

8.1.9

EE24BTECH11005 - Arjun Pavanje

Question: Find the area of the region bounded by the parabola $y = x^2$, and $y = |x|$.

Solution:

Equation for area enclosed is given by,

$$\int_{-1}^1 (|x| - x^2) dx \quad (1)$$

$$= \int_{-1}^0 (x + x^2) dx + \int_0^1 (x - x^2) dx \quad (2)$$

$$= 2 \int_0^1 (x - x^2) dx \quad (3)$$

There are two ways to solve the above integral, Theoretically and Computationally (trapezoid method). We shall compare the results obtained by both methods.

Theoretical Solution:

$$2 \int_0^1 (x - x^2) dx \quad (4)$$

$$= 2 \left(\left[\frac{x^2}{2} \right]_{x=0}^{x=1} - \left[\frac{x^3}{3} \right]_{x=0}^{x=1} \right) \quad (5)$$

$$= 2 \left(\frac{1}{2} - \frac{1}{3} \right) \quad (6)$$

$$= \frac{1}{3} \quad (7)$$

Computational Solution:

Taking trapezoid shaped strips of small area and adding them all up. Say we have to find the area of y_x from $x = x_0$ to $x = x_n$, discretize points on the x axis $x_0, x_1, x_2, \dots, x_n$ such that they are equally spaced with step-size h .

Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2}h(y(x_1) + y(x_0)) + \frac{1}{2}h(y(x_2) + y(x_1)) + \dots + \frac{1}{2}h(y(x_n) + y(x_{n-1})) \quad (8)$$

$$= h \left[\frac{1}{2}(y(x_0) + y(x_n)) + y(x_1) + \dots + y(x_{n-1}) \right] \quad (9)$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, (x_0, x_1, \dots, x_n) be equidistant points with step-size h .

$$A(x_n + h) = A(x_n) + \frac{1}{2}h(y(x_n + h) + y(x_n)) \quad (10)$$

We can repeat this till we get required area.

Discretizing the steps, making $A(x_n) = A_n$, $y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h(y_{n+1} + y_n) \quad (11)$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h((y_n + hy'_n) + y_n) \quad (12)$$

$$A_{n+1} = A_n + \frac{1}{2}h(2y_n + hy'_n) \quad (13)$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \quad (14)$$

$$x_{n+1} = x_n + h \quad (15)$$

In the given question, $y_n = x_n + x_n^2$ and $y'_n = 1 - 2x_n$
General Difference Equation will be given by,

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \quad (16)$$

$$= A_n + h(x_n + x_n^2) + \frac{1}{2}h^2(1 - 2x_n) \quad (17)$$

$$= A_n + x_n(h - h^2) + x_n^2(h) + \frac{h^2}{2} \quad (18)$$

$$x_{n+1} = x_n + h \quad (19)$$

Iterating till we reach $x_n = 1$ will return required area. Note, Area obtained is to be multiplied by 2 as the calculated area only accounts for one half of the graph.

Area obtained computationally: 0.33340024948120117 sq. units

Area obtained theoretically: $\frac{1}{3} = 0.33333 \dots$ sq. units

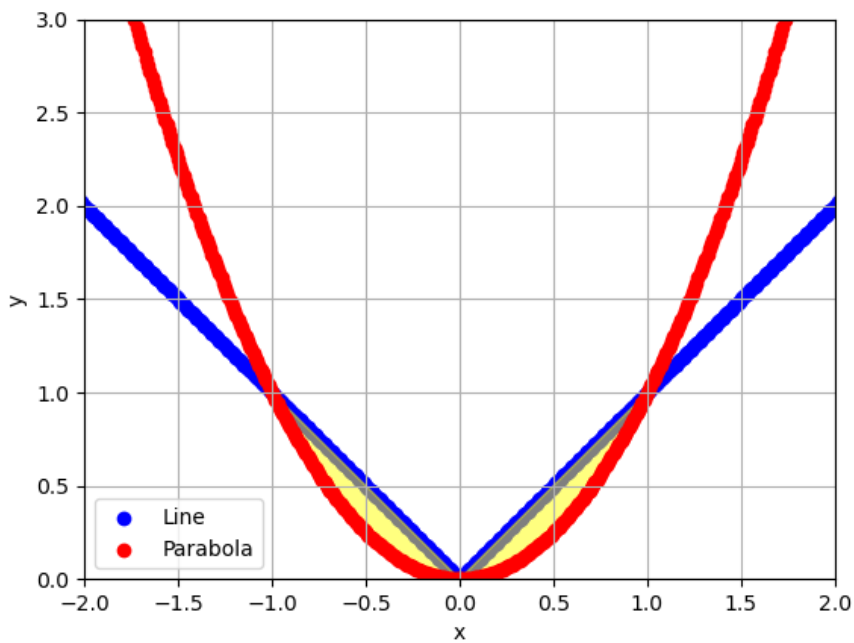


Fig. 1: Graph of the parabola $y = x^2$ and $y = |x|$ and the area enclosed between them