9.1.6

EE24BTECH11005 - Arjun Pavanje

Question: Solve the differential equation $(y''')^2 + (y'')^3 + (y')^4 + (y)^5 = 0$, with initial conditions $y''(x) = 0, y'(x) = 0, y(x) = 1$

Solution: An exact theoretical solution using known methods of solving differential equations was not found; however, it can be approximated to a pretty good degree of precision. Euler's method will be used to obtain a plot of the solution

Computational Solution:

By first principle of derivatives,

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \tag{1}$$

$$y(t+h) = y(t) + hy'(t) \tag{2}$$

Let y^i be the i^{th} derivative of the function, m be the order of the differential equation. Set $y_1 = y, y_2 = y^1, y_3 = y^2 \dots$ so on.

We obtain the system,

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{m-1}' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \end{pmatrix} \tag{3}$$

$$y_m' = f(x, y_1, y_2, \dots, y_m) \tag{4}$$

Generalizing the system according to Euler's form

$$\begin{pmatrix} y_m(x+h) \\ y_{m-1}(x+h) \\ \vdots \\ y_1(x+h) \end{pmatrix} = \begin{pmatrix} y_m(x) \\ y_{m-1}(x) \\ \vdots \\ y_1(x) \end{pmatrix} + h \begin{pmatrix} f(x, y_1, y_2, \dots, y_m) \\ y_m(x) \\ \vdots \\ y_2(x) \end{pmatrix} \tag{5}$$

Discretizing the steps we get,

$$\begin{pmatrix} (y_m)_{n+1} \\ (y_{m-1})_{n+1} \\ \vdots \\ (y_2)_{n+1} \\ (y_1)_{n+1} \end{pmatrix} = \begin{pmatrix} (y_m)_n \\ (y_{m-1})_n \\ \vdots \\ (y_2)_n \\ (y_1)_n \end{pmatrix} + h \begin{pmatrix} f(x, y_1, y_2, \dots, y_m) \\ (y_m)_n \\ \vdots \\ (y_3)_n \\ (y_2)_n \end{pmatrix} \tag{6}$$

$$x_{n+1} = x_n + h \tag{7}$$

Smaller values of step size h will give more precise plots. We obtain points to plot by iterating repeatedly.

Given differential equation can be written as,

$$y''' (x) = \pm \sqrt{-\left((y'' (x))^3 + (y' (x))^4 + (y (x))^5\right)} \tag{8}$$

Here, there are two possible functions so we need to take two cases. On substituting given initial conditions we see that we only get valid values for $y''' (x) = + \sqrt{-\left((y'')^3 + (y')^4 + (y)^5\right)}$. In the other case we observe that we get imaginary values.

$$\begin{pmatrix} (y_3)_{n+1} \\ (y_2)_{n+1} \\ (y_1)_{n+1} \end{pmatrix} = \begin{pmatrix} (y_3)_n \\ (y_2)_n \\ (y_1)_n \end{pmatrix} + h \begin{pmatrix} \sqrt{-\left((y'')^3 + (y')^4 + (y)^5\right)} \\ (y_3)_n \\ (y_2)_n \end{pmatrix} \tag{9}$$

Below is the plot for given curve based on initial conditions, obtained by iterating through the above equation.

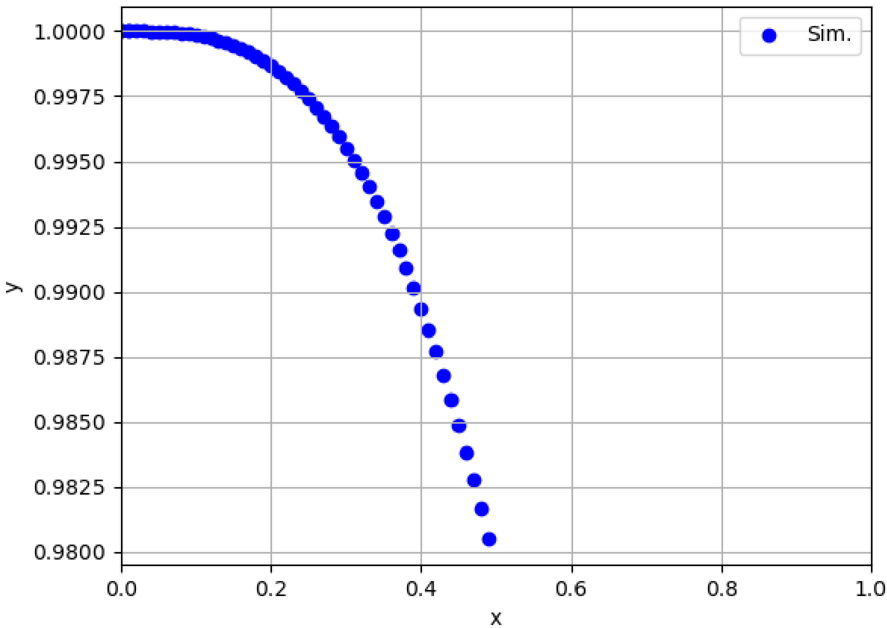

Fig. 1: Computational solution of $(y''')^2 + (y'')^3 + (y')^4 + (y)^5 = 0$