $\mathfrak{R} - \{0, -1\}$

$(f \circ f)(x) = \dfrac{x^4 + 3x^2 + 1}{x(x^2+1)}$ $\quad$ domain is $\mathfrak{R} - \{0\}$

$(g \circ g)(x) = \dfrac{2x+3}{3x+5}$ $\quad$ domain is $\mathfrak{R} - \{-2, -5/3\}$

(v) $(f \circ g)(x) = \sqrt{\dfrac{1}{x} + 1}$ $\quad$ domain is $\mathfrak{R} - (-1, 0]$

$(g \circ f)(x) = \dfrac{1}{\sqrt{x+1}}$ $\quad$ domain is $\mathfrak{R} - (-1, \infty)$

$(f \circ f)(x) = (x+1)^{1/4}$ $\quad$ domain is $(-1, \infty)$

$(g \circ g)(x) = x$ $\quad$ domain is $\mathfrak{R} - \{0\}$

(vi) $(f \circ g)(x) = 1 + x - 2\sqrt{x}$ domain is $[0, \infty)$

$(g \circ f)(x) = 1 - x$ domain is $(-\infty, \infty)$

$(f \circ f)(x) = x^4$ domain is $(-\infty, \infty)$

$(g \circ g)(x) = 1 - \sqrt{1 - \sqrt{x}}$ domain is $[0, 1]$

3. (i) $(f \circ g \circ h)(x) = 3\sin x^2 - 2$

(ii) $(f \circ g \circ h)(x) = |2^{\sqrt{x}} - 4|$

4. (i) $f(x) = x^4$ and $g(x) = 2x + x^2$

(ii) $f(x) = x^2$, $g(x) = \cos x$

(iii) $f(t) = \sec t \tan t$ and $g(t) = t^2$

5. (i) $f(x) = \sqrt{x}$, $g(x) = x - 1$ and $h(x) = \sqrt{x}$

(ii) $f(x) = \sqrt[8]{x}$, $g(x) = 2 + x$ and $h(x) = |x|$

(iii) $f(x) = x^4$, $g(x) = \sec x$ and $h(x) = \sqrt{x}$

6. (i) $y = f(x) + 3$ (ii) $y = f(x - 2)$ (iii) $y = f(-x)$ (iv) $y = 3f(x)$ (v) $y = f(x)$

7. (i) shift distance 8 unit upward

(ii) shift 8 unit distance left

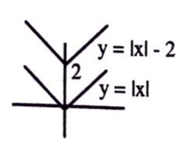
(iii) compress horizontally by factor of 8

(iv) stretch horizontally by a factor of 8 followed by stretch vertically by a factor of 8.

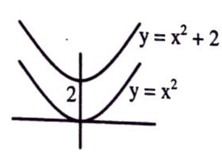
8. (i) $f(x) = -\sqrt{x - 3}$ (ii) $y = 2x$ (iii) $y = 3x^2 - 3$ (iv) $y = \sqrt{4x + 1}$ (v) $\frac{1}{2}x + 8$

(vi) $f(x) = \frac{1}{(x+2)^2} - 1$ (vii) $y = -\frac{1}{2}x^3 + 2x^2 + 5$

9. (i) Original function $f(x) = |x|$, shift the distance 2 unit downward

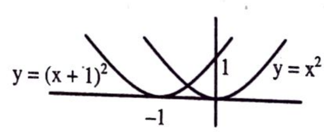


(ii) Original function, $f(x) = x^2$, shift the distance 2 unit upward



(iii) Original function $f(x) = x^2$;

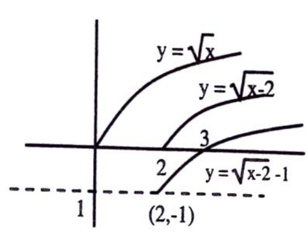
shifting the graph at the distance 1 unit left



(iv) Original function $y = \sqrt{x}$

shifting a distance 2 unit right

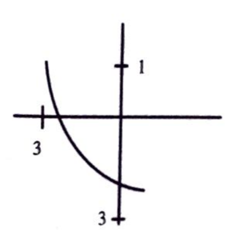
followed by shifting 1 unit down



(v) $y = 1 - 2\sqrt{x + 3}$ shift left by 3 unit,

stretch vertically by 2, reflect about

x-axis and shift 1 unit upward.



1.4 Rational, Trigonometric, Exponential and Logarithmic Function

A rational function f is ratio of two polynomials

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial.

The domain of rational function is set of all real value x such that $Q(x) \neq 0$. For example, $f(x) = \frac{x^3 + 4x - 3}{x^2 - 9}$ is rational function and its domain is set of all real number except 3 and -3.

Algebric function f is such function which can be contructed by using algebric operations (such as addition, subtraction, multiplication, divisions and taking roots) starting with polynmials.

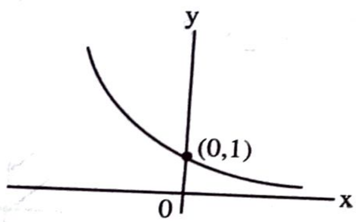
For example $f(x) = \sqrt{x^4 + x^3 + 1}$, $g(x) = \frac{x^3 - 3x^2}{x + 4} + (x - 2)\sqrt{x + 5}$

Function: f(x) = sinx, f(x) = cosx etc. are trigonometric function.

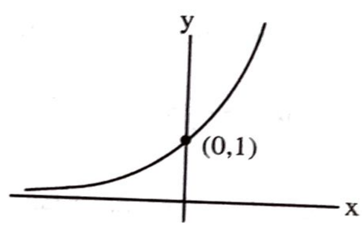
Exponential function are function of type $f(x) = a^x$ where a is positive constant. For example $f(x) = 2^x$, $f(x) = e^x$ are exponential function.

Function of form $f(x) = \log_a x$ is called **logarithmic function,** where a is base and is positive constant. Logarithmic function is inverse of exponential function. So if $y = a^x$ then $x = \log_a y$.

The graph of $y = a^x$, shown as below.



$y = a^x;\ 0 < a < 1$



$y = a^x;\ a > 1$

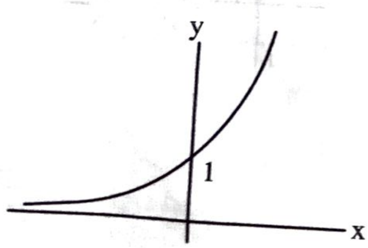
Note that the reflection of graph of $y = a^x$ about y-axis is $y = a^{-x}$.

Example 1: Find the original function and transformations are applied so obtained new function $y = 3 - 2^x$. Sketch is graph and find the domain and range.

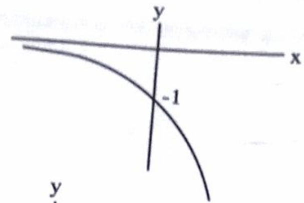
Solution: The original function is $f(x) = y = 2^x$

Reflect the graph of $y = 2^x$ about x axis (f(x) = - f(x), we get $y = -2^x$, then we shift graph $y = -2^x$ upward 3 unit so obtained $y = 3 - 2^x$.

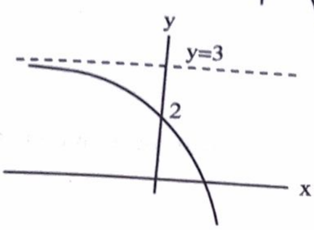
Graph of $y = 2^x$ is



Graph of $y = -2^x$ is



Graph of $y = 3 - 2^x$ is



Here domain is set of all real number i.e. $(-\infty, \infty)$, and range is $(-\infty, 3)$.

Example 2: Graph the function $y = \frac{1}{2}e^{-x} - 1$ and state the domain and range.

The original function is $f(x) = y = e^x$

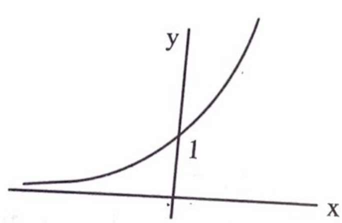
Reflect the graph $y = e^x$ about y-axis, $(f(x) = f(-x))$.

We get $y = e^{-x}$

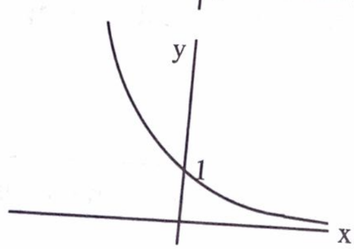
Now we compress the graph $y = e^{-x}$ vertically by a factor 2 (i.e. $f(x) = \frac{1}{2}f(x)$). So we obtain $y = \frac{1}{2}e^{-x}$.

Finally, we shift the graph down ward 1 unit, we get $y = \frac{1}{2}e^{-x} - 1$.

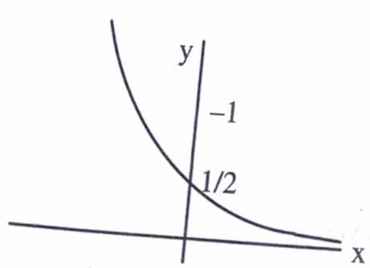
Graph of $y = e^x$ is



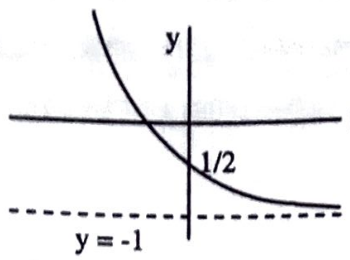
Graph of $y = e^{-x}$



Graph of $y = \frac{1}{2}e^{-x}$



Graph of $y = \frac{1}{2}e^{-x} - 1$



The domain of function is $(-\infty, \infty)$, and Range is $(-1, \infty)$.

One to One Function

A function is said to be one to one if no two different elements of domain has same image.

Example 3: Is function $f(x) = x^2$ one to one?

This function is not one to one, because

$1 \neq -1$ but $f(1) = f(-1) = 1$.

Inverse Function

Let f be a one to one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by $f^{-1}(y) = x \Leftrightarrow f(x) = y$.

Note that domain f = Range of f^{-1} and domain of f^{-1} = Range of f and the graph of f^{-1} is obtained by reflecting the graph of f about line $y = x$.

Example 4: Find the inverse function of $f(x) = x^3 + 2$. How f^{-1} is related to f and what is relation about graph of f and f^{-1}?

Solution: Since $y = x^3 + 2$

or $x^3 = y - 2$

or $x = (y - 2)^{1/3}$

or $f^{-1}(y) = (y - 2)^{1/3}$

$\therefore$ $f^{-1}(x) = (x - 2)^{1/3}$ is required formula for inverse function.

The domain of f is range of f^{-1} and range of f is domain f^{-1} f and the graph of f^{-1} is obtained by reflecting the graph of f about line $y = x$.

Exercise 1.4

1. Starting with the graph of $y = e^x$ write the equation of the graph that result from

   (i) shifting 2 units downward

   (ii) shifting 2 unit to the right

   (iii) reflecting about the x-axis

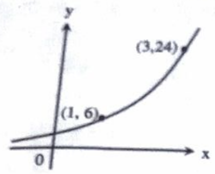
   (iv) reflecting about the y-axis

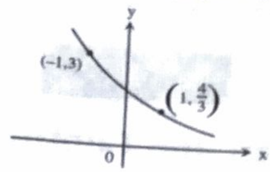
   (v) reflecting about x-axis and the about y-axis.

2. Using the transformation, make rough sketch of graph of function

   (i) $y = -2^{-x}$ (ii) $y = \left(\frac{1}{2}\right)^x - 2$ (iii) $y = 1 - \frac{1}{2}e^{-x}$

3. Find the exponential function $f(x) = Ca^x$ whose graph is given.





4. A function is given by formula. Determine whether it is one to one

(i) $f(x) = x^2 - 2x$

(ii) $f(x) = 10 - 3x$

(iii) $g(x) = \frac{1}{x}$

(iv) $h(x) = 2 + |x|$

5. Find the formula for inverse of the function:

(i) $f(x) = 1 + \sqrt{2 + 3x}$

(ii) $f(x) = \frac{4x - 1}{2x} + 3$

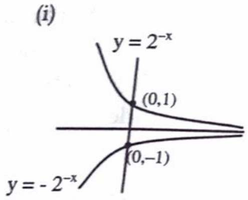
(iii) $f(x) = e^{2x - 1}$

(iv) $y = \frac{e^x}{1 + 2e^x}$

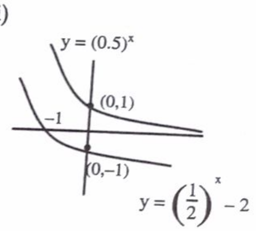
Answers:

1. (i) $y = e^x - 2$ (ii) $y = e^{x - 2}$ (iii) $y = -e^x$ (iv) $y = e^{-x}$ (v) $y = -e^{-x}$

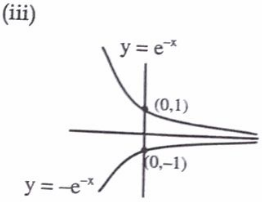
2. (i)

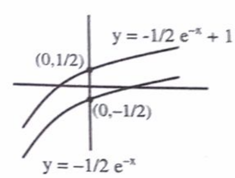


(ii)



(iii)





3. $y = 3.2^x$ and $y = 2\left(\frac{2}{3}\right)^x$

4. (i) One to one (ii) one to one (iii) one to one (iv) not one to one

5. (i) $f(x) = \frac{x^2 - 2x - 1}{3}$ (ii) $f^{-1}(x) = \frac{1}{10 - 2x}$ (iii) $f^{-1}(x) = \frac{1}{2}(\ln x - 1)$ (iv) $f^{-1}(x) = \ln\left(\frac{y}{1 - 2y}\right)$

Limit and Continuity

Introduction

The concept of the limit is one of the most crucial things to understand in order to prepare for calculus. A limit is a number that a function approaches as the independent variable of the function approaches to a given value. For example, given the function $f(x) = 3x$, you could say, "The limit of $f(x)$ as x approaches 2 is 6." Symbolically, this is written $\lim_{x \to 2} f(x) = 6$. In the following sections, we will more carefully define a limit, as well as give examples of limits of functions to help clarify the concept.

Continuity is another far-reaching concept in calculus. A function can either be continuous or discontinuous. One easy way to test for the continuity of a function is to see whether the graph of a function can be traced with a pen without lifting the pen from the paper. For the math that we are doing in precalculus and calculus, a conceptual definition of continuity like this one is probably sufficient, but for higher math, a more technical definition is needed. Using limits, we'll learn a better and far more precise way of defining continuity as well. With an understanding of the concepts of limits and continuity.

Continuity can be defined conceptually in a few different ways. A function is continuous, for example, if its graph can be traced with a pen without lifting the pen from the page. A function is continuous if its graph is an unbroken curve; that is, the graph has no holes, gaps, or breaks. But terms like "unbroken curve" and "gaps" aren't technical mathematical terms and at best, only provide a reader with a description of continuity, not a definition.

The more formal definition of continuity is this: a function $f(x)$ is continuous at a point $x = a$, if and only if the following three conditions are met. 1) $f(a)$ is defined. 2) $\lim_{x \to a} f(x)$ exists. 3) $\lim_{x \to a} f(x) = f(a)$. Otherwise, the function is discontinuous.

A function can be continuous at a point, continuous over a given interval, or continuous everywhere. We have already defined continuity at a given point. For a function to be continuous over an interval $[a, b]$, that function must be continuous at each point in the interval, as well as at both a and b. For a function to be continuous everywhere, it must be continuous for every real number.

The intuitive definition of a limit is inadequate for some purposes because such phrases as "x approach to a f(x) gets closer and closer to L" is vague. In order to be able to prove conclusively that

$$\lim_{x \to 0}\left(x^3 + \frac{\cos 5x}{10000}\right) = 0.0001 \text{ or } \lim_{x \to 0}\frac{\sin x}{x} = 1$$

We must make the definition of limit precise.

For the motivation to the precise definition of a limit, let's consider the function.

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then f(x) is close to 5 and so, $\lim_{x \to 3} f(x) = 5$.

To obtain more detailed information about how f(x) varies when x is close to 3, we have to think over with the following question.

How close to 3 does x have to be so that f(x) differs from 5 by less than 0.1?

Here, the distance from x to 3 is $|x - 3|$ and the distance from f(x) to 5 is $|f(x) - 5|$, so our problem is to find a number δ such that $|f(x) - 5| < 0.1$ if $|x - 3| < \delta$ but $x \neq 3$. If $|x - 3| > 0$, then $x \neq 3$, so an equivalent formulation to our problem is to find a number δ such that $|f(x) - 5| < 0.1$ if $0 < |x - 3| < \delta$.

Here, we can observe that if $0 < |x - 3| < \frac{0.1}{2} = 0.05$,

then $|f(x) - 5| = |2x - 1 - 5| = |2x - 6|$

$= 2|x - 3| < 2 \times 0.05 = 0.1.$ [If $\delta > 0.05$ then it does not work for $\varepsilon = 0.1$]

i.e. $|f(x) - 5| < 0.1$ if $0 < |x - 3| < 0.05$.

Thus, we have answer of above question: If x is within a distance of $0.05 = \delta$ from 3, then f(x) will be within a distance of $0.1 = \varepsilon$, from 5.

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that f(x) will differ from 5 by less than 0.01 provided that x differs from 3 by less than $\frac{0.01}{2} = 0.005$. [If $\delta > 0.005$ then it does not work for $\varepsilon = 0.01$, If $\delta = 0.05$ for $\varepsilon = 0.01$ then $|f(x) - 5| = |2x - 1 - 5| = 2|x - 3| < 2 \times 0.05 = 0.1 \geq 0.01 = \varepsilon$]

$\therefore \quad |f(x) - 5| < 0.01$ if $0 < |x - 3| < 0.005$

Similarly,

$$|f(x) - 5| < 0.001 \text{ if } 0 < |x - 3| < 0.0005.$$

The numbers 0.1, 0.01 and 0.001 that we have considered are error tolerances that we might allow. For 5 to be the precise limit of f(x) as x approaches 3, we must not only be able to bring the difference between f(x) and 5 below each of these three numbers; we must also be able to bring it below any positive number. And, by the same reasoning, we can if we write ε for any arbitrary positive number, then we find as before that

$$|f(x) - 5| < \varepsilon \text{ if } 0 < |x - 3| < \delta = \frac{\varepsilon}{2} \qquad \text{.......(1)}$$

which is precise way of saying that f(x) is close to 5 when x is close to 3. Precisely, we can make the values of f(x) within an arbitrary distance ε from 5 by taking the values of x within a distance $\frac{\varepsilon}{2}$ from 3 (but $x \neq 3$).

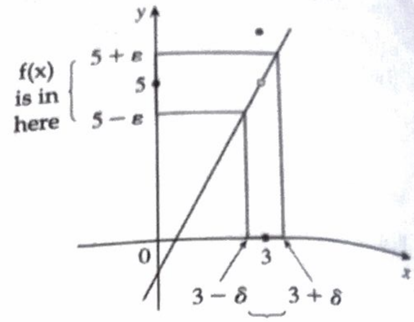


Figure 1

Note that (1) can be written as

If $3 - \delta < x < 3 + \delta$ $(x \neq 3)$ then $5 - \varepsilon < f(x) < 5 + \varepsilon$ and this is explained by the figure 1. By taking the values of $x \neq 3$ to lie in the interval $(3 - \delta, 3 + \delta)$ we can make the value of f(x) lie in the interval $(5 - \varepsilon, 5 + \varepsilon)$ with this concept, a precise definition of a limit.

2.1 Precise Definition of Limit

Definition: Let f be a function defined on some open interval that contains the number a except possibly at itself. Then we say that the limit of f(x) on x approaches a is L, and we write

$\lim_{x \to a} f(x) = L$. If for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Since $|x - a|$ is the distance from x to a and $|f(x) - L|$ is the distance from f(x) to L, and since ε can be arbitrarily small, the definition of a limit can be expressed in words as follows:

$\lim_{x \to a} f(x) = L$ means that the distance between f(x) and L can be made arbitrarily small by taking the distance from x to a is sufficiently small (but not zero).

Alternatively, $\lim_{x \to a} f(x) = L$ means that the values of f(x) can be made as close as we please to L by taking x close enough to a (but not equal to a).

Alternatively, $\lim_{x \to a} f(x) = L$ means for every $\varepsilon > 0$ (however it is small enough) we can find $\delta > 0$ such that x lies in the open interval $(a - \delta, a + \delta)$, then f(x) lies in the open interval $(L - \varepsilon, L + \varepsilon)$, which is illustrated in the following figures.

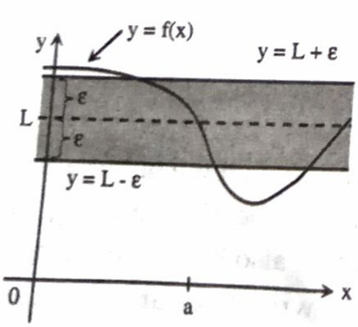


Figure 1

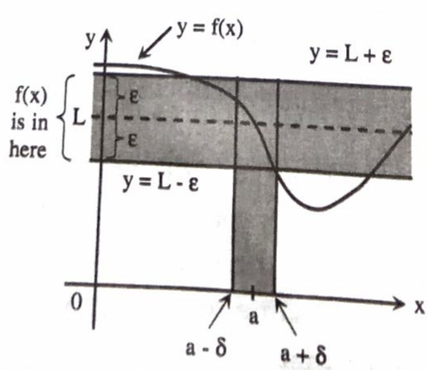


Figure 2

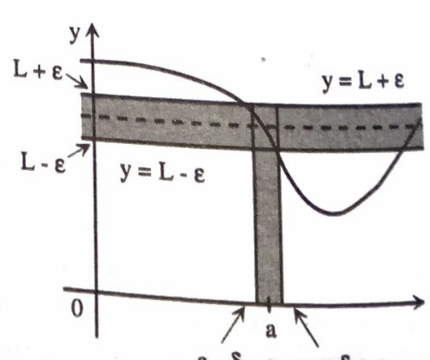


Figure 3

Example 1: Use a graph to find a number δ such that:

If $|x - 1| < \delta$ then $|(x^3 - 5x + 6) - 2| < 0.2$.

Alternatively,

Find a number δ that corresponds to $\varepsilon = 0.2$, in the definition of a limit for the function $f(x) = x^3 - 5x + 6$ with $a = 1$ and $L = 2$.

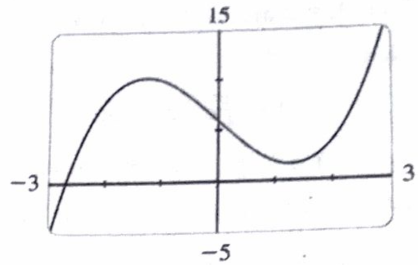


Fig. 1

Solution. A graph of f is shown in Fig. 1 our concentration is near the point (1.2)

Since,

$$|(x^3 - 5x + 6) - 2| < 0.2$$

or $\quad 1.8 < x^3 - 5x + 6 < 2.2$

We need to find the values of x for which the curve $y = x^3 - 5x + 6$ lies between the horizontal lies $y = 1.8$ and $y = 2.2$. Therefore we graph the curve $y = x^3 - 5x + 6$, $y = 1.8$, and $y = 2.2$ near the point (1, 2) in fig. 2. Then we use the cursor to estimate that the x-coordinate of the point of intersection of the line $y = 2.2$ and the curve $y = x^3 - 3x + 6$ is about 0.911. Similarly, $y = x^3 - 5x + 6$ intersects the line $y = 1.8$ at about $x = 1.124$. So rounding to be safe, we can say that

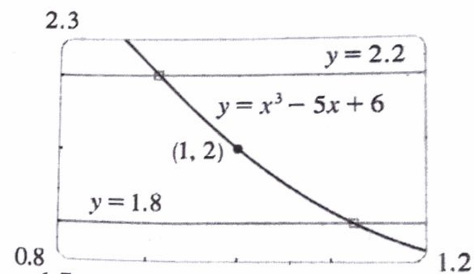


Fig. 2

If $0.92 < x < 1.12$ then $1.8 < x^3 - 5x + 6 < 2.2$

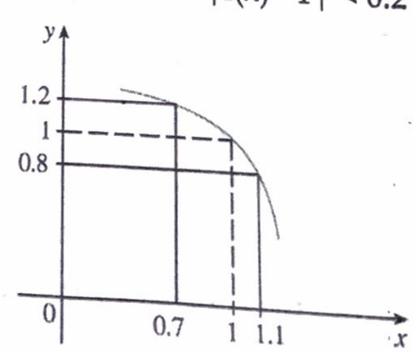
This interval (0.92, 1.12) is not symmetry about $x = 1$. The distance from $x = 1$ to the left end point is $1 - 0.92 = 0.08$ and the distance to the right end point is $1.12 - 1 = 0.12$. We can choose δ to the smaller of these numbers, that is $\delta = 0.08$. Then we can rewrite our inequalities in terms of distances as follows:

If $|x - 1| < 0.08$ then $|(x^3 - 5x + 6) - 2| < 0.2$.

Conclusion: By keeping x within 0.08 of 1, we are able to keep f(x) within 0.2 of 2.

Example 2: Use the given graph of f to find a number δ such that if $|x - 1| < \delta$ then $|f(x) - 1| < 0.2$

Solution: Here $a = 1$, $\varepsilon = 0.2$, $L = 1$, $\delta = ?$



Since if $|f(x) - 1| < 0.2$

or $\quad 0.8 < f(x) < 1.2$

∴ $\quad$ f(x) varies from 0.8 to 1.2

From the figure for $x = 0.7$, $f(x) = 1.2$

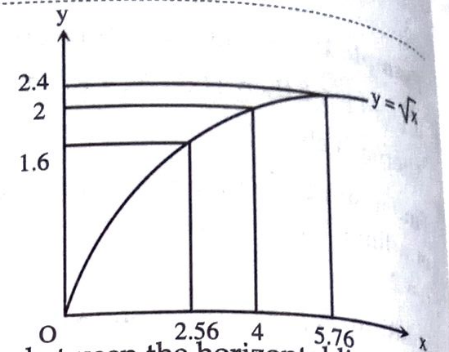
and for $x = 1.1$, $f(x) = 0.8$

∴ $\quad$ x varies from 0.7 to 1.1 which are not at symmetric from $x = 1$. The distance from $x = 1$ and $x = 0.7$ is 0.3 and distance from 1 to 1.1 is 0.1. So $\delta = 0.1$.

Example 3: Use the graph of $f(x) = \sqrt{x}$ to find a number δ such that if

$$|x - 4| < \delta \text{ then } |\sqrt{x} - 2| < 0.4.$$

Solution: Here limit of the function $f(x) = \sqrt{x}$ at $x = 4$ is 2 i.e. $L = 2$ and $a = 4$, $\varepsilon = 0.4$.



Now,

$$-0.4 < \sqrt{x} - 2 < 0.4$$

$$\Rightarrow \quad 1.6 < \sqrt{x} < 2.4 \text{ whenever } |x - 4| < \delta.$$

We need to concentrate near the region of the point (4, 2).

We need to find the values of x for which the curve $y = \sqrt{x}$ lies between the horizontal line $y = 1.6$ and $y = 2.4$. Therefore, we graph the curve $y = \sqrt{x}$, $y = 1.6$ and $y = 2.4$ near the point (4, 2).

Now the point of intersection of $y = \sqrt{x}$, $y = 1.6 \Rightarrow x = 2.56$.

Again the point of intersection of $y = \sqrt{x}$ and $y = 2.4$ is $x = 5.76$.

Now rounding to be safe we have if $2.56 < x < 5.76$ then $1.6 < \sqrt{x} < 2.4$.

This interval (2.56, 5.76) is not symmetric about $x = 4$. The distance from $x = 4$ to the left end point is $4 - 2.56 = 1.44$ and the distance of $x = 4$ from the right end point

$x = 5.76$ is $5.76 - 4 = 1.76$

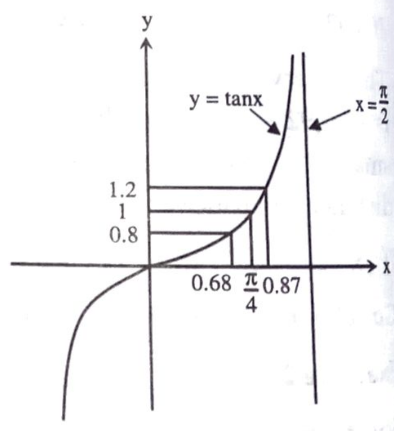
$\therefore \quad \delta$ = smallest of two numbers = 1.44 (or any positive smaller number)

Hence we can write $|x - 4| < 1.44$ then $|\sqrt{x} - 2| < 0.4$.

Example 4: Use a graph to find a number δ such that if

$$\left|x - \frac{\pi}{4}\right| < \delta \text{ then } |\tan x - 1| < 0.2.$$

Solution: Here the limit of $f(x) = \tan x$ at $x = \frac{\pi}{4}$ is 1



i.e. $L = 1$ and $a = \frac{\pi}{4}$, $\varepsilon = 0.2$, $\delta = ?$

Since $\quad |\tan x - 1| < 0.2$

$\Rightarrow \quad 0.8 < \tan x < 1.2$

Now the point of intersection of $y = 0.8$ and $y = \tan x$

$\Rightarrow \quad x = \tan^{-1}(0.8) \approx 0.674 \approx 0.68$

Again, solving $y = 1.2$ and $y = \tan x$

$x = \tan^{-1}(1.2) \approx 0.876 \approx 0.87$

Hence the value of x varies in the interval $\approx (0.68, 0.87)$

The points $x = 0.68$ and $x = 0.87$ are not at symmetric distance from $x = \frac{\pi}{4}$.

Hence the value of δ = smallest of $\left\{\frac{\pi}{4} - 0.68, 0.87 - \frac{\pi}{4}\right\} \approx \min\{0.105, 0.085\} = 0.08$ or any smaller positive number.

$\therefore \quad \delta = 0.091$ (or any smaller positive number).

Example 5: Prove that $\lim\limits_{x\to 3} 4x - 5 = 7$.

Solution: To verify this problem we need show that: for any ε (given positive number).

We have to find a positive number δ such that, if $0 < |x-3| < \delta$, then $|4x-5-7| < \varepsilon$

$\Rightarrow \quad |4x-12| < \varepsilon$

$\Rightarrow \quad |x-3| < \frac{\varepsilon}{4}$

$\therefore \quad \delta = \frac{\varepsilon}{4}$ (or any smaller positive number)

Conclusion: For any positive number ε, we can choose $\delta = \frac{\varepsilon}{4}$ such that $0 < |x-3| < \delta$

$\Rightarrow \quad |4x-5-7| < \varepsilon.$

$\therefore \quad \lim\limits_{x\to 3} 4x - 5 = 7.$

Definition: Left-Hand Limit

Left f(x) be a function of x, then a number L is called the left hand limit of f(x) at $x = a$ if, f(x) approach to L when x approaches to a with the value less than a. In this case we write $\lim\limits_{x\to a^-} f(x) = L$.

Precisely, if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a - \delta < x < a$ then $|f(x) - L| < \varepsilon$.

Definition: Right-hand Limit

Left f(x) be a function of x, then a number L is called the right hand limit of f(x) at x = a if, f(x) approaches to L when x approaches to a with the value greater than a. In this case we write $\lim\limits_{x\to a^+} f(x) = L$.

Precisely, if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $a < x < a + \delta$ then $|f(x) - L| < \varepsilon$.

Example 6: Use the definition: to prove $\lim\limits_{x\to 0+} \sqrt{x} = 0$.

Solution: Let ε be given positive number. Here a = 0 and L = 0, we want to find a number δ such that,

If $\quad 0 < x < \delta$ then $|\sqrt{x} - 0| < \varepsilon$

i.e. $\quad 0 < x < \delta$ then $\sqrt{x} < \varepsilon$

or squaring both sides of the inequality

We have of $0 < x < \delta$ then $x < \varepsilon^2$ which suggests that $\delta = \varepsilon^2$ (or any smaller positive number).

Now, observation (How this δ works for the definition of right hand limit)

Given $\varepsilon > 0$, let $\delta = \varepsilon^2$ of $0 < x < \delta$, then

$$\sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

So $|\sqrt{x} - 0| < \varepsilon$

$\therefore \lim_{x \to 0+} \sqrt{x} = 0$

Definition: Infinite Limits:

Let f be a function defined on some open interval that contains the number a, except possibly at itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that if

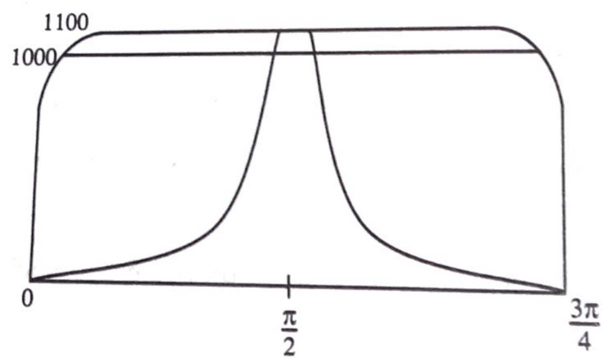
$0 < |x - a| < \delta$ then $|f(x)| > M$.

Example 7: Given that $\lim_{x \to \pi/2} \tan^2 x = \infty$. Illustrate the definition of infinite limits by finding values of δ that corresponds to $M = 1000$.

Solution.

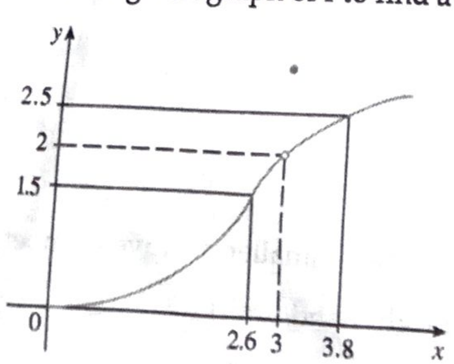
From the figure we can see that $y = \tan^2 x = 1000$ when $x \approx 1.539$ and $x \approx 1.602$ for x near $\frac{\pi}{2}$, thus we have $\delta \approx 1.602 - \frac{\pi}{2} \approx 0.031$ for $M = 1000$.

Since $0 < \left|x - \frac{\pi}{2}\right| < \delta$ then $\tan^2 x > 1000 \Rightarrow \tan x > \sqrt{1000} \Rightarrow x \approx 1.539$.

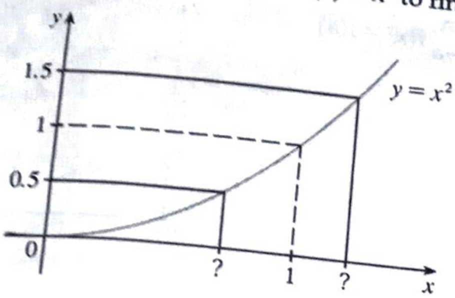


Exercise 2.1

1. Use the given graph of f to find a number δ such that if $0 < |x - 3| < \delta$ then $|f(x) - 2| < 0.5$.



2. Use the given graph of $f(x) = x^2$ to find a number δ such that if $|x - 1| < \delta$ then $|x^2 - 1| < \frac{1}{2}$.



3. Evaluate δ if $|x - 1| < \delta$ then $\left|\frac{2x}{x^2+1} - 0.4\right| < 0.1$.

4. For the limit $\lim_{x\to 2} (x^3 - 3x + 4) = 6$, illustrate the precise definition of limit by finding the values of δ that correspond to $\varepsilon = 0.2$ and $\varepsilon = 0.1$.

5. For the limit $\lim_{x\to 0} \frac{e^{2x} - 1}{x} = 2$. Illustrate the precise definition of limit by finding values of δ that corresponds to $\varepsilon = 0.5$ and $\varepsilon = 0.1$.

6. Given that $\lim_{x\to \pi/2} \tan^2 x = \infty$. Illustrate the definition of infinite limits by finding values of δ that corresponds to $M = 10000$.

7. A mechanist is required to manufacture a circular metal disk with area 100 cm^2.
   (a) What radius produces such a disk?
   (b) If the Mechanist is allowed an error tolerance of $\pm$ 5 cm^2 in the area of the disk, how close to the ideal radius in part (a) must the machinist control the radius?

8. (a) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$ (b) Find a number δ such that if $|x - 2| < \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.01$.

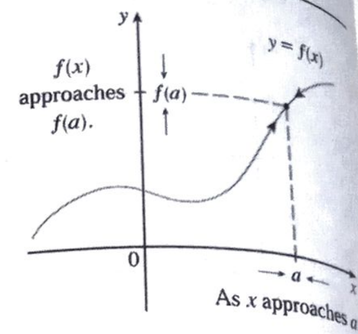
9. Given that $\lim_{x\to 2} (5x - 7) = 3$, illustrate definition of ε. δ by finding values of δ that corresponds to $\varepsilon = 0.1$, $\varepsilon = 0.05$ and $\varepsilon = 0.01$.

Answers:

1. 0.4 (or any smaller positive number)
2. 0.224 (or any smaller positive number)
3. $\frac{1}{3}$ (or any smaller positive number)
4. $\delta_1 = 0.02$, $\delta_2 = 0.01$
5. 0.2 and 0.05
6. a. 0.031 b. 0.010
7. (a) $r \approx 17.84$ (b) $\delta = 0.043$ (or any smaller positive number)
8. (a) 0.025 (b) 0.0025
9. $\delta = 0.02$, 0.01 and 0.002

2.2 Continuity

Definition: A function f(x) is continuous at x = a if $\lim_{x \to a} f(x) = f(a)$



[functional value = limiting value]

Precisely, a function f(x) is continuous at x = a if

(a) f(a) is defined (i.e. a is in the domain of f)

(b) $\lim_{x \to a} f(x)$ exists

(c) $\lim_{x \to a} f(x) = f(a)$

If any one of (a), (b), (c) is not true then the function f(x) is discontinuous at x = a.

If limit exists but not continuous then it is called removable discontinuity otherwise it is called irremovable discontinuity.

Definition: A function f is continuous functions from the right at a number a if $\lim_{x \to a+} f(x) = f(a)$

and f is continuous from the left at a if

$$\lim_{x \to a-} f(x) = f(a).$$

Definition: A function f is continuous on an interval if it is continuous at every number in the interval. If f is defined only one side of an end point of the interval, we understand continuous at the end point to mean continuous from the right or continuous from the left.

Theorem 1: If f and g are continuous at a and c is constant, then the following functions are also continuous at a.

(i) f + g (ii) f - g (iii) cg (iv) fg (v) $\frac{f}{g}$ if g (a) $\neq$ 0.

Proof (i):

Since f and g both are continuous at a then $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$.

$$\lim_{x \to a} (f + g)(x)$$

Now

$$= \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f + g)(a)$$

which shows that f + g is continuous at a.

Similarly we can show that f - g is continuous at a.

Proof (iii)

$$\lim_{x \to a} (cf)(x) = \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cf(a)$$

which shows that cf is continuous at a.

Proof (iv)

$$\lim_{x\to a}[fg](x) = \lim_{x\to a}[f(x)\,g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = f(a)\,g(a) = (fg)(a)$$

which shows that fg is continuous at a.

Proof (v)

$$\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$$

which shows that $\frac{f}{g}$ is also continuous at 'a' provided that $g(a) \neq 0$.

Theorem 2:

(a) **Any polynomial is continuous everywhere; that is, it is continuous on $R = (-\infty, \infty)$.**

(b) **Any rational function is continuous wherever it is defined; that is it is continuous on it's domain.**

Proof (a): Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \dots\dots + c_1 x + c_0$

Where $c_0, c_1, \dots, c_n$ are constants. We know that $\lim_{x\to a} c_0 = c_0$, and $\lim_{x\to a} x^m = a^m$, $m = 1, 2, \dots, n$. Since the function $f(x) = x^m$, $m = 0, 1, 2, \dots, n$ is a continuous function. Hence, the new function $g(x) = cx^m$ is also continuous at $x = a$, since $P(x)$, the sum of continuous functions and constant function (which is always continuous), is continuous at $x = a$ [$\because$ using theorem 1].

Proof (b): A rational function is a function of the form $f(x) = \frac{P(x)}{Q(x)}$ where P and Q are polynomials. The domain of f is $D = \{x \in R\}\ Q(x) \neq 0\}$. Since by part (a) both $P(x)$ and $Q(x)$ are continuous everywhere; and hence the quotient $\frac{P(x)}{Q(x)}$ is also continuous everywhere in their domain [using part (v) of theorem 1]

Example 1: Find $\lim_{x\to -2}\frac{x^3 + 2x^2 - 1}{5 - 3x}$

Solution.

Here the given function is rational, so by using theorem 2(b) it is continuous on it's domain which is $\{x : x \pm 5/3\}$. Therefore $\lim_{x\to -2} f(x) = f(-2) = \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3 \times (-2)} = \frac{1}{11}$.

Theorem 3: The following types of functions are continuous at every number in their domains: (a) Polynomials (b) Rational functions (c) Root functions (d) Trigonometric functions (e) Inverse trigonometric functions (f) Exponential functions (g) Logarithmic function.

Example 2: Where is the function $f(x) = \frac{\ln x + \tan^{-1}x}{x^2 - 1}$ continuous?

Solution. By using theorem 3 the function $y = \ln x$ is continuous for $x > 0$ and $y = \tan^{-1}x$ is continuous on **R**. Thus by part (i) of theorem 1

$y = \ln x + \tan^{-1}x$ is continuous on $(0, \infty)$.

Again the denominator $x^2 - 1$ being a polynomial is continuous by using theorem 2(a) everywhere. Now by using the part (v) of theorem 1 the quotient $f(x) = \dfrac{\ln x + \tan^{-1}x}{x^2 - 1}$ is continuous everywhere except where $x^2 - 1 = 0$. Therefore, f is continuous in $(0, 1) \cup (1, \infty)$.

Theorem 4: If f is continuous at b and $\lim\limits_{x \to a} g(x) = b$, then $\lim\limits_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Example 3: Use the property of continuous function to evaluate $\lim\limits_{x \to 1} \arcsin \dfrac{(1 - \sqrt{x})}{(1 - x)}$.

Solution.

Since arcsin is continuous in it's domain [using theorem 3] and $\lim\limits_{x \to 1} \dfrac{(1 - \sqrt{x})}{(1 - x)}$ exists so by using theorem 4 we have,

$$\Rightarrow \quad \lim_{x \to 1} \arcsin \frac{(1 - \sqrt{x})}{(1 - x)} = \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}\right)$$

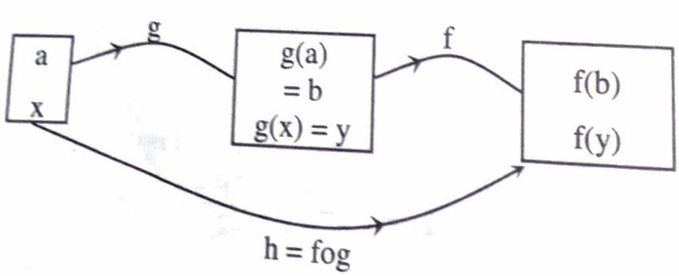
$$= \arcsin\left[\lim_{x \to 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})}\right]$$

$$= \arcsin\left(\lim_{x \to 1} \frac{1}{1 + \sqrt{x}}\right)$$

$$= \arcsin \frac{1}{2} = \frac{\pi}{6}$$

Theorem 5: If g is continues at a and f is continuous at g(a), then the composite function fog given by $(fog)(x) = f(g(x))$ is continuous at a.

Proof:



Since f is continuous at $g(a) = b$ then for every $\varepsilon > 0$ there exist $\delta' > 0$ such that $0 < |y - b| < \delta'$.

$|g(x) - g(a)| < \delta' \Rightarrow |f(y) - f(b)| < \varepsilon$

$|g(x) - g(a)| < \delta' \Rightarrow |f(g(x)) - f(g(a))| < \varepsilon$ ....(i)

Again g is continuous at a then for every $\delta' > 0$

$\exists\, \delta > 0$ such that $0 < |x - a| < \delta$

$|g(x) - g(a)| < \delta'$ ......(ii)

Using (i) and (ii) we have $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$

such that $0 < |x - a| < \delta \Rightarrow |f(g(x)) - f(g(a))| < \varepsilon$

$\Rightarrow |(fog)(x) - (fog)(a)| < \varepsilon$

which shows that h = fog is also continuous at a.

Theorem 6: The Intermediate Value Theorem

Suppose that f is continuous on the closed interval [a, b] and let N be any number between f(a) and f(b), where $f(a) \neq f(b)$ then there exists a number c in (a, b) such that f(c) = N.

The intermediate value theorem states that a continuous function takes on every intermediate values between the function values f(a) and f(b). It is illustrated by following figure. Note that the value N can be taken on once [as in fig. (a)] or more than once [as in fig. (b)].

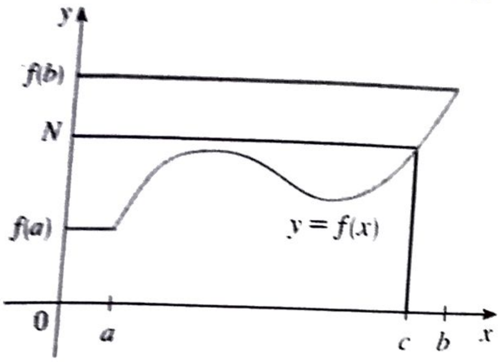


Fig. (a)

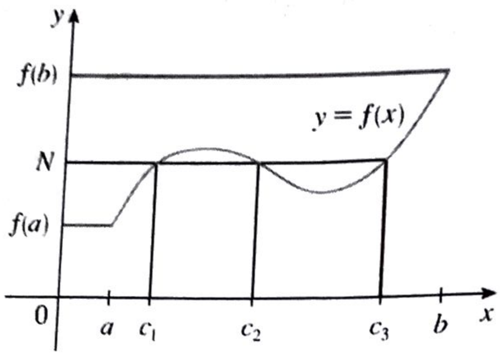


Fig. (b)

Remarks: If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the intermediate value theorem is true. In geometric terms it says that if any horizontal line y = N is given between y = f(a) and y = f(b) as in figure (c), then the graph of f can't jump over the line. It must intersect y = N somewhere.

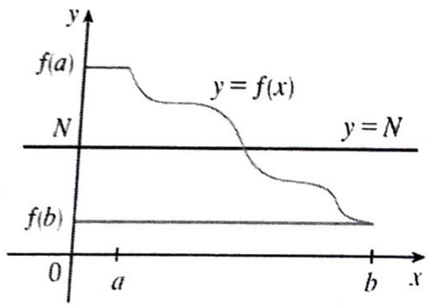


Fig. (c)

Note: One of the application of intermediate value theorem is to locate the roots of equations as in the following example.

Example 4: Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

Solution: Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solutions of the given equation, that is, a number c between 1 and 2 such that f(c) = 0. Therefore, we take a = 1, b = 2 and N = 0 in the intermediate value theorem. We have

$f(a) = f(1) = 4 - 6 + 3 - 2 = -1 < 0$

and $f(b) = f(2) = 32 - 24 + 6 - 2 = 12 > 0$

Thus $f(1) < 0 < f(2)$; that is $N = 0$ is a number between f(1) and f(2). By intermediate theorem there exists a number $c \in (1, 2)$.

Such that $f(c) = N = 0$, which shows that $x = c$ is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$.

Example 5: Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

a. $f(x) = \frac{2x+3}{x-2}$, $(2, \infty)$ b. $f(x) = 2\sqrt{3-x}$, $(-\infty, 3]$

Solution.

a. f(x) is continuous of x = a if and only if $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{2x+3}{x-2} = \frac{\lim_{x \to a} 2x+3}{\lim_{x \to a} x-2} \quad \text{[Division law of limit]}$$

$$= \frac{\lim_{x \to a} 2x + \lim_{x \to a} 3}{\lim_{x \to a} x - \lim_{x \to a} 2} \quad [\because \text{ Additional law of limits}]$$

$$= \frac{2\lim_{x \to a} x + \lim_{x \to a} 3}{\lim_{x \to a} x - \lim_{x \to a} 2} \quad [\because \text{ Constant law of limit}]$$

$$= \frac{2a+3}{a-2} \quad \left[\because \lim_{x \to a} x = a, \quad \lim_{x \to a} c = c\right]$$

$$= f(a)$$

Hence f(x) is continuous for all $a \neq 2$.

b. $f(x) = 2\sqrt{3-x}$, $(-\infty, 3]$

Polynomial and root functions are continuous over their domain. So, the polynomial $3 - x$ is continuous for all real numbers. So it is continuous and also ≥ 0 for the given interval.

The domains of root functions are the values of x for which the radicand is ≥ 0.

The composition of $3 - x$ and the root function will also be continuous by theorem 5.

Again, 2 can be consider as constant function which is also continuous. Since the product of two continuous functions is also continuous. Hence $f(x) = 2\sqrt{3-x}$ is continuous in the interval $(-\infty, 3]$.

Example 6: Explain using theorems 1, 2, 3, 5, why the function is continuous at every number in its domain state the domain.

(a) $f(x) = \frac{2x^2 - x - 1}{x^2 + 1}$ (b) $G(x) = \frac{x^2+1}{2x^2 - x - 1}$ (c) $R(t) = \frac{e^{\sin t}}{2 + \cos \pi t}$ (d) $B(x) = \frac{\tan x}{\sqrt{4 - x^2}}$

Solution.

and $f(b) = f(2) = 32 - 24 + 6 - 2 = 12 > 0$

Thus $f(1) < 0 < f(2)$; that is $N = 0$ is a number between f(1) and f(2). By intermediate theorem there exists a number $c \in (1, 2)$.

Such that $f(c) = N = 0$, which shows that $x = c$ is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$.

Example 5: Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

a. $f(x) = \dfrac{2x+3}{x-2}$, $(2, \infty)$

b. $f(x) = 2\sqrt{3-x}$, $(-\infty, 3]$

Solution.

a. f(x) is continuous of $x = a$ if and only if $\lim\limits_{x \to a} f(x) = f(a)$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{2x+3}{x-2} = \frac{\lim\limits_{x \to a} 2x+3}{\lim\limits_{x \to a} x-2} \quad \text{[Division law of limit]}$$

$$= \frac{\lim\limits_{x \to a} 2x + \lim\limits_{x \to a} 3}{\lim\limits_{x \to a} x - \lim\limits_{x \to a} 2} \quad [\because \text{ Additional law of limits}]$$

$$= \frac{2\lim\limits_{x \to a} x + \lim\limits_{x \to a} 3}{\lim\limits_{x \to a} x - \lim\limits_{x \to a} 2} \quad [\because \text{ Constant law of limit}]$$

$$= \frac{2a+3}{a-2} \quad \left[\because \lim_{x \to a} x = a, \quad \lim_{x \to a} c = c\right]$$

$$= f(a)$$

Hence f(x) is continuous for all $a \neq 2$.

b. $f(x) = 2\sqrt{3-x}$, $(-\infty, 3]$

Polynomial and root functions are continuous over their domain. So, the polynomial $3 - x$ is continuous for all real numbers. So it is continuous and also ≥ 0 for the given interval.

The domains of root functions are the values of x for which the radicand is ≥ 0.

The composition of $3 - x$ and the root function will also be continuous by theorem 5.

Again, 2 can be consider as constant function which is also continuous. Since the product of two continuous functions is also continuous. Hence $f(x) = 2\sqrt{3-x}$ is continuous in the interval $(-\infty, 3]$.

Example 6: Explain using theorems 1, 2, 3, 5, why the function is continuous at every number in its domain state the domain.

(a) $f(x) = \dfrac{2x^2 - x - 1}{x^2 + 1}$ (b) $G(x) = \dfrac{x^2+1}{2x^2 - x - 1}$ (c) $R(t) = \dfrac{e^{\sin t}}{2 + \cos \pi t}$ (d) $B(x) = \dfrac{\tan x}{\sqrt{4 - x^2}}$

Solution.

a. $f(x) = \frac{2x^2 - x - 1}{x^2 + 1}$

This function is rational so it si continuous on it's domain; theorem 3 states that all rational functions are continuous on their domain. The domain of f(x) is every value x for which $x^2 + 1 \neq 0$; $x^2 + 1$ is always greater than 0, so the domain of f(x) is $(-\infty, \infty)$.

b. $G(x) = \frac{x^2 + 1}{2x^2 - x - 1}$

By using the similar argument of theorem 2 the functions $x^2 + 1$ and $2x^2 - x - 1$ are continuous in $(-\infty, \infty)$ but the quotient function $G(x) = \frac{x^2 + 1}{2x^2 - x - 1}$ is discontinuous at the points for which $2x^2 - x - 1 = 0$ i.e.

$2x^2 - 2x + x - 1 = 0$

or, $2x(x - 1) + (x - 1) = 0$

$\therefore$ $x = -½, 1$

Hence the domain of G(x) is $D = (-\infty, -1/2) \cup (-1/2, 1) \cup (1, \infty)$.

$\therefore$ The function G(x) is continuous in the domain D.

c. $R(t) = \frac{e^{\sin t}}{2 + \cos\pi t}$

Here $e^{\sin t}$ being an exponential function is continuous everywhere in $(-\infty, \infty)$ (using theorem 3). The function $2 + \cos\pi t$ is the sum of two continuous functions is continuous in $(-\infty, \infty)$. Hence the quotient function $R(t) = \frac{e^{\sin t}}{2 + \cos\pi t}$ is continuous in its domain (using theorem 1).

Now the domain of R(t)

The function R(t) is not defined for

$2 + \cos\pi t = 0$, since, $-1 < \cos\pi t < 1$, i.e. $\cos\pi t$ never gives -2 and hence $2 + \cos\pi t$ cannot be zero in R. Hence R(t) is continuous in it's domain $D = (-\infty, \infty)$.

d. $B(x) = \frac{\tan x}{\sqrt{4 - x^2}}$

Here $B(x) = \frac{\tan x}{\sqrt{4 - x^2}}$

Both tan x and $\sqrt{4 - x^2}$ are continuous in their domain using theorem 3. Again by using theorem (1) the quotient $\frac{\tan x}{\sqrt{4 - x^2}}$ of two continuous functions is also continuous for $x \neq \pm 2$ including their domain.

Now tanx has domain of all real numbers except $\frac{\pi}{2} + n\pi$, where n is any integer.

$\sqrt{4 - x^2}$ has domain $[-2, 2]$ since any numbers outside $[-2, 2]$ will make $4 - x^2$ negative.

hence the domain of B(x) is the combination of the portions of their individual domains that wrote for both functions, and also excluding values that make the denominator 0. So we must exclude -2 and 2. Also $\pm\frac{\pi}{2} \neq \pm 1.57$ falls within (-2, 2). So we exclude those numbers. Hence $D = \left(-2, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, 2\right)$.

Example 7: Show that the function

$f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval [-1, 1].

Solution. Let $a \in (-1, 1)$ i.e. $-1 < a < 1$ then

$$\lim_{x \to a} f(x) = \lim_{x \to a} (1 - \sqrt{1 - x^2})$$

$$= \lim_{x \to a} 1 - \lim_{x \to a} \sqrt{1 - x^2} = 1 - \sqrt{1 - a^2} = f(a)$$

which shows that f(x) is continuous at $x = a \in (-1, 1)$.

For the end points i.e. $x = -1$

$$\lim_{x \to -1+} f(x) = 1 - \sqrt{1 - (4)^2}$$

$$= f(-1) = 1 = f(-1)$$

Continuous from right at the left end point $x = -1$.

Similarly for the end point $x = 1$

$$\lim_{x \to 1-} f(x) = 1 - \sqrt{1 - 1}$$

$$= 1 = f(1)$$

which shows that f(x) is continuous from the left at $x = 1$.

Hence f(x) is continuous at [-1, 1].

Example 8: Evaluate $\lim_{x \to 3} \frac{x^2 + 5x + 6}{2x - 1}$. Hence show that $f(x) = \frac{x^2 + 5x + 6}{2x - 1}$ is continuous everywhere in it's domain which does not contains $x = \frac{1}{2}$.

Solution: $f(x) = \frac{g(x)}{h(x)} = \frac{x^2 + 5x + 6}{2x - 1}$.

Here both $x^2 + 5x + 6$ and $2x - 1$ are continuous everywhere in their domain and hence the quotient function f(x) is also continuous in it's domain ($x \neq ½$).

Example 9: $f(x) = \begin{cases} \frac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$ a = 3. Check the continuity of f(x) at x = 3. [Using theorem 2(a) and theorem 1(v)]

Solution: $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 + 5x + 6}{2x - 1}$

$$= \lim_{x\to 3} \frac{x^2 - 6x + x - 3}{x - 3}$$

$$= \lim_{x\to 3} \frac{2x(x - 3) + 1(x - 3)}{x - 3}$$

$$= \lim_{x\to 3} 2x + 1$$

$$= 7$$

Functional value $f(3) = 6$.

Here $\lim_{x\to 3} f(x) \neq f(3)$

Hence $f(x)$ is discontinuous at $x = 3$.

This type of discontinuity is called removable discontinuous. Because we can make the function continuous by redefining $f(x) = 7$ for $x = 3$.

Example 10: Use continuity to evaluate limit.

(a) $\lim_{x\to 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}}$

Solution:

Since the function $f(x) = \frac{g(x)}{h(x)}$ is being a quotient of two continuous functions $g(x) = 5 + \sqrt{x}$ and $h(x) = \sqrt{5 + x}$ everywhere in their domain. In particular at $x = 4$ and hence the quotient function $f(x)$ is also continuous at $x = 4$.

$$\lim_{x\to 4} f(x) = f(4) = \frac{5 + \sqrt{4}}{\sqrt{5 + 4}} = \frac{7}{3}. \quad \text{[Using theorem 3(c) and 1(v)]}$$

$$\therefore \quad \lim_{x\to 4} \frac{5 + \sqrt{4}}{\sqrt{5 + 4}} = \frac{7}{3}.$$

(b) $\lim_{x\to 2} \arctan\left(\frac{x^2 - 4}{3x^2 - 6x}\right)$

Since arc tan is a continuous function is it's domain and $\lim_{x\to 2} \frac{x^2 - 4}{3x^2 - 6x}$ exists. Using the theorem 4,

$\arctan\left(\frac{x^2 - 4}{3x^2 - 6x}\right)$ is continuous at $x = 2$.

Hence $\lim_{x\to 2} \tan^{-1}\left(\frac{x^2 - 4}{3x^2 - 6x}\right)$

$$= \tan^{-1}\left[\lim_{x\to 2} \frac{(x - 2)(x + 2)}{3x(x - 2)}\right]$$

$$= \tan^{-1}\left(\frac{4}{6}\right) = \tan^{-1}\left(\frac{2}{3}\right).$$

Example 11: Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$

(b) $F(x) = \ln(1 + \cos x)$

Solution:

(a) Let $g(x) = x^2$, $f(x) = \sin x$; here both $f(x)$ and $g(x)$ are continuous everywhere in R and hence their composite function $\sin(x^2)$ is also continuous in $(-\infty, \infty)$.

(b) Since $f(x) = \ln x$ is continuous in its domain and $g(x) = 1 + \cos x$ is also continuous, then $\ln(1 + \cos x)$ is the composite of two continuous function is continuous wherever it is defined.

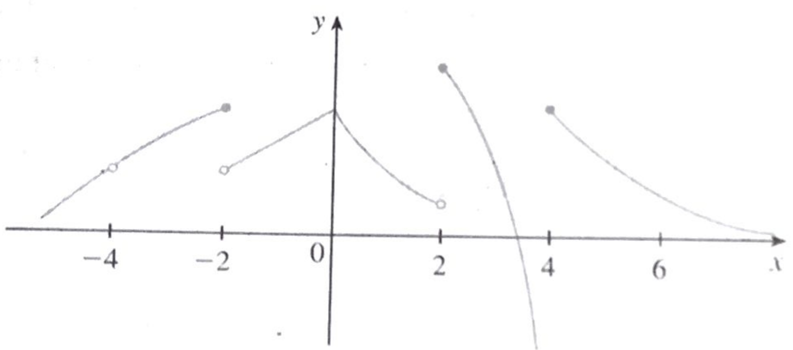
Exercise 2.2

1. Use the continuity to evaluate $\lim\limits_{x \to \pi} \dfrac{\sin x}{2 + \cos x}$.

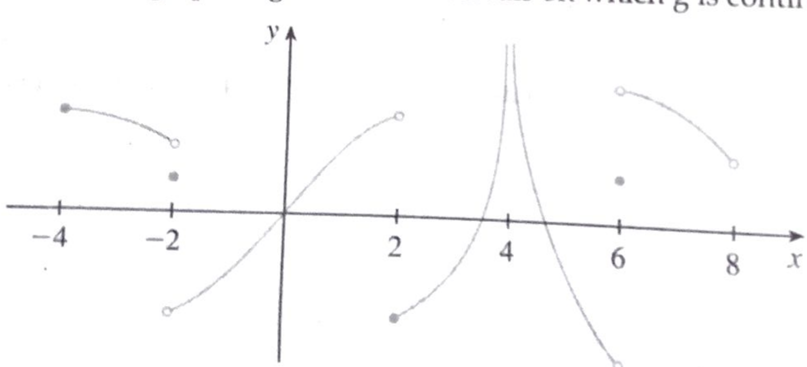
2. If f is continuous on $(-\infty, \infty)$, what can you say about it's graph?

3. (a) From the graph of f. State the numbers at which f is discontinuous and explain why.

   (b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left.



4. From the graph of g. State the intervals on which g is continuous.



Sketch graph of a function f that is continuous expect for the stated discontinuity.

a. Discontinuous, but continuous from the right at 2.

b. Removable discontinuity at 3, Jump discontinuing at 5.

5. Use the definition of continuity and the properties of limits to show that the function is continuous at the given number *a*.

(i) $f(x) = 3x^4 - 5x + 3\sqrt{x^2+4}$, $a = 2$ (ii) $f(x) = (x + 2x^3)^4$, $a = -1$

(iii) $h(t) = \dfrac{2t - 3t^2}{1 + t^3}$, $a = 1$

6. Explain, using the theorem of continuous functions why the functions are continuous of every number in it's domain. State the domain.

a. $G(x) = \dfrac{x^2 + 1}{2x^2 - x - 1}$ b. $f(x) = \dfrac{\sqrt[3]{x-2}}{x^3 - 2}$

Answers:

1. 0

3. (a) f(−4) is not defined and $\lim\limits_{x \to a} f(x)$ for a = − 2.2 and 4 does not exists (b) −4, neither; − 2 left; 2, right; 4, right

6. (a) $D = (-\infty, \infty) - \{1, -1/2\}$ (b) $D = (-\infty, \sqrt[3]{2}) \cup (\sqrt[3]{2}, \infty)$

2.3 Limit at Infinity: Horizontal Asymptotes

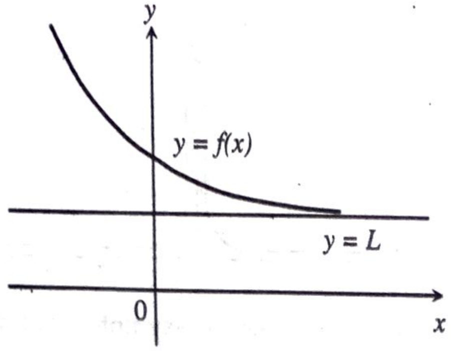
In this section we discuss about the long term behavior of functions i.e. what happens as x gets really big (positive or negative); sometimes the function will approach a specific number as x gets big. If a function f approaches a specific number L as x gets larger and larger (positive), we say that the limit f(x) as x approaches infinity is L and write,

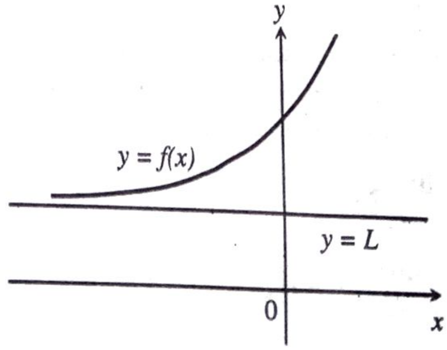
$$\lim_{x \to \infty} f(x) = L$$

Similarly, if f approaches L as x becomes larger and larger negative,

$$\lim_{x \to -\infty} f(x) = L$$

In general, we use the notation $\lim\limits_{x \to \infty} f(x) = L$ to indicate that the values of f(x) approaches L as x becomes larger and larger.





Another notation for $\lim\limits_{x \to \infty} f(x) = L$ is $f(x) \to L$ as $x \to \infty$.

Definition: The line y = L is called a horizontal asymptote of the curve y = f(x) if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to -\infty} f(x) = L.$$

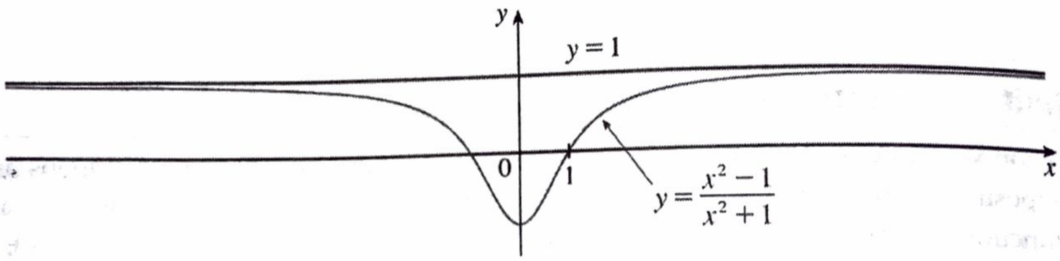
In the first case, the graph of f(x) flattens out on the right, and it approaches the line y = L. In the second case, the graph flattens out on the left, approaching to the line y = L. In either case this line is called a horizontal asymptote of f(x).

The following example illustrates the definition of limit at infinity and horizontal asymptote of the curve y = f(x).

$$f(x) = \frac{x^2-1}{x^2+1}$$

As x grows larger and larger we can see that the values of f(x) get closer and closer to 1. In fact, it seems that we can make the values of f(x) as close as we like to 1 by taking x sufficiency (may be negative or positive). This situation is expressed symbolically by writing

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$$



Example 1: Find the horizontal asymptotes of the curve y = $\tan^{-1}x$.

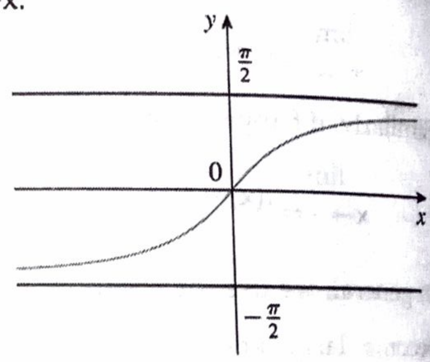
Solution: Now

$$\lim_{x\to\infty} \tan^{-1}x = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$\therefore$ $y = \frac{\pi}{2}$ is a horizontal asymptote.

Again $\lim_{x\to -\infty} \tan^{-1}x = \tan^{-1}(-\infty) = -\frac{\pi}{2}$ which shows that

$y = -\frac{\pi}{2}$ is also another horizontal asymptotes of the same function y = $\tan^{-1}x$.

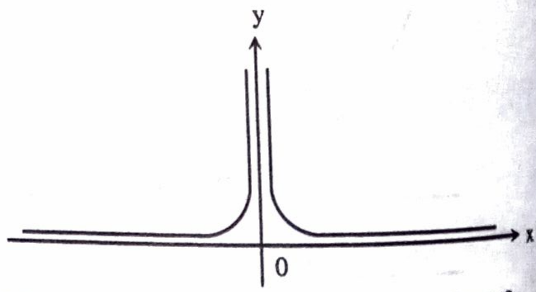


Definition of vertical asymptote: Let f(x) be a given function. If there exists a number a such that any one of the following is true;

a. $\lim_{x\to a} f(x) = \infty$ or $-\infty$

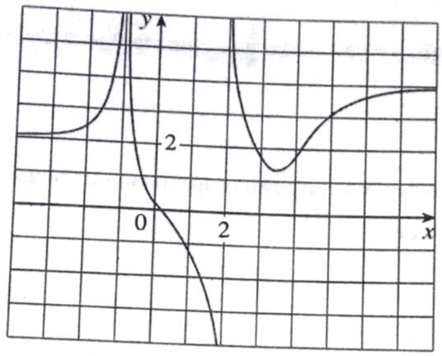
b. $\lim_{x\to a^-} f(x) = \infty$ or $-\infty$

c. $\lim_{x\to a^+} f(x) = \infty$ or $-\infty$



Here x = 0 is vertical asymptote and y = 0 is horizontal asymptote

Example 2: Find the infinite limits, limits at infinity and asymptotes for the function f whose graph is figure.



Solution: We see that the values of f(x) becomes large as $x \to -1$ from both sides, so $\lim\limits_{x \to -1} f(x) = \infty$.

Notice that f(x) becomes large negative as x approaches 2 from the left, but large positions as x approaches 2 from the right. So

$$\lim_{x \to 2-} f(x) = -\infty \text{ and } \lim_{x \to 2+} f(x) = \infty$$

∴ Both the lines $x = -1$ and $x = 2$ has infinite limits so are the vertical asymptotes.

Again, as x becomes large, it appears that f(x) approaches 4. But as x decreases through negative values, f(x) approaches to 2. So

$$\lim_{x \to \infty} f(x) = 4 \text{ and } \lim_{x \to -\infty} f(x) = 2.$$

∴ $y = 4$ and $y = 2$ are the limits at infinity and hence are the horizontal asymptotes.

Remarks:

(1) If for a given function f(x) and $x = a$ one of the following

(i) $\lim\limits_{x \to a} f(x) = \infty$ or $-\infty$ (ii) $\lim\limits_{x \to a-} f(x) = \infty$ or $-\infty$

(iii) $\lim\limits_{x \to a+} f(x) = \infty$ or $-\infty$

is true then the limit is called infinite limits and the line $x = a$ is called vertical asymptotes.

(2) If for a given function f(x) and if either $\lim\limits_{x \to -\infty} f(x) = L$ or $\lim\limits_{x \to \infty} f(x) = L_1$ then the lines $y = L$ or $y = L_1$ are called the horizontal asymptote of the curve. If both cases hold good then both $y = L$ and $y = L_1$ are called limits at infinity and the lines $y = L$ and $y = L_1$ are called horizontal asymptotes of the curves.

(3) Infinite limits claim vertical asymptotes and limits at infinity claim horizontal asymptotes.

Example 3: Find $\lim\limits_{x \to \infty} \dfrac{1}{x}$ and $\lim\limits_{x \to -\infty} \dfrac{1}{x}$

Here we can observe that when x is larger then $\frac{1}{x}$ becomes smaller. For instance $\frac{1}{100} = 0.01$

$\frac{1}{10{,}000} = 0.0001, \frac{1}{1{,}000{,}000} = 0.000001$

In fact, by taking x large enough, we can make $\frac{1}{x}$ as close to 0 as we please. Therefore we have

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large in magnitude but negative in sign then $\frac{1}{x}$ is small in magnitude but negative in sign i.e.

$$\lim_{x\to -\infty} \frac{1}{x} = 0$$

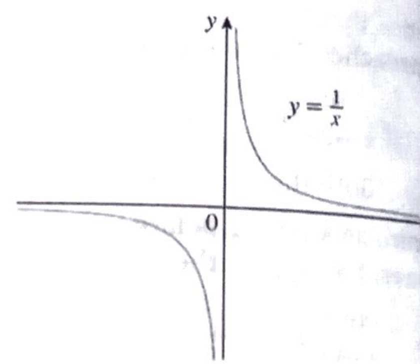
It follows that the line y = 0 (x-axis) is a horizontal asymptote of the curve $y = \frac{1}{x}$ (which is an equilateral by parabola).

Again in this problem we can observe that for

$$\lim_{x\to 0+} \frac{1}{x} = \infty \text{ and } \lim_{x\to 0-} = -\infty$$

Hence x = 0 is the vertical asymptote.

Moreover y = 0 is called the limit at infinity and x = 0 is gives the infinite limit for the given function.



Theorem: If r > 0 is a rational number then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

If r > 0 is a rational number such that x^n is defined for all x, then $\lim_{x\to -\infty} \frac{1}{x^r} = 0$

Example 4: Find the horizontal and vertical asymptotes of the graph of the function $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$

Solution: Here, $\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{3}\right)} = \frac{\sqrt{2}}{3}$

∴ $y = \frac{\sqrt{2}}{3}$ is the limit at infinity and a horizontal asymptote of the curve.

In computing the limit as $x \to -\infty$, we must remember for x < 0, we have $\sqrt{x^2} = |x| = -x$. So when we divide the numerator by x, for x < 0 we get

$$\frac{1}{x}\sqrt{2x^2+1} = -\frac{1}{\sqrt{x^2}}\sqrt{2x^2+1} = -\sqrt{2+\frac{1}{x^2}}$$

Therefore,

$$\lim_{x\to -\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to -\infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = -\frac{\sqrt{2}}{3}$$

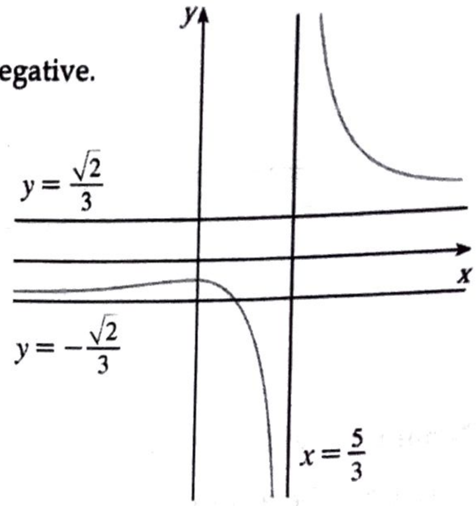
Thus the line $y = -\frac{\sqrt{2}}{3}$ is also limit at infinity and a horizontal asymptote of the curve.

For the vertical asymptote we can test at the point for which the denominator is zero. For $x = \frac{5}{3}$. If x is close to $\frac{5}{3}$ and $x > \frac{5}{3}$ then the denominator is close to 0 and positive. The numerator is positive. So f(x) becomes positive. Therefore $\lim\limits_{x\to 5/3+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$.

If x is close to $\frac{5}{3}$ but $x < \frac{5}{3}$ then $3x - 5 < 0$ and so f(x) is large negative.

$\therefore \lim\limits_{x\to 5/3-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty$.

$\therefore$ The vertical asymptote is $x = \frac{5}{3}$



Note: $\lim\limits_{x\to\infty} e^x = \infty$, $\lim\limits_{x\to -\infty} e^x = 0$, $\lim\limits_{x\to\infty} e^{-x} = 0$, $\lim\limits_{x\to -\infty} e^{-x} = \infty$

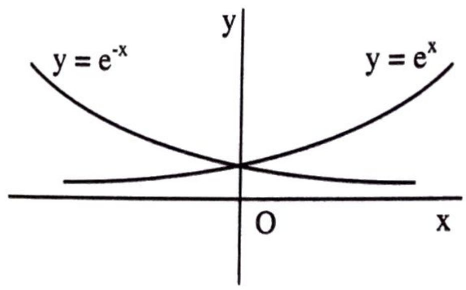
which is seen in the figure below.

$\therefore \lim\limits_{x\to\infty} e^{-x} = 0$

i.e. y = 0 is the limit at infinity and hence is horizontal asymptote of $y = e^{-x}$.

Similarly y = 0 is the limit at infinity for the function $y = e^{+x}$ and hence is horizontal asymptote of $y = e^x$.

$\therefore$ y = 0 is the common horizontal asymptote of the curve $y = e^x$ and $y = e^{-x}$.



Definition: Infinite Limits at Infinity

Let f(x) be a given function of x then the notation $\lim\limits_{x\to\infty} f(x) = \infty$ is used to indicate that the values of f(x) become large as x becomes large. Similarly $\lim\limits_{x\to -\infty} f(x) = \infty$, $\lim\limits_{x\to\infty} f(x) = -\infty$, $\lim\limits_{x\to -\infty} f(x) = -\infty$.

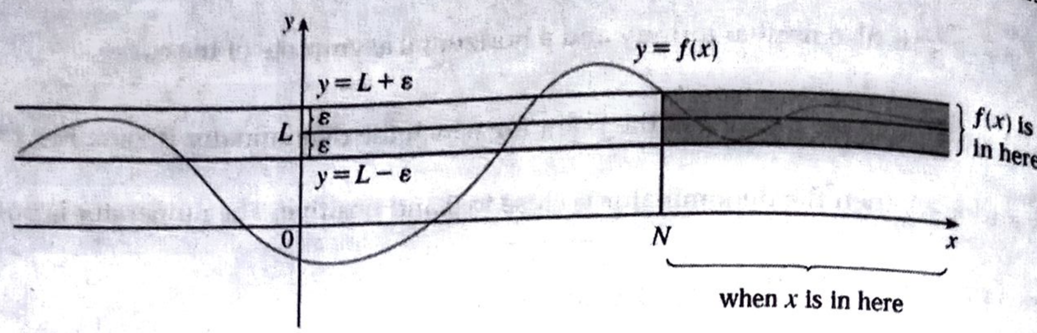
Above mentioned all limits are called infinite limits at infinity. For example $\lim\limits_{x\to\infty} e^x = \infty$, $\lim\limits_{x\to -\infty} e^{-x} = \infty$ are called infinite limits at infinity.

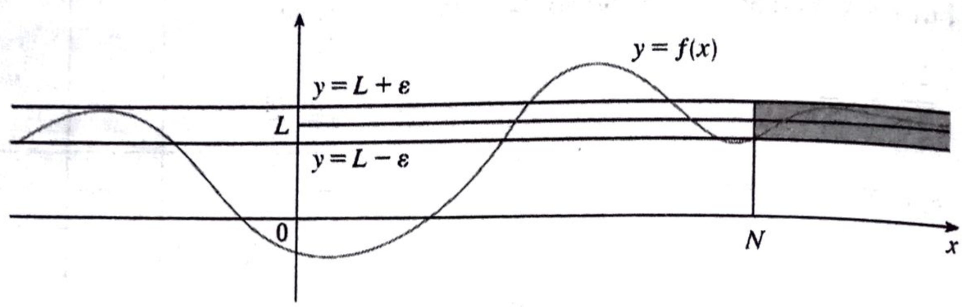
Similarly $\lim\limits_{x\to\infty} x^3$ and $\lim\limits_{x\to -\infty} x^3$ are also called infinite limits at infinity.

Precise Definition of Limit at Infinity

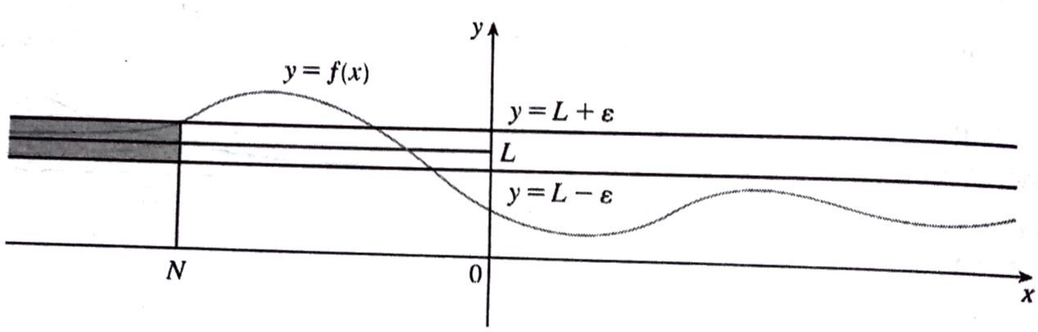
Let f be a function defined on some interval (a, ∞). Then $\lim\limits_{x\to\infty} f(x) = L$ means that for every $\varepsilon > 0$ there corresponds a number N sufficiently large positive number such that

If $x > N$ then $|f(x) - L| < \varepsilon$.





Definition: Let f be a function defined on some interval $(-\infty, \infty)$. Then $\lim_{x \to -\infty} f(x) = L$ means for every $\varepsilon > 0$ there is a corresponding N sufficiently large in magnitude but negative in sign such that of $x < N$ then $(f(x) - L) < \varepsilon$.



Example 5: Use a graph to find a number N such that if $x > N$ then $\left|\frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6\right| < 0.1$.

Here, L = 0.6 and the problem is

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = 0.6, N = ?$$

Since $-0.1 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 < 0.1$

$$\Rightarrow \quad 0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

Solving y = 0.5 and $y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$ we have x = 6.7. To the right of this number it seems that the curve stays between the lines y = 0.5 and 0.7. Rounding to be safe, we can say that

If $x > 7$ then $\left|\frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6\right| < 0.1$

In other words, for $\varepsilon = 0.1$, we can choose $N = 7$ or any larger number.

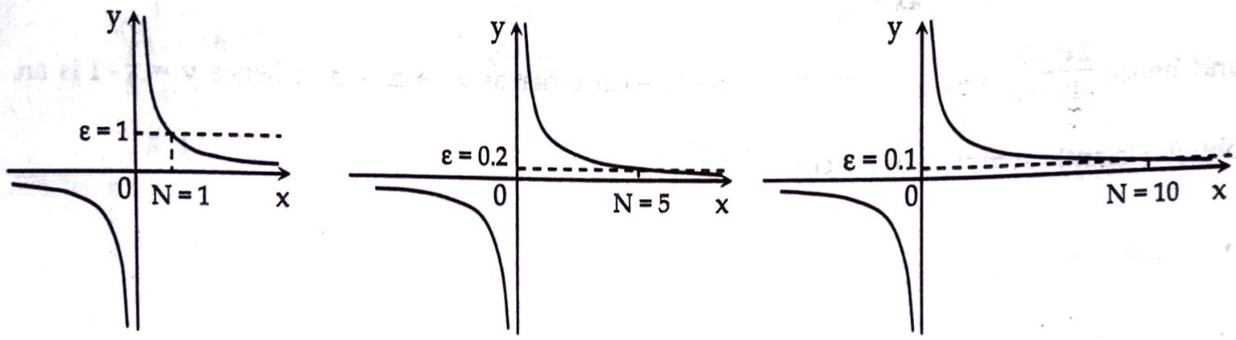
Example 6: Use definition limit at infinity to prove that: $\lim\limits_{x \to \infty} \frac{1}{x} = 0$

Solution: Given $\varepsilon > 0$, we want to find N such that if $x > N$ then $\left|\frac{1}{x} - 0\right| < \varepsilon$.

In computing this limit we may assume that $x > 0$. Then $\frac{1}{x} < \varepsilon \Leftrightarrow x > \frac{1}{\varepsilon}$. Lets choose $N = \frac{1}{\varepsilon}$. So

If $x > N = \frac{1}{\varepsilon}$ then $\left|\frac{1}{x} - 0\right| = \frac{1}{x} < \varepsilon$.

From the definition $\lim\limits_{x \to \infty} \frac{1}{x} = 0$.



Definition: Let f be a function defined on some interval (a, ∞). Then $\lim\limits_{x \to \infty} f(x) = \infty$

means that for every positive number M there corresponds a positive number N such that of $x > N$ then $f(x) > M$.

Definition: Let f be a function define on some interval $(-\infty, a)$. Then $\lim\limits_{x \to -\infty} f(x) = -\infty$

means that for every negative number M there corresponds a negative number N such that of $x < N$ then $f(x) < M$.

In this case we write $\lim\limits_{x \to -\infty} f(x) = -\infty$.

Slant (Oblique) Asymptote

Let $f(x) = \frac{p(x)}{d(x)}$ be a rational function. The function f(x) will have a horizontal asymptote of the degree of the numerator p is less than or equal to the degree of the denominator d. In particular, if the degree of p is strictly less than that of d, then the x-axis will be the horizontal asymptote. The geometrical condition that can be expressed analytically by saying $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$.

If the degree of p is greater than or equal to the degree of d, then long division use to obtain $p(x) = d(x) . q(x) + r(x)$ where q(x) is a quotient and r(x) is a remainder such that $\deg r(x) < \deg d(x)$.

Now we have $f(x) = q(x) + \frac{r(x)}{d(x)}$. Because of the degree condition on r, it is clear that $\frac{r(x)}{d(x)}$ approaches zero as $x \to \pm \infty$, so that f(x) and q(x) are close to each other when x is large. Thus the graph of the rational function f(x) is asymptote to the graph of the polynomial function q(x) as $x \to \pm\infty$. In other words, the two graphs are close to each other as $x \to \infty$ and as $x \to -\infty$. In the special case where the degree of p is one more than the degree of d, the quotient is a linear function, whose graph is a non-horizontal and non-vertical line in the plane. That line is called an oblique or slant asymptote to the curve of the particular rational function.

Examples 7: (i) $y = \frac{x^2 - 3x/2}{2x + 1} = \frac{p(x)}{d(x)}$

Here deg p(x) is one more than degree of d(x) so it has slant asymptote.

Since $\frac{2x^2 - 3x}{4x + 2} = \frac{2x^2 - 3x}{4x + 2} = \frac{x}{2} - 1 + \frac{2}{4x + 2}$

Taking $x \to \pm\infty$ we have $\frac{2}{4x + 2} \to 0$

and hence $\frac{2x^2 - 3x}{4x + 2}$ and $\frac{x}{2} - 1$ are very close to each other as $x \to \pm\infty$ and hence $y = \frac{x}{2} - 1$ is an oblique (slant) asymptote of the curve $\frac{x^2 - 3x/2}{2x + 1}$.

In addition, since, $\lim_{x \to -1/2} \frac{x^2 - 3x/2}{2x + 1} = \infty$

Hence the line $x = -\frac{1}{2}$ is a vertical asymptote.

Example 8: $y = \frac{(1 - x)^3}{x^2} = \frac{p(x)}{d(x)}$

Solution: Here the degree of p(x) is exactly 1 more of degree of d(x) so it has slant asymptotes.

Since $\frac{(1 - x)^3}{x^2} = (3 - x) + \frac{1 - 3x}{x^2}$

Taking $x \to \pm\infty$, $\frac{1 - 3x}{x^2} \to 0$ and hence $y = \frac{(1 - x)^3}{x^2}$ approaches with the straight line $y = 3 - x$ as $x \to \pm\infty$.

Hence $y = 3 - x$ is an oblique asymptote more over we can see that for $\lim_{x \to 0} \frac{(1 - x)^3}{x^2} = \infty$, hence $x = 0$ y-axis is an vertical asymptote of the curve.

Exercise 2.3

1. Explain in your own words the meaning of each of the following

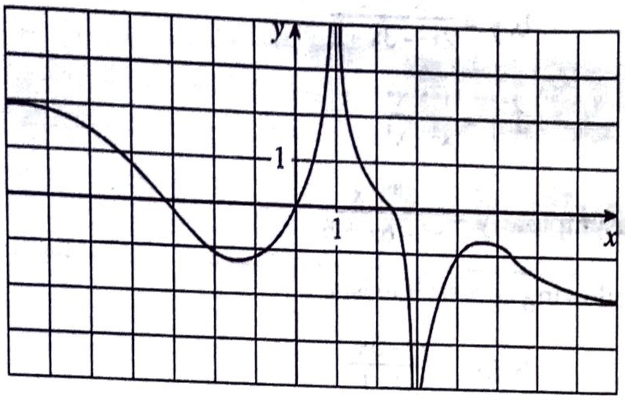
   (a) $\lim_{x \to \infty} f(x) = 5$ (b) $\lim_{x \to \infty} f(x) = 3$

2. For the function f whose graph is given, state the following:

   (a) $\lim_{x \to \infty} f(x)$ (b) $\lim_{x \to -\infty} f(x)$ (c) $\lim_{x \to 1} f(x)$

(d) $\lim_{x \to 3} f(x)$

(e) the equation of the asymptotes?



3. For the function of whose graph is given, state the following:

(a) $\lim_{x \to \infty} g(x)$

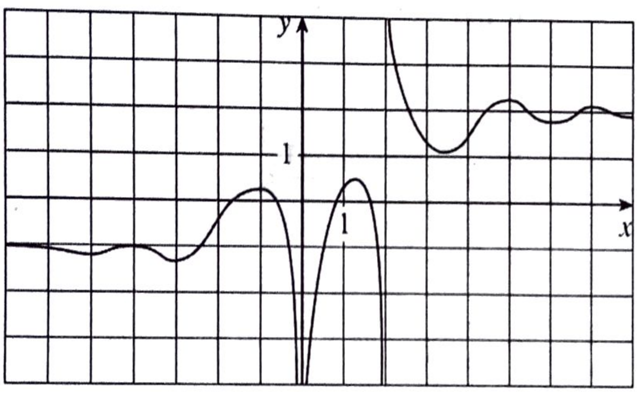
(b) $\lim_{x \to -\infty} g(x)$

(c) $\lim_{x \to 0} g(x)$

(d) $\lim_{x \to 2^-} g(x)$

(e) $\lim_{x \to 2^+}$

(f) the equations of asymptotes



4. Sketch the graph of an example of a function f that satisfies all of the given conditions

(a) $\lim_{x \to 0} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = 5$, $\lim_{x \to \infty} f(x) = -5$

5. Guess the value of limit $\lim_{x \to \infty} \frac{x^2}{2^x}$ by evaluating the function $f(x) = \frac{x^2}{2^x}$ for x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50 and 100. Then use the graph to support your guess.

6. Evaluate the limit and justify each step by indicating the properties of limits

a. $\lim_{x \to \infty} \frac{3x^2 - x + 14}{2x^2 + 5x - 8}$

b. $\lim_{x \to \infty} \sqrt{\frac{12x^3 - 5x + 2}{1 + 4x^2 + 3x^3}}$

7. Find the limit or show that it does not exists:

a. $\lim_{x \to \infty} \frac{3x - 2}{2x + 1}$

b. $\lim_{t \to \infty} \frac{\sqrt{t} + t^2}{2t - t^2}$

c. $\lim_{x \to \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$

d. $\lim_{x \to \infty} (\sqrt{9x^2 + x} - 3x)$

e. $\lim_{x \to \infty} \arctan (e^x)$

8. Find the horizontal and vertical asymptotes of each curve.

a. $y = \frac{2x+1}{x-2}$ b. $y = \frac{x^2+1}{2x^2-3x-2}$

c. $y = \frac{2x^2+x-1}{x^2+x-2}$ d. $y = \frac{1+x^4}{x^2-x^4}$

9. Find the vertical and horizontal asymptote: $y = \frac{x^3-x}{x^2-6x+5}$

10. Find the slant asymptote of the following curves: if exists.

a. $f(x) = \frac{2x^2}{1-x}$ b. $f(x) = \frac{x^3-3x^2}{x^2-1}$

c. $f(x) = \frac{(2+x)(2-3x)}{(2x+3)^2}$ d. $f(x) = \frac{x^3-1}{x^2-x-2}$

e. $f(x) = \frac{x^2-2x}{x^3+1}$ f. $f(x) = \frac{1-x^3}{x}$

g. $f(x) = \frac{x^3-1}{2(x^2-1)}$ h. $f(x) = \frac{x^4-2x^3+1}{x^2}$

11. Evaluate the following limits:

a. $\lim_{x\to\infty} \sqrt{x^2+1} - x$ b. $\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)$

c. $\lim_{x\to 0^-} e^{1/x}$ d. $\lim_{x\to\infty} \sin x$

e. $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$ f. $\lim_{x\to\infty} (x^2 - x)$

g. $\lim_{x\to\infty} \frac{x^2+x}{3-x}$

Answers:

1. (a) As x increases toward positive infinity, the value of f(x) becomes very close to 5 yet never reaches it.
2. (a) $\lim_{x\to\infty} f(x) = -2$ (b) $\lim_{x\to-\infty} f(x) = 2$ (c) $\lim_{x\to 1} f(x) = \infty$ (d) $\lim_{x\to 3} f(x) = -\infty$ (e) $x = 1, x = 3, y = 2, y = -2$
3. (a) 2 (b) -1 (c) $-\infty$ (d) $-\infty$ (e) ∞ (f) $x = 0, x = 2, y = -1, y = 2$
6. (a) 3/2 (b) 2
7. (a) 3/2 (b) -1 (c) 3 (d) 1/6 (e) $\pi/2$
8. (a) Vertical asymptote; $x = 2$, Horizontal asymptote $y = 2$
   (b) Vertical asymptote; $x = -1/2, x = 2$, Horizontal asymptote $y = 1/2$
   (c) Vertical asymptote; $x = 1, x = -2$, Horizontal asymptote $y = 2$
   (d) Vertical asymptote; $x = 0, x = 1, x = -1$, Horizontal asymptote $y = -1$
9. Vertical asymptote; $x = 5$, No horizontal asymptote
10. (a) $y + 2x + 2 = 0$ (b) $y = x - 3$ (c) Does not exist (d) $y = x + 1$ (e) Does not exist (f) $y = -x^2$ (g) $y = \frac{x}{2}$ (h) $y^2 = x^2 - 2x$
11. (a) 0 (b) $\pi/2$ (c) 0 (d) Does not exist (e) $\infty, -\infty$ (f) ∞ (g) $-\infty$