University of California Berkeley
MATH228B: Numerical Solution of Differential Equations – Spring 2020

Project 5

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1 Problem 1

Consider the illustration given in figure 1 representing the 6-vertex triangle element. Consider that $v(x, y) \in \mathbb{P}^2(K)$ a generic polynomial of degree at most 2. Then v has a representation,

$$v(x, y) = V_1 + V_2 x + V_3 y + V_4 xy + V_5 x^2 + V_6 y^2 \tag{1}$$

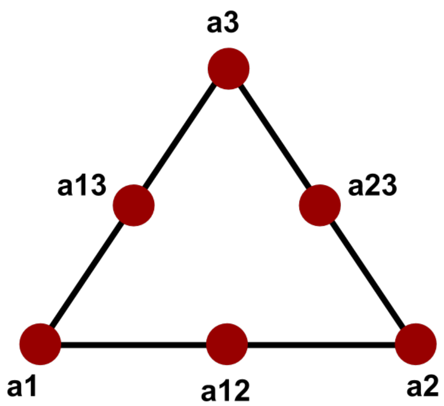

Figure 1: Illustration of a quadratic polynomial triangular element.

So $\mathbb{P}^2(K)$ has 6 degrees of freedom. We have six linear equations to determine these coefficients V_i if we know the value of $v(x, y)$ at the points $\{a_1, a_2, a_3, a_{12}, a_{23}, a_{13}\}$, i.e.,

$$\begin{aligned} v(a_1) &= v_1 \\ v(a_2) &= v_2 \\ v(a_3) &= v_3 \\ v(a_{12}) &= v_{12} \\ v(a_{13}) &= v_{13} \\ v(a_{23}) &= v_{23} \end{aligned} \tag{2}$$

1.1 Uniqueness of solution

In order to prove that the above system of equations is invertible, first we consider the representation of quadratic polynomials $v(x, y)$ in terms of the linear basis $\{\phi_1, \phi_2, \phi_3\} \in \mathbb{P}^1(K)$ as,

$$v(x, y) = \sum_{i=1}^{3} v(a_i)\phi_i(2\phi_i - 1) + \sum_{i,j=1;i<j}^{3} v(a_{ij})4\phi_i\phi_j \tag{3}$$

Now consider the situation where all of the interpolated values $v(a_i) = v(a_{ij}) = 0$ in the right hand side. Then by eq. 3, we have,

$$v(x, y) = 0 \qquad \forall x, y$$

Thus, by eq. 1 we have that all of the coefficients $V_i = 0$. So the unique solution to the linear system of equations with zero right hand side is the zero vector. This means that the zero vector is the only vector in the null space of the linear system, implying that the linear system has full rank and is invertible. This gives the uniqueness result.

1.2 Continuity of interpolant across element edges

For continuity, consider edge $1 - 2$ in fig. 1. The argument is the same for the other edges. Consider the nodal basis function ϕ_3. The restriction of ϕ_3 to the edge $1 - 2$ is a quadratic polynomial. By the interpolation property of a nodal basis we have that $\phi_3(a_1) = \phi_3(a_{12}) = \phi_3(a_2) = 0$. Since we have a quadratic polynomial equal to zero at three points, it must be zero everywhere on this edge. A similar argument holds for the basis functions ϕ_{23}, ϕ_{13}. Thus the value of an interpolated function on the edge $1 - 2$ is completely determined by the coefficients of the basis functions $\phi_1, \phi_{12}, \phi_2$ which are the only non-zero functions on this edge. Since these coefficients are shared by any adjoining element, the interpolant is continuous across this edge.

2 Problem 2

$$\begin{aligned} \Delta u &= f \qquad \text{in } \Omega \\ \frac{\partial u}{\partial n} = \nabla u \cdot n &= g \qquad \text{on } \Gamma \end{aligned} \tag{4}$$

2.1 Uniqueness of solution

Let u be a solution to eq. 4. Consider $v = u + c$ for some constant c. Then,

$$
\begin{aligned}
\Delta v &= \Delta(u + c) = \Delta u = f \\
\nabla v &= \nabla(u + c) = \nabla u \\
\implies \nabla v \cdot n &= \nabla u \cdot n = g
\end{aligned}
$$

So v is also a solution to the same boundary value problem.

2.2 Guaranteeing uniqueness via integral constraint

With the added constraint, the boundary value problem is,

$$
\begin{aligned}
\Delta u &= f \qquad \text{in } \Omega \\
\nabla u \cdot n &= g \qquad \text{on } \Gamma \\
\int_\Omega u \text{ dv} &= 0
\end{aligned}
\tag{5}
$$

Let $v = u + c$. Then as per the previous section,

$$
\begin{aligned}
\Delta v &= \Delta u = f \\
\nabla v \cdot n &= \nabla u \cdot n = g
\end{aligned}
$$

But according to the constraint we require,

$$
\begin{aligned}
\int_\Omega v \text{ dv} &= \underbrace{\int_\Omega u \text{ dv}}_{=0} + \int_\Omega c \text{ dv} \\
&= c \text{ Vol}(\Omega) = 0
\end{aligned}
$$

Since $\text{Vol}(\Omega) \neq 0$, we have $c = 0$. Thus $v \equiv u$. Thus with the added constraint the solution is unique.

2.3 Variational formulation

The given system is,

$$
\begin{aligned}
\Delta u &= f \qquad \text{in } \Omega \\
\nabla u \cdot n &= g \qquad \text{on } \Gamma \\
\int_\Omega u \text{ dv} &= 0
\end{aligned}
\tag{6}
$$

Consider the function space $V = \{v \in H^1(\Omega) : \int_\Omega v \text{ dv} = 0\}$. Multiplying the governing equation 6 by v and integrating over Ω we have,

$$\begin{aligned}
\int_\Omega v\Delta u \text{ dv} &= \int_\Omega vf \text{ dv} \\
\int_\Omega \nabla \cdot (v\nabla u) \text{ dv} - \int_\Omega \nabla v \cdot \nabla u \text{ dv} &= \int_\Omega vf \text{ dv} \\
\int_\Gamma v\nabla u \cdot n \text{ da} - \int_\Omega \nabla v \cdot \nabla u \text{ dv} &= \int_\Omega vf \text{ dv} \\
\implies -\int_\Omega \nabla v \cdot \nabla u \text{ dv} &= \int_\Omega vf \text{ dv} - \int_\Gamma vg \text{ da}
\end{aligned}$$

Thus in abstract form we have,

$$a(v,u) = L(v) \tag{7}$$

where,

$$\begin{aligned}
a(v,u) &= -\int_\Omega \nabla v \cdot \nabla u \text{ dv} \\
L(v) &= \int_\Omega vf \text{ dv} - \int_\Gamma vg \text{ da}
\end{aligned} \tag{8}$$

The above operators satisfy the property of ellipticity, namely,

Symmetry

$$a(v,u) = -\int_\Omega \nabla v \cdot \nabla u \text{ dv} = a(u,v)$$

Continuity

$$\int_\Omega \nabla v \cdot \nabla u \text{ dv} \leq ||\nabla u||_{L^2} \; ||\nabla v||_{L^2} \leq ||u||_{H^1} \; ||v||_{H^1}$$

V-ellipticity

$$\begin{aligned}
a(v,v) &= \int_\Omega |\nabla v|^2 \text{ dv} \\
&= \frac{1}{2}\int_\Omega |\nabla v|^2 \text{ dv} + \underbrace{\frac{1}{2}\int_\Omega |\nabla v|^2 \text{ dv}}_{\geq C \int_\Omega |u|^2 \text{ dv}} \\
&\geq \alpha \left(\int_\Omega |\nabla v|^2 \text{ dv} + \int_\Omega |v|^2 \text{ dv} \right) \\
&\geq \alpha ||v||^2_{H^1_0}
\end{aligned}$$

Continuity of $L(v)$

$$\begin{aligned} |L(v)| &= \left|\int_\Omega vf \text{ dv}\right| \\ &\leq ||f||_{L^2}||v||_{L^2} \\ &\leq \Lambda||v||_{H_0^1}(\Omega) \end{aligned}$$

where $\Lambda = ||f||_{L^2}$

3 Problem 3

3.1 Proof of minimizer

We have that,

$$\begin{aligned} (v, u_h)_\Omega &= (v, u)_\Omega \\ (v, u - u_h)_\Omega &= 0 \qquad \forall v \in V_h \end{aligned}$$

So the function $u - u_h$ is orthogonal to *all* functions $v \in V_h$. Thus, u_h is the closest approximation to u in V_h. Thus,

$$||u - u_h||_{L^2} \leq \inf_{v \in V_h} ||u - v||_{L^2}$$

Further, since $\Pi_h u \in V_h$,

$$||u - u_h||_{L^2} \leq Ch^{r+1}|u|_{H^{r+1}}$$

3.2 Prove $||u_h|| \leq ||u||$

We note that u can be decomposed orthogonally into,

$$u = u_h + (u - u_h) \tag{9}$$

Where $u - u_h$ is orthogonal to u_h. Thus, by the pythogorean theorem,

$$||u|| = ||u_h|| + \underbrace{||u - u_h||}_{\geq 0} \tag{10}$$

$$\implies ||u_h|| \leq ||u|| \tag{11}$$