Chapter 3

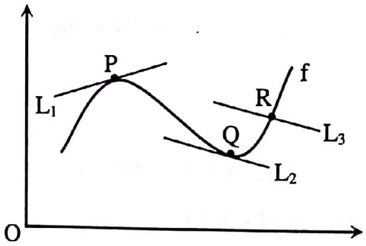
Derivatives

We interpret derivatives as slopes and rate of change. In this chapter, we develop rules for finding derivatives. These differentiation rules enables us to calculate the different types of functions and we then use these rules to solve problems involving rate of change, Mean Value Theorems and indeterminate forms.

3.1 Tangent and Velocity

The word tangent is originating from the Latin word *tangens*. In Latin, *tangens* means touching. In mathematics, a tangent line to the curve is a line that touches the curve at a point. This does not mean that if the line intersects the curve at a point, is a tangent line.

In above figure L_1 and L_2 are two straight lines that are passing by touching to the curve f at a single point P and Q respectively, therefore L_1 and L_2 are tangents to f. But the line L_3 intersects to f at R, so it is not a tangent to f.



Example 1: Find an equation of the tangent line to the parabola $y = x^2$ at the point P(1, 1).

Solution: It will be easy to find the equation of tangent line if the slope of the line is known. But we have only a single point. With the help of given curve $y = x^2$ we observe and choose a point $Q(x, y) = Q(x, x^2)$ near to P(1, 1) then the slope of the line PQ is

$$m_{PQ} = \frac{x^2 - 1}{x - 1}.$$

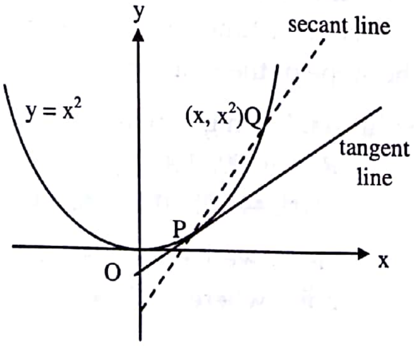
$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

So, the limit of the slope of PQ is

$$\lim_{Q \to P} m_{PQ} = m = \lim_{Q \to P} \frac{(x - 1)(x + 1)}{(x - 1)}$$

$$= \lim_{Q \to P} (x + 1) = 2.$$

Therefore, the equation of the tangent line i.e. passing through the point P(1, 1) and having slope 2 is



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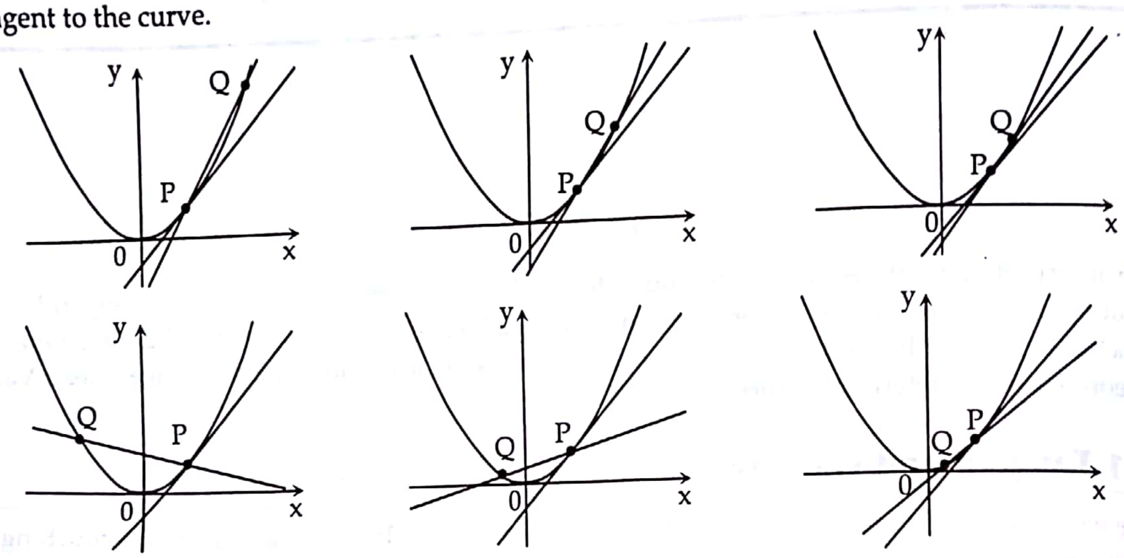
$$y - 1 = 2(x - 1).$$

i.e. $y = 2x - 1$.

Note: A straight line to the given curve is known as secant line if it cuts the curve into two points (may more than two), like as in above figure (the line PQ is secant line). The word secant is originated from Latin word *secans* where *secans* means cutting.

The following figure helps to understand, how Q approaches to P along the curve and becomes a tangent to the curve.



Definition: The *tangent* line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line though P and having the slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \text{ provided that the limit exists.}$$

Therefore, the above example can be treated as:

Example 2: Here, $a = 1$ and $f(x) = y = x^2$. Then the slope is,

$$m = \lim_{x\to 1}\left(\frac{f(x) - f(1)}{x - 1}\right) = \lim_{x\to 1}\left(\frac{x^2 - 1^2}{x - 1}\right) = \lim_{x\to 1}(x + 1) = 2.$$

Now, the equation of the tangent line to $y = x^2$ at $P(1, 1)$ is,

$$y - 1 = 2(x - 1)$$

i.e. $y = 2x - 1$.

Example 3: The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data in the table describe the charge Q remaining on the capacitor at time (measured in seconds). Use the data to estimate the slope of the tangent line at the point where $t = 0.04$.

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

Solution: From given table, we obtain the coordinate of points are R = (0.00, 100.00), (0.02, 81.87), (0.04, 67.08), (0.06, 54.88), (0.08, 44.93), (0.10, 36.76).

Here, we have to estimate the slope of tangent line at the point P(0.04, 67.03) (i.e. at the point where $t = 0.04$).

Now, the slope of the secant line PR is

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

At R = (0.00, 100.00), $m_{PR} = \dfrac{100.00 - 67.03}{0.00 - 0.04} = -824.25$

At R = (0.02, 81.87) $m_{PR} = \dfrac{81.87 - 67.03}{0.02 - 0.04} = -742.00$

At R = (0.04, 67.03), $m_{PR} = \dfrac{67.03 - 67.03}{0.04 - 0.01} = 0$

At R = (0.06, 54.88), $m_{PR} = \dfrac{54.88 - 67.03}{0.06 - 0.04} = -607.50$

At R = (0.08, 44.93), $m_{PR} = \dfrac{44.93 - 67.03}{0.08 - 0.04} = -552.50$

At R = (0.10, 36.76), $m_{PR} = \dfrac{36.76 - 67.03}{0.10 - 0.04} = -504.50$

This shows that the tangent at t = 0.04 should lie somewhere in between – 742.00 and – 607.50. The average of the slopes is

$$m = \frac{-742.00 - 607.50}{2} = -674.75$$

Thus, we estimate the slope of the tangent line is $-674.75 \approx -675$.

Suppose an object moves along a straight line according to motion of s = f(t) where s is the displacement of the object from the origin at time t. Then the function f that describes the motion, is called the ***position function*** of the object. And, the change position in the time t = a to = x is f(x) – f(a) then the ***average velocity*** is,

$$\text{Average velocity} = V_{avg} = \frac{\text{Displacement}}{\text{Time}} = \frac{f(t) - f(a)}{t - a}$$

The limit of such average velocities is called the ***velocity*** of the object. That is,

$$\text{Velocity} = V(a) = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}$$

at time t = a.

Note: From this relation we clearly observe that the velocity at time t = a is equal to the slope of the tangent line at the point.

The following two examples explain how the instantaneous velocity is obtained by periodwise observation of traveled distance.

Example 4: Suppose that a ball is dropped from the upper observation deck of a tower of 450 m above the ground. Find the velocity of the ball after 5 seconds when the distance fallen after t seconds is given by $s(t) = 4.9\,t^2$.

Solution: Given that the distance fallen by the ball after t times is given by

$$s(t) = 4.9\,t^2 \qquad \ldots\ldots \text{(i)}$$

And we have find the velocity of the ball after 5 seconds and no time interval is involved, so we observe the average velocity at time t = 5.1, 5.01, 5.001 that helps to obtain the instantaneous velocity at t = 5. The following table shows the coordinate of the ball's position under (i) at the specified time. Since $P(a = 5, 4.9a^2 = 122.5)$ (i.e. P(5, 122.5) and

Time (t)	s(t)	Q(t, s(t))
5.1	127.449	(5.1, 127.449)
5.01	122.991	(5.01, 122.911)
5.001	122.590	(5.001, 122.590)

Then the slope of the secant PQ is

when Q(5.1, 127.449), $m_{PQ} = \frac{127.449 - 122.5}{5.1 - 5} = 49.49$

when Q(5.01, 122.991), $m_{PQ} = \frac{122.991 - 122.5}{5.01 - 5} = 49.049$

when Q(5.001, 122.590), $m_{PQ} = \frac{122.590 - 122.5}{5.001 - 5} = 49.0049$

This process shows that the instantaneous velocity after t = 5 seconds is v = 49 m/s.

Example 5: The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion s = 2 sinπt + 3sinπt where t is measured in seconds. Find the average velocity during the period [1, 1.1].

Solution: Given that $s(t) = 2\sin\pi t + 3\sin\pi t$

and the time period is [1, 1.1]

We know the average velocity of a partical at time t is

$$v_{avg} = \frac{s_f - s_u}{t_f - t_a} \quad \text{.........(i)}$$

where t_f is final time and t_a is initial time. And s_f by functional value at t_f and s_a be functional value at t_a.

Here,

t_f	t_a	$t_f - t_a$	$s_f = 2\sin\pi\,(t_f) + 3\cos\pi\,(t_f)$	$s_a = 2\sin\pi t_a + 3\cos\pi t_a$	$s_f - s_a$	v_{avg}
1.1	1	0.1	– 3.47120	– 3	– 0.47120	– 4.712
1.01	1	0.01	– 3.06134	– 3	– 0.06134	– 6.134
1.001	1	0.001	– 3.00627	– 3	– 0.00627	– 6.270

Thus, the table shows that the instantaneous velocity approach – 6.3 cm/s as the interval becomes smaller.

The above example -4 can be solved as:

Example 6: Given that t = 5 and s = f(t) = 4.9 t².

Then the velocity of the ball is

$$v(5) = \lim_{t\to 5} \frac{f(t) - f(5)}{t - 5}$$

$$= \lim_{t\to 5} \left[\frac{4.9\,t^2 - 4.9(5)^2}{t - 5}\right]$$

$$= \lim_{t\to 5} \left[\frac{4.9\,(t^2 - 5^2)}{t - 5}\right]$$

$$= \lim_{t\to 5} \left[\frac{4.9\,(t-5)\,(t+5)}{(t-5)}\right] = \lim_{t\to 5} [4.9\,(t+5)] = 4.9\,(10) = 49$$

Therefore the velocity of the ball is after 5 seconds is 49 m/s.

Exercise 3.1

1. Find an equation of the tangent line to the curve at the given point.

   a. $y = 4x - 3x^2$, $(2, -4)$ b. $y = x^3 - 3x + 1$, $(2, 3)$

   c. $y = \sqrt{x}$, $(1, 1)$ d. $y = \frac{2x+1}{x+2}$, $(1, 1)$

2. (a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.

   (b) Find equations of the tangent lines at the points $(1, 5)$ and $(2, 3)$.

3. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

4. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

5. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.

   (a) Find the average velocity over each time interval:

   (i) $[3, 4]$ (ii) $[3.5, 4]$ (iii) $[4, 5]$ (iv) $[4, 4.5]$

   (b) Find the instantaneous velocity when $t = 4$.

6. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

7. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$.

Answers"

1. (a) $y = -8x + 12$ (b) $y = 9x - 15$ (c) $y = \frac{x}{2} + \frac{1}{2}$ (d) $y = \frac{x}{3} + \frac{2}{3}$

2. (a) $8a - 6a^2$ (b) $y = 2x + 3$ and $y = -8x + 19$

3. -24 ft/sec

4. $\frac{-2}{a^3}$ m/sec, -2 m/sec, $-\frac{2}{27}$ m/sec

5. (a) (i) -1 m/sec (ii) -0.5 m/sec (iii) 1 m/sec (iv) 0.5 m/sec

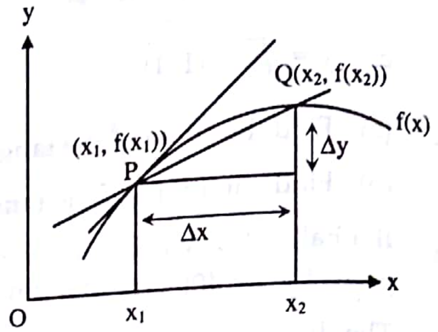
   (b) (i) velocity = 0 at $t = 4$

6. $f(4) = 3$, $f(4) = \frac{1}{4}$

7. $f'(1) = 3$, $y = 3x - 1$

3.2 Rate of Change

Suppose y be a function of x and we write $y = f(x)$. If x changes from a point x_1 to another point x_2 then the change in x is defined as



$$\Delta x = x_2 - x_1$$

which we called the increment of x.

And, the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

Then the quotient of those increments i.e.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \ldots\ldots (i)$$

is the average rate of change of y with respect to x over the interval $[x_1, x_2]$.

Geometrically, the average rate (i) be the slope of the secant line PQ.

And, the limit of the average rates (i) is called the ***rate of change*** of y with respect to x at $x = x_1$.

Mathematically,

Rate of change of y with respect to x at $x = x_1$ is,

$$\text{Rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \ldots\ldots (ii)$$

We recognize this limit (ii) as the derivative $f'(x_1)$.

Recall that the derivative $f'(x_1)$ is the slope of the tangent line to the curve $y = f(x)$ at $x = x_1$. And also $f'(x_1)$ is the rate of change of y with respect to x at $x = x_1$.

Example 1: The table shows the number of passengers P that arrived in a country by air, in millions.

Year	2001	2003	2005	2007	2009
P	8.49	9.65	11.78	14.54	12.84

a. Find the average rate of increase of P

(i) from 2001 to 2005 (ii) from 2005 to 2007.

b. Estimate the instantaneous rate of growth in 2005 by taking the average of two average rates of change. What are its units?

Solution: (a) (i) Here the average rate from 2001 to 2005 is

$$\text{Average rate} = \frac{11.78 - 8.49}{2005 - 2001} = \frac{3.29}{4} = 0.8225$$

Thus, in 2001 to 2005, the average rate of increase of P is 0.8225 millions.

(ii) Here the average rate from 2005 to 2007 is,

$$\text{Average rate} = \frac{14.54 - 11.78}{2007 - 2005} = \frac{2.76}{2} = 1.38$$

Thus, in 2005 to 2007, the average rate of increase of P is 1.38 millions.

(b) Here the base year is 2005. Here the average rate from 2005 to 2005 is,

$$\text{Average rate} = \frac{11.78 - 9.65}{2005 - 2003} = \frac{2.13}{2} = 1.065$$

By (a) (ii), the average rate from 2005 to 2007 is, 1.38. Therefore, the instantaneous rte of growth in 2005 is,

$$\text{Rate of growth} = \frac{1.065 + 1.38}{2} = 1.2225$$

If s = f(t) is the position function of a particle that moves along a straight line then f(a) be the velocity of the particle at time t = a and its magnitude i.e. $|f'(a)|$ is the speed of the particle. On the other hand, f'(a) is called marginal cost of f at t = a.

Example 2: A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x meters of the fabric is c = f(x) dollars.

(a) What is the meaning of the derivative f'(x)? What are its units?

(b) In particular terms what does it mean to say that f'(1000) = g?

(c) Which do you think is greater f'(50) or f'(500)? What about f'(5000)?

Solution: (a) In the sense of economics, f(x) is the production cost and f'(x) be the rate of change of production cost with respect to the produced meter. Therefore,

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta c}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$$

where c = f(x), be units for f'(x).

(b) The statement f'(1000) = 9 means that after the production of 1000 m fabric, the rate of the production cost is increasing by 9 dollars per meter.

(c) With the help of principle of economics, the rate of production cost will be decrease (here we assume that 500 m fabric is under demand scale). So,

$$f'(50) > f'(500).$$

But by (b), 1000 m fabric crosses the demand scale (because f'(1000) = positive). So, the expanded production becomes inefficient and might be overtime costs. Therefore, the rate of increase of costs will eventually start to rise. Thus,

$$f'(5000) > f'(500).$$

Exercise 3.2

1. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given.)

Year	2004	2005	2006	2007	2008
N	8569	10241	12440	15011	16680

a. Find the average rate of growth

(i) from 2006 to 2008 (ii) from 2006 to 2007 (iii) from 2005 to 2006.

b. Estimate the instantaneous rate of growth in 2006 by taking the average of two average rates of change. What are its units?

c. Estimate the instantaneous rate of growth in 2007 and compare it with the growth rate in 2006. What do you conclude?

2. The cost (in Rs.) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.

(a) Find the average rate of change of C with respect to x when the production level is changed.

(i) from $x = 100$ to $x = 105$ (ii) from $x = 100$ to $x = 101$

(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. This is called the marginal cost.

3. The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ rupees.

a. What is the meaning of the derivative $f'(x)$? What are its units?

b. What does the statement $f'(800) = 17$ mean?

c. Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

4. The number of bacteria after t hours in a controlled laboratory experiment is $n = f(t)$.

(a) What is the meaning of the derivative $f'(5)$? What are its units?

(b) Suppose there is an unlimited amount of space and nutrients for the bacteria. Which do you think larger, $f'(5)$ or $f'(10)$? If the supply of nutrients is limited, would that affect your conclusion? Explain.

5. The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p rupees per pound is $Q = f(p)$.

(a) What is the meaning of the derivative $f'(8)$? What are its units?

(b) Is $f'(8)$ positive or negative? Explain.

Answers:

1. (a) (i) 2120 locations/years (ii) 2571 locations/years (iii) 2199 locations/years

(b) 2385 locations/years

(c) 2120 locations/years; the growth rate is smaller than the growth of 2006.

2. (a) (i) 20.25/unit (ii) 20.05 /unit (b) Rs. 20/unit

3. (a) rate of change of production cost. Unit is rupees/paisa

(b) After 800 ounces of gold have to be produced the rate at which the production cost is increasing at Rs. 17/ounces.

(c) Short term: decrease, long term: increase

4. (a) Rate of change of number of bacteria is a laboratory with respect to 5 hours.

(b) $f'(10) > f'(5)$. The population is unstable equilibrium.

5. (a) rate of change in quality at Rs. 8/lb, lb/Rs. (unit)

(b) negative

3.3 Derivative as a Function

In previous section, we considered the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

as the slope of a tangent line to the curve f(x) at x = a or velocity of an object moving along f(x) at x = a or the marginal cost. This limit is also known as derivative of f(x) at the point x = a, in mathematics.

Definition: The ***derivative*** of a function f at a point a is denoted by f'(a) and is defined as

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \qquad \dots (1)$$

if the limit exists.

Suppose h = |x − a|. Then (1) can be rewrite as

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Example 1: If f(x) = x³ − x, then find a formula for f'(x).

Solution: Let f(x) = x³ − x.

Then,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\\ &= \lim_{h\to 0}\left(\frac{(x+h)^3-(x+h)-x^3+x}{h}\right)\\ &= \lim_{h\to 0}\left(\frac{x^3+h^3+3x^2h+3xh^2-x-h-x^3+x}{h}\right)\\ &= \lim_{h\to 0}\frac{h\,(3x^2+3xh+h^2-h)}{h}\\ &= \lim_{h\to 0}(3x^2+3xh+h^2-1)\\ &= 3x^2-1. \end{aligned}$$

Thus, $f'(x) = 3x^2 - 1.$

Example 2: Find f' where $f(x) = \dfrac{1-x}{2+x}$.

Solution: Let $f(x) = \dfrac{1-x}{2+x}$.

Then,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\left[\frac{1}{h}\left(\frac{1-(x+h)}{2+(x+h)}-\frac{1-x}{2+x}\right)\right] \end{aligned}$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\left(\frac{(1-x-h)(2+x)-(1-x)(2+x+h)}{(2+x)(2+x+h)}\right)\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\left(\frac{h(-3)}{(2+x)(2+x+h)}\right)\right] = \frac{-3}{(2+x)(2+x)} = \frac{-3}{(2+x)^2}.$$

Thus, $$f'(x) = \frac{-3}{(2+x)^2}.$$

The symbol of derivative of f(x) is often used f'(x). Here, x is independent variable and f is dependent variable that depends upon x. And, f'(x) means f is derivable with respect to x. The other common notations for derivative are

$$f'(x) = \frac{df}{dx} = Df(x)$$

The symbol D or $\frac{d}{dx}$ are called the differentiation operator.

Definition: A function f is called ***differentiable*** at a point 'a' if f'(a) exists.

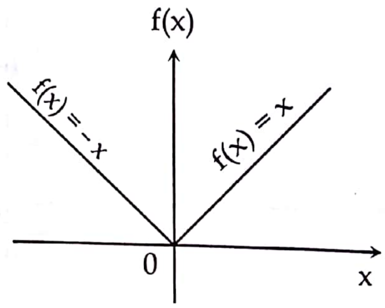
Note that, the differentiability of any function exists (if the function is differentiable) only on an open interval.

Example 3: Show that f(x) = |x| is not differentiable at x = 0.

Solution: For x > 0, we have |x| = x. Therefore, for h > 0, |x + h| = x + h.

Here, for x > 0,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{|x+h|-|x|}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{x+h-x}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{h}{h}\right) \\ &= 1. \end{aligned}$$



This means f is differentiable for x > 0.

And for x < 0, we have |x| = –x. Then choose h is so small such that |x + h| < 0 and |x + h| = – (x + h).

Here for x < 0,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{-(x+h)-(-x)}{h}\right) \\ &= \lim_{h\to 0}\left(\frac{-h}{h}\right) \\ &= -1. \end{aligned}$$

This means f is differentiable for x < 0.

But, $f'(x)|_{\text{for } x>0} \neq f'(x)|_{\text{for } x<0}$.

So, f is not differentiable at $x = 0$.

Geometrically, the curve f(x) does not have a tangent line at origin i.e. at (0, 0).

Theorem: If f is differentiable at a point 'a' then f is continuous at that point 'a'.

Proof: Suppose that f is differentiable at a point 'a'. Therefore,

$$f'(a) = \lim_{h\to 0} \frac{f(x) - f(a)}{x - a}$$

exists.

Then to prove f is continuous at a, we have to show that

$$\lim_{x\to a} f(x) = f(a)$$

Here,

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a)$$

Therefore,

$$\lim_{x\to a} [f(x) - f(a)] = \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a} \quad (x - a)\right]$$

$$= \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a}\right) \lim_{x\to a}(x - a)$$

$$= f'(a) . 0$$

$$= 0.$$

$$\Rightarrow \lim_{x\to a} f(x) = f(a).$$

This shows f is continuous at a.

Note: Remember that the converse of this theorem need not hold. For instance, consider a function,

$$f(x) = |x|.$$

Since, $\lim_{x\to a} f(x) = \lim_{x\to a} |x| = 0.$

So, f is continuous at $x = 0$ but it is not differentiable at 0 (see example 29).

Exercise 3.3

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

1. $f(x) = \frac{1}{z}x - \frac{1}{3}$
2. $f(x) = mx + b$
3. $f(x) = x^2 - 2x^3$
4. $g(t) = \frac{1}{\sqrt{t}}$
5. $g(x) = \sqrt{9 - x}$
6. $f(x) = \frac{x^2 - 1}{2x - 3}$
7. $f(x) = x^{3/2}$
8. $f(x) = x^4$

Answers

1. $\frac{1}{2}, \mathcal{R}, \mathcal{R}$
2. $f'(x) = m, \mathcal{R}, \mathcal{R}$
3. $2x - 6x^2, \mathcal{R}, \mathcal{R}$
4. $\frac{-1}{2\sqrt{t}}, (0, \infty), (0, \infty)$
5. $g'(t) = \frac{-1}{\sqrt{9-x}}, (-\infty, 9), (-\infty, 9)$
6. $f'(x) = \frac{1}{2} - \frac{5}{2(2x-3)^2}, \mathcal{R} - \left\{\frac{3}{2}\right\}, \mathcal{R} - \left\{\frac{3}{2}\right\}$
7. $\frac{3}{2}\sqrt{x}, [0, \infty), [0, \infty)$
8. $f'(x) = 4x^3 \mathcal{R}, \mathcal{R}$

3.4 Review of Derivative

Algebra of Derivative of Function

In the study of derivative of function, sometimes we observe that the function will not in simple and singular form, we impose some rules to make simple to such function. Here we will study about such rules.

Constant Multiple Rule

If c be a constant and f is a differentiable function then

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x)).$$

Example 1: (i) $\frac{d}{dx}(2x^3) = 2\frac{d}{dx}(x^3)$. (ii) $\frac{d}{dx}\left(\frac{4}{3}\tan^2 x\right) = \frac{4}{3}\frac{d}{dx}(\tan^2 x)$.

Sum and difference rule:

If f and g are both differentiable functions then

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

and $$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x)).$$

Example 2: $\frac{d}{dx}(2x^2 - 5\sin x + 3\ln x) = 2\frac{d}{dx}(x^2) - 5\frac{d}{dx}(\sin x) + 3\frac{d}{dx}(\ln(x))$.

Product Rule:

If f and g are both differentiable functions then

$$\frac{d}{dx}(f(x)\,g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(g(x)).$$

Example 3: (i) $\frac{d}{dx}(xe^x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$.

(ii) $\frac{d}{dt}(\sqrt{t}\cot t) = \sqrt{t}\frac{d}{dt}(\cot t) + \cot t\frac{d}{dt}(\sqrt{t})$.

Quotient Rule:

If f and g are both differentiable functions and $g(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{e^x}{1+x^2}\right) = \frac{(1+x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

Chain Rule:

If g is differentiable at x and f is differentiable at g(x). Then the composite function f(g(x)) is differentiable at x and is differentiate as

$$\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(f(g(x)))\,\frac{d}{dx}g(x)$$
$$= f'(g(x)) \cdot g'(x)$$

Example 4:

(i) $\frac{d}{dx}(\sqrt{1+x^2}) = \frac{d(\sqrt{1+x^2})}{d(1+x^2)} \cdot \frac{d(1+x^2)}{dx}$

(ii) $\frac{d}{dx}(\sin(x^2)) = \frac{d(\sin(x^2))}{d(x^2)} \cdot \frac{d(x^2)}{dx}$

(iii) $\frac{d}{dx}(\sin^2 x) = \frac{d(\sin^2 x)}{d(\sin x)} \cdot \frac{d(\sin x)}{dx}$

(iv) $\frac{d}{dx}(\sin(\cos(\tan x))) = \frac{d(\sin(\cot(\tan x)))}{d(\cos(\tan x))} \cdot \frac{d(\cos(\tan x))}{d(\tan x)} \cdot \frac{d(\tan x)}{dx}$

Derivative of a Function

In this section we learn how to differentiate the different type of functions.

A. Derivative of a constant function:

Suppose $f(x) = c$ is a constant function. Then $f(x+h) = c$. So that the derivative of f(x) is,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0.$$

Thus the derivative of any constant function is zero. Therefore,

$$\frac{d}{dx}(c) = 0.$$

B. Derivative of power function:

Suppose $f(x) = x^n$ where n is any real number. Then the derivative of x^n is

$$f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}.$$

Justification: Here,

$$f'(a) = \lim_{x\to a}\frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1})}{x - a}$$

$$= \lim_{x \to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x\,a^{n-2} + a^{n-1})$$

$$= a^{n-1} + a^{n-2} + a^{n-3}a^2 + \ldots + a.a^{n-2} + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1}$$

$$= n\,a^{n-1}$$

Therefore, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

Example 5: Find the derivative of

(i) 5 (ii) 100 (iii) x^n (iv) x^n (v) $x^{5/3}$.

Solution: Here,

(i) Let $f(x) = 5$. So, $f'(x) = 0$

(ii) Let $f(x) = 100$. So, $f'(x) = 0$

(iii) Let $f(x) = x^9$. So, $f'(x) = 9x^8$

(iv) Let $f(x) = x^2$. So, $f'(x) = 2x$

(v) Let $f(x) = x^{5/3}$. So, $f'(x) = \frac{5}{3}x^{5/3} - 1 = \frac{5}{3}x^{2/3}$.

Note that the power rules enables to find the tangent lines and normal lines at a point (fixed but any) to the given curve. The following example includes this concept.

Example 6: Find the equation of tangent line and normal line to the curve $y = x^{5/2}$ at the point P(1, 1).

Solution: Let $f(x) = y = x^{5/2}$. Then

$$f'(x) = y' = \frac{5}{2}x^{3/2}$$

At P(1, 1),
$$f'(1) = y' = \frac{5}{2}(1)^{3/2} = \frac{5}{2}.$$

So, the slope of the tangent line to $f(x) = y = x^{5/2}$ at P(1, 1) is $\frac{5}{2}$. Therefore, the equation of the tangent line at P(1, 1) is

$$y - 1 = -\frac{2}{5}(x - 1).$$

i.e. $5y - 5 = -2x + 2$.

$\Rightarrow$ $2x + 5y - 7 = 0$.

C. Derivative of Exponential Function

The derivative of an exponential function $f(x) = a^x$ is

$$f'(x) = a^x \,.\, f'(0)$$

In particular if $f(x) = e^x$ then

$$f'(x) = e^x f'(0).$$
$$= e^x e^0.$$
$$= e^x.$$

Therefore, $\frac{d}{dx}(e^x) = e^x$.

Example 7: If $f(x) = e^{x+1} + 1$ then find f' and f''.

Solution: Let

$$f(x) = e^{x+1} + 1.$$

Then,

$$f'(x) = \frac{d}{dx}(e^{x+1} + 1) = e^{x+1}\, e^{0+1} + 0 = e^{x+1} . e = e^{x+2}.$$

And,

$$f''(x) = e^{x+2} . e^{0+2} = e^{x+2} . e^2 = e^{x+4}.$$

Example 8: At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

Solution: Given curve is

$$y = e^x \qquad \ldots\ldots (i)$$

Then $\qquad y' = e^x.$

Let the x-coordinate of the point where the tangent touches to $y = e^x$, be a. Then the coordinate of the point be (a, e^a). And the slope of the tangent line to the curve at (a, e^a) is e^a. Also, given that this tangent line is parallel to the line

$$y = 2x \qquad \ldots\ldots (ii)$$

Clearly slope of the line (ii) is 2.

Since the tangent line to (i) is parallel to (ii). So, the slope of these lines should be equal. That is,

$$e^a = 2$$
$$\Rightarrow a = \ln(2)$$

Therefore, the coordinate of the point on $y = e^x$ is the tangent line parallel to $y = 2x$ is $(\ln(2), 2)$.

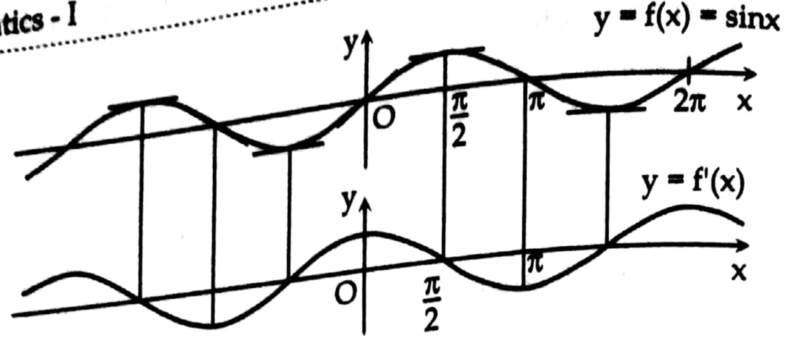
Derivative of Trigonometric Functions

We have studied in Chapter -1 the there are six different trigonometric functions and these are sinx, cosx, tanx, cotx, secx, cosecx. Here, it is understood that the variable x is the angle of the trigonometric function which is measured in radian.

Recall that all the trigonometric functions are continuous at every number in their domains. In limit, we have the rational value of sine function and its angle is 1. That is,

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

If we sketch the graph of sinx and sketch of graph of derivative of sinx then we find the graph of $\frac{d}{dx}(\sin x)$ is same as the graph of cosx.



This means

$$\frac{d}{dx}(\sin x) = \cos x.$$

Likewise we may observe

$$\frac{d}{dx}(\cos x) = -\sin x. \qquad \frac{d}{dx}(\tan x) = \sec^2 x.$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x. \qquad \frac{d}{dx}(\sec x) = \sec x \tan x.$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$$

Example 9: If $f(x) = \frac{\sec x}{1 + \tan x}$ then find $f'(x)$.

Solution: Let

$$f(x) = \frac{\sec x}{1 + \tan x}$$

Then,

$$f'(x) = \frac{(1 + \tan x)\sec x \tan x - \sec x(\sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1 + \tan x)^2}.$$

Example 10: Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $\left(\frac{\pi}{2}, \pi\right)$.

Solution: Let

$$y = 2x \sin x$$

Then,

$$y' = 2\sin x + 2x \cos x$$

At the point $\left(\frac{\pi}{2}, \pi\right)$, $y' = 2\sin\left(\frac{\pi}{2}\right) + 2\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 2 + 0 = 2.$

This shows the curve has slope 2 at the point $\left(\frac{\pi}{2}, \pi\right)$. Since the point $\left(\frac{\pi}{2}, \pi\right)$, is the common point of the curve and the tangent line to $y = 2x \sin x$ at $\left(\frac{\pi}{2}, \pi\right)$. Therefore the slope of the tangent line is 2.

Thus, the equation of tangent line to $y = 2x \sin x$ at the point $\left(\frac{\pi}{2}, \pi\right)$ is,

$$y - \pi = 2\left(x - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad y - \pi = 2x - \pi.$$

$$\Rightarrow \quad 2x - y = 0.$$

Example 11: A mass on a string vibrates horizontally on a smooth level surface. Its equation of motion is $x(t) = 8\sin t$ where t is in seconds and x in cm.

(a) Find the velocity and acceleration at time t.

(b) Find the position, velocity and acceleration of the mass at time $t = \frac{2\pi}{3}$. In what direction is it moving at the time?

(c) Find the speed of the mass at $t = \frac{\pi}{6}$.

Solution: Let the equation of motion is,

$$x(t) = 8 \sin t$$

Therefore, $\frac{d}{dt}(x(t)) = 8 \cos t$ and $\frac{d^2}{dt^2}(x(t)) = -8 \sin t.$

(a) Thus, the velocity of mass at time t is

$$v = \frac{d}{dt}(x(t)) = 8 \cos t.$$

and the acceleration of mass at time t is

$$a = \frac{d}{dt}(x(t)) = -8 \sin t.$$

(b) At $t = \frac{2\pi}{3}$,

(i) the position of the mass is

$$x\left(\frac{2\pi}{3}\right) = 8 \sin\left(\frac{2\pi}{3}\right) = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}.$$

(ii) the velocity of the mass is

$$\frac{d}{dt}\left(x\left(\frac{2\pi}{3}\right)\right) = 8\cos\left(\frac{2\pi}{3}\right) = 8\left(-\frac{1}{2}\right) = -4.$$

(iii) the acceleration of the mass is

$$\frac{d^2}{dt^2}\left(x\left(\frac{2\pi}{3}\right)\right) = -8 \sin\left(\frac{2\pi}{3}\right) = -4\sqrt{3}.$$

By (ii) we have velocity of the mass is negative at $t = \frac{2\pi}{3}$. Therefore, the mass is moving to negative direction i.e. to left.

(c) Since, at $t = \frac{\pi}{6}$,

$$|v| = \left|\frac{d}{dt}\left(x\left(\frac{\pi}{6}\right)\right)\right| = \left|8 \cos\left(\frac{\pi}{6}\right)\right| = \left|8\left(\frac{\sqrt{3}}{2}\right)\right| = 4\sqrt{3}$$

Therefore the speed of the mass at $t = \frac{\pi}{6}$ is $4\sqrt{3}$.

Exercise 3.4

1. Differentiate the function

a. $f(x) = 2^{40}$ b. $f(x) = e^5$ c. $F(x) = \frac{3}{4}x^8$

d. $f(t) = 1.4t^5 - 2.5t^2 + 6.7$ e. $h(x) = (x - 2)(2x + 3)$ f. $y = x^{5/3} - x^{2/3}$

2. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

3. Show that the curve $y = 2x^e + 3x + 5x^3$ has no tangent line with slope 2.

4. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

5. Find an equation of the normal line to the parabola $y = x^2 - 5x + 4$ that is parallel to the line $x - 3y = 5$.

6. (a) Find equations of both lines through the point (2, -3) that are tangent to the parabola $y = x^2 + x$.

   (b) Show that there is no line through the point (2, 7) that is tangent to the parabola. Then draw a diagram to see why?

7. Differentiate:

a. $g(x) = \sqrt{x}\, e^x$ b. $y = \frac{e^x}{1 - e^x}$ c. $G(x) = \frac{x^2 - 2}{2x + 1}$

d. $y = \frac{x + 1}{x^3 + x - 2}$ e. $f(t) = \frac{2t}{2 + \sqrt{t}}$ f. $f(x) = \frac{1 - xe^x}{x + e^x}$

g. $f(x) = \frac{x^2}{1 + 2x}$

8. Differentiate:

a. $f(x) = 3x^2 - 2\cos x$ b. $g(\theta) = e^\theta(\tan\theta - \theta)$ c. $y = \frac{x}{2 - \tan x}$

d. $y = \frac{\cos x}{1 - \sin x}$ e. $y = \frac{1 - \sec x}{\tan x}$

9. Find an equation of the tangent line to the curve $y = 3x + 6\cos x$ at the point $(\pi/3, \pi+3)$

10. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu\sin\theta + \cos\theta}$$

where μ is a constant called the coefficient of friction.

(a) Find the rate of change of F with respect to θ.

(b) When is this rate of change equal to 0?

11. An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time t = 0. Its position at time t is $s = f(t) = 4\cos t$. Find the velocity and acceleration at time t and use them to analyze the motion of the object.

Answers

1. (a) 0 (b) e^5 (c) $6x^7$ (d) $7t^4 - 5t$ (e) $4x - 1$ (f) $\frac{5}{3}x^{2/3} - \frac{2}{3}x^{-1/3}$
2. (-2, 21) and (1, -6)
3. no tangent line with slope 2
4. $y = 3x - 4$
5. $y = \frac{x}{3} - \frac{1}{3}$
6. (a) $x + y + 1 = 0$, $y = 11x - 25$
7. (a) $e^x\left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)$ (b) $\frac{e^x}{(1-e^x)^2}$ (c) $\frac{2x^2+2x+4}{(2x+1)^2}$ (d) $\frac{-2x^3-3x^2-3}{(x^3+x-2)^2}$ (e) $\frac{4+\sqrt{t}}{(2+\sqrt{t})^2}$
   (f) $\frac{-(x^2+1)e^x - e^{2x} - 1}{(x+e^x)^2}$ (g) $\frac{2x+2x^2}{(1+2x)^2}$
8. (a) $6x + 2\sin x$ (b) $e^\theta(\tan^2\theta + \tan\theta - \theta)$ (c) $\frac{2 - \tan x + x\sec^2 x}{(2-\tan x)^2}$ (d) $\frac{1}{1-\sin x}$ (e) $\frac{-1}{1+\cos x}$
9. $y = (\pi + 3) + (3 - 3\sqrt{3})\left(x - \frac{\pi}{3}\right)$
10. (a) $\frac{dF}{d\theta} = \frac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}$ (b) $\theta = \tan^{-1}\mu$
11. Velocity = $-4\sin t$, acceleration = $-4\cos t$

Implicit Differentiation

Sometimes the functions may define implicitly by a relation between two different variables (like between x and y) as

$$x^2 + y^2 = 6xy$$

In such case, we can use the method of implicit differentiation. This consists the derivative of both sides with respect to x and the solve the result for final solution of $\frac{dy}{dx}$. The following example helps to understand the precise concept.

Example 1: If $x^2 + y^2 = 6xy$ then find $\frac{dy}{dx}$.

Solution:

Let, $x^2 + y^2 = 6xy$.

Differentiating w.r.t. x then,

$$2x + 2y\frac{dy}{dx} = 6\left(x.\frac{dy}{dx} + y\right).$$

$$\Rightarrow (2y - 6x)\frac{dy}{dx} = 6y - 2x.$$

$$\Rightarrow \frac{dy}{dx} = \frac{6y - 2x}{2y - 6x} = \frac{3y - x}{y - 3x}.$$

Example 2: Find y' if $\sin(x + y) = y^2 \cos x$.

Solution:

Let $\sin(x + y) = y^2 \cos x$

Differentiating w.r.t. x then

$$\cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) = 2y\cos x\frac{dy}{dx} + y^2(-\sin x)$$

$$\Rightarrow \quad (\cos(x+y) - 2y\cos x)\frac{dy}{dx} = -\cos(x+y) - y^2\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\cos(x+y) - y^2\sin x}{\cos(x+y) - 2y\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos(x+y) + y^2\sin x}{2y\cos x - \cos(x+y)}.$$

Derivative of Logarithmic Function:

In chapter –1, we studied the behavior of logarithmic function. Here we develop the derivative of such function. We know,

$$\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)}.$$

Justification: Let $y = \log_a(x)$. Then $a^y = x$.

Differentiate $a^y = x$ with respect to x then

$$a^y \ln(a)\frac{dy}{dx} = 1.$$

$$\Rightarrow \quad \frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)}.$$

Note: If we choose a = e then

$$y = \log_e(x) = \ln(x) \quad \text{and} \quad \frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

Example 3: Find $\frac{d}{dx}(\ln \sin x)$

Solution: Here,

$$\frac{d}{dx}(\ln\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x.$$

Example 4: Find $\frac{d}{dx}\left(\ln\left(\frac{x+1}{\sqrt{x-2}}\right)\right)$.

Solution: Here,

$$\frac{d}{dx}\left(\ln\left(\frac{x+1}{\sqrt{x-2}}\right)\right) = \frac{d}{dx}(\ln(x+1) - \ln(\sqrt{x-2}))$$

$$= \frac{d}{dx}\left(\ln(x+1) - \frac{1}{2}\ln(x-2)\right)$$

$$= \frac{1}{x+1}(1) - \frac{1}{2}\left(\frac{1}{x-2}\right)(1)$$

$$= \frac{2(x-2)-(x+1)}{2(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}.$$

Example 5: Differentiate $y = x^{\sqrt{x}}$.

Solution: Let $y = x^{\sqrt{x}}$. Then

$$\ln(y) = \sqrt{x}\ln(x)$$

Differentiate it with respect to x then

$$\frac{1}{y}y' = \sqrt{x}\frac{1}{x} + \ln(x)\frac{1}{2}x^{-1/2}. (1)$$

$$\Rightarrow \quad y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln(x)}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln(x)}{2\sqrt{x}}\right)$$

Derivative of Inverse Trigonometric Function

The inverse trigonometric function is discussed in chapter –1. We have already reviewed the continuity of such functions. Here we have to going to find the derivative of these functions. Recall that the inverse trigonometric functions are

$\sin^{-1}x$ (arc sinx), $\cos^{-1}x$ (arc cosx), $\tan^{-1}x$ (arc tanx),

$\cot^{-1}x$ (arc cotx), $\sec^{-1}x$ (arc secx), $\operatorname{cosec}^{-1}x$ (arc cosecx).

We are clear about the arcsine function is only defined over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Let

$$y = \sin^{-1}x \quad \text{i.e.} \quad x = \sin y.$$

Differentiating,

$$1 = \cos y\frac{dy}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

Therefore,

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}.$$

Likewise we may observe that

$$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

Example 6: Find the derivative of $\tan^{-1}(\sqrt{x})$.

Solution:

Let $y = \tan^{-1}\sqrt{x}$. Then $\tan y = \sqrt{x}$.

Differentiating

$$\sec^2 y \frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}\sec^2 y} = \frac{1}{2\sqrt{x}\,(1+\tan^2 y)} = \frac{1}{2\sqrt{x}\,(1+x)}.$$

Thus, $\frac{d}{dx}(\tan^{-1}\sqrt{x}) = \frac{1}{2\sqrt{x}\,(1+x)}$.

Example 7: Find $\frac{d}{dt}\left(\cot^{-1}(t) + \cot^{-1}\left(\frac{1}{t}\right)\right)$.

Solution:

Let, $f(t) = \cot^{-1}t + \cot^{-1}\left(\frac{1}{t}\right)$.

Differentiate,

$$f'(t) = \frac{-1}{1+t^2} + \frac{-1}{1+(1/t)^2}\frac{d}{dt}\left(\frac{1}{t}\right) = \frac{-1}{1+t^2} - \frac{t^2}{t^2+1}\left(\frac{-1}{t^2}\right) = \frac{-1}{1+t^2} + \frac{1}{1+t^2} = 0.$$

Thus,

$$\frac{d}{dt}\left(\cot^{-1}(t) + \cot^{-1}\left(\frac{1}{t}\right)\right) = 0.$$

Exercise 3.5

1. Write the composite function in the form f(g(x)). [Identify the inner function u = g(x) and the outer function y = f(u).] Then find the derivative dy/dx.

   a. $y = \sqrt{4+3x}$ b. $y = \tan(\sin x)$ c. $F(x) = (4x - x^2)^{100}$

   d. $f(z) = \frac{1}{(z^2+1)}$ e. $y = \cos(a^3 + x^3)$ f. $y = a^3 + \cos^3 x$

   g. $h(t) = (t+1)^{2/3}(2t^2 - 1)^3$ h. $y = \left(\frac{x^2+1}{x^2-1}\right)^3$ i. $y = \frac{e^u - e^{-u}}{e^u + e^{-u}}$

2. Find an equation of the tangent line to the curve $y = 2/(1 + e^{-x})$ at the point (0, 1).

3. Find the x-coordinates of all points on the curve $y = \sin 2x - 2\sin x$ at which the tangent line is horizontal.

4. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

5. (a) Find y' by implicit differentiation.

   (b) Solve the equation explicitly for y and differentiate to get y' in terms of x.

   (c) Check that your solutions to parts (a) and (b) are consistent by substituting the expression for y into your solution for part (a).

   (i) $xy + 2x + 3x^2 = 4$ (ii) $\frac{1}{x} + \frac{1}{y} = 1$ (iii) $\cos x + \sqrt{y} = 5$

6. Find $\frac{dy}{dx}$ by implicit differentiation.

   a. $2x^3 + x^2y - xy^3 = 2$ b. $xe^y = x - y$ c. $x^2 y^2 + x \sin y = 4$

   d. $4 \cos x \sin y = 1$ e. $x \sin y + y \sin x = 1$ f. $\tan(x - y) = \frac{y}{1 + x^2}$

7. Find an equation of the tangent line to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

8. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

9. Find the derivative of the function. Simplify where possible

   a. $y = \tan^{-1}(x^2)$ b. $g(x) = \sqrt{x^2 - 1}\sec^{-1} x$ c. $y = \sin^{-1}(2x + 1)$

   d. $y = \tan^{-1}(x - \sqrt{1 + x^2})$ e. $F(\theta) = \arcsin\sqrt{\sin\theta}$ f. $y = x\sin^{-1}x + \sqrt{1 - x^2}$

   g. $y = \arctan\sqrt{\frac{1 - x}{1 + x}}$

10. Use implicit differentiation to find y' if $x^2 + xy + y^2 + 1 = 0$.

11. Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point (12, 3).

12. Differentiate the function

    a. $f(x) = x\ln(x) - x$ b. $f(x) = \sin(\ln x)$ c. $f(x) = \ln(\sin^2 x)$

    d. $f(x) = \sin x \ln(5x)$ e. $y = \ln|1 + t - t^3|$.

13. Find an equation of the tangent line to the curve at the given point.

    a. $y = \ln(x^2 - 3x + 1)$, (3, 0) b. $y = x^2 \ln x$, (1, 0)

Answers

1. (a) $\frac{dy}{dx} = \frac{4}{3}(1 + 3x)^{-2/3}$ (b) $\frac{dy}{dx} = -\operatorname{cosec}^2 x \cos(\cot x)$ (c) $\frac{dy}{dx} = 100(4x - x^2)^{99}(4 - 2x)$

   (d) $\frac{df}{dz} = \frac{-2z}{(z^2 + 1)^2}$ (e) $\frac{dy}{dx} = -3x^2 \sin(a^3 + x^3)$ (f) $\frac{dy}{dx} = -3 \sin x \cos^2 x$

   (g) $\frac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^2(20t^2 + 18t - 1)$ (h) $\frac{dy}{dx} = \frac{-12x(x^2 + 1)^2}{(x^2 - 1)^4}$ (i) $\frac{4}{(e^u + e^{-u})^2}$

2. $y = \frac{x}{2} + 1$

3. $x = \frac{2\pi}{3} + 2\pi n$, $\frac{4\pi}{3} + 2\pi n$, $2\pi n$ for n is integer

4. $F'(t) = 24$

5. (i) a. $\frac{9x}{y}$ b. $\pm\frac{9x}{\sqrt{9x^2 - 1}}$ c. From (a) $y = \pm\sqrt{9x^2 - 1}$ and from (b) $\frac{9x}{y}$

   (ii) a. $\frac{-y^2}{x^2}$ b. $\frac{-1}{(x - 1)^2}$ c. $\frac{-1}{(x - 1)^2}$

   (iii) a. $2\sqrt{y}$ b. $2(5 - \cos x)\sin x$ c. $y' = 2(5 - \cos x)\sin x$

6. (a) $\frac{dy}{dx} = \frac{y^3 - 6x^2 - 2xy}{x^2 - 3xy^2}$ (b) $\frac{1 - e^y}{x e^y + 1}$ (c) $y' = \frac{2x + y \sin x}{\cos x - 2y}$ (d) $\tan x \tan y$

(e) $\frac{dy}{dx} = \frac{-\sin y - y\cos x}{x\cos y + \sin x}$ (f) $\frac{dy}{dx} = \frac{2xy + (1 + x^2)^2 \sec^2(x - y)}{(1 + x^2)^2 \sec^2(x - y) + (1 + x^2)}$

7. $\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$

9. (a) $\frac{2x}{1 + x^4}$ (b) $\frac{1}{x} + \frac{x \sec^{-1} x}{\sqrt{x^2 - 1}}$ (c) $\frac{1}{\sqrt{-x - x^2}}$ (d) $\frac{1}{2(x^2 + 1)}$ (e) $\frac{\cos\theta}{2\sqrt{\sin\theta - \sin^2\theta}}$ (f) $\sin^{-1}x$

(g) $\frac{-1}{2\sqrt{1 - x^2}}$ $(-1 < x < 1)$

10. $\frac{-2x - y}{x + 2y}$

11. $y = \frac{2x}{3} - 5$

12. (a) $\ln(x)$ (b) $\frac{\cos(\ln(x))}{x}$ (c) $2\cot x$ (d) $\frac{\sin x}{x} + \cos x \ln(5x)$ (e) $\frac{1 - 3t^2}{1 + t - t^3}$

13. (a) $y = 3x - 9$ (b) $y = x - 1$

3.5 Mean Value Theorem

We know the derivative of any constant function is zero, but could there be a more complicated function whose derivative is always zero? If two functions have identical derivatives over an interval, how are the functions related? We answer these questions by Mean Value Theorem. But to arrive at the Mean Value Theorem, we first prove the Rolle's Theorem. This theorem was first published in 1691 by French mathematician Michel Rolle (1652–1719) in a book entitled '*Methode pour resoudre les Egalitez*'.

The Extreme Value Theorem:

If f is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum f(d) at some points c and d in [a, b].

Fermat's Theorem:

If f has a local maximum (or minimum) at c and if f is differentiable at c then f'(c) = 0.

Rolle's Theorem:

Let f be a function that satisfies the following three hypotheses:

(i) f is continuous on the closed interval [a, b].

(ii) f is differentiable on the open interval (a, b).

(iii) f(a) = f(b).

Then there is a number c in (a, b) such that f '(c) = 0.

Proof:

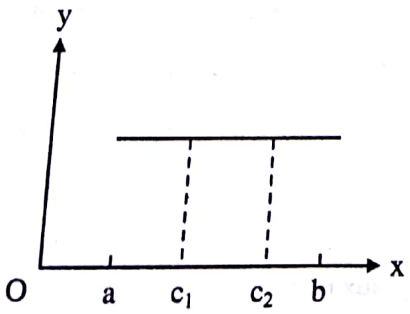


Fig. (1)

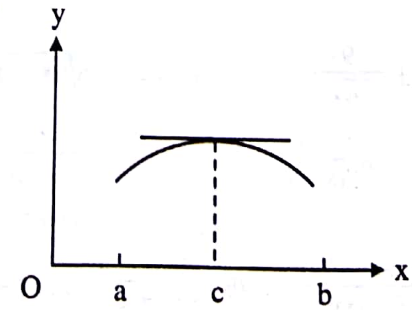


Fig. (2)

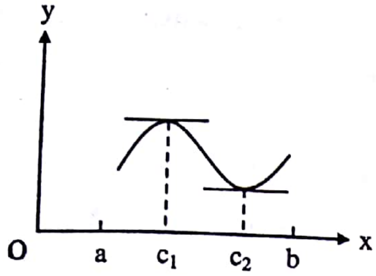


Fig. (3)

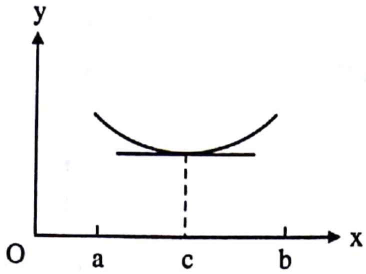


Fig. (4)

Let f satisfies the given hypotheses (i), (ii) and (iii).

There are three cases:

Case I: f(x) = k (k is a constant) (as in figure 1).

If f(x) = k (k is a constant) for x in [a, b] then f'(x) = 0, for x in (a, b). So, at x = c for any x in (a, b), f'(c) = 0.

Case II: f(x) > f(a) for some x in (a, b) (as in figure 2 and 3).

Let f(x) > f(a) for some x in (a, b) then by the Extreme Value Theorem, f has a maximum value somewhere in [a, b] (as hypothesis (i)). Since by (iii) f(a) = f(b), so f has maximum value somewhere in (a, b). Let f has local maximum at c in (a, b). Since by hypothesis (ii) f is differentiable at c. Therefore, by Fermat's theorem, f'(c) = 0.

Case III: f(x) < f(a) for some x in (a, b) (as figure 3 and 4).

Let f(x) < f(a) for some x in (a, b) then by the Extreme Value Theorem f has a minimum value in [a, b]. Since f(a) = f(b), let f attains its minimum value at c in (a, b). Therefore by Fermat's Theorem f'(c) = 0.

Thus, in either case, if f satisfies the hypothesis (i), (ii) and (iii) then f'(c) = 0 for some c in (a, b).

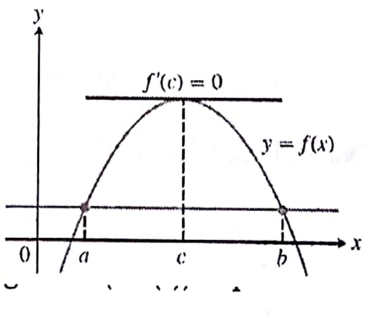
Geometrical Meaning of Rolle's Theorem

If a function $f(x)$ is

(i) Continuous in the closed interval $[a, b]$;

(ii) Differentiable in the open interval (a, b); and

(iii) $f(a) = f(b)$.



Then, there exist at least one real number $c \in (a, b)$ such that ta tangent at x = c is parallel to x–axis.

Example 1: State Roll's theorem for differentiable function. Support with examples that the hypothesis are essential to hold the theorem.

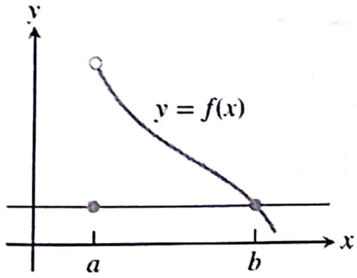
Solution:

In figure (a), given function $f(x)$ is discontinuous at end point $x = a$ of $[a, b]$, so it is discontinuous in $[a, b]$.

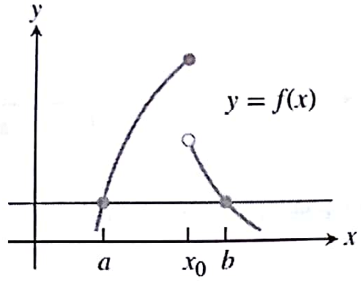
In figure (b); given function $f(x)$ is discontinuous at interior point $x = x_0$ of $[a, b]$. In figure (c), given function $f(x)$ is continuous in $[a, b]$ but not differentiable at an interior point $x = x_0$ of $[a, b]$.

In each case above, there has not horizontal tangent (i.e. $f'(c) = 0$ not exist).

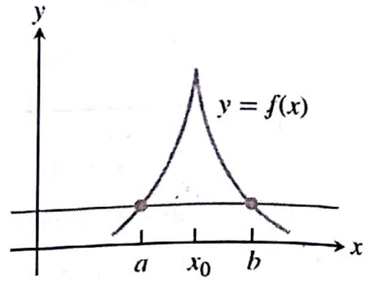
Thus; hypothesis are essential to hold the Rolle's theorem.



(a) Discontinuous at an endpoint of $[a, b]$



(b) Discontinuous at an interior point of $[a, b]$



(c) Continuous on $[a, b]$ but not differentiable at an interior point

Example 2: Verify the Rolle's theorem for the function $y = \sqrt{1 - x^2}$, $[-1, 1]$.

Solution: Given $f(x) = \sqrt{1 - x^2}$.

Since, $\lim\limits_{x \to -1^+} f(x) = f(-1)$ and $\lim\limits_{x \to 1^-} f(x) = f(1)$.

Thus, $f(x)$ is continuous at end points and all interior points of $[-1, 1]$

(i) $f(x)$ is continuous in $[-1, 1]$

(ii) $f'(x) = \dfrac{-2x}{2\sqrt{1 - x^2}} = \dfrac{-x}{\sqrt{1 - x^2}}$ exist in $(-1, 1)$. So, $f(x)$ is differentiable in $(-1, 1)$.

(iii) $f(1) = f(-1) = 0$.

Thus, all three conditions are holds on $f(x)$. Hence, $\exists$ a point $c \in (-1, 1)$ such that

$$f'(c) = 0.$$

$$\Rightarrow \quad \frac{-c}{\sqrt{1 - c^2}} = 0$$

$$\Rightarrow \quad c = 0 \in (-1, 1).$$

So, Rolle's theorem is verified.

Example 3: Discuss the applicability of Rolle's theorem to function $f(x) = \dfrac{x^2 - x - 6}{x - 1}$ on $[-2, 3]$.

Solution: Here $f(1)$ does not exist; thus $f(x)$ is not continuous at $x = 1$. Hence, $f(x)$ is not continuous on $[-2, 3]$. So, condition (i) for Rolle's theorem, is not satisfied.

Thus, in this function Rolle's theorem is not applicable.

This theorem is true when a ball is thrown directly upward.

Example 4: Let S = f(t) be position function of a moving object. If the object is in the same plane at *two distinct instants* t = a and t = b then f(a) = f(b). Then Rolle's theorem says that there is some instant of time t = c between a and b when the velocity is zero i.e. f'(c) = 0.

The main importance of the Rolle's Theorem is that it is useful in the prove the Mean Value Theorem which was first formulated by a French Mathematician (born in Italy) Joseph-Louis Lagrange *(1736–1813)*.

Mean Value Theorem:

Let f be a function that satisfies the following hypothesis:

(i) f is continuous on the closed interval [a, b].

(ii) f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{i.e. } f(b) - f(a) = (b - a) f'(c).$$

Proof:

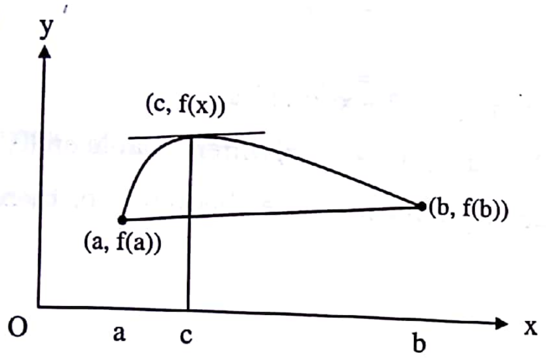


Fig. (i)

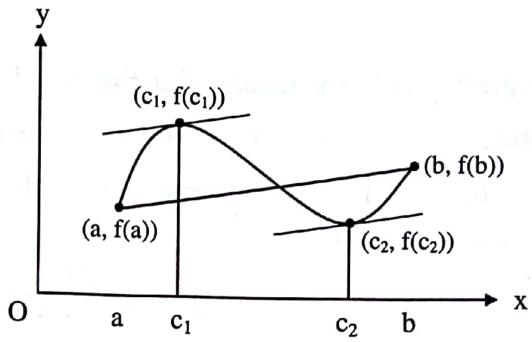


Fig. (ii)

Proof: We define a new function $g(x)$ involving $f(x)$ so that $g(x)$ satisfy the Rolle's theorem in $[a, b]$. So, consider

$$g(x) = f(x) + Ax, \qquad \text{where A is constant}$$

such that $g(a) = g(b)$.

Since,

$$\begin{aligned} g(a) = g(b) &\Rightarrow f(a) + Aa = f(b) + Ab \\ &\Rightarrow Aa - Ab = f(b) - f(a) \\ &\Rightarrow A(a - b) = f(b) - f(a) \\ &\Rightarrow -A = \frac{f(b) - f(a)}{b - a} \end{aligned}$$

Since, Ax is continuous everywhere hence is continuous on $[a, b]$. Also, $f(x)$ is continuous on $[a, b]$ by given condition. Therefore $f(x) + Ax = g(x)$ also continuous on $[a, b]$ Similarly, $f(x) + Ax = g(x)$ also differentiable on (a, b).

Moreover, we have $g(a) = g(b)$.

Thus, $g(x)$ satisfies all three condition of Rolle's theorem, so by the Rolle's theorem there exist at least a point $c \in (a, b)$ such that

$$\begin{aligned} & g'(c) = 0 \\ \Rightarrow\ & f'(c) + A = 0 \\ \Rightarrow\ & f'(c) = -A \\ \Rightarrow\ & f'(c) = \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

This completes the proof.

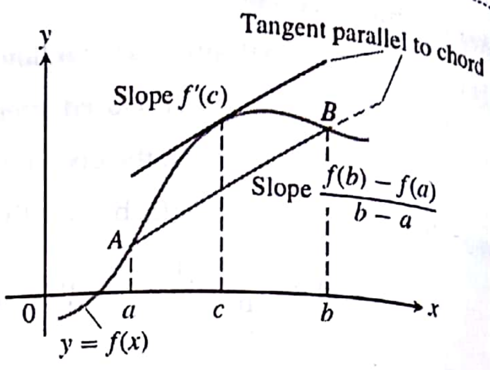
Geometrical Meaning of Mean Value Theorem

If a function $f(x)$ is

(i) Continuous in the closed interval $[a, b]$;

(ii) Differentiable in the open interval (a, b);

Then, there exist at least a real number $c \in (a, b)$ such that the slope of chord joining the points $(a, f(a))$ and $(b, f(b))$ is equal to slope of tangent drawn at $(c, f(c))$.



Example 5: Verify the mean value theorem for function $f(x) = 1 - x^2$ on $[0, 2]$.

Solution: Since, $f(x) = 1 - x^2$ is continuous on $[0, 2]$ and $f'(x) = -2x$ so, differentiable on $(0, 2)$.

Thus, $f(x) = 1 - x^2$ satisfy the both conditions for mean value theorem. So, there exist $c \in (0, 2)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad -2c = \frac{f(2) - f(0)}{2 - 0}$$

$$\Rightarrow \quad -2c = \frac{-3 - 1}{2}$$

$$\Rightarrow \quad -2c = -1.$$

$$\Rightarrow \quad c = \frac{1}{2} \in (0, 2).$$

Hence, mean value theorem satisfied.

Example 6: Discuss the applicability of mean value theorem to function $f(x) = x^{\frac{2}{3}}$ on $[-1, 8]$.

Solution: Since, $$f(x) = x^{\frac{2}{3}}$$

Then $$f'(x) = \frac{2}{3} x^{\frac{2}{3} - 1} = \frac{2}{3x^{\frac{1}{3}}}$$

Here, $f'(0) = \infty$. So, $f(x)$ is not differentiable on $(-1, 8)$.

So, condition (ii) for mean value theorem is not satisfied.

Thus, in this function mean value theorem is not applicable.

Example 7: Find the point 'c' of mean value theorem for the function $f(x) = \sqrt{x - 1}$ on $[1, 3]$.

Solution: Since, $\lim_{x \to 1^+} f(x) = f(1)$. So $f(x)$ is continuous at $x = 1$.

Also, $\lim_{x \to 3^-} f(x) = f(3)$, so $f(x)$ is continuous at $x = 3$.

Thus, $f(x)$ is continuous at end point and interior points of $[1, 3]$. So,

(i) $f(x)$ is continuous on $[1, 3]$

(ii) $f'(x) = \frac{1}{2\sqrt{x - 1}}$ exist on $(1, 3)$. So, $f(x)$ is differentiable on $(1, 3)$.

Thus, both conditions for mean value theorem are satisfied, hence $\exists\, c \in (1, 3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2} - 0}{2}$$

$$\Rightarrow \quad \sqrt{c-1} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad c = \frac{3}{2}.$$

Corollary 1: Functions with zero derivatives are constant.

OR

If $f'(x) = 0,\ \forall\ x \in I$; then $f(x) = c\ \forall\ x \in I$, where c is constant.

Proof: To show $f(x) = c\ \forall\ x$ in I (i.e. function is constant), where $f'(x) = 0\ \forall\ x \in I$, we show

$$f(x_1) = f(x_2) \qquad \text{for any } x_1, x_2 \in I \text{ with } x_1 < x_2.$$

Let, $x_1, x_2 \in I$ with $x_1 < x_2$. Then, $f(x)$ is differentiable on $[x_1, x_2]$ because given that $f'(x) = 0\ \forall\ x \in I$. Also, $f(x)$ is continuous on $[x_1, x_2]$ because every differentiable function is continuous.

Thus, both condition of mean value theorem is satisfied on $f(x)$, hence $\exists\ c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow \quad 0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow \quad f(x_2) - f(x_2) = 0$$

$$\Rightarrow \quad f(x_1) = f(x_2).$$

Therefore, for $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$. So $f(x)$ is constant function.

Corollary 2: Functions with same derivative differ by a constant.

OR

If $f'(x) = g'(x)$ at each $x \in I$ then there exist a constant c such that $f(x) - g(x) = c\ \forall\ x \in I$.

Proof: Let, $h(x) = f(x) - g(x)\ \forall\ x \in I$. Then,

$$h'(x) = f'(x) - g'(x) \qquad \forall\ x \in I$$

$$\Rightarrow \quad h'(x) = 0 \qquad [\text{given that } f'(x) = g'(x)]$$

$$\Rightarrow \quad h(x) = \text{constant} = c \qquad [\text{using cor. 1}]$$

$$\Rightarrow \quad f(x) - g(x) = c \qquad \forall\ x \in I$$

Corollary 3: Suppose that f is continuous on [a, b] and differentiable on (a, b).

(i) If $f'(x) > 0$ at each $x \in (a, b)$ then f is increasing on $[a_1\ b]$

(ii) If $f'(x) < 0$ at each $x \in (a, b)$ then f is decreasing on $[a_1\ b]$

Proof: Let x_1 and x_2 in [a, b] with $x_1 < x_2$. Here, f(x) is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) so by mean value theorem of f(x) on $[x_1, x_2]$; $\exists\, c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) - f(x_1) = f'(c)\,(x_2 - x_1) \qquad \ldots (A)$$

Since, $x_1 < x_2 \Rightarrow x_2 - x_1 > 0.$

(i) If $f'(x) > 0$ at each $x \in (a, b) \Rightarrow f'(c) > 0$ then RHS of (A) is greater than 0.

Hence, from (A),

$$f(x_2) - f(x_1) > 0$$

$$\Rightarrow f(x_2) > f(x_1).$$

Thus for $x_1 < x_2$ we find $f(x_2) > f(x_1)$. This means $f(x)$ is increasing.

(ii) If is $f'(x) < 0$ at each $x \in (a, b) \Rightarrow f'(c) < 0$ then RHS of (A) is less than 0.

Hence, from (A),

$$f(x_2) - f(x_1) < 0.$$

$$\Rightarrow f(x_2) < f(x_1).$$

Thus for $x_1 < x_2$ we find $f(x_2) < f(x_1)$. This means $f(x)$ is decreasing.

The Mean Value Theorem can be used to establish the following fact:

Theorem: If f'(x) = 0 for all x in (a, b) then f is constant in (a, b).

Note: We must careful in applying above theorem.

Let

$$f(x) = \frac{|x|}{x} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Then the domain of f(x) is $D = \{x : x \neq 0\}$ and f'(x) = 0 for all x in D. Clearly f is not a constant function on D. This does not contradict to above theorem because D is not an interval. Notice that f is a constant function in $(0, \infty)$ and also in $(-\infty, 0)$.

Example 8: Prove that $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$.

Solution: Let

$$f(x) = \tan^{-1}x + \cot^{-1}x$$

Then

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1-x^2} = 0 \qquad \text{for all } x.$$

Therefore,

$$f(x) = c \text{ (a constant value).}$$

Set x = 1 then

$$f(1) = \tan^{-1}(1) + \cot^{-1}(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

and f(1) = c.

Thus,

$$c = f(x) = \frac{\pi}{2}.$$

Thus, $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$.

Example 9: Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x. How large can f(2) possibly be?

Solution: Suppose that

$$f(0) = -3 \quad \text{and} \quad f'(x) \le 5 \quad \text{for all values of } x.$$

This implies f is differentiable (and therefore is continuous) everywhere.

In particular, we choose f is defined on [0, 2]. Then by Mean Value Theorem on [0, 2], there exists a number c in (0, 2) such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{f(2) + 3}{2}.$$

$$\Rightarrow \quad f(2) = 2f'(c) - 3 \le 2(5) - 3 = 10 - 3 = 7.$$

This shows the largest possible value for f(2) is 7.

Exercise 3.6

1. Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

   a. $f(x) = 5 - 12x + 3x^2$, in [1, 3] b. $f(x) = x^3 - x^2 - 6x + 2$, in [0, 3]

   c. $f(x) = \cos 2x$, in $[\pi/8, 7\pi/8]$

2. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in (–1, 1) such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

3. Let $f(x) = \tan x$. Show that $f(0) = f(\pi)$ but there is no number c in $(0, \pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

4. Verify that the function satisfies the three hypotheses the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

   a. $f(x) = 2x^2 - 3x + 1$, in [0, 2]. b. $f(x) = x^3 + x - 1$ [0, 2]

   c. $f(x) = e^{-2x}$, in [0, 3] d. $f(x) = \frac{x}{x+2}$ in [1, 4].

5. Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in (1, 4) such that $f(4) - f(1) = f'(c)(4 - 1)$. Why does this not contradict the Mean Value Theorem?

6. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval [–2, 2].

7. If $f(1) = 10$ and $f'(x)$ 2 for $1 \le x \le 4$, how small can f(4) possibly be?

8. Use the Mean Value Theorem to prove the inequality $|\sin a - \sin b| \le |a - b|$ for all a and b.

9. Use the method of Example 3 to prove the identity $2\sin^{-1}x = \cos^{-1}(1 - 2x^2)$ for $x \ge 0$.

Answers

1. (a) $c = 2$ (b) $c = \frac{1+\sqrt{19}}{3}$ (c) $= \frac{\pi}{2}$
2. f is not differentiable at $0 \in (-1, 1)$
3. f is not continuous at $\frac{\pi}{2} \in [0, \pi]$
4. (a) $c = 1$ (b) $c = \pm\frac{2}{\sqrt{3}}$ (c) $c = \frac{3}{\ln(4)}$ (d) $c = \sqrt{3}$
5. f is not continuous at 3
7. Smallest value for f(4) is 16.

3.6 Indeterminate Forms and L'Hospital Rule

Suppose a function

$$f(x) = \frac{\ln(x)}{x-1} \qquad \dots\dots (1)$$

Clearly f(x) is undefined at $x = 1$. But if x approaches to 1 then what value takes by f? In this case we can not apply the limit directly (being the denominator is zero). Here, if the denominator leads then $f(x) \to \infty$ as $x \to 1$ and if the numerator leads then $f(x) \to 0$ as $x \to 1$. In fact it is not clear that which case will win.

In general, if a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

with both $f(x) \to 0$, $g(x) \to 0$ as $x \to a$ then this limit may or may not exist, is called an **indeterminate form** of type $\frac{0}{0}$.

To evaluation of indeterminate forms, we introduce here a systematic method, known as L'Hospital Rule.

L'Hospital Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a number 'a' (except possibly at a). And suppose that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side, exists.

Note: L'Hospital Rule is also valid for one-sided limits and for limits at infinity (may positive or negative).

Example 1: Find $\lim_{x\to 1} \frac{\ln(x)}{x-1}$.

Solution: Here,

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} \quad \left[\text{This form is in } \frac{0}{0} \text{ type as } x\to 1\right]$$

$$= \lim_{x\to 1} \frac{1/x}{1}$$

$$= \lim_{x\to 1} \frac{1}{x} = \frac{1}{1} = 1$$

Thus, $\lim_{x\to 1} \frac{\ln(x)}{x-1} = 1$.

Different Types of Indetermiane Forms

A. Indeterminate forms of type $\frac{\infty}{\infty}$.

If we choose $x\to\infty$ then (1) implies $f(x)\to\infty$ if denominator leads and $f(x)\to\infty$ if numerator leads.

In general, if the limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

with both $f(x)\to\infty$, $g(x)\to\infty$ as $x\to a$ then the limit may exist or may not exist, is called an indeterminate form of type $\frac{\infty}{\infty}$.

As above the L'Hospital rule is applicable to evaluation of indeterminate forms of type $\frac{\infty}{\infty}$.

L'Hospital Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a number 'a' (except possibly at a). And, suppose that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side, exists.

Example 2: Calculate $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution: Here,

$$\lim_{x\to\infty} \frac{e^x}{x^2} \quad \left[\text{This form is in } \frac{\infty}{\infty} \text{ type as } x\to\infty\right]$$

$$= \lim_{x\to\infty} \frac{e^x}{2x} \quad \left[\text{This form is in } \frac{\infty}{\infty} \text{ type as } x\to\infty\right]$$

$$= \lim_{x\to\infty} \frac{e^x}{2}$$

$$= \infty.$$

Thus $\lim_{x\to\infty} \frac{e^x}{x^2} = \infty$.

B. Indeterminate forms of type $0\times\infty$.

Sometimes we observe the limit behaves as $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$. Then the product of such functions under the limit, gives 0 value if f wins and ∞ if g wins. This kind of limit is called indeterminate form of type $0\times\infty$. We deal it by writing the product f.g as quotient as

$$\lim_{x\to a} fg = \lim_{x\to a} \frac{f}{1/g} \quad \text{or} \quad \lim_{x\to a} fg = \lim_{x\to a} \frac{g}{1/f}$$

so that the type $0\times\infty$ converts into the form either $\frac{0}{0}$ or $\frac{\infty}{\infty}$. After then we can use L'Hospital rule.

Example 3: Evaluate $\lim_{x\to 0^+} x \ln(x)$.

Solution: Here

$$\lim_{x\to 0^+} x \ln(x) \qquad [\text{This form is in } 0\times\infty \text{ type as } x\to 0^+]$$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \qquad \left[\text{This form is in } \frac{\infty}{\infty} \text{ type as } x\to 0^+\right]$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} \frac{1}{-1/x}$$

$$= \lim_{x\to 0^+} (-x)$$

$$= 0.$$

Note: If we unable to choose right path for conversion then the solution processes may more complicate and sometimes the solution process may divert to answer. In above example if we choose another possible option.

Remark: Here,

$$\lim_{x\to 0+} x \ln(x) \qquad [\text{This form is in } 0\times\infty \text{ type as } x\to 0^+]$$

$$= \lim_{x\to 0+} \frac{x}{1/\ln(x)} \qquad \left[\text{This form is in } \frac{\infty}{\infty} \text{ type as } x\to 0^+\right]$$

$$= \lim_{x\to 0+} \frac{1}{-1/(\ln(x))^2 \cdot 1/x}$$

$$= \lim_{x\to 0+} \frac{-x}{1/(\ln(x))^2} \qquad \left[\text{This form is in } \frac{0}{0} \text{ type as } x\to 0^+\right]$$

This shows the process more complicated expression rather than initial step.

Example 4: Evaluate $\lim_{x\to\infty} x \tan(1/x)$.

Solution: Here,

$$\lim_{x\to\infty} x\tan(1/x) \qquad \text{[This form is in } 0\times\infty \text{ type as } x\to\infty]$$

$$= \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} \qquad \left[\text{This form is in } \frac{0}{0} \text{ type as } x\to\infty\right]$$

$$= \lim_{x\to\infty} \frac{\sec^2(1/x)\,(-1/x^2)}{(-1/x^2)}$$

$$= \lim_{x\to\infty} \sec^2(1/x)$$

$$= \sec^2(0)$$

$$= 1.$$

Thus, $\lim_{x\to\infty} x\tan(1/x) = 1$.

Alternative method:

Here,

$$\lim_{x\to\infty} x\tan(1/x) \qquad \text{[This form is in } 0\times\infty \text{ type as } x\to\infty]$$

Put $x = \frac{1}{y}$ for $y \neq 0$ then $y\to 0$ as $x\to\infty$. So that

$$= \lim_{y\to 0} \frac{1}{y}\cdot\tan(y)$$

$$= \lim_{y\to 0} \frac{\tan(y)}{y} \qquad \left[\text{This form is in } \frac{0}{0} \text{ type as } x\to\infty\right]$$

$$= \lim_{y\to 0} \frac{\sec^2 y}{1}$$

$$= \sec^2(0)$$

$$= 1.$$

Thus $\lim_{x\to\infty} x\tan(1/x) = 1$.

C. Indeterminate forms of type $\infty - \infty$.

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an indeterminate form of type $\infty - \infty$.

To evaluate such indeterminate form by converting the difference into a quotient so that above form takes the form of type either $\frac{0}{0}$. After then we can use L'Hospital rule.

Remember: We convert the difference into quotient as:

$$\lim_{x\to a} [f(x) - g(x)] \qquad [\text{in } \infty - \infty \text{ form}]$$

$$= \lim_{x\to a}\left[\frac{11}{1/f} - \frac{1}{1/g}\right] \qquad \left[\text{in}\left(\frac{1}{0}-\frac{1}{0}\right)\text{form}\right]$$

$$= \lim_{x\to a}\left[\frac{1/g - 1/f}{1/fg}\right] \qquad \left[\text{in}\frac{0-0}{0}\text{ i.e. }\frac{0}{0}\text{ form}\right].$$

Example 5: Compute $\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$

Solution: Here,

$$\lim_{x\to(\pi/2)^-} (\sec x - \tan x) \qquad [\text{This form is in } \infty - \infty \text{ type as } x \to (\pi/2)^-]$$

$$= \lim_{x\to(\pi/2)^-}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to(\pi/2)^-}\left(\frac{1-\sin x}{\cos x}\right) \qquad [\text{This form is in } \tfrac{0}{0} \text{ type as } x\to(\pi/2)^-]$$

$$= \lim_{x\to(\pi/2)^-}\left(-\frac{\cos x}{\sin x}\right)$$

$$= \frac{-\cos(\pi/2)}{\sin(\pi/2)}$$

$$= \frac{-0}{1}$$

$$= 0.$$

Thus, $\lim_{x\to(\pi/2)^-} (\sec x - \tan x) = 0.$

D. Indeterminate forms of type power form.

Consider $f(x) \to 0$ or $f(x) \to \infty$ and $g(x) \to 0$ or $g(x) \to \infty$ as $x \to a$, then from the limit

$$\lim_{x\to a} [f(x)]^{g(x)}$$

We observe several forms like 0^0, 0^∞, ∞^0, ∞^∞. Also, consider $f(x) \to 1$ and $g(x) \to \infty$ as $x \to a$ then the limit

$$\lim_{x\to a} [f(x)]^{g(x)}$$

takes the form 1^∞.

Recall that among these five forms 0^∞ and ∞^∞ are not an indeterminate forms because multiple of infinite times of zero(s) is zero and multiple of multiple of infinite times of ∞ is again ∞.

Therefore, the forms 0^0, ∞^0 and 1^∞ are indeterminate forms.

Solution Process of each of these three cases:

Let $y = [f(x)]^{g(x)}$

Then

$$\ln(y) = g^{(x)} \ln [f(x)]$$

So, $$\lim_{x\to a} \ln (y) = \lim_{x\to a} g(x).\ln (f(x)) \quad \ldots\ldots \text{(i)}$$

The right part of (i) will takes the form of type $0\times\infty$ and we solve it (as given B), and we get the solution β (say). Therefore,

$$\lim_{x\to a} \ln(y) = \beta$$

$$\Rightarrow \ln\left(\lim_{x\to 0^+} y\right) = \beta$$

$$\Rightarrow \lim_{x\to a} y = e^{\beta}$$

$$\Rightarrow \lim_{x\to a} [f(x)]^{g(x)} = e^{\beta}$$

Example 6: Calculate $\lim_{x\to 0^+} x^x$.

Solution: Here,

$$\lim_{x\to 0^+} x^x \qquad \text{[This form is in } 0^\circ \text{ type as } x\to 0^+]$$

Put $y = x^x$. Then

$$\lim_{x\to 0^+} \ln (y) = \lim_{x\to 0^+} [x \ln (x)] \qquad \text{[This form is in } 0\times\infty \text{ type as } x\to 0^+]$$

$$= \lim_{x\to 0^+} \left(\frac{\ln(x)}{1/x}\right) \qquad \left[\text{This form is in } \frac{\infty}{\infty} \text{ type as } x\to 0^+\right]$$

$$= \lim_{x\to 0^+} \left(\frac{1/x}{-1/x^2}\right)$$

$$= \lim_{x\to 0^+} (-x)$$

$$= 0.$$

Thus, $\lim_{x\to 0^+} \ln (y) = 0.$

$$\Rightarrow \ln\left(\lim_{x\to 0^+} y\right) = 0$$

$$\Rightarrow \lim_{x\to 0^+} y = e^0 = 1$$

Therefore, $\lim_{x\to 0^+} x^x = 1.$

Example 7: Compute $\lim_{x\to 0^+} (\cos x)^{1/x^2}$

Solution: Here,

$$\lim_{x\to 0^+} (\cos x)^{1/x^2}. \qquad \text{[This form is in } 1^\infty \text{ type as } x\to 0^+]$$

Put $y = (\cos x)^{1/x^2}$. Then

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \frac{1}{x^2}\cdot \ln(\cos x) \qquad \left[\text{This form is in } \frac{0}{0} \text{ type as } x\to 0^+\right]$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

$$= \left(\frac{-1}{2}\right) \lim_{x\to 0^+} \frac{\tan x}{x} \qquad \left[\text{This form is in } \frac{0}{0} \text{ type as } x\to 0^+\right]$$

$$= \left(\frac{-1}{2}\right) \lim_{x\to 0^+} \frac{\sec^2 x}{1}$$

$$= \left(\frac{-1}{2}\right) \sec^2(0)$$

$$= \frac{-1}{2} \qquad (\because \sec^2 0 = 1)$$

Thus,

$$\lim_{x\to 0^+} \ln(y) = \frac{-1}{2}.$$

$$\Rightarrow \ln\left(\lim_{x\to 0^+} y\right) = \frac{-1}{2}$$

$$\Rightarrow \lim_{x\to 0^+} (y) = e^{-1/2}$$

Therefore, $\lim_{x\to 0^+} (\cos x)^{1/x^2} = e^{-1/2}$.

Exercise 3.7

1. Find the limit. Use L'Hospital Rule where appropriate. If there is a more elementary method consider using it. If L'Hospital Rule does not apply, explain why.

a. $\lim_{x\to 1} \frac{x^2 - 1}{x^2 - x}$ b. $\lim_{x\to 1/2} \frac{6x^2 + 5x - 4}{4x^2 + 16x - 9}$ c. $\lim_{x\to(\pi/2)^+} \frac{\cos x}{1 - \sin x}$

d. $\lim_{\theta\to\pi/2} \frac{1 - \sin\theta}{1 + \cos 2\theta}$ e. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ f. $\lim_{x\to 0^+} \frac{\ln x}{x}$

g. $\lim_{x\to\infty} \frac{\ln\sqrt{x}}{x^2}$ h. $\lim_{x\to 0} \frac{\tanh x}{\tan x}$ i. $\lim_{x\to 0} \frac{\sin^{-1} x}{x}$

j. $\lim_{x\to\infty} x \sin(\pi/x)$ k. $\lim_{x\to\infty} \sqrt{x}\, e^{-x/2}$ l. $\lim_{x\to 0} \cot 2x \sin 6x$

m. $\lim_{x \to 0^+} \sin x \ln x$ n. $\lim_{x \to \infty} x^3 e^{-x^2}$ o. $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

p. $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$ q. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$ r. $\lim_{x \to 0} \left(\cot x - \frac{1}{x}\right)$

s. $\lim_{x \to \infty} (x - \ln x)$ t. $\lim_{x \to 0^+} x^{\sqrt{x}}$ u. $\lim_{x \to 0^+} (\tan 2x)^x$

v. $\lim_{x \to 0} (1 - 2x)^{1/x}$ w. $\lim_{x \to \infty} x^{1/x}$ x. $\lim_{x \to \infty} (e^x + x)^{1/x}$

2. Prove that $\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$ for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.
3. Prove that: $\lim_{x \to \infty} \frac{\ln x}{x^p} = 0$ for any number $p > 0$. This shows that the logarithmic function approaches ∞ more slowly than any power of x.
4. What happens if you try to use L'Hospital's Rule to find the limit? Evaluate the limit using another method.

a. $\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}$ b. $\lim_{x \to (\pi/2)^-} \frac{\sec x}{\tan x}$

Answers

1. (a) 2 (b) $\frac{11}{20}$ (c) $-\infty$ (d) $\frac{1}{4}$ (e) 0 (f) $-\infty$ (g) 1 (h) 1 (i) 1 (j) π (k) 0 (*l*) 3 (m) 0 (n) 0 (o) $\frac{1}{2}$
(p) 0 (q) $\frac{1}{2}$ (r) 0 (s) ∞ (t) 1 (u) 1 (v) e^{-2} (w) 1 (x) e
2. ∞
3. 0
4. (a) 1 (b) 1

Chapter 4

Application of Derivative

4.1 Curve Sketching

In this section we sketch graph of given function. To sketch the graph of a function, we should know information about the given function, which we have already learn in previous chapters.

Guidelines for Sketching a Curve

To sketch the curve of given function y = f(x) by hand, following information about curve must know:

A. Domain: Find the domain of the given function, it means value of x so that f(x) is exist.

B. Intercepts: It means where curve meet x-axis and y-axis. For x-intercept (i.e. it meet x-axis) put y = 0 and find value of x and for y-intercept (i.e. it meet y-axis) put x = 0 and find value of y.

C. Symmetry: For curve symmetric about y-axis, f(x) = f(–x). For curve symmetric about origin, f(–x) = – f(x).

D. Asymptote: [In general rational function and logarithmic and exponential functions has asymptote].

Horizontal asymptotes: If either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$ then y = L is horizontal asymptote of curve y = f(x).

Vertical asymptotes: A line x = a is vertcal asymptote of y = f(x) if either $\lim_{x \to a^+} f(x) = \pm \infty$ or $\lim_{x \to a^-} f(x) = \pm \infty$.

Note that graph of asymptotes must be dashed lines.

E. Interval of increasing and decreasing: Find the critical points by using f'(x) = 0 and f'(x) = ∞ and find the interval of increasing and decreasing.

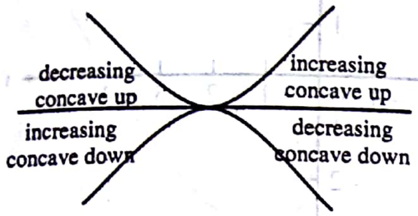
F. Local maxima and minima points. Find the points where local maximum and minimum value of f(x) occurs. This is obtained from table formed from E. Note that local maximum point is occure at that point where sign is changed from + to - ve and local minimum is obtained when sign is changed from -ve to +ve.

G. Concavity and point of inflection: By computing f''(x) = 0 and f''(x) = ∞, find the interval of concave up and concave down and find the point of inflection. Note that point where concavity changes (changes from +ve to -ve or -ve to +ve), is point of inflection.

H. Using the above information A-G. Sketch the graph by free hand.

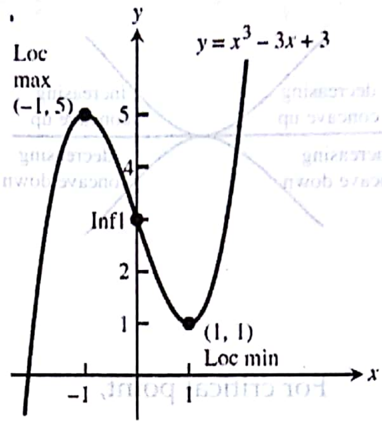
Example 1: Sketch the graph of $f(x) = x^3 - 3x + 3$.

Solution: Given curve is,



$$f(x) = x^3 - 3x + 3$$

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$$



For critical points;

$$f'(x) = 0.$$

$$\Rightarrow 3(x-1)(x+1) = 0.$$

$$\Rightarrow x = 1, -1.$$

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of f'	+ve	-ve	+ve
Nature of f	Increasing	decreasing	Increasing

Thus, the maxima occur at $x = -1$, and $(-1, f(-1))) = (-1, 5)$ is maximum point. And minima occur at $x = 1$ i.e. $(1, f(1)) = (1, 1)$ is minimum point.

Again, $f''(x) = 6x$.

For point of inflection,

$$f''(x) = 0$$

$$\Rightarrow 6x = 0$$

$$\Rightarrow x = 0.$$

Interval	$(-\infty, 0)$	$(0, \infty)$
Sign of f''	-ve	+ve
Nature of f	Concave down	Concave up

Thus, point of inflection occur at $x = 0$. So $(0, f(0)) = (0, 3)$ is the point of inflection.

Summarizing above two tables,

$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Increasing	Decreasing	Decreasing	Increasing
Concave down	Concave down	Concave up	Concave up

Example 2: Sketch the graph of $f(x) = x^{4/3} - 4x^{1/3}$

Solution: Given curve is,

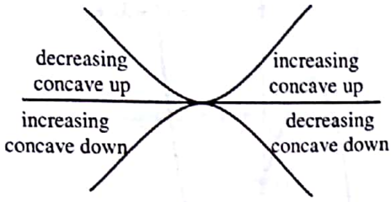
$$f(x) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$$

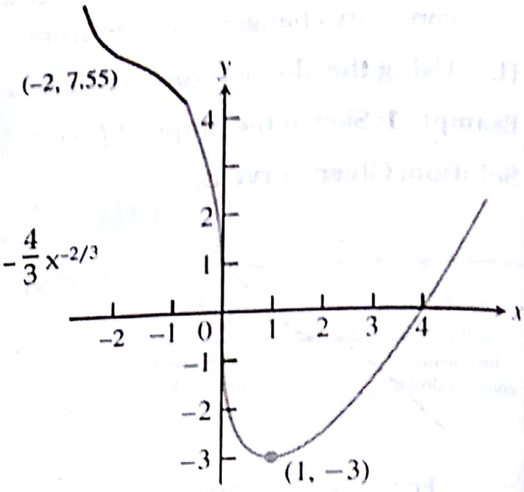
$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{1/3} . x^{-2/3} . x^{2/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}[x-1]$$

$$= \frac{4(x-1)}{3x^{2/3}}$$





For critical point,

$$f'(x) = 0 \text{ and } f'(x) = \infty.$$

Therefore, the critical points are $x = 0, 1$.

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f'	-ve	-ve	+ve
Nature of f	Decreasing	Decreasing	Increasing

This shows the minima occur at $x = 1$. So, the minima point is $(1, f(1)) = (1, -3)$.

Again, $$f''(x) = \frac{4}{9} \times x^{-2/3} + \frac{8}{9}x^{-5/3}$$

$$= \frac{4}{9}x^{-2/3}\, x^{-5/3} . x^{5/3} + \frac{8}{9}x^{-5/3}$$

$$= \frac{4}{9}x^{-5/3}(x+2)$$

$$= \frac{4(x+2)}{9x^{5/3}}$$

For point of inflection,

$$f''(x) = 0 \text{ and } f''(x) = \infty$$

which gives $x = -2$ and $x = 0$.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
Sign of f''	+ve	-ve	+ve
Nature of f	Concave up	Concave down	Concave up

Here, point of inflections are at $x = -2$ and at $x = 0$

$\therefore$ Point of inflections are $(0, 0)$ and $(-2, 7.56)$

Summarizing above tables

$(-\infty, -2)$	$(-2, 0)$	$(0, 1)$	$(1, \infty)$
Decreasing	Decreasing	Decreasing	Increasing
Concave up	Concave down	Concave up	Concave up

Example 3: Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. For domain, set of all real number except 1 and -1.

i.e. domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

B. For the x-intercept: Put $y = 0$ we get $x = 0$

For the y-intercept: Put $x = 0$ we get $y = 0$

Thus curve meet the x-axis and y-axis at origin.

C. Since $f(-x) = f(x)$, so it is symmetrical about x-axis.

D. To find asymptotes;

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x\to\pm\infty} \frac{2}{1 - 1/x^2} = 2.$$

Thus $y = 2$ is vertical asymtote.

Here $\lim\limits_{x\to 1} f(x) = \infty$ and $\lim\limits_{x\to -1} f(x) = \infty$.

Thus $x = +1$ and $x = -1$ are. Horizontal asymptote.

E. For interval of increasing and decreasing,

$$f'(x) = \frac{(x^2 + 1)\,4x - 2x^2 \times 2x}{(x^2 + 1)^2} = -\frac{4x}{(x^2 - 1)^2}$$

For critical points $f'(x) = 0$ and $f'(x) = \infty$

We get $x = 0$ and $x = \pm 1$

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Sign of $f'(x)$	+ve	+ve	–ve	–ve
Nature of $f(x)$	Increasing	Increasing	Decreasing	Decreasing

F. Sign is changing from +ve to -ve at point $x = 0$, so

Maxima occur at $x = 0$, so maximum point is $(0, y(0))$, i.e. $(0, 0)$.

G. For concavity and point of inflection.

$$f''(x) = \frac{(x^2 - 1)^2 \times -4 + 4x\,2(x^2 - 1) \times 2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

For point of inflection $f''(x) = 0$ and $f''(x) = \infty$, we get $x = \pm 1$.

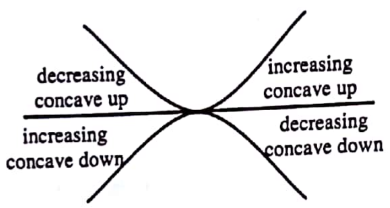
Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of $f'(x)$	+ve	-ve	+ve
Nature of $f(x)$	concave up	concave down	concave up

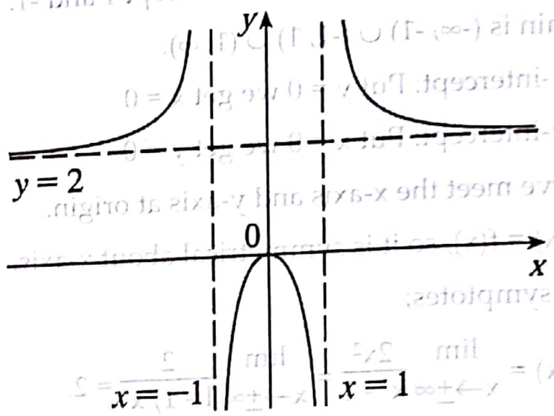
Even signs are changes at x = 1 and x = -1, but they are not point of inflection because they are not lies on domain.

H. Summarizing above two table in E and G.

Interval	(-∞, -1)	(-1, 0)	(0, 1)	(1, ∞)
Nature of f(x)	Increasing	Increasing	Decreasing	Decreasing
f(x)	Concave up	Concave down	Concave down	Concave up

Using this table with informations we can sketch the graph as follows:





Example 4: Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$ by using the guidelines.

A. Domain for the function is $x + 1 \geq 0$

i.e. $x \geq -1$

So domain is $[-1, \infty)$.

B. Intercepts: The x-intercept is x = 0 (Put y = 0 in equation $y = \dfrac{x^2}{\sqrt{x^2+1}}$)

y-intercept is y = 0 (Put x = 0 in equation $y = \dfrac{x^2}{\sqrt{x^2+1}}$)

∴ Curve meet x-axis and y-axis at only one point (0, 0).

C. Symmetry: It is not symmetrical about y-axis and origin.

(Here $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$.

D. Asymptote:

$$\text{Since} \lim_{x\to\infty} \frac{x^2}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^3}+\frac{1}{x^4}}} = \infty \text{ (not finite)}$$

∴ There is no horizonal asymptote.

$$\text{Since} \lim_{x\to 1^+} + \frac{x^2}{\sqrt{x+1}} = \infty$$

Thus x = - 1 is vertical asymptote.

E. Interval of increasing and decreasing:

$$f'(x) = \frac{2x\sqrt{x+1} - x^2 \cdot \frac{1}{2\sqrt{x+1}}}{(x+1)} = \frac{x(3x+4)}{2(x+1)^{3/2}}$$

Let f'(x) = 0 then x = 0 and $x = -\frac{4}{3}$ (This lies outside of domain so no need to take)

$f'(x) = \infty$ then x = - 1.

Thus, x = 0, -1.

Interval	(-1, 0)	(0, ∞)
Sign of f'(x)	-ve	+ve
Nature of f(x)	Decreasing	Increasing

F. Minima occure at x = 0. So minimum point is (0, f(0)) i.e. (0, 0).

G. Concavity and point of inflection:

$$f''(x) = \frac{2(x+1)^{3/2} \cdot (6x+4) - x(3x+4) \cdot 3(x+1)^{1/2}}{4(x+1)^3}$$

$$= \frac{3x^2 + 8x + 8}{4(x+1)^{5/2}}$$

Let f''(x) = 0 then $3x^2 + 8x + 8 = 0$ gives not real number ($b^2 - 4ac = -32$)

$f''(x) = \infty$ then x = - 1

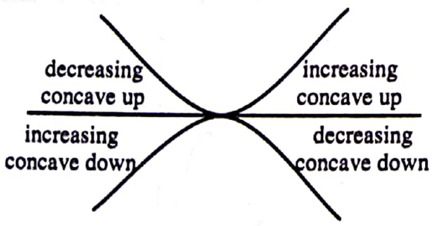
Thus for x = - 1

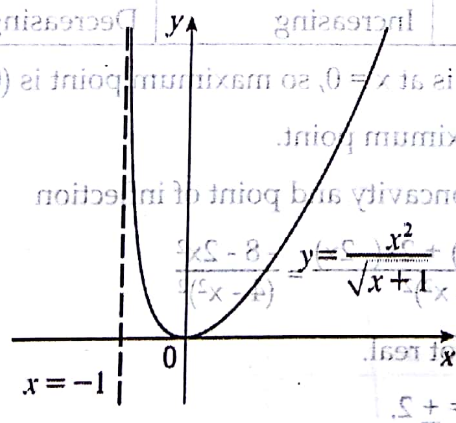
Interval	(-1, ∞)
Sign of f''(x)	+ve
Nature of f(x)	Concave upward

H. Summarizing tables in E and G

Interval	(-1, 0)	(0, ∞)
Nature of	Decreasing	Increasing
f(x)	Concave	Concave

Using this table with information can sketch the graph as follows:





Example 5: Use the guideline to sketch the curve $y = \ln(4 - x^2)$.

A. Domain, $4 - x^2 > 0$

i.e. $x^2 - 4 < 0$

$(x - 2)(x + 2) < 0$

$\therefore$ Domain is $(-2, 2)$

B. Intercepts: For x-intercept; Put $y = 0$ then $\ln(4 - x^2) = 0$

$$4 - x^2 = 1$$

$$x = \pm\sqrt{3}$$

So curve meet x-axis at $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$

For y-intercept, put $x = 0$, we get $y = \ln 4$.

So curve meet y-axis at $(0, \ln 4)$.

C. Symmetry: Here $f(-x) = f(x)$, so curve is symmetrical about y-axis.

D. For vertical asymptote;

$\lim_{x\to 2^-} \ln(4 - x^2) = -\infty$ and

$\lim_{x\to -2^+} \ln(4 - x^2) = -\infty$

Thus $x = -2$ and $x = 2$ are vertical asymptotes.

There is not horizontal asymptotes because $\lim_{x\to\pm\infty} \ln(4 - x^2) = \infty$.

E. For interval of increasing and decreasing

$$f'(x) = \frac{-2x}{4 - x^2}$$

For $f'(x) = 0$ then $x = 0$

and $f'(x) = \infty$ then $x = \pm 2$

So, $x = 0, 2$ and -2.

Interval	(-2, 0)	(0, 2)
Sign of f'(x)	+ve	-ve
Nature of f(x)	Increasing	Decreasing

F. Maximum point is at $x = 0$, so maximum point is $(0, f(0))$

So, $(0, \ln 4)$ is maximum point.

G. For interval of concavity and point of inflection

$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

$f''(x) = 0$ then x not real.

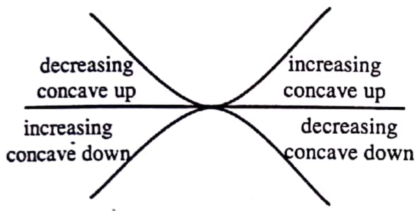
$f''(x) = \infty$ then $x = \pm 2$.

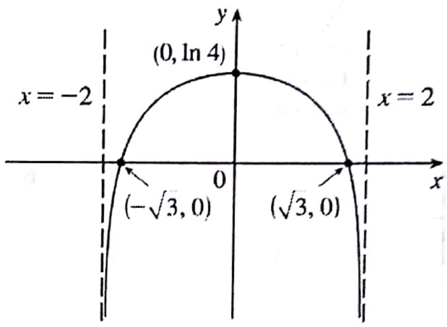
Interval	(-2, 2)
Sign of f''(x)	-ve
Nature of f(x)	Concave down

H: Summarizing the table on E and G

Interval	(-2, 0)	(0, 0)
Nature of	Increasing	Decreasing
f(x)	Concave down	Concave down

Using this table with information we can sketch the graph:





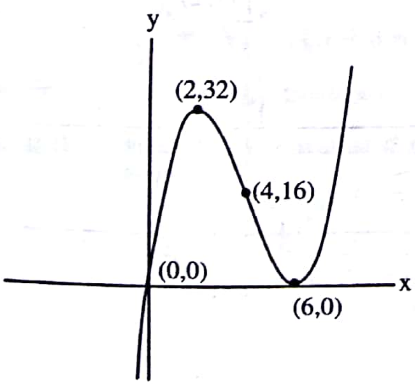
Exercise 4.1

Use the guidelines to sketch the curve

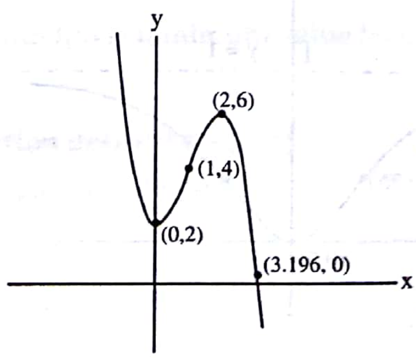
1. $y = x^3 - 12x^2 + 36x$
2. $y = 2 + 3x^2 - x^3$
3. $y = x^4 - 4x^3 + 10$
4. $y = x - 3x^{1/3}$
5. $y = \frac{5}{2}x^{2/3} - x^{5/3}$
6. $y = x^{5/3} - 5x^{2/3}$
7. $y = \frac{x}{x-1}$
8. $y = \frac{x^2}{x^2+9}$
9. $y = \frac{x^2}{x^2+3}$
10. $f(x) = \frac{(x+1)^2}{1+x^2}$
11. $y = \sqrt{x^2 + x - 2}$
12. $y = xe^x$
13. $y = (1 - x)e^x$

Answers:

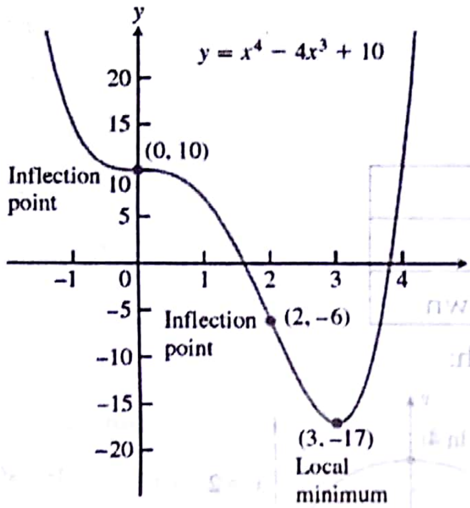
1.



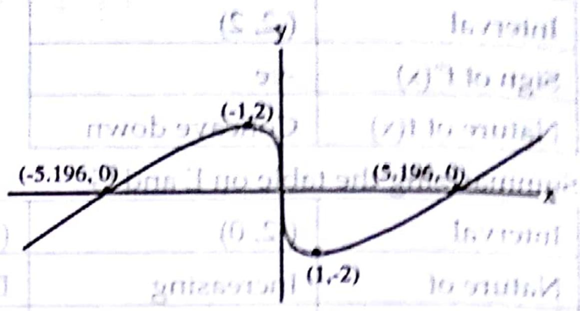
2.



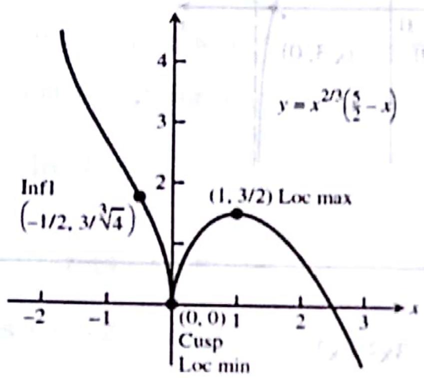
3.


y
$y = x^4 - 4x^3 + 10$
20
15
(0, 10)
Inflection point
10
5
-1
0
1
2
3
4
x
-5
-10
-15
-20
Inflection point
(2, -6)
(3, -17)
Local minimum

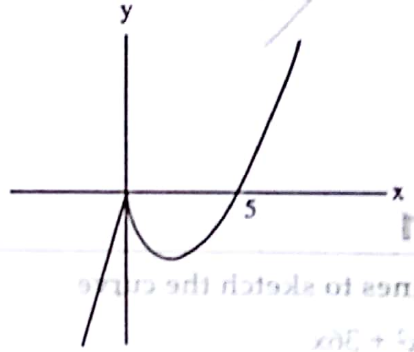
4.


y
(-1,2)
(-5.196, 0)
(5.196, 0)
x
(1,-2)

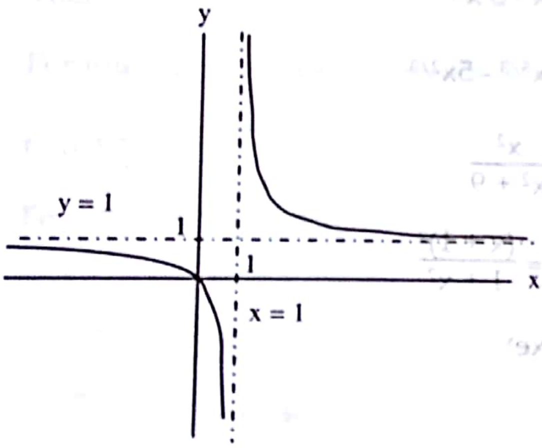
5.


4
3
2
$y = x^{2/3}\left(\frac{5}{2} - x\right)$
Infl
$\left(-1/2, 3/\sqrt[3]{4}\right)$
(1, 3/2) Loc max
-2
-1
(0, 0)
1
2
3
x
Cusp
Loc min

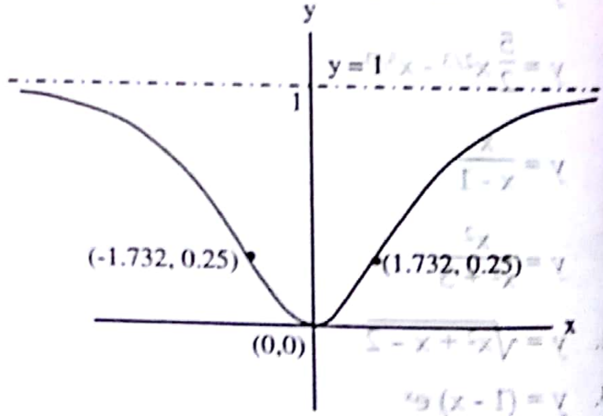
6.


y
x
5

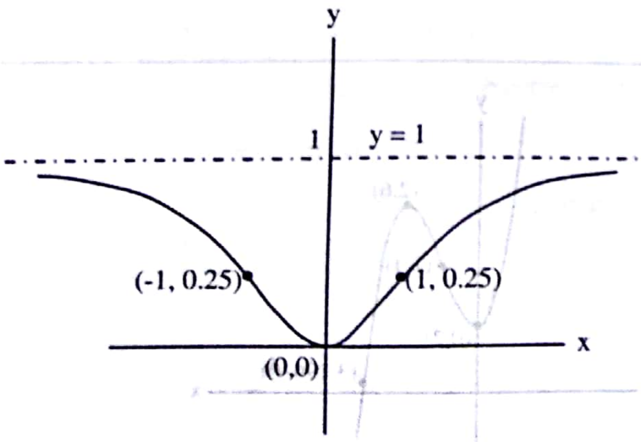
7.


y
y = 1
1
1
x
x = 1

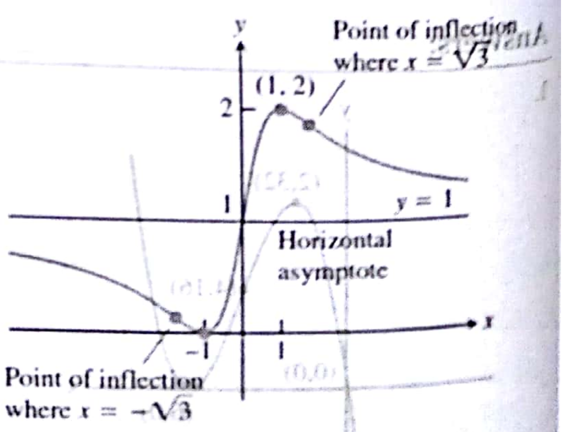
8.


y
y = 1
1
(-1.732, 0.25)
(1.732, 0.25)
(0,0)
x

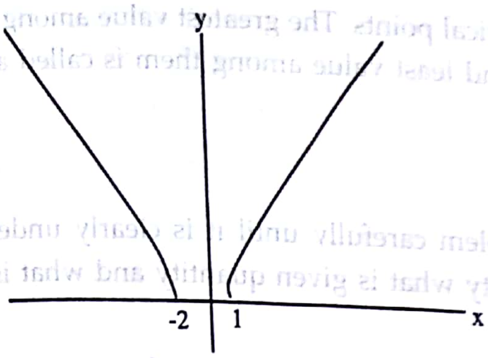
9.


y
1
y = 1
(-1, 0.25)
(1, 0.25)
(0,0)
x

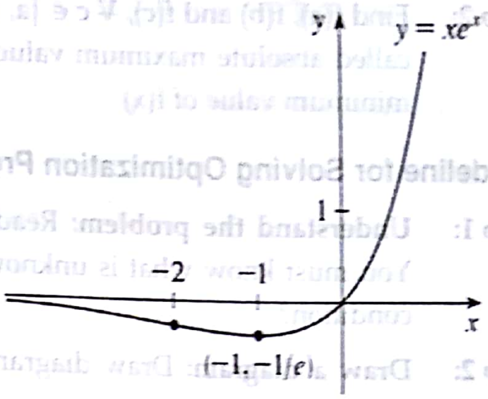
10.


y
Point of inflection
where $x = \sqrt{3}$
(1, 2)
2
1
y = 1
Horizontal asymptote
-1
1
x
Point of inflection
where $x = -\sqrt{3}$

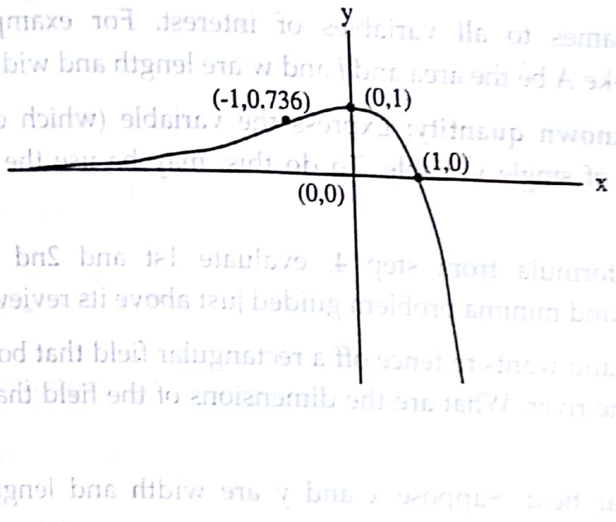
11.



12.



13.



4.2 Optimization Problems

To optimize something means to maximize or minimize some aspect of it; so in this unit we have learned to find the maximum and minimum values of a function of interest (for Example maximum profit, minimum cost). In this section we solve the optimization problems from business. Mathematics and economics. Before enter on optimization problems, we must review how find the local and absolute maximum and minima.

To find the local maxima and minima of function.

Step 1: Find the critical point: Evaluate f'(x) and solve f'(x) = 0. Say x = a, b, c .... are solutions (There are critical points).

Step 2: Evaluate f''(x) and find f''(a).

If f'' (a) < 0 then at x = a maxima occures and f(a) is maximum value of f(x).

If f''(a) > 0 then at x = a minima occures and f(a) is minimum value f(x).

Repeat this for all critical points x = a, b, c, ...

To find the absolute maxima and minima of function defined on [a, b].

Step 1: Find the critical point: Evaluate f'(x) and solve f'(x) = 0.

Take all those critical points which lies on [a, b].

Step 2: Find f(a), f(b) and f(c), $\forall$ c $\in$ [a, b], c is critical points. The greatest value among them is called absolute maximum value of f(x) and least value among them is called absolute minimum value of f(x).

Guideline for Solving Optimization Problems

Step 1: **Understand the problem:** Read the problem carefully until it is clearly understand. You must know what is unknown quantity what is given quantity and what is given condition?

Step 2: **Draw a diagram:** Draw diagram and level the given and required quantities on the diagram.

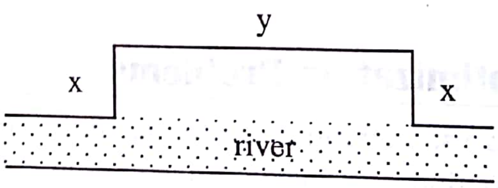
Step 3: **Introduce variables:** Assign names to all variables of interest. For example, to maximize the area of rectangle take A be the area and l and w are length and width.

Step 4: **Write an equation for the unknown quantity:** Express the variable (which can be maximize or minimize) in term of single variable. To do this, may be use the given condition.

Step 5: After knowing the functional formula from step 4, evaluate 1st and 2nd order derivative and solve the maxima and minima problem guided just above its review.

Example 1: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution: Let A be the area of rectangular field. Suppose x and y are width and length of rectangle field, so A = xy



Also $2x + y = 2400$

$\therefore \quad y = 2400 - 2x$

Thus $A = x(2400 - 2x)$

$A = 2400x - 2x^2$

Now,

$A' = 2400 - 4x$

For A is maximum,

$A' = 0$

$\therefore \quad 2400 - 4x = 0$

i.e. $x = 600$

Again,

$A'' = -4 < 0$

$\therefore$ A is maximum when x = 600 ft

and y = 2400 - 2 × 600 = 1200 ft.

$\therefore$ Length of rectangular field is 1200 ft and width is 600 ft, so that 2400 ft fencing enclosed maximum area.

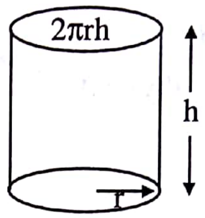
Example 2: A cylinder can is to be made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution: We know that as small as possible the surface area, cost of material is minimize. So

Surface area of can = Area of top and bottom + Curved surface area

$= 2\pi r^2 + 2\pi rh$, where r is radius and h is height of cylinder

$$A = 2\pi r^2 + 2\pi rh$$



Given,

$\pi r^2 h = 1000$ $(1L = 100\ cm^3)$

$$\therefore\quad h = \frac{1000}{\pi r^2}$$

Hence, $A = 2\pi r^2 + 2\pi r \times \dfrac{1000}{\pi r^2}$

$$A = 2\pi r^2 + \frac{2000}{r}$$

Now, $A' = 4\pi r - \dfrac{2000}{r^2}$

For A minimum, $A' = 0$

i.e. $4\pi r - \dfrac{2000}{r^2} = 0$

$$\therefore\quad r^3 = \frac{2000}{4\pi}$$

$$r = \left(\frac{500}{\pi}\right)^{1/3} \text{ cm}$$

Again, $A'' = 4\pi + \dfrac{4000}{r^3} > 0$ when $r = \left(\dfrac{500}{\pi}\right)^{1/3}$. Thus A is minimum when $r = \left(\dfrac{500}{\pi}\right)^{1/3}$.

From $h = \dfrac{1000}{\pi r^2}$

$$h = \frac{1000}{\pi} \times \frac{1}{\left(\frac{500}{\pi}\right)^{2/3}}$$

$$h = \frac{1000}{\pi} \times \left(\frac{\pi}{500}\right)^{2/3}$$

$$= \frac{1000}{(500)^{2/3}}\, \pi^{2/3 - 1} = \frac{1000}{(500)^{2/3}} (\pi)^{-1/3} = 2(500)^{1-2/3} (\pi)^{-1/3} = 2(500)^{1/3} (\pi)^{-1/3} = 2\left(\frac{500}{\pi}\right)^{1/3}$$

$\therefore$ When $r = \left(\dfrac{500}{\pi}\right)^{1/3}$ cm and $h = 2\left(\dfrac{500}{\pi}\right)^{1/3} = 2r$, the cost will be minimum.

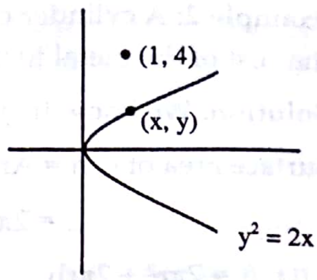
Example 3: Find the point on the parabola $y^2 = 2x$ that is closet to the point (1, 4).

Solution: Let (x, y) be any point on the parabola, so dis. from (1, 4) is

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

$d = \sqrt{(y^2/2 - 1)^2 + (y - 4)^2}$

$d^2 = (y^2/2 - 1)^2 + (y - 4)^2 = f(y)$.



To minimize d, we minimize d^2. So let, $d^2 = f(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y - 4)^2$

Thus, $f'(y) = 2\left(\frac{y^2}{2} - 1\right)^2 . y + 2(y - 4)$

$= y^3 - 8$

For f(y) minimum, $f'(y) = 0$

i.e. $y^3 - 8 = 0$

So $y = 2$

Again, $f''(y) = 3y^2$

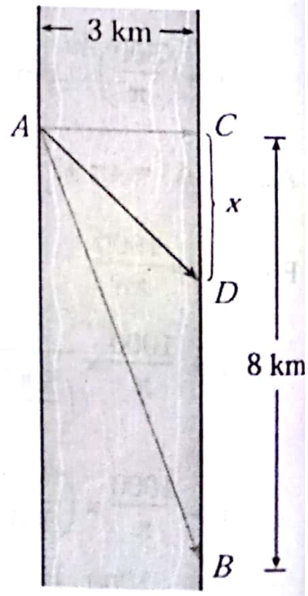
Here $y'' > 0$ when $y = 2$.

Thus f(y) is minimum, when $y = 2$ and $x = \frac{y^2}{2} = \frac{4}{2} = 2$

Thus closest point from (1, 4) to parabola $y^2 = 2x$ is (2, 2).

Example 4: A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows).

Solution: If we let x be the distance from C to D, then the running distance is $|DB| = 8 - x$ and the Pythagorean Theorem gives the rowing distance as $|AD| = \sqrt{x^2 + 9}$. We use the equation



$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

Then the rowing time is $\sqrt{x^2 + 9/6}$ and the running time is $(8 - x)/8$, so the total time T as a function of x is

$$T(x) = \frac{\sqrt{x^2 + 9}}{6} + \frac{8 - x}{8}$$

The domain of this function T is [0, 8]. Notice that if $x = 0$, he rows to C and if $x = 8$, he rows directly to B. The derivative of T is

$$T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}$$

Thus, using the fact that $x \geq 0$, we have

$$T'(x) = 0 \Leftrightarrow \frac{x}{6\sqrt{x^2 + 9}} = \frac{1}{8} \quad \Leftrightarrow 4x = 3\sqrt{x^2 + 9}$$

$$\Leftrightarrow 16x^2 = 9(x^2 + 9) \quad \Leftrightarrow 7x^2 = 81$$

$$\Leftrightarrow x = \frac{9}{\sqrt{7}}$$

The only critical number is $x = 9/\sqrt{7}$. To see whether the minimum occurs at this critical number or at an endpoint of the domain [0, 8], we evaluate T at all three points:

$$T(0) = 1.5 \qquad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \qquad T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

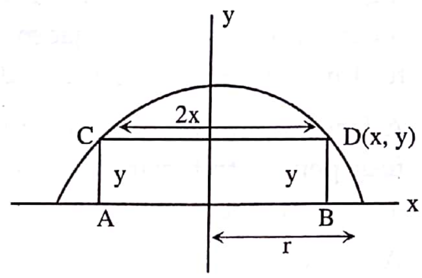
Since the smallest of these values of T occurs when $x = 9/\sqrt{7}$, the absolute minimum value of T must occur there. Fig. illustrates this calculation by showing the graph of T.

Thus the man should land the boat at a point $9/\sqrt{7}$ km ($\approx$ 3.4 km) downstream from his starting point.

Example 5: Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

Solution: Let ABCD be the inscribed rectangle whose length is 2x and width y where point D(x, y) lies on semicircle of radius r. Equation of semicircle is $x^2 + y^2 = r^2$.

Area A of inscribed rectangle is



$$A = 2xy$$

$$A = 2x\sqrt{r^2 - x^2}$$

$$A' = 2\sqrt{r^2 - x^2} + 2x \cdot \frac{1}{2\sqrt{r^2 - x^2}}\, x - 2x$$

$$= 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}}$$

$$= \frac{2(r^2 - x^2) - 2x^2}{\sqrt{r^2 - x^2}}$$

$$A' = \frac{2r^2 - 4x^2}{\sqrt{r^2 - x^2}}$$

For A maximum, A' = 0

$$x = \pm\frac{r}{\sqrt{2}}$$

Since domain of A is $0 \leq x \leq r$ so take $x = \frac{r}{\sqrt{2}}$

Thus, $A = 2x\sqrt{r^2 - x^2}, 0 \leq x \leq r$

$$A(0) = 0,\ A(r) = 0$$

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}}$$

$$= \sqrt{2}\, r\sqrt{\frac{r^2}{2}} = \sqrt{2}\, r\frac{r}{\sqrt{2}} = r^2$$

Maximum value of A is r^2 at $x = \frac{r}{\sqrt{2}}$.

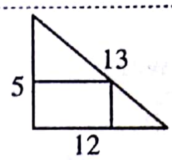
Thus, maximum area is $A = r^2$.

Exercise 4.2

1. Find two numbers whose difference is 100 and whose product is a minimum.
2. Find two positive numbers whose product is 100 and whose sum is a minimum.
3. Find the dimensions of a rectangle with perimeter 100 m whose area is as large as possible.
4. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.
5. (a) Show that of all the rectangles with a given area, the one with smallest perimeter is a square.

   (b) Show that of all the rectangles with a given perimeter, the one with greatest area is a square.
6. The highway department is planning to built a picnic park for motorists along a major highway. The park is to be rectangular with an area 5000 square yards and is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing required for this job? How long and wide should the park be for the fencing to be maximized?
7. A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
8. A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
9. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the amount of material used.
10. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.
11. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs $10 per square meter. Material for the sides cost $6 per square meter. Find the cost of materials for the cheapest such container.
12. Find the point on the line $y = 2x + 3$ that is closest to the origin.
13. Find the point on the curve $y = \sqrt{x}$ that is closest to the point (3, 0).
14. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point (1, 0).
15. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.
16. Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.
17. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.
18. Find the area of the largest trapezoid that can be inscribed in a circle of radius 1 and whose base is a diameter of the circle.
19. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.

20. A rectangle inscribed in a right angled, as shown in the figure. If the triangle has side of length 5, 12 and 13, what are the dimension of inscribed rectangle of greatest area?



21. A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible surface area of such a cylinder.

22. A Norman window has the shape of a rectangle surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width of the rectangle). If the perimeter of the window is 30 ft, find the dimensions of the window so that the greatest possible amount of light is admitted.

23. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) a minimum?

24. Answer Exercise 23 if one piece is bent into a square and the other into a circle.

25. A cylindrical can without a top is made to contain V cm^3 of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

26. Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.

27. At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

28. A boat leaves a dock at 2:00 PM and travels due south at a speed of 20 km/h. Another boat has been heading due east at 15 km/h and reaches the same dock at 3:00 PM. At what time were the two boats closest together?

29. A man launches his boat from point A on a bank of a straight river, 5 km wide, and wants to reach point B, 5 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows).

30. A cylinder can is to be constructed to hold a fixed volume of liquid. The cost of the material used for the top and bottom of the can is 5 cents per square inch and the cost of the material used for the curved side is 2 cents per square inch. Use the calculus to derive a simple relationship between the radius and height of the can that is the least costly to construct.

31. A cable is to be run from a power plant on one side of a river 900 meters wide to a factory on the other side; 3000 meters downstream. The cost of running the cable under the water is \$5 per meter, while the cost over land is \$4 per meter. What is the most economical route over which to run the cable?

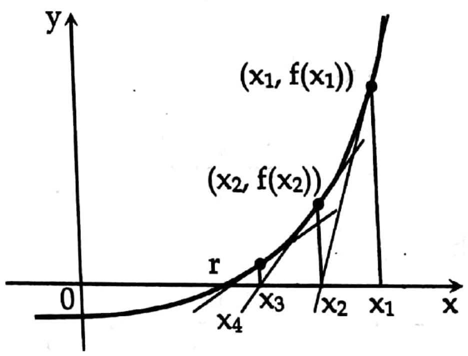
Answers:

1. 50, -50
2. 10, 10
3. Rectangle with maximum area is square with length 25 m
4. $x = 10\sqrt{10}$ m, $y = 10\sqrt{10}$ m, perimeter $= 40\sqrt{10}$ m
6. 100 yards long and 50 yards wide
7. 14062.5 ft^2
8. 2 ft^3
9. Side length of square base is 40 cm and height is 20 cm
10. Max volume 4000 cm^3
11. Minimum cost = \$163.54
12. (- 1.2, 0.6)
13. $\left(\frac{5}{2}, \frac{\sqrt{10}}{2}\right)$
14. $\left(-\frac{1}{3}, \pm\frac{\sqrt{32}}{3}\right)$ or $\left(-\frac{1}{3}, \pm 1.88\right)$
15. Maximum area $2r^2$
16. 2ab
17. length $= \frac{L}{2}$ Width $= \sqrt{3}\frac{L}{4}$
18. Area $\frac{3\sqrt{3}}{4}$
19. Area 3 cm^2
20. Length 6 and width 2.5
21. Area $\pi r^2 (1 + \sqrt{5})$
22. Radius of semi-circle is $\frac{30}{4 + \pi}$ and length 8.4 ft, width 4.2 ft
23. (a) For maximum area all the wire should be used to make the square
    (b) For minimum area 4.35 m should be used for square and 5.65 m should be used for equilateral triangle.
24. (a) For maximum area all the wire should be used to make circle
    (b) For minimum area 5.60 m should be used for square and 4.4 m should be used for circle
25. $r = h = \left(\frac{v}{\pi}\right)^{1/3}$
26. $y = -\frac{5}{3}x + 10$
27. (2, 225) and (-2, -233)
28. 2 : 21 : 36 PM
29. At point 5 km downstream from his starting point
30. h = 3r
31. Minimal installation cost is \$14,700, which will occur if the cable reaches the opposite bank 1200 meter downstream from the power plant.

4.3 Newton's Method

For the quadratic equation $ax^2 + bx + c = 0$, roots are obtained by well-known formula. To find the solution of cubic and quatric equation there has also formulas which are complicated. But there is no any formula to find the roots of the polynomials of degree five and more. Also there is no any formula to find roots of equation type $\sin x = x^2$. For such type of equations we can find an approximation solution by using tangent lines in place of graph of $y = f(x)$ near the points where f is zero. (A value x where $f(x) = 0$ is root of $f(x) = 0$). Method to find the solution of polynomial $f(x) = 0$ of degree 5 and more and type of $\sin x = x^2$ is called Newton's method or Newton-Ramphson method. Brief explanation of this method is as follows:

Let root of $f(x) = 0$ is at point r shown in figure (Because cure $y = f(x)$ meet the x-axis at $x = r$). We begin with first approximation x_1, which is obtained by gussing then find the equation of tangent to curve $y = f(x)$ at $(x_1, f(x_1))$ and found the point where tangent at $(x_1, f(x_1))$ meet the x-axis say it is x_2 and is better approximation to the solution than x_1). Now find the equation of tangent at $(x_2, f(x_2))$ which meet the x-axis at point x_3 which is close to cure i.e. x_3 is near to r as compare to x_1 and x_2 so x_3 is better approximation to the solution than x_1 and x_2. This way we repeat process and get best approximation x_{n+1} which is obtained as follows.



Slope of tangent at $(x_1, f(x_1))$ is $f'(x_1)$ so it equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

To find where it meet x axis, put $y = 0$

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \text{for } f'(x_1) \neq 0$$

If we repeat this process, we obtain a sequence of approximation $x_1, x_2, \ldots$ and n^{th} approximation is x_n and for $f'(x_n) \neq 0$. The $n+1^{th}$ approximation is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$; $f'(x_n) \neq 0$

Example 1: Use Newton's method with initial approximation $x_1 = 1$ find third approximation x_3 to the root of the equation

$$x^5 - x - 1 = 0$$

Solution: Given $f(x) = x^5 - x - 1$

So, $f'(x) = 5x^4 - 1$

By Newtons method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^5 - x_n - 1}{5x_n^4 - 1}$$

When $x_1 = 1$

$$\text{Then } x_2 = x_1 - \frac{x_1^5 - x_1 - 1}{5x_1^4 - 1}$$

$$x_2 = 1 - \frac{1-1-1}{5-1} = 1 + \frac{1}{4} = 1.25$$

Then $x_3 = x_2 - \dfrac{x_2^5 - x_2 - 1}{5x_2^4 - 1}$

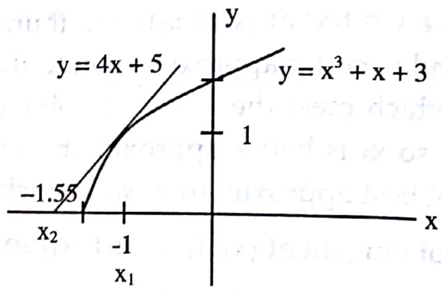
$$= 1.25 - \frac{(1.25)^5 - (1.25) - 1}{5(1.25)^4 - 1} = 1.1785$$

$\therefore \quad x_3 = 1.1785$

Example 2: Use Newton's method with initial approximation $x_1 = -1$ to find x_2, the second approximation to the root of the equation $x^3 + x + 3 = 0$. Explain how the method works by first graphing the function and its tangent line at (-1, 1).

Solution: Here, $f(x) = x^3 + x + 3$

$\therefore \quad f'(x) = 3x^2 + 1$



By Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore \quad x_2 = x_1 - \frac{f(x_n)}{f'(x_n)}$$

$$= x_1 - \frac{x_1^3 + x_1 + 3}{3x_1^2 + 1}$$

When $x_1 = -1$

$$= -1 - \frac{(-1)^3 + (-1) + 3}{3(-1)^2 + 1} = -1 - \frac{1}{4} = -\frac{5}{4} = -1.25$$

Since $f'(-1) = 3(-1)^2 + 1 = 4$. So tangent line to $f(x) = x^3 + x + 3$ at (-1, 1) is $y - 1 = 4(x + 1)$ i.e. $y = 4x + 5$ and its x intercept $-\frac{5}{4} = -1.25$

Example 3: Find the $\sqrt[6]{2}$ correct to eight decimal places by using Newton's method.

Solution: Let $x = \sqrt[6]{2}$

So $x^6 - 2 = 0$

Hence, $f(x) = x^6 - 2$

Thus $f'(x) = 6x^5$.

The Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

i.e. $\quad x_{n+1} = x_n - \dfrac{x_n^6 - 2}{6x_n^5}$

Result of applying Newton's method to $f(x) = x^6 - 2$.

x_n	$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5}$
$x_1 = 1$	$x_2 = 1 - \frac{(1-2)}{6} = 1.6666667$
$x_2 = 1.6666667$	$x_3 = 1.6666667 - \frac{(1.6666667)^6 - 2}{6(1.666667)^5} = 1.2644368$
$x_3 = 1.26444368$	$x_4 = 1.2644368 - \frac{(1.2644368)^6 - 2}{6(1.26444368)^5} = 1.2249707$
$x_4 = 1.2249707$	$x_5 = 1.2249707 - \frac{(1.2249707)^6 - 2}{6(1.2249707)^5} = 1.12246205$
$x_5 = 1.2246205$	$x_6 = 1.2246205 - \frac{(1.2249707)^6 - 2}{6(1.2249070)^5} = 1.12246205$

$\therefore \quad \sqrt[6]{2} = 1.2246205$

Example 4: Find correct to six decimal places, the root of the equation cosx = x.

Solution: Sinc cosx - x = 0, so f(x) = cosx - x.

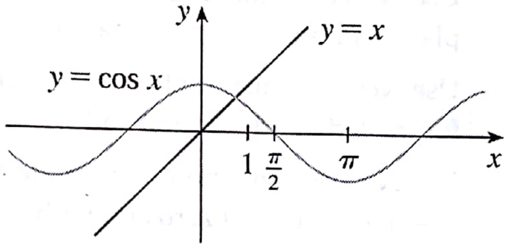
Thus, f '(x) = - sinx - 1

Hence, Newton formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1}$$

$$= x_n + \frac{\cos x_n - x_n}{1 + \sin x_n}$$



To guess the suitable value x_1, if we draw rough graph of f(x) = cosx and f(x) = x; x is less than one so put x = 1.

Result of applying	Newton Method to f(x) = cosx - x
x_n	$x_{n+1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$
$x_1 = 1$	$x_2 = 1 + \frac{\cos 1 - 1}{\sin 1 + 1} = 0.75036387$
$x_2 = 0.75036387$	$x_3 = 0.75036387 + \frac{\cos 0.75036387 - 0.75036387}{\sin 0.75036387 + 1} = 0.73911289$
$x_3 = 0.73911289$	$x_4 = 0.73911289 + \frac{\cos 0.73911289 - 0.73911289}{\sin 0.73911289 + 1} = 0.73908513$
$x_4 = 0.73908513$	$x_5 = 0.73908513 + \frac{\cos 0.73908513 - 0.73908513}{\sin 0.73908513 + 1} = 0.73908513$

$\therefore$ root of cosx = x is 0.73908513

Here, x_4 and x_5 agree to six decimal place (infact is eight decimal place.)

Exercise 4.4

1. Suppose the tangent line to the curve $y = f(x)$ at the point (2, 5) has the equation $y = 9 - 2x$. If Newton's method is used to locate a root of the equation $f(x) = 0$ and the initial approximation is $x_1 = 2$, find the second approximation x_2.

2. Use Newton's method with the specified initial approximation x_1 to find x_3, the third approximation to the root of the given equation. (Give your answer to four decimal places).

   (i) $\frac{1}{3}x^3 + \frac{1}{2}x^2 + 3 = 0, \quad x_1 = -3$

   (ii) $x^7 + 4 = 0, \quad x_1 = -1$

3. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

4. Use Newton's method with initial approximation $x_1 = 1$ to find x_2, the second approximation to the root of the equation $x^4 - x - 1 = 0$. Explain how the method works by first graphing the function and its tangent line at (1, -1).

5. Use Newton's method to approximate the given number correct to eight decimal places.

   (a) $\sqrt[5]{20}$ (b) $\sqrt[100]{100}$

6. Use Newton's method to find all roots of the equation $3\cos x = x + 1$ correct to six decimal place with initial approximation $x_1 = -3.5$

7. Use Newtons method to estimate the solutions of the equation $x^2 + x - 1 = 0$ start with $x_1 = -1$ for the left hand solution and with $x = 1$ for the solution on the right. Then each case find x_3.

8. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$ start with $x_1 = 0$ for left hand zero and with $x = 2$ for the zero on the right then each case find x_3.

Answers:

1. 9/2
2. (i) $x_3 \approx -2.7186$ (ii) $x_3 \approx -1.29172$
3. $x_3 \approx 2.0946$
4. $\frac{4}{3}$
5. (a) $\sqrt[5]{20} \approx 1.8205642$ (b) $\sqrt[100]{100} \approx 1.04712855$
6. $x = -3.637958$
7. $-\frac{5}{3}, \frac{13}{21}$
8. $-\frac{5}{12}, \frac{29}{12}$

Chapter 5

Integration

One of the great achievements of classical geometry was to obtain formulas for the area and volumes of triangles, spheres, and cones. In this chapter we study a method to calculate the areas and volumes of these and other more general shapes. The method we develop, called integration is a tool for calculating much more than areas and volumes. The integral has many applications in statistics, economics, the sciences and engineering.

The idea behind integration is that we can effectively compute many quantities by breaking them into small pieces, and the summing the contributions from each small part.

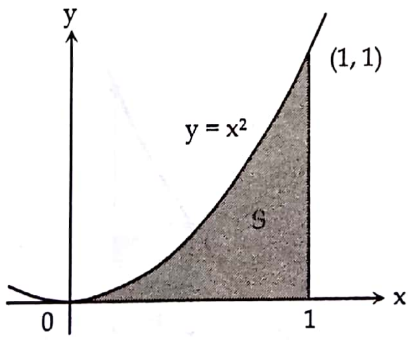
5.1 Area and Distances

In this section we discover that in trying to find the area under a curve or the distance traveled by a car, we end up with the same special type of limit.

Area

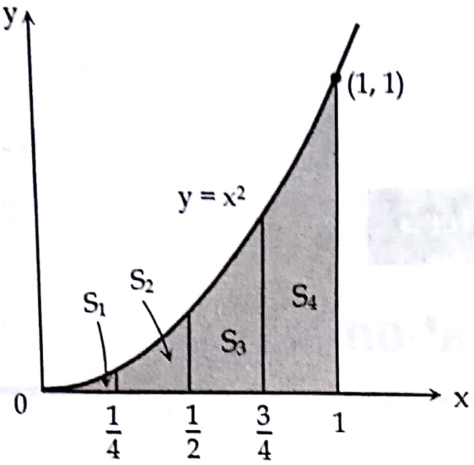
The area of a region with a curved boundary can be approximated by summing the areas of a collection of rectangles. Using more rectangles can increase the accuracy of the approximation.

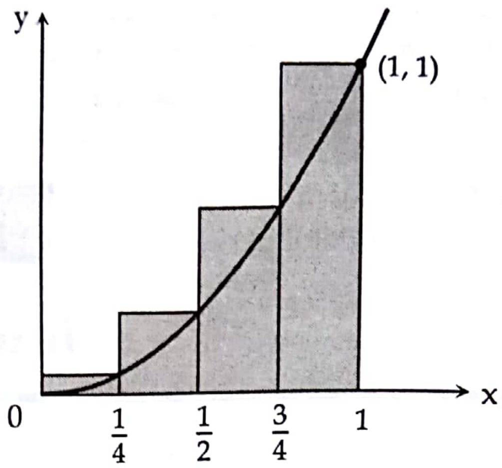
An illustration is as follows: Estimate the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in figure).



We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that.

Suppose we divide S into four strips S_1, S_2, S_3 and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$ and $x = \frac{3}{4}$ as in figure.





We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip (see figure). In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right end points of the subintervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$, and $\left[\frac{3}{4}, 1\right]$.

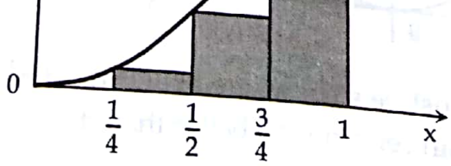
Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2 = \frac{15}{32} = 0.46875$$

From figure we see that the area A of S is less than R_4, so

$$A < 0.46875$$

Instead of using the rectangles in figure we could use the smaller rectangles in following figure whose heights are the values of f at the left end points of the subintervals. (The leftmost rectangle has collapsed because its height is 0.) The sum of the areas of these approximating rectangles is

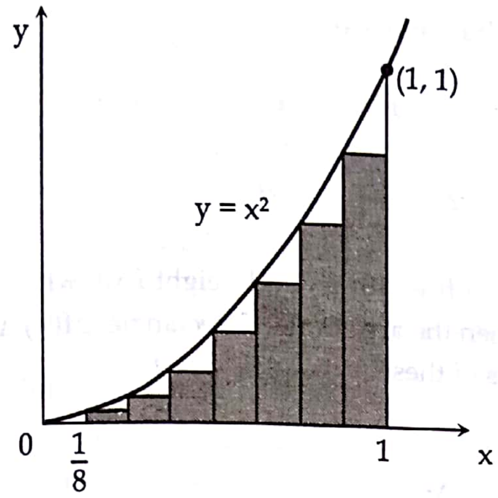


$$L_4 = \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$$

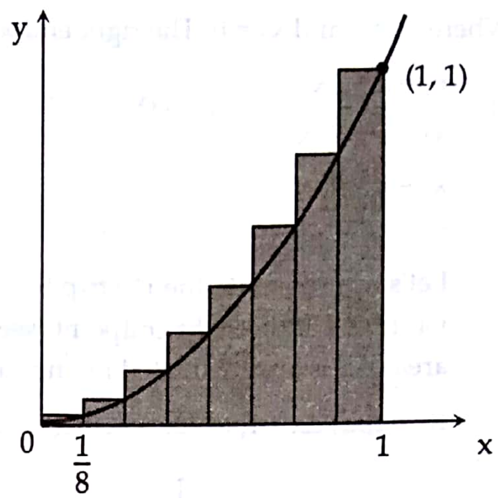
We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips. Following figure shows what happens when we divide the region S into eight strips of equal width.



(a) Using left endpoints



(b) Using right endpoints

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

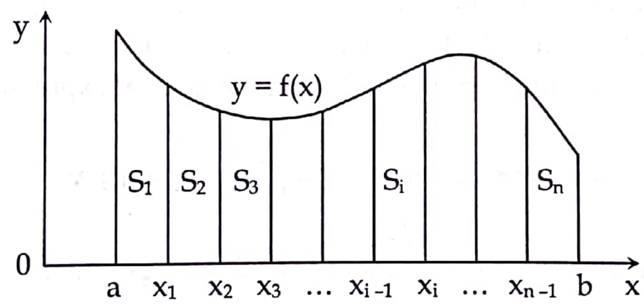
By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

We could obtain better estimates by increasing the number of strips. The table at the left shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_n). In particular, we see by using 50 strips that the area lies between 0.3234 and 0.3434. With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335. A good estimate is obtained by averaging these numbers: $A \approx 0.3333335$.

Let's apply idea of example 1 to the more general region S of figure. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in the following figure.



The width of the interval [a, b] is b – a, so the width of each of the n strips is

$$\Delta x = \frac{b-a}{n}$$

These strips divide the interval [a, b] into n subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$$

Where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals are

$x_1 = a + \Delta x,$

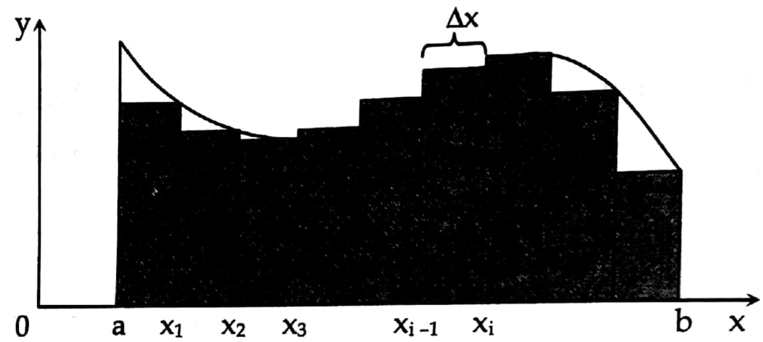
$x_2 = a + 2\Delta x,$

$x_3 = a + 3\Delta x$

$\vdots$

Let's approximate the i^{th} strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see in figure). Then the area of the i^{th} rectangle is $f(x_i)\,\Delta x$. The area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \ldots + f(x_n)\,\Delta x$$



Definition: The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \ldots + f(x_n)\,\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

It can be proved that the limit in definition always exists, since we are assuming that f is continuous. It can also be shown that we get the same value if we use left endpoints:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + \ldots + f(x_{n-1})\,\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the i^{th} rectangle to be the value of f at any number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ the sample points. Following figure shows approximating rectangles when the sample points are not chosen to be endpoints. So a more general expression for the area of S is

$$A = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \ldots + f(x_n^*)\,\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Example 1: Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using four sub-intervals and then ten subintervals.

Solution.

a. Since a = 0 and b = 2, the width of a subinterval is

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

So $x_1 = 2/n$, $x_2 = 4/n$, $x_3 = 6/n$, $x_i = 2i/n$, and $x_n = 2n/n$. The sum of the areas of the approximating rectangles is

$$R_n = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \ldots + f(x_n)\,\Delta x$$
$$= e^{-x_1}\,\Delta x + e^{-x_2}\,\Delta x + \ldots + e^{-x_n}\,\Delta x$$
$$= e^{-2/n}\left(\frac{2}{n}\right) + e^{-4/n}\left(\frac{2}{n}\right) + \ldots + e^{-2n/n}\left(\frac{2}{n}\right)$$

According to definition, the area is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\left(e^{-2/n} + e^{-4/n} + e^{-6/n} + \ldots + e^{-2n/n}\right)$$

$$A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$

b. With n = 4 the subintervals of equal width are $\Delta x = 0.5$ are [0, 0.5], [0.5, 1], [1, 1.5], and [1.5, 2]. The midpoints of these subintervals are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$, and $x_4^* = 1.75$, and the sum of the areas of the four approximating rectangles is

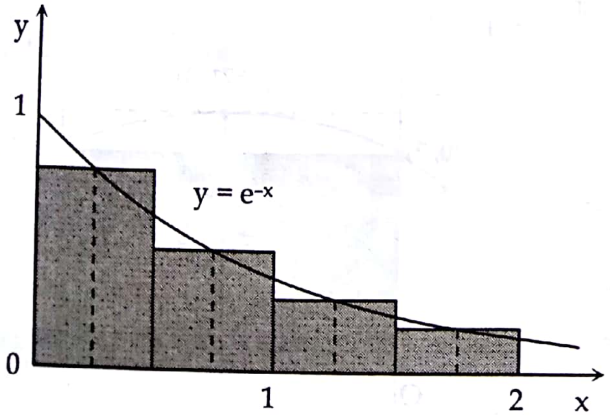
$$M_4 = \sum_{i=1}^{4} f(x_1^*)\,\Delta x$$
$$= f(0.25)\,\Delta x + f(0.75)\,\Delta x + f(1.25)\,\Delta x + f(1.75)\,\Delta x$$
$$= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5)$$
$$= \frac{1}{2}\left(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}\right) \approx 0.8557$$

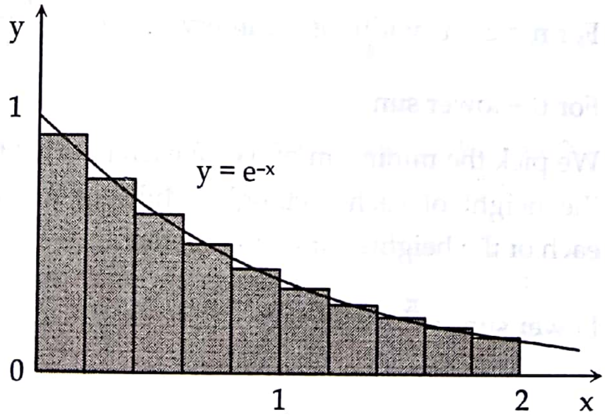
So an estimate for the area is

$$A \approx 0.8557$$

With n = 10 the subintervals are [0, 0.2], [0.2, 0.4], …, [1.8, 2] and the midpoints are $x_1^* = 0.1$, $x_2^* = 0.3$, $x_3^* = 0.5$, …, $x_{10}^* = 1.9$. Thus

$$A \approx M_{10} = f(0.1)\,\Delta x + f(0.3)\,\Delta x + f(0.5)\,\Delta x + \ldots + f(1.9)\,\Delta x$$
$$= 0.2\left(e^{-0.1} + e^{-0.3} + e^{-0.5} + \ldots + e^{-1.9}\right) \approx 0.8632$$



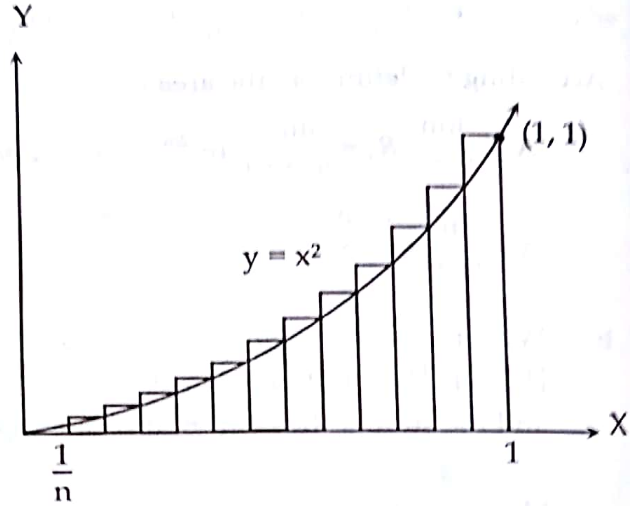


Example 2: Show that the sum of the areas of the upper approximating rectangle approaches $\frac{1}{3}$ under the parabola $y = x^2$ from 0 to 1.

Solution.

Let R_n be the sum of the area of the n rectangles as shown in figure. Each rectangle has width $\frac{1}{n}$ and the heights are the values of the function $f(x) = x^2$ at the points $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}$

i.e. the heights are $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \ldots, \left(\frac{n}{n}\right)^2$



Thus,

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n^3}(1 + 2^2 + 3^2 + \ldots + n^2)$$

$$= \frac{1}{n^3} \times \frac{n(n+1)(2n+1)}{6}$$

$$R_n = \frac{(n+1)(2n+1)}{6n^2}$$

Thus, we have,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6} \times 1 \times 2 = \frac{1}{3}$$

Example 3: Evaluate the upper and lower sums for $f(x) = 2 + \sin x$, $0 \le x \le \pi$, with $n = 2, 4$ and 8. Illustrate with diagrams.

Solution. Given,

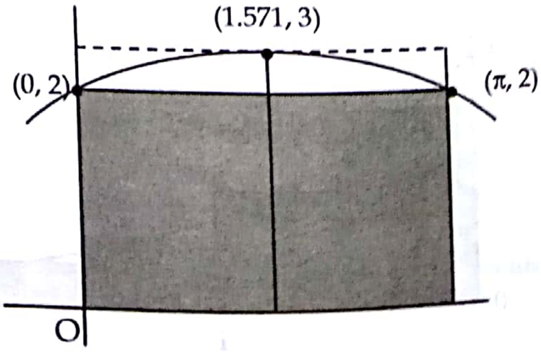
$$f(x) = 2 + \sin x, \quad 0 \le x \le \pi$$

Here, $a = 0$, $b = \pi$

For $n = 2$, the width of the intervals is $\Delta x = \frac{b-a}{n} = \frac{\pi}{2} \approx 1.571.$

For the lower sun

We pick the minimum of $f(x)$ in each interval to be the height of each rectangle. Multiply Δx times each of the heights summed together.



$$\text{Lower sum} = \frac{\pi}{2} \times 2 + \frac{\pi}{2} \times 2$$

$= 2\pi$

≈ 6.283

For the upper sum we pick the maximum of f(x) in each interval to be the height.

Upper sum $= \frac{\pi}{2} \times 3 + \frac{\pi}{2} \times 3 = 3\pi \approx 9.425$

For $n = 4$, $\Delta x = \frac{\pi}{4} \approx 0.785$

$$\text{Lower sum} = \frac{\pi}{4} \times f(0) + \frac{\pi}{4} \times f\left(\frac{\pi}{4}\right) + \frac{\pi}{4} \times f\left(\frac{\pi}{4}\right) + \frac{\pi}{4} \times f(\pi)$$

$$= \frac{\pi}{4}\left[2 + \sin 0 + 2 + \sin\frac{\pi}{4} + 2 + \sin\frac{\pi}{4} + 2 + \sin\pi\right]$$

$$= \frac{\pi}{4}\left(8 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

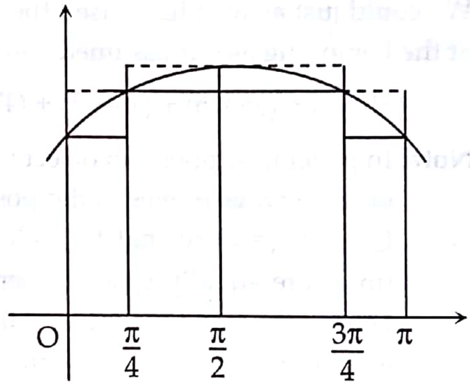
$$= 7.39$$

$$\text{Upper sum} = \frac{\pi}{4} \times f\left(\frac{\pi}{4}\right) + \frac{\pi}{4} \times f\left(\frac{\pi}{2}\right) + \frac{\pi}{4} \times f\left(\frac{\pi}{2}\right) + \frac{\pi}{4} \times f\left(\frac{3\pi}{2}\right)$$

$$= \frac{\pi}{4}\left(2 + \sin\frac{\pi}{4} + 2 + \sin\frac{\pi}{2} + 2 + \sin\frac{\pi}{2} + 2 + \sin\frac{3\pi}{4}\right)$$

$$= \frac{\pi}{4}\left(8 + \frac{1}{\sqrt{2}} + 1 + 1 + \frac{1}{\sqrt{2}}\right)$$

$$= 8.96$$



Similarly for $n = 8$, we get

Lower sum = 7.86

Upper sum = 8.65

The Distance Problem

Now let's consider the *distance problem:* Find the distance traveled by an object during a certain time period if the velocity of the object is known at all times. If the velocity remains constant, then the distance problem is easy to solve by means of the formula

Distance = Velocity × Time

But if the velocity varies, it's not so easy to find the distance traveled. We investigate the problem in the following example.

Example 4: Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

In order to have the time and the velocity in consistent units, let's convert the velocity readings to feet per second (1 mi/h = 5280/3600 ft/s):

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

During the first five seconds the velocity doesn't change very much, so we can estimate the distance traveled during that time by assuming that the velocity is constant. If we take the velocity during that time interval to be the initial velocity (25 ft/s), then we obtain the approximate distance traveled during the first five seconds:

$$25 \text{ ft/s} \times 5\text{s} = 125 \text{ ft}$$

Similarly, during the second time interval the velocity is approximately constant and we take it to be the velocity when t = 5s. So our estimate for the distance traveled from t = 5s to t = 10s is

$$31 \text{ ft/s} \times 5\text{s} = 155 \text{ ft}$$

If we add similar estimates for the other time intervals, we obtain an estimate for the total distance traveled:

$$(25 \times 5) + (31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (46 \times 5) = 1135 \text{ ft}$$

We could just as well have used the velocity at the *end* of each time period instead of the velocity at the beginning as our assumed constant velocity. Then our estimate becomes

$$(31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (46 \times 5) + (41 \times 5) = 1215 \text{ ft}$$

Note: In general, suppose an object moves with velocity $v = f(t)$, where $a \le t \le b$ and $f(t) \ge b$ (so the object always moves in the positive direction). We take velocity readings at time t_0 (= a), t_1, t_2, ..., t_n (= b) so that the velocity is approximately constant on each subinterval. If these times are equally spaced, then the time between consecutive readings is $\Delta t = (b - a)/n$. During the first time interval the velocity is approximately $f(t_0)$ and so the distance traveled is approximately $f(t_0)\,\Delta t$. Similarly, the distance traveled during the second time interval is about $f(t_1)\,\Delta t$ and the total distance traveled during the time interval [a, b] is approximately

$$f(t_0)\,\Delta t + f(t_1)\,\Delta t + \ldots + f(t_{n-1})\,\Delta t = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t$$

If we use the velocity at right endpoints instead of left endpoints, our estimate for the total distance becomes

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \ldots + f(t_n)\,\Delta t = \sum_{i=1}^{n} f(t_i)\,\Delta t$$

The more frequently we measure the velocity, the more accurate our estimates become, so it seems plausible that the exact distance d traveled is the limit of such expressions:

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

Exercise 5.1

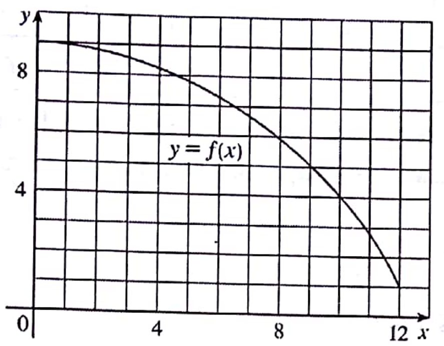
1. (a) Use six rectangles to find estimates of each type for the area under the given graph of f from x = 0 to x = 12.

   (i) L_6 (ii) R_6 (iii) M_6

   (b) Is L_6 an underestimate or overestimate of the true of area?

   (c) Is R_6 and underestimate or overestimate of the true area?

   (d) Which of the numbers L_6, R_6 or M_6 gives the best estimate? Explain.



2. (a) Estimate the area under the graph of f(x) = cos x from x = 0 to x = π/2 using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

   (b) Repeat part (a) using left endpoints.

3. (a) Estimate the area under the graph of $f(x) = \sqrt{x}$ from x = 0 to x = 4 using four approximating rectangles and right endpoints. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?

   (b) Repeat part (a) using left endpoints.

4. (a) Estimate the area under the graph of $f(x) = 1 + x^2$ from x = −1 to x = 2 using three rectangles and right endpoints. Then improve your estimate by using six rectangles. Sketch the curve and the approximating rectangles.

   (b) Repeat part (a) using left endpoints.

   (c) Repeat part (a) using midpoints.

   (d) From your sketches in parts (a) – (c), which appears to be the best estimate?

5. Evaluate the upper and lower sums for $f(x) = 1 + x^2$, $-1 \le x \le 1$, with n = 3 and 4. Illustrate with diagrams.

6. The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Find lower and upper estimates for the distance that she traveled during these three seconds.

t(s)	0	0.5	1.0	1.5	2.0	2.5	3.0
v(ft/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

7. Speedometer readings for a motorcycle at 12 second intervals are given in the table.

   (a) Estimate the distance traveled by the motorcycle during this time period using the velocities at the beginning of the time intervals.

(b) Give another estimate using the velocities at the end of the time periods.

(c) Are your estimates in parts (a) and (b) upper and lower estimates? Explain.

t(s)	0	12	24	36	48	60
v(ft/s)	30	28	25	22	24	27

8. Oil leaked from a tank at a rate of r(t) liters per hour. The rate decreased as time passed and values of the rate at two hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

t(h)	0	2	4	6	8	10
r(t) (L/h)	8.7	7.6	6.8	6.2	5.7	5.3

9. Let A be the area of the region that lies under the following graphs of f(x) then, find an expression for the area under the graph of f as a limit. Do not evaluate the limit.

(a) $f(x) = \frac{2x}{x^2+1},\ 1 \le x \le 3$

(b) $f(x) = x^2 + \sqrt{1+2x},\ 4 \le x \le 7$

(c) $f(x) = \sqrt{\sin x},\ 0 \le x \le \pi$

10. Determine a region whose area is equal to the given limit. Do not evaluate the limit.

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}$ (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$

Answers:

1. (a) (i) 86.6 (ii) 70.6 (iii) 79.2 (b) Overestimate (c) Underestimate d) M_6
2. (a) 0.79, underestimate (b) 1.18, overestimate
3. (a) $3+\sqrt{2}+\sqrt{3}$ (b) $1+\sqrt{2}+\sqrt{3}$
4. (a) 8, 6.875 (b) 5, 5.375 (c) 5.75, 5.9375 (d) C
5. Lower limit = 2.148 Upper limit = 3.407; Lower limit = 2.25 Upper limit = 3.25
6. 34.7 ft, 44.8 ft
7. (a) 1548 ft (b) 1512 ft (c) neither
8. Lower limit = 63.2, Upper limit = 70 *l*
9. (a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2+1}\right] \times \frac{2}{n}$ (b) $\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(4+\frac{3i}{n}\right)^2 + \sqrt{9+\frac{6i}{n}}\right]$

   (c) $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\sin\frac{\pi i}{n}} \times \frac{\pi}{n}$
10. (a) $y = x^{10}$ for $5 \le x \le 7$ (b) $y = \tan x$ for $0 \le x \le \frac{\pi}{4}$

5.2 The Definite Integral

We saw in section 5.1 that a limit of the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} [f(x_1^*)\,\Delta x + f(x_2^*) + \ldots + f(x_n^*)\,\Delta x]$$

arises when we compute an area. We also saw that it arises when we try to find the distance traveled by an object. It turns out that this same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. We therefore give this type of limit a special name and notation.

Definition of a Definite Integral If *f* is function defined for $a \le x \le b$, we divide the interval [a, b] into *n* subintervals of equal width $\Delta x = (b - a)/n$. We let x_0 (= a), x_1, x_2, ..., x_n (= b) be the endpoints of these subintervals and we let x_1^*, x_2^*, ..., x_n^* be any sample points in these subintervals, so x_i^* lies in the *i*th subinterval $[x_{i-1}, x_i]$. Then the definite integral of *f* from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that *f* is integrable on [a, b].

The precise meaning of the limit that defines the integral is as follows:

For every number $\varepsilon > 0$ there is an integer *N* such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$

for every integer $n > N$ and for every choice of x_i^* in $[x_{i-1}, x_i]$.

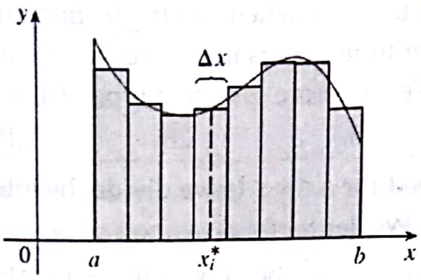
the sum that occurs in definition is called a Riemann system.

Note 1: The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation is called the **integrand** and a and b are called the **limits of integration**; is the **lower limit** and b is the **upper limit**. For now, the symbol *dx* has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The dx simply indicates that the independent variable is x. The procedure of calculating an integral is called **integration**.

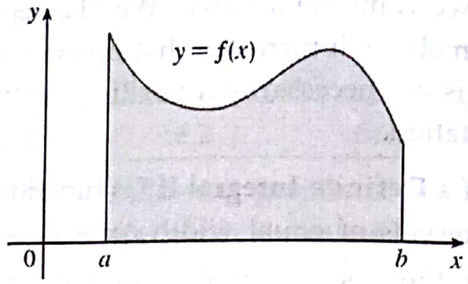
Note 2: The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

Note 3: If f takes on both positive and negative values then the Riemann sum is the sum of the area of the rectangles that lie above the x-axis and the negatives of the area of rectangles that lie below the x-axis.



If $f(x) \geq 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles



If $f(x) \geq 0$, the integral $\int_a^b dx$ is the area under the curve y = f(x) from a to b.

1. Theorem: (The existence of definite integrals) If f is continuous on [a, b] or if f has only a finite number of jump discontinuities, then f is integrable on [a, b] i.e., the definite integral $\int_a^b f(x)\,dx$ exists.

Note: If f is integrable on [a, b], then the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$ exists and gives the same value no matter how we choose the sample points x_i^*, to simplify the calculation of the integral we often take the sample points to be right end points. Then $x_i^* = x_i$ and the definition of the integral simplifies as follows.

2. Theorem: If f is integrable on [a, b] then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \text{ where } \Delta x = \frac{b-a}{n},\ x_i = a + i\,\Delta x$$

Example 1: Express $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x$ as an integral on the interval $[0, \pi]$

Solution. Comparing the given limit with theorem (2)

We see that $f(x) = x^3 + x\sin x$, $a = 0$, $b = \pi$

∴ by theorem 2, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i \sin x_i)\,\Delta x = \int_0^{\pi} (x^3 + x\sin x)\,dx$$

Properties of the definite integral

If f and g are integrable, and k is any number then

1. Order of integration $\int_a^b f(x)\,dx = -\int_a^b f(x)\,dx$

2. $\int_a^b k\,dx = k\,(b-a)$

3. Zero width interval $\int_a^a f(x)\,dx = 0$

4. Constant multiple $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

5. Sum and difference $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

6. Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

7. Domination: $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

   $f(x) \geq 0$ on $[a, b] \rightarrow \int_a^b f(x)\,dx \geq 0$

8. Max – Min inequality: If f has maximum and minimum value M and m respectively on [a, b] then

   $m\,.\,(b-a) \leq \int_a^b f(x)\,dx \leq M.\,(b-a)$

Proof 4: $\int_a^b [f(x) \pm g(x)]\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) \pm g(x_i)]\,\Delta x$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\,\Delta x \pm \sum_{i=1}^{n} g(x_i)\,\Delta x\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \pm \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\,\Delta x$$

$$= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

Proof 8: Since, $m \le f(x) \le M$ then using property 7 we get $\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx.$

Using property (2) we get

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Example 2: If $\int_0^{12} f(x)\,dx = 17$ and $\int_0^{8} f(x)\,dx = 15$. Find $\int_8^{12} f(x)\,dx.$

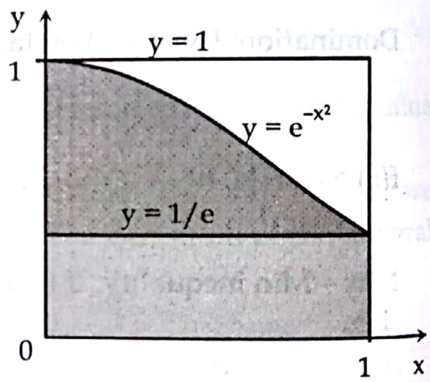
Solution. We have using the property

$$\int_0^{8} f(x)\,dx + \int_8^{12} f(x)\,dx = \int_0^{12} f(x)\,dx$$

$$\therefore \quad \int_8^{12} f(x)\,dx = \int_0^{12} f(x)\,dx - \int_0^{8} f(x)\,dx = 17 - 15 = 2$$

Example 3: Use Max-min inequality estimate $\int_0^1 e^{-x^2}$

Solution.



Because $f(x) = e^{-x^2}$ is a decreasing function on [0, 1], its absolute maximum value is $M = f(0) = 1$ and its absolute minimum value is $m = f(1) = e^{-1}$. Thus, by property,

$$e^{-1}(1-0) \le \int_0^1 e^{-x^2}\,dx \le 1\,(1-0)$$

or $\quad e^{-1} \le \int_0^1 e^{-x^2}\,dx \le 1$

Since $e^{-1} \approx 0.3679$, we can write

$$0.367 \le \int_0^1 e^{-x^2}\,dx \le 1$$

Example 4: using max-min inequality estimate the value of the integral $\int_\pi^{2\pi} (x - 2\sin x)\,dx$. Take the derivative of $f(x) = x - 2\sin x$, So we can find the maximum and minimum values. So, $f'(x) = 1 - 2\cos x$.

If $f'(x) = 0$ then, $\cos x = \frac{1}{2} \Rightarrow x = \frac{5\pi}{3}$

So, $f(\pi) = \pi$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$$

$f(2\pi) = 2\pi$

$\therefore$ Minimum value (m) $= \pi$

Maximum value (M) $= \frac{5\pi}{3} + \sqrt{3}$

So by using Max-min inequality, we have

$$m\,(b-a) \le \int_a^b f(x)\,dx \le M.\,(b-a)$$

$$\pi\,(2\pi - \pi) \le \int_\pi^{2\pi} (x - 2\sin x)\,dx \le \left(\frac{5\pi}{3} + \sqrt{3}\right)(2\pi - \pi)$$

$$\therefore \quad \pi^2 \le \int_\pi^{2\pi} (x - 2\sin x)\,dx \le \pi\left(\frac{5\pi}{3} + \sqrt{3}\right)$$

Example 5:

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$

Solution.

(a) With $n = 6$ the interval width is

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

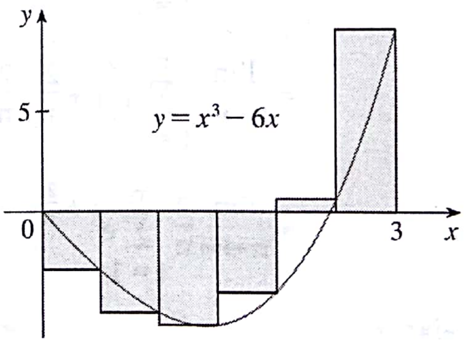
And the right endpoints are $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$, $x_5 = 2.5$, and $x_6 = 3.0$. So the Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$$

$$= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x + f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.9)\,\Delta x$$

$$= \frac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9)$$

$$= -3.9375$$



Notice that f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas above the x-axis minus the sum of the areas below the x-axis in figure.

(b) With n subintervals we have,

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}$$

Thus, $x_0 = 0$, $x_1 = 3/n$, $x_2 = 6/n$, $x_3 = 9/n$, and, in general, $x_i = 3i/n$. Since we are using right endpoints, we can use theorem 4:

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] \qquad \text{(equation 9 with } c = 3/n\text{)}$$

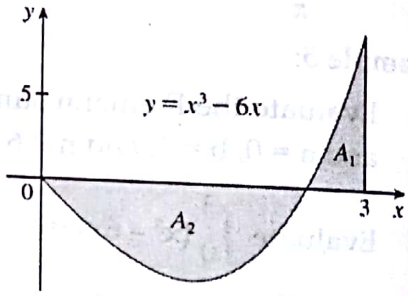
$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{27}{n^3}\right) i^3 - \left(\frac{18}{n} i\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right] \qquad \text{(equations 11 and 9)}$$

$$= \lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right\} \qquad \text{(equations 7 and 5)}$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75 = (A_1 - A_2)$$



Example 6: Set up an expression for $\int_1^3 e^x\,dx$ as a limit of sums.

Solution.

(a) Here we have $f(x) = e^x$, $a = 1$, $b = 3$, and

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}$$

So $x_0 = 1$, $x_1 = 1 + \frac{2}{n}$, $x_2 = 1 + \frac{4}{n}$, $x_3 = 1 + \frac{6}{n}$, and $x_t = 1 + \frac{2i}{n}$

From theorem, we get

$$\int_1^3 e^x\,dx = \lim_{n\to\infty} \sum_{t=1}^{n} f(x_t)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{t=1}^{n} f\left(1 + \frac{2i}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{t=1}^{n} e^{1 + \frac{2i}{n}}$$

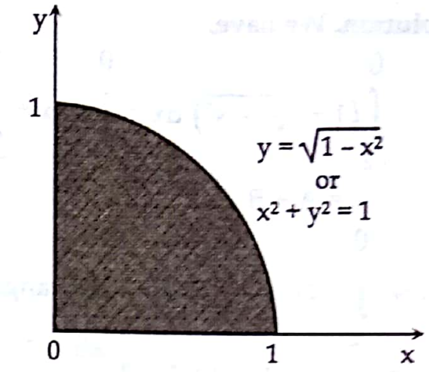
Example 7: Evaluate the following integrals by interpreting each in terms of areas.

$\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\int_0^3 (x-1)\,dx$

Solution.

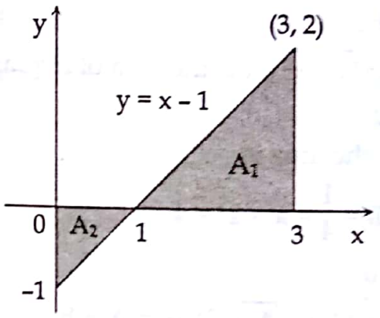
a. Since $f(x) = \sqrt{1-x^2} \geq 0$, we can interpret this integral as the area under the curve $y = \sqrt{1-x^2}$ from 0 to 1. But, since $y^2 = 1 - x^2$, we get $x^2 + y^2 = 1$, which shows that the graph of f is the quarter-circle with radius 1 in figure. Therefore

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}\pi\,(1)^2 = \frac{\pi}{4}$$



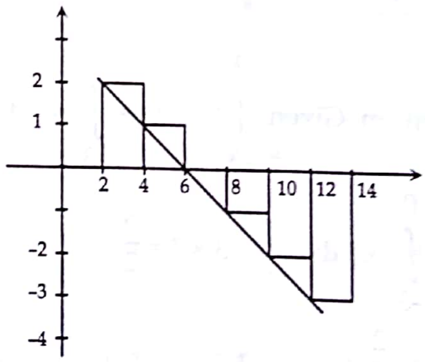
b. The graph of $y = x - 1$ is the line with slope 1 shown in figure. We compute the integral as the difference of the areas of the two triangles:

$$\int_0^3 (x-1)\,dx\ A_1 - A_2 = \frac{1}{2}(2\,.\,2) = \frac{1}{2}(1\,.\,1) = 1.5$$



Example 8: Evaluate the Riemann sum for $f(x) = 3 - \frac{x}{2}$, $2 \leq x \leq 14$, with six subintervals, taking the sample points to be left end-points . Explain with the aid of a diagram, what the Riemann sum represents.

Solution. Given $f(x) = 3 - \frac{x}{2}$, $\quad 2 \leq x \leq 14$



$$a = 2, b = 14, \Delta x = \frac{b-a}{n} = \frac{14-2}{6} = 2$$

∴ The intervals are (2, 2), (4, 1), (6, 0), (8, −1), (10, −2), (12, −3) and (14, −4).

Now, by using formula

$$\sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$= f(x_0)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + f(x_5)\,\Delta x$

$= 2 \times 2 + 2 \times 1 + 0 \times 2 + 2 \times (-1) + 2 \times (-2) + 2 \times (-3)$

$= 4 + 2 - 2 - 4 - 6$

$= -6$

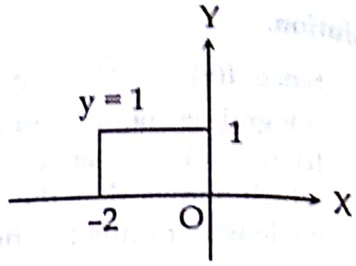
The Riemann sum represents an estimate of the area between the line and the x-axis. Although when some of the rectangles are below the x-axis, their areas are negative so we have to take their absolute values if we find actual area.

Example 9: Evaluate the integral by interpreting it in terms of areas of $\int_{-2}^{0}\left(1+\sqrt{4-x^2}\right)dx$.

Solution. We have,

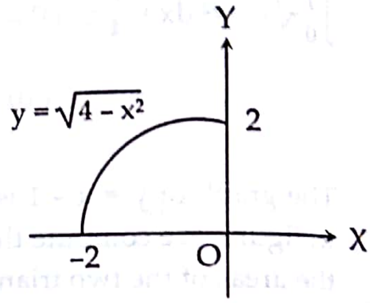
$$\int_{-2}^{0}\left(1+\sqrt{4-x^2}\right)dx = \int_{-2}^{0} 1\,dx + \int_{-2}^{0}\sqrt{4-x^2}\,dx \quad \dots (1)$$

$$= A + B$$



Now, $\int_{-2}^{0} 1\,dx$ is the area of a rectangle with base = 2, and height = 1.

Thus, area of A = 2 × 1 = 2

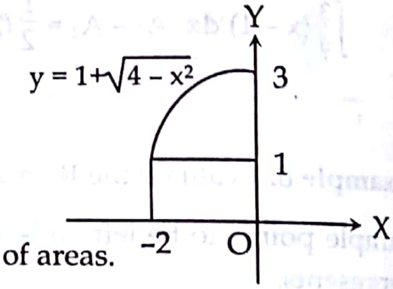


For $\int_{-2}^{0}\sqrt{4-x^2}\,dx$ is the area of a quarter-circle with radius = 2.

Thus, the area is

$$B = \frac{1}{4}\times\pi\times 2^2 = \pi$$

$$\therefore \quad \int_{-2}^{0}\left(1+\sqrt{4-x^2}\right)dx = A + B = 2 + \pi$$



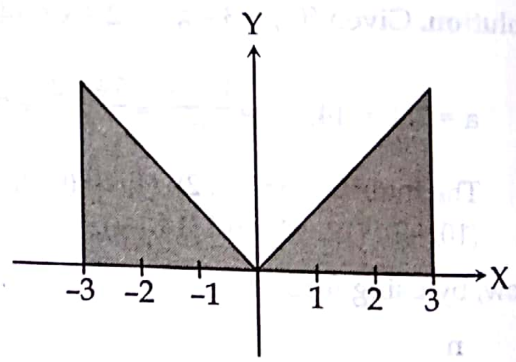
Example 10: Evaluate the integral by interpreting $\int_{-3}^{2} |x|\,dx$ in terms of areas.

Solution. Given, $\int_{-3}^{2}|x|\,dx = \int_{-3}^{0}|x|\,dx + \int_{0}^{2}|x|\,dx$

For $\int_{-3}^{0}|x|\,dx = \frac{1}{2}\times 3\times 3 = \frac{9}{2}$

and $\int_{0}^{2}|x|\,dx = \frac{1}{2}\times 2\times 2 = 2$

$$\therefore \quad \int_{-3}^{2}|x|\,dx = \frac{9}{2} + 2 = \frac{13}{2}$$



The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the i^{th} subinterval because it is convenient for computing the limit. But if the purpose is to find an approximation to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\bar{x}_i$. Any

Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x\,[f(\bar{x}_1) + \dots + f(\bar{x}_n)]$$

where $\Delta x = \dfrac{b-a}{n}$

and $\bar{x}_i = \dfrac{1}{2}(x_{i-1} + x_i)$ = midpoint of $[x_{i-1}, x_i]$

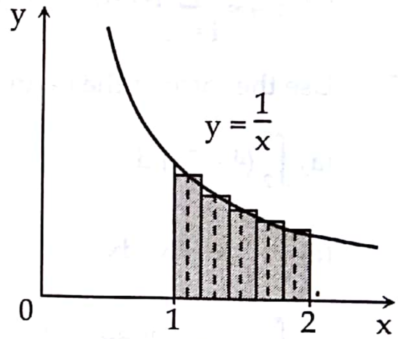
Example 11: Use the Midpoint Rule with n = 5 to approximate $\int_1^2 \frac{1}{x}\,dx$.

Solution. The endpoints of the five subintervals are 1, 1.2, 1.4, 1.6, 1.8, and 2.0, so the midpoints are 1.1, 1.3, 1.5, 1.7, and 1.9. The width of the subintervals is $\Delta x = \dfrac{(2-1)}{5} = \dfrac{1}{5}$, so the Midpoint Rule gives.

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\,[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx 0.691908$$



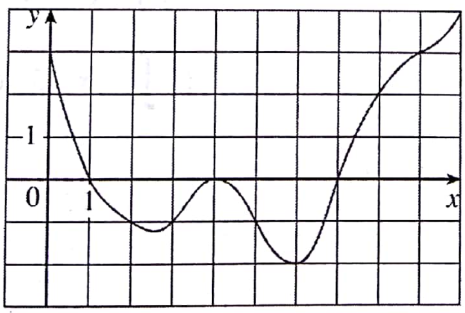
Since $f(x) = 1/x > 0$ for $1 \le x \le 2$, the integral represents an area, and the approximation given by the Midpoint Rule is the sum of the areas of the rectangles shown in figure.

Exercise 5.2

1. If $f(x) = x^2 - 2x$, $0 \le x \le 3$, evaluate the Riemann sum with n = 6, taking the sample points to be right endpoints. What does the Riemann sum represent?

2. If $f(x) = e^x - 2$, $0 \le x \le 2$, find the Riemann sum with n = 4 correct to six decimal places, taking the sample points to be midpoints. What does the Riemann sum represent?

3. The graph of a function f is given. Estimate $\int_0^{10} f(x)$ using five subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.



4. A table of values of an increasing function f is shown. Use the table to find lower and upper estimates for $\int_{10}^{30} f(x)\,dx$.

x	10	14	18	22	26	30
f(x)	-12	-6	-2	1	3	8

5. Use the Midpoint Rule with the given value of to approximate the integral.

(a) $\int_0^8 \sin\sqrt{x}\,dx,\ n = 4$ (b) $\int_0^{\pi/2} \cos^4 x\,dx,\ n = 4$

(c) $\int_0^2 \frac{x}{x+1}\,dx,\ n = 5$ (d) $\int_1^5 x^2 e^{-x}\,dx,\ n = 4$

6. Express the limit as a definite integral on the given interval.

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} x_i \ln(1 + x_i^2)\,\Delta x,\ [2, 6]$ (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i}\,\Delta x,\ [\pi, 2\pi]$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} [5(x_i^*)^3 - 4x_i^*]\,\Delta x,\ [2, 7]$ (d) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2 + 4}\,\Delta x,\ [1, 3]$

7. Use the form of the definition of the integral. Evaluate the integral.

(a) $\int_2^5 (4 - 2x)\,dx$ (b) $\int_1^4 (x^2 - 4x + 2)\,dx$

(c) $\int_{-2}^0 (x^2 + x)\,dx$ (d) $\int_0^2 (2x - x^3)\,dx$

(e) $\int_0^1 (x^3 - 3x^2)\,dx$

8. Prove that

(a) $\int_a^b x\,dx = \frac{b^2 - a^2}{2}$ (b) $\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}$

9. Evaluate the integral by interpreting it in terms of areas.

(a) $\int_{-1}^2 (1 - x)\,dx$ (b) $\int_0^9 \left(\frac{1}{3}x - 2\right) dx$

(c) $\int_{-3}^0 \left(1 + \sqrt{9 - x^2}\right) dx$ (d) $\int_{-5}^5 \left(1 + \sqrt{25 - x^2}\right) dx$

(e) $\int_{-1}^2 |x|\,dx$ (f) $\int_0^{10} |x - 5|\,dx$

Answers:

1. 0.87
2. 2.32
3. (a) 6 (b) 4 (c) 2
4. - 64, 16
5. (a) 6.18 (b) 0.589 (c) 0.9 (d) 1.6
6. (a) $\int_2^6 x \ln(1+x^2)\,dx$ (b) $\int_\pi^{2\pi} \frac{\cos x}{x}\,dx$ (c) $\int_2^7 (5x^3-4x)\,dx$ (d) $\int_1^3 \frac{x}{x^2+4}\,dx$
7. (a) - 9 (b) - 3 (c) $\frac{2}{3}$ (d) 0 (e) $\frac{-3}{5}$
9. (a) $\frac{3}{2}$ (b) $\frac{-9}{2}$ (c) 10.07 (d) $\frac{-25\pi}{2}$ (e) $\frac{5}{2}$ (f) 25

5.3 The Fundamental Theorem of Calculus

In this section we present the fundamental theorem of calculus which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals using an antiderivative of the integrand function rather than by taking limits of Riemann Sums.

Theorem: The mean value theorem for definite integrals

If f is continuous on [a, b} then at some point C in [a, b], $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

Example 1: Find the average value of $f(x) = 3 - \frac{3x}{2}$ on [0, 2].

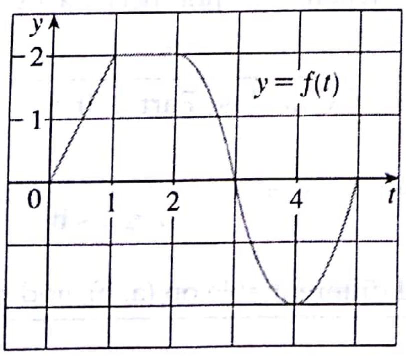
Solution.

$$\text{avf} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-0}\int_0^2 \left(3 - \frac{3}{2}x\right)dx$$

$$= \frac{1}{2}\left(\int_0^2 3dx - \frac{3}{2}\int_0^2 x\,dx\right) = \frac{1}{2}(6-3) = \frac{3}{2}$$

$\therefore$ The average value of $f(x) = 3 - \frac{3}{2}x$ over [0, 2] is $\frac{3}{2}$.

Example 2: If f is the function whose graph is shown in figure and $g(x) = \int_0^2 f(t)\,dt$, find the values of g(0), g(1), g(2), g(3), g(4), and g(5). Then sketch a rough graph of g.



Solution. First we notice that $g(0) = \int_0^0 f(t)\,dt = 0$. From figure we see that g(1) is the area of a triangle:

$$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2}(1 \,.\, 2) = 1$$

To find g(2) we add to g(1) the area of a rectangle:

$$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 1 + (1 \,.\, 2) = 3$$

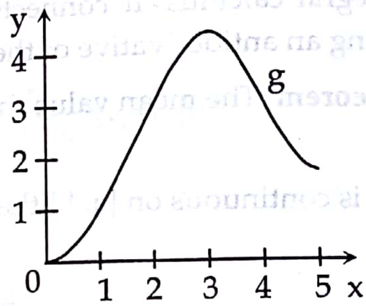
We estimate that the area under f from 2 to 3 is about 1.3, so

$$g(3) = g(2) + \int_2^3 f(t)\,dt \approx 3 + 1.3 = 4.3$$

For t > 3, f(t) is negative and so we start subtracting areas:

$$g(4) = g(3) + \int_3^4 f(t)\,dt \approx 4.3 + (-1.3) = 3.0$$

$$g(5) = g(4) + \int_4^5 f(t)\,dt \approx 3 + (-1.3) = 1.7$$



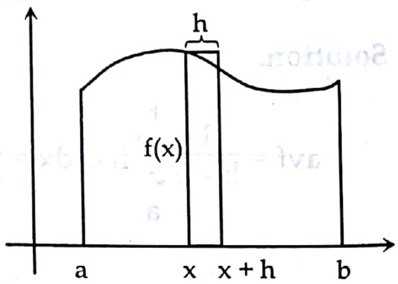
We use these values to sketch the graph of g in figure. Notice that, because f(t) is positive for t < 3, we keep adding area for t < 3 and so g is increasing up to x = 3, where it attains a maximum value. For x > 3, g decreases because f(t) is negative.

$$g(x) = \int_a^x f(t)\,dt$$

Fundamental Theorem, Part 1

If we consider any continuous function f with $f(x) \geq 0$.

Then $g(x) = \int_a^x f(t)\,dt$ can be interpreted as the area under the graph of f from a to x.



In order to compute g'(x) from the definition of a derivative we first observe that, for h > 0, g(x + h) – g(x) is obtained by subtracting areas, so it is the area under the graph of f from x to x + h (as in figure). This area is approximately equal to the area of the rectangle with height f(x) and width h.

$$g(x + h) - g(x) \approx h\, f(x)$$

$$\frac{g(x + h) - g(x)}{h} = f(x) = g'(x)$$

The fact that this is true, even when f is not necessarily positive, is the first part of the fundamental theorem of calculus.

> The fundamental theorem of Calculus, Part 1 If f is continuous on [a, b], then the function g defined by
>
> $$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$
>
> is continuous on [a, b] and differentiable on (a, b), and $g'(x) = f(x)$.

Proof: If x and x + h are in (a, b), then

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt$$

$$= \left(\int_a^{x} f(t)\,dt + \int_x^{x+h} f(t)\,dt\right) - \int_a^{x} f(t)\,dt$$

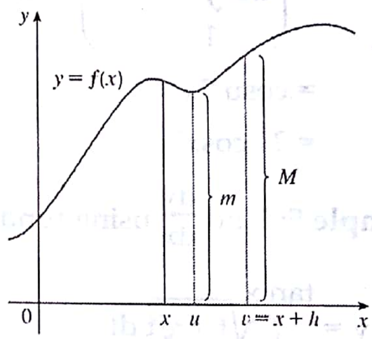
$$= \int_x^{x+h} f(t)\,dt$$

and so, for h ≠ 0,

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

For now let's assume that h > 0. Since f is continuous on [x, x + h], the Extreme Value Theorem says that there are numbers u and v in [x, x + h] such that f(u) = m and f(v) = M, where m and M are the absolute minimum and minimum values of f on [x, x + h]. (see in figure)

By Property of integrals, we have



$$mh \le \int_a^{x+h} f(t)\,dt \le Mh \qquad \text{.... (2)}$$

that is, $f(u)\,h \le \int_a^{x+h} f(t)\,dt \le f(v)\,h$

Since h > 0, we can divide this inequality by h

$$f(u) \le \frac{1}{h}\int_a^{x+h} f(t)\,dt \le f(v)$$

$$f(u) \le \frac{g(x+h) - g(x)}{h} \le f(v) \qquad \text{.... (1)}$$

Inequality 1 can be proved in a similar manner for the case where h < 0.

Now we let h → 0. Then u → x and v → x, since u and v lie between x and x + h.

Therefore

$$\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x) \text{ and } \lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x)$$

because f is continuous at x. We conclude, from 3 and the Squeeze Theorem, that

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f(x) \qquad \text{.... (2)}$$

If x = a or b, the equation 2 can be interpreted as a one-sided limit.

Example 3: Find the derivative of the function $g(x) = \int_0^x \frac{1}{1+t^2}\,dt$

Solution. Since $f(t) = \frac{1}{1+t^2}$ is continuous, so part 1 of the fundamental theorem of calculus gives

$$g'(x) = \frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$$

Example 4: Find $\frac{dy}{dx}$ if $y = \int_1^{x^2} \cos t\,dt$

Solution. The upper limit of integration is not x but x^2. So, let $u = x^2$ then $y = \int_1^{u} \cos t\,dt$ by using chain rule we have,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \left(\frac{d}{du}\int_1^u \cos t\,dt\right) \times \frac{du}{dx}$$

$$= \cos u\ 2x$$

$$= 2x \cos x^2$$

Example 5: Find $\frac{dy}{dx}$ using fundamental theorem of

a. $y = \int_0^{\tan x} \sqrt{t + \sqrt{t}}\,dt$

b. $y = \int_{1-3x}^{1} \frac{u^3}{1+u^2}\,du$

Solution.

a. Given $y = \int_0^{\tan x} \sqrt{t+\sqrt{t}}\,dt$

Put $\tan x = t$ then $\sec^2 x\,dx = dt$

so, $y = \int_0^{t} \sqrt{\tan x + \sqrt{\tan x}}\ \sec^2 x\,dx$

So by using fundamental theorem 1st $y'(x) = \sqrt{\tan x + \sqrt{\tan x}}\ \sec^2 x$

$\therefore \quad y'(x) = \sec^2 x \sqrt{\tan x + \sqrt{\tan x}}$

b. Given, $y = \int_{1-3x}^{1} \frac{u^3}{1+u^2}\,du$

Let $1 - 3x = v$ then $-3dx = dv$

So,

$$\frac{dy}{dx} = \frac{d}{dx}\int_v^1 \frac{u^3}{1+u^2}\,du = \frac{d}{dv}\left[\int_v^1 \frac{u^3}{1+u^2}\right]\frac{dv}{dx} = \frac{-v^3}{1+v^2} \times (-3) = \frac{3v^3}{1+v^2} = \frac{3(1-3x)^3}{1+(1-3x)^2}$$

The fundamental Theorem of Calculus, Part 2 If f is continuous on [a, b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Proof: Let $g(x) \int_a^x f(t)\,dt$. We know from part 1 that $g'(x) = f(x)$; that is, g is an antiderivative of f. If F is any other antiderivative of f on [a, b], the we know that F and g differ by a constant:

$$F(x) = g(x) + C \qquad \text{....... (1)}$$

for $a < x < b$. But both F and g are continuous on [a, b] and so, by taking limits of both sides of equation 1 (as $x \to a^+$ and $x \to b^-$), we see that it also holds when $x = a$ and $x = b$.

If we put $x = a$ in the formula for g(x), we get

$$g(a) = \int_a^a f(t)\,dt = 0$$

So, using equation 1 with $x = b$ and $x = a$, we have

$$F(b) - F(a) = [g(b) + C] - [g(a) + C]$$

$$= g(b) - g(a) = a(b) = \int_a^b f(t)\,dt$$

Example 6: If $f(1) = 12$, f' is continuous and $\int_1^4 f'(x)\,dx = 17$, find f(4).

Solution. We know $\int_a^b f'(x)\,dx = f(b) - f(a)$ by fundamental part 2.

So, $\int_1^4 f'(x)\,dx = 17$

$f(4) - f(1) = 17$

$f(4) - 12 = 17$

$f(4) = 29$

Example 7: Evaluate $\int_0^\pi f(x)\,dx$ where $f(x) = \begin{cases} \sin x \text{ if } 0 \le x < \frac{\pi}{2} \\ \cos x \text{ if } \frac{\pi}{2} \le x \le \pi \end{cases}$.

Solution. Given, $\int_0^\pi f(x)\,dx = \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^\pi f(x)\,dx$

$$= \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^\pi \cos x\,dx$$

$$= [-\cos x]_0^{\pi/2} + [\sin x]_{\pi/2}^{\pi}$$

$$= -\left(\cos\frac{\pi}{2} - \cos 0\right) + \left(\sin\pi - \sin\frac{\pi}{2}\right)$$

$$= -(0-1) + (0-1)$$

$$= 0$$

Example 8: Find the derivative of $y = \int_{\cos x}^{\sin x} \ln(1+2v)\,dv$

Solution. Given, $y = \int_{\cos x}^{\sin x} \ln(1+2v)\,dv$

Let, $f(v) = \ln(1+2v)$ and we know $\int_a^b f(x)\,dv = F(b) - F(a)$

Then,

$$y = \int_{\cos x}^{\sin x} \ln(1+2v)\,dv = F(\sin x) - F(\cos x)$$

Differentiating we get

$$y' = [F(\sin x) - F(\cos x)]'$$

$$= F(\sin x)(\sin x)' - F(\cos x)(\cos x)' \text{ (using chain rule)}$$

$$= \ln(1 + 2\sin x)\cos x - \ln(1 + 2\cos x)(-\sin x)$$

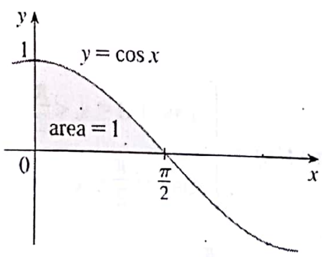
$$= \cos x \ln(1 + 2\sin x) + \sin x \ln(1 + 2\cos x)$$

Example 9: Find the area under the cosine curve from 0 to b, where $0 \le b \le \pi/2$.

Solution. Since an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$, we have

$$A = \int_0^b \cos x\,dx = \sin x]_0^b = \sin b - \sin 0 = \sin b$$

In particular, taking $b = \pi/2$, we have proved that the area under the cosine curve from 0 to $\pi/2$ is $\sin(\pi/2) = 1$.



The Fundamental theorem of Calculus

Suppose f is continuous on [a, b]

1. If $g(x) = \int_a^x f(t)\,dt$ then $g'(x) = f(x)$

2. $\int_a^b f(x)\,dx = F(b) . F(a)$, where F is any antiderivative of F, i.e., $F' = f$.

Example 10: Evaluate the limit by first recognizing the sum as a Riemann sum for a function defined on [0, 1].

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$$

Solution. We know $\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(a + i\,\Delta x)$; where $\Delta x = \frac{b-a}{n}$.

So, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{i^3}{n^3}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(0 + i . \frac{1}{h}\right)^3$$

Here, $\Delta x = \frac{1}{h}$, a = 0, So $\Delta x = \frac{b-a}{n}$

$$\frac{1}{n} = \frac{b}{n}$$

$$b = 1$$

$$f(x) = x^3$$

$$\therefore \quad \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

Exercise 5.3

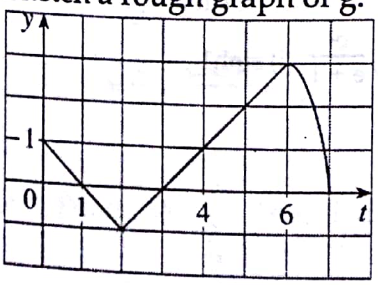
1. Let $\int_0^x f(t)\,dt$, where f is the function whose graph is shown in following figure.

   (a) Evaluate g(x) for x = 0, 1, 2,3 ,4 , 5, and 6.

   (b) Estimate g(7)

   (c) Where does g have a maximum value and minimum value?

   (d) Sketch a rough graph of g.



2. Use part 1 of the fundamental theorem of calculus to find the derivative of the function

(a) $g(x) = \int_1^x \frac{1}{t^3+1} dt$

(b) $g(x) = \int_3^x e^{t^2-t} dt$

(c) $g(s) = \int_5^s (t - t^2)^8 dt$

(d) $g(r) = \int_0^r \sqrt{x^2+4}\, dx$

(e) $F(x) = \int_x^\pi \sqrt{1 + \sec t}\, dt$

(f) $G(x) = \int_x^1 \cos \sqrt{t}\, dt$

(g) $h(x) = \int_1^{e^x} \ln t\, dt$

(h) $h(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4+1} dz$

(i) $y = \int_1^{x^4} \cos^2\theta\, d\theta$

(j) $y = \int_{\sin x}^1 \sqrt{1+t^2}\, dt$

(k) $g(x) = \int_{1-2x}^{1+2x} t \sin t\, dt$

(*l*) $y = \int_{\cos x}^{\sin x} \ln (1 + 2v)\, dv$

3. Evaluate the integral

(a) $\int_1^4 (5 - 2t + 3t^2)\, dt$

(b) $\int_0^1 \left(1 + \frac{1}{2}u^4 - \frac{2}{5}u^9\right) du$

(c) $\int_{-5}^5 e\, dx$

(d) $\int_0^1 (u + 2)(u - 3)\, du$

(e) $\int_1^9 \frac{x-1}{\sqrt{x}} dx$

(f) $\int_0^{\pi/4} \sec\theta \tan\theta\, dt$

(g) $\int_0^3 (2 \sin x - e^x)\, dx$

(h) $\int_0^1 (x^e + e^x)\, dx$

(i) $\int_0^1 \cosh t\, dt$

(j) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} dx$

(k) $\int_{-2}^2 f(x)\, dx$ where $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2 \end{cases}$

4. Evaluate as a Riemann sum: $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \ldots + \sqrt{\frac{n}{n}}\right)$.

Answers:

1. (a) $\frac{1}{2}, 0, -\frac{1}{2}, 0, \frac{3}{2}, 4$ (b) 6.2 (c) maximum at t = 7, minimum at t = 3

2. (a) $\frac{1}{x^3+1}$ (b) $e^{x^2} - x$ (c) $(s - s^2)^8$ (d) $\sqrt{r^2+4}$ (e) $-\sqrt{1 + \sec x}$ (f) $-\cos\sqrt{x}$ (g) $x e^x$

(h) $\frac{\sqrt{x}}{2(x^2+1)}$ (i) $4x^3 (\cos x^4)^2$ (j) $-\cos x \sqrt{1 + \sin^2 x}$

(k) $(2 - 4x) \sin(1 - 2x) + (2 + 4x) \sin (1 + 2x)$ (*l*) $\sin x \ln (1 + 2 \cos x) + \cos x \ln (1 + 2 \sin x)$

3. (a) 63 (b) $\frac{53}{50}$ (c) 10e (d) $-\frac{37}{6}$ (e) $\frac{40}{3}$ (f) $\sqrt{2} - 1$ (g) $-2 \cos 3 - e^3 + 3$ (h) $\frac{e^2}{e+1}$ (i) sinh1

(j) $\frac{4\pi}{3}$ (k) $\frac{28}{3}$

4. $\frac{2}{3}$

5.4 Indefinite Integrals and The Net Change Theorem

Indefinite Integrals

Both parts of the fundamental theorem establish connection between antiderivatives and definite integrals.

We need a convenient notation for antiderivatives that makes them easy to work with because of the relation given by the Fundamental theorem between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an indefinite integral.

Thus,

$$\int f(x)\,dx = F(x) \text{ means } F'(x) = f(x)$$

Definition: The set of all antiderivatives of the function f is called the indefinite integral of f with respect to x and we write

$$\int f(x)\,dx$$

Note: A definite integral $\int_a^b f(x)\,dx$ is a number whereas an indefinite integral $\int f(x)\,dx$ is a function.

Example 1: Find the general indefinite integral $\int (3x^2 - 5\sin x)\,dx$

Solution.

$$\int (3x^2 - 5\sin x)\,dx$$

$$= \int 3x^2\,dx - \int 5\sin x\,dx$$

$$= 3\int x^2\,dx - 5\int \sin x\,dx$$

$$= 3\frac{x^3}{3} + 5\cos x + C$$

Example 2: Evaluate $\int \sin^2 x\,dx$

Solution.

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{x}{2} - \frac{1}{2} \times \frac{\sin 2x}{2} + C$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

Example 3: Evaluate $\int_0^{3\pi/2} |\sin x|\,dx$

Solution. Given $\int_0^{3\pi/2} |\sin x|\, dx$ since, $|\sin x| = \begin{cases} \sin x \text{ for } 0 \le x \le \pi \\ -\sin x \text{ for } \pi \le x \le \frac{3\pi}{2} \end{cases}$

$$= \int_0^{\pi} \sin x\, dx + \int_{\pi}^{3\pi/2} (-\sin x)\, dx$$

$$= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{3\pi/2}$$

$$= (-\cos\pi + \cos 0) + \left(\cos\frac{3\pi}{2} - \cos\pi\right)$$

$$= (1+1) + (0+1)$$

$$= 3$$

Applications

Part 2 of the fundamental theorem says that if f is continuous on [a, b], then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is any antiderivative of f i.e., $F'(x) = f(x)$. So above equation can be written as

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

we know that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to x and $F(b) - F(a)$ is the change in y when x changes from a to b. (Note that y could, for instance, increase, then decrease, then increase again. Although y might change in both directions, $F(b) - F(a)$ represents the net change in y). So we can reformulate

FTC 2 in words as follows

Net Change Theorem: The integral of a rate of change is the net change

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

This principle can be applied to all of the rates of change in the natural and social sciences that we discussed in section 3.7. Here are a few instances of this idea:

- If V(t) is the volume of water in a reservoir at time t, then its derivative V'(t) is the rate at which water flows into the reservoir at time t. So

  $$\int_{t_1}^{t_2} V'(t)\, dt = V(t_2) - V(t_1)$$

  is the change in the amount of water in the reservoir between time t_1 and time t_2.
- If [C](t) is the concentration of the product of a chemical reaction at time t, then the rate of reaction is the derivative $\frac{d[C]}{dt}$. So

$$\int_{t_1}^{t_2} \frac{d[C]}{dt}\, dt = [C](t_2) - [C](t_1)$$

is the change in the amount of water in the reservoir between time t_1 and time t_2.

- If the mass of a rod measured from the left end to a point x is m(x), then the linear density is $\rho(x) = m'(x)$. So

$$\int_a^b \rho(x)\, dx = m(b) - m(a)$$

is the mass of the segment of the rod that lies between x = a and x = b.

- If the rate of growth of a population is $\frac{dn}{dt}$, then

$$\int_{t_1}^{t_2} \frac{dn}{dt}\, dt = n(t_2) - n(t_1)$$

is the net change in population during the time period from t_1 to t_2. (The population increases when births happen and decreases when deaths occur. The net change takes into account both births and deaths.)

- If C(x) is the cost of producing x units of a commodity, then the marginal cost is the derivative C'(x). So,

$$\int_{x_1}^{x_2} C'(x)\, dx = C(x_2) - C(x_1)$$

is the increase in cost when production is increased from x_1 units to x_2 units.

- If an object moves along a straight line with position function s(t), then its velocity is v(t) = s'(t), so

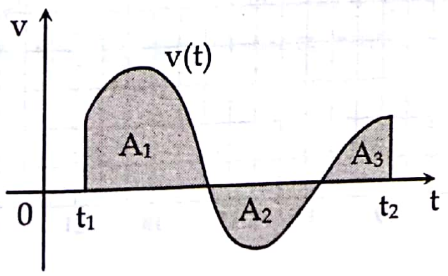
$$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)$$

is the net change of position, or displacement, of the particle during the time period from t_1 to t_2. In section 5.1 we guessed that this was true for the case where the object moves in the positive direction, but now we have proved that it is always true.

- If we want to calculate the distance the object travels during the time interval, we have to consider the intervals when $v(t) \geq 0$ (the particle moves to the right) and also the intervals when $v(t) \leq 0$ (the particle moves to the left). In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore,

$$\int_{t_1}^{t_2} |v(t)|\, dt = \text{total distance traveled}$$

Following figure shows how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve.



- The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$ is the change in velocity from time t_1 to time t_2.

Example 4: A particle moves along a line so that its velocity at time t is $v(t)\ t^2 - t - 6$ (measured in meters per second).

a. Find the displacement of the particle during the time period $1 \le t \le 4$.

b. Find the distance traveled during this time period.

Solution.

a. The displacement is

$$s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}$$

This means that the particle moved 4.5 m toward the left.

b. Note that $v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \le 0$ on the interval $[1, 3]$ and $v(t) \ge 0$ on $[3, 4]$. Thus, the distance traveled is

$$\int_1^4 |v(t)|\,dt = \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt$$

$$= \int_1^3 (-t^2 + t + 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 = \frac{61}{6} \approx 10.17 \text{ m}$$

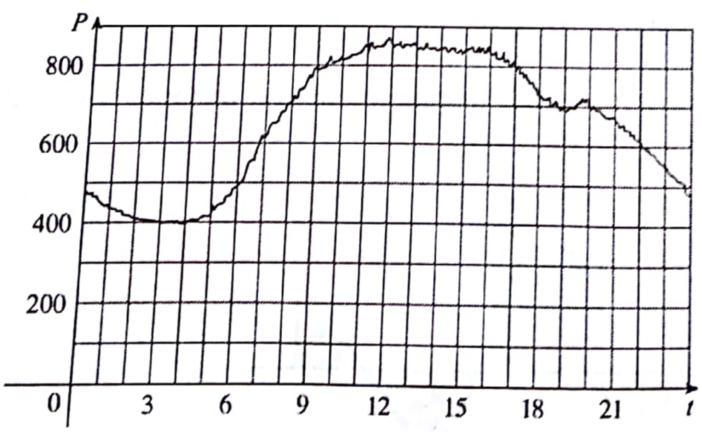
Example 5: Water flows from the bottom of a storage tank of a rate of $r(t) = 200 - 4t$ 1/m, where *$0 \le t \le 50$. Find the amount of water that flows from the tank during the first 10 m.*

Solution. *Given* $r(t) = 200 - 4t\ l/m$ where $0 \le t \le 50$

According to net change theorem the amount of water in 1st 10 m is

$$= \int_0^{10} r(t)\,dt = \int_0^{10} (200 - 4t)\,dt = [200t - 2t^2]_0^{10} = 1800\ l$$

Example 6: Following figure shows the power consumption in the city of Dharan for a day in September (P is measured in megawatts; t is measured in hours starting at midnight). Estimate the energy used on that day.



Solution. Power is the rate of change of energy: $P(t) = E'(t)$. So, by the Net Change Theorem,

$$\int_0^{24} P(t)\,dt = \int_0^{24} E'(t)\,dt = E(24) - E(0)$$

is the total amount of energy used on that day. We approximate the value of the integral using the Midpoint Rule with 12 subintervals and $\Delta t = 2$

$$\int_0^{24} P(t)\,dt \approx [P(1) + P(3) + P(5) + \ldots + P(21) + P(23)]\,\Delta t$$

$$\approx (440 + 400 + 420 + 620 + 790 + 840 + 850 + 840 + 810 + 690 + 670 + 550) \quad \ldots (2)$$

$$= 15{,}840$$

The energy used was approximately 15,840 megawatt-hours.

Exercise 5.4

1. Evaluate the integral

(a) $\int (x^2 + x^{-2})\,dx$

(b) $\int \left(\sqrt{x^3} + \sqrt[3]{x^2}\right) dx$

(c) $\int \left(x^2 + 1 + \frac{1}{x^2+1}\right) dx$

(d) $\int (\sin x + \sinh x)\,dx$

(e) $\int (1 + \tan^2\alpha)\,d\alpha$

(f) $\int \frac{\sin 2x}{\sin x}\,dx$

(g) $\int_0^1 (x^{10} + 10^x)\,dx$

(h) $\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,dx$

(i) $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta$

(j) $\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\,dx$

(k) $\int_1^2 \frac{(x-1)^3}{x^2}\,dx$

(*l*) $\int_0^2 |2x - 1|\,dx$

(m) $\int_{-1}^2 (x - 2|x|)\,dx$

2. Use a graph to estimate the x-intercepts of the curve $y = 1 - 2x - 5x^4$. Then use this information to estimate the area of the region that lies under the curve and above the x-axis.

3. The area of the region that lies to the right of the y-axis and to the left of the parabola $x = 2y - y^2$.

4. Find the area of region bounded by y-axis, $y = \sqrt[4]{x}$ and $y = 1$.

5. The velocity function (in meters per second) is given for a particle moving along a line. Find (1) the displacement and (2) the distance traveled by the particle during the given time interval.

(a) $v(t) = 3t - 5,\ 0 \le t \le 3$

(b) $v(t) = t^2 - 2t - 8,\ 1 \le t \le 6$

6. The acceleration function (in m/s²) and the initial velocity are given for a particle moving along a line. Find (1) the velocity at time t and (2) the distance traveled during the given time interval.

   (a) $a(t) = t + 4,\ v(0) = 5,\ 0 \le t \le 10$ (b) $a(t) = 2t + 3,\ v(0) = -4,\ 0 \le t \le 3$

7. The linear density of a rod of length 4 m is given by $\rho(x) = 9 + 2\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from one end of the rod. Find the total mass of the rod.

8. The marginal cost of manufacturing x yards of a certain fabric is $C'(x) = 3 - 0.01x + 0.000006x^2$ (in dollars per yard). Find the increase in cost if the production level is raised from 2000 yards to 4000 yards.

9. A bacteria population is 4000 at time $t = 0$ and its rate of growth is 1000×2^t bacteria per hour after t hours. What is the population after one hour?

Answers:

1. (a) $\frac{x^3}{3} - \frac{1}{x} + C$ (b) $\frac{2}{5}x^{5/2} + \frac{3}{5}x^{5/3} + C$ (c) $\frac{x^3}{3} + x + \tan^{-1}x + C$ (d) $-\cos x + \cosh x + C$

   (e) $\tan\alpha + C$ (f) $2\sin x + C$ (g) $\frac{1}{11} + \frac{9}{\ln 10}$ (h) $1 + \frac{\pi}{4}$ (i) $\frac{1}{2}$ (j) 40 (k) $-2 + 3\ln 2$

   (*l*) $\frac{5}{2}$ (m) $\frac{-7}{2}$

2. 1.36
3. 4/3
4. 1/5
5. (a) (1) – 1.5 (2) $\frac{41}{6}$ (b) (1) – 10/3 (2) 98/3
6. (a) (1) $\frac{t^2}{2} + 4t + 5$ (2) $\frac{1250}{3}$ (b)(1) $t^2 + 3t - 4$ (2) 89/6
7. 140/3
8. 58000
9. 5442.7

5.5 Techniques of Integration

In this section we develop technique for using the basic integration formulas to obtain integral of more complicated function.

Some Formulas and Rules of Integrals

1. $\int c\, f(x)\, dx = c \int f(x)\, dx$
2. $\int k\, dx = kx + c$
3. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C\ (n \neq 1)$
4. $\int e^x\, dx = e^x + C$
5. $\int \frac{1}{x}\, dx = \ln|x| + C$
6. $\int a^x\, dx = \frac{a^x}{\ln a} + C$
7. $\int \sin x\, dx = -\cos x + C$
8. $\int \cos x\, dx = \sin x + C$
9. $\int \sec^2 x\, dx = \tan x + C$
10. $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$
11. $\int \sec x \tan x\, dx = \sec x + C$
12. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$
13. $\int \frac{1}{x^2+1}\, dx = \tan^{-1} x + C$
14. $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$
15. $\int \sinh x\, dx = \cosh x + C$
16. $\int \cosh x\, dx = \sinh x + C$

Note: $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$

Algebraic Substitution Rule

Example 1: Find $\int x^2 \sin x^3\, dx$

Solution.

Let $x^3 = u$ then $3x^2\, dx = du$ i.e., $x^2\, dx = \frac{1}{3} du$

So, $\int x^2 \sin x^3\, dx = \int \sin u \cdot \frac{1}{3} du$

$$= \frac{1}{3} \int \sin u\, du$$

$$= \frac{1}{3} \times -\cos u + C$$

$$= -\frac{\cos x^3}{3} + C$$

Example 2: Find $\int \sqrt{1+x^2}\, x^5\, dx$

Solution. Let $u = 1 + x^2$ then $du = 2x\, dx$

i.e., $x\, dx = \frac{1}{2} du$

also, $x^2 = u - 1$ so $x^4 = (u-1)^2$

$$\int \sqrt{1+x^2}\, x^5\, dx$$

$$\int \sqrt{1+x^2} \cdot x^4 \cdot x\, dx$$

$$= \int \sqrt{u} \times (u-1)^2 \cdot \frac{1}{2} du$$

$$= \frac{1}{2} \int \sqrt{u}\,(u^2 - 2u + 1)\, du$$

$$= \frac{1}{2} \int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$

$$= \frac{1}{2}\left[\frac{u^{\frac{7}{2}}}{\frac{7}{2}} - \frac{2u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + C$$

$$= \frac{1}{7}(1 + x^2)^{7/2} - \frac{2}{5}(1 + x^2)^{5/2} + \frac{1}{3}(1 + x^2)^{3/2} + C$$

Example 3: Evaluate $\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

Solution. Let $1 + \sqrt{x} = u$

Then,

$$\frac{1}{2\sqrt{x}} dx = du$$

$$dx = 2\sqrt{x}\, du$$

$$dx = 2\,(u - 1)\, du$$

if $x = 0$, then $u = 1$,

if $x = 1$, then $u = 2$

$$\therefore \int_0^1 \frac{1}{(1+\sqrt{x})^4} dx = \int_1^2 \frac{1}{u^4} \times 2(u-1)\, du$$

$$= 2\int_1^2 \left(\frac{1}{u^3} - \frac{1}{u^4}\right) du$$

$$= 2\left[\frac{u^{-2}}{-2} - \frac{u^{-3}}{-3}\right]_1^2 = 2\left[\left(-\frac{1}{8} + \frac{1}{24}\right) - \left(-\frac{1}{2} + \frac{1}{3}\right)\right] = 2\left(\frac{-2}{24} + \frac{1}{6}\right) = 2 \times \frac{2}{24} = \frac{1}{6}$$

Integration by parts

$$\int uv\, dx = u\int v\, dx - \int \left[\frac{du}{dx}\int v\, dx\right] dx$$

Example 4: Evaluate: $\int x \sin x\, dx$

Solution: $\int x \sin x\, dx$

$$= x\int \sin x\, dx - \int \left[\frac{dx}{dx} \int \sin dx\right] dx$$

$$= x\,(-\cos x) - \int (-\cos x)\, dx$$

$$= -x \cos x + \sin x + C$$

Example 5: $\int t^2 e^t\, dt$

Solution: $\int t^2 e^t\, dt$

$= t^2\int e^t\, dt - \int \left[\frac{dt^2}{dt}\int e^t\, dt\right] dt$

$= t^2 e^t - \int 2t\, e^t\, dt$

$= t^2 e^t - 2\{t\int e^t\, dt - \int\left[\frac{dt}{dt}\int e^t\, dt\right] dt$

$= t^2 e^t - 2\,\{te^t - \int e^t\, dt\}$

$= t^2 e^t - 2t\, e^t + 2e^t + C$

Example 6: Evaluate: $\int_0^1 \tan^{-1}x\, dx$

Solution: $\int_0^1 \tan^{-1}x\, dx = [x \tan^{-1} x]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$

$= (1 \tan^{-1}1 - 0 \tan^{-1}0) - \int_0^1 \frac{x}{1+x^2}\, dx$

$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\, dx \quad \text{......(1)}$

For $\int_0^1 \frac{x}{1+x^2}\, dx$

Let $1 + x^2 = t$

then $2x\, dx = dt$ when $x = 0, t = 1, x = 1, t = 2$

$\therefore \int_0^1 \frac{x}{1+x^2}\, dx$

$= \int_1^2 \frac{1}{t} \times \frac{dt}{2} = \frac{1}{2}[\ln|t|]_1^2 = \frac{1}{2}(\ln 2 - \ln 1) = \frac{\ln 2}{2}$

$\therefore \int_0^1 \tan^{-1} x\, dx = \frac{\pi}{4} - \frac{\ln 2}{2}.$

Trigonometric Integrals

Example 7: Evaluate: $\int \cos^3 x\, dx$

Solution. $\int \cos^3 x\, dx = \int (1 - \sin^2 x) \cos x\, dx$

Let $\sin x = u$ then $\cos x\, dx = du$

So $\int (1 - \sin^2 x) \cos x \, dx$

$$= \int (1 - u^2)\, du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C$$

Example 8: Find $\int \sin^5 x \cos^2 x \, dx$

Solution: $\int \sin^5 x \cos^2 x \, dx$

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx$$

Let $\cos x = t$ then $- \sin x \, dx = dt$

So $\int \sin^5 x \cos^2 x \, dx = - \int (1 - u^2)^2 \times u^2 \, du$

$$= - \int (u^2 - 2u^4 + u^6)\, du$$

$$= - \left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C$$

$$= - \frac{1}{3} \cos^3 x + \frac{2}{5} \cos^5 x - \frac{1}{7} \cos^7 x + C$$

Example 9: Evaluate $\int x \sin^3 x \, dx$

Solution. $\int x \sin^3 x \, dx$

$$= x \int \sin^3 x \, dx - \int \left[\frac{d}{dx} x \int \sin^3 x \, dx\right] dx$$

$$= x \int \sin^3 x - \int \left(\int \sin^3 x \, dx\right) dx \qquad \text{......(1)}$$

We know,

$$\int \sin^3 x \, dx = \int (1 - \cos^2 x) \sin x \, dx$$

Let $\cos x = t$ then $- \sin x \, dx = dt$

So $\int \sin^3 x \, dx = \int (1 - t^2)\, x - dt$

$$= - t + \frac{t^3}{3}$$

$$= - \cos x + \frac{\cos^3 x}{3} \text{ using in (1)}$$

$$x \sin^3 x \, dx = x\left(- \cos x + \frac{\cos^3 x}{3}\right) - \int \left(- \cos x + \frac{\cos^3 x}{3}\right) dx$$

$$= - x \cos x + \frac{x \cos^3 x}{3} + \sin x - \frac{1}{3} \int \cos^3 x \, dx$$

milarly we get

$$\int \cos^3 x \, dx = \sin x - \frac{1}{3} \sin^3 x$$

e get

$$\int x \sin^3 x \, dx = - x \cos x + \frac{x \cos^3 x}{3} + \sin x - \frac{1}{3}\left(\sin x - \frac{1}{3} \sin^3 x\right) + C$$

$$= - x \cos x + \frac{x \cos^3 x}{3} + \frac{2 \sin x}{3} + \frac{1}{9} \sin^3 x + C$$

Example 10: Evaluate: $\int_0^{\pi/4} \tan^4 t \, dt$

Solution: $\int_0^{\pi/4} \tan^4 t \, dt$

$$= \int_0^{\pi/4} (\sec^2 t - 1) \tan^2 t \, dt$$

$$= \int_0^{\pi/4} \sec^2 t \tan^2 t \, dt - \int_0^{\pi/4} \tan^2 t \, dt \quad ....(1)$$

Since $\int_0^{\pi/4} \tan^2 t \, dt = \int_0^{\pi/4} (\sec t - 1) \, dt$

$$= [\tan t - t]_0^{\pi/4}$$

$$= 1 - \frac{\pi}{4}$$

Also for $\int_0^{\pi/4} \sec^2 t \tan^2 t \, dt$

Let $\tan t = u$

$\sec^2 t \, dt = du$

If $t = 0$, $u = 0$, $t = \frac{\pi}{4}$, $u = 1$.

So, $\int_0^{\pi/4} \sec^2 t \tan^2 t \, dt$

$$= \int_0^1 u^2 \, du$$

$$= \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}$$

$\therefore$ Equation (1) is

$$\int_0^{\pi/4} \tan^4 t \, dt = \frac{1}{3} - 1 + \frac{\pi}{4} = -\frac{2}{3} + \frac{\pi}{4}$$

Trigonometric Substitution

Note: For $\sqrt{a^2 - x^2}$, suppose $x = a \sin\theta$; $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

For $\sqrt{a^2 + x^2}$, $x = a \tan\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

For $\sqrt{x^2 - a^2}$, suppose $x = a \sec\theta$, $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta \le \frac{3\pi}{2}$

Example 11: Evaluate $\int \frac{\sqrt{9 - x^2}}{x^2} dx$

Solution.

Let $x = 3 \sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ then $dx = 3 \cos\theta \, d\theta$

So, $$\int \frac{\sqrt{9 - x^2}}{x^2} dx = \int \frac{\sqrt{9 - 9 \sin^2\theta}}{9 \sin^2\theta} \times 3 \cos\theta \, d\theta$$

$$= \int \frac{9 \cos^2\theta \, d\theta}{9 \sin^2\theta}$$

$$= \int \cot^2\theta \, d\theta$$

$$= \int (\mathrm{cosec}^2\theta - 1) \, d\theta$$

$$= -\cot\theta - \theta + c$$

$$= -\frac{\cos\theta}{\sin\theta} - \theta + c$$

$$= \frac{-\sqrt{1 - \sin^2\theta}}{\sin\theta} - \theta + c$$

$$= \frac{-\sqrt{1 - \frac{x^2}{9}}}{\frac{x}{3}} - \sin^{-1}\left(\frac{x}{3}\right) + c$$

$$= \frac{-\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + c$$

Example 12: $\int \frac{x^2 + 1}{(x^2 - 2x + 2)^2} dx$

Solution.

$$= \int \frac{x^2 + 1}{(x^2 - 2x + 2)^2} dx$$

$$= \int \frac{x^2 + 1}{[(x - 1)^2 + 1]^2}$$

Let $x - 1 = \tan\theta$. Then $dx = \sec^2\theta \, d\theta$

So,

$$\int \frac{x^2 + 1}{(x^2 - 2x + 2)^2} dx = \int \frac{(\tan\theta + 1)^2 + 1}{(\tan^2\theta + 1)^2} \times \sec^2\theta \, d\theta$$

$$= \int \frac{(\tan^2\theta + 2\tan\theta + 2)\sec^2\theta}{\sec^4\theta} d\theta$$

$$= \int \frac{\tan^2\theta + 2\tan\theta + 2}{\sec^2\theta} d\theta$$

$$= \int \sin^2\theta \, d\theta + \int 2 \sin\theta \cos\theta \, d\theta + \int 2\cos^2\theta \, d\theta$$

$$= \frac{1}{2} \int (1 - \cos 2\theta) \, d\theta + \int \sin 2\theta \, d\theta + \int (1 + \cos 2\theta) \, d\theta$$

$$= \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta - \frac{1}{2}\cos 2\theta + \theta + \frac{\sin 2\theta}{2} + C$$

$$= \frac{3}{2}\theta - \frac{1}{4}\sin 2\theta - \frac{1}{2}\cos 2\theta + C$$

Integration of Rational Functions by Partial Fractions

Example 13: Evaluate $\int \frac{x^5 + x - 1}{x^3 + 1}\,dx$

Solution. Since we can write $\frac{x^5 + x - 1}{x^3 + 1} = x^2 + \frac{(-x^2 + x - 1)}{x^3 + 1}$

So we have,

$$\int \frac{x^5 + x - 1}{x^3 + 1}\,dx = \int x^2\,dx + \int \frac{(-x^2 + x - 1)}{x^3 + 1}\,dx$$

$$= \int x^2\,dx - \int \frac{(x^2 - x + 1)}{(x+1)(x^2 - x + 1)}\,dx$$

$$= \frac{x^3}{3} - \int \frac{1}{x+1}\,dx$$

$$= \frac{x^3}{3} - \ln|x+1| + c$$

Example 14: $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx$

Solution. Since, $2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x - 1)(x + 2)$

So let $\frac{x^2 + 2x - 1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$

or $x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$

or $x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$

Comparing coefficients of x^2, x and constant respectively,

$2A + B + 2C = 1$ .... (1)

$3A + 2B - C = 2$ .... (2)

$-2A = -1$ .... (3)

Solving we get $A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$

So we get

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\,dx = \int \left[\frac{1}{2x} + \frac{1}{5(2x-1)} - \frac{1}{10(x+2)}\right]dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

Example 15: Evaluate: $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$

Solution. Since we have $\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$

and $x^3 - x^2 - x + 1 = (x-1)(x^2-1) = (x-1)^2(x+1)$

So we have,

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \int \left(x + 1 + \frac{4x}{x^3 - x^2 - x + 1}\right) dx$$

$$= \int x\,dx + \int 1\,dx + \int \frac{4x}{(x+1)(x-1)^2}\,dx$$

$$= \frac{x^2}{2} + x + \int \frac{4x}{(x+1)(x-1)^2}\,dx \qquad \text{.... (A)}$$

For $\int \frac{4x}{(x+1)(x-1)^2}\,dx$

Let $\frac{4x}{(x+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

or $4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

Solving and comparing we get

$A = 1, B = 2, C = -1$

So we get

$$\int \frac{4x}{(x+1)(x-1)^2}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

$$= -\frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C$$

∴ Equation (A) becomes

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C$$

Method of Partial Fraction $\left(\frac{f(x)}{g(x)}\right)$ proper

1. Let $(x - r)$ be a linear fractor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $(x - r)$ then to this factor assign the sum of the m partial fractions

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \dots + \frac{A_m}{(x-r)^m}$$

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor then to this factor assign the sum of the n partial fractions

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{x^2 + px + q} + \dots\dots\dots + \frac{B_nx + C_n}{x^2 + px + q}$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

Exercise 5.5

1. Evaluate:

(a) $\int ln\sqrt[3]{x}\ dx$

(b) $\int \sin^{-1}x\ dx$

(c) $\int s2^s\ dx$

(d) $\int z^3 e^z\ dz$

(e) $\int x\tan^2 x\ dx$

(f) $\int (\sin^{-1}x)^2\ dx$

(g) $\int_0^{2\pi} t^2 \sin 2t\ dt$

(h) $\int_1^2 x^4 (ln\ x)^2\ dx$

2. Evaluate:

(a) $\int \sin^2 x \cos^3 x\ dx$

(b) $\int_0^{\pi} \sin^2 t \cos^4 t\ dt$

(c) $\int \cos\theta \cos^5(\sin\theta)\ d\theta$

(d) $\int \cos^2 x\ \tan^3 x\ dx$

(e) $\int \tan x \sec^3 x\ dx$

(f) $\int \tan^2\theta \sec^4\theta\ d\theta$

(g) $\int \tan^4 x \sec^6 x\ dx$

(h) $\int_0^{\pi/4} \tan^4 t\ dt$

(i) $\int \sin 8x \cos 5x\ dx$

(j) $\int \frac{\cos x + \sin x}{\sin 2x}\ dx$

(k) $\int_0^1 x\tan^2 x\ dx$

3. Evaluate:

(a) $\int \frac{dx}{x^2\sqrt{4-x^2}}$

(b) $\int \frac{x^3}{\sqrt{x^2+4}}\ dx$

(c) $\int \frac{\sqrt{x^2-4}}{x}\ dx$

(d) $\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\ dt$

(e) $\int \frac{t^5}{\sqrt{t^2+2}}\ dt$

(f) $\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\ dx$

(g) $\int \sqrt{5+4x-x^2}\ dx$

(h) $\int \frac{x}{\sqrt{x^2+x+1}}$

4. Evaluate:

(a) $\int \frac{5x+1}{(2x+1)(x-1)}\ dx$

(b) $\int_0^1 \frac{2}{2x^2+3x+1}\ dx$

(c) $\int \frac{ax}{x^2-bx}\ dx$

(d) $\int \frac{x^2+1}{(x-3)(x-2)^2}\ dx$

(e) $\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\ dx$

(f) $\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\ dx$

(g) $\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}\ dx$

Answers

1. (a) $x \ln \sqrt[3]{x} - \frac{x}{3} + C$

(b) $x \sin^{-1}x + \sqrt{1-x^2} + C$

(c) $\frac{s\,2^s}{\ln 2} - \frac{2^s}{(\ln 2)^2} + C$

(d) $z^3 e^z - 3z^2 e^z + 6ze^z - 6e^z + C$

(e) $x \tan x + \ln |\cos x| - \frac{x^2}{2} + C$

(f) $-2 \sin^{-1}x \sqrt{1-x^2} + 2x + C$

(g) $-2\pi^2$

(h) $\frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln^2 + \frac{62}{125}$

2. (a) $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

(b) $\frac{\pi}{16}$

(c) $\sin(\sin\theta) - \frac{2\sin^3(\sin\theta)}{3} + \frac{\sin^5(\sin\theta)}{5} + C$

(d) $\frac{\cos^2 x}{2} - \ln|\cos x| + C$

(e) $\frac{\sec^3 x}{3} + C$

(f) $\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + C$

(g) $\frac{\tan^5 x}{5} + \frac{2\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$

(h) $\frac{\pi}{4} - \frac{2}{3}$

(i) $\frac{1}{2}\left[-\frac{\cos 3x}{3} - \frac{\cos 13x}{13}\right] + C$

(j) $\frac{1}{2}\ln|\text{cosec}x + \cot x| + \frac{1}{2}\ln|\sec x + \tan x| + C$

(k) $x\tan x - \ln|\sec x| - \frac{x^2}{2} + C$

3. (a) $\frac{-\sqrt{4-x^2}}{4x} + C$

(b) $\frac{(\sqrt{4+x^2})^3}{3} + 4\sqrt{1+x^2} + C$

(c) $\sqrt{x^2-4} - 2\sec^{-1}\frac{x}{2} + C$

(d) $\frac{1}{2}\left(\frac{\pi}{2} + \frac{\sqrt{13}}{4} - \frac{1}{2}\right)$

(e) $\frac{\sqrt{t^2+2}}{15}(3t^4 - 8t^2 + 32) + C$

(f) $\frac{9\pi}{500}$

(g) $\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{x-2}{2}\sqrt{5+4x-x^2} + C$

(h) $\sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\sqrt{x^2+x+1} - \left(x+\frac{1}{2}\right)\right| + C$

4. (a) $\frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$

(b) $\ln\frac{9}{4}$

(c) $a\ln|x-b| + C$

(d) $10\ln|x-3| - 9\ln|x-2| + \frac{5}{x-2} + C$

(e) $\frac{3}{2}\ln|2x+1| - \ln|x-2| - \frac{2}{x-2} + C$

(f) $\ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$

(g) $\frac{1}{4}\ln\frac{8}{3}$

5.6 Improper Integral

Definition (Improper Integral)

The definite integral $\int_a^b f(x)\,dx$, it is assumed that either

(i) a or b or both are infinite or

(ii) $f(x)$ becomes infinite at some interior point of the interval $[a, b]$.

The integral of these types are called **improper integral.**

Type I: Improper Integral

The integrals

(i) $\int_a^{\infty} f(x)\,dx = \lim_{h\to\infty} \int_a^{h} f(x)\,dx$ (ii) $\int_{-\infty}^{b} f(x)\,dx = \lim_{h\to-\infty} \int_h^{b} f(x)\,dx$

(iii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$ where c is any real number are called **improper** integral of type I. If limit exist, we say that improper integral converges and that limit is the value of the improper integral. If limit fail to exist then the improper **integral** diverges.

Useful Formula

(i) $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}x$ (ii) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left|\frac{x-a}{x+a}\right|$

(iii) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left|\frac{a+x}{a-x}\right|$ (iv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

(v) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln \left|x + \sqrt{x^2 + a^2}\right|$ (vi) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left|x + \sqrt{x^2 - a^2}\right|$

Example 1: Determine whether the integral $\int_1^{\infty} \frac{1}{x}\,dx$ is convergent or divergent.

Solution.

We have $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{h\to\infty} \int_1^{h} \frac{1}{x}\,dx$

$$= \lim_{h\to\infty} [\ln|x|]_1^h$$

$$= \lim_{h\to\infty} [\ln h - \ln 1]$$

$$= \lim_{h\to\infty} \ln h = \infty$$

Since the limit does not exist as a finite number so it divergent.

Example 2: Evaluate $\int_{-\infty}^{0} x\, e^x \, dx$

Solution. We have

$$\int_{-\infty}^{0} x\, e^x\, dx = \lim_{t\to -\infty} \int_{t}^{0} x\, e^x\, dx$$

$$= \lim_{t\to -\infty} \left[[xe^x]_t^0 - \int_t^0 e^x\, dx \right]$$

$$= \lim_{t\to -\infty} (-t\, e^t - 1 + e^t) \quad \left[\because \lim_{t\to -\infty} t\, e^t = \lim_{t\to -\infty} \frac{t}{e^{-t}} = \lim_{t\to -\infty} \frac{1}{-e^{-t}} = 0 \text{ using L-Hospital Rule} \right]$$

$$= -0 - 1 + 0$$

$$= -1$$

Example 3: Evaluate, $\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1}v)}$.

Solution: Here,

$$\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1}v)} = \lim_{h\to\infty} \int_0^{h} \frac{dv}{(1+v^2)(1+\tan^{-1}v)} \quad \text{....... (1)}$$

Put, $1 + \tan^{-1}v = x$ then $\frac{d(1+\tan^{-1}v)}{dv} = \frac{dx}{dv} \Rightarrow \frac{1}{1+v^2} = \frac{dx}{dv}$.

$$\Rightarrow \frac{1}{1+v^2} dv = dx.$$

Then,

$$\int \frac{dv}{(1+v^2)(1+\tan^{-1}v)} = \int \frac{dx}{x} = \ln(x) = \ln(1+\tan^{-1}v).$$

Therefore, (1) gives

$$\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1}v)} = \lim_{h\to\infty} \int_0^{h} \frac{dv}{(1+v^2)(1+\tan^{-1}v)}$$

$$= \lim_{h\to\infty} [\ln(1+\tan^{-1}v)]_0^h$$

$$= \lim_{h\to\infty} [\ln(1+\tan^{-1} h) - \ln 1]$$

$$= \lim_{h\to\infty} [\ln(1+\tan^{-1}h)]$$

$$= \ln(1+\tan^{-1}(\infty))$$

$$= \ln(1+\pi/2).$$

Example 4: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Solution: Here,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{1} \frac{dx}{1+x^2} + \int_{1}^{\infty} \frac{dx}{1+x^2} = I_1 + I_2 \qquad \ldots (1)$$

Since,

$$I_1 = \int_{-\infty}^{1} \frac{dx}{1+x^2} = \lim_{h\to -\infty} \int_{h}^{1} \frac{dx}{1+x^2}$$

$$= \lim_{h\to -\infty} [\tan^{-1} x]_h^1$$

$$= \lim_{h\to -\infty} [\tan^{-1}1 - \tan^{-1}h]$$

$$= \frac{\pi}{4} - (\tan^{-1}(-\infty)) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{4} + \frac{\pi}{2}.$$

Again,

$$I_2 = \int_{1}^{\infty} \frac{dx}{1+x^2} = \lim_{h\to\infty} \int_{1}^{h} \frac{dx}{1+x^2}$$

$$= \lim_{h\to\infty} [\tan^{-1}x]_1^h$$

$$= \lim_{h\to\infty} (\tan^{-1}h - \tan^{-1}1)$$

$$= \tan^{-1}\infty - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4}$$

Thus, From (1), $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = I_1 + I_2 = \frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} = \pi.$

Example 5: For what value of p is the integral $\int_{1}^{\infty} \frac{1}{x^p} dx$ convergent?

Solution. If p = 1, then the integral is divergent. So let $p \neq 1$ then

$$\int_{1}^{\infty} \frac{1}{x^p} dx = \lim_{t\to\infty} \int_{1}^{t} x^{-p}\, dx = \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right]$$

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If $p > 1$ then $p - 1 > 0$ so $t^{p-1} \to \infty$ as $t \to \infty$. So, $\frac{1}{t^{p-1}} \to 0$.

$$\therefore \quad \int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{p-1} \text{ if } p > 1$$

If $p < 1$ then $p - 1 < 0$. So

$$\frac{1}{t^{p-1}} \to \infty \text{ as } t \to \infty$$

$\therefore$ The above integral is convergent if $p > 1$.

Example 6: Evaluate $\int_{-1}^{\infty} \frac{d\theta}{\theta^2 + 5\theta + 6}$.

Solution: Here,

$$\int_{-1}^{\infty} \frac{d\theta}{\theta^2 + 5\theta + 6} = \lim_{h\to\infty} \int_{-1}^{h} \frac{d\theta}{\theta^2 + 5\theta + 6} \qquad \text{........ (1)}$$

Since,

$$\int \frac{d\theta}{\theta^2 + 5\theta + 6} = \int \frac{d\theta}{\theta^2 + 3\theta + 2\theta + 6}$$

$$= \int \frac{d\theta}{\theta(\theta + 3) + 2(\theta + 3)}$$

$$= \int \frac{d\theta}{(\theta + 2) + (\theta + 3)}$$

$$= \int \left[\frac{1}{\theta + 2} - \frac{1}{\theta + 3}\right] d\theta$$

$$= ln(\theta + 2) - ln(\theta + 3)$$

$$= ln\left(\frac{\theta + 2}{\theta + 3}\right)$$

Now from (1)

$$\int_{-1}^{\infty} \frac{d\theta}{\theta^2 + 5\theta + 6} = \lim_{h\to\infty} \int_{-1}^{h} \frac{d\theta}{\theta^2 + 5\theta + 6}$$

$$= \lim_{h\to\infty} \left[ln\left(\frac{\theta + 2}{\theta + 3}\right)\right]_{-1}^{h}$$

$$= \lim_{h\to\infty} \left[ln\left(\frac{h + 2}{h + 3}\right) - ln\left(\frac{1}{2}\right)\right]$$

$$= \lim_{h\to\infty} \left[ln\frac{\left(1 + \frac{2}{h}\right)}{\left(1 + \frac{3}{h}\right)} - ln\left(\frac{1}{2}\right)\right]$$

$$= ln\,1 - ln\frac{1}{2} = -ln\frac{1}{2} = ln\,2.$$

Example 7: Evaluate $\int_{-\infty}^{1} \frac{2x}{(x^2+1)^2}\,dx$.

Solution: Here,

$$\int_{-\infty}^{1} \frac{2x}{(x^2+1)^2}\,dx = \lim_{h\to-\infty} \int_{h}^{1} \frac{2x}{(x^2+1)^2}\,dx \quad \text{..... (1)}$$

Take,

$$\int \frac{2x}{(x^2+1)^2}\,dx.$$

Put $x^2 + 1 = y$. Then, $2x\,dx = dy$. So that,

$$\int \frac{2x}{(x^2+1)^2}\,dx = \int \frac{dy}{y^2} = -\frac{1}{y} = -\frac{1}{(x^2+1)}$$

Thus, from (i)

$$\int_{-\infty}^{1} \frac{2x}{(x^2+1)^2}\,dx = \lim_{h\to-\infty} \int_{h}^{1} \frac{2x}{(x^2+1)^2}\,dx$$

$$= \lim_{h\to-\infty} \left[-\frac{1}{(x^2+1)}\right]_h^1$$

$$= \lim_{h\to-\infty} \left[-\frac{1}{2} + \frac{1}{h^2+1}\right]$$

$$= -\frac{1}{2}.$$

Type II: Improper Integral

The integrals

(i) $\int_{a}^{b} f(x)\,dx = \lim_{h\to a^+} \int_{h}^{b} f(x)\,dx$; if $f(x) \to \infty$ as $x = a$.

(ii) $\int_{a}^{b} f(x)\,dx = \lim_{h\to b^-} \int_{a}^{h} f(x)\,dx$; if $f(x) \to \infty$ as $x = b$.

(iii) $\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$; if $f(x) \to \infty$ as $x = c$, where $a < c < b$,

are called improper integral of type II.

If the limit is finite or exist, then such improper integral converges and that the limit is the value of the improper integral. If the limit doesn't exist, then the improper integral diverges.

Example 8: Evaluate: $\int_{0}^{4} \frac{dx}{\sqrt{4-x}}$

Solution: Since, at $x = 4, f(x) = \frac{1}{\sqrt{4-x}} = \infty$, so it is improper integral. So

$$\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{h\to 4^-} \int_0^4 \frac{dx}{(4-x)^{1/2}}$$

$$= \lim_{h\to 4^-} \left[\frac{(4-x)^{-1/2+1}}{-1(-1/2+1)}\right]_0^h$$

$$= \lim_{h\to 4^-} \left[\frac{(4-x)^{1/2}}{-1/2}\right]_0^h$$

$$= \lim_{h\to 4^-} [-2\sqrt{4-x}]_0^h$$

$$= \lim_{h\to 4^-} (-2\sqrt{4-h} + 2\sqrt{4})]$$

$$= -2\times 0 + 4$$

$$= 4$$

Example 9: Determine whether $\int_0^{\pi/2} \sec x \, dx$ converges or diverges.

Solution. We have $\int_0^{\pi/2} \sec x \, dx = \lim_{t\to\frac{\pi}{2}^-} \int_0^t \sec x \, dx$

$$= \lim_{t\to\frac{\pi}{2}^-} [ln\ |\sec x + \tan x|]_0^t$$

$$= \lim_{t\to\frac{\pi}{2}^-} [ln\ |\sec t + \tan t| - ln\ 1]$$

$$= \infty$$

∴ *It is divergent.*

Example 10: Evaluate $\int_0^3 \frac{dx}{(x-1)^{2/3}}$

Solution: Since, $0 < 1 < 3$ at at $x = 1, \frac{1}{(x-1)^{2/3}} = \infty$. So, it is improper integral.

Thus,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} = I_1 + I_2 \quad \text{...... (1)}$$

For I_1;

$$I_1 = \int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{h\to 1^-} \int_0^h \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{h\to 1^-} \left[\frac{(x-1)^{-2/3+1}}{-2/3+1}\right]_0^h$$

$$= \lim_{h\to 1^-} [3(x-1)^{1/3}]_0^h$$

$$= \lim_{h\to 1^-} [3(h-1)^{1/3} - 3(-1)^{1/3}].$$

$$= 0 - 3 \times (-1)$$

$$= 3.$$

For I_2:

$$I_2 = \int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{h\to 1^+} \int_h^3 (x-1)^{-2/3}\, dx$$

$$= \lim_{h\to 1^+} [3(x-1)^{1/3}]_h^3$$

$$= \lim_{h\to 1^+} [3(2)^{1/3} - 3(h-1)^{1/3}]$$

$$= 3(2)^{1/3}$$

Therefore, (1) becomes

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = I_1 + I_2 = 3 + 3(2)^{1/3} = 3[1 + 2^{1/3}].$$

Example 11: Evaluate $\displaystyle\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\, dx$

Since, at $x = 0$, $\dfrac{x+1}{\sqrt{x^2+2x}} = \infty$. So, it is improper integral.

Now,

$$\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\, dx = \lim_{h\to 0^+} \int_h^1 \frac{x+1}{\sqrt{x^2+2x}}\, dx \qquad \text{........ (1)}$$

Take $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x}}\, dx$

Put $x^2 + 2x = y$, so $2(x+1)\, dx = dy \Rightarrow (x+1)\, dx = \dfrac{dy}{2}$. Then,

$$\int \frac{x+1}{\sqrt{x^2+x}}\, dx = \frac{1}{2}\int \frac{dy}{\sqrt{y}} = \frac{1}{2}\int y^{-1/2}\, dy = \sqrt{y} = \sqrt{x^2+2x}$$

Thus (1) becomes

$$\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\, dx = \lim_{h\to 0^+} \left[\sqrt{x^2+2x}\right]_h^1$$

$$= \lim_{h\to 0^+} [\sqrt{3} - \sqrt{h^2+2h}]$$

$$= \sqrt{3}$$

Hence, $\displaystyle\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\, dx = \sqrt{3}$.

Comparison Theorem: Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$ then,

a. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is also convergent.

b. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is also divergent.

Example 12: Show that $\int_0^\infty e^{-x^2}\,dx$ is convergent.

Solution. We have, $\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$

We know, $x \geq 1$, $x^2 \geq x$, so $-x^2 \leq -x$

$\therefore \quad e^{-x^2} \leq e^{-x}$

Since, $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx = \lim_{t\to\infty} (e^{-1} - e^{-t}) = e^{-1}$

Also since,

$\int_0^1 e^{-x^2}\,dx$ is definite integral.

Since $\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx$ and $\int_1^\infty e^{-x}\,dx$ is convergent. So, by comparison test

$\therefore \quad \int_1^\infty e^{-x^2}\,dx$ is convergent

Hence, $\int_0^\infty e^{-x^2}\,dx$ is convergent.

Example 13: Show that $\int_1^\infty \frac{1 + e^{-x}}{x}\,dx$ is divergent.

Solution. Since, $\frac{1 + e^{-x}}{x} > \frac{1}{x}$ for $x \in [1, \infty)$ and we know that $\int_1^\infty \frac{1}{x}\,dx$ is divergent.

$\therefore \quad \int_1^\infty \frac{1 + e^{-x}}{x}\,dx$ is also divergent by comparison theorem.

Exercise 5.6

1. Determine whether each integral is convergent or divergent. Evaluate for convergent.

(a) $\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\,dx$

(b) $\int_{-\infty}^{0} \frac{1}{3-4x}\,dx$

(c) $\int_2^{\infty} e^{-5p}\,dp$

(d) $\int_0^{\infty} \frac{x^2}{\sqrt{1+x^3}}\,dx$

(e) $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

(f) $\int_0^{\infty} \sin^2\alpha\,d\alpha$

(g) $\int_2^{\infty} \frac{dv}{v^2+2v-3}$

(h) $\int_1^{\infty} \frac{\ln x}{x}\,dx$

(i) $\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx$

(j) $\int_0^{\infty} \frac{x \arctan x}{(1+x^2)^2}\,dx$

(k) $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

(*l*) $\int_{-2}^{3} \frac{1}{x^4}\,dx$

(m) $\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$

(n) $\int_0^{9} \frac{1}{\sqrt[3]{x-1}}\,dx$

(o) $\int_{\pi/2}^{\pi} \operatorname{cosec} x\,dx$

(p) $\int_{-1}^{0} \frac{e^{1/x}}{x^3}\,dx$

2. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

(a) $\int_0^{\infty} \frac{x}{x^3+1}\,dx$

(b) $\int_1^{\infty} \frac{2+e^{-x}}{x}\,dx$

(c) $\int_1^{\infty} \frac{x+1}{\sqrt{x^4-x}}\,dx$

(d) $\int_0^{\infty} \frac{\tan^{-1}x}{2+e^x}\,dx$

(e) $\int_0^{1} \frac{\sec^2 x}{x\sqrt{x}}\,dx$

(f) $\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}}\,dx$

Answers

1. (a) converges 2 (b) diverges (c) converges $\frac{e^{-10}}{5}$ (d) diverges (e) converges 0 (f) diverges (g) converges $\frac{\ln 5}{4}$ (h) diverges (i) converges $\frac{\pi}{9}$ (j) converges $\frac{\pi}{8}$ (k) converges $\frac{32}{3}$ (*l*) diverges (m) converges $\frac{\pi}{2}$ (n) converges $\frac{9}{2}$ (o) diverges (p) converges $\frac{-2}{e}$

2. (a) converges (b) diverges (c) converges (d) converges (e) diverges (f) converges

Chapter 6

Application of Antiderivatives

In this unit we explain some of the applications of definite integral by using it to find area bounded between curves and volumes of solids.

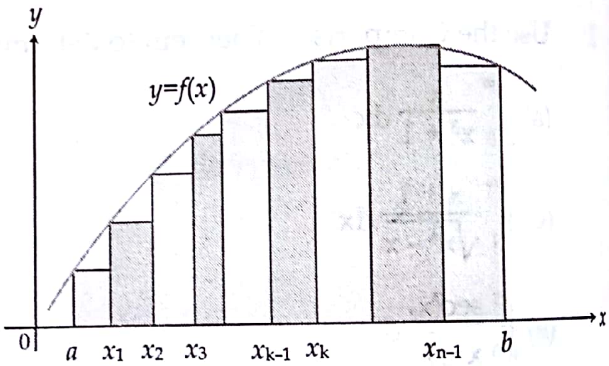
6.1 Area between Two Curves

Riemann Sum

Let function $f(x)$ is continuous in $[a, b]$. We divided the interval $[a, b]$ into n sub-interval and x_k be the a point on each $[x_{k-1}, x_k]$, where k = 1, 2, ..., n. The sum $\sum_{k=1}^{n} f(x_k)\,\Delta x_k$ is called the Riemann sum i.e.

$$R = \sum_{k=1}^{n} f(x_k)\,\Delta x_k$$

is Riemann sum.



Definite integral as a Riemann sum

Let $f(x)$ be continuous function on $[a, b]$. The definite integral of $f(x)$ from a to b is a limit of Riemann sum i.e.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k$$

Area between Two Curves

If f and g are continuous functions with $f(x) \geq g(x)$ throughout $[a, b]$ then area of region between curve

$y = f(x)$ and $y = g(x)$ from a to b is

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Example 1 : Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution: We first find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$. Thus $2x(x - 1) = 0$, so $x = 0$ or 1. The points of intersection are $(0, 0)$ and $(1, 1)$.

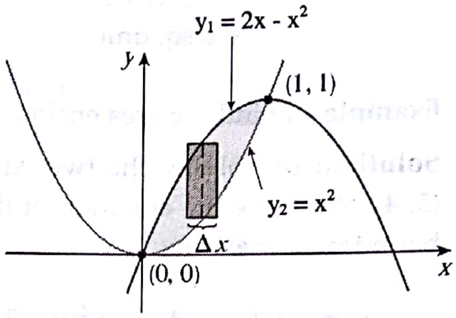
We see from figure that the top and bottom boundaries are

$$y_1 = 2x - x^2 \quad \text{and} \quad y_2 = x^2$$

The area of a typical rectangle is

$$(y_1 - y_2)\,\Delta x = (2x - x^2 - x^2)\,\Delta x$$

and the region lies between $x = 0$ and $x = 1$. So the total area is



$$A = \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

Example 2 : Find the area of the region in the first quadrant that is bounded above $y = \sqrt{x}$ and below by the x–axis and line $y = x - 2$.

Solution: Solving the given curves $y = x - 2$ and $y = \sqrt{x}$ we observe as

$$\sqrt{x} = x - 2$$

$$\Rightarrow \quad x = x^2 - 4x + 4$$

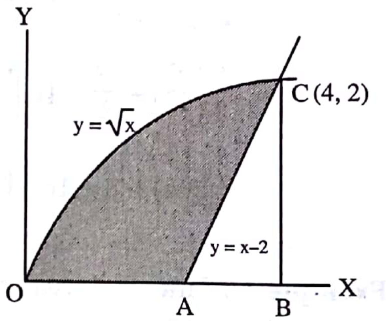
$$\Rightarrow \quad x^2 - 5x + 4 = 0$$

$$\Rightarrow \quad x^2 - 4x - x + 4 = 0$$

$$\Rightarrow \quad x(x - 4) - 1(x - 4) = 0$$

$$\Rightarrow \quad (x - 1)(x - 4) = 0$$

$$\Rightarrow \quad x = 1, 4.$$



Here, $x = 4$ satisfies $\sqrt{x} = x - 2$ but $x = 1$ not satisfies it. Thus point of interaction of line and curve is $(4, 2)$.

Now,

Required area = (Area between curve $y = \sqrt{x}$, $0 \le x \le 4$) – Area of triangle ABC

$$= \int_0^4 \sqrt{x}\,dx - \frac{1}{2}\times 2\times 2$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^4 - 2 = \frac{2}{3}(8) - 0 - 2 = \frac{10}{3} \text{ sq. unit}$$

Example 3 : Show that the area under the arch of the curve $y = \sin x$ is 2.

Solution: Required area is

$$A = \int_0^{\pi} y\,dx$$

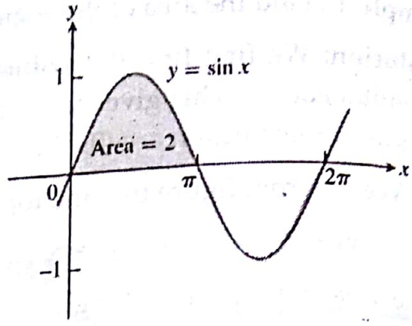
$$= \int_0^{\pi} \sin x \, dx$$

$$= [-\cos x]_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$= -(-1) + 1$$

$$= 2 \text{ sq. unit}$$



Example 4 : Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution: By solving the two equations we find that the points of intersection are (-1, -2) and (5, 4). We solve the equation of the parabola for x and notice from Figure that the left and right boundary curves are

$$x_1 = y + 1 \quad \text{and} \quad x_2 = \frac{1}{2}y^2 - 3 .$$

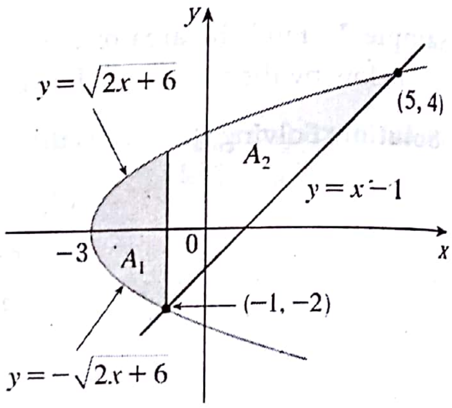
We must integrate between the appropriate y-values, $y = -2$ and $y = 4$. Thus

$$A = \int_{-2}^{4} (x_1 - x_2)\, dy = \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3\right)\right] dy$$

$$= \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4\right) dy$$

$$= -\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y \Bigg]_{-2}^{4}$$

$$= -\frac{1}{6}(64) + 8 + 16 - \left(\frac{4}{3} + 2 - 8\right) = 18$$



Example 5 : Find the area of the region between x axis and the graph of $f(x) = x^3 - x^2 - 2x$, for $-1 \le x \le 2$.

Solution: For zeros of $f(x)$, let

$$f(x) = 0$$

$$\Rightarrow x^3 - x^2 - 2x = 0$$

$$\Rightarrow x(x^2 - x - 2) = 0$$

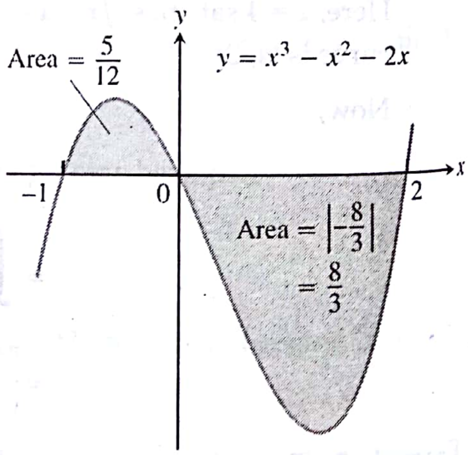
Either $x = 0$ or, $x^2 - x - 2 = 0$

$$\Rightarrow x^2 - 2x + x - 2 = 0$$

$$\Rightarrow x(x-2) + 1(x-2) = 0$$

$$\Rightarrow (x+1)(x-2) = 0$$

$$\Rightarrow x = -1, 2.$$



Hence, zeros of $f(x)$ are $x = 0, -1, 2$.

Zero's sub-divided the interval [–1, 2] into two sub-intervals [–1, 0] and [0, 2].

Now,

$$\int_{-1}^{0} f(x)\,dx = \int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} \left[0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right)\right] = \frac{5}{12}$$

And,

$$\int_{0}^{2} f(x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

Thus, the total area is

$$A = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{5}{12} + \frac{8}{3} = \frac{37}{12} \text{ sq. unit}$$

Theorem - Substitution in Definite Integrals

If g' is continuous on the interval [a, b] and f is continuous on the range of g(x) = u, then

$$\int_{a}^{b} f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 6 : Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

Solution: Let's transform the integral and evaluate the transformed integral with the transformation $u = x^3 + 1 \Rightarrow u\,du = 3x^2\,dx$, when x = -1, u = 0 and when x = 1, u = 2, hence

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

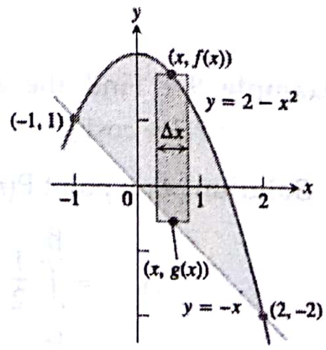
$$= \int_{0}^{2} \sqrt{u}\,du = \frac{2}{3}\left[u^{3/2}\right]_{0}^{2} = \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

xample 7: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

olution: First we sketch the two curves in Figure. The limits of integration are found by solving $= 2 - x^2$ and $y = -x$ simultaneously for x.

$$2 - x^2 = -x$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1, x = 2$$

The region runs from x = - 1 to x = 2. The limits of integration are a = - 1, b = 2. The area between the curves is



$$A = \int_{a}^{b} [f(x) - g(x)]\,dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx$$

$$= \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

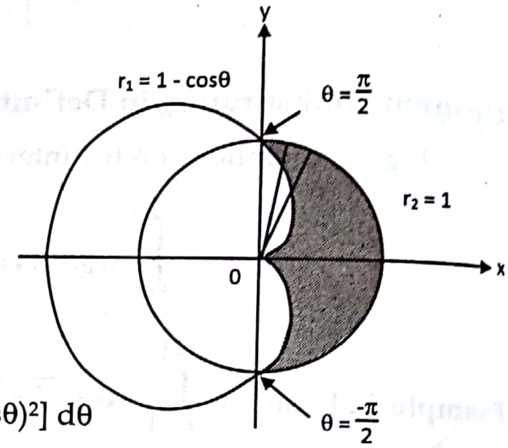
Area of the Region in polar form:

Theorem: Area of region $0 \le r_1(\theta) \le r(\theta) \le r_2(\theta)$, $\alpha \le \theta \le \beta$,

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta.$$

Example 8 : Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Solution: To determine the boundaries and find the limit of integration, we sketch the graph as shown in figure.



The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$ and θ runs from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Thus area of the region is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(r_2^2 - r_1^2)\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2}[1 - (1 - \cos\theta)^2]\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}[1 - 1 + 2\cos\theta - \cos^2\theta]\, d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(2\cos\theta - \frac{1}{2} - \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left[2\sin\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{4}\left(\sin 2\frac{\pi}{2}\right) - \left\{2\sin\left(-\frac{\pi}{2}\right) + \frac{\pi}{4} - \frac{1}{4}\left(\sin\left(2\frac{-\pi}{2}\right)\right)\right\}\right]$$

$$= \frac{1}{2}\left[2 - \frac{\pi}{4} - 0 + 2 - \frac{\pi}{4} - 0\right]$$

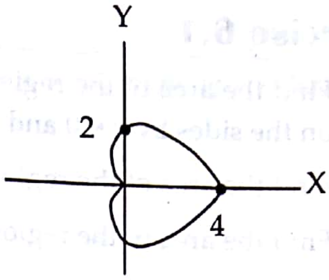
$$= \frac{1}{2}\left(-\frac{2\pi}{4} + 4\right) \quad = -\frac{\pi}{4} + 2 \quad = 2 - \frac{\pi}{4}.$$

Example 9 : Find the area of the region that lies in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution: Any point $P(r, \theta)$ determines the curve as θ runs from 0 to 2π. Then the area is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2\, d\theta = \int_{0}^{2\pi} \frac{1}{2}[2(1 + \cos\theta)]^2\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}4(1 + 2\cos\theta + \cos^2\theta)\right] d\theta$$



$$= 2\int_0^{2\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 2\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

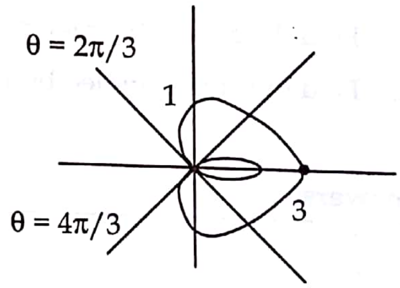
$$= 2\left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= 2\left[\frac{3}{2}2\pi + 2\sin 2\pi + \frac{\sin 4\pi}{4} - 0 + 2\sin 0 + \frac{\sin 0}{4}\right] = 2(3\pi) = 6\pi.$$

Example 10 : Find the area inside the smaller loop of the limacine $r = 2\cos\theta + 1$.

Solution: The smaller loop is traced by the point P(r, θ) as θ increases from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$. Since the curve is symmetric about the x-axis, then the area of the shaded half of the inner loop by integrating from $\frac{2\pi}{3}$ to π. So

$$A = \int_\alpha^\beta \frac{1}{2}r^2\, d\theta = 2\int_{2\pi/3}^{\pi} \frac{1}{2}(2\cos\theta + 1)^2\, d\theta$$



$$= 2\int_{2\pi/3}^{\pi} \frac{1}{2}(4\cos^2\theta + 4\cos\theta + 1)\, d\theta$$

$$= \int_{2\pi/3}^{\pi} \left[4\left(\frac{1+\cos 2\theta}{2}\right) + 4\cos\theta + 1\right] d\theta$$

$$= \int_{2\pi/3}^{\pi} [2 + 2\cos 2\theta + 4\cos\theta + 1]\, d\theta$$

$$= \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta$$

$$= \left[3\theta + \frac{2\sin 2\theta}{2} + 4\sin\theta\right]_{2\pi/3}^{\pi}$$

$$= 3\pi + \sin 2\pi + 4\sin\pi - \left(3\frac{2\pi}{3} + \sin 2\frac{2\pi}{3} + 4\sin\frac{2\pi}{3}\right)$$

$$= 3\pi + 0 + 0 - 2\pi + \frac{\sqrt{3}}{2} - \frac{4\sqrt{3}}{2}$$

$$= \pi - \frac{3\sqrt{3}}{2}.$$

Exercise 6.1

1. Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.
2. Find the area of the region enclosed by $x + y^2 = 0$ and $x + 3y^2 = 2$.
3. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.
4. Find the area between two curves $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$.
5. Find the area between two curves $x = \tan^2 y$ and $x = -\tan^2 y$, $\frac{-\pi}{4} \le y \le \frac{\pi}{4}$.
6. Find the area between two curves $y = \sec^2 x$ and $y = \tan^2 x$, $\frac{-\pi}{4} \le x \le \frac{\pi}{4}$.
7. Find the area of region between the curve and x-axis

   (i) $f(x) = -x^2 - 2x$, $[-3, 2]$.   (ii) $f(x) = x^2 - 6x + 8$, $[0, 3]$.

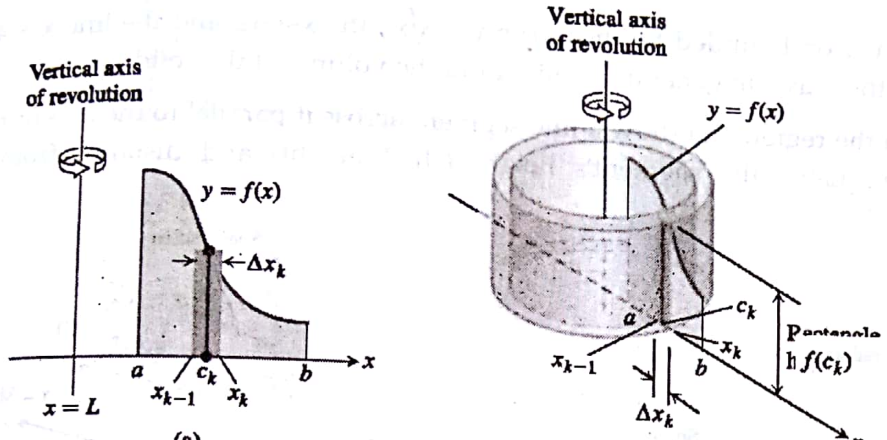
   (iii) $y = x^3 - 4x$, $-2 \le x \le 2$.
8. Find the area of region enclosed by the parabola $y = 2 - x^2$ and line $y = -x$.
9. Find the area of the region enclosed by parabola $x = y^2$ and line $x = y + 2$ in first quadrant.
10. Find the area of the region enclosed by parabola $y^2 - 4x = 4$ and line $4x - y = 16$.
11. Find the area of the region bounded by curve $x = 2y^2$, $x = 0$ and $y = 3$.
12. Find the area bounded by x-axis and curve $y = 4 - x^2$.

Answers:

1. $e - 1.5$
2. $\frac{8}{3}$ sq. unit
3. $2\sqrt{2} - 2$
4. $\frac{1}{\sqrt{2}}$
5. $4 - \pi$
6. $\frac{\pi}{2}$
7. (i) $\frac{28}{3}$ (ii) $\frac{22}{3}$ (iii) 8
8. $\frac{9}{2}$
9. $\frac{10}{3}$
10. $\frac{243}{8}$
11. 18
12. $\frac{32}{3}$

6.2 Volumes of Cylindrical Shells

The volume of the solid generated by revolving the region between x-axis and the graph of $y = f(x) \geq 0$; $a \leq x \leq b$ about a vertical line $x = L \leq a$ is given by



(a) (b)

Here, $\Delta V_k = 2\pi\,(c_k - L)\, f(c_k)\, \Delta x_k$

Using Riemann Sum, $$V = \int_a^b 2\pi\,(x - L)\, f(x)\, dx$$

about L = 0, $$V = \int_a^b 2\pi x\, f(x)\, dx$$

for simplicity,

$$V = \int_a^b 2\pi\,(\text{radius of shell})\,(\text{height of shell})\, dx$$

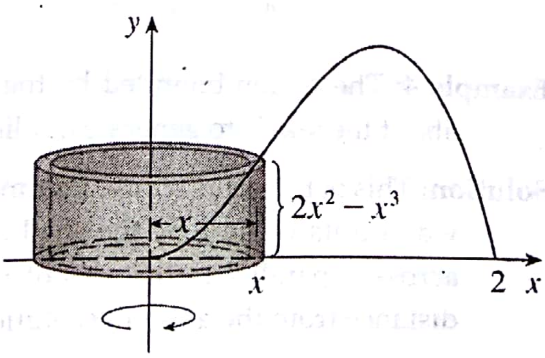
R, $$V = \int_a^b (\text{circumference})\,(\text{height})\,(\text{thickness})$$

ample 1: Find the volume of the solid obtained by rotating about the y-axis the region bounded $y = 2x^2 - x^3$ and $y = 0$.

ution: From the sketch in Figure, we see that a typical shell has radius x, circumference $2\pi x$, l height $f(x) = 2x^2 - x^3$. So by the shell method, the volume is

$$V = \int_0^2 (2\pi x)\,(2x^2 - x^3)\, dx$$

$$= 2\pi \int_0^2 (2x^3 - x^4)\, dx$$

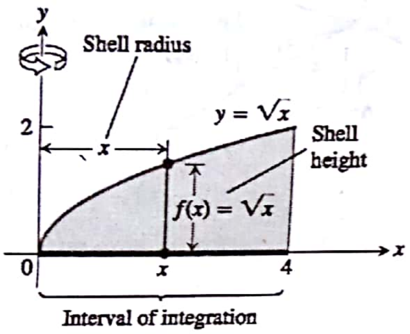


$$= 2\pi\left[\frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^2 = 2\pi\left(8 - \frac{32}{5}\right)$$

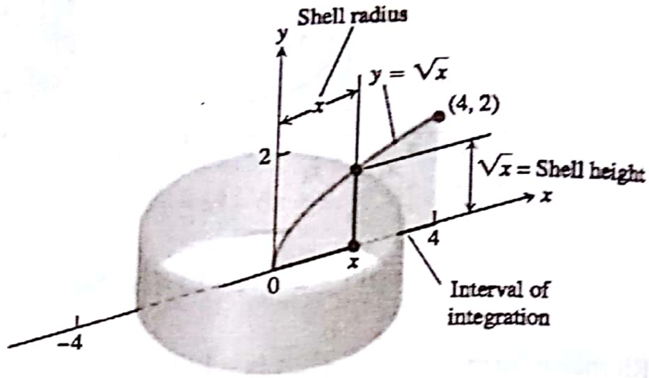
$$= \frac{16}{5}\pi$$

Example 2: The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolve about the y-axis to generate a solid. Find the volume of the solid.

Solution: Sketch the region and draw a line segment across it parallel to the axis of revolution (i Figure). Label the segment's height (shell height) and distance from the axis o revolution (shell radius).

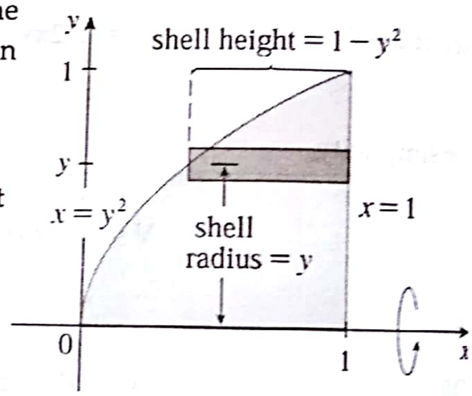


(a)



(b)

Example 3: Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



Solution: This problem was solved using disks in Example To use shells we relabel the curve $y = \sqrt{x}$ (in the figure in that example) as $x = y^2$ in Figure. For rotation about the x-axis we see that a typical shell has radius y, circumference $2\pi y$, and height $1 - y^2$. So the volume is

$$V = \int_0^1 (2\pi y)(1 - y^2)\, dy$$

$$= 2\pi \int_0^1 (y - y^3)\, dy$$

$$= 2\pi\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}$$

Example 4: The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid by the shell method.

Solution: This is the solid whose volume was found by the disk method in Example below. Now we find its volume by the shell method. First, sketch the region and draw a line segment across it parallel to the axis of revolution. Label the segment's length (shell height) and distance from the axis of revolution (shell radius).

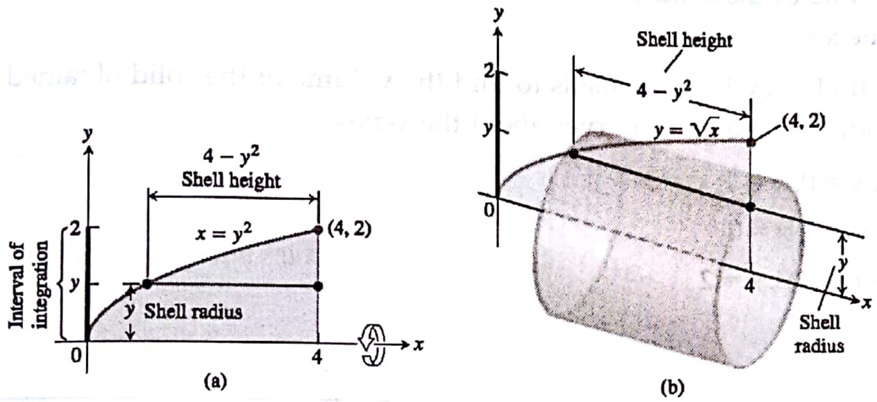
In this section, the shell thickness variable is y, so the limits of integration for the shell formula method are a = 0 and b = 2 (along the y-axis in figure). The volume of the solid is

$$V = \int_a^b 2\pi \text{ (shell radius) (shell height) } dy$$

$$= \int_0^2 2\pi (y)(4 - y^2)\, dy$$

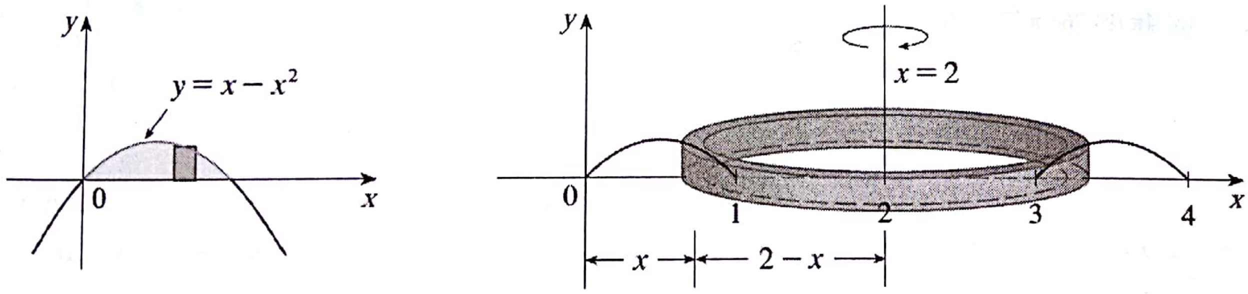
$$= 2\pi \int_0^2 (4y - y^3)\, dy$$

$$= 2\pi \left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi$$



Example 5: Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Solution: Figure shows the region and a cylindrical shell formed by rotation about the line $x = 2$. It has radius $2 - x$, circumference $2\pi(2 - x)$, and height $x - x^2$.



The volume of the given solid is

$$V = \int_0^1 2\pi (2 - x)(x - x^2)\, dx$$

$$= 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\, dx$$

$$= 2\pi \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$

Exercise 6.2

1. The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid. Find the volume of solid.
2. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.
3. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.
4. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.
5. Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.
6. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.
7. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

   a. $xy = 1, x = 0, y = 1, y = 3$

   b. $y = x^3, y = 8, x = 0$

   c. $x = 1 + (y - 2)^2, x = 2$

Answers:

1. $\frac{45\pi}{2}$
2. $\frac{\pi}{6}$
3. $\frac{16}{5}\pi$
4. $\frac{\pi}{6}$
5. $\frac{\pi}{2}$
6. $\frac{\pi}{2}$
7. (a) 4π (b) $768\,\pi/7$ (c) $16\,\pi/3$

6.3 Approximate Integration

A definite integral is sometimes impossible to find its exact value or it is difficult to find the value even possible because of its indefinite integral is not integrable or it is difficult to integrate. In these cases we need to find approximate values of definite integral. In such cases, we use Riemann sum as an approximation to the integral and divide [a, b] into n subintervals of equal length given by

$\Delta x = \frac{b-a}{n}$ *to get*

$$\int_b^a f(x)\,dx = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

As a mid point rule, it can be written as,

$$\int_b^a f(x)\,dx \approx \Delta x\left[f(\overline{x_1}) + f(\overline{x_2}) + \dots + f(\overline{x_n})\right]$$

Where, $\Delta x = \frac{b-a}{n}$

and $\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)$, i.e. $\overline{x_i}$ is the mid point of $[x_{i-1}, x_i]$

Trapezoidal Rule

$$\int_b^a f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\,\Delta x$; $x_n = b,\ x_0 = a$

Example 1: Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with n = 5 to approximate the integral $\int_1^2 (1/x)\,dx$.

Solution

a. With n = 5, a = 1, and b = 2, we have $\Delta x = (2-1)/5 = 0.2$, and so the Trapezoidal Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx T_5 = \frac{0.2}{2}\left[f(1) + 2\,f(1.2) + 2\,f(1.4) + 2\,f(1.6) + 2\,f(1.8) + f(2)\right]$$

$$= 0.1\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

$$= 0.695635$$

b. The mid points of the five subintervals are 1.1, 1.3, 1.5, 1.7, and 1.9, so the Midpoint Rule gives

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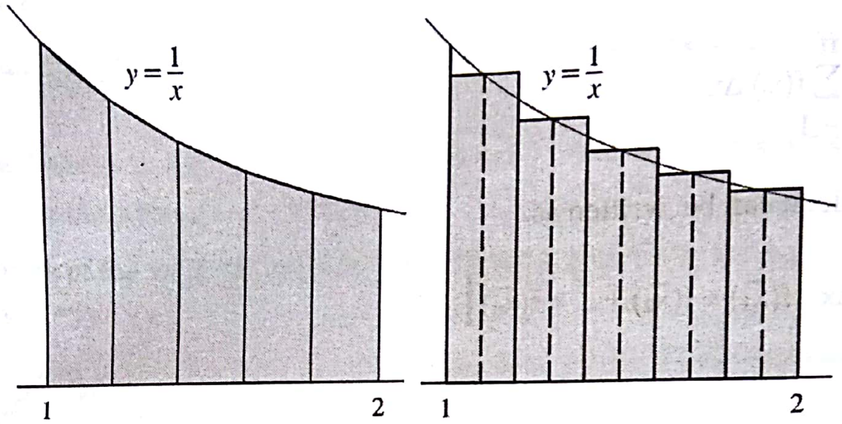
$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x = [f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx 0.691908$$



(a) (b)

Error Bounds

Suppose $|f''(x)| \le K$ for $a \le x \le b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

Let's apply this error estimate to the Trapezoidal Rule approximation in example 1. If $f(x) = 1/x$, then $f'(x) = -1/x^2$ and $f''(x) = 2/x^3$. Since $1 \le x \le 2$, we have $1/x \le 1$, so

$$|f''(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{1^3} = 2$$

Therefore, taking $K = 2$, $a = 1$, $b = 2$, and $n = 5$, we see that

$$|E_T| \le \frac{2(2-1)^3}{12\,(5)^2} = \frac{1}{150} \approx = 0.006667$$

Next we seek, how large should we take n in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int_1^2 (1/x)\,dx$ are accurate to within 0.0001?

Notice that, $|f''(x)| \le 2$ for $1 \le x \le 2$, so we can take $K = 2$, $a = 1$, and $b = 2$; accuracy to within 0.0001 means that the size of the error should be less than 0.0001. Therefore we choose n so that

$$\frac{2(1)^3}{12\,n^2} < 0.0001$$

Solving the inequalivity for n, we get

$$n^2 > \frac{2}{12\,(0.0001)}$$

or $n > \dfrac{1}{\sqrt{0.0006}} \approx 40.8$

Thus, n = 41 will ensure the desired accuracy.

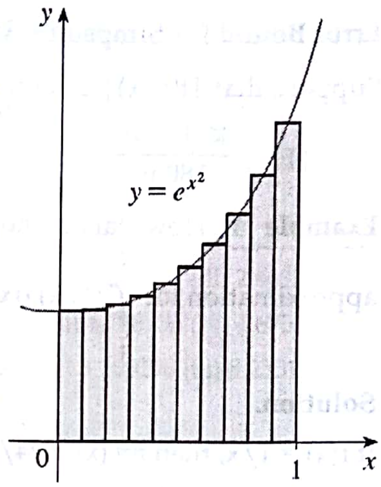
For the same accuracy with the Midpoint Rule we choose n so that

$$\frac{2(1)^3}{24\,n^2} < 0.0001 \text{ and so } n > \frac{1}{\sqrt{0.0012}} \approx 29$$

Example 2:

a. Use the Midpoint Rule with n = 10 to approximate the integral $\int_0^1 e^{x^2}\,dx$.

b. Give an upper bound for the error involved in this approximation.



Solution.

a. Since a = 0, b = 1, and n = 10, the Midpoint Rule gives

$$\int_0^1 e^{x^2}\,dx \approx \Delta x\,[f(0.05) + f(0.15) + \ldots + f(0.85) + f(0.95)]$$

$$= 0.1\,[e^{0.0025} + e^{0.0225} + e^{0.0625} + e^{0.1225} + e^{0.2025} + e^{0.3025} + e^{0.4225} + e^{0.5625} + e^{0.7225} + e^{0.9025}]$$

$$\approx 1.460393$$

Figure illustrates this approximation.

b. Since $f(x) = e^{x^2}$, we have $f'(x)\ 2x\,e^{x^2}$ and $f''(x) = (2 + 4x^2)\,e^{x^2}$. Also, since $0 \le x \le 1$, we have $x^2 \le 1$ and so

$$0 \le f''(x) = (2 + 4x^2)\,e^{x^2} \le 6e$$

Taking K = 6e, a = 0, b = 1, and n = 10, we see that an upper bound for the error is

$$\frac{6e\,(1)^3}{24\,(10)^2} = \frac{e}{400} \approx 0.007$$

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\,[f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + \ldots + 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)]$$

Where n is even and $\Delta x = (b - a)/n$

Example 3: Use Simpson's Rule with n = 10 to approximate $\int_1^2 (1/x)\,dx$.

Solution.

Putting $f(x) = 1/x$, $n = 10$, and $\Delta x = 0.1$ in Simpson's Rule, we obtain

$$\int_1^2 \frac{1}{x}\,dx \approx S_{10}$$

$$= \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \ldots + 2f(1.8) + 4f(1.9) + f(2)]$$

$$= \frac{0.1}{3}\left(\frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{2}{1.6} + \frac{4}{1.7} + \frac{2}{1.8} + \frac{4}{1.9} + \frac{1}{2}\right)$$

$$\approx 0.693150$$

Error Bound for Simpson's Rule

Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_S is the error involved in using Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180\,n^4}$$

Example 4: How large should we take n in order to guarantee that the Simpson's Rule approximation for $\int_1^2 (1/x)\,dx$ is accurate to within 0.0001?

Solution.

If $f(x) = 1/x$, then $f^{(4)}(x) = 24/x^5$. Since $x \geq 1$, we have $1/x \leq 1$ and so

$$|f^{(4)}(x)| = \left|\frac{24}{x^5}\right| \leq 24$$

Therefore we can take K = 24. Thus, for an error less than 0.0001, we should choose n so that

$$\frac{24(1)^5}{180\,n^4} < 0.0001$$

This gives

$$n^4 > \frac{24}{180(0.0001)}$$

or $$n > \frac{1}{\sqrt[4]{0.00075}} \approx 6.04$$

Therefore n = 8 (n must be even) gives the desired accuracy.

Example 5: Use Simpson's Rule with n = 10 to approximate the integral $\int_0^1 e^{x^2}\,dx$.

Solution. If n = 10, then $\Delta x = 0.1$ and Simpson's Rule gives

$$\int_0^1 e^{x^2}\,dx \approx \frac{\Delta x}{3}[f(0) + 4f(0.1) + 2f(0.2) + \ldots + 2f(0.8) + 4f(0.9) + f(1)]$$

$$= \frac{0.1}{3}[e^0 + 4e^{0.01} + 2e^{0.04} + 4e^{0.09} + 2e^{0.16} + 4e^{0.25} + 2e^{0.36} + 4e^{0.49} + 2e^{0.64} + 4e^{0.81} + e^1]$$

$$\approx 1.462681$$

Exercise 6.3

Use (a) the Trapezoidal Rule (b) the Mid-point Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of n. (Round your answers to six decimal places).

a. $\int_1^2 \sqrt{x^3 - 1}\, dx,\ n = 10$

b. $\int_0^2 \frac{e^x}{1 + x^2}\, dx,\ n = 10$

c. $\int_1^4 \sqrt{\ln x}\, dx,\ n = 6$

d. $\int_0^4 e^{\sqrt{t}} \sin t\, dt,\ n = 8$

e. $\int_1^5 \frac{\cos x}{x}\, dx,\ n = 8$

f. $\int_{-1}^1 e^{e^x}\, dx,\ n = 10$

Answers:

(a) (i) 1.506361 (ii) 1.518362 (iii) 1.511519

(b) (i) 2.660833 (ii) 2.664377 (iii) 2.663244

(c) (i) 2.591334 (ii) 2.681046 (iii) 2.631976

(d) (i) 4.513618 (ii) 4.748256 (iii) 4.675111

(e) (i) - 0.495333 (ii) - 0.543321 (iii) - 0.526123

(f) (i) 8.363853 (ii) 8.163298 (iii) 8.235114

6.4 Arc Length

Finding the length of the curve whose graph is continuous defined over an interval I, we use the following formulae with concept of partition on I.

The Length of a Curve

1. Let $y = f(x)$ is continuous first order derivative. Then the total length of curve from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Note if $\frac{dy}{dx}$ at $x = a$ or $x = b$ does not exist, then we observe $\frac{dx}{dy}$. Then the total length of curve from $y = a$ to $y = b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

2. If $x = f(t)$ and $y = g(t)$ be two continuous function of t, $a \le t \le b$, then

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

3. If $r = f(\theta)$ be continuous first order derivative for $a \le \theta \le b$ and if the point $P(r, \theta)$

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example 1: Find the length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \quad 0 \le x \le 1.$$

Solution. We use Eq. (3) with a = 0, b = 1, and

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \qquad x = 1, y \approx 0.89$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2} x^{1/2} = 2\sqrt{2}x^{½}$$

$$\left(\frac{dy}{dx}\right)^2 = (2\sqrt{2}x^{1/2})^2 = 8x$$

The length of the curve over x = 0 to x = 1 is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}dx = \left[\frac{2}{3} \bullet \frac{1}{8} (1 + 8x)^{3/2}\right]_0^1 = \frac{13}{6} \approx 2.17.$$

Example 2: Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8).

Solution. For the top half of the curve we have

$$y = x^{3/2} \qquad \frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

and so the arc length formula gives

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4} x}\, dx$$

If we substitute $u = 1 + \frac{9}{4}x$, then $du = \frac{9}{4}dx$. When x = 1, $u = \frac{13}{4}$; when x = 4, u = 10.

Therefore,

$$L = \frac{4}{9} \int_{13/4}^{10} \sqrt{u}\, du = \left[\frac{4}{9} \bullet \frac{2}{3} u^{3/2}\right]_{13/4}^{10}$$

$$= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$$

Example 3: Find the length of the arc of the parabola $y^2 = x$ from (0, 0) to (1, 1).

Solution: Since $x = y^2$, we have dx/dy = 2y, and formula gives

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dy = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

We make the trigonometric substitution $y = \frac{1}{2}\tan\theta$, which gives $dy = \frac{1}{2}\sec^2\theta\, d\theta$ and $\sqrt{1 + 4y^2} = \sqrt{1 + \tan^2\theta} = \sec$. When y = 0, tan θ = 0, so θ = 0, when y = 1, tan θ = 2, so $\theta = \tan^{-1}2 = \alpha$, say. Thus

$$L = \int_0^{\alpha} \sec\theta \frac{1}{2}\sec^2\theta \, d\theta = \frac{1}{2}\int_0^{\alpha} \sec^3\theta \, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}[\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|]_0^{\alpha} = \frac{1}{4}(\sec\alpha \tan\alpha + \ln|\sec\alpha + \tan\alpha|)$$

Since $\tan\alpha = 2$, we have $\sec^2\alpha = 1 + \tan^2\alpha = 5$, so $\sec\alpha = \sqrt{5}$ and

$$L = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5} + 2)}{4}$$

Example 4: (a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $(2, \frac{1}{2})$.

(b) Use Simpson's Rule, with $n = 10$ to estimate the arc length.

Solution: (a) We have

$$y = \frac{1}{x} \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

and so the arc length is

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx = \int_1^2 \frac{\sqrt{x^4 + 1}}{x^2}\, dx$$

b. Using Simpson's Rule with $a = 1$, $b = 2$, $n = 10$, $\Delta x = 0.1$ and $f(x) = \sqrt{1 + 1/x^4}$, we have

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx$$

$$\approx \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + ... + 2f(1.8) + 4f(1.9) + f(2)]$$

$$\approx 1.1321$$

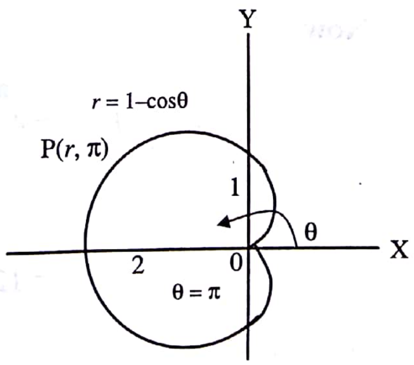
Example 5: Find the length of the cardioid $r = 1 - \cos\theta$.

Solution: Let the point $P(r, \theta)$ traces the curve anticlockwise as θ runs from 0 to 2π. To find the limit of integration put $r = 0$, then

$$1 - \cos\theta = 0$$

$$\Rightarrow \cos\theta = 1$$

$$\Rightarrow \theta = 0, 2\pi.$$



Given equation of cardioid is

$$r = 1 - \cos\theta.$$

Then, $$\frac{dr}{d\theta} = \sin\theta.$$

Now, the length of cardioid is,

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{[(1-\cos\theta)^2 + \sin^2\theta]}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad \left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \left[\because \sin\frac{\theta}{2} \geq 0 \text{ for } 0 \leq \theta \leq 2\pi\right]$$

$$= -4\left[\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= -4\left[\cos\frac{2\pi}{2} - \cos 0\right]$$

$$= -4[-1-1]$$

$$= 8 \text{ units}$$

Example 6: Find the arc length of asteroid $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq 2\pi$.

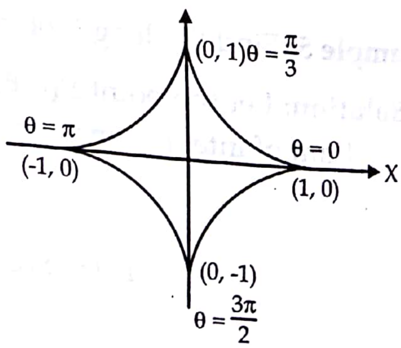
Solution: The curve's symmetry with respect to the coordinate axes, so its length is four times the length of the first quadrant portion we have

$$x = \cos^3 t,\ y = \sin^3 t$$

Then,

$$\frac{dx}{dt} = \frac{d}{dt}(\cos^3 t) = -3\cos^2 t \sin t.$$

$$\frac{dy}{dt} = \frac{d}{dt}(\sin^3 t) = 3\sin^2 t \cos t$$



Now,

$$L = 4\int_0^{\pi/2} \sqrt{(3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt$$

$$= 12\int_0^{\pi/2} \sqrt{\cos^2 t \sin^2 t\, (\cos^2 t + \sin^2 t)}\, dt$$

$$= 12\int_0^{\pi/2} \cos t \sin t \, dt$$

$$= 6\int_0^{\pi/2} 2\sin t \cos t \, dt$$

$$= 6\int_0^{\pi/2} \sin 2t \, dt$$

$$= 6\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2}$$

$$= -3\,[\cos\theta - \cos 0]$$

$$= -3\,[-1-1].$$

$$= 6 \text{ units.}$$

Exercise 6.4

1. Find the length of cardioids $r = 1 + \cos\theta$
2. Find the length of the graph of

   $f(x) = \frac{x^3}{12} + \frac{1}{x}, \quad 1 \le x \le 4.$
3. Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.
4. Find the arc length function for the curve in Example 2 taking $A = (1, 13/12)$ as the starting point.
5. Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1, 1) and (4, 8).
6. Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1)$ as the starting point.

Answers:

1. 8	2. 6
3. 2.27	4. 6
5. $\frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$	6. 8.1373

6.5 Area of Surface of Revolution

Definition: If the function $f(x) \geq 0$ is continuously differentiable on [a, b] the area of the surface generated by revolving the graph of $y = f(x)$

About the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx.$$

Example 1: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis (Fig.)

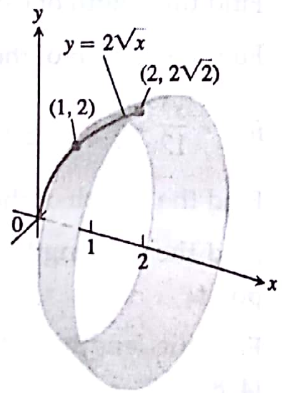
Solution. We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

with

$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

First, we perform some algebraic manipulation on the radical in the integrand to transform it into an expression that is easier to integrate.



$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}$$

With these substitutions, we have

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\, \frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 2\pi \int_1^2 \sqrt{x+1}\, dx$$

$$= 4\pi \bullet \left[\frac{2}{3}(x+1)^{3/2}\right]_1^2 = \frac{8\pi}{3}(3\sqrt{3} - 2\sqrt{2}).$$

Revolution about the y-axis

For revolution about the y-axis, we interchange x and y in Eq. (3)

Surface Area for Revolution about the y-axis

If $x = g(y) \geq 0$ is continuously differentiable on [c, d], the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2}\, dy$$

Example 2: The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y-axis to generate the cone in Fig. Find its lateral surface area (which excludes the base year).

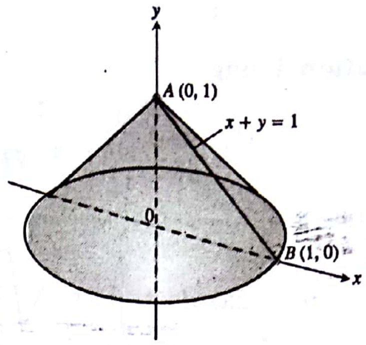
Solution. Here we have a calculation we can check with a formula from geometry:

Lateral surface area $= \dfrac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}$.

To see how Eq. (4) gives the same result, we take

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$



and calculate

$$S = \int_c^d 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dy = \int_0^1 2\pi(1-y)\sqrt{2}\, dy$$

$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1-\frac{1}{2}\right) = \pi\sqrt{2}$$

Example 3: The curve $y = \sqrt{4 - x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2.)

Solution: We have

$$\frac{d}{dx}y = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4-x^2}}$$

and, the surface are is

$$S = \int_{-1}^{1} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi\int_{-1}^{1}\sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\, dx = 2\pi\int_{-1}^{1}\sqrt{4-x^2}\frac{2}{\sqrt{4-x^2}}\, dx = 4\pi\int_{-1}^{1} 1dx = 4\pi(2) = 8\pi$$

Example 4: The arc of the parabola $y = x^2$ from (1, 1) to (2, 4) is rotated about the y-axis. Find the area of the resulting surface.

Solution: Using $y = x^2$ and $\dfrac{dy}{dx} = 2x$

We have,

$$S = \int 2\pi x\, ds$$

$$= \int_1^2 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi\int_1^2 x\sqrt{1+4x^2}\, dx$$

Substituting $u = 1 + 4x^2$, we have $du = 8x\, dx$. Remembering to change the limits of integration, we have

$$S = \frac{\pi}{4}\int_5^{17}\sqrt{u}\,du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_5^{17} = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$$

Solution: Using

$$x = \sqrt{y} \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

We have

$$S = \int 2\pi x\,ds = \int_1^4 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dy$$

$$= 2\pi\int_1^4 \sqrt{y}\sqrt{1 + \frac{1}{4y}}\,dy = \pi\int_1^4 \sqrt{4y + 1}\,dy$$

$$= \frac{\pi}{4}\int_5^{17}\sqrt{u}\,du \qquad (\text{where } u = 1 + 4y)$$

$$= \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$$

Exercise 6.5

1. Find the exact area of the surface obtained by rotating the curve about the x-axis.
   a. $y = x^3,\ 0 \le x \le 2$
   b. $y = \sqrt{1 + 4x},\ 1 \le x \le 5$
   c. $y = \sin \pi x,\ 0 \le x \le 1$
   d. $x = \frac{1}{3}(y^2 + 2)^{3/2},\ 1 \le y \le 2$
2. The following curve is rotated about the y-axis. Find the area of the resulting surface.
   a. $y = \sqrt[3]{x},\ 1 \le y \le 2$
   b. $x = \sqrt{a^2 - y^2},\ 0 \le y \le a/2$

Answers:

1. a. $\frac{1}{27}\pi(145\sqrt{145} - 1)$ b. $\frac{98}{3}\pi$ c. $2\sqrt{1 + \pi^2} + (2/\pi)\ln(\pi + \sqrt{1 + \pi^2})$ d. $\frac{21}{2}\pi$
2. (a) $\frac{1}{27}\pi(145\sqrt{145} - 10\sqrt{10})$ (b) πa^2