10.4.3.5

EE24BTECH11005 - Arjun Pavanje

Question: In a class test, the sum of Shefali's marks in Mathematics and English is 30. Had she got 2 marks more in Mathematics and 3 marks less in English, the product of their marks would have been 210. Find her marks in the two subjects.

Solution:

Let x, y be the marks obtained in Mathematics and English respectively.

$$x + y = 30 \tag{1}$$

$$(x + 2)(y - 3) = 210 \tag{2}$$

On combining the above two equations we get,

$$x^2 - 25x + 156 \tag{3}$$

There are a few ways to solve this,

Newton Ralphson Method:

Start with an initial guess x_0, and then run the following logical loop,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{4}$$

where,

$$f(x) = x^2 - 25x + 156 \tag{5}$$

$$f'(x) = 2x - 25 \tag{6}$$

The update equation will be

$$x_{n+1} = x_n - \frac{{x_n}^2 - 25x_n + 156}{2x_n - 25} \tag{7}$$

$$\tag{8}$$

The problem with this method is if the roots are complex but the coeffcients are real, x_n either converges to an extrema or grows continuously without any bound. In the case of complex solutions, we can just take our initial guess as a complex number, and that will return the required roots. The output of a program written to find roots is shown below:

$$x = 12.000000000000014 \tag{9}$$

$$x = 12.999999999999964 \tag{10}$$

Eigenvalue solution

For a polynomial equation of form $x_n + c_{n-1}x^{n-1} + \cdots + c_2x^2 + c_1x + c_0 = 0$ we construct a matrix called companion matrix of form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \tag{11}$$

The eigenvalues of the companion matrix are the roots of the polynomial equation. Thus, the quadratic equation and companion matrix are related by the characteristic polynomial equation. For the given question, the companion matrix comes out to be,

$$\begin{pmatrix} 0 & -156 \\ 1 & 25 \end{pmatrix} \tag{12}$$

Using QR decomposition algorithm, we will now solve for the eigenvalues of the above companion matrix.

Basic principle behind iterative QR decomposition is similar matrices. Two square matrices A and B of size $n \times n$ are said to be *similar* if there exists an invertible matrix P such that:

$$B = P^{-1}AP. \tag{13}$$

Similar matrices turn out to have the same eigenvalues. This can be easily proved,

Given that A, B are similar matrices, and P is an invertible matrix such that $B = P^{-1}AP$ By definition of eigenvalues,

$$\begin{aligned} \det(B - \lambda I) &= \det(P^{-1}AP - \lambda I) && (14) \\ &= \det(P^{-1}AP - \lambda P^{-1}P) && (15) \\ &= \det(P^{-1}) \det(A - \lambda I) \det(P) && (16) \\ &= \det(P^{-1}P) \det(A - \lambda I) && (17) \\ &= \det(A - \lambda I) && (18) \end{aligned}$$

Basic idea is to use similarity transforms to perform Schur Decomposition of matrix.

$$A = QUQ^{-1} \tag{19}$$

for some unitary matrix Q (so that the inverse Q^{-1} is also the conjugate transpose Q^* of Q), and some upper triangular matrix U. The diagonal entries of the U matrix are the eigenvalues.
Steps to perform QR decomposition and accelerate its convergence,

1) Convert to Upper Hessenberg form via Householder Reflections
2) Performing QR decomposition via Givens Rotations with shifts
3) Read off diagonal elements

Step 1: Performing Householder Reflections
A square matrix A of order $n \times n$ is said to be in upper Hessenberg form if all the entries

below the first subdiagonal are zero. For example:

$$H = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}. \tag{20}$$

Applying QR decomposition to reach Schur-form to the matrix after it is in Upper Hessenberg form greatly accelerates rate of convergence.
Householder transformations are used to reduce a general matrix A to Hessenberg form. A Householder reflector is an orthogonal matrix defined as:

$$P = I - 2\mathbf{u}\mathbf{u}^\top \tag{21}$$

$$\tag{22}$$

where $||\mathbf{u}|| = 1$
If the entries of the matrix are complex, transpose is to replaced with conjugate transpose. Vector $\mathbf{u}$ must be carefully chosen such that the resultant matrix P obtained from it must zero out all elements below the first subdiagonal for that particular column while maintaining similarity to preserve eigenvalues. For a given column vector $x \in \mathbb{R}^n$, the vector u is chosen as:

$$\mathbf{u} = \frac{\mathbf{x} - ||\mathbf{x}||\rho\mathbf{e_1}}{||\mathbf{x} - ||\mathbf{x}||\rho\mathbf{e_1}||} \tag{23}$$

where:

- $\mathbf{e_1}$ is impulse vector of appropriate dimensions, first element 1
- ρ is something we have a degree of freedom in choosing as long as $|\rho| = 1$

Usually, $\rho = -sign(x_1)$, but here for ease of calculation $\rho = -e^{j\phi}$ where $x_1 = |x_1|e^{j\phi}$

We then construct Householder reflector matrix P_1 as,

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & I_5 - 2\mathbf{u_1}\mathbf{u_1}^\top \end{bmatrix} \tag{24}$$

For a general case P_i (for i^{th} column) would be to create $I_{n-i} - 2u_iu_i^\top$ and expand it from the top left such that it is like an $n \times n$ identity matrix with it as the bottom right submatrix. Visualizing the process,

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{bmatrix} \xrightarrow{P_2} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}. \tag{25}$$

Step 2: Constructing Matrix for Givens Rotation

A Givens rotation matrix ($G(i, j, \theta)$ zeroes out the element a_{ij} by rotating in the (i, j)-plane. It is defined as:

$$G(i, j, \theta) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\overline{s} & \cdots & \overline{c} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

where $\cos\theta$ and $\sin\theta$ are chosen such that the target element is eliminated.
To choose the values of c and s for the Givens rotation in QR decomposition, let a_j be the element we wish to null out (i.e. make 0). Pick an arbitrary non-zero pivot element a_i (on a different row). Usually, if we wish to null a particular sub-diagonal element, we pick the principal diagonal element above it as a pivot.

$$c = \frac{\overline{a_i}}{\sqrt{a_i^2 + a_j^2}}, \quad s = \frac{-\overline{a_j}}{\sqrt{a_i^2 + a_j^2}}$$

Givens rotation essentially rotates the two rows that a_i and a_j are on such that $a_j = 0$ after rotation, other rows remain unaffected.

These values rotate the vector formed by a_i and a_j to eliminate a_j while maintaining the orthogonality of the rotation matrix. The choice of c and s ensures that the resulting transformed matrix has zeros below the diagonal in the desired locations. Visualizing the process,

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(3,2,\theta_1)} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(4,3,\theta_2)} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}. \tag{26}$$

After all Givens rotations, the resulting matrix is upper triangular:

$$R = \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}. \tag{27}$$

For the Companion matrix in the given question,

$$c_k = \frac{0}{\sqrt{0^2 + 1^2}} = 0 \tag{28}$$

$$s_k = \frac{1}{\sqrt{0^2 + 1^2}} = 1 \tag{29}$$

$$\tag{30}$$

The sequence of Givens rotations $G_1, G_2, \ldots, G_m$ satisfies:

$$G_m \cdots G_2 G_1 A = R, \tag{31}$$

where R is upper triangular. The QR decomposition is obtained by combining the transposes of the Givens rotations into Q:

$$A = QR, \quad Q = G_1^\top G_2^\top \cdots G_m^\top. \tag{32}$$

$$\begin{aligned} A_{k+1} &= R_k Q_k &(33)\\ &= (G_n \ldots G_2 G_1) A_k (G_1^\top G_2^\top \ldots G_n^\top) &(34)\\ &= (G_n \ldots G_2 G_1) A_k (G_n \ldots G_2 G_1)^\top &(35) \end{aligned}$$

Introducing a shift to the matrix can drastically improve its convergence rate. If σ is the shift amount, before applying QR decomposition on a matrix shift the matrix by σ, perform QR on shifted matrix, shift the matrix back by σ

$$\begin{aligned} A'_k &= A - \sigma I &(36)\\ A'_{k+1} &= (Q'_k R'_k) + \sigma I &(37) \end{aligned}$$

Iteratively repeating this process causes the matrix to converge to upper triangular. Where σ is given by element $H_{n-1,n-1}$ of the matrix.
Special Case: Jordan Blocks

Jordan blocks are used to represent non-diagonalizable matrices. A Jordan block occurs when the Matrix on which eigenvalue alorithm is applied does not converge to Upper-Triangular. One example is when the enteries of the matrix are real, but they some of the eigenvalues are complex. Presence of a Jordan block indicates the presence of a pair of complex conjugate eigenvalues.

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix}$$

Jordan blocks can be handled by identifying where subdiagonal element is non-zero and solvinbg the 2×2 submatrix obtained.

Running the eigenvalue code for our companion matrix we get,

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Eigenvalues:
(11.999964 + 0.000000i)
(13.000036 + 0.000000i)
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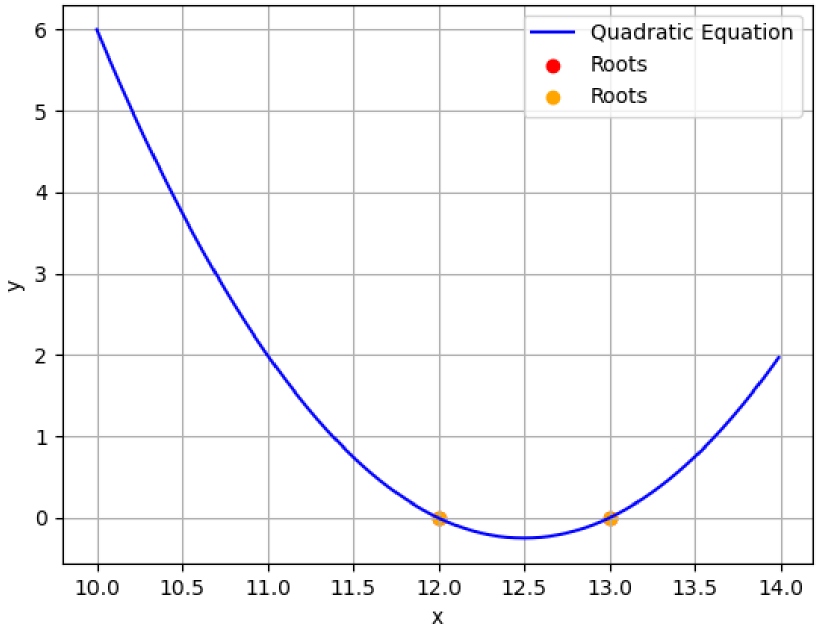

Fig. 1: Solving quadratic equation $x^2 - 25x + 156$