8.1.9

EE24BTECH11005 - Arjun Pavanje

Question: Find the area of the region bounded by the parabola $y = x^2$, and $y = |x|$.
Solution:
The curve $y = x^2$ and the line $y = |x|$ meet at two points $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ Equation for area enclosed is given by,

$$\int_{-1}^{1} \left(|x| - x^2\right) dx \tag{1}$$

$$= \int_{-1}^{0} \left(x + x^2\right) dx + \int_{0}^{1} \left(x - x^2\right) dx \tag{2}$$

$$= 2\int_{0}^{1} \left(x - x^2\right) dx \tag{3}$$

There are two ways to solve the above integral, Theoretically and Computationally (trapezoid method).We shall compare the results obtained by both methods.
Theoretical Solution:

$$2\int_{0}^{1} \left(x - x^2\right) dx \tag{4}$$

$$= 2\left(\left[\frac{x^2}{2}\right]_{x=0}^{x=1} - \left[\frac{x^3}{3}\right]_{x=0}^{x=1}\right) \tag{5}$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) \tag{6}$$

$$= \frac{1}{3} \tag{7}$$

Computational Solution:

Taking trapezoid shaped strips of small area and adding them all up. Say we have to find the area of y_x from $x = x_0$ to $x = x_n$, discretize points on the x axis $x_0, x_1, x_2, \dots, x_n$ such that they are equally spaced with step-size h.
Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2}h\left(y\left(x_1\right) + y\left(x_0\right)\right) + \frac{1}{2}h\left(y\left(x_2\right) + y\left(x_1\right)\right) + \cdots + \frac{1}{2}h\left(y\left(x_n\right) + y\left(x_{n-1}\right)\right) \tag{8}$$

$$= h\left[\frac{1}{2}\left(y\left(x_0\right) + y\left(x_n\right)\right) + y\left(x_1\right) + \cdots + y\left(x_{n-1}\right)\right] \tag{9}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \dots x_n)$ be equidistant points with step-size h.

$$A(x_n + h) = A(x_n) + \frac{1}{2}h\left(y(x_n + h) + y(x_n)\right) \tag{10}$$

We can repeat this till we get required area.
Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h\left(y_{n+1} + y_n\right) \tag{11}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h\left((y_n + hy'_n) + y_n\right) \tag{12}$$

$$A_{n+1} = A_n + \frac{1}{2}h\left(2y_n + hy'_n\right) \tag{13}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{14}$$

$$x_{n+1} = x_n + h \tag{15}$$

In the given question, $y_n = x_n + x_n^2$ and $y'_n = 1 - 2x_n$
General Difference Equation will be given by,

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{16}$$

$$= A_n + h\left(x_n + x_n^2\right) + \frac{1}{2}h^2\left(1 - 2x_n\right) \tag{17}$$

$$= A_n + x_n\left(h - h^2\right) + x_n^2(h) + \frac{h^2}{2} \tag{18}$$

$$x_{n+1} = x_n + h \tag{19}$$

Iterating till we reach $x_n = 1$ will return required area. Note, Area obtained is to be multiplied by 2 as the calculated area only accounts for one half of the graph.

Area obtained computationally: 0.3333195149898529 sq. units
Area obtained theoretically: $\frac{1}{3} = 0.33333\dots$ sq. units

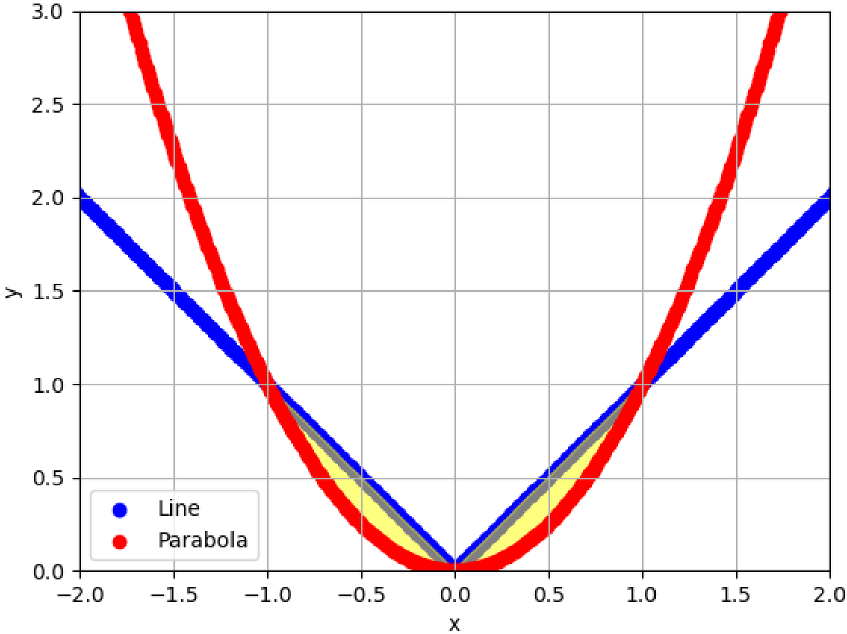

Fig. 1: Graph of the parabola $y = x^2$ and $y = |x|$ and the area enclosed between them