11.16.3.8.6

EE24BTECH11005 - Arjun Pavanje

Question:
Three coins are tossed once. Find the probability of getting 3 tails
Solution:
The sample space Ω in case of this experiment is given by,

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \tag{1}$$

Now we define a discrete random variable X = number of tails in the sequence. In this solution, we treat our random variable X as a sum of three Bernoulli random variables

$$X = X_1 + X_2 + X_3 \tag{2}$$

where,

$$X_i = \begin{cases} 0 & i^{th}\text{toss is heads} \\ 1 & i^{th}\text{toss is tails} \end{cases} \tag{3}$$

Here $p = \frac{1}{2}$. Our random variable is a sum of three Bernoulli random variables. By applying Z transform on both sides and applying properties of Z transform of PMF,

$$M_X(z) = M_{X_1}(z)\, M_{X_2}(z)\, M_{X_3}(z) \tag{4}$$

Here, the three coin tosses can be considered as independant events, which implies $M_{X_1}(z) = M_{X_2}(z) = M_{X_3}(z)$. Because of this, we can generalize this to m coin tosses.

$$M_X(z) = \prod_{k=1}^{m} M_{X_k}(z) \tag{5}$$

We now define Probability Mass Function for the Bernoulli random variable X_i be given by,

$$p_{X_i}(n) = \begin{cases} p & n = 0 \\ 1 - p & n = 1 \\ 0 & n \in \mathbb{Z} - \{0, 1\} \end{cases} \tag{6}$$

where p is the probability of getting heads

$$M_{X_1}(z) = \sum_{k=-\infty}^{\infty} p_{X_1}(k) z^{-n} = p + (1-p) z^{-1} \tag{7}$$

$$M_{X_2}(z) = \sum_{k=-\infty}^{\infty} p_{X_2}(k) z^{-n} = p + (1-p) z^{-1} \tag{8}$$

$$\vdots \tag{9}$$

$$M_{X_m}(z) = \sum_{k=-\infty}^{\infty} p_{X_m}(k) z^{-n} = p + (1-p) z^{-1} \tag{10}$$

$$M_X(z) = \left(p + (1-p) z^{-1}\right)^m \tag{11}$$

This can be simplified to,

$$M_X(z) = \sum_{n=-\infty}^{\infty} {}^mC_k p^k (1-p)^{3-k} z^{-k} \tag{12}$$

Taking $z-$inverse on both sides we get,

$$p_X(k) = {}^mC_k p^k (1-p)^{3-k} \tag{13}$$

Taking $m = 3, p = \frac{1}{2}$ we get,

$$p_X(n) = {}^3C_n \left(\frac{1}{2}\right)^3 \tag{14}$$

Here, since we require probability of 3 tails out of 3 tosses, set $k = 3$ in the above equation.
Required probability $= \frac{1}{8}$
If we redefine the problem a little bit, it can be solved using Cummulative Frequency Distribution. We can say $\Pr(3 \text{ tails}) = 1 - \Pr(\text{Atleast one head})$ since out of three coin tosses there can be at max 3 tails.
We can say that

$$\Pr(\text{m-k tails}) = \Pr(\text{k heads}) \tag{15}$$

$$\Pr(\text{atleast k heads}) = \Pr(\text{atleast m-k tails}) \tag{16}$$

For our problem $m = 3, k = 1$ Cummulative Distribution Function is given by,

$$F_X(n) = \sum_{k=-\infty}^{n} {}^3C_k \left(\frac{1}{2}\right)^3 \tag{17}$$

$$F_X(n) = \begin{cases} 0 & n < 0 \\ {}^3C_0\left(\frac{1}{2}\right)^3 = \frac{1}{8} & 0 \le n < 1 \\ {}^3C_0\left(\frac{1}{2}\right)^3 + {}^3C_1\left(\frac{1}{2}\right)^3 = \frac{1}{2} & 1 \le n < 2 \\ {}^3C_0\left(\frac{1}{2}\right)^3 + {}^3C_1\left(\frac{1}{2}\right)^3 + {}^3C_2\left(\frac{1}{2}\right)^3 = \frac{7}{8} & 2 \le n < 3 \\ {}^3C_0\left(\frac{1}{2}\right)^3 + {}^3C_1\left(\frac{1}{2}\right)^3 + {}^3C_2\left(\frac{1}{2}\right)^3 + {}^3C_3\left(\frac{1}{2}\right)^3 = 1 & 3 \le n \end{cases} \tag{18}$$

What we require is,

$$1 - \Pr((X \ge 1)) = 1 - (1 - F_X(1)) \tag{19}$$

$$= \frac{1}{8} \tag{20}$$

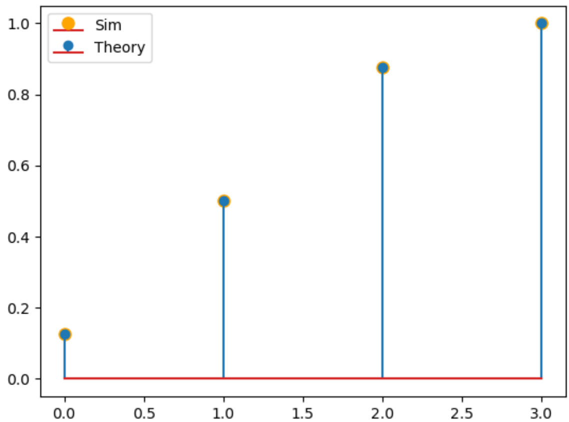

Fig. 1: CDF Plot

We notice something interesting about PMF plots for different values of m. We see that it appears to converge Gaussian Distribution curve. One more interesting fact is that area under the Gaussian Distribution Curve is 1. This further supports the fact that PMF converges to Gaussian Distribution Curve, as sum of all the probabilites ($\sum P_X(n) = 1$) is 1. As $m \to \infty$ the PMF plot onverges to the Gaussian Distribution curve. We can visually verify this,

Below are the PMF graphs for $m = 10, 25, 500, 100$ respectively

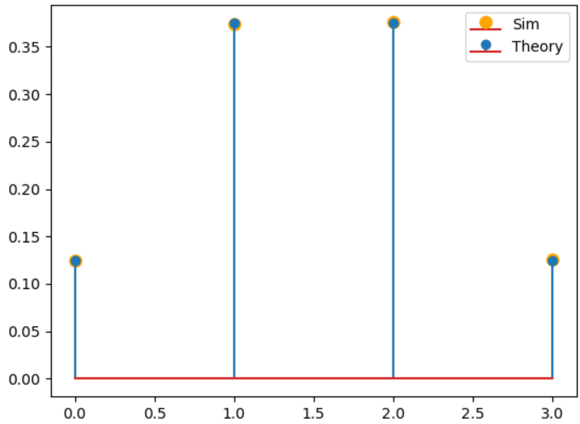

Fig. 2: PMF plot for $m = 3$