9.1.8

EE24BTECH11005 - Arjun Pavanje

Question: Solve the differential equation $y' + y = e^x$ with initial conditions $y(0) = 1$
Solution:

Theoretical Solution
Given equation is a linear first order differential equation, so laplace transform may be used to solve it. Some properties of laplace transform used are,

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{1}$$

$$\mathcal{L}(\kappa f(t)) = \kappa \mathcal{L}(f(t)) \tag{2}$$

$$\mathcal{L}(y') = sY(S) - y_0 \tag{3}$$

Where $\mathcal{L}(y) = Y(s)$ Applying Laplace Transform to given differential Equation,

$$\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(e^x) \tag{4}$$

$$sY(s) - y_0 + Y(S) = \int_0^\infty e^{-x(x-1)} dx = \frac{1}{s-1} \tag{5}$$

$$Y(s) = \left(\frac{y_0 - 0.5}{s+1}\right) + \frac{1}{2}\left(\frac{1}{s-1}\right) \tag{6}$$

Taking Laplace Inverse on both sides we get,

$$y = \mathcal{L}^{-1}\left(\frac{y_0 - 0.5}{s+1}\right) + \mathcal{L}^{-1}\frac{1}{2}\left(\frac{1}{s-1}\right) \tag{7}$$

$$y = (y_0 - 0.5)\, e^{-x} + \frac{1}{2} e^x \tag{8}$$

Putting in $y_0 = 1$ we get,

$$y = \frac{(e^x + e^{-x})}{2} \tag{9}$$

Computational Solution:
Taking Laplace transform to both sides of the Differential Equation,

$$sY(s) - y_0 + Y(s) = \int_0^\infty e^{-x(x-1)} dx = \frac{1}{s-1} \tag{10}$$

$$Y(s) = \frac{y_0}{s+1} + \frac{1}{(s+1)(s-1)} \tag{11}$$

$$Y(s) = \left(\frac{y_0 - 0.5}{s+1}\right) + \frac{1}{2}\left(\frac{1}{s-1}\right) \tag{12}$$

Applying Bilinear Transform with $T = h$. To go from the domain of Laplace transform to that of Z-transform, we transform our s. On substituting we get,

$$Y(s) = (y_0 - 0.5)\left(\frac{h\left(1+z^{-1}\right)}{2\left(1-z^{-1}\right)+h\left(1+z^{-1}\right)}\right) + \frac{1}{2}\left(\frac{h\left(1+z^{-1}\right)}{2\left(1-z^{-1}\right)-h\left(1+z^{-1}\right)}\right) \tag{13}$$

$$= (y_0 - 0.5)\left(\frac{h\left(1+z^{-1}\right)}{(2+h)+z^{-1}(h-2)}\right) + \frac{1}{2}\left(\frac{h\left(1+z^{-1}\right)}{(2-h)+z^{-1}(2+h)}\right) \tag{14}$$

$$= \left(\frac{y_0 - 0.5}{2+h}\right)\left(\frac{h\left(1+z^{-1}\right)}{1-\alpha z^{-1}}\right) + \left(\frac{h}{2(2-h)}\right)\left(\frac{h\left(1+z^{-1}\right)}{1-\alpha^{-1}z^{-1}}\right) \tag{15}$$

Where $\alpha = \frac{2-h}{2+h}$
Radius of convergence of the first term is, $|z| > |\alpha|$, for the second term it is, $|z| > \left|\alpha^{-1}\right|$.
R.O.C of the total equation turns out to be, $|z| > max\left(|\alpha|, \left|\frac{1}{\alpha}\right|\right)$
Taking $\left(1-\alpha z^{-1]}\right)$ to $R.H.S$ we get,

$$Y(s)\left(1-\alpha z^{-1}\right) = h\left(\frac{y_0 - 0.5}{2+h}\right)\left(\left(1+z^{-1}\right)\right) + \left(\frac{h}{2(2-h)}\right)\left(\frac{h\left(1+z^{-1}\right)\left(1-\alpha z^{-1}\right)}{1-\alpha^{-1}z^{-1}}\right) \tag{16}$$

After applying algebraic manipulations on the second term, above equation comes out to be,

$$Y(s)\left(1-\alpha z^{-1}\right) = h\left(\frac{y_0 - 0.5}{2+h}\right)\left(\left(1+z^{-1}\right)\right) + \tag{17}$$

$$\left(\frac{h}{2(2+h)}\right)\left(\frac{1+\alpha-\alpha^2-\alpha^3}{1-\alpha^{-1}z^{-1}} + \alpha^2 z^{-1} - \alpha\left(1-\alpha-\alpha^2\right)\right) \tag{18}$$

Applying inverse Z-transform,

$$y_{n+1} = \alpha y_n + h\left(\frac{y_0 - 0.5}{2+h}\right)(\delta[n] + \delta[n-1]) + \left(\frac{h}{2(2+h)}\right)\times \tag{19}$$

$$\left(\left(1+\alpha-\alpha^2-\alpha^3\right)\alpha^{-n}u(n) + \alpha^2\delta[n-1] - \alpha\left(1-\alpha-\alpha^2\right)\delta[n]\right) \tag{20}$$

here, δ is defined as,

$$\delta[n-n_0] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases} \tag{21}$$

Final General Difference Equation comes out to be,

$$y_{n+1} = \alpha y_n + h\left(\frac{y_0 - 0.5}{2+h}\right)(\delta[n] + \delta[n-1]) + \left(\frac{h}{2(2+h)}\right)\times \tag{22}$$

$$\left(\left(1+\alpha-\alpha^2-\alpha^3\right)\alpha^{-n}u(n) + \alpha^2\delta[n-1] - \alpha\left(1-\alpha-\alpha^2\right)\delta[n]\right) \tag{23}$$

Alternate Computational Solution

Finding the difference equation using trapezoid law,
Given Differential Equation,

$$y' = -y + e^x \tag{24}$$

$$\int_{y_n}^{y_{n+1}} dy = -\int_{x_n}^{x_{n+1}} y dx + \int_{x_n}^{x_{n+1}} e^x dx \tag{25}$$

Discretizing steps using trapezoid rule,

$$y_{n+1} = y_n - \frac{h}{2}(y_n + y_{n+1}) + e^{x_n}(e^h - 1) \tag{26}$$

$$y_{n+1} - y_n = -\frac{h}{2}(y_n + y_{n+1}) + e^{x_n}\left(e^h - 1\right) \tag{27}$$

$$y_{n+1} = y_n\left(\frac{2-h}{2+h}\right) + \frac{2e^{x_n}}{2+h}\left(e^h - 1\right) \tag{28}$$

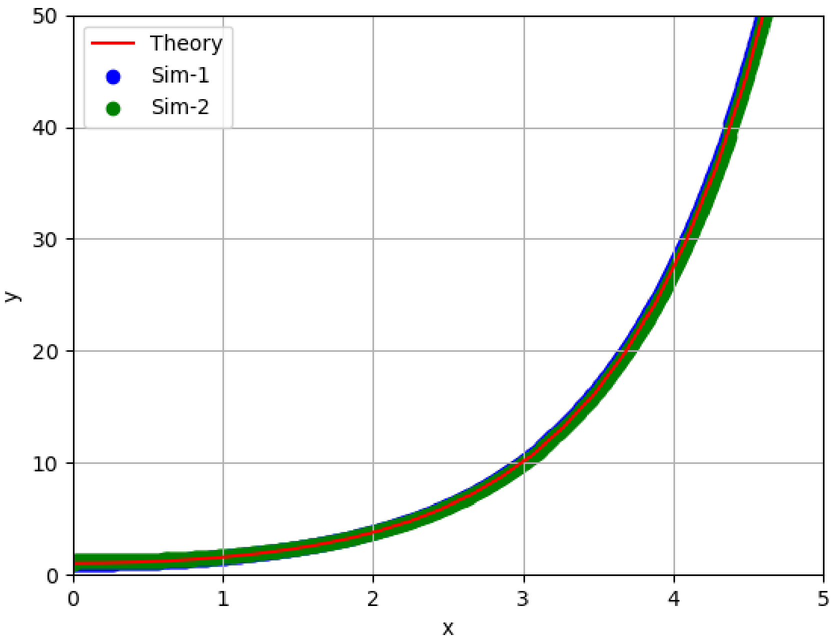

Fig. 1: Differential Equation $y' + y = e^x$ solved using Bilinear transform. Sim-1 shows plot obtained by using trapezoidal law, Sim-2 shows plot obtained by using Bilinear transform method. Theory shows plot obtained theoretically