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EE24BTECH11005 - Arjun Pavanje

Question:

Find the area of the region bounded by the curve $y^2 = 4x$, $x^2 = 4y$

Variable	Description
\mathbf{e}	Eccentricity of conic
\mathbf{F}	Focus of conic
\mathbf{I}	Identity matrix
$\mathbf{n}^\top \mathbf{x} = c$	Equation of directrix
\mathbf{n}	Slope of directrix
f	$\ \mathbf{n}\ ^2 \ \mathbf{F}\ ^2 - c^2 e^2$
\mathbf{V}	A symmetric matrix given by eigenvalue decomposition
\mathbf{u}	Vertex of conic with same directrix

TABLE I: Variables Used

Solution: The general equation of a parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$ is given by,

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (1)$$

$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n} \mathbf{n}^\top \quad (2)$$

$$\mathbf{u} = c e^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \quad (3)$$

$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \quad (4)$$

for the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} = 1$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (5)$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad (6)$$

$$f = 0 \quad (7)$$

for the parabola $x^2 = 4y$, equation of directrix is, $\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 1$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (8)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad (9)$$

$$f = 0 \quad (10)$$

The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i, i = 1, 2$ is defined as,

$$\mathbf{x}^T (\mathbf{V}_1 - \mathbf{V}_2) \mathbf{x} + 2(\mathbf{u}_1 - \mathbf{u}_2)^T \mathbf{x} + (f_1 - f_2) = 0 \quad (11)$$

On solving we get the points of intersection to be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ Area between the 2 parabolas is,

$$\int_0^4 2\sqrt{x}dx - \int_0^4 \frac{x^2}{4}dx = \frac{16}{3} \quad (12)$$

The area between the curves $y^2 = 4x, x^2 = 4y$ is, $\frac{16}{3}$ units

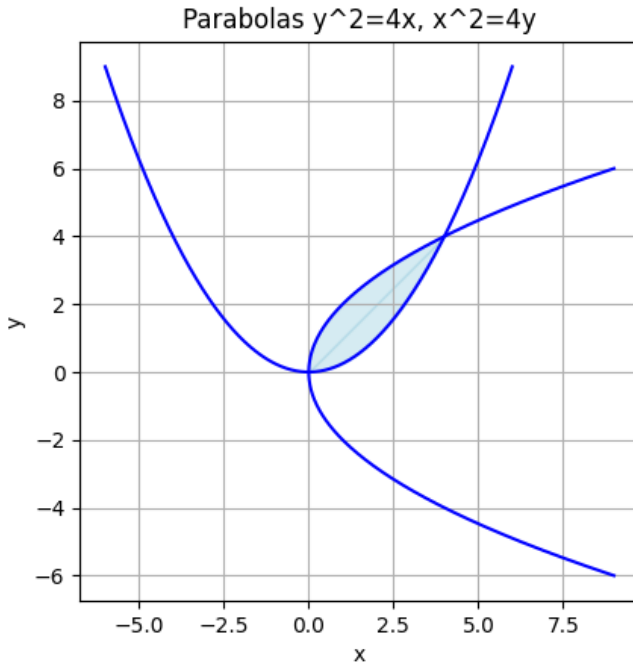


Fig. 1: Required Parabolas