7-7.2-20

EE24BTECH11005 - Arjun Pavanje

Question:
Find the area of the region bounded by the curve $y^2 = 4x, x^2 = 4y$

Variable	Description
$\mathbf{e}$	Eccentricity of conic
$\mathbf{F}$	Focus of conic
$\mathbf{I}$	Identity matrix
$\mathbf{n}^\top \mathbf{x} = c$	Equation of directrix
$\mathbf{n}$	Slope of normal to directrix
f	$\|\|\mathbf{n}\|\|^2 \|\|\mathbf{F}\|\|^2 - c^2e^2$
$\mathbf{V}$	A symmetric matrix given by eigenvalue decomposition
$\mathbf{u}$	Vertex of conic with same directrix

TABLE I: Variables Used

Solution: The general equation of a parabola with directrix $\mathbf{n}^\top \mathbf{x} = c$ is given by,

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$
$$\mathbf{V} = \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{2}$$
$$\mathbf{u} = ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{3}$$
$$f = \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{4}$$

for the parabola $y^2 = 4x$, equation of directrix is, $\begin{pmatrix} -1 & 0 \end{pmatrix} \mathbf{x} = 1$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$
$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{6}$$
$$f = 0 \tag{7}$$

for the parabola $x^2 = 4y$, equation of directrix is, $\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 1$

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8}$$
$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{9}$$
$$f = 0 \tag{10}$$

The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, f_i, i = 1, 2$ is defined as,

$$\mathbf{x}^\top (\mathbf{V}_1 - \mathbf{V}_2)\mathbf{x} + 2(\mathbf{u}_1 - \mathbf{u_2})^\top \mathbf{x} + (f_1 - f_2) = 0 \tag{11}$$

On solving we get the points of intersection to be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ Area between the 2 parabolas is,

$$\int_0^4 2\sqrt{x}dx - \int_0^4 \frac{x^2}{4}dx = \frac{16}{3} \tag{12}$$

The area between the curves $y^2 = 4x, x^2 = 4y$ is, $\frac{16}{3}$ units

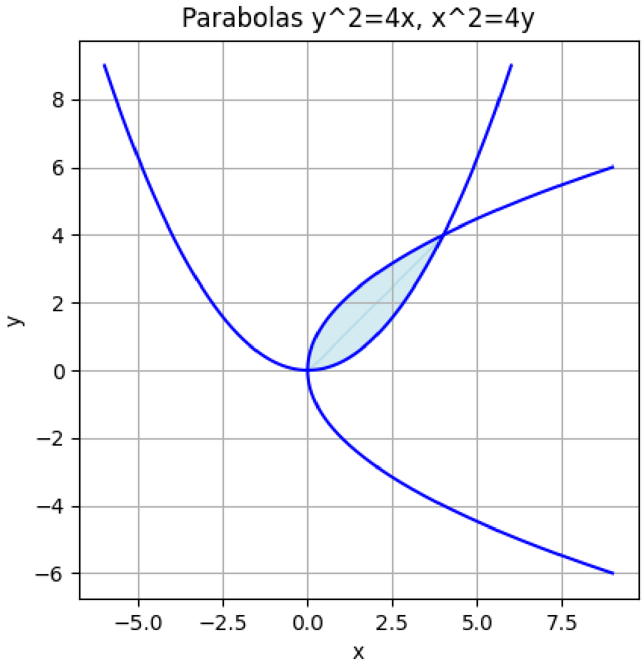

Fig. 1: Required Parabolas