AI1110: Practice Problems - Set I

January 13, 2025

Topics Covered: Preliminaries, Axiomatic Probability (Lectures 1 to 5)

1. Consider rolling a six-sided die. Let A be the set of outcomes where the roll is an even number. Let B be the set of outcomes where the roll is greater than 3. Calculate and compare the sets on both sides of De Morgan's laws.

2. Consider the following combinatorial identity $\sum_{k=1}^{n} k\binom{n}{k} = n.2^{(n-1)}$.

   (a) Present a combinatorial argument for this identity by considering a set of n people and determining in two ways the number of possible selections of a committee of any size and a chairperson for the committee.

   (b) Present a similar combinatorial proof for the identity $\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{(n-2)} n(n+1)$ as the number of different selections of a committee, its chairperson and a secretary (possibly the same as the chairperson).

   (c) Now argue that $\sum_{k=1}^{n} \binom{n}{k} k^3 = 2^{(n-3)} n^2(n+3)$

3. An investor has 20,000 USD that must be invested among 4 possible start-up companies. Each investment must be in units of 1000USD.

   (a) If the total 20,000 USD is to be invested, how many different investment strategies are possible? What if not all the money needs to be invested? (A given investment strategy need not invest in all the 4 companies.)

   (b) If there are minimal investment requirement in each of the four companies, which are 2000$, 2000$, 3000$, and 4000$, respectively, how many different investment strategies are available if an investment must be made a) in each company?, b)in at least three of the four companies?

4. Prove binomial theorem using induction.

   (a) Using binomial theorem, prove that for $n > 0$, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

5. In the first round of a knockout tournament with $n = 2^m$ players, the n players are divided into $\frac{n}{2}$ pairs, with each of these pairs then playing a game. The losers of the game are eliminated while the winners go on to the next round where the process is repeated until only a single player remains. Suppose we have a knockout tournament of 16 players.

   (a) How many possible outcomes are there for the initial round?

(b) How many outcomes of the tournament are possible, where an outcome gives complete information of all the rounds?

6. A cricket team is chosen such that there are 10 bowlers and 10 batters. The players are to be paired for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are
   (a) no bowler-batter roommate pairs?
   (b) $2i$ bowler-batter roommate pairs for $i = 1, 2, 3, 4, 5$? (A closed-form expression is sufficient. Numerical answers are not expected.)

7. The matching problem: Suppose that each of N people at a party throws their hat in the center of the room. The hats are first mixed up and then each person randomly selects a hat.
   (a) What is the probability that none of the people will select their own hat?
   (b) Intuitively, what happens to the above probability when $N \to \infty$?
   (c) Using the identity $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, calculate the above probability as $N \to \infty$.

8. Consider an experiment whose sample space consists of a countably infinite number of points $\Omega = \{\omega_1, \omega_2, \cdots, \}$.
   (a) Using axioms of probability, show that not all points can be equally likely.
   (b) Can all the points have a positive probability of occurring?

9. Players A, B, and C take turns flipping a coin. The first one to get a head wins. The sample space of this experiment can be defined by

$$\Omega = \{\ 1, 01, 001, 0001, \ldots, 0000\ldots,\ \}$$

   (a) Interpret the sample space.
   (b) Define the following events in terms of Ω:
      (i) Event $A \triangleq$ Player A wins.
      (ii) Event $B \triangleq$ Player B wins.
      (iii) $(A \cup B)^c$.

      Assume that A flips first, then B, then C, then A, and so on.

10. Bonferroni's Inequality:
   (a) Prove that for any two events A and B, we have

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

   (b) Generalize to the case of n events $A_1, A_2, \ldots, A_n$, by showing that

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1).$$

11. A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model for a single roll of this die and find the probability that the outcome is less than 4.

12. A four-sided die is rolled repeatedly, until the first time (if ever) that an even number is obtained. What is the sample space for this experiment?

13. You enter a special kind of chess tournament, in which you play one game with each of three opponents, but you get to choose the order in which you play your opponents, knowing the probability of a win against each. You win the tournament if you win two games in a row, and you want to maximize the probability of winning. Show that it is optimal to play the weakest opponent second, and that the order of playing the other two opponents does not matter.

14. A partition of the sample space Ω is a collection of disjoint events $S_1, \ldots, S_n$ such that $\Omega = \bigcup_{i=1}^n S_i$.

    (a) Show that for any event A, we have
    $$P(A) = \sum_{i=1}^{n} P(A \cap S_i).$$

    (b) Use part (a) to show that for any events A, B, and C, we have
    $$P(A) = P(A \cap B) + P(A \cap C) + P(A \cap B^c \cap C^c) - P(A \cap B \cap C).$$

15. A system is composed of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if component i is working and is equal to 0 if component i has failed.

    (a) How many outcomes are in the sample space od this experiment?

    (b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3 and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

    (c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?

    (d) List all the outcomes in the event AW.

16. Suppose that E and F are two events. Prove that the probability of exactly one of these events occurring is $P(E) + P(F) - 2P(E \cap F)$.

17. A pair of dice is rolled and the sum is computed.

    (a) Find the probability that the sum equals n for each $n = 2, 3, \ldots, 12$.

    (b) The dice are rolled until a sum of 4 or 9 appears. Find the probability that 4 appears first.

18. You and your friend are playing a game with a fair coin. The two of you will continue to toss the coin until the sequence **HH** or **TH** shows up. If **HH** shows up first, then you win, and if **TH** shows up first, your friend wins. What is the probability that your friend wins the game?

19. If you pick three numbers uniformly at random from the interval $[0, 1]$, what is the probability that the median of the three numbers is greater than 0.75?

20. Let $\mathcal{F}$ be an event space, (i.e., a subset of 2^Ω satisfying the event axioms) and suppose that $B \in \mathcal{F}$. Show that $\mathcal{G} = \{A \cap B : A \in \mathcal{F}\}$ is an event space for B.

21. Let A and B be events with probabilities $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Show that $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$, and give examples to show that both extremes are possible. Find corresponding bounds for $P(A \cup B)$.

22. In each packet of cornflakes from a brand ABC may be found a plastic bust of the first five prime ministers of India, the probability that any given packet contains any specific PM being $\frac{1}{5}$, independently of all other packets. What is the probability that the busts of each of the first three PMs is obtained in a bulk purchase of six packets of ABC cornflakes?

23. You are given that at least one of the events $A_r, 1 \leq r \leq n$, is certain to occur, but certainly no more than two occur. If $P(A_r) = p$, $P(A_r \cap A_s) = q$, $r \neq s$, show that $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n}$.

24. (Selection of a point in a square): Take Ω to be the square region in the plane, $\Omega = \{(x, y) : 0 \leq x < 1, 0 \leq y < 1\}$. It can be shown that there is a probability space $(\Omega, \mathcal{F}, P)$ such that any rectangular region that is a subset of Ω of the form $R = \{(u, v) : a \leq u < b, c \leq v < d\}$ is an event, and $P(R) =$ area of $R = (b - a)(d - c)$.

    Let T be the triangular region $T = \{(x, y) : x \geq 0, y \geq 0, x + y < 1\}$. Since T is not rectangular, it is not immediately clear whether T is an event. Show that T is an event, and find $P(T)$, using the axioms of probability.

25. (Possible probability assignments): Suppose A and B are events for some probability space such that $P(A \cap B) = 0.3$ and $P(A \cup B) = 0.6$. Find the set of possible values of the pair $(P(A), P(B))$ and sketch this set, as a subset of the plane.

26. A fair coin is tossed repeatedly until the first heads shows up; the outcome of the experiment is the number of tosses required until the first head occurs. Find a probability law for this experiment. (Show that the chosen probability law satisfies all the axioms of probability).

References

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3. G. Grimmett and D. Stirzaker, “One Thousand Exercises in Probability”, Indian Edition, Oxford University Press, 2001.
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Acknowledgement

Thanks are due to Ms. Chaithanya, Course TA, for her timely help in scribing the questions.