Mathematics Extended Essay

Title: An investigation into the chaotic behavior of an n-pendulum

Research question: How does the chaotic behavior of an n-pendulum change as the value of n is increased?

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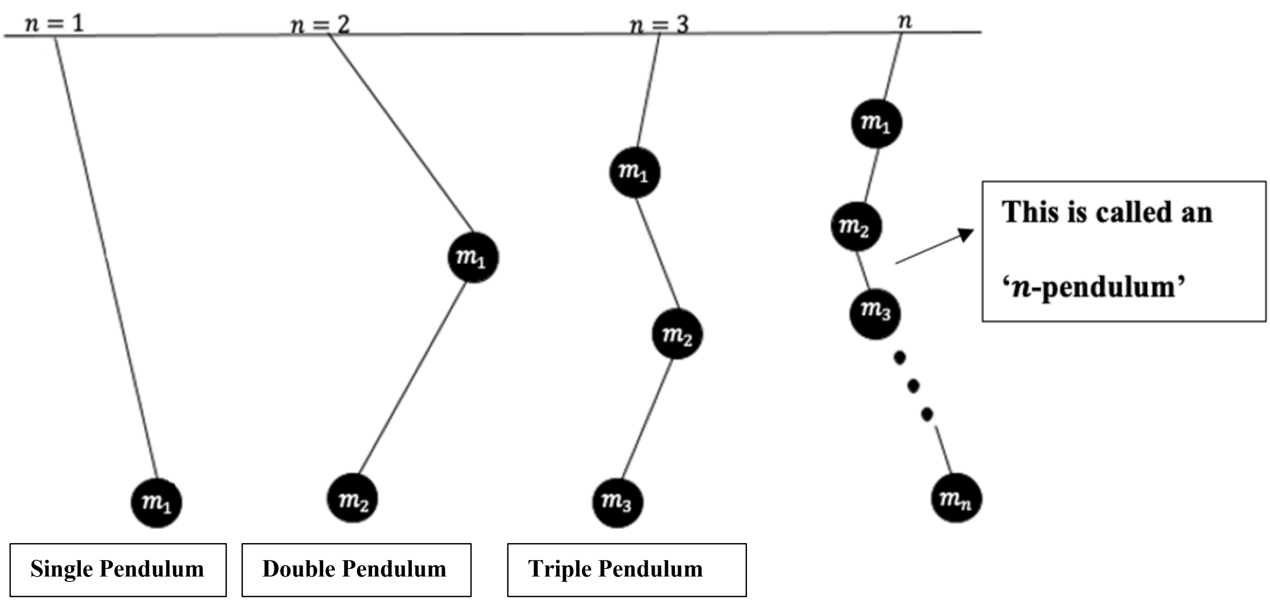
Introduction

Chaos theory, originally discovered by Edward Lorenz in the 1960's, is a branch of mathematics which focuses on the study of deterministic dynamical systems that *exhibit sensitive dependence on initial conditions* (Oestreicher). This means that infinitesimally small changes to their initial conditions, such as those due to errors in measurements or rounding errors during numerical computation, yield widely diverging outcomes – rendering long-term prediction of their behaviour impossible (Kellert 32). Few examples of chaotic systems include the weather, ocean turbulence, and a double pendulum (D. Straussfogel 151).

The double pendulum grabbed my attention. This is because, if simply adding another simple pendulum at the end of the first results in chaos, then what would happen if one keeps adding pendulums (figure 1.1)? Will the chaotic behaviour keep 'increasing'? This question became the base of this essay. **This essay seeks to explore how the chaotic behavior of an n-pendulum changes as n is increased.**

Figure 1.1: Sketch of the ***n*-pendulum** system for $n = 1$, $n = 2$, $n = 3$, and arbitrary n



Approach

To answer the research question, the essay will be divided into 3 sections.

In section-1, the general equation of motion for an n-pendulum will be derived using Lagrangian Mechanics.

In section-2, the equation will be simulated for different values of n using numerical methods.

In section-3, using the numerical simulation developed in sections 1 and 2, a quantitative analysis of how chaotic behavior of an n-pendulum changes with increasing value of n will be performed. Thus, the research question will be answered.

Section-1: Deriving the general equation of motion for an $\boldsymbol{n}$-pendulum

In this section, an equation describing the motion of an n-pendulum will be derived. This step is crucial in answering the research question because to quantify the relationship between the value of n and the chaotic behavior of an n-pendulum, a general equation which can describe the motion of an n-pendulum is necessary.

To obtain this equation, we first need to examine the mechanics that governs the system. There are two models of mechanics that can be used for this problem, Newtonian and Lagrangian (Rubenzahl 2). In this essay, the Lagrangian approach shall be used because in Lagrangian Mechanics, we do not need to deal with vector quantities like tension (Morin, ch. 6.1). Instead, the equations of motion come from a single scalar function, i.e. The Lagrangian, which itself is a function of the system's potential and kinetic energy (scalar quantities) (Morin, ch. 6.1).

1.1 - Lagrangian Procedure to Derive the Equations of Motion

Step 1 – The Lagrangian (L) equals the total kinetic energy (T) minus the total potential energy of the system (V) (Morin, Ch. 6.1).

$$L = T - V \tag{1}$$

Taking the example of a single pendulum:

Kinetic Energy $(T) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

Potential Energy $(V) = mgy$

Using (1):

$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$

Where $\dot{x} = \frac{d}{dt}(x)$ *and* $\dot{y} = \frac{d}{dt}(y)$

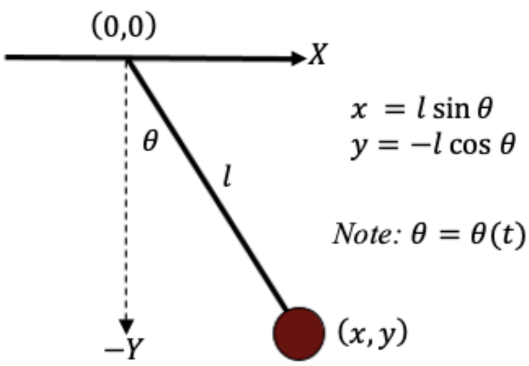


Figure 1.2: Diagram of a single pendulum

Step 2 - The Lagrangian (L) is substituted into the Euler-Lagrange equation to obtain the equations of motion (Morin, Ch. 6.1). Note: *equation (2) is equivalent to Newtons Second Law, $F = ma$. Proof is provided in appendix B. It was not included in the main body because it deviates from the research question.*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{2}$$

Variable	Meaning
q_j	Generalized Coordinate
$\dot{q}_j$	$\frac{d}{dt}(q_j)$

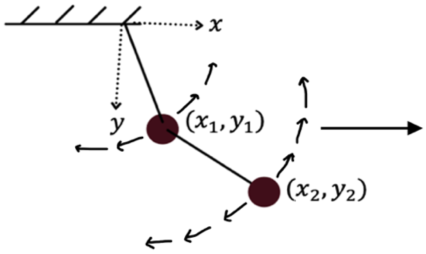
Generalized coordinates are any set of coordinates (not necessarily cartesian), which can be used to completely describe a system in space (Vandiver). In Lagrangian Mechanics, any system of coordinates can be used, as long as it satisfies the following 3 conditions (Vandiver).

Table 1: Conditions for choosing Generalized Coordinates in Lagrangian Mechanics

Condition	Explanation
Condition A – System must be Independent	If one fixes all but one coordinate, the remaining coordinate should still have a continuous range of movement
Condition B – System must be Complete	Using the generalized coordinates, one should be able to locate all parts of the system at all times
Condition C – System must be Holonomic	The number of degrees of freedom in the dynamical system must be equal to the number of coordinates used to describe the system.

The number of degrees of freedom is the number of independent motions allowed to a body (Gary, slide 4). A simple pendulum has 1 degree of freedom because it can only oscillate about a vertical plane (refer to assumption 2 and 5 in section-1.2). Similarly, a double pendulum has 2 degrees of freedom because the first pendulum oscillates in the vertical plane and has its own degree of freedom, while the second pendulum has its own degree of freedom with respect to the first (Gary, slide 4). Hence, we can generalize by saying that an n-pendulum will have n degrees of freedom.

We cannot use cartesian coordinates as generalized coordinates because it does not satisfy condition A; it is not independent. (figure 1.3)



If x_1 and x_2 are fixed, y_1 and y_2 will not have a continuous range of motion (rod is rigid)

Figure 1.3: Demonstration of why Cartesian Coordinates cannot be used as the coordinate system

In contrast, using angular displacements like θ_1, θ_2 (figure 1.4) satisfies all conditions. It is independent because if θ_1 is fixed, θ_2 still exhibits a continuous range of motion. It is complete because the two angles can be used to describe all possible states of the system. It is holonomic because the number of coordinates equals the number of degrees of freedom (two). **Hence, throughout this essay, angular displacements will be used as generalized coordinates.**

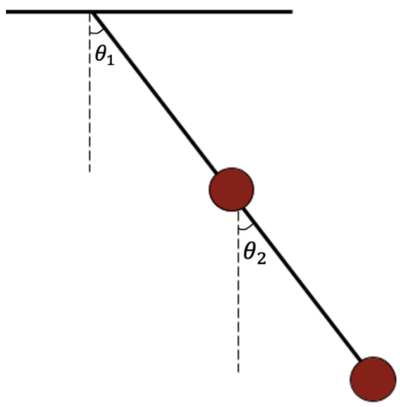


Figure 1.4: Demonstration of why Angular Displacements will be used as the coordinate system

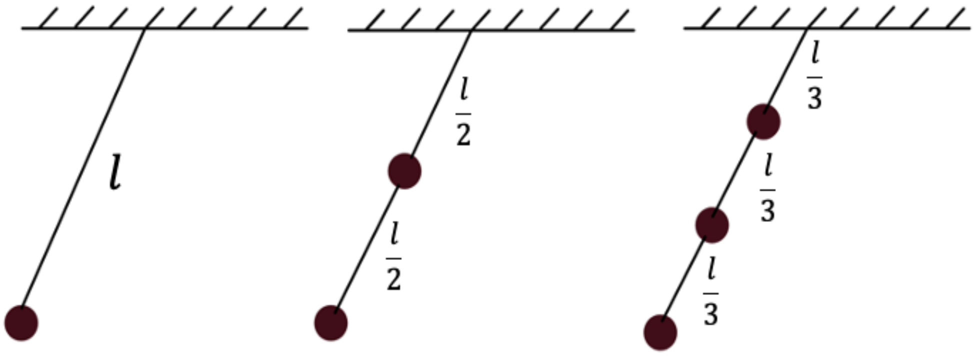
The Euler-Lagrange-Equation will be applied to each generalized coordinate (Vandiver). For instance, a double pendulum has 2 degrees of freedom and hence 2 generalized coordinates (θ_1 *and* θ_2), so we will have 2 equations of motion. One will be found using $\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_1}}\right) - \frac{\partial L}{\partial \theta_1} = 0\right)$ and the other using $\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_2}}\right) - \frac{\partial L}{\partial \theta_2} = 0\right)$. Thus, for an n-pendulum, we would have n equations of motion (Vandiver)

Now, the equations of motion for a single and double pendulum will be derived. Then, a general equation of motion for an n-pendulum will be derived.

1.2 – Assumptions made during the derivation

1. Mass of the rod is negligible compared to mass of the bobs
2. The rod connecting the pendulums is rigid
3. The length of the whole pendulum is fixed as l and all bobs are equidistant from each other. Hence, the distance between two adjacent bobs in an n-pendulum is $\frac{l}{n}$.

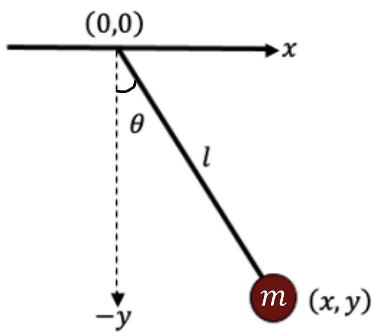
Figure 1.5: Diagrammatic representation of Assumption 3



4. No energy loss to the surroundings through heat, air resistance, etc.

5. The motion will take place in a 2-dimensional space/a single vertical plane.

1.3 - Deriving the equation of motion for a single pendulum ($n = 1$)



Angle (θ) continuously changes with time.

Therefore, $\theta = \theta(t)$

Figure 1.6: Diagram of a Single Pendulum

Cartesian Coordinates (Displacement)

$$x = l \sin \theta \tag{3}$$

$$y = -l \cos \theta \tag{4}$$

Calculating Total Potential Energy (V)

Substituting (4) into Potential Energy (P.E) formula

$$V = mgy$$

$$V = -mgl \cos \theta \tag{5}$$

Calculating Total Kinetic Energy (T)

Calculating velocities (time derivative of displacement) - using (3) and (4)

$$\dot{x} = \dot{\theta} l \cos\theta \tag{6}$$

$$\dot{y} = \dot{\theta} l \sin\theta \tag{7}$$

Substituting (6) and (7) into Kinetic Energy (K.E) Formula

$$T = \frac{1}{2} m v^2$$

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$T = \frac{1}{2} m\left(\left[\dot{\theta} l \cos\theta\right]^2 + \left[\dot{\theta} l \sin\theta\right]^2\right)$$

$$T = \frac{1}{2} m\left(\dot{\theta}^2 l^2 [\cos^2\theta + \sin^2\theta]\right)$$

$$\therefore\ T = \tfrac{1}{2} m\dot{\theta}^2 l^2 \tag{8}$$

Calculating the Lagrangian –

Substituting (5) and (8) into (1)

$$L\ =\ T - V$$

$$L = \tfrac{1}{2} m\dot{\theta}^2 l^2 + mgl\cos\theta \tag{9}$$

Solving the Euler-Lagrange Equation for θ $\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0\right)$ – Using (9):

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

Thus:

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\therefore \ddot{\theta} = \frac{-g\sin\theta}{l} \tag{10}$$

Solving (10) gives us $\theta(t)$, which completely describes the motion because angular displacements are generalized coordinates fulfilling condition B (table 1).

1.4 - Deriving the equations of motion for a double pendulum $(n = 2)$

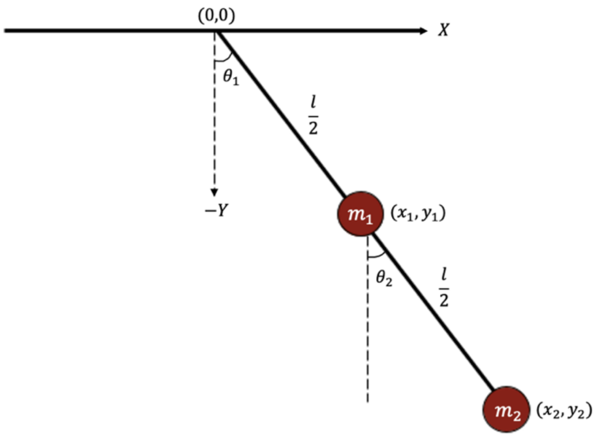


Figure 1.7: Diagram of a Double Pendulum

Cartesian Coordinates (Displacement)

$$x_1 = \frac{l}{2}\sin\theta_1 \quad (11) \qquad\qquad y_1 = -\frac{l}{2}\cos\theta_1 \quad (12)$$

$$x_2 = \frac{l}{2}\sin\theta_1 + \frac{l}{2}\sin\theta_2 \quad (13) \qquad\qquad y_2 = -\frac{l}{2}\cos\theta_1 - \frac{l}{2}\cos\theta_2 \quad (14)$$

Calculating Total Potential Energy (V)

Substituting (12) and (14) into P.E Formula

$$V = m_1 g y_1 + m_2 g y_2$$

$$V = \frac{-l}{2} g\big(m_1 cos\theta_1 + m_2(cos\theta_1 + cos\theta_2)\big) \quad (15)$$

Calculating Total Kinetic Energy (T)

Calculating Velocities using (11) (12) (13) and (14)

$$\dot{x}_1 = \dot{\theta}_1 \frac{l}{2}\cos\theta_1 \quad (16) \qquad\qquad \dot{x}_2 = \dot{\theta}_1 \frac{l}{2}\cos\theta_1 + \dot{\theta}_2 \frac{l}{2}\cos\theta_2 \quad (17)$$

$$\dot{y}_1 = \dot{\theta}_1 \frac{l}{2}\sin\theta_1 \quad (18) \qquad\qquad \dot{y}_2 = \dot{\theta}_1 \frac{l}{2}\sin\theta_1 + \dot{\theta}_2 \frac{l}{2}\sin\theta_2 \quad (19)$$

Substituting (16), (17), (18), and (19) into K.E Formula[1] *–*

$$T = \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2$$

$$T = \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2)$$

$$\therefore T = \frac{1}{2}\frac{l^2}{4}(m_1\dot{\theta_1}^2 + m_2(\dot{\theta_1}^2 + \dot{\theta_2}^2 + 2\dot{\theta_1}\dot{\theta_2}(\cos(\theta_1 - \theta_2)\,) \tag{20}$$

Calculating the Lagrangian (L)

Substituting (15) and (20) into (1)

$$L = \frac{1}{2}\frac{l^2}{4}\left(m_1\dot{\theta_1}^2 + m_2\left[\dot{\theta_1}^2 + \dot{\theta_2}^2 + 2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1 - \theta_2)\,\right]\right)$$
$$+\frac{l}{2}g(m_1 cos\theta_1 + m_2[cos\theta_1 + cos\theta_2]) \tag{21}$$

Then, by solving the Euler-Lagrange Equation for θ_1 $\left(\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_1}}\right) - \frac{\partial L}{\partial\theta_1} = 0\right)$, *we get:*

$$\frac{l^2}{4}\ddot{\theta_2}[m_1 + m_2] + \frac{l^2}{4}m_2\left[\ddot{\theta_2}\cos(\theta_1 - \theta_2) + \dot{\theta_2}^2\sin(\theta_1 - \theta_2)\right] + \frac{l}{2}gsin\theta_1[m_1 + m_2] = 0 \tag{22}$$

And for θ_2 $\left(\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_2}}\right) - \frac{\partial L}{\partial\theta_2} = 0\right)$, *we get*[2]*:*

$$\frac{l^2}{4}m_2\ddot{\theta_2} + \frac{l^2}{4}m_2\ddot{\theta_1}\cos(\theta_1 - \theta_2) - \frac{l^2}{4}m_2\dot{\theta_1}^2\sin(\theta_1 - \theta_2) + \frac{l}{2}gm_2\sin(\theta_2) = 0. \tag{23}$$

Solving (22) and (23) will give us $\theta_1(t)$ and $\theta_2(t)$, which completely describes the motion of a double pendulum.

[1] Full working to find (20) is in appendix C.

[2] Full derivation of (22) and (23) is in appendix C. It is not included in the main body as it deviates from the research focus

1.5 – Deriving the equations of motion for an n-pendulum

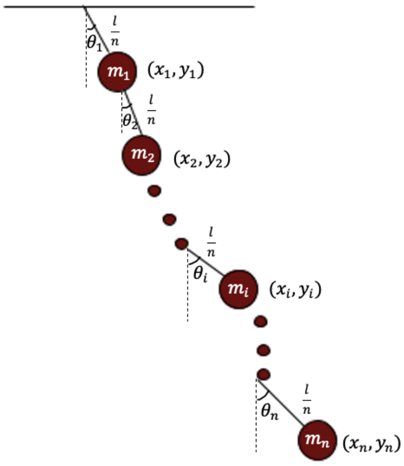


Figure 1.8: Diagram of an n-pendulum

Cartesian Coordinates (Displacement)

$$x_i = \frac{l}{n}sin\theta_1 + \frac{l}{n}sin\theta_2 + \cdots + \frac{l}{n}sin\theta_i$$

$$\therefore x_i = \sum_{j=1}^{i} \frac{l}{n} \sin\theta_j \tag{24}$$

$$y_i = -\frac{l}{n}\cos\theta_1 - \frac{l}{n}\cos\theta_2 \ldots - \frac{l}{n}\cos\theta_i$$

$$\therefore y_i = \sum_{j=1}^{i} -\frac{l}{n}\cos\theta_j \tag{25}$$

Calculating Total Potential Energy (V)

Substituting (25) into P.E Formula –

$$V = \sum_{i=1}^{n} m_i g y_i$$

$$V = g\sum_{i=1}^{n} m_i \left(\sum_{j=1}^{i} -\frac{l}{n}\cos\theta_j \right)$$

$$V = -g\frac{l}{n}\left(\sum_{i=1}^{n} m_i \sum_{j=1}^{i} \cos\theta_j \right) \tag{26}$$

Calculating Total Kinetic Energy (T)

Calculating Velocities (time derivative of displacement) – using (24) and (25)

$$\dot{x}_i = \frac{l}{n}\sum_{j=1}^{i} \dot{\theta}_j \cos\theta_j \tag{27}$$

$$\dot{y}_i = \frac{l}{n}\sum_{j=1}^{i} \dot{\theta}_j \sin\theta_j \tag{28}$$

Substituting (27) and (28) into K.E Formula –

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\left(\dot{x}_i^{\,2} + \dot{y}_i^{\,2}\right)$$

$$T = \frac{1}{2}\frac{l^2}{n^2}\sum_{i=1}^{n} m_i\left(\left[\sum_{j=1}^{i}\dot{\theta}_j\cos\theta_j\right]^2 + \left[\sum_{j=1}^{i}\dot{\theta}_j\sin\theta_j\right]^2\right) \tag{29}$$

Calculating the Lagrangian

Substituting (26) and (29) into (1) –

$$L = \frac{1}{2}\frac{l^2}{n^2}\left[\sum_{i=1}^{n} m_i\left[\left(\sum_{j=1}^{i}\dot{\theta}_j\, cos\theta_j\right)^2 + \left(\sum_{j=1}^{i}\dot{\theta}_j\, sin\theta_j\right)^2\right]\right] + g\frac{l}{n}\left(\sum_{i=1}^{n} m_i \sum_{j=1}^{i} cos\theta_j\right) \tag{30}$$

Calculating final equation of motion using Euler-Lagrange equation ($\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$)

For an n-pendulum, we will get n equations of motion. However, we desire one general equation which can be used to find all n equations of motion. Hence, a variable k has been defined which refers to the equation number required. Hence, an equation will be found in the following format –

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_k}\right) - \frac{\partial L}{\partial \theta_k} = 0 \tag{31}$$

$$k = (1, 2 \ldots n)$$

Finding $\frac{\partial L}{\partial \dot{\theta}_k}$

Using (30):

$$\frac{\partial L}{\partial \dot{\theta}_k} = \frac{\partial}{\partial \dot{\theta}_k}\left\{\frac{1}{2}\frac{l^2}{n^2}\left[\sum_{i=1}^{n} m_i\left[\left(\sum_{j=1}^{i}\dot{\theta}_j\cos\theta_j\right)^2 + \left(\sum_{j=1}^{i}\dot{\theta}_j\sin\theta_j\right)^2\right]\right] + g\frac{l}{n}\left(\sum_{i=1}^{n} m_i\sum_{j=1}^{i}\cos\theta_j\right)\right\}$$

Since the second term of the equation does not contain any $\dot{\theta}$ terms, we can discard it. We can use the following substitutions to simplify the equation:

$$\lambda = \frac{1}{2}\frac{l^2}{n^2} \qquad \alpha = \sum_{j=1}^{i}\dot{\theta}_j\cos\theta_j \qquad \beta = \sum_{j=1}^{i}\dot{\theta}_j\sin\theta_j$$

The remaining equation becomes:

$$\frac{\partial L}{\partial \dot{\theta}_k} = \frac{\partial}{\partial \dot{\theta}_k}\left\{\lambda\sum_{i=1}^{n} m_i\left[(\alpha)^2 + (\beta)^2\right]\right\}$$

$\frac{\partial}{\partial \dot{\theta}_k}$ moves inside the sigma sign since λ and $\sum_{i=1}^{n} m_i$ are constants.

The lower limit of the sigma sign changes to $(i = k)$ because for iterations where $(i < k)$, there are no $\dot{\theta}_k$ terms.

$$\frac{\partial L}{\partial \dot{\theta}_k} = \lambda\sum_{i=k}^{n} m_i\left[\frac{\partial}{\partial \dot{\theta}_k}(\alpha)^2 + \frac{\partial}{\partial \dot{\theta}_k}(\beta)^2\right]$$

Differentiating using power and chain rule –

$$\frac{\partial L}{\partial \dot{\theta}_k} = \lambda\sum_{i=k}^{n} m_i\left[2\alpha\left(\frac{\partial}{\partial \dot{\theta}_k}\alpha\right) + 2\beta\left(\frac{\partial}{\partial \dot{\theta}_k}\beta\right)\right]$$

Substituting back α and β –

$$\frac{\partial L}{\partial \dot{\theta}_k} = \lambda\sum_{i=k}^{n} m_i\left[2\alpha\left(\frac{\partial}{\partial \dot{\theta}_k}\left[\sum_{j=1}^{i}\dot{\theta}_j\cos\theta_j\right]\right) + 2\beta\left(\frac{\partial}{\partial \dot{\theta}_k}\left[\sum_{j=1}^{i}\dot{\theta}_j\sin\theta_j\right]\right)\right]$$

Since $\frac{\partial}{\partial \dot{\theta_k}}\left[\sum_{j=1}^{i} \dot{\theta}_{\mathrm{j}}\, cos\theta_{\mathrm{j}}\right] = \; cos\theta_{\mathrm{k}}$ and $\frac{\partial}{\partial \dot{\theta_k}}\left[\sum_{j=1}^{i} \dot{\theta}_{\mathrm{j}}\, sin\theta_{\mathrm{j}}\right] = sin\theta_{\mathrm{k}}$, this can be simplified:

$$\frac{\partial L}{\partial \dot{\theta_k}} = \lambda \sum_{i=k}^{n} m_i \left[2\alpha(cos\theta_{\mathrm{k}}) \; + \; 2\beta(sin\theta_{\mathrm{k}})\right]$$

Substituting back λ, α, and β -

$$\frac{\partial L}{\partial \dot{\theta_k}} = \frac{l^2}{n^2} \sum_{i=k}^{n} m_i \left[\left(\sum_{j=1}^{i} \dot{\theta}_{\mathrm{j}}\, cos\theta_{\mathrm{j}}\right) cos\theta_{\mathrm{k}} \; + \; \left(\sum_{j=1}^{i} \dot{\theta}_{\mathrm{j}}\, sin\theta_{\mathrm{j}}\right) sin\theta_{\mathrm{k}}\right] \tag{32}$$

$$n = number\ of\ pendulums$$

$$k = (1, 2, 3, \dots n)$$

Finding $\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\boldsymbol{\partial}\mathbf{L}}{\boldsymbol{\partial}\dot{\boldsymbol{\theta}}_{\mathbf{k}}}\right)$

Using (32):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_k}}\right) = \frac{d}{dt}\left\{\frac{l^2}{n^2} \sum_{i=k}^{n} m_i \left[\left(\sum_{j=1}^{i} \dot{\theta}_{\mathrm{j}}\, cos\theta_{\mathrm{j}}\right) cos\theta_{\mathrm{k}} \; + \; \left(\sum_{j=1}^{i} \dot{\theta}_{\mathrm{j}}\, sin\theta_{\mathrm{j}}\right) sin\theta_{\mathrm{k}}\right]\right\}$$

Making the following substitutions to simplify the equation –

$$\lambda = \frac{l^2}{n^2} \qquad \alpha \; = \sum_{j=1}^{i} \dot{\theta}_{\mathrm{j}}\, cos\theta_{\mathrm{j}} \qquad \beta \; = \sum_{j=1}^{i} \dot{\theta}_{\mathrm{j}}\, sin\theta_{\mathrm{j}}$$

The remaining equation becomes –

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_k}}\right) = \frac{d}{dt}\left\{\lambda \sum_{i=k}^{n} m_i \left[(\alpha) cos\theta_{\mathrm{k}} \; + \; (\beta) sin\theta_{\mathrm{k}}\,\right]\right\}$$

$\frac{d}{dt}$ goes inside the sigma sign since $\frac{l^2}{n^2}$ and $\sum_{i=k}^{n} m_i$ are constants –

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_k}}\right) = \lambda \sum_{i=k}^{n} m_i \left[\frac{d}{dt}(\alpha cos\theta_{\mathrm{k}}) \; + \; \frac{d}{dt}(\beta sin\theta_{\mathrm{k}})\right]$$

Using product rule –

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_k}}\right) = \lambda \sum_{i=k}^{n} m_i \left[cos\theta_{\mathrm{k}} \frac{d}{dt}(\alpha) + \alpha \frac{d}{dt}(cos\theta_{\mathrm{k}}) + \; sin\theta_{\mathrm{k}} \frac{d}{dt}(\beta) + \beta \frac{d}{dt}(sin\theta_{\mathrm{k}})\right]$$

Substituting back α and β, and using product rule:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_k}\right) = \lambda \sum_{i=k}^{n} m_i \left[\begin{array}{l} cos\theta_k \left(\sum_{j=1}^{i} \cos\theta_j \frac{d}{dt}(\dot{\theta}_j) + \dot{\theta}_j \frac{d}{dt}(\cos\theta_j)\right) + \alpha\left(-\dot{\theta}_k sin\theta_k\right) \\ +sin\theta_k \left(\sum_{j=1}^{i} \sin\theta_j \frac{d}{dt}(\dot{\theta}_j) + \dot{\theta}_j \frac{d}{dt}(\sin\theta_j)\right) + \beta\left(\dot{\theta}_k cos\theta_k\right) \end{array}\right]$$

Simplifying:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_k}\right) = \lambda \sum_{i=k}^{n} m_i \left[\begin{array}{l} cos\theta_k \left(\sum_{j=1}^{i} \ddot{\theta}_j \, cos\theta_j - \dot{\theta}_j^2 sin\theta_j\right) + \alpha\left(-\dot{\theta}_k sin\theta_k\right) \\ +sin\theta_k (\sum_{j=1}^{i} \ddot{\theta}_j \, sin\theta_j + \dot{\theta}_j^2 cos\theta_j) + \beta\left(\dot{\theta}_k cos\theta_k\right) \end{array}\right]$$

Substituting back constants:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_k}\right) = \frac{l^2}{n^2} \sum_{i=k}^{n} m_i \left\{\begin{array}{l} -\dot{\theta}_k sin\theta_k \left(\sum_{j=1}^{i} \dot{\theta}_j cos\theta_j\right) + cos\theta_k \left(\sum_{j=1}^{i} \ddot{\theta}_j cos\theta_j - \dot{\theta}_j^{\,2} sin\theta_j\right) + \\ \dot{\theta}_k cos\theta_k \left(\sum_{j=1}^{i} \dot{\theta}_j sin\theta_j\right) + sin\theta_k \left(\sum_{j=1}^{i} \ddot{\theta}_j sin\theta_j + \dot{\theta}_j^{\,2} cos\theta_j\right) \end{array}\right\} \quad (33)$$

We have now found $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_k}\right)$. Next, we need to find $\frac{\partial L}{\partial \theta_k}$. Then the equations can be substituted into (31) to find the k[th] equation of motion for an n-pendulum.

Finding $\frac{\partial \mathrm{L}}{\partial \theta_k}$

Using (30):

$$\frac{\partial \mathrm{L}}{\partial \theta_k} = \frac{\partial}{\partial \theta_k}\left\{\frac{1}{2}\frac{l^2}{n^2}\left[\sum_{i=1}^{n} m_i \left[\left(\sum_{j=1}^{i} \dot{\theta}_j \, cos\theta_j\right)^2 + \left(\sum_{j=1}^{i} \dot{\theta}_j \, sin\theta_j\right)^2\right]\right] + g\frac{l}{n}\left(\sum_{i=1}^{n} m_i \sum_{j=1}^{i} cos\theta_j\right)\right\}$$

Using the following substitutions:

$$\lambda = \frac{1}{2}\frac{l^2}{n^2} \qquad \alpha = \sum_{j=1}^{i} \dot{\theta}_j \, cos\theta_j \qquad \beta = \sum_{j=1}^{i} \dot{\theta}_j \, sin\theta_j$$

The equation becomes:

$$\frac{\partial \mathrm{L}}{\partial \theta_{\mathrm{k}}} = \frac{\partial}{\partial \theta_{\mathrm{k}}}\left\{\lambda\left[\sum_{i=1}^{n} m_i\,[(\alpha)^2 + (\beta)^2]\right] + g\frac{l}{n}\left(\sum_{i=1}^{n} m_i \sum_{j=1}^{i} cos\theta_j\right)\right\}$$

Moving $\frac{\partial}{\partial \theta_{\mathrm{k}}}$ inside the bracket.

The lower limits of the sigma signs change to $(i = k)$ because for iterations where $(i < k)$, there are no θ_k terms.

$$\frac{\partial \mathrm{L}}{\partial \theta_{\mathrm{k}}} = \lambda\sum_{i=k}^{n} m_i\left[\frac{\partial}{\partial \theta_{\mathrm{k}}}(\alpha)^2 + \frac{\partial}{\partial \theta_{\mathrm{k}}}(\beta)^2\right] + g\frac{l}{n}\sum_{i=k}^{n} m_i\left(\frac{\partial}{\partial \theta_{\mathrm{k}}}\sum_{j=1}^{i} cos\theta_j\right)$$

Differentiating using chain rule and using the fact $\frac{\partial}{\partial \theta_{\mathrm{k}}}\sum_{j=1}^{i} cos\theta_j = -\sin\theta_k$ we get

$$\frac{\partial \mathrm{L}}{\partial \theta_{\mathrm{k}}} = \lambda\sum_{i=k}^{n} m_i\left[2\alpha\frac{\partial}{\partial \theta_{\mathrm{k}}}(\alpha) + 2\beta\frac{\partial}{\partial \theta_{\mathrm{k}}}(\beta)\right] + g\frac{l}{n}\sum_{i=k}^{n} m_i\,(-\sin\theta_k)$$

Using the fact that $\frac{\partial}{\partial \theta_{\mathrm{k}}}(\alpha) = -\dot{\theta}_k sin\theta_k$ and $\frac{\partial}{\partial \theta_{\mathrm{k}}}(\beta) = \dot{\theta}_k cos\theta_k$:

$$\frac{\partial \mathrm{L}}{\partial \theta_{\mathrm{k}}} = \lambda\sum_{i=k}^{n} m_i\left[2\alpha(-\dot{\theta}_k sin\theta_k) + 2\beta(\dot{\theta}_k cos\theta_k)\right] - g\frac{l}{n}\sum_{i=k}^{n} m_i\sin\theta_k$$

Substituting Constants back –

$$\frac{\partial \mathrm{L}}{\partial \theta_{\mathrm{k}}} = \frac{l^2}{n^2}\sum_{i=k}^{n} m_i\left[\left(\sum_{j=1}^{i}\dot{\theta}_j cos\theta_j\right)(-\dot{\theta}_k sin\theta_k) + \left(\sum_{j=1}^{i}\dot{\theta}_j sin\theta_j\right)(\dot{\theta}_k cos\theta_k)\right] - g\frac{l}{n}\sum_{i=k}^{n} m_i\sin\theta_k \quad (34)$$

Final Equation

Substituting (33) and (34) into (31) –

$$\frac{l^2}{n^2}\left\{\begin{matrix} cos\theta_k\left(\sum_{i=k}^{n} m_i\left[\sum_{j=1}^{i}\ddot{\theta}_j cos\theta_j - \sum_{j=1}^{i}\dot{\theta}_j^{\,2} sin\theta_j\right]\right) + \\ sin\theta_k\left(\sum_{i=k}^{n} m_i\left[\sum_{j=1}^{i}\ddot{\theta}_j sin\theta_j + \sum_{j=1}^{i}\dot{\theta}_j^{\,2} cos\theta_j\right]\right) \end{matrix}\right\} + g\frac{l}{n}\left(\sum_{i=k}^{n} m_i\,(sin\theta_k)\right) = 0 \quad (35)$$

$$n = number\ of\ pendulums$$

$$k = (1, 2, \ldots n)$$

Section-1 Conclusion

In this section, the general equation for an n-pendulum was derived (equation 35). In section-2, this equation will be numerically simulated. In section-3, an investigation into how the chaotic behaviour of an n-pendulum changes as n increases will be conducted, which will answer the research question.

Section-2: Numerically simulating the $\boldsymbol{n}$-pendulum equation

In this section, equation (35) will be numerically simulated. A numerical method was chosen because the equation is a second-order ordinary differential equation (ODE), and solving it analytically is impractical, especially for larger values of n.

2.1 - Choosing an ODE Solver

For the numerical simulation, MATLAB will be used. The 7 types of numerical ODE solvers available on MATLAB are presented in table 2.1 (MathWorks "Choose").

Table 2.1: Comparison table of the 7 ode solvers available on MATLAB

Solver	**Order of Accuracy**	**When To Use**
Ode45	Medium	Most of the time. This should be the first solver you try
Ode23	Low	If using crude error tolerances or solving moderately stiff problems
Ode113	Low to High	If using stringent error tolerances or solving a computationally intensive ODE file
Ode15s	Low to Medium	If ode45 is slow because the problem is stiff
Ode23s	Low	If using crude error tolerances to solve stiff systems and the mass matrix is constant
Ode23t	Low	If the problem is only moderately stiff and you need a solution without numerical damping.

The only ODE solver whose order of accuracy ranged till high was ode113. Moreover, ode113 could deal with "computationally intensive ODE files" and had "stringent error tolerances". This was needed for a chaotic system like the n-pendulum, because slight errors whilst simulating numerically could lead to inaccurate results.

A range of accuracies, "low to high" was present because the local error of the ode113 solver could be manually modified through the functions *"RelTol"* and *"AbsTol"*. The local error at every time step was calculated by the equation (Deval):

$$|e(i)| \leq \max\left(RelTol \times abs\big(y(i)\big), AbsTol(i)\right) \qquad (36)$$

Variable	Meaning
$e(i)$	Local Error
$RelTol$	Relative Error Tolerance
$AbsTol$	Absolute Error Tolerance
$y(i)$	Output of the ode solver – in our case it will be the value of angles $(\theta_1, \ldots \theta_n)$ and the velocities $\left(\dot{\theta}_1, \ldots \dot{\theta}_n\right)$. More will be said about it below.

Equation (36) showed a direct correlation between the values of $RelTol$ and $AbsTol$, and the local error. The default values for 'RelTol' and 'AbsTol' were 10^{-3} and 10^{-6} respectively. However, they could be manually set as 2.2205×10^{-14} each (least possible value), which reduced the local error to the order of 10^{-14}.

2.2 - Inputs of ode113

- **Time span**: must be in the format $[t_0, t_1, t_2, \ldots . t_f]$. This will return the solution evaluated at the given times (MathWorks, "ode113")

- **System of differential equations**: ode113 can only solve systems of first-order differential equations (MathWorks, "ode113"). However, the general pendulum equation (equation 35) resulted in second-order differential equations. Hence, we must convert them into first-order equations through variable substitution.

- Taking the example of a single pendulum. Recalling equation (10):

$$\ddot{\theta} = \frac{-g\ \sin\theta}{l}$$

- This can be converted into two first-order ODE's by making the substitutions:

$$y_1 = \theta$$

$$y_2 = \dot{\theta}$$

- The two first-order differential equations become:

$$y_1{}' = y_2 \tag{37}$$

$$y_2{}' = \frac{-g\ \sin y_1}{l} \tag{38}$$

- (37) and (38) can be inputted into ode113 to numerically simulate a single pendulum. Similarly, for a double pendulum, we would obtain 4 first-order ODE's, etc.

- We do not have to do this process manually because the MATLAB function, *odeToVectorField* takes the second-order ODE's generated by equation (35) as input (eqn_1, …., eqn_N), and returns a set of first order ODE's ready to be inputted into ode113 (MathWorks "odeToVectorField").

- **Initial conditions**: We need to input initial angles and velocity in the format: $(\theta_1, \ldots \theta_n, \dot{\theta}_1, \ldots \dot{\theta}_n)$, along with the mass of each bob and the length of pendulum (MathWorks, "ode113").

Table 2.2: Example inputs into ode113 for a single pendulum

Input	Value
Time Span	[0, 0.001, 0.002, 0.003… 50]
System of first order ODE's (recall equations 37 and 38)	$y_1{}' = y_2$ $y_2{}' = \frac{-g \sin y_1}{l}$
Initial Conditions	$\left[\frac{\pi}{2}; 0\right], m = 1kg, l = 1m$

2.3 - Output of ode113

The output is a matrix containing the value of $(\theta_1, \dots \theta_n, \dot{\theta}_1, \dots \dot{\theta}_n)$ for each time step entered as input. (MathWorks, "ode113")

Figure 2.1: Format of matrix showing output of ode113

Time Steps	θ_1	θ_2	$\cdots$	θ_n	$\dot{\theta}_1$	$\dot{\theta}_2$	$\cdots$	$\dot{\theta}_n$
t_0								
t_1								
t_2								
$\cdots$								
t_n								

Using ode113, numerical simulations for n-pendulums with various values of n were created[3]

[3] Video links for simulations are in appendix A

Section-3: Investigating the chaotic behaviour of an n-pendulum

In this section, “chaotic behaviour” will be quantified. Then, a quantitative analysis will be conducted on how chaotic behaviour of an n-pendulum changes as the value of n increases (which will answer the research question)

3.1 - Introduction to Chaos

“Chaos is **aperiodic long-term behaviour** in a **deterministic system** that exhibits **sensitive dependence on initial conditions”** (Strogatz 331). Key terms in the definition have been examined below:

3.1.1 - Aperiodic Long-Term Behaviour

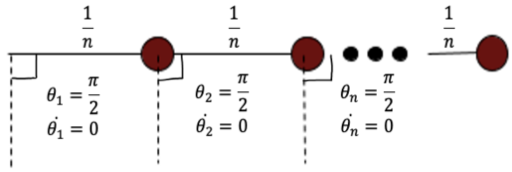
“Aperiodic long-term behaviour” means that the motion of a system does not settle down to a fixed point or periodic orbit (Strogatz 331).

To test whether an n-pendulum exhibits aperiodic long-term behaviour, a simulation of n-pendulums with $(n = 1, 2, 3, 4)$ was run and $(\theta_1, \dots \theta_n)$ were plotted with time[4].

Table 3: Input into ode113 to test for Aperiodic Long-Term Behaviour

Time Interval	[0 50] – 10,000 total time steps
Initial Conditions (figure 3.1)	$\frac{\pi}{2}$ for all initial θ 0 for all initial $\dot{\theta}$
Mass	1kg for all bobs

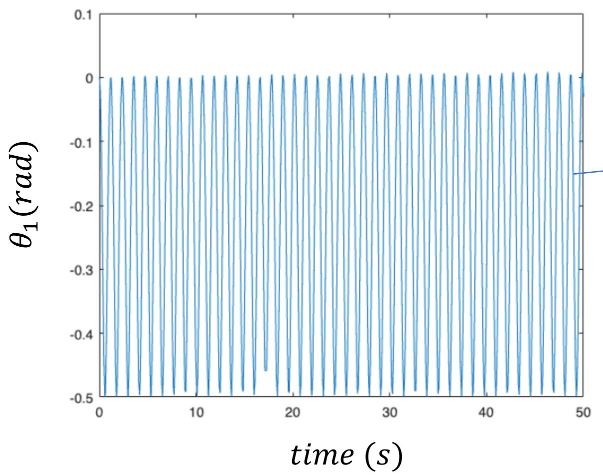
Figure 3.1: Initial Conditions used to test for aperiodic long-term behaviour



[4] Code used is in appendix A

Length	1m for the entire pendulum

Figure 3.2: Plot of θ_1 with time for $n = 1$



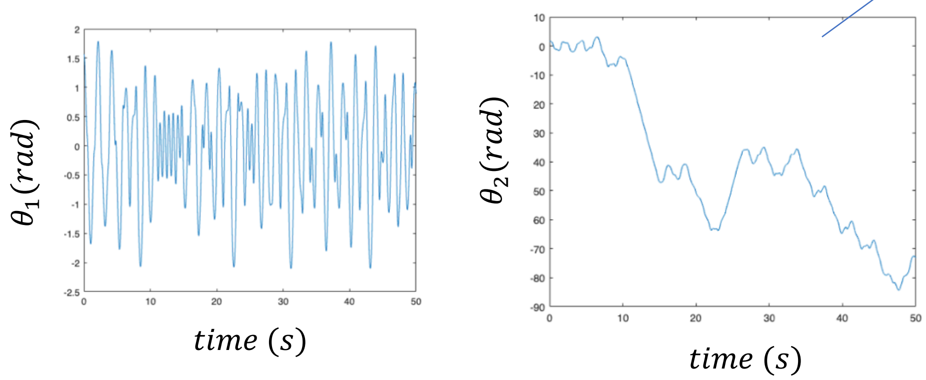
Periodic Long-Term Behavior

Hence, a simple pendulum is **non-chaotic**

θ_2 becomes negative

because of rotation

Figure 3.3: Plots for θ_1 *and* θ_2 with time for $n = 2$



Both plots show

Aperiodic Long-

Term Behavior

All 3 plots show Aperiodic

Long-Term Behavior

Figure 3.4: Plots for θ_1, θ_2 *and* θ_3 with time for $n = 3$

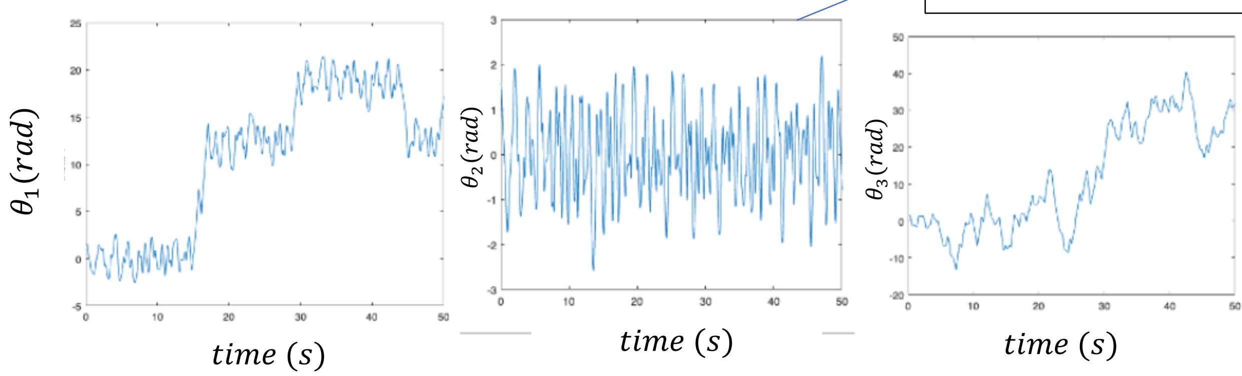
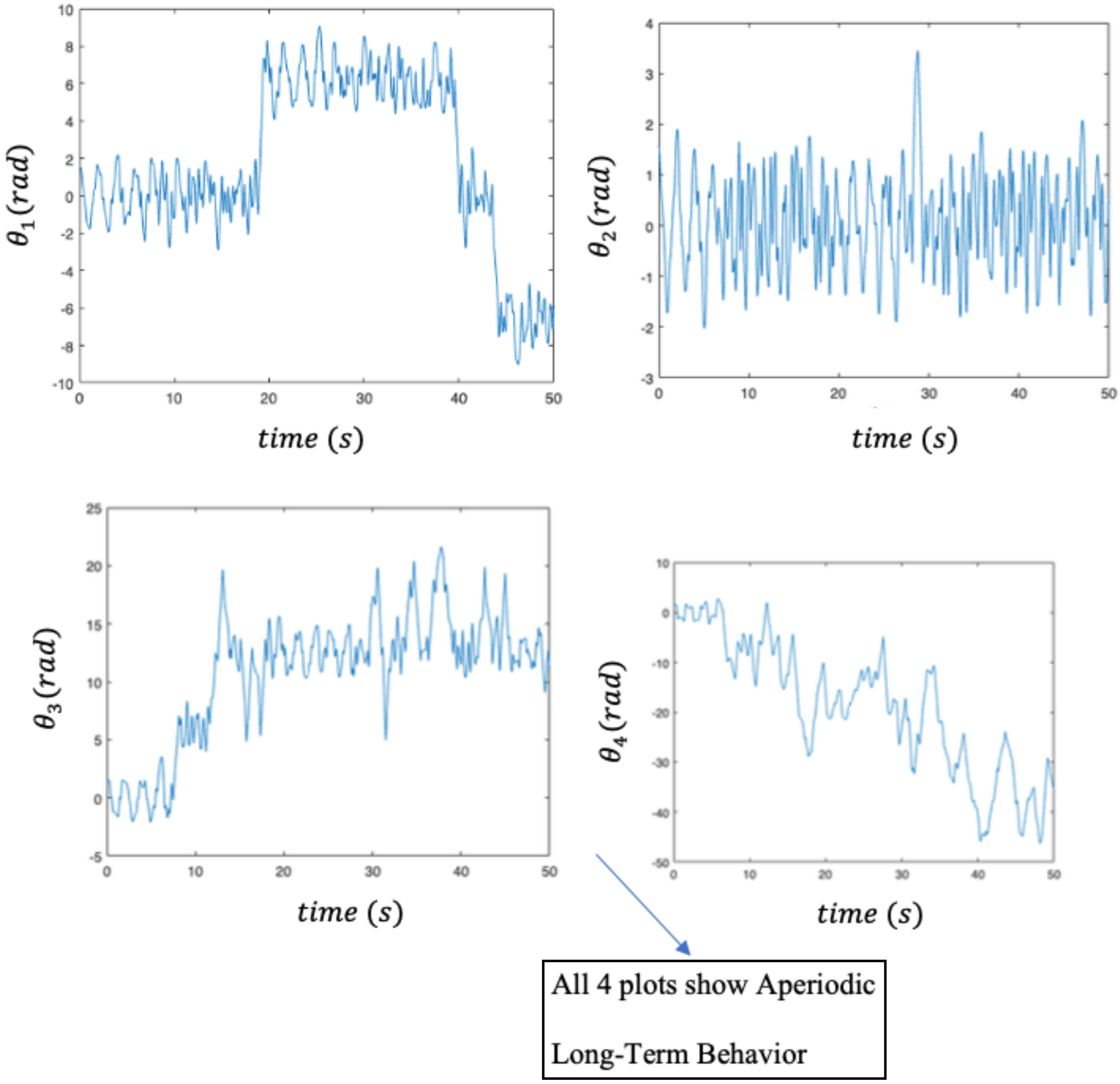


Figure 3.5: Plots for $\theta_1, \theta_2, \theta_3$ *and* θ_4 with time for $n = 4$



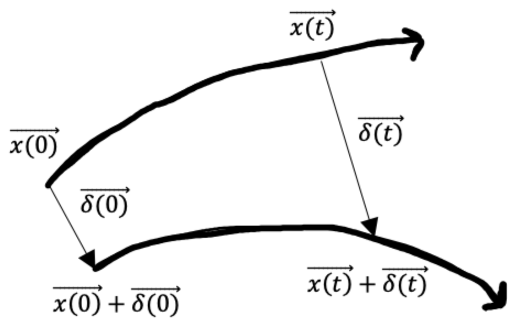
3.1.2 – Deterministic System

A deterministic system always produces the same output from a given set of initial conditions, because there are no random variables involved (Dronka). The pendulum is deterministic. (Shvets et Al. 1)

3.1.3 – Sensitive Dependence on Initial Conditions

"Sensitive dependence on initial conditions" means that nearby trajectories separate exponentially fast (Strogatz 331). This makes it impossible to predict the path of chaotic systems in the real world, because even an infinitesimally small perturbation from the initial conditions results in a totally different trajectory.

Figure 3.6: Illustrating sensitive dependence on initial conditions



In physical measurements, there is always an error/uncertainty while measuring quantities like angles. Assume that the magnitude of this uncertainty is $\left|\overrightarrow{\delta(0)}\right|$.

Suppose $\overrightarrow{x(0)}$ is a set of initial conditions which a chaotic system takes, and consider another set of initial conditions $\overrightarrow{x(0)} + \overrightarrow{\delta(0)}$, where $\overrightarrow{\delta(0)}$ is a small perturbation. For example, of magnitude 10^{-10}. After time t, the trajectories are represented by $\overrightarrow{x(t)}$ and $\overrightarrow{x(t)} + \overrightarrow{\delta(t)}$ (figure 3.6). Sensitive dependence on initial conditions means $\left|\overrightarrow{\delta(t)}\right|$ grows exponentially fast

To demonstrate this concept, a simulation of two double pendulums was run[5], one from initial conditions $\left[\frac{\pi}{2}; \frac{\pi}{2}; 0; 0\right]$, and another with initial conditions $\left[\frac{\pi}{2} + 10^{-10}; \frac{\pi}{2}; 0; 0\right]$[6]. *In both cases, mass of bobs=1kg and length of pendulum=1m.*

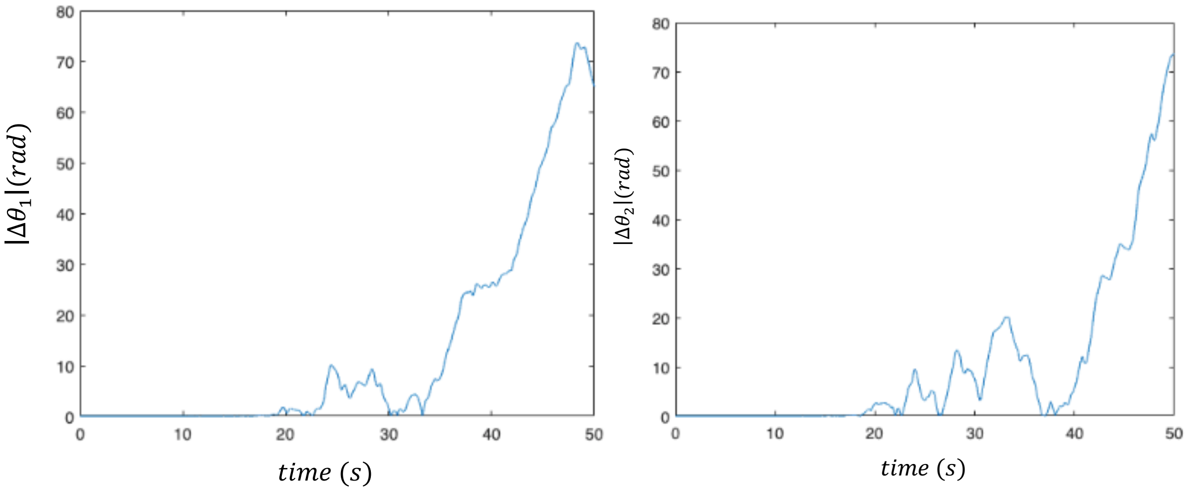
---

[5] Code used is in appendix A

[6] Recall the format of initial conditions from section-2.2

The difference in θ_1 and θ_2 between the respective trajectories was then plotted.

Figure 3.7: Plotting the difference between θ_1 and θ_2 of two pendulums $\left(\left|\overrightarrow{\delta(0)}\right| = 10^{-10}\right)$



Result: the divergence between the two trajectories was growing exponentially even though the difference in initial conditions was just 10^{-10}. This illustrated sensitive dependence on initial conditions.

3.2 - Time horizon of a chaotic system

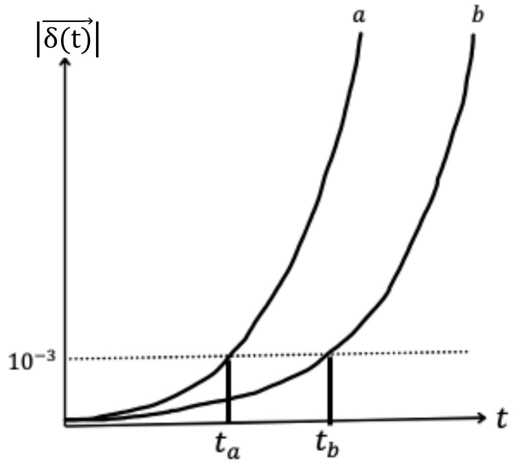
To quantify chaos in this essay, a property called the **"time horizon"** of the system will be used.

Time Horizon: The time step t_h at which the difference between two nearby trajectories, $\overrightarrow{x(t)}$, and $\overrightarrow{x(t)} + \overrightarrow{\delta(t)}$, becomes greater than 10^{-3} (10^{-3} was an arbitrary choice)

This is because, when $\left|\overrightarrow{\delta(t)}\right| > 10^{-3}$, the path $\overrightarrow{x(t)} + \overrightarrow{\delta(t)}$ is no longer an accurate prediction for the path $\overrightarrow{x(t)}$, all because of the initial perturbation $\overrightarrow{\delta(0)}$.

In chaotic systems, lower the time horizon, the more chaotic the system is (Strogatz 330). If $|\overrightarrow{\delta(t)}|$ of two chaotic systems, A, and B, with same magnitude of $\overrightarrow{\delta(0)}$ are plotted, then whichever graph crosses 10^{-3} first is more chaotic.

Figure 3.8: Illustrating Time Horizon for Systems A and B



Time Horizon of System A = t_a

Time Horizon of System B = t_b

Since $t_a < t_b$, system A is predictable for a lesser amount of time, and hence is more chaotic

3.3 - How will “chaotic behaviour” of a system be quantified?

In this essay, chaotic behaviour of a system will be quantified by plotting the relationship between the magnitude of initial perturbation $\overrightarrow{\delta(0)}$, and the resulting time horizon. The plot will be drawn for an n-pendulum with $n = 2, 3, 4, 5, and\ 6$, and the result will be analysed to quantify how chaotic behaviour of an n-pendulum changes as n is increased. Thus, the research question will be answered.

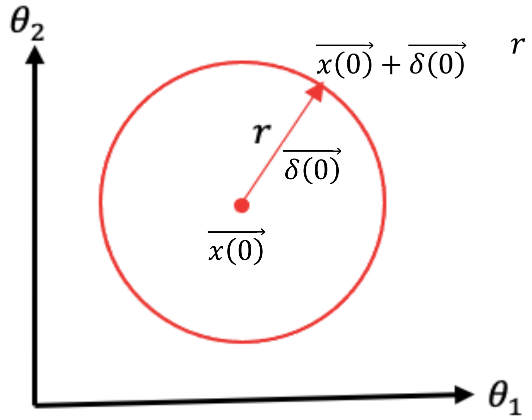
3.4 - Calculating Time Horizon Mathematically

To plot the relationship between $|\overrightarrow{\delta(0)}|$ and the resulting time horizon for an n-pendulum, we first need to formulate a method to mathematically calculate the time horizon for any $|\overrightarrow{\delta(0)}|$.

First, it will be formulated for a double pendulum ($n = 2$).

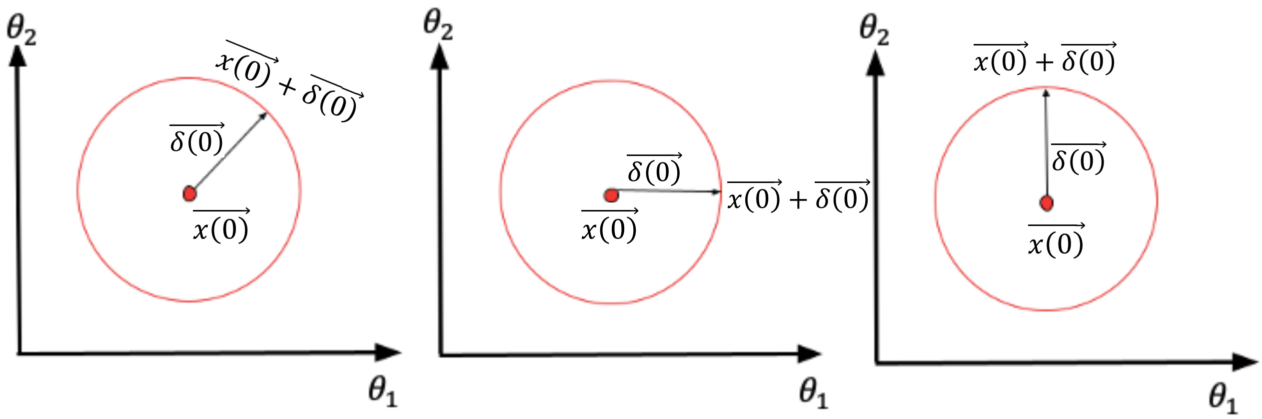
Firstly, it is important to note that $\overrightarrow{\delta(0)}$ is a **vector** because a perturbation of magnitude $\left|\overrightarrow{\delta(0)}\right|$ can occur in any direction. To generate the plots in figure 3.6, the initial perturbation of $\left|\overrightarrow{\delta(0)}\right|$ was made in the θ_1 direction. However, since a double pendulum has 2 degrees of freedom (θ_1 and θ_2), the perturbation could be in any combination of the two directions.

Figure 3.9: Various directions of initial perturbation $\overrightarrow{\delta(0)}$, when $n = 2$



In figure 3.9, the centre of the circle represents any set of initial conditions, $\overrightarrow{x(0)}$, and radius equals $\left|\overrightarrow{\delta(0)}\right|$. The circumference represents all possible states of $\overrightarrow{x(0)} + \overrightarrow{\delta(0)}$. Figure 3.10 shows three possible directions of $\overrightarrow{\delta(0)}$

Figure 3.10: Three possible directions of initial perturbation $\overrightarrow{\delta(0)}$, when $n = 2$



Different directions of $\overrightarrow{\delta(0)}$ will have different time horizons because the difference between trajectories $\overrightarrow{x(0)}$ and $\overrightarrow{x(0)} + \overrightarrow{\delta(0)}$ will grow at a different rate (Strogatz 329).

However, while taking measurements in real life, we cannot control the direction of the uncertainty $\left[\overrightarrow{\delta(0)}\right]$. Thus we must compute the time horizons for all directions of $\overrightarrow{\delta(0)}$ and choose the **minimum time horizon,** so that the system is predictable for *at least* that amount of time. Thus, an approach which considers all directions of $\overrightarrow{\delta(0)}$ needs to be applied to calculate the minimum time horizon.

Note: from now, "time horizon" will automatically refer to the "minimum time horizon"

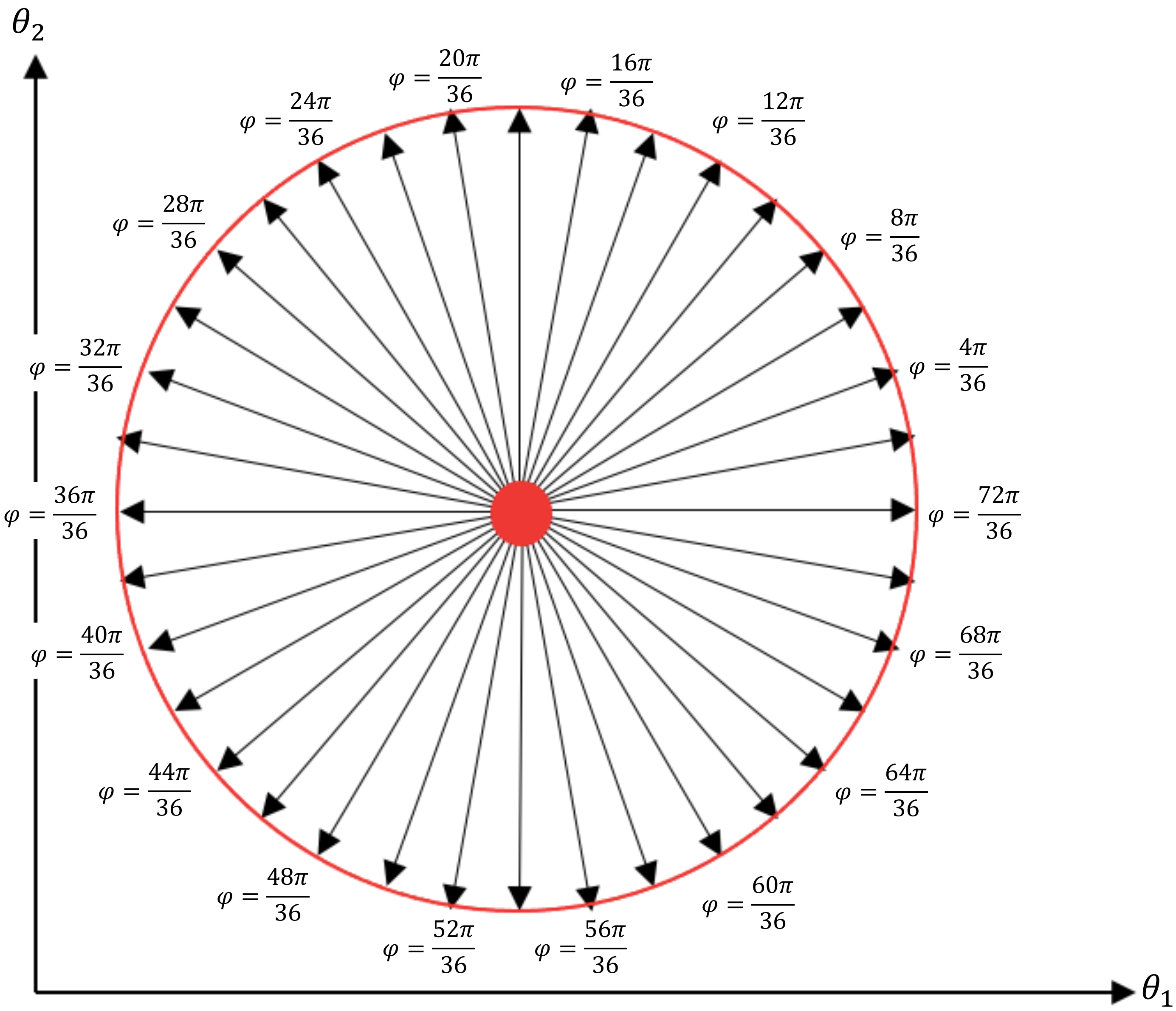
3.4.1 – The Distance Formula Approach

3.4.1.1 – Calculating for $n = 2$

To calculate the minimum time horizon for a double pendulum, we could consider 36 different directions of $\delta(0)$[7] (figure 3.11)

[7] Only 36 directions were considered due to computational restrictions

Figure 3.11: Demonstration of how 36 different directions of $\overrightarrow{\delta(0)}$ were simulated $(n = 2)$



Then we will run ode113 37 different times.

Once with initial conditions (without perturbation) - $\left[\frac{\pi}{2}; \frac{\pi}{2}; 0; 0\right]$. *Note: one can choose any value for initial angles θ and initial velocity $\dot{\theta}$. In this essay $\frac{\pi}{2}$ and 0 were arbitrary choices*

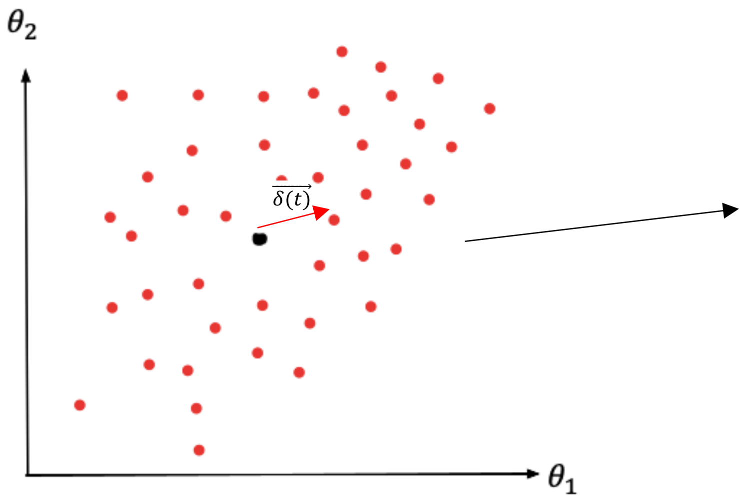
36 times for different directions of $\overrightarrow{\delta(0)}$, with initial conditions –

$$\left[\frac{\pi}{2} + \left(\left|\overrightarrow{\delta(0)}\right| \times \cos\varphi\right); \frac{\pi}{2} + \left(\left|\overrightarrow{\delta(0)}\right| \times \sin\varphi\right); 0; 0\right] \; where\ \varphi = \left(\frac{2\pi}{36}, \frac{4\pi}{36}, \frac{6\pi}{36} \dots \frac{72\pi}{36}\right)$$

After time t, we can plot the coordinates (θ_1 *and* θ_2) of the double pendulum for the initial conditions without perturbation, and the 36 initial conditions perturbed with different directions of $\overrightarrow{\delta(0)}$.

The generated plot may look like –

Figure 3.12: Example of location of 36 different directions of $\overrightarrow{\delta(0)}$ after time t $(n = 2)$



Black dot – Represents coordinates of pendulum for initial conditions $x(0)$ (without perturbation) after time t

Red dots – Represents coordinates of pendulums with initial conditions $x(0) + \overrightarrow{\delta(0)}$ after time t (with perturbation). There are 36 points because we have considered 36 different directions of $\overrightarrow{\delta(0)}$

Red arrow – Can be drawn for all points, equals $\overrightarrow{\delta(t)}$

Now, $\left|\overrightarrow{\delta(t)}\right|$ can be found by applying the distance formula to each red point individually

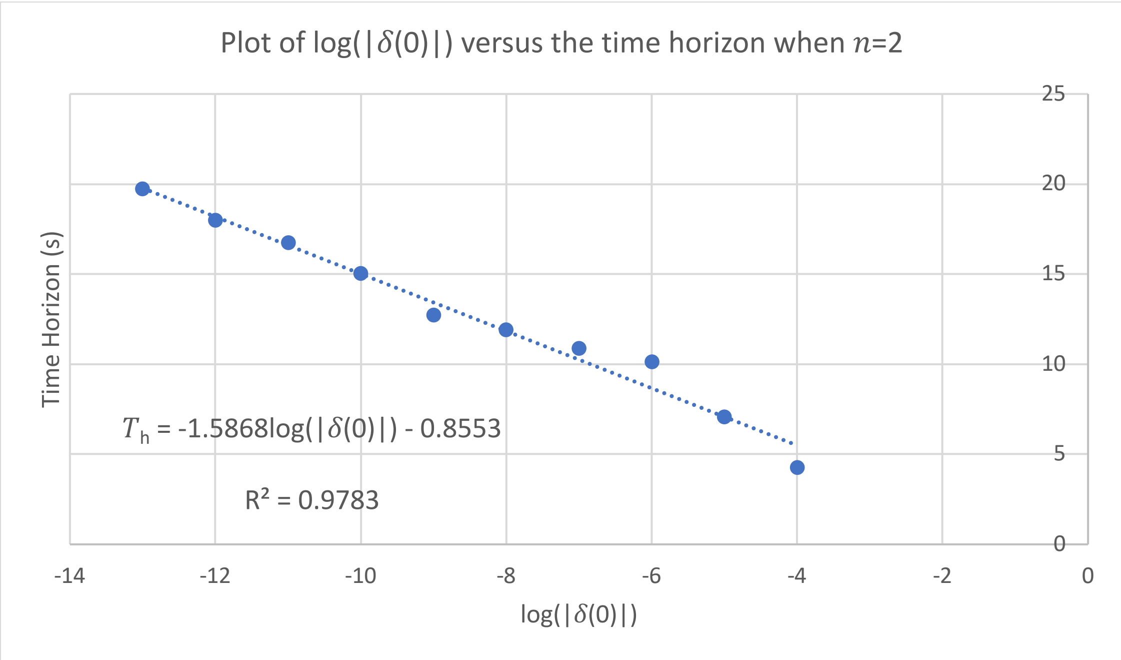
$$\left|\overrightarrow{\delta(t)}\right| = \sqrt{(\theta_{1f} - \theta_{1i})^2 + (\theta_{2f} - \theta_{2i})^2}$$

VARIABLE	MEANING
θ_{1f}	θ_1 of the red point
θ_{1i}	θ_1 of the black point
θ_{2f}	θ_2 of the red point
θ_{2i}	θ_2 of the black point

The time step at which $\left|\overrightarrow{\delta(t)}\right|$ for **any of the red points** becomes greater than 10^{-3} is the **minimum time horizon.**

Using this MATLAB code[8], the time horizons for $\left|\overrightarrow{\delta(0)}\right| = (10^{-13}, 10^{-12} \dots, 10^{-4})$ for a double pendulum ($n = 2$) were plotted.

Figure 3.13: Plot of log ($\left|\overrightarrow{\delta(0)}\right|$) versus the time horizon when $n = 2$



Equation of Trendline –

$$Time\ horizon\ (T_h) = -1.5868 \log \left|\overrightarrow{\delta(0)}\right| - 0.8553$$

This equation demonstrated the logarithmic dependence of the time horizon on the initial perturbation (result was reliable as $R^2 = 0.9783$). The trendline shows that even if one decreases $\left|\overrightarrow{\delta(0)}\right|$ by 10^{-10} times, the time horizon only increases by 15.868 seconds. This aligns with Chaos Theory: no matter how precisely one makes the initial measurement, they can only predict the future state of the system for a short time.

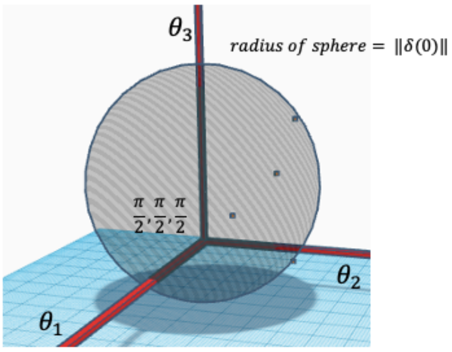
[8] Code can be found in appendix A

3.4.1.2 – Calculating for $n = 3$

The plot in figure 3.13 will now be generated for $n = 3$: The range of $\overrightarrow{\delta(0)}$ will be kept constant $(10^{-13}, 10^{-12}, 10^{-11}, \ldots 10^{-4})$

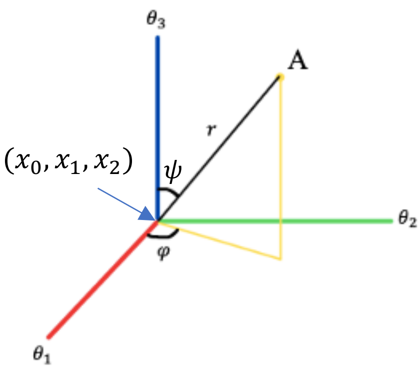
For a triple pendulum, $\overrightarrow{\delta(0)}$ is now 3-dimensional because there are three degrees of freedom, $\theta_1, \theta_2, and\ \theta_3$.

Figure 3.14: Demonstration of the directions the vector $\overrightarrow{\delta(0)}$ can be situated in when $n = 3$



All initial perturbations $\overrightarrow{\delta(0)}$ lie on the surface of this sphere with radius $\left|\overrightarrow{\delta(0)}\right|$. We can describe all $\overrightarrow{\delta(0)}$'s using spherical coordinates.

Figure 3.15: Spherical Coordinate System in 3-D



If point A is a possible direction of $\overrightarrow{\delta(0)}$, then it can be described in spherical coordinates as (r, ψ, φ) where $r = \left|\overrightarrow{\delta(0)}\right|$, ψ ranges from $[0, \pi]$, and φ ranges from $[0, 2\pi]$ (Blumenson 63-66)

The spherical coordinates can be converted into cartesian coordinates:

$(x_0 + \left|\overrightarrow{\delta(0)}\right| \cos\psi, x_1 + \left|\overrightarrow{\delta(0)}\right| \sin\psi\cos\varphi,\ x_2 + \left|\overrightarrow{\delta(0)}\right| \sin\psi\sin\varphi)$ (Blumenson 63-66)

Now, we can take 36 linearly spaced points for ψ *and* φ. So, the total number of directions of $\overrightarrow{\delta(0)}$ considered are $36^2 = 1296$

Then, ode113 will be run 1297 times:

Once with initial conditions (without perturbation) – $\left[\frac{\pi}{2}; \frac{\pi}{2}; \frac{\pi}{2}; 0; 0; 0\right]$

$\frac{\pi}{2}$ *and 0 have been chosen to remain consistent with* $n = 2$

1296 times for different directions of $\overrightarrow{\delta(0)}$, with initial conditions –

$\left[\frac{\pi}{2} + \left(|\overrightarrow{\delta(0)}| \times \cos\psi\right); \frac{\pi}{2} + \left(|\overrightarrow{\delta(0)}| \times \sin\psi\cos\varphi\right); \frac{\pi}{2} + \left(|\overrightarrow{\delta(0)}| \times \sin\psi\sin\varphi\right); 0; 0; 0\right]$

where $\varphi = (\frac{2\pi}{36}, \frac{4\pi}{36}, \frac{6\pi}{36}, \ldots \frac{72\pi}{36})$ and $\psi = (\frac{\pi}{36}, \frac{2\pi}{36}, \frac{3\pi}{36}, \ldots \frac{36\pi}{36})$

The time step when distance ($\left|\overrightarrow{\delta(t)}\right|$) between **any** of the 1296 perturbed points and the non-perturbed point becomes greater than 10^{-3} will be the time horizon.

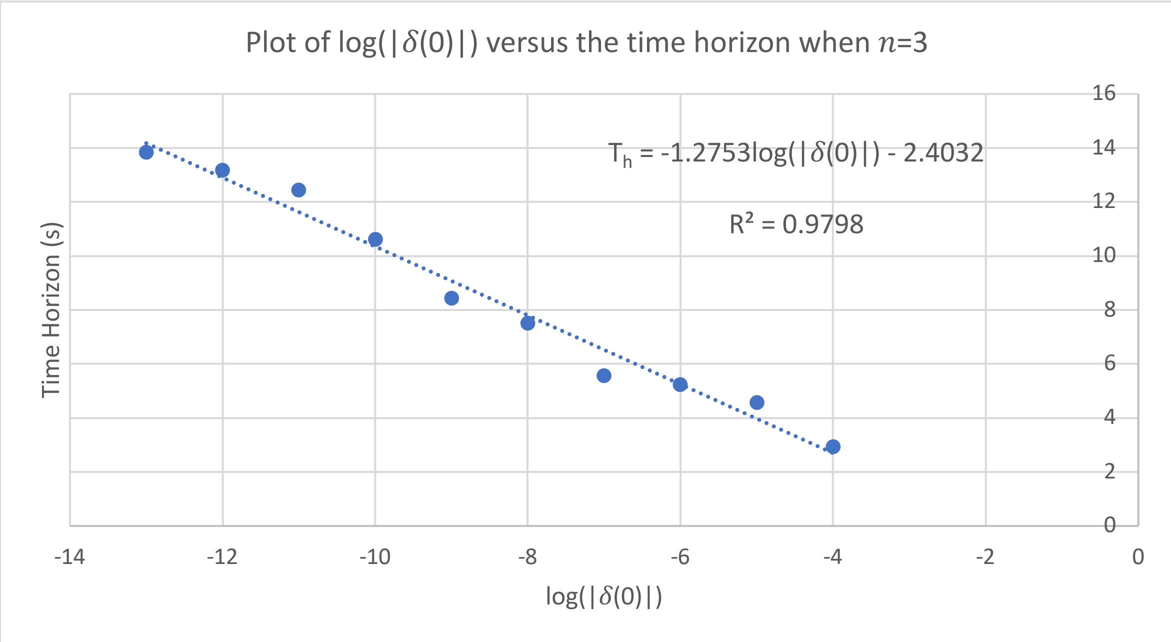
$\left|\overrightarrow{\delta(t)}\right|$ will be calculated by the 3D distance formula:

$$\left|\overrightarrow{\delta(t)}\right| = \sqrt{(\theta_{1f} - \theta_{1i})^2 + (\theta_{2f} - \theta_{2i})^2 + (\theta_{3f} - \theta_{3i})^2}$$

VARIABLE	**MEANING**
θ_{1f}	θ_1 of the point with perturbation
θ_{1i}	θ_1 of point without perturbation
θ_{2f}	θ_2 of the point with perturbation
θ_{2i}	θ_2 of point without perturbation
θ_{3f}	θ_3 of the point with perturbation
θ_{3i}	θ_3 of point without perturbation

The plot for $\left|\overrightarrow{\delta(0)}\right| = (10^{-13}, 10^{-12}, \dots 10^{-4})$, and the respective time horizon[9] is given below

Figure 3.16: Plot of log $\left(\left|\overrightarrow{\delta(0)}\right|\right)$ versus the time horizon when $n = 3$



Equation of Trendline -

$$T_h = -1.2753 \log\left|\overrightarrow{\delta(0)}\right| - 2.4032$$

3.4.1.3 – Calculating for $n = 4, 5,$ and 6

For $n = 4, 5,$ *and* 6, the degrees of freedom are $4, 5,$ *and* 6 respectively. Hence, the initial perturbation vector $\overrightarrow{\delta(0)}$ will be situated in $4, 5,$ *and* 6 dimensions respectively. This is impossible to represent diagrammatically. However, the same approach which was used for $n = 2$ and $n = 3$ will be used.

---

[9] Code is in appendix A

Let us make that approach more abstract so it can be applied in higher dimensions –

The number of angles (A) we define to describe $\overrightarrow{\delta(0)}$ in an n-dimensional space using spherical coordinates is $A = n - 1$. For instance, when $n = 3$ we used 2 angles.

Next, we describe any $\overrightarrow{\delta(0)}$ in an n-dimensional space using spherical coordinates: $(r, \varphi_1, \varphi_2, \varphi_3, \dots \varphi_{n-1})$ where $r = \left|\overrightarrow{\delta(0)}\right|$, the angles $\varphi_1, \varphi_2 \dots \varphi_{n-2}$ range over $[0, \pi]$ and the angle φ_{n-1} ranges over $[0, 2\pi]$ (Blumenson 63-66)

These spherical coordinates can be converted into cartesian coordinates (Blumenson 63-66):

$$x_1 = r\cos\varphi_1$$

$$x_2 = r\sin\varphi_1\cos\varphi_2$$

$$x_3 = r\sin\varphi_1\sin\varphi_2\cos\varphi_3$$

$$\dots$$

$$x_{n-1} = r\sin\varphi_1 \dots \sin\varphi_{n-2}\cos\varphi_{n-1}$$

$$x_n = r\sin\varphi_1 \dots \sin\varphi_{n-2}\sin\varphi_{n-1}$$

Next, we will take 36 linearly spaced points on each angle. So, the number of directions of $\overrightarrow{\delta(0)}$ considered in n-dimensions will be 36^{n-1} (there are $n - 1$ angles.)

Then ode113 will be run $36^{n-1} + 1$ times, once with initial conditions:

$$\left[\underbrace{\frac{\pi}{2}; \frac{\pi}{2}; \frac{\pi}{2} \dots \frac{\pi}{2}}_{n\ times}; \underbrace{0; 0; 0 \dots; 0}_{n\ times}\right] \longrightarrow$$

$\frac{\pi}{2}$ and 0 have been chosen to remain consistent with $n = 2$ and $n = 3$.

And 36^{n-1} times with initial conditions:

$$\begin{bmatrix} \frac{\pi}{2} + \|\delta(0)\| \ \cos\varphi_1; \\ \frac{\pi}{2} + \|\delta(0)\| \ \sin\varphi_1 \cos\varphi_2 \ ; \\ \frac{\pi}{2} + \|\delta(0)\| \ \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\ \dots \\ \frac{\pi}{2} + \|\delta(0)\| \ \sin\varphi_1 \dots \sin\varphi_{n-2} \cos\varphi_{n-1} \ ; \\ \frac{\pi}{2} + \|\delta(0)\| \ \sin\varphi_1 \dots \sin\varphi_{n-2} \sin\varphi_{n-1}; \\ 0; \\ 0; \\ 0; \\ \dots \\ 0 \end{bmatrix}^T$$

Where

$$\varphi_1, \varphi_2 \dots \varphi_{n-2} = \left[\frac{\pi}{36}, \frac{2\pi}{36} \dots \frac{36\pi}{36}\right]$$

$$\varphi_{n-1} = \left[\frac{2\pi}{36}, \frac{4\pi}{36} \dots \frac{72\pi}{36}\right]$$

Then, we will use the distance formula in the n-dimensional space for all 36^{n-1} points to calculate $\left|\overrightarrow{\delta(t)}\right|$, and the time step at which **any** $\left|\overrightarrow{\delta(t)}\right| > 10^{-3}$ will be the time horizon.

Distance formula in the nth dimension –

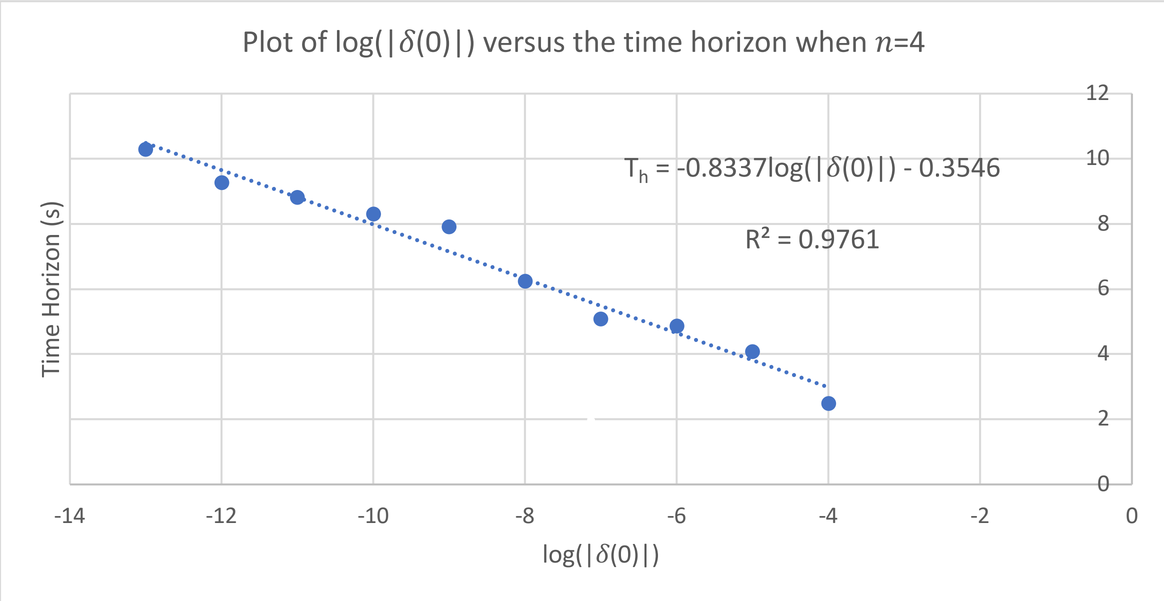
$$\left|\overrightarrow{\delta(t)}\right| = \sqrt{\sum_{k=1}^{n} \left(\theta_{kf} - \theta_{ki}\right)^2}$$

VARIABLE	MEANING
θ_{kf}	θ_k of the perturbed point ($k = 1, 2 \dots n$)
θ_{ki}	θ_k of the non-perturbed point ($k = 1, 2 \dots n$)

The plots for $n = 4, 5, and\ 6$ are given below[10] –

[10] Code is in appendix A

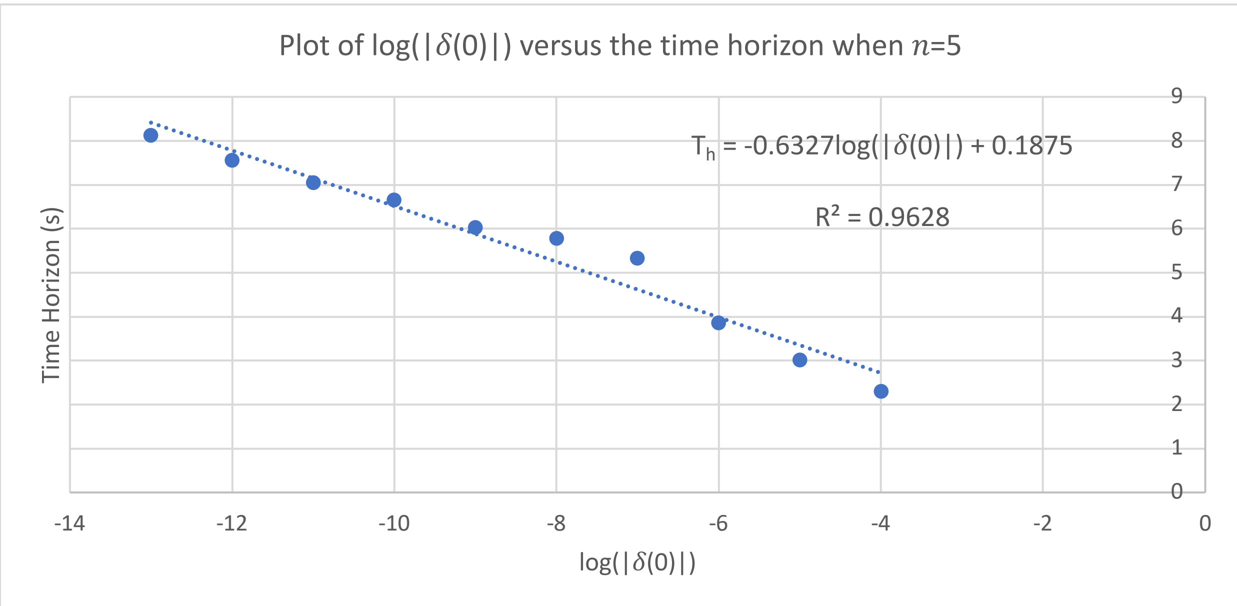
Figure 3.17: Plot of log $\left(\left|\overrightarrow{\delta(0)}\right|\right)$ versus the time horizon when $n = 4$



Equation of Trendline:

$$T_h = -0.8337 \log\left(\left|\overrightarrow{\delta(0)}\right|\right) - 0.3546$$

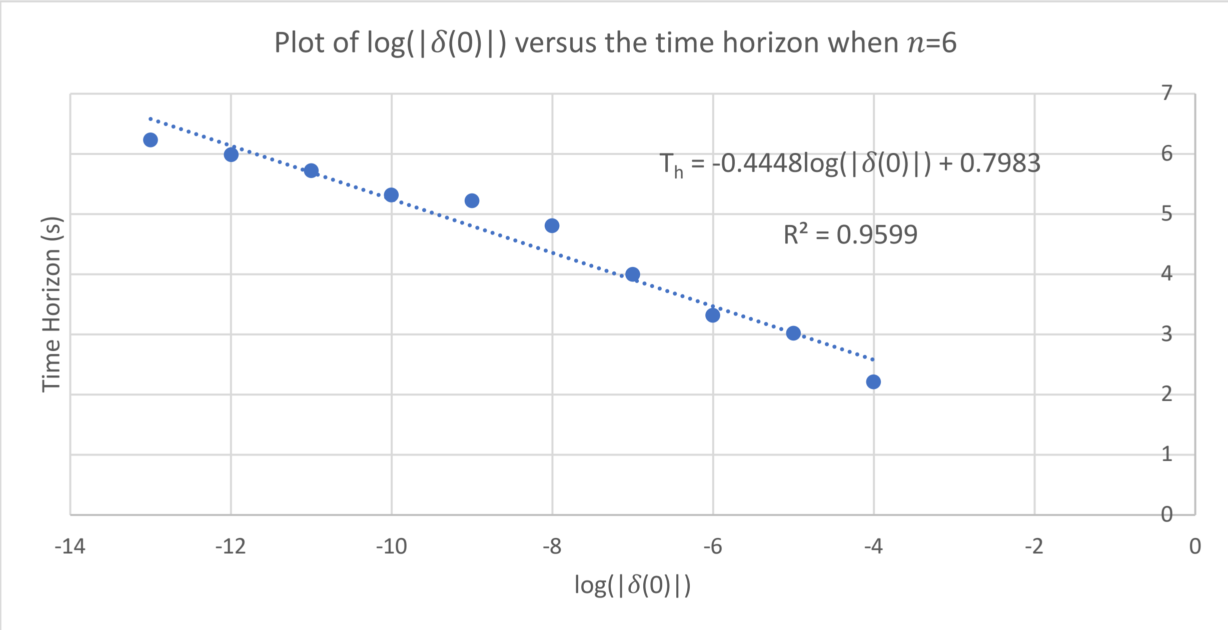
Figure 3.18: Plot of log $\left(\left|\overrightarrow{\delta(0)}\right|\right)$ versus the time horizon when $n = 5$



Equation of Trendline:

$$T_h = -0.6327 \log\left|\overrightarrow{\delta(0)}\right| + 0.1875$$

Figure 3.19: Plot of log $\left(\left|\overrightarrow{\delta(0)}\right|\right)$ versus the time horizon when $n = 6$



Equation of Trendline:

$$T_h = -0.4448 \log\left|\overrightarrow{\delta(0)}\right| + 0.7983$$

3.5 – Comparison of how chaotic behaviour of an n-pendulum changes as the value of n is increased

In this section, the plots for log $\left(\left|\overrightarrow{\delta(0)}\right|\right)$ versus time horizon for pendulums with $n = 2, 3, 4, 5, and\ 6$ will be compared (figures $3.13, 3.16, 3.17, 3.18, 3.19$) and conclusions regarding the research question will be drawn

Table 3.1: Table showcasing all data points obtained for the time horizons for $n = 2, 3, 4, 5, and\ 6$ for different values of $\log(|\overrightarrow{\delta(0)}|)$

		Time Horizon				
$\lvert\overrightarrow{\delta(0)}\rvert$	$\log(\lvert\overrightarrow{\delta(0)}\rvert)$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
10^{-13}	-13	19.703	13.836	10.292	8.125	6.232
10^{-12}	-12	17.966	13.179	9.26	7.553	5.981
10^{-11}	-11	16.751	12.449	8.809	7.042	5.723
10^{-10}	-10	15.023	10.622	8.298	6.651	5.312
10^{-9}	-9	12.706	8.441	7.916	6.019	5.213
10^{-8}	-8	11.881	7.511	6.235	5.778	4.801
10^{-7}	-7	10.849	5.572	5.074	5.326	3.991
10^{-6}	-6	10.126	5.245	4.861	3.852	3.312
10^{-5}	-5	7.057	4.585	4.078	3.009	3.012
10^{-4}	-4	4.26	2.929	2.494	2.3	2.212

The data for different values of n can be plotted on a single axis to graphically compare the chaotic behaviour

Figure 3.20: Plot of time horizon versus $\log\left(\left|\overrightarrow{\delta(0)}\right|\right)$ for different values of n on the same axis

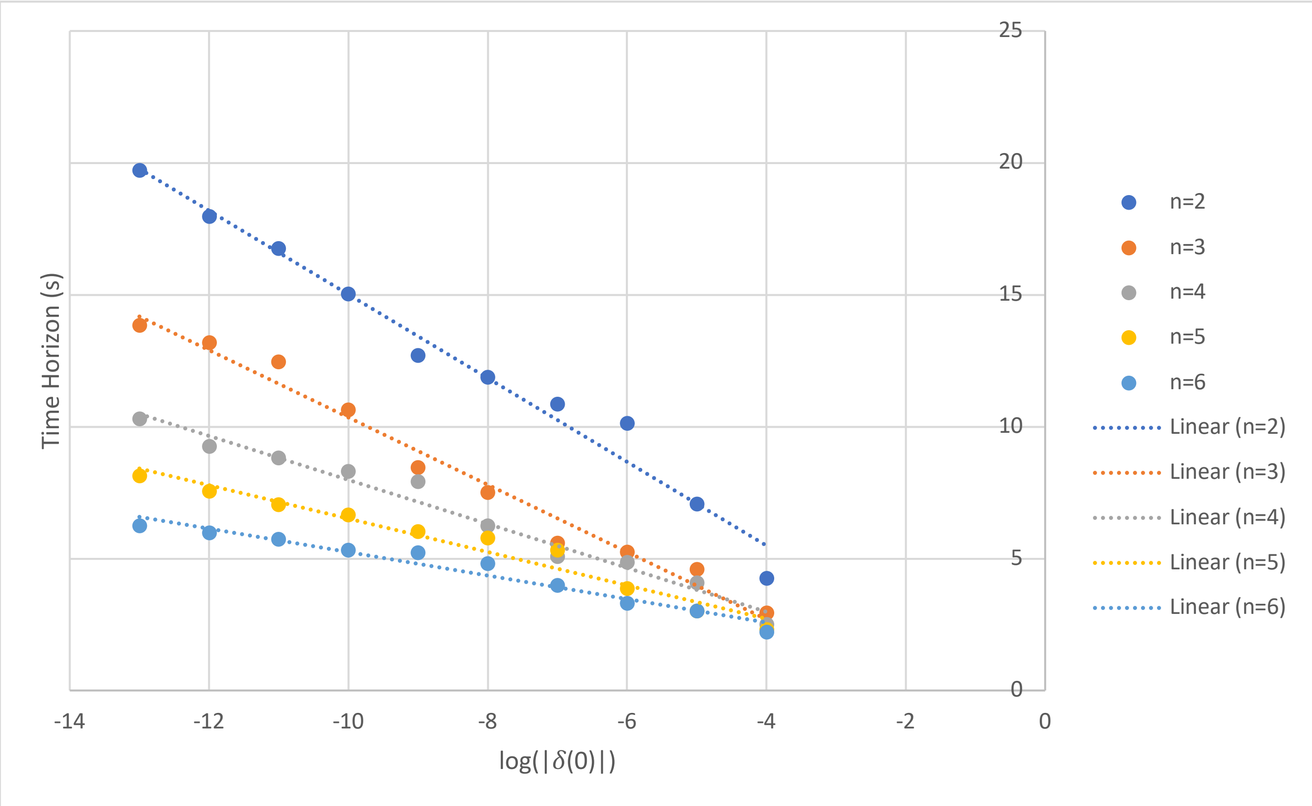


Figure 3.20 suggests that as n increases, the gradient of the trendline decreases, which indicates that the same decrease in initial perturbation $\left|\overrightarrow{\delta(0)}\right|$ leads to a smaller increase in time horizon. **This shows that as n increases, an n-pendulum becomes more chaotic.** In the following paragraphs, we will quantify exactly how much 'more' chaotic it gets.

Figure 3.20 shows that as n increases, there is roughly a proportional decrease in the time horizon of the n-pendulum for a fixed value of $\left|\overrightarrow{\delta(0)}\right|$

Thus, for any $\left|\overrightarrow{\delta(0)}\right|$, the relationship between Time horizon for any value of n (T_n) and the time horizon for when $n = 2$ (T_2) can be approximated through the equation:

$$T_n = \frac{1}{k_n} \times T_2$$

Where k_n is a proportionality constant

Hence,

$$k_n = \frac{T_2}{T_n} \tag{39}$$

Using (39), we can calculate the proportionality constant for any value of n and $\left|\overrightarrow{\delta(0)}\right|$. For example, when $n = 3$ and $\left|\overrightarrow{\delta(0)}\right| = 10^{-13}$, the proportionality constant becomes:

$$k_3 = \frac{T_2}{T_3}$$

Substituting values from table 3.1:

$$k_3 = \frac{19.703}{13.836} = 1.424$$

Using the same process, the proportionality constants of each value of n at each value of $|\overrightarrow{\delta(0)}|$ were calculated.

Table 3.2: Table showcasing the proportionality constants for each value of n

	Proportionality Constants				
$\log(\lvert\overrightarrow{\delta(0)}\rvert)$	$k_2 = \left(\frac{T_2}{T_2}\right)$	$k_3 = \left(\frac{T_2}{T_3}\right)$	$k_4 = \left(\frac{T_2}{T_4}\right)$	$k_5 = \left(\frac{T_2}{T_5}\right)$	$k_6 = \left(\frac{T_2}{T_6}\right)$
-13	1	1.424	1.914	2.425	3.162
-12	1	1.363	1.94	2.379	3.004
-11	1	1.345	1.902	2.379	2.927
-10	1	1.414	1.81	2.259	2.828
-9	1	1.505	1.605	2.111	2.437
-8	1	1.582	1.906	2.056	2.474
-7	1	1.947	2.138	2.037	2.718
-6	1	1.931	2.083	2.629	3.057
-5	1	1.539	1.731	2.345	2.343
-4	1	1.455	1.709	1.852	1.926
MEAN PROPORTIONALITY CONSTANT	**1**	**1.551**	**1.874**	**2.247**	**2.688**
STANDARD DEVIATION	**0**	**0.206**	**0.157**	**0.218**	**0.367**

The mean proportionality Constants were then plotted with n

Figure 3.21: Plot of the mean proportionality constant versus the value of n

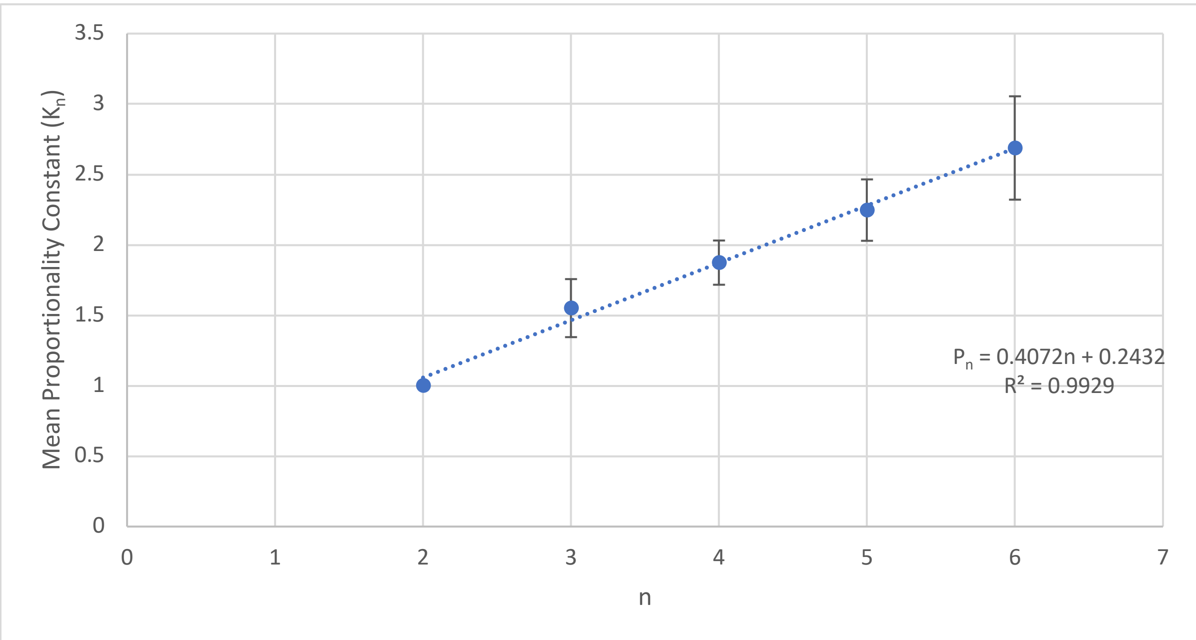


Figure 3.21 shows that the mean proportionality constants have a linear correlation with n, indicating that **as the value of n increases, the chaotic behaviour of pendulums increases linearly.** The result is reliable as the plot has an R^2 value of 0.9929. **This answers the research question.**

Conclusion, Extensions, and Limitations of Research

Through the investigation, **the research question can be answered: in an n-pendulum, as the value of n increases, the chaotic behaviour of the system increases linearly.** However, the result is **provisional** because in this essay, only pendulums with $2 \leq n \leq 6$ $(n \in Z^+)$ were investigated (due to computational restrictions). Hence, the linear relationship may not continue for larger values of n. Conducting a similar investigation for larger values of n would make the result more reliable.

Moreover, to generate data in table 3.1, initial values of θ_i and $\dot{\theta}_i$ were taken as $\frac{\pi}{2}$ and 0 respectively $[i = (1, 2 \dots n)]$, while the mass and length were assumed as 1kg and 1m. Different combinations of the same could lead to different answers. For instance, if the mass of one bob was significantly larger than the others, would the chaotic behaviour change? Such questions could be explored as an extension. They were not investigated in this essay because despite renting a cloud computer from Microsoft Azure, sufficient computational power was not available.

Furthermore, the results in table 3.1 could be improved by considering a larger number of directions of $\overrightarrow{\delta(0)}$ per $\left|\overrightarrow{\delta(0)}\right|$. However, we considered 36^{n-1} directions which is sufficient.

Additionally, this essay used ‘time horizons’ to quantify chaotic behaviour. As an extension, other methods like Lyapunov Exponents could be explored to further testify the result.

Overall, despite certain limitations, in this essay, by developing a numerical simulation for an n-pendulum, quantifying ‘chaotic behaviour’, and then determining its relationship with the value of n, I could successfully answer the research question: As the value of n increases, the chaotic behaviour of an n-pendulum increases **linearly.** Although, one must note the provisional nature of this conclusion as only pendulums with $2 \leq n \leq 6$ were investigated.

APPENDIX A: Relevant Links

The code used throughout this essay can be found on the following link -

https://github.com/Anonymous-EE/extended-essay

The google drive link for the simulations of the pendulums:

https://drive.google.com/drive/folders/1QHTV9whHUUKzC2tDfo_vFMq-BTdpFwjl?usp=sharing

APPENDIX B: Why is substituting the Lagrangian into the Euler Lagrange Equation a valid step?

We can perform this step, because substituting the Lagrangian into the Euler-Lagrange Equation is equivalent to Newton's second law, $F = ma$.

Proof: Consider a one dimensional system, such as a ball being thrown straight up in the air

The Lagrangian (L) equals the total kinetic energy (T) minus the total potential energy (V) of the system.

$$L = T - V$$

$$L = \frac{1}{2}m\dot{y}^2 - V(y)$$

Recalling the Euler Lagrange Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

Thus,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y}$$

Substituting the Lagrangian, this becomes

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2 - V(y)\right)\right) = \frac{\partial}{\partial y}\left(\frac{1}{2}m\dot{y}^2 - V(y)\right)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right)\right) = \frac{\partial}{\partial y}\big(-V(y)\big)$$

Simplifying the equation using chain rule:

$$\frac{d}{dt}(m\dot{y}) = \frac{\partial}{\partial y}\big(-V(y)\big)$$

$$m\ddot{y} = -\frac{dV}{dy}$$

Since $\ddot{y}$ = acceleration in the y-direction, and $F = -\frac{dV}{dy}$ for a one-dimensional system (Morin, ch. 6.1), this can be written as

$$ma = F$$

This proves that substituting the Lagrangian into the Euler-Lagrange equation is equivalent to $F = ma$, at least for a one dimensional system represented by cartesian coordinates.

The equivalence to $F = ma$ can be proven for higher dimensional systems, and even for generalized coordinates (Morin, ch. 6.4) (which satisfy conditions in table 1 in the main body of the essay). However, these proofs are high in complexity and are beyond the scope of this investigation.

APPENDIX C: Full Algebraic Working for a few calculations:

To find Equation (20):

$$T = \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2)$$

Substituting (16), (17), (18), and (19):

$$T = \frac{1}{2}m_1((\dot{\theta}_1\frac{l}{2}\cos\theta_1)^2 + (\dot{\theta}_1\frac{l}{2}\sin\theta_1)^2) + \frac{1}{2}m_2((\dot{\theta}_1\frac{l}{2}\cos\theta_1 + \dot{\theta}_2\frac{l}{2}\cos\theta_2)^2$$

$$+(\dot{\theta}_1\frac{l}{2}\sin\theta_1 + \dot{\theta}_2\frac{l}{2}\sin\theta_2)^2)$$

Expanding the Brackets:

$$T = \frac{1}{2}m_1(\dot{\theta}_1{}^2\frac{l^2}{4}cos^2\,\theta_1 + \dot{\theta}_1{}^2\frac{l^2}{4}sin^2\,\theta_1) + \frac{1}{2}m_2(\dot{\theta}_1{}^2\frac{l^2}{4}cos^2\,\theta_1 + \dot{\theta}_2{}^2\frac{l^2}{4}cos^2\theta_2$$

$$+ 2\dot{\theta}_1\dot{\theta}_2\frac{l^2}{4}\cos\theta_1\cos\theta_2 + + \dot{\theta}_1{}^2\frac{l^2}{4}sin^2\,\theta_1 + \dot{\theta}_2{}^2\frac{l^2}{4}sin^2\,\theta_2 + 2\dot{\theta}_1\dot{\theta}_2\frac{l^2}{4}\sin\theta_1\sin\theta_2)$$

Simplifying:

$$T = \frac{1}{2}m_1(\dot{\theta}_1{}^2\frac{l^2}{4}(cos^2\,\theta_1 + sin^2\,\theta_1) + \frac{1}{2}m_2(\dot{\theta}_1{}^2\frac{l^2}{4}(cos^2\,\theta_1 + sin^2\,\theta_1) +$$

$$\dot{\theta}_2{}^2\frac{l^2}{4}(cos^2\theta_2 + sin^2\,\theta_2) + 2\dot{\theta}_1\dot{\theta}_2\frac{l^2}{4}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2))$$

Using Trigonometric identities:

$$T = \frac{1}{2}m_1(\dot{\theta}_1{}^2\frac{l^2}{4}) + \frac{1}{2}m_2(\dot{\theta}_1{}^2\frac{l^2}{4} + \dot{\theta}_2{}^2\frac{l^2}{4} + 2\dot{\theta}_1\dot{\theta}_2\frac{l^2}{4}(\cos(\theta_1 - \theta_2))$$

Finally, we get:

$$T = \frac{1}{2}\frac{l^2}{4}(m_1\dot{\theta}_1{}^2 + m_2(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2(\cos(\theta_1 - \theta_2)) \qquad (20)$$

To find Equation (22):

Recalling (21):

$$L = \frac{1}{2}\frac{l^2}{4}\left(m_1\dot{\theta_1}^2 + m_2\left[\dot{\theta_1}^2 + \dot{\theta_2}^2 + 2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1 - \theta_2)\right]\right) + \frac{l}{2}g(m_1 cos\theta_1 + \ m_2[cos\theta_1 + cos\theta_2]) \quad (21)$$

Solving the Euler-Lagrange equation for θ_1

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_1}}\right) - \frac{\partial L}{\partial\theta_1} = 0\right) \quad \text{(a)}$$

Using (21):

$$\frac{\partial L}{\partial\dot{\theta_1}} = \frac{1}{2}\frac{l^2}{4}(2m_1\dot{\theta_1} + \ 2m_2\dot{\theta_1} + \ 2m_2\dot{\theta_2}\cos(\theta_1 - \theta_2))$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_1}}\right) = \frac{l^2}{4}(m_1\ddot{\theta_1} + m_2\ddot{\theta_1} + m_2\ddot{\theta_2}\cos(\theta_1 - \theta_2) - \ m_2\dot{\theta_2}\sin(\theta_1 - \theta_2)(\dot{\theta_1} - \ \dot{\theta_2})) \quad (b)$$

$$\frac{\partial L}{\partial\theta_1} = \ -m_2\frac{l^2}{4}\dot{\theta_1}\dot{\theta_2}sin(\theta_1 - \theta_2) - \frac{l}{2}gm_1sin\theta_1 - \frac{l}{2}gm_2sin\theta_1 \quad (c)$$

Substituting (b) and (c) into (a), we obtain the first equation of motion

$$\frac{l^2}{4}\ddot{\theta_2}[m_1 + m_2] + \frac{l^2}{4}m_2\left[\ddot{\theta_2}\cos(\theta_1 - \theta_2) + \ \dot{\theta_2}^2\sin(\theta_1 - \theta_2)\right] + \frac{l}{2}gsin\theta_1[m_1 + m_2] = 0 \quad (22)$$

To find Equation (23):

Solving the Euler-Lagrange equation for θ_2

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_2}}\right) - \frac{\partial L}{\partial\theta_2} = 0\right) \quad \text{(d)}$$

Using (21)

$$\frac{\partial L}{\partial \dot{\theta_2}} = \frac{l^2}{4}(m_2\dot{\theta}_2 + m_2\dot{\theta}_1\cos(\theta_1 - \theta_2))$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_2}}\right) = \frac{l^2}{4}m_2\ddot{\theta_2} + \frac{l^2}{4}m_2\ddot{\theta_1}\cos(\theta_1 - \theta_2) - \frac{l^2}{4}m_2\dot{\theta_1}\sin(\theta_1 - \theta_2)(\dot{\theta_1} - \dot{\theta_2}) \quad (e)$$

$$\frac{\partial L}{\partial \theta_2} = +\frac{l^2}{4}m_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - \frac{l}{2}gm_2\sin(\theta_2) \quad (f)$$

Substituting (e) and (f) into (d), we obtain the second equation of motion:

$$\frac{l^2}{4}m_2\ddot{\theta_2} + \frac{l^2}{4}m_2\ddot{\theta_1}\cos(\theta_1 - \theta_2) - \frac{l^2}{4}m_2\dot{\theta_1}^2\sin(\theta_1 - \theta_2) + \frac{l}{2}gm_2\sin(\theta_2) = 0 \quad (23)$$

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