Matgeo Presentation

Arjun Pavanje,
EE24BTECH11005,
IIT Hyderabad.

January 9, 2025

Table of Contents

Problem

Problem Statement

Solve the differential equation

$$\left(y'''\right)^2 + \left(y''\right)^3 + \left(y'\right)^4 + \left(y\right)^5 = 0 \tag{2.1}$$

with initial conditions

$$y''(x) = 0, y'(x) = 0, y(x) = 1 \tag{2.2}$$

Solution

Theoretical Solution

An exact theoretical solution using known methods of solving differential equations was not found; however, it can be approximated to a pretty good degree of precision. Euler's method will be used to obtain a plot of the solution

Computational Solution

By first principle of derivatives,

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \tag{3.1}$$

$$y(t+h) = y(t) + hy'(t) \tag{3.2}$$

Let y^i be the ith derivative of the function, m be the order of the differential equation. Set $y_1 = y, y_2 = y^1, y_3 = y^2 \ldots$ so on. We obtain the system,

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{m-1}' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \end{pmatrix} \tag{3.3}$$

$$y_m' = f(x, y_1, y_2, \ldots, y_m) \tag{3.4}$$

Computational solution

Generalizing the system according to Euler's form

$$
\begin{pmatrix} y_1 \left(x + h \right) \\ \vdots \\ y_{m-1} \left(x + h \right) \\ y_m \left(x + h \right) \end{pmatrix} = \begin{pmatrix} y_1 \left(x \right) \\ \vdots \\ y_{m-1} \left(x \right) \\ y_m \left(x \right) \end{pmatrix} + h \begin{pmatrix} y_2 \left(x \right) \\ \vdots \\ y_m \left(x \right) \\ f \left(x, y_1, y_2, \ldots, y_m \right) \end{pmatrix} \tag{3.5}
$$

$$
\mathbf{y} \left(x + h \right) = \mathbf{y} \left(x \right) + h \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{f(x, y_1, y_2, \ldots, y_m)}{y_m(x)} \end{pmatrix} \mathbf{y} \left(x \right) \tag{3.6}
$$

Computational Solution

$$\mathbf{y}(x+h) = \begin{pmatrix} 1 & h & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 + \frac{f(x,y_1,y_2,\dots,y_m)}{y_m(x)} \end{pmatrix} \mathbf{y}(x) \quad (3.7)$$

Discretizing the steps we get,

$$\mathbf{y}_{n+1} = \begin{pmatrix} 1 & h & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 + \frac{f(x,y_1,y_2,\dots,y_m)}{(y_m)_n} \end{pmatrix} \mathbf{y}_n \quad (3.8)$$

Computational Solution

Where,

$$\mathbf{y}_n = \begin{pmatrix} y_1\left(x_n\right) \\ y_2\left(x_n\right) \\ \vdots \\ y_m\left(x_n\right) \end{pmatrix} \tag{3.9}$$

$$x_{n+1} = x_n + h \tag{3.10}$$

Smaller values of step size h will give more precise plots. We obtain points to plot by iterating repeatedly.

Computational Solution

Given differential equation can be written as,

$$y'''(x) = \pm\sqrt{-\left((y''(x))^3 + (y'(x))^4 + (y(x))^5\right)} \qquad (3.11)$$

Here, order m is 3, there are two possible functions so we need to take two cases. On substituting given initial conditions we see that we only get valid values for $y'''(x) = +\sqrt{-\left((y'')^3 + (y')^4 + (y)^5\right)}$.
In the other case we observe that we get imaginary values.

$$\mathbf{y}_{n+1} = \begin{pmatrix} 1 & h & 0 \\ 0 & 1 & h \\ 0 & 0 & 1 + \frac{\sqrt{-\left((y_3)_n^3 + (y_2)_n^4 + (y_1)_n^5\right)}}{(y_3)_n} \end{pmatrix} \mathbf{y}_n \qquad (3.12)$$

Computational Solution

Note, here the vector **y** is not to be confused with y_i which represents a function, namely the $i + 1^{th}$ derivative of $y(x)$ Below is the plot for given curve based on initial conditions, obtained by iterating through the above equation.

Computational Solution