University of California Berkeley
MATH228B: Numerical Solution of Differential Equations – Spring 2020

Project 4

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1 Problem 1

For the first problem, please look at the file `fempoitst.m` which produces the following contour plots for the difference meshes:

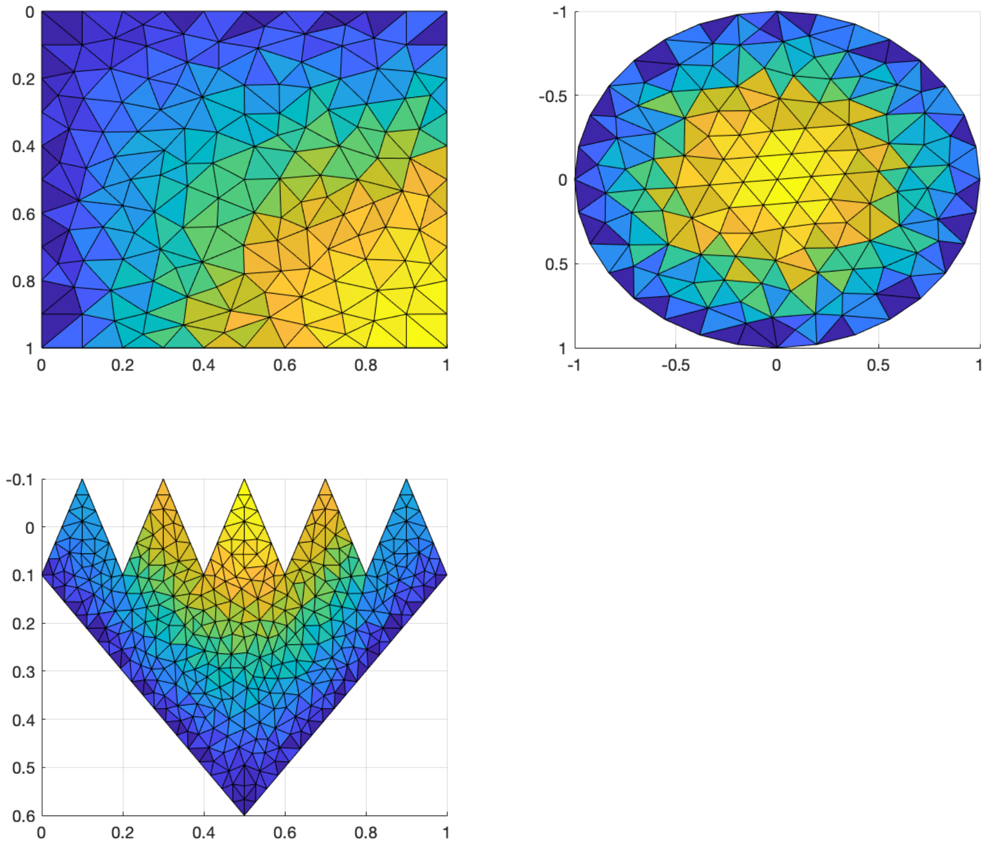

Figure 1: Solution of the poisson equation on the given meshes.

2 Problem 2

For the convergence study, please look at the file `poiconvtst.m`. Executing this file produces a convergence rate estimate of **2.0** for the square domain and **1.1** for the generic polygon domain.

The convergence plot is shown below,

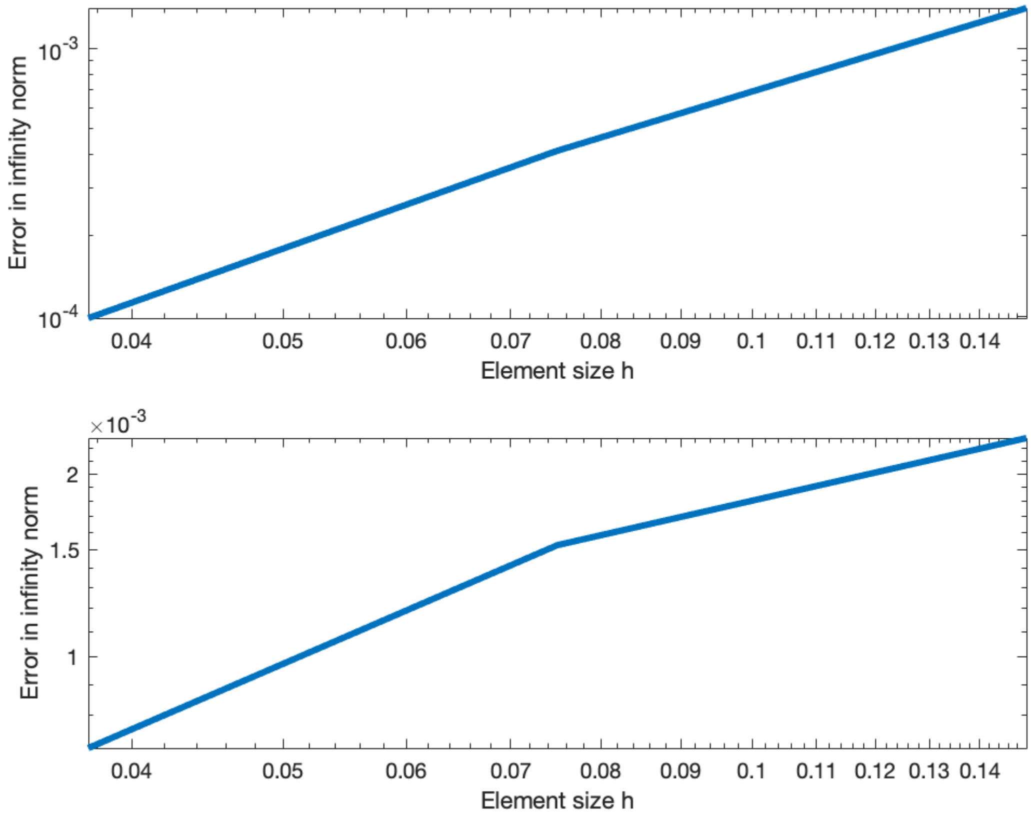

Figure 2: Convergence plots for the poisson problem. Top figure is convergence on the square domain. Bottom figure is convergence on the general polygon domain.

3 Problem 3

3.1 a) Weak form of the equation

We want the solution to,

$$\begin{aligned} \frac{d^4u}{dx^4} &= f(x) \qquad x \in (0,1) \\ u(0) = u(1) = \frac{du}{dx}(0) &= \frac{du}{dx}(1) = 0 \end{aligned} \tag{1}$$

Consider the space of functions,

$$V_h = \left\{ v \in H^1 : v(0) = v(1) = \frac{dv}{dx}(0) = \frac{dv}{dx}(1) = 0 \right\} \tag{2}$$

then multiplying eq. 1 by $v \in V_h$ and integrating over $(0, 1)$,

$$\begin{aligned}
\int_0^1 vf dx &= \int_0^1 v\frac{d^4u}{dx^4}dx \\
&= \underbrace{\left[v\frac{d^3u}{dx^3}\right]_0^1}_{=0} - \int_0^1 \frac{dv}{dx}\frac{d^3u}{dx^3}dx \\
&= -\underbrace{\left[\frac{dv}{dx}\frac{d^2u}{dx^2}\right]_0^1}_{=0} + \int_0^1 \frac{d^2v}{dx^2}\frac{d^2u}{dx^2}dx
\end{aligned}$$

So the weak form is to seek $u_h \in V_h$ such that,

$$\int_0^1 v''(x)u''(x)\ dx = \int_0^1 vf\ dx \qquad \forall v \in V_h \tag{3}$$

3.2 b) Defining a basis

For the given triangulation, we are looking for cubic polynomials over the given domain $[0, 0.5]$ and $[0.5, 1]$ with continuous first derivative at $x = 0.5$

First consider a basis over the reference domain $[-1, 1]$. A generic cubic polynomial can be written as,

$$v(x) = \sum_{j=0}^{3} V_j x^j$$

for some coefficients V_j. Thus, the function space over a given element has 4 degrees of freedom. Thus we need 4 independent functions to construct a basis.

The polynomial can alternatively be represented in terms of its value and the value of its derivative at the element boundaries $-1, +1$.

Consider the basis $\{N_0, N_1, N_2, N_3\}$ such that,

$$N_i = \sum_{j=0}^{3} C_{ij} x^j$$

Now we place the following conditions on the basis functions to determine their coefficients,

$$\begin{aligned}
N_0(-1) &= 1 \qquad & N_1(-1) = N_2(-1) = N_3(-1) = 0 \\
N_1'(-1) &= 1 \qquad & N_0'(-1) = N_2'(-1) = N_3'(-1) = 0 \\
N_2(+1) &= 1 \qquad & N_0(+1) = N_1(+1) = N_3(+1) = 0 \\
N_3'(+1) &= 1 \qquad & N_0'(+1) = N_1'(+1) = N_2'(+1) = 0
\end{aligned}$$

This gives us the following system of equations,

$$\begin{bmatrix} C_{00} & C_{10} & C_{20} & C_{30} \\ C_{01} & C_{11} & C_{21} & C_{31} \\ C_{02} & C_{12} & C_{22} & C_{32} \\ C_{03} & C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix}^{-1}$$

This gives the basis functions in coefficient form as,

$$\begin{aligned} N_0(x) &= \frac{1}{4}(2 - 3x + x^3) \\ N_1(x) &= \frac{1}{4}(1 - x - x^2 + x^3) \\ N_2(x) &= \frac{1}{4}(2 + 3x - x^3) \\ N_3(x) &= \frac{1}{4}(-1 - x + x^2 + x^3) \end{aligned} \tag{4}$$

These are the hermite polynomials on $[-1, 1]$. To get the basis functions on the elements $[0, 0.5]$ and $[0.5, 1]$ an appropriate transformation is performed.

3.3 c) Numerical solution

To solve the problem, please run the file `solve_prob3.m` which will solve the given problem with 2 elements and generate the plot below,

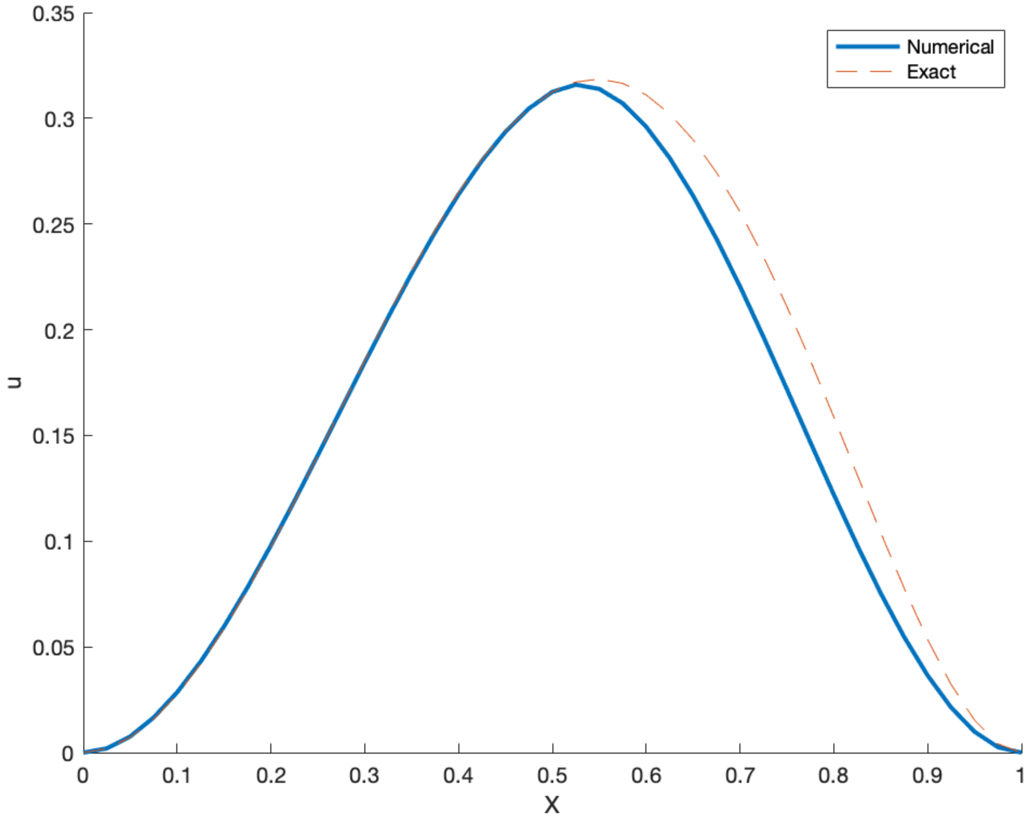

Figure 3: Solution to eq. 3 with two elements over $[0, 0.5], [0.5, 1]$