NCERT Presentation

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Problem

Problem Statement

Solve the differential equation $y' + y = e^x$ with initial conditions $y\left(0\right) = 1$

Solution

Theoretical Solution

Given equation is a linear first order differential equation, so laplace transform may be used to solve it. Some properties of laplace transform used are,

$$\mathcal{L}\left(f\left(t\right)\right) = \int_0^\infty e^{-st} f\left(t\right)\, dt \tag{3.1}$$

$$\mathcal{L}\left(\kappa f\left(t\right)\right) = \kappa \mathcal{L}\left(f\left(t\right)\right) \tag{3.2}$$

$$\mathcal{L}\left(y'\right) = sY\left(S\right) - y_0 \tag{3.3}$$

Where $\mathcal{L}\left(y\right) = Y\left(s\right)$

Applying Laplace Transform to given differential Equation,

$$\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(e^x) \tag{3.4}$$

$$sY(s) - y_0 + Y(S) = \int_0^{\infty} e^{-x(x-1)} dx = \frac{1}{s-1} \tag{3.5}$$

$$Y(s) = \left(\frac{y_0 - 0.5}{s+1}\right) + \frac{1}{2}\left(\frac{1}{s-1}\right) \tag{3.6}$$

Taking Laplace Inverse on both sides we get,

$$y = \mathcal{L}^{-1}\left(\frac{y_0 - 0.5}{s+1}\right) + \mathcal{L}^{-1}\frac{1}{2}\left(\frac{1}{s-1}\right) \tag{3.7}$$

$$y = (y_0 - 0.5)\, e^{-x} + \frac{1}{2} e^x \tag{3.8}$$

Putting in $y_0 = 1$ we get,

$$y = \frac{(e^x + e^{-x})}{2} \tag{3.9}$$

Computational Solution 1

Solving given differential equation using bilinear transform. Taking Laplace transform to both sides of the Differential Equation,

$$sY(s) - y_0 + Y(s) = \int_0^{\infty} e^{-x(x-1)}dx = \frac{1}{s-1} \tag{3.10}$$

$$Y(s) = \frac{y_0}{s+1} + \frac{1}{(s+1)(s-1)} \tag{3.11}$$

$$Y(s) = \left(\frac{y_0 - 0.5}{s+1}\right) + \frac{1}{2}\left(\frac{1}{s-1}\right) \tag{3.12}$$

Applying Bilinear Transform with $T = h$. To go from the domain of Laplace transform to that of Z-transform, we transform our s. On substituting we get,

Computational solution 1

$$Y\left(s\right)=\left(y_0-0.5\right)\left(\frac{h\left(1+z^{-1}\right)}{\left(2+h\right)+z^{-1}\left(h-2\right)}\right)+\frac{1}{2}\left(\frac{h\left(1+z^{-1}\right)}{\left(2-h\right)+z^{-1}\left(2+h\right)}\right) \tag{3.13}$$

$$=\left(\frac{y_0-0.5}{2+h}\right)\left(\frac{h\left(1+z^{-1}\right)}{1-\alpha z^{-1}}\right)+\left(\frac{h}{2\left(2-h\right)}\right)\left(\frac{h\left(1+z^{-1}\right)}{1-\alpha^{-1}z^{-1}}\right) \tag{3.14}$$

Where $\alpha=\frac{2-h}{2+h}$
Radius of convergence of the first term is, $|z|>|\alpha|$, for the second term it is, $|z|>\left|\alpha^{-1}\right|$. R.O.C of the total equation turns out to be, $|z|>max\left(|\alpha|,\left|\frac{1}{\alpha}\right|\right)$
Taking $\left(1-\alpha z^{-1]}\right)$ to $R.H.S$ we get,

$$Y\left(s\right)\left(1-\alpha z^{-1}\right)=h\left(\frac{y_0-0.5}{2+h}\right)\left(\left(1+z^{-1}\right)\right)+\left(\frac{h}{2\left(2-h\right)}\right)\left(\frac{h\left(1+z^{-1}\right)\left(1-\alpha z^{-1}\right)}{1-\alpha^{-1}z^{-1}}\right) \tag{3.15}$$

Computational Solution 1

After applying algebraic manipulations on the second term, above equation comes out to be,

$$Y(s)\left(1-\alpha z^{-1}\right)=h\left(\frac{y_0-0.5}{2+h}\right)\left(\left(1+z^{-1}\right)\right)+ \tag{3.16}$$

$$\left(\frac{h}{2(2+h)}\right)\left(\frac{1+\alpha-\alpha^2-\alpha^3}{1-\alpha^{-1}z^{-1}}+\alpha^2z^{-1}-\alpha\left(1-\alpha-\alpha^2\right)\right) \tag{3.17}$$

Applying inverse Z-transform,

$$y_{n+1}=\alpha y_n+h\left(\frac{y_0-0.5}{2+h}\right)(\delta[n]+\delta[n-1])+\left(\frac{h}{2(2+h)}\right)\times \tag{3.18}$$

$$\left(\left(1+\alpha-\alpha^2-\alpha^3\right)\alpha^{-n}u(n)+\alpha^2\delta[n-1]-\alpha\left(1-\alpha-\alpha^2\right)\delta[n]\right) \tag{3.19}$$

Computational Solution 1

here, δ is defined as,

$$\delta[n - n_0] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases} \tag{3.20}$$

Final General Difference Equation comes out to be,

$$y_{n+1} = \alpha y_n + h\left(\frac{y_0 - 0.5}{2 + h}\right)(\delta[n] + \delta[n-1]) + \left(\frac{h}{2(2+h)}\right) \times \tag{3.21}$$

$$\left(\left(1 + \alpha - \alpha^2 - \alpha^3\right)\alpha^{-n}u(n) + \alpha^2\delta[n-1] - \alpha\left(1 - \alpha - \alpha^2\right)\delta[n]\right) \tag{3.22}$$

Computational Solution 2

Finding the difference equation using trapezoid law,

Given Differential Equation,

$$y' = -y + e^x \tag{3.23}$$

$$\int_{y_n}^{y_{n+1}} dy = -\int_{x_n}^{x_{n+1}} y dx + \int_{x_n}^{x_{n+1}} e^x dx \tag{3.24}$$

Discretizing steps using trapezoid rule,

$$y_{n+1} = y_n - \frac{h}{2}(y_n + y_{n+1}) + e^{x_n}(e^h - 1) \tag{3.25}$$

$$y_{n+1} - y_n = -\frac{h}{2}(y_n + y_{n+1}) + e^{x_n}\left(e^h - 1\right) \tag{3.26}$$

$$y_{n+1} = y_n\left(\frac{2-h}{2+h}\right) + \frac{2e^{x_n}}{2+h}\left(e^h - 1\right) \tag{3.27}$$

Computational Solution