EE1204 Assignment 1

Arjun Pavanje

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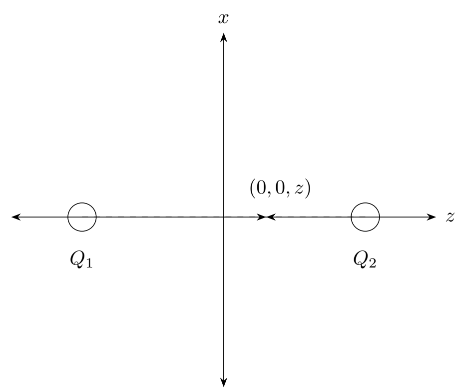
1. Two point charges of equal magnitude q are positioned at $z = \pm\frac{d}{2}$. We aim to determine:

(a) The electric field everywhere on the z-axis.

(b) The electric field everywhere on the x-axis.

(c) The results of (a) and (b) if the charge at $z = -\frac{d}{2}$ is $-1C$ instead of $1C$.

Solution:
(a) Field on z-axis (E): $E = E_1 + E_2$



- E_1: Field due to Q_1
- E_2: Field due to Q_2

$$E_1 = \frac{q}{4\pi\epsilon_0}\frac{|x-\frac{d}{2}|}{(x-\frac{d}{2})^3}\hat{k}, \quad E_2 = -\frac{q}{4\pi\epsilon_0}\frac{|x+\frac{d}{2}|}{(x+\frac{d}{2})^3}\hat{\mathbf{k}}$$

$$E = E_1 + E_2 = \frac{q}{4\pi\epsilon_0}\left[\frac{|x-\frac{d}{2}|}{(x-\frac{d}{2})^3} - \frac{|x+\frac{d}{2}|}{(x+\frac{d}{2})^3}\right]\hat{\mathbf{k}}$$

Case 1: $z > \frac{d}{2}$

$$E = \frac{q}{2\pi\epsilon_0}\left[\frac{z^2 + (\frac{d}{2})^2}{(z^2 - (\frac{d}{2})^2)^2}\right](\hat{\mathbf{k}})$$

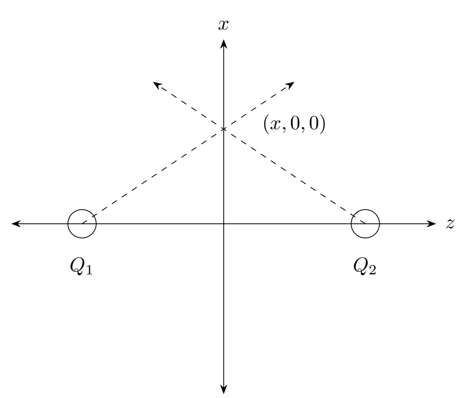
Case 2: $z < -\frac{d}{2}$

$$E = \frac{q}{2\pi\epsilon_0}\left[\frac{z^2 + (\frac{d}{2})^2}{(z^2 - (\frac{d}{2})^2)^2}\right](-\hat{\mathbf{k}})$$

Case 3: $-\frac{d}{2} < z < \frac{d}{2}$

$$\begin{aligned} E &= \frac{q}{2\pi\epsilon_0}\left[\frac{1}{(z + \frac{d}{2})^2} - \frac{1}{(z - \frac{d}{2})^2}\right](\hat{\mathbf{k}}) \\ &= \frac{q}{2\pi\epsilon_0}\left[\frac{zd}{(z^2 - (\frac{d}{2})^2)^2}\right](-\hat{\mathbf{k}}) \end{aligned}$$

(b)



Field on x-axis (E): $E_1 + E_2$

E_1 : Field due to q_1, $\quad E_2$: Field due to q_2

$$E = \frac{q}{4\pi\epsilon_0}\left(\frac{x\hat{i} - \frac{d}{2}\hat{k}}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}} + \frac{\left(x\hat{i} + \frac{d}{2}\hat{k}\right)}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\right)\hat{\mathbf{i}}$$

$$E = \frac{q}{2\pi\epsilon_0}\frac{x}{\left(x^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}\hat{\mathbf{i}}.$$

(c) Result obtained in (a) becomes,
Case 1: $z > \frac{d}{2}$,

$$E = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{(z-\frac{d}{2})^2} - \frac{1}{(z+\frac{d}{2})^2}\right)\hat{\mathbf{k}}$$
$$E = \frac{q}{2\pi\epsilon_0}\frac{zd}{(z^2-(\frac{d}{2})^2)^2}\hat{\mathbf{k}}.$$

Case 2: $z < -\frac{d}{2}$,

$$E = +\frac{q}{4\pi\epsilon_0}\left(\frac{zd}{(z^2-(\frac{d}{2})^2)^3})\right) - \hat{\mathbf{k}}$$

Case 3: $-\frac{d}{2} < z < \frac{d}{2}$

$$E = -\frac{q}{4\pi\epsilon_0}\left(\frac{1}{(z+\frac{d}{2})^2} + \frac{1}{(z-\frac{d}{2})^2}\right)\hat{\mathbf{k}}$$
$$= -\frac{q}{4\pi\epsilon_0}\left(\frac{z^2+(\frac{d}{2})^2}{(z^2-(\frac{d}{2})^2)^2}\right)\hat{\mathbf{k}}$$

Result obtained in (b) becomes,

$$E = \frac{q}{4\pi\epsilon_0}\left(\frac{x\hat{i} - \frac{d}{2}\hat{k}}{\left(x^2+\left(\frac{d}{2}\right)^2\right)^{3/2}} - \frac{\left(x\hat{i} + \frac{d}{2}\hat{k}\right)}{\left(x^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}\right)\hat{\mathbf{i}}$$
$$E = \frac{q}{4\pi\epsilon_0}\frac{d}{\left(x^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}(-\hat{\mathbf{i}}).$$

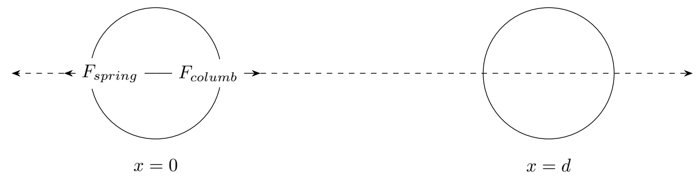
2. A crude device for measuring charge consists of two small insulating spheres of radius a, one of which is fixed in position. The other is movable along the x-axis and is subject to a restraining force kx, where k is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of Q/C, obtain the expression by which Q may be found as a function of x. Determine the maximum charge that can be measured in terms of ϵ_0, k, and d, and then state the separation of the spheres. What happens if a larger charge is applied?

Solution:

Say $Q_1 = +Q$ is the movable charge, $Q_2 = -Q$ is the stationary charge.

At equilibrium, moving force on Q is balanced by electrostatic force:

$$\frac{1}{4\pi\epsilon_0}\frac{Q^2}{(x-d)^2} = kx$$



$$Q = \sqrt{4\pi\epsilon_0 kx(x-d)^2}$$

Maximum measurable charge:

$$\frac{dQ}{dx} = 0$$

Differentiating,

$$(x-d)^2\left(x - \frac{d}{3}\right) = 0$$

$$x = d \quad \text{(minimum)}$$

$$x = \frac{d}{3} \quad \text{(maximum)}$$

$$Q_{\max} = \sqrt{\frac{16\pi\epsilon_0 kd^3}{9}}$$

Seperation,

$$x = \frac{2d}{3}$$

3. A flux density field is given as

$$\mathbf{F}_1 = 5\mathbf{a_z}.$$

The task is to evaluate the outward flux of $\mathbf{F}_1$ through the hemispherical surface defined by $r = a$, $0 < \theta < \pi/2$, and $0 < \phi < 2\pi$. Next, consider what simple observation would have saved a lot of work in the previous part. Identifying symmetries or using alternative methods can simplify the calculations significantly.

Now suppose the field is given by

$$\mathbf{F}_2 = 5z\mathbf{a_z}.$$

Using the appropriate surface integrals, determine the net outward flux of $\mathbf{F}_2$ through the closed surface consisting of the hemisphere from the first part and its circular base in the xy plane. Finally, repeat the previous calculation by applying the divergence theorem and evaluating an appropriate volume integral. This approach should confirm the result obtained through direct surface integration.

Solution:

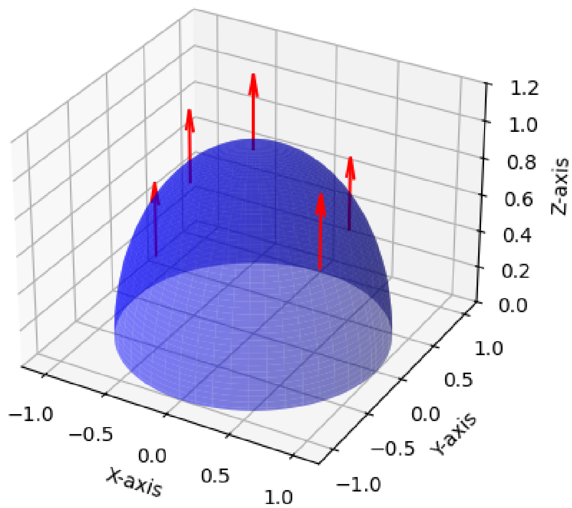


Figure 1: Hemisphere, Field lines

Flux due to electric field over a surface S:

$$\iint \vec{E} \cdot \vec{n} \, dS$$

Here, the surface is a hemispherical surface. Using spherical coordinates:

$$\vec{E} = 5z\hat{k},$$
$$\vec{n} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

In spherical coordinates,

$$\begin{aligned} x &= \rho \sin(\theta)\cos(\phi) \\ y &= \rho \sin(\theta)\sin(\phi) \\ z &= \rho\cos(\theta) \end{aligned}$$

Thus,

$$dS = a^2 \sin\theta \, d\theta \, d\phi$$

The flux becomes:

$$= \iint 5a^2 \cos\theta \sin\theta \, d\theta \, d\phi$$

Separating the integrals:

$$= 5a^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

For the θ integral:

$$\begin{aligned} &= 5a^2 \int_0^{2\pi} d\phi \left[-\frac{\cos^2\theta}{2} \right]_0^{\pi/2} \\ &= 5a^2 \int_0^{2\pi} d\phi \cdot \frac{1}{2} \\ &= 5\pi a^2 \end{aligned}$$

There is an easier way to find flux, we can say that flux passing through the hemisphere is the same as that passing through a disc of radius a centered at the origin as both surfaces subtend the same solid angle.

$$\begin{aligned} \text{Flux} &= \iint \vec{E} \cdot \vec{n} \, dS \\ \vec{E} \cdot \vec{n} &= 5 \\ \text{Flux } &= 5 \iint dS \\ &= 5(\pi a^2) \end{aligned}$$

This is the same result obtained above by integration. Now if field changes to $E = 5z\hat{k}$,

$$\vec{E} = 5z\hat{k}, \quad \hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Flux:

$$\text{Flux} = \iint_S \vec{E} \cdot \hat{n} \, dS$$

Substituting:

$$\text{Flux} = \int_0^{2\pi} \int_0^{\pi} 5a^3 \cos^2\theta \, a^2 \sin\theta \, d\theta d\phi$$

Simplifying,

$$= \int_0^{2\pi} \int_0^{\pi/2} 5a^3 \cos^2\theta \sin\theta \, d\theta d\phi = \int_0^{2\pi} \left[-\frac{5a^3}{3} cos(\theta) \right]_0^{\pi/2} d\phi$$

Using substitution:

$$= 5a^3 \cdot 2\pi \cdot \frac{1}{3} = \frac{10}{3}\pi a^3$$

Applying Divergence Theorem:
Flux:

$$\text{Flux} = \iint_S (\vec{E} \cdot \hat{n}) dS = \iiint_V (\nabla \cdot \vec{E}) dV$$

Calculate divergence:

$$\nabla \cdot \vec{E} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(5z) = 5$$

Volume integral:

$$\text{Flux} = 5 \iiint_V dV = 5 \left[\frac{2}{3}\pi a^3 \right] = 5a^3 = \frac{10}{3}\pi a^3$$

This verifies the result obtained by direct integration.

4. An infinitely long cylindrical dielectric of radius b contains charge within its volume with a charge density given by

$$\rho_v = a\rho^2,$$

where a is a constant. The goal is to determine the electric field strength **E** both inside and outside the cylinder.

Solution:

We can solve this problem using Gauss's law,

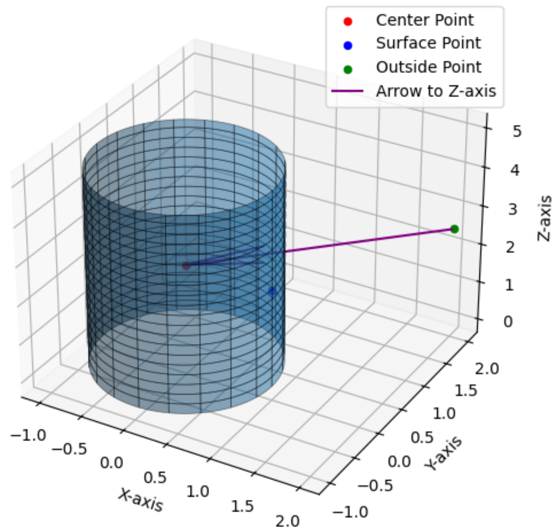


Figure 2: Cylinder

$$\iint \vec{E} \cdot d\vec{S} = \frac{Q_{\text{enc}}}{\epsilon_0}.$$

For Q_{enc}:

$$Q_{\text{enc}} = \int_0^l \int_0^{2\pi} \int_0^r ax^3 r \, dr \, d\phi \, dz,$$

where $x = r$.

Simplifying:

$$Q_{\text{enc}} = \int_{z=0}^l \int_{\phi=0}^{2\pi} \int_{r=0}^r ar^4 \, dr \, d\phi \, dz.$$

$$Q_{\text{enc}} = \int_0^l \int_0^{2\pi} \frac{ar^5}{5} \Big|_0^r d\phi dz.$$

$$Q_{\text{enc}} = \int_0^l \int_0^{2\pi} \frac{ar^5}{5} d\phi dz.$$

$$Q_{\text{enc}} = (2\pi l)\frac{ar^4}{4}.$$

Next, we will take our Gaussian surface as a cylinder of length l, radius r. For the inside cylinder:

$$\iint \vec{E} \cdot d\vec{S} = E(2\pi r l) = (2\pi l)\frac{ar^4}{4}.$$

Thus:

$$E = \frac{ar^3}{4\epsilon_0}\hat{r}.$$

For the outside cylinder:

$$\iint \vec{E} \cdot d\vec{S} = E(2\pi r l) = (2\pi l)al^4/4.$$

Thus:

$$E = \frac{al^4}{4\epsilon_0 r}.$$

$$E = \begin{cases} \dfrac{ab^3}{4\epsilon_0} & \text{if } r \leq b, \\ \dfrac{ab^4}{4\epsilon_0 r} & \text{if } r > b. \end{cases}$$

5. Given the vector field in cylindrical coordinates:

$$\mathbf{F} = \left[\frac{40}{s^2+1} + 3(\cos\phi + \sin\phi)\right]\hat{s} + 3(\cos\phi - \sin\phi)\hat{\phi} - 2\hat{z}$$

(a) Compute and plot the magnitude $|\mathbf{F}|$ as a function of ϕ for $s = 3$.

(b) Compute and plot the magnitude $|\mathbf{F}|$ as a function of s for $\phi = 45°$.

(c) Calculate the divergence $\nabla \cdot \mathbf{F}$.

(d) Calculate the curl $\nabla \times \mathbf{F}$ and verify whether the field is conservative.

Solution:

(a) $\vec{F}$ at $s = 3$:

$$\vec{F}\Big|_{s=3} = [4 + 3(\cos\phi + \sin\phi)]\,\hat{s} + 3(\cos\phi - \sin\phi)\hat{\phi} + 4\hat{z}.$$

$$
\begin{aligned}
|\vec{F}|^2 &= 16 + \left[(\cos\phi + \sin\phi)^2 + (\cos\phi - \sin\phi)^2\right] + 4, \\
&= 16 + \left[\cos^2\phi + 2\cos\phi\sin\phi + \sin^2\phi + \cos^2\phi - 2\cos\phi\sin\phi + \sin^2\phi\right] + 4, \\
&= 16 + (2) + 4, \\
&= 38 + 24(\cos\phi + \sin\phi).
\end{aligned}
$$

(b) $\vec{F}$ at $\phi = \frac{\pi}{4}$:

$$
\vec{F}_\phi = \left[\frac{40}{s^2+1} + 3\sqrt{2}\right]\hat{s} + s\hat{\phi} - 2\hat{z}.
$$

Magnitude of $\vec{F}$:

$$
|\vec{F}|^2 = \frac{160}{(s^2+1)^2} + 22 + \frac{240\sqrt{2}}{s^2+1}.
$$

(c) Divergence in cylindrical coordinates: The divergence in cylindrical coordinates is given by:

$$
\begin{aligned}
\nabla \cdot \vec{F} &= \frac{1}{s}\frac{\partial}{\partial s}(s f_s) + \frac{1}{s}\frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z}, \\
F &= f_s + f_\phi + f_z
\end{aligned}
$$

$$
\begin{aligned}
\nabla \cdot \vec{F} &= \frac{1}{s}\frac{\partial}{\partial s}\left(40s\frac{1}{s^2+1} + 3s(\cos\phi + \sin\phi)\right) + \frac{1}{s}\frac{\partial}{\partial \phi}\left(3s(\cos\phi + \sin\phi)\right) + \frac{\partial}{\partial z}(-2) \\
&= \frac{1}{s}\left[40\frac{\left(1-s^2\right)}{\left(1+s^2\right)^2} + 3(\cos\phi + \sin\phi)\right] + \frac{1}{s}\left[-3(\cos\phi + \sin\phi)\right] \\
&= \frac{40(1-s^2)}{s(1+s^2)^2}.
\end{aligned}
$$

(d) Curl in cylindrical coordinates is given by:

$$
\nabla \times \mathbf{F} = \frac{1}{r}\left(\frac{\partial f_z}{\partial r} - \frac{\partial f_r}{\partial z}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial (r f_\phi)}{\partial z} - \frac{\partial f_z}{\partial \phi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{\partial f_r}{\partial \phi} - \frac{\partial (r f_\phi)}{\partial r}\right)\hat{\mathbf{z}}
$$

$$
\begin{aligned}
beginalign{*}10pt] &= \frac{1}{r}[3(\cos\phi - \sin\phi) + 3(\cos\phi - \sin\phi)]\hat{\mathbf{z}} \\
beginalign{*}10pt] &= 0
\end{aligned}
$$

Field is conservative.

6.

Solution:

(a) The electric field due to each charge at a point in space can be expressed as follows:

1. Electric field due to the charge $+1\,\mathrm{C}$ at $(0, 1)$:

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}} + (y-1)\hat{\mathbf{j}}}{(x^2 + (y-1)^2)^{\frac{3}{2}}}$$

2. Electric field due to the charge $+1\,\mathrm{C}$ at $(0, -1)$:

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}} + (y+1)\hat{\mathbf{j}}}{(x^2 + (y+1)^2)^{\frac{3}{2}}}$$

3. Electric field due to the charge $-1\,\mathrm{C}$ at $(1, 0)$:

$$\vec{E}_1 = \frac{-1}{4\pi\epsilon_0}\frac{(x-1)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x-1)^2 + y^2)^{\frac{3}{2}}}$$

4. Electric field due to the charge $-1\,\mathrm{C}$ at $(-1, 0)$:

$$\vec{E}_1 = \frac{-1}{4\pi\epsilon_0}\frac{(x+1)\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{((x+1)^2 + y^2)^{\frac{3}{2}}}$$

To find the total electric field at a point in space due to these charges, you add

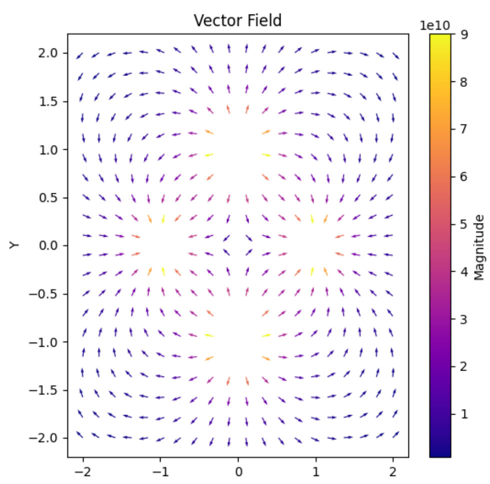


Figure 3: Field Lines

the individual fields (by superposition):

$$E_{\text{total}} = E_1 + E_2 + E_3 + E_4$$

(b) Electric Potential may be derived from Electric field by the relation $E = -\nabla V$

1. Electric Potential due to the charge $+1\,\mathrm{C}$ at $(0, 1)$:

$$V_1 = \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + (y-1)^2}}$$

2. Electric Potential due to the charge $+1\,\mathrm{C}$ at $(0, -1)$:

$$V_2 = \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + (y+1)^2}}$$

3. Electric Potential due to the charge $-1\,\mathrm{C}$ at $(1, 0)$:

$$V_3 = -\frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{(x-1)^2 + y^2}}$$

4. Electric Potential due to the charge $-1\,\mathrm{C}$ at $(-1, 0)$:

$$V_4 = -\frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{(x+1)^2 + y^2}}$$

To find the total electric potential at a point in space due to these charges, you

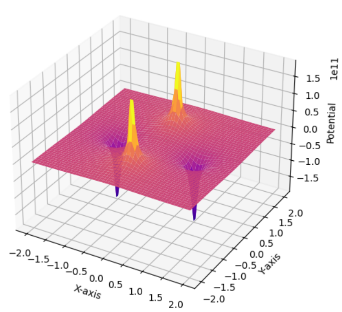


Figure 4: Potential Plot

add the individual potentials (by superposition):

$$V_{\text{total}} = V_1 + V_2 + V_3 + V_4$$

(c) The potential energy of a system of point charges is given by:

$$U = \frac{1}{4\pi\epsilon_0} \sum_{i<j} \frac{q_i q_j}{r_{ij}}.$$

In the give question there are 4 charges,

- $1C$ at $(0,1)$ and $(0,-1)$.
- $-1C$ at $(1,0)$ and $(-1,0)$.

Summing up their contributions toward Potential Energy of system,

$$U = \frac{1}{4\pi\epsilon_0} \left[\frac{(1C)(-1C)}{\sqrt{2}} + \frac{(1C)(-1C)}{\sqrt{2}} + \frac{(1C)(-1C)}{\sqrt{2}} + \right.$$
$$\left. \frac{(1C)(-1C)}{\sqrt{2}} + \frac{(1C)(1C)}{2} + \frac{(-1C)(-1C)}{2} \right].$$

Simplifying:

$$\begin{aligned} U &= \frac{1}{4\pi\epsilon_0} \left[-\frac{4}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2} \right] \\ &= \frac{q^2}{4\pi\epsilon_0} \left[1 - 2\sqrt{2} \right]. \end{aligned}$$

Thus, the potential energy of the configuration is:

$$U = \frac{q^2}{4\pi\epsilon_0}(1 - 2\sqrt{2}).$$

(d) Divergence of the electric field
The electric field $\mathbf{E}$ due to a point charge q at position $\mathbf{r}_0$ is given by:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3}.$$

By the superposition principle, the total electric field $\mathbf{E}$ is the sum of the contributions from all four charges.

The divergence of the electric field is given by Gauss's law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0},$$

where $\rho(\mathbf{r})$ is the charge density.

Since we have point charges, we can use the Dirac delta function to represent charge density:

$$\rho(\mathbf{r}) = q\delta(x)\delta(y-1) + q\delta(x)\delta(y+1) - q\delta(x-1)\delta(y) - q\delta(x+1)\delta(y).$$

Applying Gauss's law,

$$\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0}\left[\delta(x)\delta(y-1) + \delta(x)\delta(y+1) - \delta(x-1)\delta(y) - \delta(x+1)\delta(y)\right].$$

This expression shows that the divergence of field $\mathbf{E}$ is nonzero only at the locations of the charges. This verifies the fact that field due to point charges satisfy Gauss law (in differential form)

(e) The curl of a vector field $\mathbf{E} = (E_x, E_y, E_z)$ is given by,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

For a single point charge, the electric field components in Cartesian coordinates are:

$$E_x = \frac{q}{4\pi\varepsilon_0}\frac{x-x_0}{r^3},$$

$$E_y = \frac{q}{4\pi\varepsilon_0}\frac{y-y_0}{r^3},$$

$$E_z = \frac{q}{4\pi\varepsilon_0}\frac{z-z_0}{r^3}.$$

where,

$$r = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{1/2}$$

The curl becomes,

$$\left(\frac{q}{4\pi\varepsilon_0}\left[\frac{-3(z-z_0)(y-y_0)}{(r^2)^{5/2}}\right] - \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(y-y_0)(z-z_0)}{(r^2)^{5/2}}\right]\right)\hat{i} -$$
$$\left(\frac{q}{4\pi\varepsilon_0}\left[\frac{-3(z-z_0)(x-x_0)}{(r^2)^{5/2}}\right] - \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(x-x_0)(z-z_0)}{(r^2)^{5/2}}\right]\right)\hat{j} +$$
$$\left(\frac{q}{4\pi\varepsilon_0}\left[\frac{-3(y-y_0)(x-x_0)}{(r^2)^{5/2}}\right] - \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(x-x_0)(y-y_0)}{(r^2)^{5/2}}\right]\right)$$

Thus, we get,

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

Since the curl of the field due to a single point charge is zero, and curl operator is linear, the total electric field (which is a superposition of fields of the induvidual charges) also has zero curl.

$$\nabla \times \mathbf{E} = 0.$$

This shows that the electric field due to the system of charges is conservative, i.e. its curl is 0

7. Given the spherically symmetric potential field in free space,

$$V(r) = V_0 e^{-r/a},$$

determine the following:

(a) Find the charge density ρ_v at $r = a$.

(b) Calculate the electric field $\mathbf{E}$ at $r = a$.

(c) Compute the total charge.

Solution:
(a) Poisson's Equation is given by,

$$\nabla^2 V(r) = -\frac{\rho(r)}{\epsilon_0}$$

In spherical coordiantes,

$$\nabla^2 \phi(r, \theta, \phi) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \phi}{\partial \phi^2} = -\frac{\rho}{\epsilon_0}$$

Simplifying:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = -\frac{\rho(r)}{\epsilon_0}.$$

Substituting $V(r) = V_0 e^{-\frac{r}{a}}$:

$$-\frac{V_0 e^{-r/a}}{a r^2} \left(\frac{-r^2}{a} + 2r \right) = -\frac{\rho(r)}{\epsilon_0}.$$

$$\rho(r) = -\frac{V_0 \epsilon_0 e^{-r/a}}{a} \left(\frac{1}{a} - \frac{2}{r} \right)$$

At $r = a$

$$\rho = \frac{V_0 \epsilon_0 e^{-\frac{r}{a}}}{a^2}$$

(b)

$$E = -\nabla\Phi,$$

where Φ: Potential, E: Electric Field.
Gradient in spherical coordinates,

$$\nabla f(r,\theta,\phi) = \hat{e}_r\frac{\partial f}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

Substituting:

$$E(r) = \frac{V_0 e^{-\frac{r}{a}}}{a}\hat{r}.$$

At $r = a$:

$$E(r=a) = \frac{V_0 e^{-1}}{a}\hat{r}.$$

(c) Total charge is given by:

$$Q = \iiint \rho\, dV,$$

where $\rho =$ charge density.

$$\rho = -V_0\frac{\epsilon_0 e^{-\frac{r}{a}}}{a}\left(\frac{1}{a} - \frac{2}{r}\right).$$

Substitute into the integral:

$$Q = \iiint -V_0\frac{\epsilon_0}{a}\left(\frac{1}{a} - \frac{2}{r}\right) dV.$$

Using spherical coordinates:

$$Q = \int_0^\infty \int_0^\pi \int_0^{2\pi} -V_0\frac{\epsilon_0}{a}\left(\frac{1}{a} - \frac{2}{r}\right) r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

Analysing the first integral seperately, and applying integration by-parts on the first part

$$= -V_0\frac{\epsilon_0}{a}\left[\frac{1}{a}\int_0^\infty r^2 e^{-\frac{r}{a}}\, dr - 2\int_0^\infty r e^{-\frac{r}{a}}\, dr\right]$$
$$= -V_0\frac{\epsilon_0}{a}\left[\left(\frac{1}{a}r^2(-a)e^{-\frac{r}{a}}\right)_0^\infty + \frac{1}{a}\int_0^\infty 2r(-a)e^{-\frac{r}{a}}\, dr - 2\int_0^\infty r e^{-\frac{r}{a}}\, dr\right] \quad = 0$$

Thus, the total charge is zero.

8. A parallel-plate capacitor has plates located at $z = 0$ and $z = d$. The region between the plates is filled with a material that contains a uniform volume charge density ρ_0 C/m^3 and has permittivity ϵ. Both plates are held at ground potential.

(a) Determine the potential field between the plates.

(b) Determine the electric field intensity **E** between the plates.

(c) Repeat parts (a) and (b) for the case where the plate at $z = d$ is raised to a potential V_0, with the plate at $z = 0$ grounded.

Solution:
(a) Solving Poisson's Equation

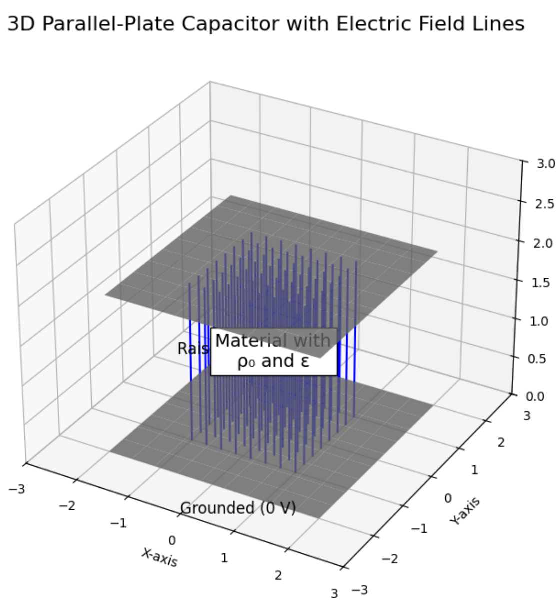


Figure 5: capacitor

$$\nabla^2 V = \frac{\rho}{\epsilon_0}$$

Here V only has a z component:

$$\frac{d^2V}{dz^2} = \frac{\rho}{\epsilon_0}$$

$$\frac{dV}{dz} = \frac{\rho z}{\epsilon_0} + C_1$$

$$V = \frac{\rho z^2}{2\epsilon_0} + C_1 z + C_2$$

Substituting $V(z=0) = V(z=d) = 0$:

$$V = \frac{\rho z(z-d)}{2\epsilon_0}\hat{k}$$

Electric Field Intensity (E)

$$E = -\nabla V$$

$$E = -\left(-\frac{\rho}{\epsilon_0}\left(z - \frac{d}{2}\right)\right) \quad \hat{k}$$

(c) If $V(z=d) = V_0$
From the first part we know,

$$V = \frac{\rho z^2}{2\epsilon_0} + C_1 z + C_2$$

Boundary conditions:

$$V(0) = 0 \implies C_2 = 0$$

$$V(d) = V_0 \implies \frac{\rho d^2}{2\epsilon_0} + C_1 d = V_0$$

Solving for C_1 :

$$C_1 = \frac{V_0}{d} - \frac{\rho_0 d}{2\epsilon_0}$$

We finally get potential between the plates to be,

$$V = \left(\frac{\rho_0 z(z-d)}{2\epsilon_0} + \frac{V_0 z}{d}\right)\hat{r}$$

Electric Field Intensity $(\vec{E})$ is given by,

$$\vec{E} = -\nabla V$$
$$\vec{E} = -\left[\frac{\rho_0}{\epsilon_0}\left(z - \frac{d}{2}\right) + \frac{V_0}{d}\right]$$