The Limits of Sorting
Comparison Sorts III

CS 311 Data Structures and Algorithms

Lecture Slides

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Unit Overview
Algorithmic Efficiency & Sorting

Major Topics

- ✓ Analysis of Algorithms
- ✓ Introduction to Sorting
- ✓ Comparison Sorts I
- ✓ Asymptotic Notation
- ✓ Divide and Conquer
- ✓ Comparison Sorts II
- The Limits of Sorting
- Comparison Sorts III
- Non-Comparison Sorts
- Sorting in the C++ STL

Review

Review
Analysis of Algorithms

Using Big-O	In Words
$O(1)$	Constant time
$O(\log n)$	Logarithmic time
$O(n)$	Linear time
$O(n \log n)$	Log-linear time
$O(n^2)$	Quadratic time
$O(c^n)$, for some $c > 1$	Exponential time

Cannot read all of input $\uparrow$

Probably not scalable $\downarrow$

$\uparrow$ Faster

Slower $\downarrow$

Useful Rules

- When determining big-O, we can collapse any constant number of steps into a single step.
- **Rule of Thumb.** For nested "real" loops, order is $O(n^t)$, where t is the number of nested loops.
- **Addition Rule.** $O(f(n)) + O(g(n))$ is either $O(f(n))$ or $O(g(n))$, *whichever is larger*. And similarly for Θ. This works when adding up any *fixed*, *finite* number of terms.

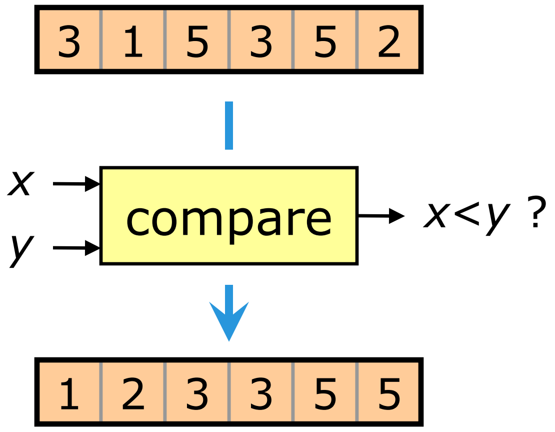
Review
Introduction to Sorting — Basics, Analyzing

Sort: Place a list in order.

Key: The part of the item we sort by.

Comparison sort: Sorting algorithm that only gets information about item by comparing them in pairs.

A **general-purpose comparison sort** places no restrictions on the size of the list or the values in it.



Analyzing a general-purpose comparison sort:

- (Time) Efficiency
- Requirements on Data
- Space Efficiency ← **In-place** = no large additional space required.
- Stability ← **Stable** = never reverses the relative order of equivalent items.
- Performance on Nearly Sorted Data ← 1. All items close to proper places, OR 2. few items out of order.

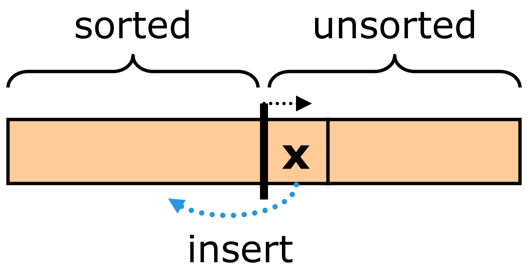
Review
Introduction to Sorting — Overview of Algorithms

Sorting Algorithms Covered

- Quadratic-Time [$O(n^2)$] Comparison Sorts
  - ✓ Bubble Sort
  - ✓ Insertion Sort
  - Quicksort
- Log-Linear-Time [$O(n \log n)$] Comparison Sorts
  - ✓ Merge Sort
  - Heap Sort (mostly later in semester)
  - Introsort
- Special Purpose—Not Comparison Sorts
  - Pigeonhole Sort
  - Radix Sort

Review
Comparison Sorts I — Insertion Sort

Insertion Sort repeatedly does this:



Analysis

- (Time) Efficiency: $O(n^2)$. Average case same. ☹
- Requirements on Data: Works for Linked Lists, etc. ☺
- Space Efficiency: In-place. ☺
- Stability: It is stable. ☺
- Performance on Nearly Sorted Data: $O(n)$ for both kinds. ☺

See `insertion_sort.cpp`.

Notes

- Too slow for general-purpose use.
- Fast in special cases: *nearly sorted data* and *small lists*.
- Thus, often used as part of other algorithms.

Review
Asymptotic Notation

$g(n)$ is:

- $O(f(n))$ if $g(n) \le k \times f(n)$ …
- $\Omega(f(n))$ if $g(n) \ge k \times f(n)$ …
- $\Theta(f(n))$ if both are true—possibly with different values of k.

Θ is very useful!

Ω not as much.

	1	n	$n \log n$	n^2	$5n^2$	$n^2 \log n$	n^3	n^4
$O(n^2)$	YES	YES	YES	YES	YES	no	no	no
$\Omega(n^2)$	no	no	no	YES	YES	YES	YES	YES
$\Theta(n^2)$	no	no	no	YES	YES	no	no	no

In an algorithmic context, $g(n)$ might be:

- The maximum number of basic operations performed by the algorithm when given input of size n.
- The maximum amount of additional space required.

In-place means using $O(1)$ additional space.

Review
Divide-and-Conquer [1/2]

A Divide/Decrease-and-Conquer algorithm needs analysis.

- It splits its input into *b* **nearly equal-sized** parts.
- It makes *a* recursive calls, each taking one part.
- It does other work requiring *f*(*n*) operations.

To Analyze

- Find *b*, *a*, *d* so that *f*(*n*) is $\Theta(n^d)$—or $O(n^d)$.
- Compare *a* and b^d.
- Apply the appropriate case of the Master Theorem.

The **Master Theorem**

Suppose $T(n) = a\,T(n/b) + f(n)$; $a \geq 1$, $b > 1$, $f(n)$ is $\Theta(n^d)$.

- "*n*/*b*" can be a nearby integer.

Then:

- Case 1. If $a < b^d$, then $T(n)$ is $\Theta(n^d)$.
- Case 2. If $a = b^d$, then $T(n)$ is $\Theta(n^d \log n)$.
- Case 3. If $a > b^d$, then $T(n)$ is $\Theta(n^k)$, where $k = \log_b a$.

We may also replace each "Θ" above with "*O*".

Review
Divide-and-Conquer [2/2]

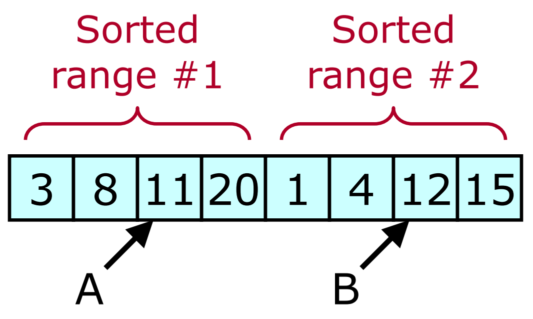
Algorithm *C* splits its input into two nearly equal-sized parts, makes two recursive calls, each taking one the parts, and does other work requiring constant time.

Analyze Algorithm *C* using the Master Theorem:

- Find b, a, d.
  - Algorithm *C* splits its input into 2 nearly-equal-sized parts.
    - $b = 2$.
  - Algorithm *C* makes 2 recursive calls.
    - $a = 2$.
  - In addition, Algorithm *C* does other work requiring constant time.
    - $f(n)$ is $O(1)$. 1 is n^0. So $d = 0$.
- Which Case?
  - Compare a (= 2) with b^d ($= 2^0 = 1$). $a > b^d \rightarrow$ Case 3.
- Conclusion
  - By Case 3 of the Master Theorem, $T(n)$ is $O(n^k)$, where $k = \log_b a$.
  - $k = \log_2 2 = 1$. So $T(n)$ is $O(n^1)$.
  - Simplify. Algorithm *C* is $O(n)$: linear time.

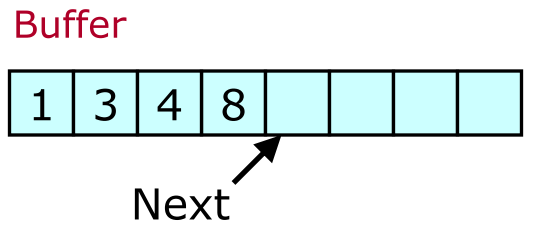
Review
Comparison Sorts II — Merge Sort [1/2]

Stable Merge: two iterators go through input, one for each sorted part. Repeat: determine which referenced item comes first; put this item into the merged list.



For a Linked List, we can do this in place.

General-purpose Stable Merge, which works on arrays, generally uses a separate buffer—Θ(n) space—to hold the merged list.

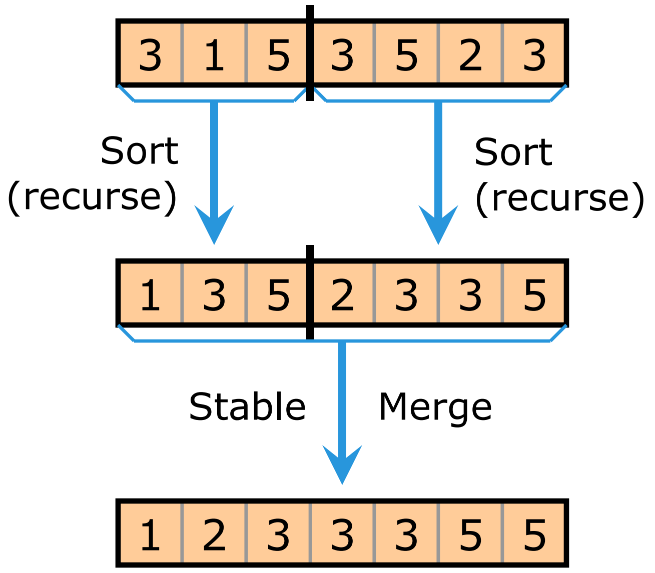


In both cases, the merge can be done in linear time, without reversing the relative order of equivalent items (so, **stable**).

Review
Comparison Sorts II — Merge Sort [2/2]

Merge Sort splits the data in half, recursively sorts both, merges.

See `merge_sort.cpp`.



Analysis

- Efficiency: Θ(*n* log *n*). Avg same. ☺
- Requirements on Data: Works for Linked Lists, etc. ☺
- Space Efficiency: Θ(log *n*) space for recursion. Iterative version is in-place for Linked List. Θ(*n*) space for array. 😐/☺/☹
- Stable: Yes. ☺
- Performance on Nearly Sorted Data: Not better or worse. 😐

See `iterative_merge_sort.cpp`.

Notes

- Practical & often used.
- Fastest known for (1) stable sort, (2) sorting a Linked List.

The Limits of Sorting

The Limits of Sorting
Introduction

We have mentioned that most sorting algorithms fall into one of two categories:

- Slow: $\Theta(n^2)$—e.g., Bubble Sort, Insertion Sort.
- Fast: $\Theta(n \log n)$—e.g., Merge Sort.

Can we sort even faster than that?

No, we cannot—not with a general-purpose comparison sort.

Fact. A general-purpose comparison sort that lies in any time-efficiency category faster than $\Theta(n \log n)$ is *impossible*. (Remember: worst-case analysis.)

More precisely: we can *prove* that the worst-case number of comparisons performed by a general-purpose comparison sort must be $\Omega(n \log n)$. $\longleftarrow$ Here is what Ω is good for: statements that say, "You cannot do better than this."

The Limits of Sorting
Proof — Background

Sorting is determining the ordering of a list. Many orderings are possible.

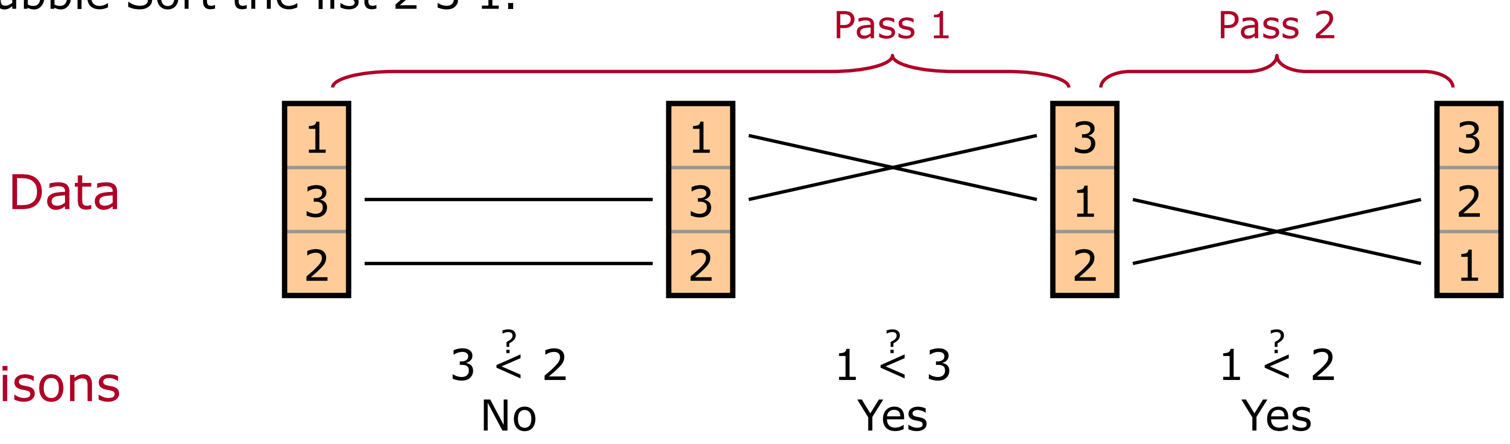
Each time we do a comparison, we find the relative order of two items.

Say x < y; we can throw out all orderings in which y comes before x.

We cannot stop until only one possible ordering is left.

Example

- Bubble Sort the list 2 3 1.



Alternate View

Possible orderings:			Possible orderings:			Possible orderings:			Possible orderings:	
123	132	→	123	~~132~~	→	123	~~132~~	→	$\boxed{123}$	~~132~~
213	231		213	231		213	~~231~~		~~213~~	~~231~~
312	321		~~312~~	~~321~~		~~312~~	~~321~~		~~312~~	~~321~~

The Limits of Sorting
Proof — Outline [1/2]

We prove that the worst-case number of comparisons performed by a general-purpose comparison sort must be $\Omega(n \log n)$.

As on the previous slide:

- We are given a list of n items to be sorted.
- There are $n! = n \times (n-1) \times \ldots \times 3 \times 2 \times 1$ orderings of n items.
- Start with all $n!$ orderings. Do comparisons, throwing out orderings that do not match what we know, until just one ordering is left.

How many comparisons are required?

- With each comparison, we cannot guarantee that more than half of the orderings will be thrown out. (Remember: worst case.)
- How many times must we cut $n!$ in half, to get 1?
- Answer: $\log_2(n!)$.

Continued on next slide …

The Limits of Sorting
Proof — Outline [2/2]

We know that the worst-case number of comparisons performed by a general-purpose comparison sort cannot be less than $\log_2(n!)$.

Now we use **Stirling's Approximation**: $n! \approx \frac{n^n}{e^n}\sqrt{2\pi n}$.

Take $\log_2$ of both sides:

$$\log_2(n!) \approx n\log_2 n - n\log_2 e + \frac{1}{2}\log_2(2\pi) + \frac{1}{2}\log_2 n,$$

which is $\Theta(n \log n)$.

See `stirling.py`*.*

So $\log_2(n!)$ is $\Theta(n \log n)$.

The worst case number of comparisons done by a general-purpose comparison sort must be *at least* that big. Thus: $\Omega(n \log n)$.

The Limits of Sorting
Another View

The worst-case number of comparisons performed by a general-purpose comparison sort must be $\Omega(n \log n)$.

Another way to say this involves a different model of computation:

- Legal operations:
  - Any operation that does not depend on the values of input data items.
  - A comparison of two data items.
- Basic operation: Comparison of two data items.
- Size: Number of items in given list.

In this model of computation, comparison sorting is the only kind of sorting that can be done.

A restatement of what was proven:

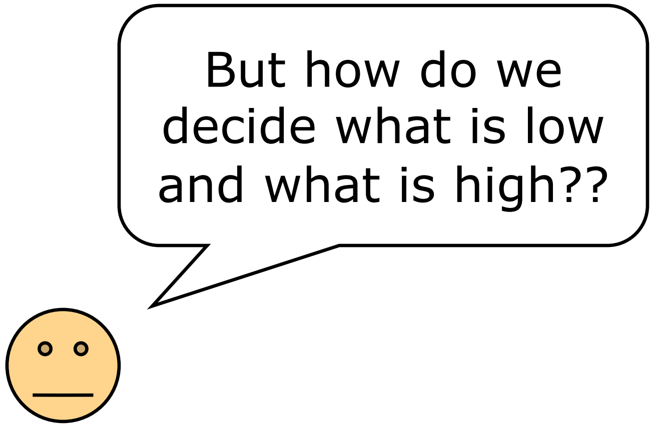
In the above model of computation, every sort is $\Omega(n \log n)$ time.

Comparison Sorts III

Comparison Sorts III
Quicksort — Introduction [1/3]

Idea

- Instead of simply splitting a list in half in the middle, try to be intelligent about it.
- Split the list into the $\boxed{\text{low}}$-valued items and the $\boxed{\text{high}}$-valued items; then recursively sort each bunch.
- Now no Merge is necessary.

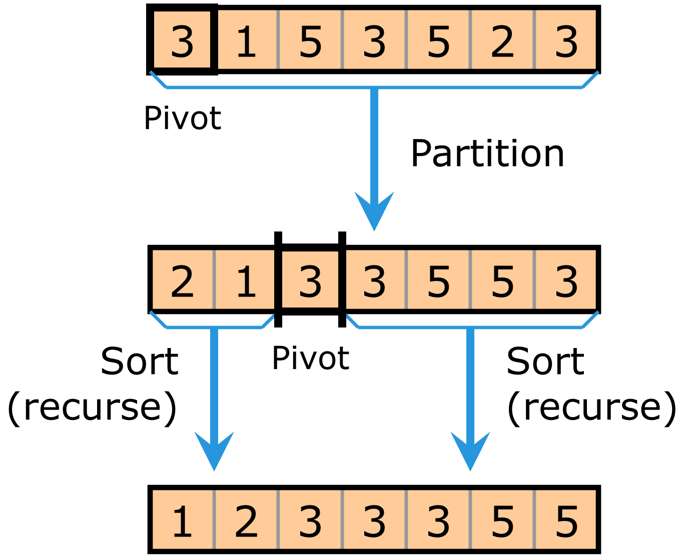


Comparison Sorts III
Quicksort — Introduction [2/3]

Let's be more precise about this algorithmic idea.

We use another Divide-and-Conquer technique:

- Pick an item in the list.
  - This first item will do—for now.
  - The chosen item is called the **pivot**.
- Rearrange the list so that the items before the pivot are all less than or equivalent to the pivot, and the items after the pivot are all greater than or equivalent to the pivot.
  - This operation is called **Partition**. It can be done in linear time.
- Recursively sort the sub-lists: items before pivot, items after pivot.

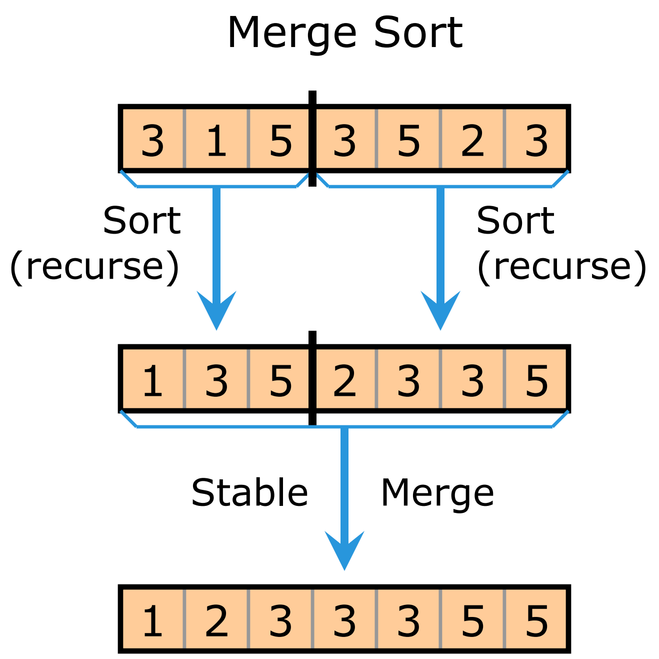


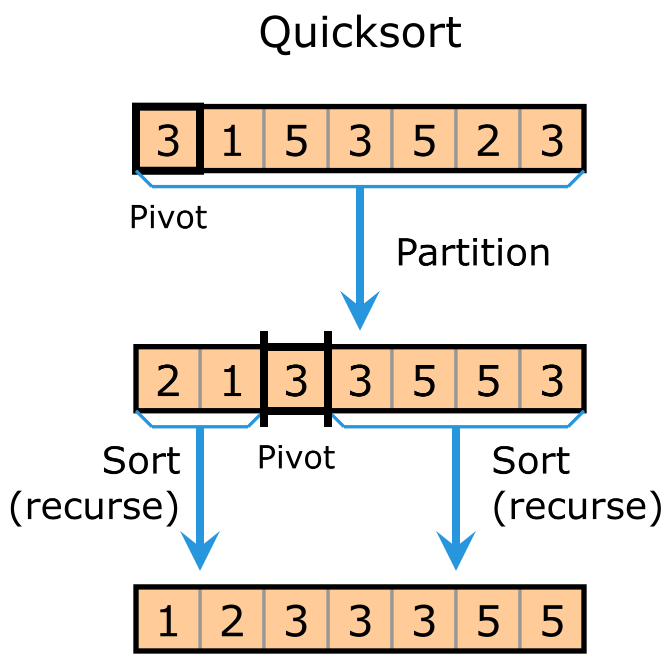
This algorithm is called **Quicksort** [C.A.R. ("Tony") Hoare, 1961].

Comparison Sorts III
Quicksort — Introduction [3/3]

Compare Merge Sort & Quicksort.

- Both use Divide-and-Conquer.
- Both have an auxiliary operation (Stable Merge, Partition) that does all modification of the data set and that takes linear time.
- Merge Sort recurses first. Quicksort recurses last.





Comparison Sorts III
Quicksort — Partition [1/2]

How do we do the Partition operation?

There are two partition algorithms used with Quicksort:

- **Hoare's Partition Algorithm**. Requires bidirectional iterators.
- **Lomuto's Partition Algorithm**. Requires forward iterators.

Both are in-place and linear time. We will discuss Hoare's. (*You may wish to look up Lomuto's.*)

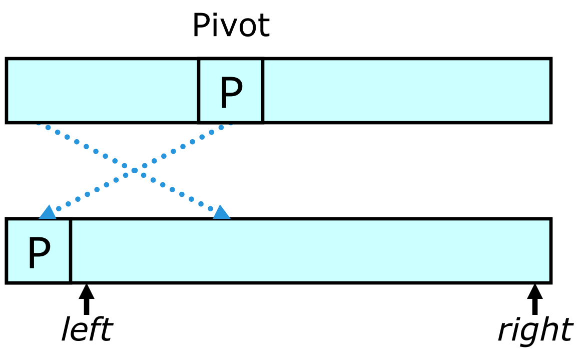
Note. Using Lomuto's, Quicksort *can* be written to work with forward iterators. In practice, however, Quicksort is only fast for random-access data. Speed is Quicksort's primary advantage, so implementations usually **require random-access data**.

Next we look at the details of Hoare's Partition Algorithm.

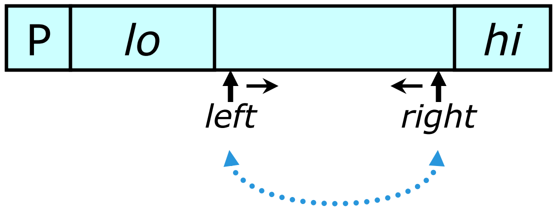
Comparison Sorts III
Quicksort — Partition [2/2]

Hoare’s Partition Algorithm

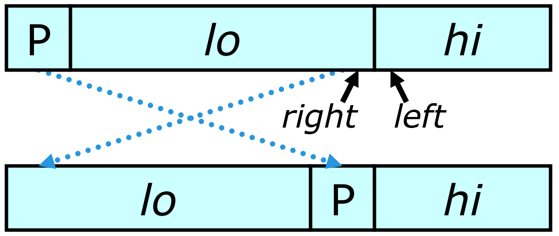
- First, get the pivot out of the way: swap it with the first list item.
- Set iterator *left* to point to the first item past the pivot and *right* to point to the last list item.



- Move iterator *left* up, leaving only low items below it. Move *right* down, leaving only high items above it.
- If both iterators get stuck—*left* points to a high item and *right* points to a low item—then swap the items and continue.



- Eventually *left* & *right* cross each other.
- Finish by swapping the pivot with the last low item.



Comparison Sorts III
Quicksort — CODE

TO DO

- Write Quicksort, with the in-place Partition being a separate function.
  - Use Hoare’s Partition Algorithm, written as a separate function.
  - Require random-access iterators.

Not done yet.
See `quicksort1.cpp`*.*

Comparison Sorts III
TO BE CONTINUED …

Comparison Sorts III will be continued next time.