9.1.8

EE24BTECH11005 - Arjun Pavanje

Question: Solve the differential equation $y' + y = e^x$ with initial conditions $y(0) = \frac{1}{2}$
Solution:

Computational Solution:
Aim is to find a computational solution to the differential equation using trapezoidal law.

$$y' + y = e^x \tag{1}$$

Integrating by setting limits after discretizing,

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} -ydx + \int_{x_n}^{x_{n+1}} e^x dx \tag{2}$$

We can solve the two integrals on the $R.H.S$ by using trapezoid method

$$\int_{x_n}^{x_{n+1}} ydx = h\left[\frac{y(x_{n+1}) + y(x_n)}{2}\right] \tag{3}$$

$$\int_{x_n}^{x_{n+1}} e^x dx = h\left[\frac{e^{x_{n+1}} + e^{x_n}}{2}\right] \tag{4}$$

Here, h is the step-size. Using smaller values of h will give a more accurate plot. Substituting into equation (11) we get,

$$y_{n+1} - y_n = -h\left(\frac{y_{n+1} + y_n}{2}\right) + h\left(\frac{e^{x_{n+1}} + e^{x_n}}{2}\right) \tag{5}$$

$$y_{n+1}(2 + h) = y_n(2 - h) + he^{x_n}\left(1 + e^h\right) \tag{6}$$

The general difference equation comes out to be,

$$y_{n+1} = y_n\left(\frac{2 - h}{2 + h}\right) + e^{x_n}h\left(\frac{1 + e^h}{2 + h}\right) \tag{7}$$

$$x_{n+1} = x_n + h \tag{8}$$

Theoretical Solution:
Let us verify the solution obtained above (using trapezoid law) theoretically to see if it matches.
This is a linear differential equation of the first order.

$$\frac{dy}{dx} + y = e^x \tag{9}$$

$$\tag{10}$$

Finding integrating factor

$$e^{\int 1dx} \tag{11}$$
$$= e^x \tag{12}$$

Multiplying both sides of (2) with integrating factor,

$$\frac{dy}{dx}(e^x) + ye^x = e^{2x} \tag{13}$$
$$\frac{d(ye^x)}{dx} = e^{2x} \tag{14}$$
$$ye^x = \frac{e^{2x}}{2} + c \tag{15}$$
$$y = \frac{e^x}{2} + ce^{-x} \tag{16}$$

On substituting initial conditions we get,

$$y = \frac{e^x}{2} \tag{17}$$

Below is a comparission between Simulated Plot and Theoretical Plot.

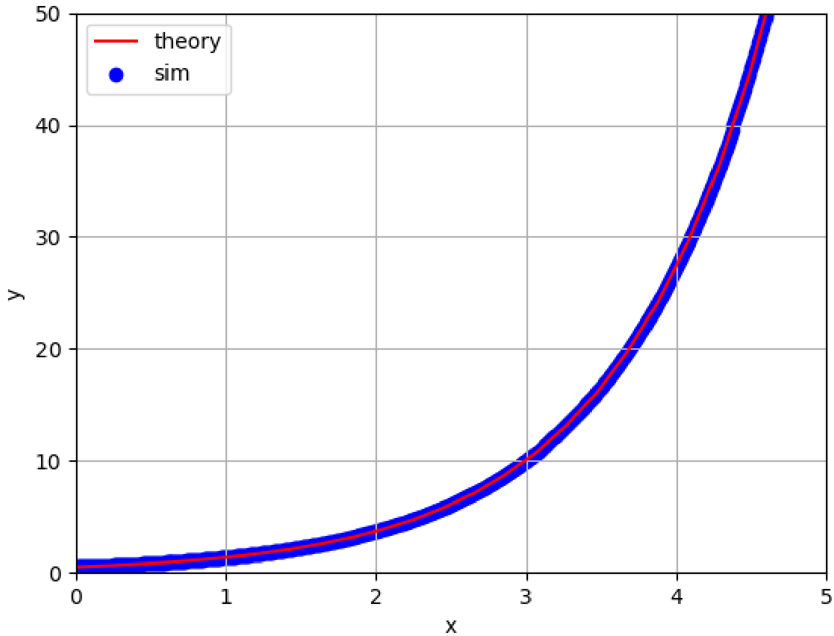

Fig. 1: Differential Equation $y' + y = e^x$ solved using trapezoid method