EE1060 Quiz 6

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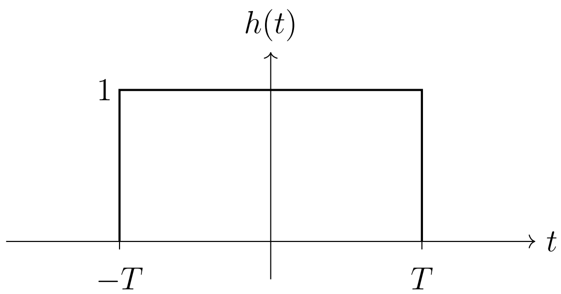
Chapter 1

Introduction

The goal of this report is to study the convolution of a signal with a rectangular kernel, analyze how shifting and modifying the kernel affects the output and understand the system's behavior under various conditions.
The rectangular kernel is defined as :

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$



1.1 Convolution :

Overview

Convolution is a powerful mathematical operation widely used in engineering, physics, and signal processing. Closely tied to Fourier transforms, convolution helps describe how systems respond to inputs over time or space.

Definition

Mathematically, the convolution of two functions $f(x)$ and $g(x)$ is defined as:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t - \tau) \, d\tau$$

Alternatively, it can be written by switching the roles of the functions (convolution is commutative),

$$(f * g)(\tau) = \int_{-\infty}^{\infty} g(\tau) \cdot f(t - \tau) \, d\tau$$

This operation effectively combines two functions by sliding one across the other, multiplying pointwise values, and integrating the result.

Intuitive Understanding

We can think of convolution as a weighted averaging process. We can imagine that one function represents a signal, and the other acts as a filter or response function. As one function slides over the other, their overlap determines the value of the output. The integral collects these weighted overlaps across the domain.

Convolution and the Fourier Transform

One of the most useful aspects of convolution is its relationship with Fourier transforms. The **Convolution Theorem** states:

- The Fourier transform of a convolution is the product of the individual Fourier transforms:
  $$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}$$
- Conversely, the Fourier transform of a product is the convolution of the transforms:
  $$\mathcal{F}\{f \cdot g\} = \mathcal{F}\{f\} * \mathcal{F}\{g\}$$

This relationship allows easier computation of convolutions in the frequency domain.

Convolution with a Rectangular Kernel

The output of an LTI system with impulse response $h(t)$ and input $f(t)$ is given by the convolution integral

$$y(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau,$$

which "slides" one function over another and integrates their product. In our case, $h(t) = 1$ for $-T \leq t \leq T$ and 0 otherwise. Thus, $h(t-\tau) = 1$ exactly when $\tau \in [t-T,\ t+T]$. The convolution becomes

$$y(t) = \int_{max(0,t-T)}^{t+T} f(\tau)\, d\tau.$$

In short, the output is the **sliding integral** (or running sum) of f over the interval of length $2T$.

1.2 Step Input ($f(t) = u(t)$)

Convolving a unit step with the symmetric box yields a **ramp** that saturates. Specifically:

- y(t)=(x * h)(t) $= \int_{max(0,t-T)}^{T+t} 1\, d\tau$

- For $t < -T$, the interval $[t-T, t+T]$ lies entirely before 0, so $y(t) = 0$.

- For $-T \leq t < T$, the box partially overlaps the step:

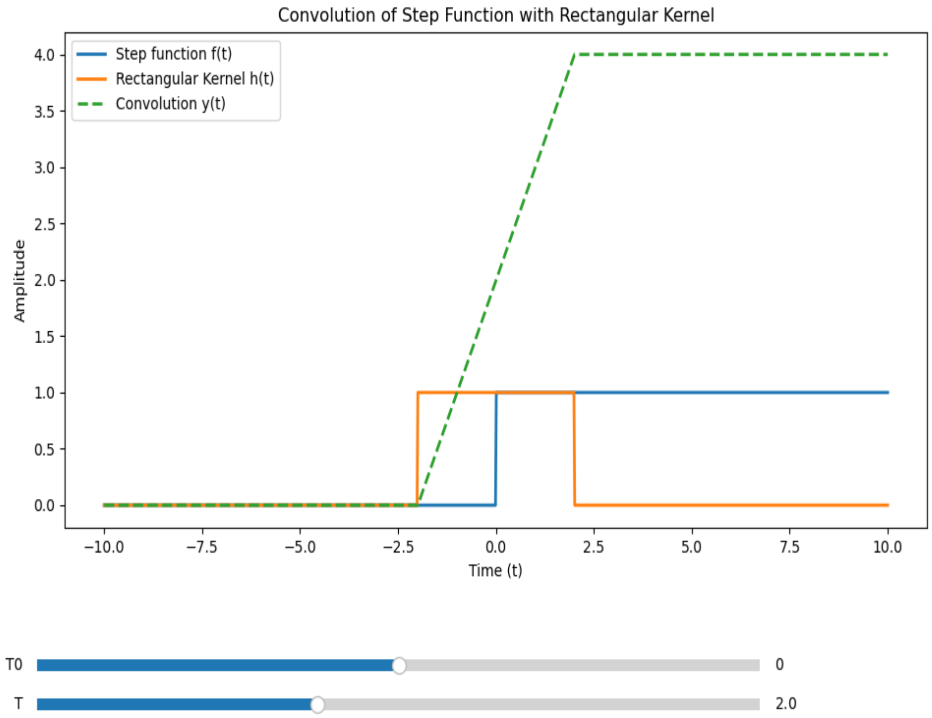
  $$y(t) = \int_{0}^{t+T} 1\, d\tau = t + T.$$

  Thus, $y(t)$ rises linearly from 0 (at $t = -T$) up to $2T$ (at $t = T$).

- For $t \geq T$, the box fully covers the step:

  $$y(t) = \int_{t-T}^{t+T} 1\, d\tau = 2T.$$

Thus, $y(t)$ is triangular-shaped: it increases from 0 to $2T$ over $[-T, T]$ and then stays constant.



1.3 Polynomial input

Let the polynomial be given by:

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

The convolution simplifies to:

$$y(t) = \int_{t-T}^{t+T} f(\tau)\, d\tau$$

Substituting $f(\tau) = a_n \tau^n + a_{n-1} \tau^{n-1} + \cdots + a_1 \tau + a_0$, we get:

$$y(t) = \int_{t-T}^{t+T} \left(a_n \tau^n + a_{n-1} \tau^{n-1} + \cdots + a_1 \tau + a_0 \right) d\tau$$

For each term in the polynomial, we compute the integral. For the τ^n term:

$$\int_{t-T}^{t+T} \tau^n \, d\tau = \left[\frac{\tau^{n+1}}{n+1} \right]_{t-T}^{t+T} = \frac{(t+T)^{n+1} - (t-T)^{n+1}}{n+1}$$

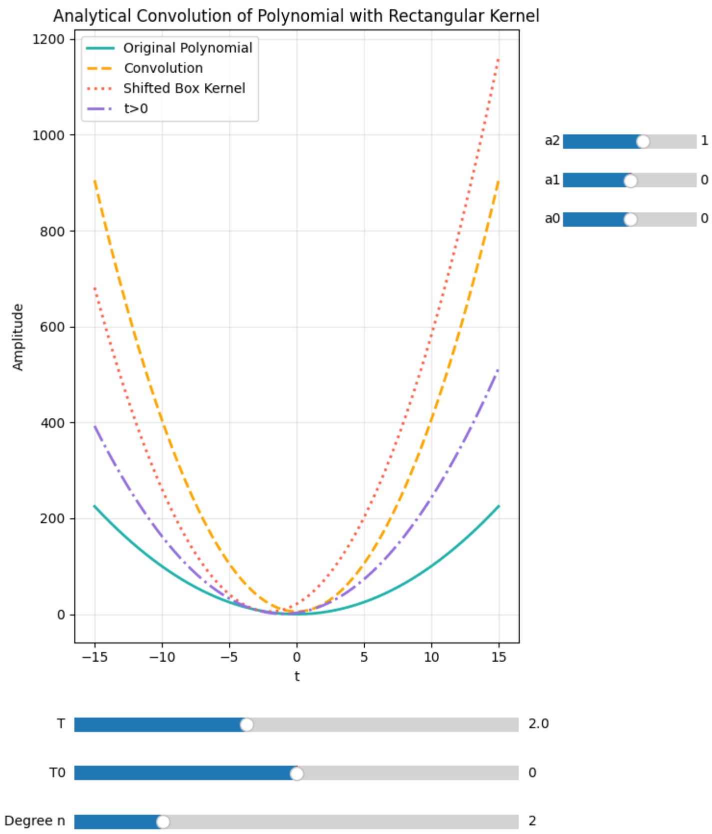
For the constant term :

$$\int_{t-T}^{t+T} a_0 \, d\tau = a_0 \cdot [\tau]_{t-T}^{t+T} = a_0 \cdot 2T$$

Overall, the result is

$$\begin{aligned} y_s(t) &= \sum_{k=0}^{n} a_k \int_{t-T}^{t+T} \tau^k \, d\tau \\ &= \sum_{k=0}^{n} a_k \cdot \frac{(t+T)^{k+1} - (t-T)^{k+1}}{k+1}. \end{aligned}$$

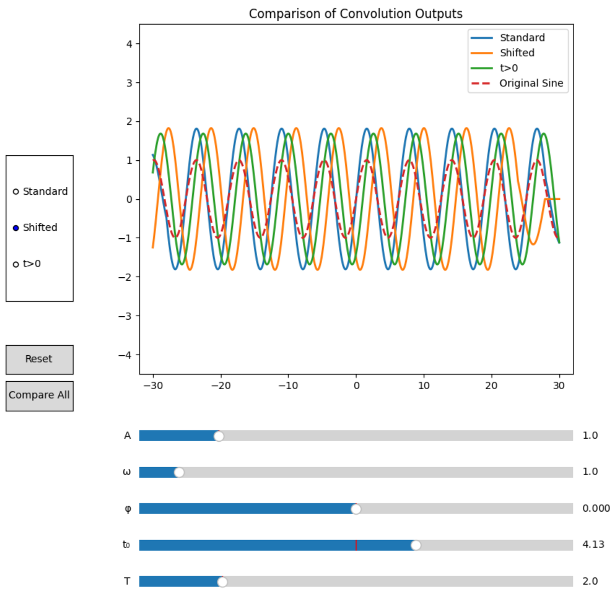
This is a polynomial of degree at most $n+1$.
This is convolution of polynomial t^2 with rectangle kernel taking $T = 2$



1.4 Sinusoidal Input

$$
\begin{aligned}
H(\omega) = (f * h)(t) &= \int_{-\infty}^{\infty} A \sin(\omega\tau + \phi)\ h(t-\tau)\ d\tau \\
&= A \int_{t-T}^{t+T} \sin(\omega\tau + \phi)\, d\tau \\
&= \frac{A}{\omega} \left[-\cos(\omega\tau + \phi)\right] i_{t-T}^{t+T} \\
&= \frac{A}{\omega} \left[\cos(\omega(t-T) + \phi) - \cos(\omega(t+T) + \phi)\right] \\
&= \frac{2A}{\omega} \sin(\omega t + \phi) sin(\omega T)
\end{aligned}
$$

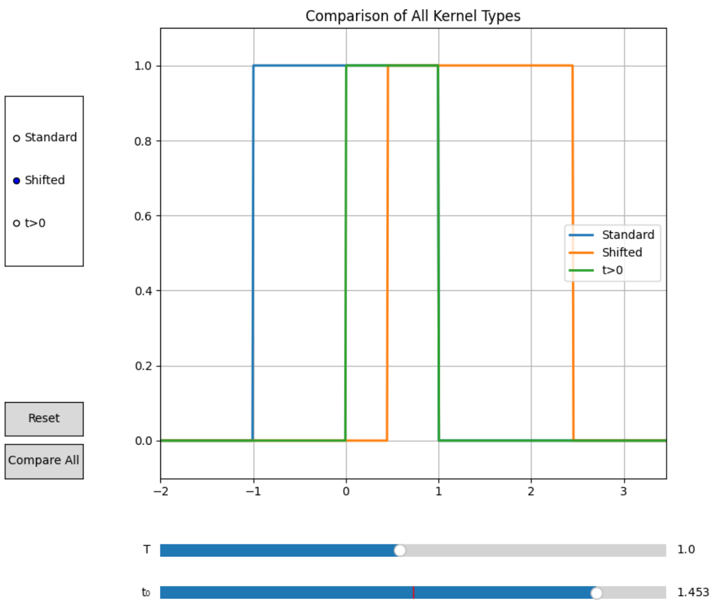
We observe, that if we convolve input signal $f(t) = sin(t)$ with box kernal, resultant signal is just scaled by a factor of $\frac{2}{\omega} \sin(\omega T)$. Time period remains unchanged. For low frequencies ($\omega \to 0$), $\sin(\omega T) \approx \omega T$, so $H(\omega) \approx 2T$. At higher frequencies, $|H(\omega)|$ falls off, and the rectangular window acts as a low-pass filter.



1.5 Impulse Input

$$
\begin{aligned}
(f * h) &= \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} \delta(\tau)\, h(t-\tau)\, d\tau \\
&= h(t)
\end{aligned}
$$

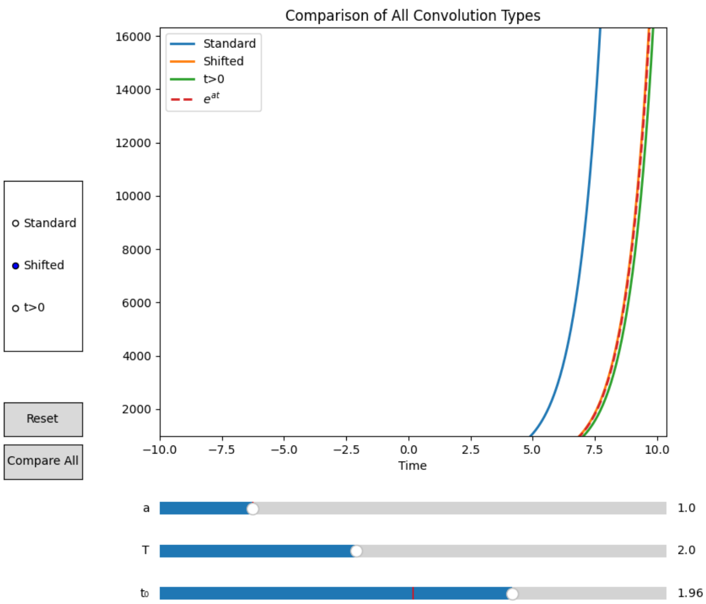
We observe that we get the rectangular kernel back as the output.



1.6 Expontial Function

$$\begin{aligned}(f * h) &= \int_{-\infty}^{\infty} e^{a\tau} h(t-\tau) d\tau \\ &= \int_{t-\tau_0-T}^{t-\tau_0+T} e^{a\tau} d\tau \\ &= \frac{1}{a} e^{a\tau} \Big|_{t-\tau_0-T}^{t-\tau_0+T} \\ &= \frac{e^{a(t-\tau_0+T)} - e^{a(t-\tau_0-T)}}{a} \\ &= \frac{2e^{a(t-\tau_0)}}{a} \sinh(aT)\end{aligned}$$

We get a scaled hyperbolic sine function corresponding to input exponential wave as the result of convolution.



Chapter 2

Behavior of the Convolution

2.1 Step input

The result of the convolution is a piecewise-linear function with three distinct regions:

- **Zero Region** ($t < T_0 - T$): No overlap between u and h, hence $(u*h)(t) = 0$.
- **Linear Ramp** ($T_0 - T \leq t \leq T_0 + T$): Partial overlap; the output grows linearly:
$$(u * h)(t) = t - (T_0 - T).$$
- **Plateau Region** ($t > T_0 + T$): Full overlap; the output is constant:
$$(u * h)(t) = 2T.$$

2.1.1 Specific Cases

- $T = 0$: The kernel becomes a Dirac delta centered at T_0, and the convolution simply shifts the step function:
$$(u * h)(t) = u(t - T_0).$$
- **Very small** T: The ramp is extremely narrow and steep; the output approximates a sharp step.
- **Very large** T: The ramp becomes extremely gradual and wide. Without normalization, the plateau height grows unbounded; with normalization, the rise becomes extremely slow.
- $T_0 = 0$: Symmetric case; ramp spans from $-T$ to T around the origin.

2.1.2 Graphical Interpretation

The convolution measures the area of overlap between the step function and the sliding rectangular window. The output is:

- 0 when no overlap,
- proportional to the length of overlap during partial overlap (ramp region),
- maximal when full overlap (plateau region).

2.2 Polynomial Input

Let $f(t)$ be a polynomial of degree n defined on the entire real line, and let $h(t)$ be a rectangular pulse of width $2T$ centered at T_0, i.e.,

$$h(t) = \begin{cases} 1, & t \in [T_0 - T, T_0 + T] \\ 0, & \text{otherwise} \end{cases}$$

The convolution $(f * h)(t)$ is given by:

$$(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau)\, d\tau = \int_{t-(T_0+T)}^{t-(T_0-T)} f(\tau)\, d\tau$$

- **Smoothness:** Since f is continuous and h is piecewise continuous, the convolution is continuous and differentiable. The result is a polynomial of degree atmost $n + 1$.
- **No Piecewise Behavior:** Because $f(t)$ has full (infinite) support, the convolution is defined for all $t \in \mathbf{R}$. The expression for $(f * h)(t)$ does not need to be split into regions, it's a single, continuous polynomial.
- **Output Degree:** If $f(t)$ is degree-n, then $(f * h)(t)$ is degree-$(n + 1)$ due to integration.
- **Moving Average:** The convolution acts as a moving average over a width $2T$, smoothing the input. The result can also be written as: $(f * h)(t) = \int_{-T}^{T} f(t - T_0 + \tau)\, d\tau$ when re-centered around the kernel.
- **Specific Case $T_0 = 0$:** The kernel is symmetric about the origin. Then:

  $$(f * h)(t) = \int_{-T}^{T} f(t + \tau)\, d\tau$$

  which yields a symmetric smoothing operation with no phase shift.

2.2.1 Graphical Interpretation

- The convolution at time t computes the area under the polynomial $f(\tau)$ inside a window $[t - T, t + T]$.
- This results in a polynomial curve $(f * h)(t)$ that smooths the original function.
- The output is continuous and globally defined (infinite support), and it inherits smoothness and general shape from $f(t)$.
- As T increases, the convolution becomes smoother and approximates a broader moving average.

Chapter 3

Causal (One-Sided) Rectangular Kernel

Suppose $h(t) = 1$ for $0 \le t \le T$ and 0 otherwise. Then,

$$y(t) = \int_{max(0,t-T)}^{t} f(\tau)\, d\tau,$$

for $t \ge 0$ (and $y(t) = 0$ for $t < 0$ if f was zero for $t < 0$).

3.1 Step Input

For $f(t) = u(t)$:

$$y(t) = \begin{cases} t, & 0 \le t < T, \\ T, & t \ge T. \end{cases}$$

Thus, $y(t)$ ramps up to T and stays there.

- For $t < 0$: output is 0 (causal).
- For $0 \le t < T$: output grows linearly (ramp).
- For $t \ge T$: output saturates to T (plateau).

3.2 Polynomial input

$$\begin{aligned} y_c(t) = (f * h_c)(t) &= \int_{-\infty}^{\infty} f(\tau) h_c(t-\tau)\, d\tau \\ &= \int_{t-T}^{t} f(\tau)\, d\tau \\ &= \sum_{k=0}^{n} a_k \int_{t-T}^{t} \tau^k\, d\tau \\ &= \sum_{k=0}^{n} a_k \cdot \frac{t^{k+1} - (t-T)^{k+1}}{k+1}. \end{aligned}$$

- As $T \to 0$, the system approaches the identity:

$$(f * h)(t) \to f(t).$$

- As T increases, the smoothing window widens, and the high-frequency content in $f(t)$ gets suppressed more.

3.3 Sinusoidal Input

Here, we take a modified $h\,(t)$, which only considers the positive half of the function $t > 0$.

$$g\,(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned} (f * h)\,(t) &= \int_{-\infty}^{\infty} A \sin\left(\omega\tau + \phi\right)\, g\,(t-\tau)\; d\tau \\ &= \left[-\frac{A}{\omega} \cos\left(\omega\tau + \phi\right) \right]_{t-T}^{t} \\ &= \frac{A}{\omega} \left(\cos\left(\omega t - \omega T + \phi\right) - \cos\left(\omega t + \phi\right) \right) \end{aligned}$$

The Fourier transform of the one-sided box is

$$H(\omega) = \int_0^T e^{-j\omega\tau}\, d\tau = \frac{1 - e^{-j\omega T}}{j\omega},$$

which in magnitude behaves similarly to the symmetric box but introduces a phase shift. The output is a sinusoid scaled by $|H(\omega)|$ with a phase shift.

3.4 Impulse Input

Here, we take a modified $h(t)$, which only considers the positive half of the function $t > 0$.

$$g(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned} (f * h) &= \int_{\infty}^{\infty} g(\tau)\, g(t-\tau)\, d\tau \\ &= \int_{\infty}^{\infty} \delta(\tau)\, g(t-\tau)\, d\tau \\ &= g(t) \end{aligned}$$

This is basically the box kernel shifted to the right by T and it's time period halved to T.

3.5 Exponential Input

Here, we take a modified $h(t)$, which only considers the positive half of the function $t > 0$.

$$g(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned} (f * h) &= \int_{t-T}^{t} e^{a\tau}\, d\tau \\ &= \frac{1}{a} e^{a\tau} \Big|_{t-T}^{t} \\ &= \frac{1}{a} \left(e^{at} - e^{a(t-T)} \right) \\ &= \frac{e^{at}}{a} \left(1 - e^{-aT} \right) \end{aligned}$$

Chapter 4

Time-Shifted Kernel

If we shift the symmetric kernel by τ_0, using $h_{\tau_0}(t) = h(t - \tau_0)$, then

$$(f * h_{\tau_0})(t) = (f * h)(t - \tau_0).$$

Thus, the output is simply a delayed version of the original output by τ_0. In the frequency domain, this introduces a phase factor $e^{-j\omega\tau_0}$, representing a linear phase delay.

4.1 Step input

Changing the center T_0 affects:

- **Horizontal Shift**: Increasing T_0 shifts the ramp and plateau to the right. Decreasing T_0 shifts them to the left.
- **Timing of Transition**:
  - Positive T_0 delays the onset of the ramp.
  - Negative T_0 causes the ramp to begin earlier.

4.2 Polynomial input

- **Horizontal Shift**: Increasing T_0 shifts the smoothed polynomial to the right, while decreasing T_0 shifts it to the left.
- **Phase Relation**:
  - Positive T_0 delays the response; the smoothing is centered at a later point in time.

- Negative T_0 advances the response; the smoothing occurs earlier.

- **Symmetry**: For $T_0 = 0$, the smoothing is centered about the origin, preserving symmetry if the input polynomial is even.

4.3 Sinusoidal Input

Here, we are dealing with a modified $h(t)$,

$$g(t) = \begin{cases} 1 & -T + t_0 < t < T + t_0 \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned}
(f * h)(t) &= \int_{-\infty}^{\infty} A \sin(\omega\tau + \phi)\; g(t - \tau)\; d\tau \\
&= \int_{t+\tau_0-T}^{t+\tau_0+T} A \sin(\omega\tau + \phi)\; d\tau \\
&= \left[-\frac{A}{\omega} \cos(\omega\tau + \phi) \right]_{t+\tau_0-T}^{t+\tau_0+T} \\
&= \frac{A}{\omega} \left(\cos(\omega(t + \tau_0 - T) + \phi) - \cos(\omega(t + \tau_0 + T) + \phi) \right) \\
&= \frac{2A}{\omega} \sin(\omega T) \sin(\omega(t + \tau_0) + \phi)
\end{aligned}$$

We observe that this is just the result obtained in the first part, but shifted in time domain by τ_0. Shift in kernal just causes a phase shift of τ_0 in the resultant sinusoid. Time period of function remains the same. After convolution, resultant is just scaled and shifted in time-domain.

4.4 Impulse Input

Here, we are dealing with a modified $h(t)$,

$$g(t) = \begin{cases} 1 & -T + t_0 < t < T + t_0 \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned}
(f * g) &= \int_{\infty}^{\infty} f(\tau)\, g(t - \tau)\, d\tau \\
&= \int_{\infty}^{\infty} \delta(\tau)\, g(t - \tau)\, d\tau \\
&= g(t) = h(t - t_0)
\end{aligned}$$

We basically get a time shifted version of the first result

4.5 Exponential Function

Here, we are dealing with a modified $h\left(t\right)$,

$$g\left(t\right) = \begin{cases} 1 & -T + t_0 < t < T + t_0 \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned} (f * h) &= \int_{-\infty}^{\infty} e^{a\tau} g(t-\tau) d\tau \\ &= \int_{t-\tau_0-T}^{t-\tau_0+T} e^{a\tau} d\tau \\ &= \frac{1}{a} e^{a\tau} \Big|_{t-\tau_0-T}^{t-\tau_0+T} \\ &= \frac{e^{a(t-\tau_0+T)} - e^{a(t-\tau_0-T)}}{a} \\ &= \frac{2e^{a(t-\tau_0)}}{a} \sinh(aT) \end{aligned}$$

We basically get a time shifted version of the first result

Chapter 5

Effect of T

A larger window ($2T$) means more averaging/smoothing. For a fixed input, increasing T broadens the integration interval. For example, if f is nonnegative, a larger T produces a larger total output (the area under f over a longer interval), and the system behaves like a stronger low-pass filter. In the limit $T \to 0$, $y(t) \approx 2T\, f(t)$ (output vanishes), whereas as T grows, the output changes more slowly.

5.1 Step input

Changing the half-width T affects:

5.1.1 Ramp Width

The ramp extends over an interval of width $2T$. Increasing T makes the ramp wider.

5.1.2 Ramp Slope

- Without normalization, the slope remains 1.
- With normalization (so that the kernel has area 1), the slope becomes $\frac{1}{2T}$, decreasing with T.

5.1.3 Plateau Height

- Without normalization, the plateau height is $2T$.
- With normalization, the plateau height is 1.

5.1.4 Smoothing

Larger T results in more smoothing (longer transition). Smaller T results in sharper transitions (more step-like).

5.2 Polynomial input

Changing the half-width T of the rectangular kernel affects:

5.2.1 Smoothing Window

The kernel averages the polynomial over an interval of width $2T$. Increasing T increases the extent of smoothing.

5.2.2 Output Detail

- Larger T averages over a wider region, reducing local fluctuations and smoothing sharp variations in the polynomial.
- Smaller T captures more local detail, making the output resemble the original polynomial more closely.

5.2.3 Scaling Effects

- Without normalization, the output is scaled by $2T$:

$$y(t) \approx 2Tf(t).$$

- With normalization (kernel area 1), the output approximates a local average of the polynomial over $[t - T, t + T]$.

5.2.4 Edge Effects

Near the boundaries of the domain or near sharp changes, a larger T introduces more smoothing and can reduce the accuracy of representing rapid changes.

5.3 Exponential

5.3.1 Smoothing Window

The rectangular kernel averages the function over an interval of width $2T$. Increasing T increases the domain of integration, which broadens the effective smoothing window. For exponential functions, this does not result in visual smoothing like in polynomials but does increase the contribution from more distant parts of the function.

5.3.2 Output Detail

In contrast, for the exponential function:

- Larger T increases the value of $\sinh(aT)$, thereby amplifying or attenuating the output significantly depending on the sign of a.
- Smaller T limits the influence of the exponential's growth or decay, keeping the output closer to a local average.

5.3.3 Scaling Effects

The output includes the scaling factor $\frac{2}{a}\sinh(aT)$, which depends non-linearly on T:

$$y(t) = \frac{2e^{at}}{a}\sinh(aT)$$

- For $a > 0$, $\sinh(aT)$ grows exponentially with T, leading to rapid amplification.
- For $a < 0$, the output is attenuated more strongly as T increases.
- When $a \to 0$, $\sinh(aT) \approx aT$, and the output approaches $2Te^{at}$, a linear dependence similar to polynomial case without normalization.

5.3.4 Edge Effects

As with polynomials, edge effects arise when t is near the boundary of the signal domain:

- A large T may cause the kernel to extend beyond the defined domain of $f(t)$, leading to underrepresented data and boundary distortion.
- This effect becomes more pronounced for exponential inputs with high curvature (large $|a|$).

Choosing an appropriate T is therefore essential, especially when convolving with rapidly changing functions such as exponentials, to balance detail retention and output stability.

5.3.5 Energy Considerations

The output energy E_y for $a < 0$:

$$E_y \propto \frac{\sinh(2|a|T)}{|a|}$$

- Grows exponentially with T
- Finite for all T when $a < 0$

5.3.6 Observation

The kernel width T dramatically alters the system's response to exponential signals:

- Small T: Preserves signal details but with weak scaling
- Large T: Dominated by exponential growth/decay characteristics
- Critical regime $aT \approx 1$: Transition between linear and exponential behavior

5.4 Sine Wave

5.4.1 Smoothing Window

The rectangular kernel acts as a moving average filter. For sinusoidal inputs, this results in frequency-dependent attenuation:

- Larger T broadens the averaging interval, allowing more cycles of the sine wave into the window, potentially reducing the output amplitude.
- Smaller T captures fewer oscillations, preserving more high-frequency detail.

5.4.2 Output Detail

The output remains a sinusoidal signal of the same frequency as the input, but with its amplitude scaled by a factor that depends on the kernel width T and the input frequency ω. This scaling reflects the low-pass filtering effect of the rectangular kernel.

5.4.3 Scaling Effects

- The output amplitude scales with $\dfrac{\sin(\omega T)}{\omega}$.
- As $T \to 0$, $\sin(\omega T) \approx \omega T \Rightarrow y(t) \approx 2AT\sin(\omega t)$.
- As $T \to \infty$, $\sin(\omega T)$ oscillates, leading to fluctuating output gain.

5.4.4 Edge Effects

- For finite-duration signals, kernel edges can extend beyond signal domain, causing amplitude loss near boundaries.
- For infinite-duration sinusoids, edge effects are theoretically negligible.

5.4.5 Energy Considerations

$$E_y \propto \left(\frac{2A\sin(\omega T)}{\omega}\right)^2$$

- Maximum energy occurs at $\omega T = (2n+1)\frac{\pi}{2}$
- Zero energy at $\omega T = n\pi$ — complete destructive interference

5.4.6 Observation

The convolution with a rectangular kernel has the effect of a low-pass filter:

- Small T: Preserves high-frequency components, behaves like a delta function
- Large T: Smooths the input, attenuates high frequencies
- Convolution amplitude is highly dependent on alignment between kernel width and sinusoid frequency

5.5 Impulse

5.5.1 Smoothing Window

The rectangular kernel in this case does not act as a smoothing filter since the input is instantaneous. Instead, it simply reveals the inherent shape of the kernel itself, as an impulse acts like an identity under convolution.

5.5.2 Output Detail

When convolved with an impulse, the output is exactly the kernel function. That is:

$$y(t) = (h * \delta)(t) = h(t)$$

The system responds by reproducing its own impulse response, which is the rectangular function defined over $-T < t < T$.

5.5.3 Scaling Effects

- The amplitude and duration of the output are determined solely by the kernel.
- As T increases, the output widens linearly and covers a broader time window.
- The area under the output remains constant if the kernel is normalized, but increases proportionally to $2T$ otherwise.

5.5.4 Edge Effects

- There are no edge effects since the impulse is localized and fully contained within the convolution domain.
- The output is symmetric and centered at the impulse location.

5.5.5 Energy Considerations

$$E_y = \int_{-\infty}^{\infty} |h(t)|^2 \, dt = \int_{-T}^{T} 1^2 \, dt = 2T$$

- The energy of the output is directly proportional to the width of the kernel.
- Wider kernels result in greater energy due to their larger time support.

5.5.6 Observation

The convolution of an impulse input with a rectangular kernel yields the kernel itself:

- Useful for characterizing system behavior (impulse response).
- No frequency filtering or distortion occurs.
- The result serves as a baseline for analyzing responses to more complex signals.

Chapter 6

Denoising

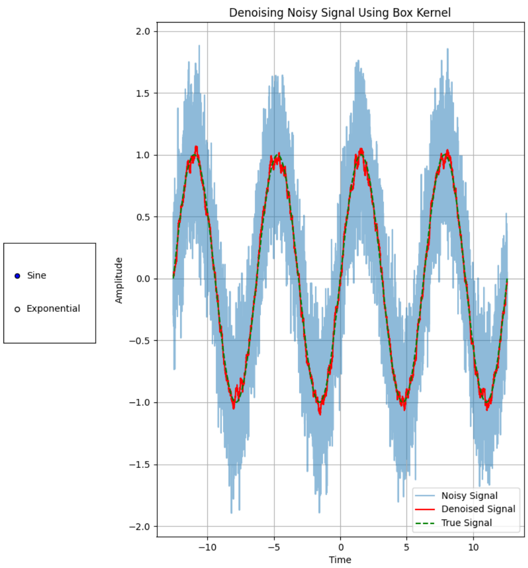
We remove noise from a noisy sine wave by applying convolution with a box kernel. The noisy signal is created by adding random noise to a clean sine wave. The box kernel $h(t)$ is defined as:

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

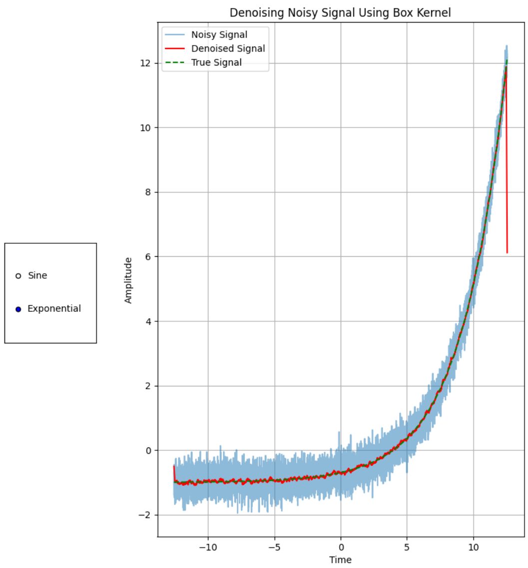
and is normalized so that the sum of its values equals 1.

Convolution with this kernel replaces each point in the signal by the average of its neighboring points. Since random noise varies quickly, it cancels out when averaged, while the smooth sine wave remains. This process reduces high-frequency noise and preserves the main shape of the sine wave. A larger T results in stronger denoising but may also smooth out important details.

(a) Sine



(b) Exponential



Chapter 7

Conclusion

The convolution of various input signals with a rectangular (box) kernel demonstrates how this operation acts as a fundamental smoothing and averaging tool in signal processing. For step, polynomial, sinusoidal, impulse, and exponential inputs, the rectangular kernel produces characteristic effects: smoothing sharp transitions, attenuating high-frequency components, and scaling or shifting the output depending on the kernel width and position.

Adjusting the kernel width 'T' controls the degree of smoothing and detail retention, while time-shifting the kernel introduces corresponding delays in the output. Overall, convolution with a box kernel provides a versatile method for analyzing and processing signals, balancing noise reduction with preservation of essential signal features.

Chapter 8

Bibliography

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