9.1.6

EE24BTECH11005 - Arjun Pavanje

Question: Solve the differential equation $(y''')^2 + (y'')^3 + (y')^4 + (y)^5 = 0$, with initial conditions $y''(x) = 0, y'(x) = 0, y(x) = 1$

Solution: An exact theoretical solution using known methods of solving differential equations was not found; however, it can be approximated to a pretty good degree of precision. Euler's method will be used to obtain a plot of the solution

Computational Solution:
By first principle of derivatives,

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \tag{1}$$

$$y(t+h) = y(t) + hy'(t) \tag{2}$$

Let y^i be the i^{th} derivative of the function, m be the order of the differential equation. Set $y_1 = y, y_2 = y^1, y_3 = y^2 \ldots$ so on.
We obtain the system,

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{m-1}' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \end{pmatrix} \tag{3}$$

$$y_m' = f(x, y_1, y_2, \ldots, y_m) \tag{4}$$

Generalizing the system according to Euler's form

$$\begin{pmatrix} y_1(x+h) \\ \vdots \\ y_{m-1}(x+h) \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) \\ \vdots \\ y_{m-1}(x) \\ y_m(x) \end{pmatrix} + h \begin{pmatrix} y_2(x) \\ \vdots \\ y_m(x) \\ f(x, y_1, y_2, \ldots, y_m) \end{pmatrix} \tag{5}$$

$$\mathbf{y}(x+h) = \mathbf{y}(x) + h \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{f(x,y_1,y_2,\ldots,y_m)}{y_m(x)} \end{pmatrix} \mathbf{y}(x) \tag{6}$$

$$\mathbf{y}(x+h) = \begin{pmatrix} 1 & h & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & h & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & h & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & h \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 + \frac{f(x,y_1,y_2,\ldots,y_m)}{y_m(x)} \end{pmatrix} \mathbf{y}(x) \tag{7}$$

Discretizing the steps we get,

$$\mathbf{y}_{n+1} = \begin{pmatrix} 1 & h & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & h & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & h & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & h \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 + \frac{f(x,y_1,y_2,\ldots,y_m)}{(y_m)_n} \end{pmatrix} \mathbf{y}_n \tag{8}$$

$$x_{n+1} = x_n + h \tag{9}$$

Smaller values of step size h will give more precise plots. We obtain points to plot by iterating repeatedly. Given differential equation can be written as,

$$y'''(x) = \pm\sqrt{-\left((y''(x))^3 + (y'(x))^4 + (y(x))^5\right)} \tag{10}$$

Here, order m is 3, there are two possible functions so we need to take two cases. On substituting given initial conditions we see that we only get valid values for $y'''(x) = +\sqrt{-\left((y'')^3 + (y')^4 + (y)^5\right)}$. In the other case we observe that we get imaginary values.

$$\mathbf{y}_{n+1} = \begin{pmatrix} 1 & h & 0 \\ 0 & 1 & h \\ 0 & 0 & 1 + \frac{\sqrt{-\left((y_3)_n^3 + (y_2)_n^4 + (y_1)_n^5\right)}}{(y_3)_n} \end{pmatrix} \mathbf{y}_n \tag{11}$$

Note, here the vector $\mathbf{y}$ is not to be confused with y_i which represents a function, namely the $i+1^{th}$ derivative of $y(x)$ Below is the plot for given curve based on initial conditions, obtained by iterating through the above equation.

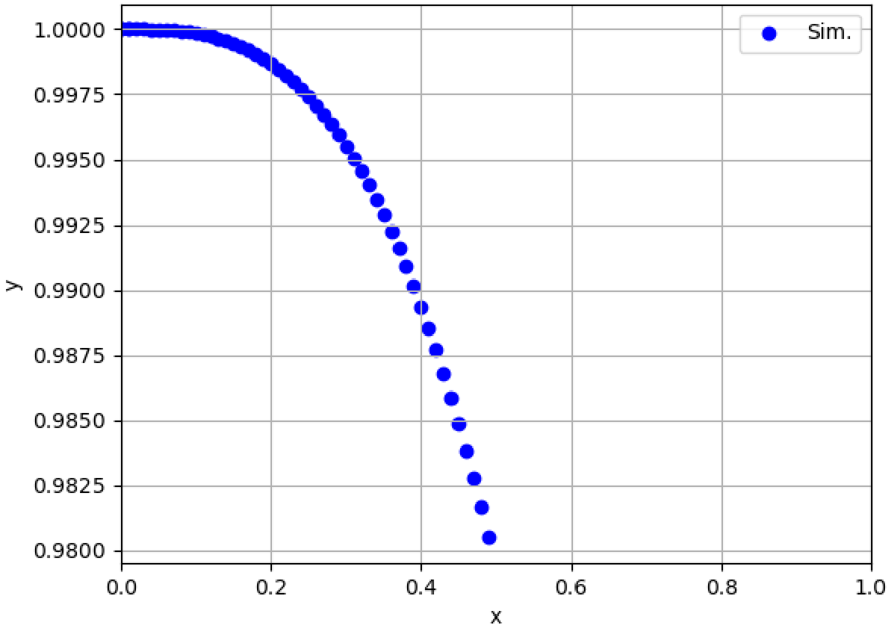

Fig. 1: Computational solution of $(y''')^2 + (y'')^3 + (y')^4 + (y)^5 = 0$