EE1204 Assignment 1

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1. Two point charges of equal magnitude q are positioned at $z = \pm\frac{d}{2}$. We aim to determine:

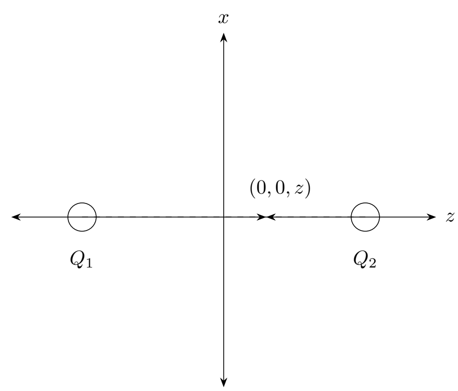
(a) The electric field everywhere on the z-axis.

(b) The electric field everywhere on the x-axis.

(c) The results of (a) and (b) if the charge at $z = -\frac{d}{2}$ is $-1C$ instead of $1C$.

Solution:

(a) Field on z-axis (E): $E = E_1 + E_2$ (By superposition)



- E_1: Field due to Q_1
- E_2: Field due to Q_2

$$E_1 = \frac{q}{4\pi\epsilon_0}\frac{|x - \frac{d}{2}|}{(x - \frac{d}{2})^3}\hat{k}, \quad E_2 = \frac{q}{4\pi\epsilon_0}\frac{|x + \frac{d}{2}|}{(x + \frac{d}{2})^3}\hat{\mathbf{k}}$$

$$E = E_1 + E_2 = \frac{q}{4\pi\epsilon_0}\left[\frac{|x - \frac{d}{2}|}{(x - \frac{d}{2})^3} + \frac{|x + \frac{d}{2}|}{(x + \frac{d}{2})^3}\right]\hat{\mathbf{k}}$$

Case 1: $z > \frac{d}{2}$

$$E = \frac{q}{2\pi\epsilon_0}\left[\frac{z^2 + (\frac{d}{2})^2}{(z^2 - (\frac{d}{2})^2)^2}\right](\hat{\mathbf{k}})$$

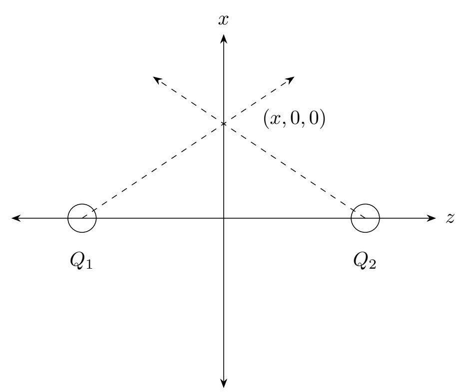
Case 2: $z < -\frac{d}{2}$

$$E = \frac{q}{2\pi\epsilon_0}\left[\frac{z^2 + (\frac{d}{2})^2}{(z^2 - (\frac{d}{2})^2)^2}\right](-\hat{\mathbf{k}})$$

Case 3: $-\frac{d}{2} < z < \frac{d}{2}$

$$\begin{aligned} E &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(z+\frac{d}{2})^2} - \frac{1}{(z-\frac{d}{2})^2}\right](\hat{\mathbf{k}}) \\ &= \frac{q}{2\pi\epsilon_0}\left[\frac{zd}{(z^2 - (\frac{d}{2})^2)^2}\right](-\hat{\mathbf{k}}) \end{aligned}$$

(b)



Field on x-axis (E): $E_1 + E_2$

E_1 : Field due to q_1, E_2 : Field due to q_2

$$E = \frac{q}{4\pi\epsilon_0}\left(\frac{x\hat{i} - \frac{d}{2}\hat{k}}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} + \frac{\left(x\hat{i} + \frac{d}{2}\hat{k}\right)}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\right)\hat{\mathbf{i}}$$

$$E = \frac{q}{2\pi\epsilon_0}\frac{x}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\hat{\mathbf{i}}.$$

(c) Result obtained in (a) becomes,
Case 1: $z > \frac{d}{2}$,

$$E = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(z-\frac{d}{2})^2} - \frac{1}{(z+\frac{d}{2})^2}\right)\hat{\mathbf{k}}$$
$$E = \frac{q}{2\pi\epsilon_0}\frac{zd}{(z^2-(\frac{d}{2})^2)^2}\hat{\mathbf{k}}.$$

Case 2: $z < -\frac{d}{2}$,

$$E = +\frac{q}{4\pi\epsilon_0}\left(\frac{zd}{(z^2-(\frac{d}{2})^2)^3})\right) - \hat{\mathbf{k}}$$

Case 3: $-\frac{d}{2} < z < \frac{d}{2}$

$$E = -\frac{q}{4\pi\epsilon_0}\left(\frac{1}{(z+\frac{d}{2})^2} + \frac{1}{(z-\frac{d}{2})^2}\right)\hat{\mathbf{k}}$$
$$= -\frac{q}{2\pi\epsilon_0}\left(\frac{z^2+(\frac{d}{2})^2}{(z^2-(\frac{d}{2})^2)^2}\right)\hat{\mathbf{k}}$$

Result obtained in (b) becomes,

$$E = \frac{q}{4\pi\epsilon_0}\left(\frac{x\hat{i} - \frac{d}{2}\hat{k}}{\left(x^2+\left(\frac{d}{2}\right)^2\right)^{3/2}} - \frac{\left(x\hat{i} + \frac{d}{2}\hat{k}\right)}{\left(x^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}\right)\hat{\mathbf{i}}$$
$$E = \frac{q}{4\pi\epsilon_0}\frac{d}{\left(x^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}(-\hat{\mathbf{i}}).$$

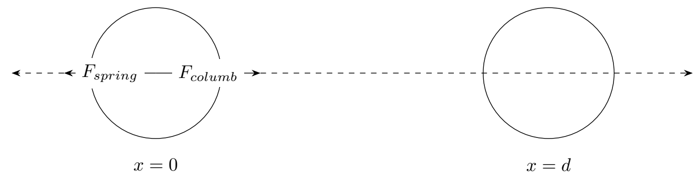
2. A crude device for measuring charge consists of two small insulating spheres of radius a, one of which is fixed in position. The other is movable along the x-axis and is subject to a restraining force kx, where k is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of Q/C, obtain the expression by which Q may be found as a function of x. Determine the maximum charge that can be measured in terms of ϵ_0, k, and d, and then state the separation of the spheres. What happens if a larger charge is applied?

Solution:

Say $Q_1 = +Q$ is the movable charge, $Q_2 = -Q$ is the stationary charge.

At equilibrium, spring force on $+Q$ is balanced by electrostatic force:

$$\frac{1}{4\pi\epsilon_0}\frac{Q^2}{(x-d)^2} = kx$$



$$Q = \sqrt{4\pi\epsilon_0 kx(x-d)^2}$$

Maximum measurable charge:

$$\frac{dQ}{dx} = 0$$

Differentiating,

$$(x-d)^2\left(x - \frac{d}{3}\right) = 0$$

$$x = d \quad \text{(minimum)}, \frac{d}{3} \quad \text{(maximum)}$$

$$Q_{\max} = \sqrt{\frac{16\pi\epsilon_0 kd^3}{27}}$$

seperation at equilibrium $= \frac{2d}{3}$

When a charge larger than Q_{max} is added, columb force will overcome spring force and the two spheres will end up colliding

3. A flux density field is given as

$$\mathbf{F}_1 = 5\mathbf{a_z}.$$

The task is to evaluate the outward flux of $\mathbf{F}_1$ through the hemispherical surface defined by $r = a$, $0 < \theta < \pi/2$, and $0 < \phi < 2\pi$. Next, consider what simple observation would have saved a lot of work in the previous part. Identifying symmetries or using alternative methods can simplify the calculations significantly.

Now suppose the field is given by

$$\mathbf{F}_2 = 5z\mathbf{a_z}.$$

Using the appropriate surface integrals, determine the net outward flux of $\mathbf{F}_2$ through the closed surface consisting of the hemisphere from the first part and its circular base in the xy plane. Finally, repeat the previous calculation

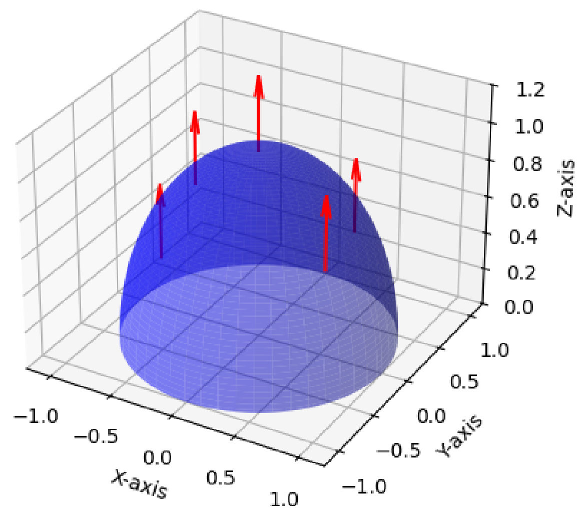


Figure 1: Hemisphere, Field lines

by applying the divergence theorem and evaluating an appropriate volume integral. This approach should confirm the result obtained through direct surface integration.

Solution:

Flux due to electric field over a surface S:

$$\iint \vec{E} \cdot \vec{n} \, dS$$

Here, the surface is a hemispherical surface. Using spherical coordinates:

$$\vec{E} = 5z\hat{k},$$
$$\vec{n} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

In spherical coordinates,

$$x = \rho \sin(\theta) \cos(\phi)$$
$$y = \rho \sin(\theta) \sin(\phi)$$
$$z = \rho \cos(\theta)$$

Thus,

$$dS = a^2 \sin\theta \, d\theta \, d\phi$$

The flux becomes:

$$= \iint 5a^2 \cos\theta \sin\theta \, d\theta \, d\phi$$

Separating the integrals:

$$= 5a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

For the θ integral:

$$\begin{aligned} &= 5a^2 \int_0^{2\pi} d\phi \left[-\frac{\cos^2\theta}{2} \right]_0^{\pi/2} \\ &= 5a^2 \int_0^{2\pi} d\phi \cdot \frac{1}{2} \\ &= 5\pi a^2 \end{aligned}$$

There is an easier way to find flux, we can say that flux passing through the hemisphere is the same as that passing through a disc of radius a centered at the origin as both surfaces subtend the same solid angle.

$$\begin{aligned} \text{Flux} &= \iint \vec{E} \cdot \vec{n} \, dS \\ \vec{E} \cdot \vec{n} &= 5 \\ \text{Flux } &= 5 \iint dS \\ &= 5(\pi a^2) \end{aligned}$$

This is the same result obtained above by integration. Now if field changes to $E = 5z\hat{k}$,

$$\vec{E} = 5z\hat{k}, \quad \hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Flux:

$$\text{Flux} = \iint_S \vec{E} \cdot \hat{n} \, dS$$

Substituting:

$$\text{Flux} = \int_0^{2\pi} \int_0^{\pi} 5a^3 \cos^2\theta \, a^2 \sin\theta \, d\theta d\phi$$

Simplifying,

$$= \int_0^{2\pi} \int_0^{\pi/2} 5a^3 \cos^2\theta \sin\theta \, d\theta d\phi = \int_0^{2\pi} \left[-\frac{5a^3}{3} cos(\theta) \right]_0^{\pi/2} d\phi$$

Using substitution:

$$= 5a^3 \cdot 2\pi \cdot \frac{1}{3} = \frac{10}{3}\pi a^3$$

Applying Divergence Theorem:
Flux:

$$\text{Flux} = \iint_S (\vec{E} \cdot \hat{n}) dS = \iiint_V (\nabla \cdot \vec{E}) dV$$

Calculate divergence:

$$\nabla \cdot \vec{E} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(5z) = 5$$

Volume integral:

$$\text{Flux} = 5 \iiint_V dV = 5 \left[\frac{2}{3}\pi a^3 \right] = 5a^3 = \frac{10}{3}\pi a^3$$

This verifies the result obtained by direct integration.

4. An infinitely long cylindrical dielectric of radius b contains charge within its volume with a charge density given by

$$\rho_v = a\rho^2,$$

where a is a constant. The goal is to determine the electric field strength $\mathbf{E}$ both inside and outside the cylinder.
Solution:
We can solve this problem using Gauss's law,

$$\iint \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enc}}}{\epsilon_0}.$$

For Q_{enc}:

$$Q_{\text{enc}} = \int_0^l \int_0^{2\pi} \int_0^r ax^3 r \, dr \, d\phi \, dz,$$

where $x = r$.

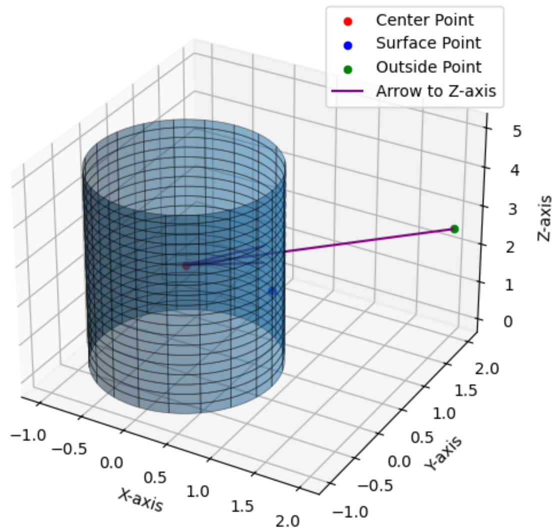


Figure 2: Cylinder

Simplifying:

$$Q_{\text{enc}} = \int_{z=0}^{l} \int_{\phi=0}^{2\pi} \int_{r=0}^{r} ar^4 \, dr \, d\phi \, dz.$$

$$Q_{\text{enc}} = \int_{0}^{l} \int_{0}^{2\pi} \frac{ar^5}{5}\Big|_0^r d\phi dz.$$

$$Q_{\text{enc}} = \int_{0}^{l} \int_{0}^{2\pi} \frac{ar^5}{5} d\phi dz.$$

$$Q_{\text{enc}} = (2\pi l)\frac{ar^4}{4}.$$

Next, we will take our Gaussian surface as a cylinder of length l, radius r. For the inside cylinder:

$$\iint \vec{E} \cdot d\vec{S} = E(2\pi r l) = (2\pi l)\frac{ar^4}{4}.$$

Thus:

$$E = \frac{ar^3}{4\epsilon_0}\hat{r}.$$

For the outside cylinder:

$$\iint \vec{E}\cdot d\vec{S} = E(2\pi rl) = (2\pi l)al^4/4.$$

Thus:

$$E = \frac{al^4}{4\epsilon_0 r}.$$

$$E = \begin{cases} \dfrac{ab^3}{4\epsilon_0} & \text{if } r \leq b, \\ \dfrac{ab^4}{4\epsilon_0 r} & \text{if } r > b. \end{cases}$$

5. Given the vector field in cylindrical coordinates:

$$\mathbf{F} = \left[\frac{40}{s^2+1} + 3(\cos\phi + \sin\phi)\right]\hat{s} + 3(\cos\phi - \sin\phi)\hat{\phi} - 2\hat{z}$$

(a) Compute and plot the magnitude $|\mathbf{F}|$ as a function of ϕ for $s = 3$.

(b) Compute and plot the magnitude $|\mathbf{F}|$ as a function of s for $\phi = 45°$.

(c) Calculate the divergence $\nabla \cdot \mathbf{F}$.

(d) Calculate the curl $\nabla \times \mathbf{F}$ and verify whether the field is conservative.

Solution:

(a) $\vec{F}$ at $s = 3$:

$$\vec{F}\Big|_{s=3} = [4 + 3(\cos\phi + \sin\phi)]\,\hat{s} + 3(\cos\phi - \sin\phi)\hat{\phi} + 4\hat{z}.$$

$$\begin{aligned} |\vec{F}|^2 &= 16 + \left[(\cos\phi + \sin\phi)^2 + (\cos\phi - \sin\phi)^2\right] + 4, \\ &= 16 + \left[\cos^2\phi + 2\cos\phi\sin\phi + \sin^2\phi + \cos^2\phi - 2\cos\phi\sin\phi + \sin^2\phi\right] + 4, \\ &= 16 + (2) + 4, \\ &= 38 + 24(\cos\phi + \sin\phi). \end{aligned}$$

(b) $\vec{F}$ at $\phi = \frac{\pi}{4}$:

$$\vec{F}_\phi = \left[\frac{40}{s^2+1} + 3\sqrt{2}\right]\hat{s} + s\hat{\phi} - 2\hat{z}.$$

Magnitude of $\vec{F}$:

$$|\vec{F}|^2 = \frac{160}{(s^2+1)^2} + 22 + \frac{240\sqrt{2}}{s^2+1}.$$

(c) Divergence in cylindrical coordinates: The divergence in cylindrical coordinates is given by:

$$\nabla \cdot \vec{F} = \frac{1}{s}\frac{\partial}{\partial s}(s f_s) + \frac{1}{s}\frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z},$$
$$F = f_s + f_\phi + f_z$$

$$\begin{aligned}
\nabla \cdot \vec{F} &= \frac{1}{s}\frac{\partial}{\partial s}\left(40s\frac{1}{s^2+1} + 3s(\cos\phi + \sin\phi)\right) + \frac{1}{s}\frac{\partial}{\partial \phi}\left(3s(\cos\phi + \sin\phi)\right) + \frac{\partial}{\partial z}(-2) \\
&= \frac{1}{s}\left[40\frac{(1-s^2)}{(1+s^2)^2} + 3(\cos\phi + \sin\phi)\right] + \frac{1}{s}\left[-3(\cos\phi + \sin\phi)\right] \\
&= \frac{40(1-s^2)}{s(1+s^2)^2}.
\end{aligned}$$

(d) Curl in cylindrical coordinates is given by:

$$\nabla \times \mathbf{F} = \frac{1}{r}\left(\frac{\partial f_z}{\partial r} - \frac{\partial f_r}{\partial z}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial (r f_\phi)}{\partial z} - \frac{\partial f_z}{\partial \phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{\partial f_r}{\partial \phi} - \frac{\partial (r f_\phi)}{\partial r}\right)\hat{\mathbf{z}}$$

$$beginalign{*}10pt] = \frac{1}{r}[3(\cos\phi - \sin\phi) + 3(\cos\phi - \sin\phi)]\hat{\mathbf{z}}$$
$$beginalign{*}10pt] = 0$$

Field is conservative.

6.

Solution:
(a) The electric field due to each charge at a point in space can be expressed as follows:
1. Electric field due to the charge +1 C at $(0, 1)$:

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}} + (y-1)\hat{\mathbf{j}}}{(x^2 + (y-1)^2)^{\frac{3}{2}}}$$

2. Electric field due to the charge +1 C at $(0, -1)$:

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}} + (y+1)\hat{\mathbf{j}}}{(x^2 + (y+1)^2)^{\frac{3}{2}}}$$

3. Electric field due to the charge $-1\,\mathrm{C}$ at $(1, 0)$:

$$\vec{E}_1 = \frac{-1}{4\pi\epsilon_0}\frac{(x-1)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-1)^2 + y^2)^{\frac{3}{2}}}$$

4. Electric field due to the charge $-1\,\mathrm{C}$ at $(-1, 0)$:

$$\vec{E}_1 = \frac{-1}{4\pi\epsilon_0}\frac{(x+1)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x+1)^2 + y^2)^{\frac{3}{2}}}$$

To find the total electric field at a point in space due to these charges, you add

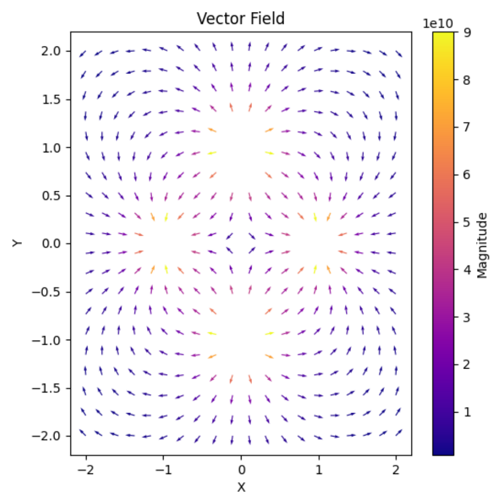


Figure 3: Field Lines

the individual fields (by superposition):

$$E_{\text{total}} = E_1 + E_2 + E_3 + E_4$$

(b) Electric Potential may be derived from Electric field by the relation $E = -\nabla V$

1. Electric Potential due to the charge $+1\,\text{C}$ at $(0, 1)$:

$$V_1 = \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + (y-1)^2}}$$

2. Electric Potential due to the charge $+1\,\text{C}$ at $(0, -1)$:

$$V_2 = \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + (y+1)^2}}$$

3. Electric Potential due to the charge $-1\,\text{C}$ at $(1, 0)$:

$$V_3 = -\frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{(x-1)^2 + y^2}}$$

4. Electric Potential due to the charge $-1\,\text{C}$ at $(-1, 0)$:

$$V_4 = -\frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{(x+1)^2 + y^2}}$$

To find the total electric potential at a point in space due to these charges, you

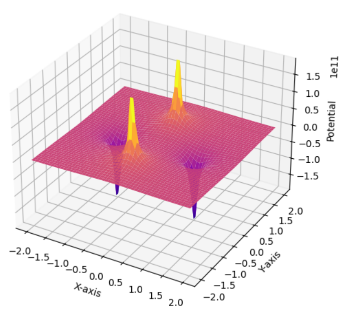


Figure 4: Potential Plot

add the individual potentials (by superposition):

$$V_{\text{total}} = V_1 + V_2 + V_3 + V_4$$

(c) The potential energy of a system of point charges is given by:

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i<j} \frac{q_i q_j}{r_{ij}}.$$

In the give question there are 4 charges,

- $1C$ at $(0, 1)$ and $(0, -1)$.
- $-1C$ at $(1, 0)$ and $(-1, 0)$.

Summing up their contributions toward Potential Energy of system,

$$U = \frac{1}{4\pi\epsilon_0} \left[\frac{(1C)(-1C)}{\sqrt{2}} + \frac{(1C)(-1C)}{\sqrt{2}} + \frac{(1C)(-1C)}{\sqrt{2}} + \right.$$
$$\left. \frac{(1C)(-1C)}{\sqrt{2}} + \frac{(1C)(1C)}{2} + \frac{(-1C)(-1C)}{2} \right].$$

Simplifying:

$$\begin{aligned} U &= \frac{1}{4\pi\epsilon_0} \left[-\frac{4}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2} \right] \\ &= \frac{q^2}{4\pi\epsilon_0} \left[1 - 2\sqrt{2} \right]. \end{aligned}$$

Thus, the potential energy of the configuration is:

$$U = \frac{q^2}{4\pi\epsilon_0} (1 - 2\sqrt{2}).$$

(d) Divergence of the electric field
The electric field $\mathbf{E}$ due to a point charge q at position $\mathbf{r}_0$ is given by:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3}.$$

By the superposition principle, the total electric field $\mathbf{E}$ is the sum of the contributions from all four charges.

The divergence of the electric field is given by Gauss's law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0},$$

where $\rho(\mathbf{r})$ is the charge density.

Since we have point charges, we can use the Dirac delta function to represent charge density:

$$\rho(\mathbf{r}) = q\delta(x)\delta(y-1) + q\delta(x)\delta(y+1) - q\delta(x-1)\delta(y) - q\delta(x+1)\delta(y).$$

Applying Gauss's law,

$$\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0}\left[\delta(x)\delta(y-1) + \delta(x)\delta(y+1) - \delta(x-1)\delta(y) - \delta(x+1)\delta(y)\right].$$

This expression shows that the divergence of field $\mathbf{E}$ is nonzero only at the locations of the charges. This verifies the fact that field due to point charges satisfy Gauss law (in differential form)

(e) The curl of a vector field $\mathbf{E} = (E_x, E_y, E_z)$ is given by,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

For a single point charge, the electric field components in Cartesian coordinates are:

$$E_x = \frac{q}{4\pi\varepsilon_0}\frac{x-x_0}{r^3},$$

$$E_y = \frac{q}{4\pi\varepsilon_0}\frac{y-y_0}{r^3},$$

$$E_z = \frac{q}{4\pi\varepsilon_0}\frac{z-z_0}{r^3}.$$

where,

$$r = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{1/2}$$

The curl becomes,

$$\left(\frac{q}{4\pi\varepsilon_0}\left[\frac{-3(z-z_0)(y-y_0)}{(r^2)^{5/2}}\right] - \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(y-y_0)(z-z_0)}{(r^2)^{5/2}}\right]\right)\hat{i} -$$
$$\left(\frac{q}{4\pi\varepsilon_0}\left[\frac{-3(z-z_0)(x-x_0)}{(r^2)^{5/2}}\right] - \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(x-x_0)(z-z_0)}{(r^2)^{5/2}}\right]\right)\hat{j} +$$
$$\left(\frac{q}{4\pi\varepsilon_0}\left[\frac{-3(y-y_0)(x-x_0)}{(r^2)^{5/2}}\right] - \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(x-x_0)(y-y_0)}{(r^2)^{5/2}}\right]\right)$$

Thus, we get,

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

Since the curl of the field due to a single point charge is zero, and curl operator is linear, the total electric field (which is a superposition of fields of the induvidual charges) also has zero curl.

$$\nabla \times \mathbf{E} = 0.$$

This shows that the electric field due to the system of charges is conservative, i.e. its curl is 0

7. Given the spherically symmetric potential field in free space,

$$V(r) = V_0 e^{-r/a},$$

determine the following:

(a) Find the charge density ρ_v at $r = a$.

(b) Calculate the electric field $\mathbf{E}$ at $r = a$.

(c) Compute the total charge.

Solution:
(a) Poisson's Equation is given by,

$$\nabla^2 V(r) = -\frac{\rho(r)}{\epsilon_0}$$

In spherical coordiantes,

$$\nabla^2 \phi(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \phi^2} = -\frac{\rho}{\epsilon_0}$$

Simplifying:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = -\frac{\rho(r)}{\epsilon_0}.$$

Substituting $V(r) = V_0 e^{-\frac{r}{a}}$:

$$-\frac{V_0 e^{-r/a}}{ar^2}\left(\frac{-r^2}{a} + 2r\right) = -\frac{\rho(r)}{\epsilon_0}.$$

$$\rho(r) = -\frac{V_0 \epsilon_0 e^{-r/a}}{a}\left(\frac{1}{a} - \frac{2}{r}\right)$$

At $r = a$

$$\rho = \frac{V_0 \epsilon_0 e^{-\frac{r}{a}}}{a^2}$$

(b)

$$E = -\nabla\Phi,$$

where Φ: Potential, E: Electric Field.
Gradient in spherical coordinates,

$$\nabla f(r,\theta,\phi) = \hat{e}_r\frac{\partial f}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Substituting:

$$E(r) = \frac{V_0 e^{-\frac{r}{a}}}{a}\hat{r}.$$

At $r = a$:

$$E(r = a) = \frac{V_0 e^{-1}}{a}\hat{r}.$$

(c) Total charge is given by:

$$Q = \iiint \rho\, dV,$$

where $\rho =$ charge density.

$$\rho = -V_0\frac{\epsilon_0 e^{-\frac{r}{a}}}{a}\left(\frac{1}{a} - \frac{2}{r}\right).$$

Substitute into the integral:

$$Q = \iiint -V_0\frac{\epsilon_0}{a}\left(\frac{1}{a} - \frac{2}{r}\right) dV.$$

Using spherical coordinates:

$$Q = \int_0^\infty\int_0^\pi\int_0^{2\pi} -V_0\frac{\epsilon_0}{a}\left(\frac{1}{a} - \frac{2}{r}\right) r^2\sin\theta\, dr\, d\theta\, d\phi.$$

Analysing the first integral seperately, and applying integration by-parts on the first part

$$= -V_0\frac{\epsilon_0}{a}\left[\frac{1}{a}\int_0^\infty r^2 e^{-\frac{r}{a}}\, dr - 2\int_0^\infty re^{-\frac{r}{a}}\, dr\right]$$
$$= -V_0\frac{\epsilon_0}{a}\left[\left(\frac{1}{a}r^2(-a)e^{-\frac{r}{a}}\right)_0^\infty + \frac{1}{a}\int_0^\infty 2r(-a)e^{-\frac{r}{a}}\, dr - 2\int_0^\infty re^{-\frac{r}{a}}\, dr\right] \quad = 0$$

Thus, the total charge is zero.

8. A parallel-plate capacitor has plates located at $z = 0$ and $z = d$. The region between the plates is filled with a material that contains a uniform volume charge density ρ_0 C/m^3 and has permittivity ϵ. Both plates are held at ground potential.

(a) Determine the potential field between the plates.

(b) Determine the electric field intensity **E** between the plates.

(c) Repeat parts (a) and (b) for the case where the plate at $z = d$ is raised to a potential V_0, with the plate at $z = 0$ grounded.

Solution:
(a) Solving Poisson's Equation

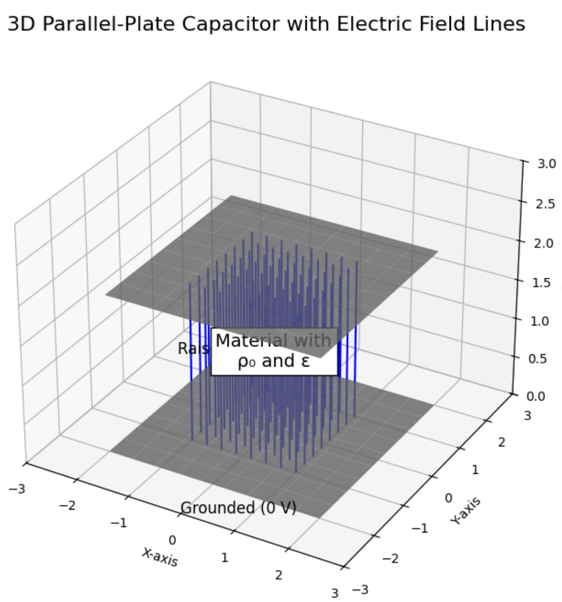


Figure 5: capacitor

$$\nabla^2 V = \frac{\rho}{\epsilon_0}$$

Here V only has a z component:

$$\begin{aligned} \frac{d^2V}{dz^2} &= \frac{\rho}{\epsilon_0} \\ \frac{dV}{dz} &= \frac{\rho z}{\epsilon_0} + C_1 \\ V &= \frac{\rho z^2}{2\epsilon_0} + C_1 z + C_2 \end{aligned}$$

Substituting $V(z=0) = V(z=d) = 0$:

$$V = \frac{\rho z(z-d)}{2\epsilon_0}\hat{k}$$

Electric Field Intensity (E)

$$E = -\nabla V$$

$$E = -\left(-\frac{\rho}{\epsilon_0}\left(z - \frac{d}{2}\right)\right) \quad \hat{k}$$

(c) If $V(z=d) = V_0$
From the first part we know,

$$V = \frac{\rho z^2}{2\epsilon_0} + C_1 z + C_2$$

Boundary conditions:

$$V(0) = 0 \implies C_2 = 0$$

$$V(d) = V_0 \implies \frac{\rho d^2}{2\epsilon_0} + C_1 d = V_0$$

Solving for C_1 :

$$C_1 = \frac{V_0}{d} - \frac{\rho_0 d}{2\epsilon_0}$$

We finally get potential between the plates to be,

$$V = \left(\frac{\rho_0 z(z-d)}{2\epsilon_0} + \frac{V_0 z}{d}\right)\hat{r}$$

Electric Field Intensity ($\vec{E}$) is given by,

$$\vec{E} = -\nabla V$$
$$\vec{E} = -\left[\frac{\rho_0}{\epsilon_0}\left(z - \frac{d}{2}\right) + \frac{V_0}{d}\right]$$