9-9.3-10

EE24BTECH11005 - Arjun Pavanje

Question:
If the area of the region bounded by the curve $y^2 = 4ax$ and the line $x = 4a$ is $\frac{256}{3}$ sq. units, then find the positive value of a, where $a > 0$

Variable	Description
$\mathbf{h}$	Point lying on the line
$\mathbf{m}$	Slope of line
$\mathbf{e}$	Eccentricity of conic
$\mathbf{F}$	Focus of conic
$\mathbf{I}$	Identity matrix
f	$\|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2$
$\mathbf{V}$	A symmetric matrix given by eigenvalue decomposition
$\mathbf{u}$	Vertex of conic with same directrix

TABLE I: Variables Used

Solution: Line equation of form $\mathbf{x} = \mathbf{h} + k\mathbf{m}$

$$\mathbf{x} = \begin{pmatrix} 4a \\ 0 \end{pmatrix} + k \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

Equation of parabola of form $\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0$ is

$$\mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix}, f = 0, \mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

If a line intersects the conic, k value of intersecting point is given by,

$$k_i = \frac{-\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(h)(\mathbf{m}^\top \mathbf{V}\mathbf{m})}}{\mathbf{m}^\top \mathbf{V}\mathbf{m}} \tag{3}$$

On substituting values of $\mathbf{u}, \mathbf{m}, \mathbf{h}, \mathbf{V}$ we get,

$$k = \pm 4a \tag{4}$$

Points of intersection with parabola are, $\begin{pmatrix} 4a \\ 4a \end{pmatrix}, \begin{pmatrix} 4a \\ -4a \end{pmatrix}$ Area bound between the parabola and the line is,

$$2\int_0^4 a2\sqrt{a}\sqrt{x}dx \tag{5}$$

$$= \frac{64a^2}{3} = \frac{256}{3} \tag{6}$$

We get $a = 2$
Required value of a is 2

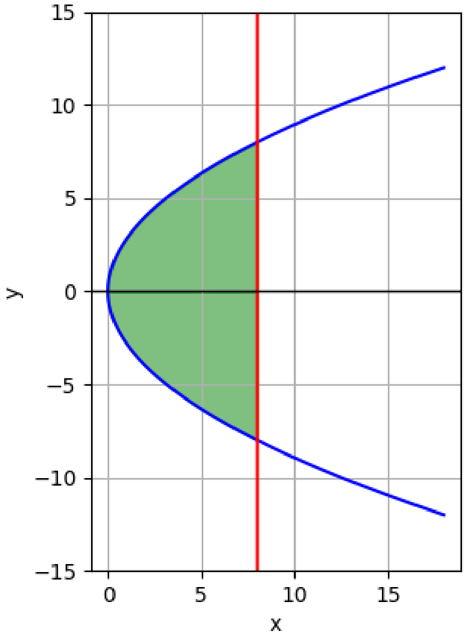

Fig. 1: Parabola $y^2 = 8x$, Line $x = 8$