

# EE1204 Assignment 1

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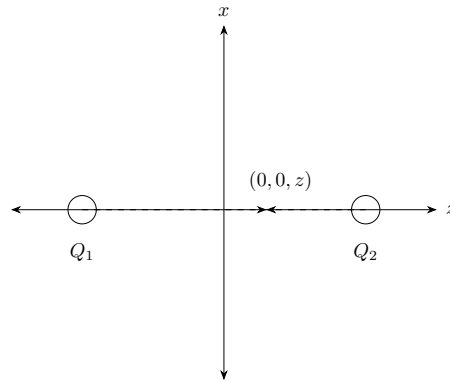
EE24BTECH11005

1. Two point charges of equal magnitude  $q$  are positioned at  $z = \pm \frac{d}{2}$ . We aim to determine,

- (a) The electric field everywhere on the  $z$ -axis.
- (b) The electric field everywhere on the  $x$ -axis.
- (c) The results of (a) and (b) if the charge at  $z = -\frac{d}{2}$  is  $-1C$  instead of  $1C$ .

**Solution,**

(a) Field on  $z$ -axis ( $E$ ),  $E = E_1 + E_2$  (By superposition)



- $E_1$ , Field due to  $Q_1$
- $E_2$ , Field due to  $Q_2$

$$E_1 = \frac{q}{4\pi\epsilon_0} \frac{|x - \frac{d}{2}|}{(x - \frac{d}{2})^3} \hat{k}, \quad E_2 = \frac{q}{4\pi\epsilon_0} \frac{|x + \frac{d}{2}|}{(x + \frac{d}{2})^3} \hat{k}$$
$$E = E_1 + E_2 = \frac{q}{4\pi\epsilon_0} \left[ \frac{|x - \frac{d}{2}|}{(x - \frac{d}{2})^3} + \frac{|x + \frac{d}{2}|}{(x + \frac{d}{2})^3} \right] \hat{k}$$

Case 1,  $z > \frac{d}{2}$

$$E = \frac{q}{2\pi\epsilon_0} \left[ \frac{z^2 + (\frac{d}{2})^2}{(z^2 - (\frac{d}{2})^2)^2} \right] (\hat{\mathbf{k}})$$

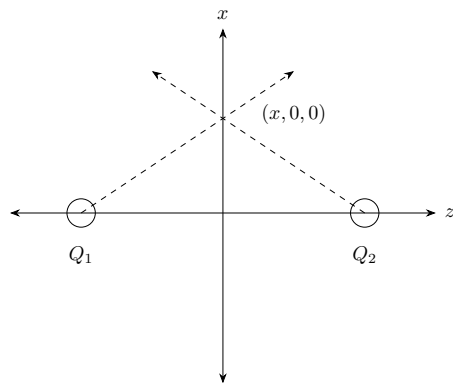
Case 2,  $z < -\frac{d}{2}$

$$E = \frac{q}{2\pi\epsilon_0} \left[ \frac{z^2 + (\frac{d}{2})^2}{(z^2 - (\frac{d}{2})^2)^2} \right] (-\hat{\mathbf{k}})$$

Case 3,  $-\frac{d}{2} < z < \frac{d}{2}$

$$\begin{aligned} E &= \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{(z + \frac{d}{2})^2} - \frac{1}{(z - \frac{d}{2})^2} \right] (\hat{\mathbf{k}}) \\ &= \frac{q}{2\pi\epsilon_0} \left[ \frac{zd}{(z^2 - (\frac{d}{2})^2)^2} \right] (-\hat{\mathbf{k}}) \end{aligned}$$

(b)



Field on x-axis (E),  $E_1 + E_2$

$E_1$ , Field due to  $q_1$ ,  $E_2$  : Field due to  $q_2$

$$\begin{aligned} E &= \frac{q}{4\pi\epsilon_0} \left( \frac{x\hat{i} - \frac{d}{2}\hat{k}}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} + \frac{\left(x\hat{i} + \frac{d}{2}\hat{k}\right)}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} \right) \hat{\mathbf{i}} \\ E &= \frac{q}{2\pi\epsilon_0} \frac{x}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} \hat{\mathbf{i}}. \end{aligned}$$

(c) Result obtained in (a) becomes,

Case 1,  $z > \frac{d}{2}$ ,

$$E = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{(z - \frac{d}{2})^2} - \frac{1}{(z + \frac{d}{2})^2} \right) \hat{\mathbf{k}}$$

$$E = \frac{q}{2\pi\epsilon_0} \frac{zd}{(z^2 - (\frac{d}{2})^2)^2} \hat{\mathbf{k}}.$$

Case 2,  $z < -\frac{d}{2}$ ,

$$E = +\frac{q}{4\pi\epsilon_0} \left( \frac{zd}{(z^2 - (\frac{d}{2})^2)^3} \right) - \hat{\mathbf{k}}$$

Case 3,  $-\frac{d}{2} < z < \frac{d}{2}$

$$E = -\frac{q}{4\pi\epsilon_0} \left( \frac{1}{(z + \frac{d}{2})^2} + \frac{1}{(z - \frac{d}{2})^2} \right) \hat{\mathbf{k}}$$

$$= -\frac{q}{2\pi\epsilon_0} \left( \frac{z^2 + (\frac{d}{2})^2}{(z^2 - (\frac{d}{2})^2)^2} \right) \hat{\mathbf{k}}$$

Result obtained in (b) becomes,

$$E = \frac{q}{4\pi\epsilon_0} \left( \frac{x\hat{\mathbf{i}} - \frac{d}{2}\hat{\mathbf{k}}}{\left(x^2 + (\frac{d}{2})^2\right)^{3/2}} - \frac{\left(x\hat{\mathbf{i}} + \frac{d}{2}\hat{\mathbf{k}}\right)}{\left(x^2 + (\frac{d}{2})^2\right)^{3/2}} \right) \hat{\mathbf{i}}$$

$$E = \frac{q}{4\pi\epsilon_0} \frac{d}{\left(x^2 + (\frac{d}{2})^2\right)^{3/2}} (-\hat{\mathbf{i}}).$$

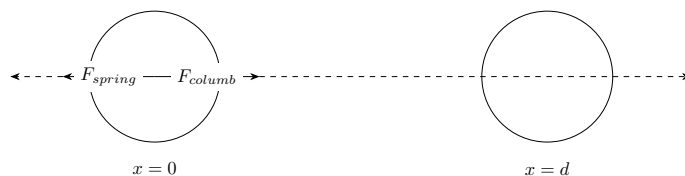
2. A crude device for measuring charge consists of two small insulating spheres of radius  $a$ , one of which is fixed in position. The other is movable along the  $x$ -axis and is subject to a restraining force  $kx$ , where  $k$  is a spring constant. The uncharged spheres are centered at  $x = 0$  and  $x = d$ , the latter fixed. If the spheres are given equal and opposite charges of  $Q/C$ , obtain the expression by which  $Q$  may be found as a function of  $x$ . Determine the maximum charge that can be measured in terms of  $\epsilon_0$ ,  $k$ , and  $d$ , and then state the separation of the spheres. What happens if a larger charge is applied?

**Solution,**

Say  $Q_1 = +Q$  is the movable charge,  $Q_2 = -Q$  is the stationary charge.

At equilibrium, spring force on  $+Q$  is balanced by electrostatic force,

$$\frac{1}{4\pi\epsilon_0} \frac{Q^2}{(x - d)^2} = kx$$



$$Q = \sqrt{4\pi\epsilon_0 kx(x-d)^2}$$

Maximum measurable charge,

$$\frac{dQ}{dx} = 0$$

Differentiating,

$$(x-d)^2 \left( x - \frac{d}{3} \right) = 0$$

$$x = d \quad (\text{minimum}), \frac{d}{3} \quad (\text{maximum})$$

$$Q_{\max} = \sqrt{\frac{16\pi\epsilon_0 k d^3}{27}}$$

seperation at equilibrium =  $\frac{2d}{3}$

When a charge larger than  $Q_{max}$  is added, columb force will overcome spring force and the two spheres will end up colliding

3. A flux density field is given as

$$\mathbf{F}_1 = 5\mathbf{a}_z.$$

The task is to evaluate the outward flux of  $\mathbf{F}_1$  through the hemispherical surface defined by  $r = a$ ,  $0 < \theta < \pi/2$ , and  $0 < \phi < 2\pi$ . Next, consider what simple observation would have saved a lot of work in the previous part. Identifying symmetries or using alternative methods can simplify the calculations significantly.

Now suppose the field is given by

$$\mathbf{F}_2 = 5z\mathbf{a}_z.$$

Using the appropriate surface integrals, determine the net outward flux of  $\mathbf{F}_2$  through the closed surface consisting of the hemisphere from the first part and its circular base in the  $xy$  plane. Finally, repeat the previous calculation

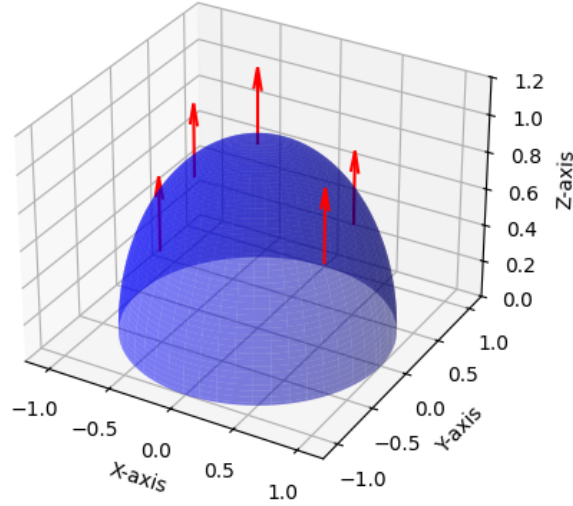


Figure 1: Hemisphere, Field lines

by applying the divergence theorem and evaluating an appropriate volume integral. This approach should confirm the result obtained through direct surface integration.

**Solution,**

Flux due to electric field over a surface  $S$ ,

$$\iint \vec{E} \cdot \vec{n} dS$$

Here, the surface is a hemispherical surface. Using spherical coordinates,

$$\vec{E} = 5z\hat{k},$$

$$\vec{n} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\hat{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\hat{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\hat{k}$$

In spherical coordinates,

$$x = \rho \sin(\theta) \cos(\phi)$$

$$y = \rho \sin(\theta) \sin(\phi)$$

$$z = \rho \cos(\theta)$$

Thus,

$$dS = a^2 \sin \theta \, d\theta \, d\phi$$

The flux becomes,

$$= \iint 5a^2 \cos \theta \sin \theta \, d\theta \, d\phi$$

Separating the integrals,

$$= 5a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta$$

For the  $\theta$  integral,

$$\begin{aligned} &= 5a^2 \int_0^{2\pi} d\phi \left[ -\frac{\cos^2 \theta}{2} \right]_0^{\pi/2} \\ &= 5a^2 \int_0^{2\pi} d\phi \cdot \frac{1}{2} \\ &= 5\pi a^2 \end{aligned}$$

There is an easier way to find flux, we can say that flux passing through the hemisphere is the same as that passing through a disc of radius  $a$  centered at the origin as both surfaces subtend the same solid angle.

$$\begin{aligned} \text{Flux} &= \iint \vec{E} \cdot \vec{n} \, dS \\ \vec{E} \cdot \vec{n} &= 5 \\ \text{Flux} &= 5 \iint dS \\ &= 5(\pi a^2) \end{aligned}$$

This is the same result obtained above by integration. Now if field changes to  $E = 5z\hat{k}$ ,

$$\vec{E} = 5z\hat{k}, \quad \hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Flux,

$$\text{Flux} = \iint_S \vec{E} \cdot \hat{n} \, dS$$

Substituting,

$$\text{Flux} = \int_0^{2\pi} \int_0^{\pi} 5a^3 \cos^2 \theta \sin \theta \, d\theta \, d\phi$$

Simplifying,

$$= \int_0^{2\pi} \int_0^{\pi/2} 5a^3 \cos^2 \theta \sin \theta \, d\theta d\phi = \int_0^{2\pi} \left[ -\frac{5a^3}{3} \cos(\theta) \right]_0^{\pi/2} d\phi$$

Using substitution,

$$= 5a^3 \cdot 2\pi \cdot \frac{1}{3} = \frac{10}{3} \pi a^3$$

Applying Divergence Theorem,  
Flux,

$$\text{Flux} = \iint_S (\vec{E} \cdot \hat{n}) dS = \iiint_V (\nabla \cdot \vec{E}) dV$$

Calculate divergence,

$$\nabla \cdot \vec{E} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(5z) = 5$$

Volume integral,

$$\text{Flux} = 5 \iiint_V dV = 5 \left[ \frac{2}{3} \pi a^3 \right] = 5a^3 = \frac{10}{3} \pi a^3$$

This verifies the result obtained by direct integration.

4. An infinitely long cylindrical dielectric of radius  $b$  contains charge within its volume with a charge density given by

$$\rho_v = a\rho^2,$$

where  $a$  is a constant. The goal is to determine the electric field strength  $\mathbf{E}$  both inside and outside the cylinder.

**Solution,**

We can solve this problem using Gauss's law,

$$\iint \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enc}}}{\epsilon_0}.$$

For  $Q_{\text{enc}}$ ,

$$Q_{\text{enc}} = \int_0^l \int_0^{2\pi} \int_0^r ax^3 r \, dr \, d\phi \, dz,$$

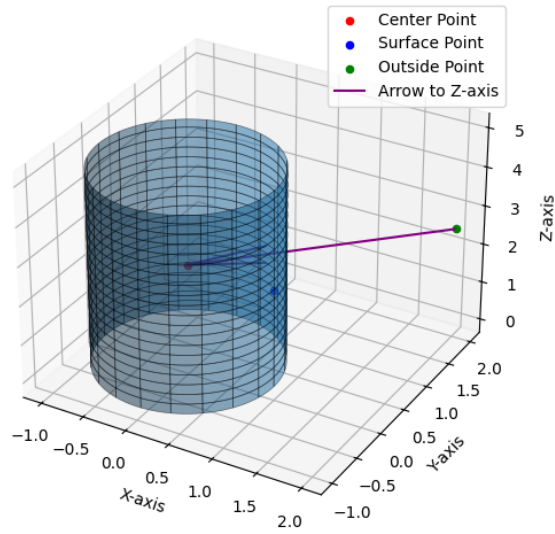


Figure 2: Cylinder

Simplifying,

$$Q_{\text{enc}} = \int_{z=0}^l \int_{\phi=0}^{2\pi} \int_{r=0}^r ar^4 dr d\phi dz.$$

$$Q_{\text{enc}} = \int_0^l \int_0^{2\pi} \left. \frac{ar^5}{5} \right|_0^r d\phi dz.$$

$$Q_{\text{enc}} = \int_0^l \int_0^{2\pi} \frac{ar^5}{5} d\phi dz.$$

$$Q_{\text{enc}} = (2\pi l) \frac{ar^4}{4}.$$

Next, we will take our Gaussian surface as a cylinder of length  $l$ , radius  $r$ . For the inside cylinder,

$$\iint \vec{E} \cdot d\vec{S} = E(2\pi rl) = (2\pi l) \frac{ar^4}{4}.$$



Thus,

$$E = \frac{ar^3}{4\epsilon_0} \hat{\mathbf{r}}.$$

For the outside cylinder,

$$\iint \vec{E} \cdot d\vec{S} = E(2\pi rl) = (2\pi l)al^4/4.$$

Thus,

$$E = \frac{al^4}{4\epsilon_0 r}.$$

$$E = \begin{cases} \frac{ab^3}{4\epsilon_0} \hat{\mathbf{r}} & \text{if } r \leq b, \\ \frac{ab^4}{4\epsilon_0 r} \hat{\mathbf{r}} & \text{if } r > b. \end{cases}$$

5. Given the vector field in cylindrical coordinates,

$$\mathbf{F} = \left[ \frac{40}{s^2 + 1} + 3(\cos \phi + \sin \phi) \right] \hat{s} + 3(\cos \phi - \sin \phi) \hat{\phi} - 2\hat{z}$$

- (a) Compute and plot the magnitude  $|\mathbf{F}|$  as a function of  $\phi$  for  $s = 3$ .
- (b) Compute and plot the magnitude  $|\mathbf{F}|$  as a function of  $s$  for  $\phi = 45^\circ$ .
- (c) Calculate the divergence  $\nabla \cdot \mathbf{F}$ .
- (d) Calculate the curl  $\nabla \times \mathbf{F}$  and verify whether the field is conservative.

**Solution,**

(a)  $\vec{F}$  at  $s = 3$ ,

$$\vec{F}|_{s=3} = [4 + 3(\cos \phi + \sin \phi)] \hat{s} + 3(\cos \phi - \sin \phi) \hat{\phi} + 4\hat{z}.$$

$$\begin{aligned} |\vec{F}|^2 &= 16 + [(\cos \phi + \sin \phi)^2 + (\cos \phi - \sin \phi)^2] + 4, \\ &= 16 + [\cos^2 \phi + 2 \cos \phi \sin \phi + \sin^2 \phi + \cos^2 \phi - 2 \cos \phi \sin \phi + \sin^2 \phi] + 4, \\ &= 16 + (2) + 4, \\ &= 38 + 24(\cos \phi + \sin \phi). \end{aligned}$$

Code used may be found at,

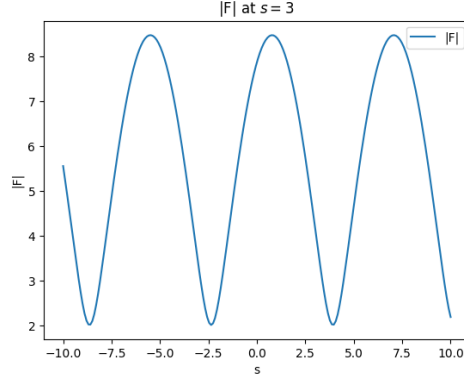


Figure 3: Field Lines

<https://github.com/ArjunPavanje/EE1204/blob/main/Assignment1/codes/py1.py>  
**(b)**  $\vec{F}$  at  $\phi = \frac{\pi}{4}$ ,

$$\vec{F}_\phi = \left[ \frac{40}{s^2 + 1} + 3\sqrt{2} \right] \hat{s} + s\hat{\phi} - 2\hat{z}.$$

Code used may be found at,

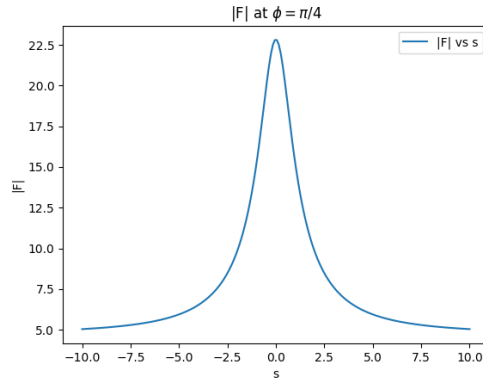


Figure 4: Field Lines

<https://github.com/ArjunPavanje/EE1204/blob/main/Assignment1/codes/py2.py>  
Magnitude of  $\vec{F}$ ,

$$|\vec{F}|^2 = \frac{160}{(s^2 + 1)^2} + 22 + \frac{240\sqrt{2}}{s^2 + 1}.$$

(c) Divergence in cylindrical coordinates, The divergence in cylindrical coordinates is given by,

$$\nabla \cdot \vec{F} = \frac{1}{s} \frac{\partial}{\partial s} (s f_s) + \frac{1}{s} \frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z},$$

$$F = f_s + f_\phi + f_z$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \frac{1}{s} \frac{\partial}{\partial s} \left( 40s \frac{1}{s^2 + 1} + 3s(\cos \phi + \sin \phi) \right) + \frac{1}{s} \frac{\partial}{\partial \phi} (3s(\cos \phi + \sin \phi)) + \frac{\partial}{\partial z} (-2) \\ &= \frac{1}{s} \left[ 40 \frac{(1 - s^2)}{(1 + s^2)^2} + 3(\cos \phi + \sin \phi) \right] + \frac{1}{s} [-3(\cos \phi + \sin \phi)] \\ &= \frac{40(1 - s^2)}{s(1 + s^2)^2}. \end{aligned}$$

(d) Curl in cylindrical coordinates is given by,

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{1}{r} \left( \frac{\partial f_z}{\partial r} - \frac{\partial f_r}{\partial z} \right) \hat{\phi} + \frac{1}{r} \left( \frac{\partial(r f_\phi)}{\partial z} - \frac{\partial f_z}{\partial \phi} \right) \hat{r} + \frac{1}{r} \left( \frac{\partial f_r}{\partial \phi} - \frac{\partial(r f_\phi)}{\partial r} \right) \hat{z} \\ &= \frac{1}{r} [3(\cos \phi - \sin \phi) + 3(\cos \phi - \sin \phi)] \hat{z} \\ &= 0 \end{aligned}$$

Field is conservative.

6.

**Solution,**

(a) The electric field due to each charge at a point in space can be expressed as follows,

1. Electric field due to the charge +1 C at (0, 1),

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0} \frac{x\hat{i} + (y-1)\hat{j}}{(x^2 + (y-1)^2)^{\frac{3}{2}}}$$

2. Electric field due to the charge +1 C at (0, -1),

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0} \frac{x\hat{i} + (y+1)\hat{j}}{(x^2 + (y+1)^2)^{\frac{3}{2}}}$$

3. Electric field due to the charge -1 C at (1, 0),

$$\vec{E}_1 = \frac{-1}{4\pi\epsilon_0} \frac{(x-1)\hat{i} + y\hat{j}}{((x-1)^2 + y^2)^{\frac{3}{2}}}$$

4. Electric field due to the charge  $-1\text{ C}$  at  $(-1, 0)$ ,

$$\vec{E}_1 = \frac{-1}{4\pi\epsilon_0} \frac{(x+1)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x+1)^2 + y^2)^{\frac{3}{2}}}$$

Code used may be found at,

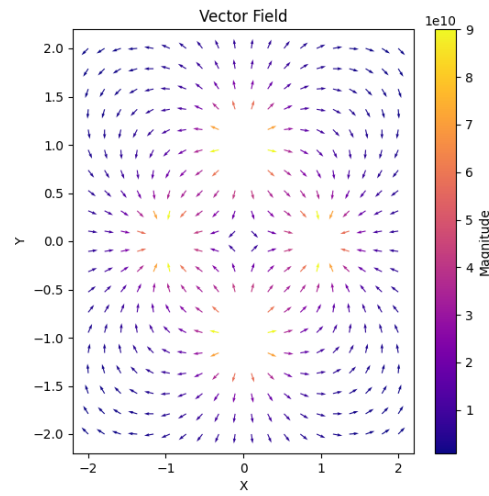


Figure 5: Field Lines

<https://github.com/ArjunPavanje/EE1204/blob/main/Assignment1/codes/field.py>

To find the total electric field at a point in space due to these charges, you add the individual fields (by superposition),

$$E_{\text{total}} = E_1 + E_2 + E_3 + E_4$$

(b) Electric Potential may be derived from Electric field by the relation  $E = -\nabla V$

1. Electric Potential due to the charge  $+1 \text{ C}$  at  $(0, 1)$ ,

$$V_1 = \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + (y-1)^2}}$$

2. Electric Potential due to the charge  $+1 \text{ C}$  at  $(0, -1)$ ,

$$V_2 = \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + (y+1)^2}}$$

3. Electric Potential due to the charge  $-1 \text{ C}$  at  $(1, 0)$ ,

$$V_3 = -\frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{(x-1)^2 + y^2}}$$

4. Electric Potential due to the charge  $-1 \text{ C}$  at  $(-1, 0)$ ,

$$V_4 = -\frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{(x+1)^2 + y^2}}$$

Code used may be found at,

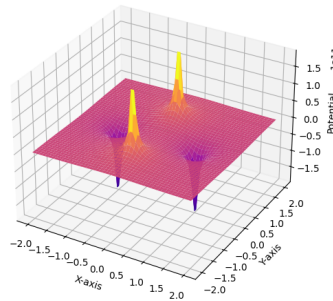


Figure 6: Potential Plot

<https://github.com/ArjunPavanje/EE1204/blob/main/Assignment1/codes/pot.py>

To find the total electric potential at a point in space due to these charges, you add the individual potentials (by superposition),

$$V_{\text{total}} = V_1 + V_2 + V_3 + V_4$$

(c) The potential energy of a system of point charges is given by,

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i < j} \frac{q_i q_j}{r_{ij}}.$$

In the give question there are 4 charges,

- $1C$  at  $(0, 1)$  and  $(0, -1)$ .
- $-1C$  at  $(1, 0)$  and  $(-1, 0)$ .

Summing up their contributions toward Potential Energy of system,

$$U = \frac{1}{4\pi\epsilon_0} \left[ \frac{(1C)(-1C)}{\sqrt{2}} + \frac{(1C)(-1C)}{\sqrt{2}} + \frac{(1C)(-1C)}{\sqrt{2}} + \frac{(1C)(-1C)}{\sqrt{2}} + \frac{(1C)(1C)}{2} + \frac{(1C)(1C)}{2} + \frac{(-1C)(-1C)}{2} + \frac{(-1C)(-1C)}{2} \right].$$

Simplifying,

$$\begin{aligned} U &= \frac{1}{4\pi\epsilon_0} \left[ -\frac{4}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2} \right] \\ &= \frac{q^2}{4\pi\epsilon_0} [1 - 2\sqrt{2}]. \end{aligned}$$

Thus, the potential energy of the configuration is,

$$U = \frac{1}{4\pi\epsilon_0} (1 - 2\sqrt{2}).$$

(d) Divergence of the electric field

The electric field  $\mathbf{E}$  due to a point charge  $q$  at position  $\mathbf{r}_0$  is given by,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3}.$$

By the superposition principle, the total electric field  $\mathbf{E}$  is the sum of the contributions from all four charges.

The divergence of the electric field is given by Gauss's law,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0},$$

where  $\rho(\mathbf{r})$  is the charge density.

Since we have point charges, we can use the Dirac delta function to represent charge density,

$$\rho(\mathbf{r}) = q\delta(x)\delta(y-1) + q\delta(x)\delta(y+1) - q\delta(x-1)\delta(y) - q\delta(x+1)\delta(y).$$

Applying Gauss's law,

$$\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0} [\delta(x)\delta(y-1) + \delta(x)\delta(y+1) - \delta(x-1)\delta(y) - \delta(x+1)\delta(y)].$$

This expression shows that the divergence of field  $\mathbf{E}$  is nonzero only at the locations of the charges. This verifies the fact that field due to point charges satisfy Gauss law (in differential form)

(e) The curl of a vector field  $\mathbf{E} = (E_x, E_y, E_z)$  is given by,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

For a single point charge, the electric field components in Cartesian coordinates are,

$$E_x = \frac{q}{4\pi\varepsilon_0} \frac{x - x_0}{r^3},$$

$$E_y = \frac{q}{4\pi\varepsilon_0} \frac{y - y_0}{r^3},$$

$$E_z = \frac{q}{4\pi\varepsilon_0} \frac{z - z_0}{r^3}.$$

where,

$$r = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}$$

The curl becomes,

$$\begin{aligned} & \left( \frac{q}{4\pi\varepsilon_0} \left[ \frac{-3(z - z_0)(y - y_0)}{(r^2)^{5/2}} \right] - \frac{q}{4\pi\varepsilon_0} \left[ \frac{-3(y - y_0)(z - z_0)}{(r^2)^{5/2}} \right] \right) \hat{i} - \\ & \left( \frac{q}{4\pi\varepsilon_0} \left[ \frac{-3(z - z_0)(x - x_0)}{(r^2)^{5/2}} \right] - \frac{q}{4\pi\varepsilon_0} \left[ \frac{-3(x - x_0)(z - z_0)}{(r^2)^{5/2}} \right] \right) \hat{j} + \\ & \left( \frac{q}{4\pi\varepsilon_0} \left[ \frac{-3(y - y_0)(x - x_0)}{(r^2)^{5/2}} \right] - \frac{q}{4\pi\varepsilon_0} \left[ \frac{-3(x - x_0)(y - y_0)}{(r^2)^{5/2}} \right] \right) \hat{k} \end{aligned}$$

Thus, we get,

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

Since the curl of the field due to a single point charge is zero, and curl operator is linear, the total electric field (which is a superposition of fields of the individual charges) also has zero curl.

$$\nabla \times \mathbf{E} = 0.$$

This shows that the electric field due to the system of charges is conservative, i.e. its curl is 0

7. Given the spherically symmetric potential field in free space,

$$V(r) = V_0 e^{-r/a},$$

determine the following,

- (a) Find the charge density  $\rho_v$  at  $r = a$ .
- (b) Calculate the electric field  $\mathbf{E}$  at  $r = a$ .
- (c) Compute the total charge.

**Solution,**

(a) Poisson's Equation is given by,

$$\nabla^2 V(r) = -\frac{\rho(r)}{\epsilon_0}$$

In spherical coordinates,

$$\nabla^2 \phi(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2} = -\frac{\rho}{\epsilon_0}$$

Simplifying,

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) = -\frac{\rho(r)}{\epsilon_0}.$$

Substituting  $V(r) = V_0 e^{-r/a}$ ,

$$-\frac{V_0 e^{-r/a}}{ar^2} \left( \frac{-r^2}{a} + 2r \right) = -\frac{\rho(r)}{\epsilon_0}.$$

$$\rho(r) = -\frac{V_0 \epsilon_0 e^{-r/a}}{a} \left( \frac{1}{a} - \frac{2}{r} \right)$$

At  $r = a$

$$\rho = \frac{V_0 \epsilon_0 e^{-1}}{a^2}$$



(b)

$$E = -\nabla\Phi,$$

where  $\Phi$ , Potential,  $E$ : Electric Field.

Gradient in spherical coordinates,

$$\nabla f(r, \theta, \phi) = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi}$$

Substituting,

$$E(r) = \frac{V_0 e^{-\frac{r}{a}}}{a} \hat{r}.$$

At  $r = a$ ,

$$E(r = a) = \frac{V_0 e^{-1}}{a} \hat{r}.$$

(c) Total charge is given by,

$$Q = \iiint \rho dV,$$

where  $\rho$  = charge density.

$$\rho = -V_0 \frac{\epsilon_0 e^{-\frac{r}{a}}}{a} \left( \frac{1}{a} - \frac{2}{r} \right).$$

Substitute into the integral,

$$Q = \iiint -V_0 \frac{\epsilon_0}{a} \left( \frac{1}{a} - \frac{2}{r} \right) dV.$$

Using spherical coordinates,

$$Q = \int_0^\infty \int_0^\pi \int_0^{2\pi} -V_0 \frac{\epsilon_0}{a} \left( \frac{1}{a} - \frac{2}{r} \right) r^2 \sin \theta dr d\theta d\phi.$$

Analysing the first integral seperately, and applying integration by-parts on the first part

$$\begin{aligned} &= -V_0 \frac{\epsilon_0}{a} \left[ \frac{1}{a} \int_0^\infty r^2 e^{-\frac{r}{a}} dr - 2 \int_0^\infty r e^{-\frac{r}{a}} dr \right] \\ &= -V_0 \frac{\epsilon_0}{a} \left[ \left( \frac{1}{a} r^2 (-a) e^{-\frac{r}{a}} \right)_0^\infty + \frac{1}{a} \int_0^\infty 2r (-a) e^{-\frac{r}{a}} dr - 2 \int_0^\infty r e^{-\frac{r}{a}} dr \right] = 0 \end{aligned}$$

Thus, the total charge is zero.

8. A parallel-plate capacitor has plates located at  $z = 0$  and  $z = d$ . The region between the plates is filled with a material that contains a uniform volume charge density  $\rho_0$  C/m<sup>3</sup> and has permittivity  $\epsilon$ . Both plates are held at ground potential.

- (a) Determine the potential field between the plates.
- (b) Determine the electric field intensity  $\mathbf{E}$  between the plates.
- (c) Repeat parts (a) and (b) for the case where the plate at  $z = d$  is raised to a potential  $V_0$ , with the plate at  $z = 0$  grounded.

**Solution,**

(a) Solving Poisson's Equation

3D Parallel-Plate Capacitor with Electric Field Lines

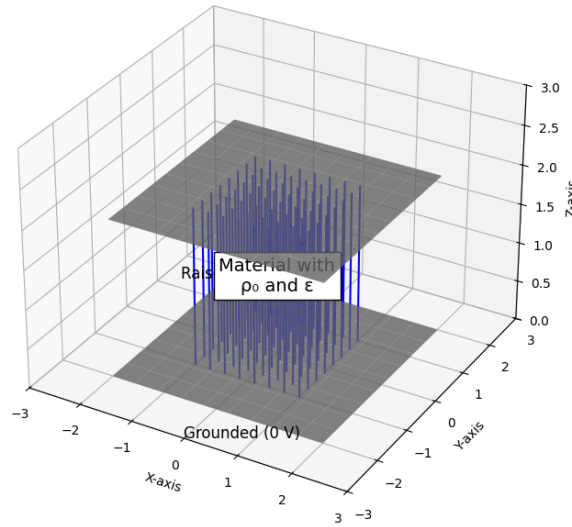


Figure 7: capacitor

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Here  $V$  only has a  $z$  component,

$$\begin{aligned}\frac{d^2 V}{dz^2} &= -\frac{\rho}{\epsilon_0} \\ \frac{dV}{dz} &= -\frac{\rho z}{\epsilon_0} + C_1 \\ V &= -\frac{\rho z^2}{2\epsilon_0} + C_1 z + C_2\end{aligned}$$

Substituting  $V(z = 0) = V(z = d) = 0$ ,

$$V = \frac{\rho z(d - z)}{2\epsilon_0}, \quad 0 \leq x \leq d$$

(b) Electric Field Intensity ( $E$ )

$$E = -\nabla V$$

$$E = \left( \frac{\rho}{\epsilon_0} \left( z - \frac{d}{2} \right) \right) \hat{k}, \quad 0 \leq x \leq d$$

(c) If  $V(z = d) = V_0$

From the first part we know,

$$V = \frac{\rho z^2}{2\epsilon_0} + C_1 z + C_2$$

Boundary conditions,

$$V(0) = 0 \implies C_2 = 0$$

$$V(d) = V_0 \implies \frac{\rho d^2}{2\epsilon_0} + C_1 d = V_0$$

Solving for  $C_1$ ,

$$C_1 = \frac{V_0}{d} + \frac{\rho_0 d}{2\epsilon_0}$$

We finally get potential between the plates to be,

$$V = \left( \frac{\rho_0 z(d - z)}{2\epsilon_0} + \frac{V_0 z}{d} \right), \quad 0 \leq x \leq d$$

Electric Field Intensity ( $\vec{E}$ ) is given by,

$$\vec{E} = -\nabla V$$

$$\vec{E} = \left[ \frac{\rho_0}{\epsilon_0} \left( z - \frac{d}{2} \right) - \frac{V_0}{d} \right] \hat{k}, \quad 0 \leq x \leq d$$

Codes used,

**5(a)**

```
import matplotlib.pyplot as plt
import numpy as np
```

```
phi = np.linspace(-10, 10, 200)
```

```
F = np.sqrt(38 + 24*(np.cos(phi) + np.sin(phi)))
```

```
plt.plot(phi, F, label="|F|" )
plt.legend()
plt.xlabel("s")
plt.ylabel("|F|")
plt.title("|F| at $s=3$ ")
plt.savefig("../figs/fig1.png")
plt.show()
```

### 5(b)

```
import matplotlib.pyplot as plt
import numpy as np
```

```
phi = np.pi/4
s = np.linspace(-10, 10, 200)
F = np.sqrt((160/((s**2+1)**2)) + 22 + (240*np.sqrt(2)/(s**2 + 1))
)
```

```
plt.plot(s, F, label="|F| vs s")
plt.legend()
plt.xlabel("s")
plt.ylabel("|F|")
plt.title("|F| at $\phi=\pi/4$")
plt.savefig("../figs/fig2.png")
plt.show()
```

### 6(a)

```
import matplotlib.pyplot as plt
import numpy as np
from matplotlib import colormaps
```

```
def E_charge(X, Y, q, position):
    X1 = X - position[0]
    Y1 = Y - position[1]
    norm = np.sqrt(X1**2 + Y1**2)
    return q/(4*np.pi*e0)*(X-position[0])/norm**3, q/(4*np.pi*e0)*(Y-position[1]
```

```
def E_field(X, Y):
    E1 = E_charge(X, Y, -q, (1, 0))
    E2 = E_charge(X, Y, q, (0, 1))
    E3 = E_charge(X, Y, -q, (-1, 0))
    E4 = E_charge(X, Y, q, (0, -1))
    return E1[0]+E2[0]+E3[0]+E4[0], E1[1]+E2[1]+E3[1]+E4[1]
```

```

q = 1
e0 = 8.854e-12

x = np.linspace(-2, 2, 20)
y = np.linspace(-2, 2, 20)
X, Y = np.meshgrid(x, y)

U, V = E_field(X, Y)

# Compute the magnitude of the vectors
magnitude = np.sqrt(U**2 + V**2)

for i in range(0, len(magnitude)):
    for j in range(0, len(magnitude)):
        if magnitude[i][j] > 1e11:
            magnitude[i][j] = 0

# Plot using quiver, setting the color using a colormap
plt.figure(figsize=(6, 6))
quiver_plot = plt.quiver(X, Y, U/magnitude, V/magnitude, magnitude, cmap='plasma')

# Add a colorbar
plt.colorbar(quiver_plot, label='Magnitude')

plt.xlabel("X")
plt.ylabel("Y")
plt.title("Vector Field")
plt.savefig("../figs/field.png")
plt.show()

```

**6(b)**

```

import matplotlib.pyplot as plt
import numpy as np
#from matplotlib import colormaps

def potential_charge(X, Y, q, position):
    X1 = X - position[0]
    Y1 = Y - position[1]
    R = (np.sqrt(X1**2 + Y1**2))
    if (R.all() == 0):
        R = 1e-9
    return q/(4*np.pi*e0)*(1/R)
def potential_field(X, Y):
    V1 = potential_charge(X, Y, -q, (1, 0))
    V2 = potential_charge(X, Y, q, (0, -1))

```

```

        V3 = potential_charge(X, Y, -q, (-1, 0))
        V4 = potential_charge(X, Y, q, (0, 1))
        #V = V1 + V2 + V3 + V4
        return V1+V2+V3+V4
q = 1
e0 = 8.854e-12

x = np.linspace(-2, 2, 50)
y = np.linspace(-2, 2, 50)
X, Y = np.meshgrid(x, y)

#V = np.zeros_like(Y)

V = potential_field(X, Y)
fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection="3d")
ax.plot_surface(X, Y, V, cmap="plasma", edgecolor="none")
ax.set_xlabel("X-axis")
ax.set_ylabel("Y-axis")
ax.set_zlabel("Potential")
plt.savefig("../figs/pot.png")
plt.show()

```