EQUILATERAL TRIANGLES PROJECT

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Equilateral Triangles Project

Problem 1. Given $\triangle ABC$, with altitudes h_A, h_B, and h_C dropped to sides a, b, and c, respectively, prove that $ah_A = bh_B = ch_C$. (Thus, the altitudes of a triangle are proportional to the reciprocals of the sides of the triangle.) *Careful*: $\triangle ABC$ could be right or obtuse.

We must consider three cases for $\triangle ABC$: whether it is right, acute, or obtuse.

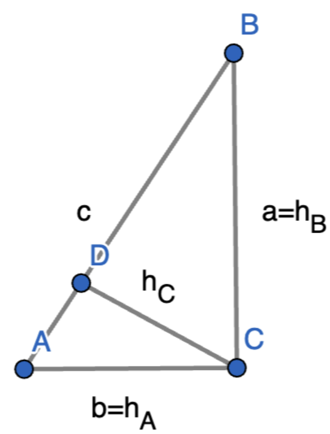
Case 1: $\triangle ABC$ is right

If $\triangle ABC$ is a right triangle, the orthocenter is at the vertex of the right angle, which we denote as $\angle C$. Let D be the intersection of the altitude h_A and side c. Since $\triangle ABC$ and $\triangle ACD$ are both right triangles and share $\angle A$, by AA similarity, they are similar. Thus, their corresponding sides are proportional:

$$\frac{b}{c} = \frac{h_C}{a}.$$

In a right triangle, $a = h_B$ and $b = h_A$. By transitivity:

$$c \cdot h_C = a \cdot h_A = b \cdot h_B.$$



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Case 2: $\triangle ABC$ is acute

If $\triangle ABC$ is acute, the orthocenter lies inside the triangle. Let D, E, and F be the feet of the altitudes h_A, h_B, and h_C, respectively. By AA similarity, $\triangle AFC \sim \triangle AEB$, so:

$$\frac{b}{c} = \frac{h_C}{h_B}.$$

This implies:

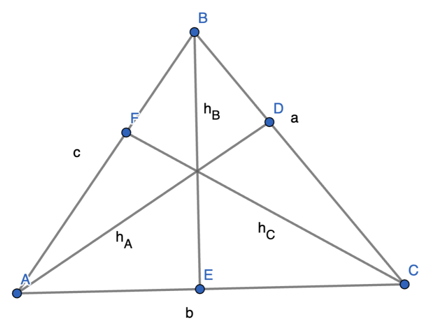
$$b \cdot h_B = c \cdot h_C.$$

Similarly, by AA similarity, $\triangle CEB \sim \triangle CDA$, so:

$$\frac{a}{b} = \frac{h_B}{h_A}.$$

By cross multiplication and transitivity:

$$b \cdot h_B = c \cdot h_C = a \cdot h_A.$$



Case 3: $\triangle ABC$ is obtuse

If $\triangle ABC$ is obtuse, the orthocenter lies outside the triangle. ∠CBD and ∠ABE are vertical angles, so by AA similarity, $\triangle BDC \sim \triangle BEA$, where D is the foot of altitude h_A , and E is the intersection of h_B and h_C. It follows that:

$$\frac{a}{c} = \frac{h_C}{h_A},$$

so:

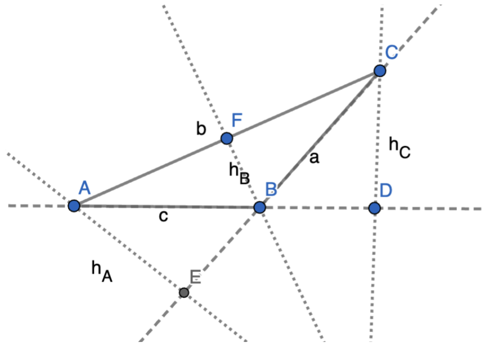
$$a \cdot h_A = c \cdot h_C.$$

Similarly, by AA similarity, $\triangle AFB \sim \triangle ADC$, so:

$$\frac{c}{b} = \frac{h_B}{h_C}.$$

By cross multiplication and transitivity:

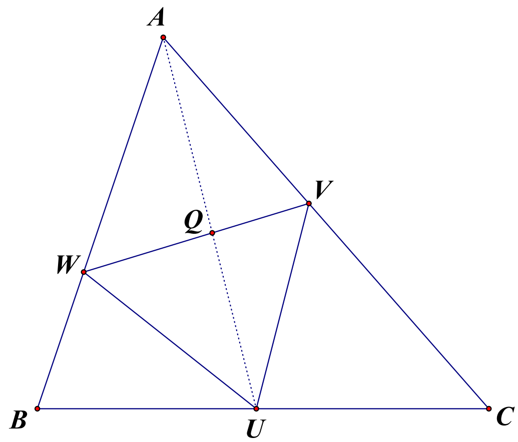
$$a \cdot h_A = b \cdot h_B = c \cdot h_C.$$



Definition. The first problem shows that the product of a side of a triangle and the altitude dropped to that side is the same for all sides (and their corresponding altitudes). Given $\triangle ABC$, let $\langle ABC\rangle$ be this product.

Problem 2. Given triangle ABC with points U, V, and W on sides BC, CA, and AB respectively, prove that

$$\langle AWV\rangle + \langle BUW\rangle + \langle CVU\rangle + \langle UVW\rangle = \langle ABC\rangle.$$



We first drop an altitude h_1 from V in $\triangle AUV$. (AUV) $\cong$AU$\cdot h_1$. Since $AU = AQ + QU$, by the distributive property:

$$(AUV) = AQ \cdot h_1 + QU \cdot h_1.$$

Here AQ $\cdot h_1$ = (AQV), and QU$\cdot h_1$ = (QUV). By transitivity:

$$(AUV) = (AQV) + (QUV).$$

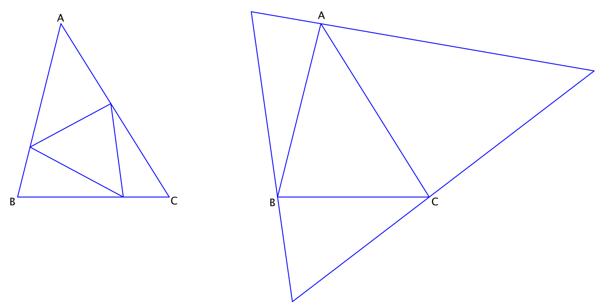
Using the same reasoning for other triangles, we find:

$$(ABU) = (BUW) + (AWU),$$
$$(AUC) = (AVU) + (VUC),$$
$$(AWU) = (AQW) + (WUQ),$$
$$(AWV) = (AQW) + (AQV),$$
$$(UVW) = (UQV) + (UQW).$$

Finally, for (ABC), we know: (ABC) = (ABU) + (AUC)
By transitivity:

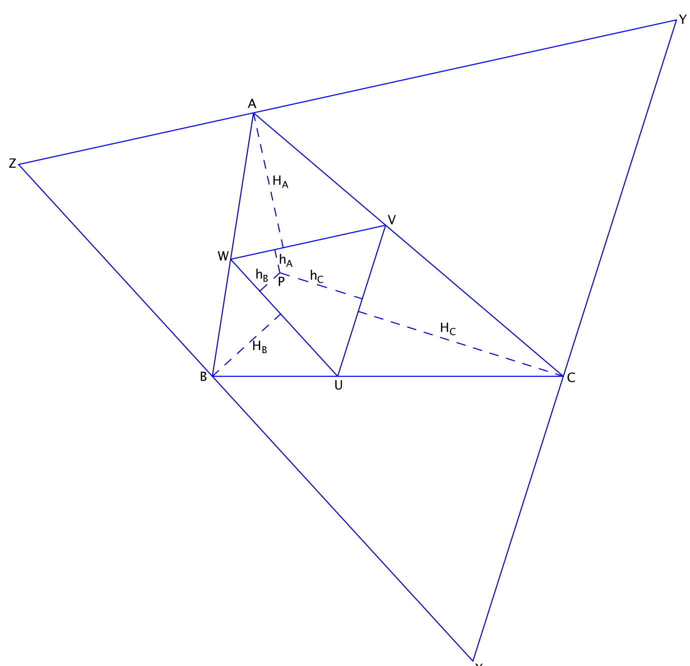
$$(ABC) = (AWV) + (BUW) + (CVU) + (UVW).$$

Definition. Given a triangle, an *inner* triangle is a triangle with vertices on the sides of the given triangle and an *outer* triangle is a triangle with sides on the vertices of the given triangle. (Inner and outer triangles of $\triangle ABC$ are pictured below.)



Problem 3. Suppose given $\triangle ABC$ and parallel inner and outer equilateral triangles $\triangle UVW$ and $\triangle XYZ$. Let $s = UV$ and suppose P is an arbitrary point inside $\triangle UVW$. The segments labeled h_A or H_A are perpendiculars dropped from P and A (respectively) to VW. Segments h_B, H_B, h_C, and H_C are defined similarly. Let H be the altitude of $\triangle XYZ$. Prove that

- $h_A + H_A + h_B + H_B + h_C + H_C = H$ and
- $sH = \langle ABC \rangle$



Part 1: First we will extend h_A, h_B, and h_C to sides YZ, XZ, and XY respectively. Since it is given that the corresponding sides of △UVW are parallel to the sides of △XYZ, h_A intersects YZ at a right angle. The same logic can be applied to altitudes h_B and h_C. Since the shortest distance between parallel lines is constant, the perpendicular from P to sides YZ, XZ, and XY can be written as H_A+h_A, H_B+h_B, and H_C+h_C respectively. As we know from problem 4.98, the perpendiculars from a point inside an equilateral triangle, P, add up to the altitude of the equilateral triangle. Therefore, H_A+h_A+H_B+h_B+ H_C+h_C≅H.

Part 2: We know from the previous problem that:

$$(AWV) + (BUW) + (VUW) + (CUV) = (ABC).$$

△VUW is equilateral, so UV ≅WU ≅WV ≅s. We also know from problem 4.98 that the altitude, a, of equilateral △WUV≅h_A+h_B+h_C. By substitution, we get s*H_A+s*H_B+s*+s*a+s*H_C≅(ABC). By the distributive property, we get s(H_A+H_B+a+H_C)≅ABC. We also know from problem 4.98 that the altitude, a, of equilateral △WUV≅h_A+h_B+h_C. By transitivity, we get sH≅(ABC).

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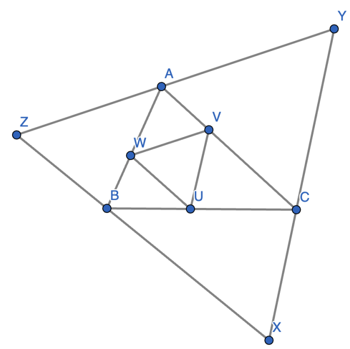
Problem 4. Given a triangle and one of its outer equilateral triangles, construct a parallel inner equilateral triangle. Then prove that the smallest inner equilateral triangle is parallel to the largest outer equilateral triangle.



Part 1: Suppose we are given $\triangle ABC$ with an outer equilateral triangle $\triangle XYZ$. From the previous problem, we know that:

$$sH \cong (ABC),$$

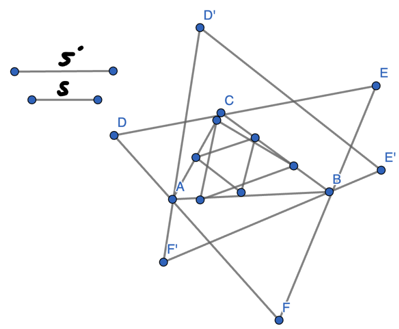
where s is the side length of the inner equilateral triangle and H is the altitude of $\triangle XYZ$. (ABC) can be written as:

$$(ABC) = BC \cdot H_A,$$

where H_A is the altitude dropped to side BC from vertex A. By transitivity, we have:

$$sH \cong BC \cdot H_A.$$

Since H, BC, and H_A are constructible, using problem 5.23, s is constructible. By problem 5.24, we can construct segment WU such that it is congruent to s and parallel to XZ. Similarly, we can construct WV and VU such that they are congruent to s and parallel to YZ and XY, respectively. Thus, $\triangle WUV$ has all sides congruent and is parallel to the outer equilateral triangle $\triangle XYZ$.



Part 2: Suppose we are given segments s' and s such that $s' > s$, and these are the side lengths of inner equilateral triangles of $\triangle ABC$. Consider $\triangle DEF$ as the parallel outer equilateral triangle of the inner equilateral triangle with side length s, and $\triangle D'E'F'$ as the parallel outer equilateral triangle of the inner equilateral triangle with side length s'. We have already proven that there exists a parallel inner equilateral triangle. We will refer to the altitude of $\triangle DEF$ as H, the altitude of $\triangle D'E'F'$ as H', and the altitude dropped from C to side AB as H_C. From Problem 5.29, we know that:

$$sH \cong H_C \cdot BC \quad \text{and} \quad s'H' \cong H_C \cdot BC.$$

By transitivity, we have:

$$sH \cong s'H'.$$

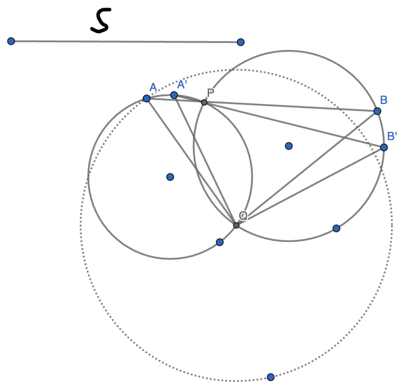
Since we know that $sH < s'H$, by transitivity, we can conclude that:

$$H' < H.$$

Since the altitude of $\triangle DEF$ is greater than the altitude of $\triangle D'E'F'$, it follows that $\triangle DEF$ is the largest equilateral triangle. We have shown that as the side length of the inner equilateral triangle decreases, the side length of the outer equilateral triangle increases. Therefore, the smallest equilateral triangle is parallel to the largest equilateral triangle.

Problem 5. Suppose given two circles intersecting at two points P and Q.

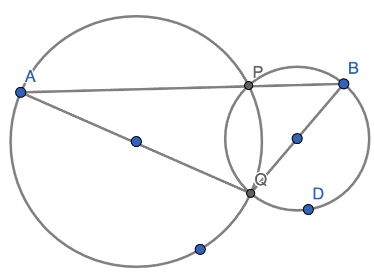
- Construct a segment AB such that A lies on one circle, B lies on the other circle, P lies on AB, and AB is congruent to the given segment.
- Construct the largest segment AB such that A lies on one circle, B lies on the other circle, and P lies on AB.



Part 1: Suppose given segment s. First we will draw a line through P and call the intersection points with the two circles A' and B'. Although we do not know the locations of A and B, we know by the Inscribed Angle Theorem that ∠QA'P' ≅ ∠QAP and ∠QB'P ≅ ∠QBP since they intersect the same arc. By AA similarity, △AQB ∼ △A'QB'. Since corresponding sides are proportional,

$$\frac{AQ}{A'Q} \cong \frac{AB}{A'B'} \cong \frac{QB}{QB'}.$$

By constructing the fourth proportional, we can get segments AQ and QB since we know AB, A'Q, A'B', and QB'. Next we will construct a circle centered at Q with radius AQ. This will intersect the circle with point A at two points. We can choose one of the two intersection points as point A and draw line AP. B is the intersection of line AP and the circle containing B. We have constructed segment AB such that P lies on it and it is congruent to s.

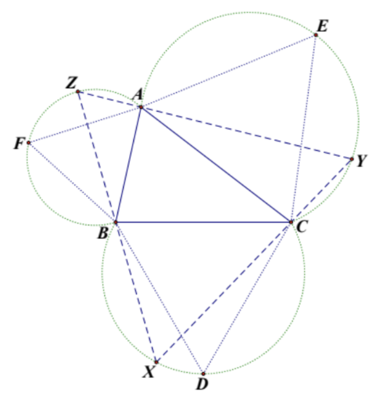


Part 2: In order for AB to maximized, we need to maximize the other two sides of △AQB, QA and QB. Since diameters are the longest chord, we will construct points A and B such that QA and QB are diameters. By Thales, since QA and QB are diameters, ∠APQ and ∠BPQ are right. Since they are adjacent and two adjacent right angles form a

linear pair, P lies on AB. We have now maximized AB such that P lies on AB.

Problem 6. Suppose given a triangle.

- Given a segment, construct an outer equilateral triangle such that the sides of the equilateral are congruent to the given segment.
- Construct the largest outer equilateral triangle.



Part 1: Problem 4.70 allows us to construct an outer equilateral triangle given a triangle, so we will use the same diagram. The first part of the previous problem allows us construct segment ZX such that it is congruent to the given segment. Segments XY and and YZ can be constructed similarly. Thus, given $\triangle$ABC, we can construct outer equilateral $\triangle$XYZ with side lengths congruent to the given segment.

Part 2: The second part of the previous problem allows us to maximize a segment with endpoints on different circle such that a certain point is contained on it. Therefore, we know how to construct the largest segment ZX such that B lies on it. Segments XY and YZ can be constructed similarly. We have constructed the largest outer equilateral $\triangle$XYZ of $\triangle$ABC.

Problem 7. Given a triangle, construct its smallest inner equilateral triangle.

From the previous problem we know how to construct the largest outer equilateral triangle given a triangle. Problem 5.30 allows us to construct the inner equilateral triangle that is parallel to the outer equilateral triangle. By problem 5.30 we also know the inner equilateral triangle that is parallel to the largest outer equilateral triangle is the smallest one. Therefore, from problems 5.30 and 5.33, we can construct a given triangle's smallest inner equilateral triangle.