DOCUMENT 1: SOLVING A SIMPLE LINEAR PDE IN ONE-DIMENSION USING THE FINITE ELEMENT METHOD

Consider the 1-D PDE

$$\frac{d}{dx}\left(K_1 \frac{dF}{dx}\right) + F = 0 \quad \text{on} \quad x = [0,1]$$

with boundary conditions

$$F = 1 \quad @ \quad x = 0$$

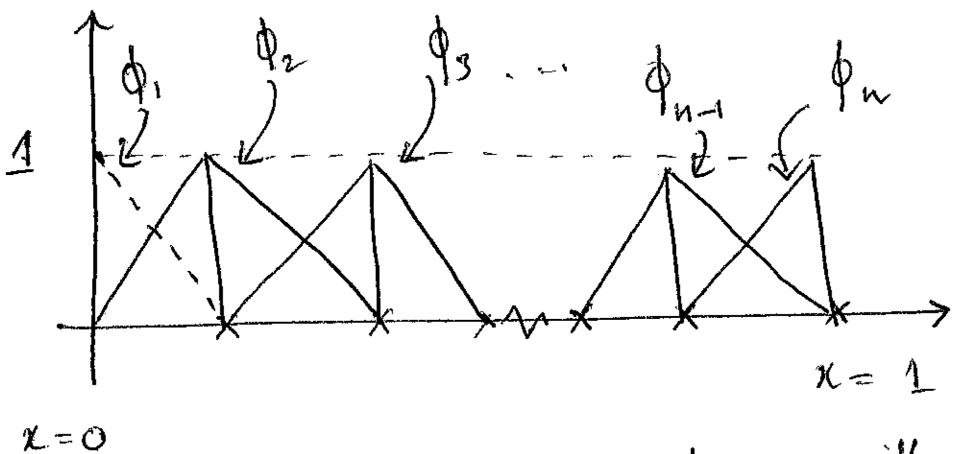
$$\frac{dF}{dx} = -1 \quad @ \quad x = 1$$

The first step is to approximate the "field variable" we are trying to solve, in this case "F". Let $\bar{F}$ be an approximation of F.

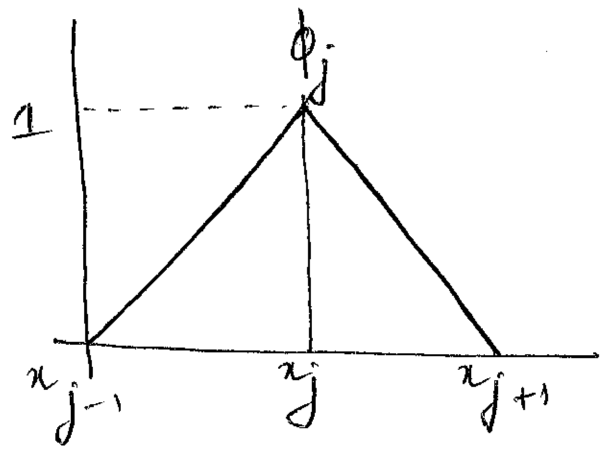
$$\text{so} \quad F \simeq \bar{F} = \sum_{j=1}^{n} \alpha_j \phi_j$$

if the domain [0, 1] was discretized into having n nodes then α_j is the nodal value of the function "F" at node j. It is also the unknown variable(s) we are trying to solve. Then ϕ_j is the nodal basis function @ j.

Consider the domain once again.



ϕ_j is the linear basis function with value unity at the nodes and 0 at all other nodes. Lets take a closer look at ϕ_j



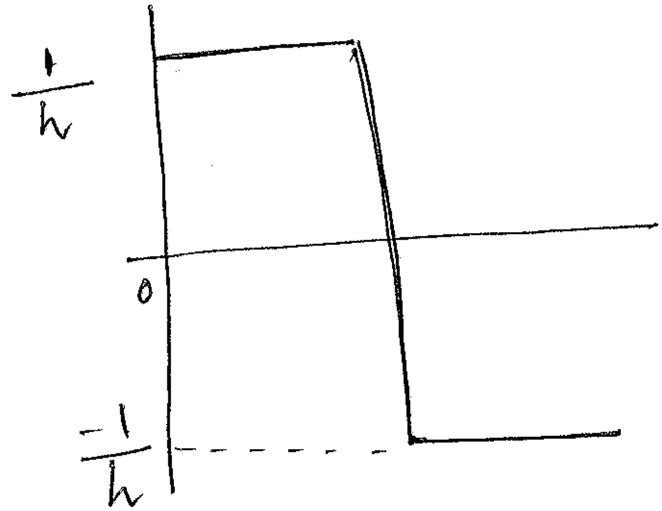
more formally in the interval $[x_{j-1}, x_j]$

$$\phi_j = \frac{x - x_j}{x_j - x_{j-1}}$$

and the interval $[x_{j+1}, x_j]$

$$\phi_j = \frac{x_{j+1} - x}{x_{j+1} - x_j}$$

And the derivatives



if $h = x_{j+1} - x_j = x_j - x_{j-1}$
Essentially the flux of the field variable will be constant in any given element.

Lets get back to the PDE at this point

$$\frac{d}{dx}\left(K_1 \frac{dF}{dx}\right) + F = 0$$

lets replace F with $\bar{F}$ and multiply with the basis function ϕ

$$\frac{d}{dx}\left(K_1 \frac{d\bar{F}}{dx}\right) \times \phi + \bar{F} \cdot \phi = 0$$

now integrating over the whole domain [0,1] we have.

$$\underbrace{\int_0^1 \frac{d}{dx}\left(K_1 \frac{d\bar{F}}{dx}\right) \cdot \phi \cdot dx}_{\text{Applying integration by parts we have}} + \int_0^1 \bar{F} \cdot \phi \cdot dx = 0$$

$$-\int_0^1 K_1 \frac{d\bar{F}}{dx} \cdot \phi' \cdot dx + [\phi \cdot \bar{F}']_0^1 + \int_0^1 \bar{F} \cdot \phi \cdot dx = 0$$

$$-\int_0^1 K_1 \frac{d\bar{F}}{dx} \cdot \phi' \cdot dx + \phi \bar{F}'\Big|_{x=1} - \phi \bar{F}'\Big|_{x=0} + \int_0^1 \bar{F} \cdot \phi \cdot dx = 0$$

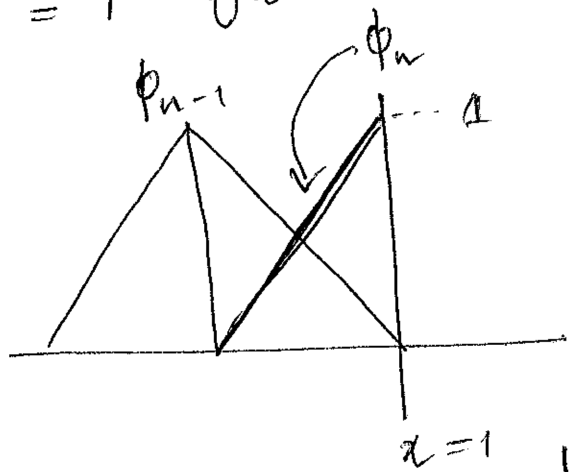
here ϕ' 4 $\bar{F}'$ are the respective first derivatives of ϕ 4 $\bar{F}$.

Since at $(x = 0)$ we have a dirichlet boundary condition $\phi_{x=0}$ is eliminated, since its nodal value is already known

Rewriting we have:-

$$-\int_0^1 K_1 \frac{d\bar{F}}{dx} \cdot \phi' \cdot dx + \phi \bar{F}\Big|_{x=1} + \int_0^1 F \cdot \phi \cdot dx = 0$$

recall $\phi_{x=1} = 1$ from the previous definition.



$\therefore$

$$-\int_0^1 K_1 \frac{d\bar{F}}{dx} \cdot \phi' \cdot dx + \bar{F}\Big|_{x=1} + \int_0^1 \bar{F} \cdot \phi \cdot dx = 0$$

this is the neumann boundary condition which comes naturally in the formulation of the FEM.

now substitute $\bar{F} = \sum_{j=1}^{n} \alpha_j \phi_j$

$$\cancel{\int_0^1 K_1 \, d\phi_j'}$$

$$-\int_0^1 K_1 \sum_{j=1}^{n} \alpha_j \phi_j' \cdot \phi_i' \cdot dx + \phi_n \bar{F} \Big|_{\substack{@ \\ x=1 \\ \text{or} \\ i=n}} + \int_0^1 \bar{F} \cdot \phi_i \cdot dx = 0 \quad \to (1)$$

ϕ_i = multiplied weight function

ϕ_j = from the formulation of F

But we need $(n-1)$ equations 4 $(n-1)$ unknowns, how do we do this. Equation (1) is just one equation. ϕ_i also varies from $(1 \to n-1)$ giving us $(n-1)$ equations. Put generally we have:-

$$-\sum_{i=2}^{n} \left(\sum_{i=1}^{n} \int_0^1 K_1 \phi_j' \phi_i' \cdot dx + \phi_n \bar{F} \Big|_{i=n} + \int_0^1 \bar{F} \cdot \phi_i \cdot dx = 0 \right.$$

because we eliminate the first equation (dirichlet node)

because at $i=1$ we just substitute α_1 which is known, rest are unknowns

but we can exploit a very interesting property
writing the PDE again we have:-

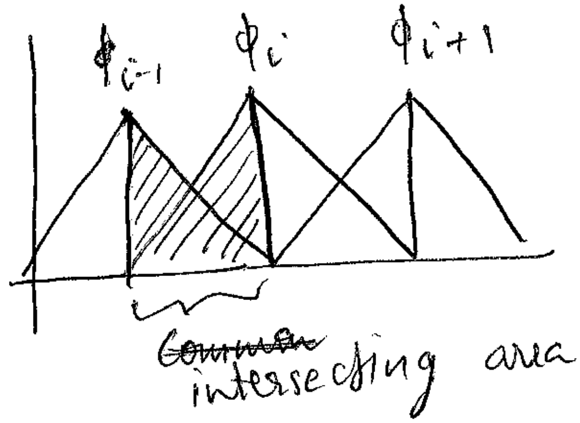
(without $\sum_i$)

$$-\sum_{j=1}^{n}\alpha_j\int_0^1 K_1\,\phi_j'\cdot\phi_i'\cdot dx + \sum_{i=1}^{n}\alpha_j\int_0^1\phi_j\cdot\phi_i\cdot dx = 0$$

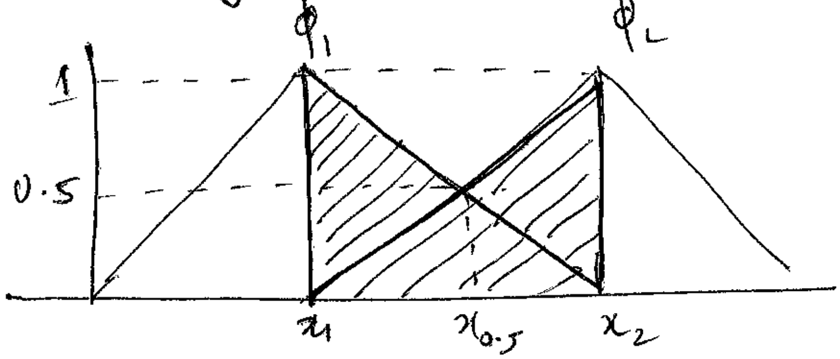
we eliminated $\bar{F}\Big|_{i=n}$ temporarily because it will occur only in one equation. So the above description of the PDE is what is solved for. There is an interesting property of these integrals

$$\int_0^1 \phi_j'\,\phi_i'\cdot dx \quad \& \quad \int_0^1 \phi_j\cdot\phi_i\cdot dx$$

Lets look at the linear ~~fun~~ basis functions once again



Only those integrals will ~~be~~ have values that have common intersecting area. for instance ϕ_1 & ϕ_2 will have a value but ϕ_1 & ϕ_3 will be zero. Lets look at a simple integration between ϕ_1 & ϕ_2 using the midpoint rule



$$\int \phi_1 \phi_2 \cdot dx = \phi_1(x_{0.5}) \times \phi_2(x_{0.5}) \times (x_2 - x_1)$$

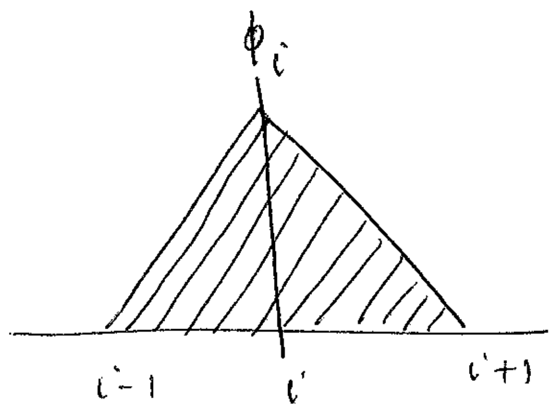
$$= 0.5 \times 0.5 \times h$$

by the midpoint rule

$$\int_{x_1}^{x_2} f \cdot dx = f(m).\,(x_2 - x_1)$$

where $m = \dfrac{x_2 - x_1}{2}$

All the integrations can be carried out easily by this rule similarly $\int \phi_i' \, \phi_j' \cdot dx$ is evaluated the same way. For integrals of type $\int \phi_i \, \phi_j \, dx$ where $i = j$



the integral can be split from $[i-1, i]$ + $[i, i+1]$

From a programming point of view:-

```
for i = 2, n    ← loop over φi
    for j = 1, n    ← loop over φj
        ______
        ______
    end for
end for
```

i starts from 2 because ϕ_1 is removed from the solution domain. $j = [1, n]$ because when j corresponds to a dirichlet node the value of ϕ_j is just substituted. The rest is explained in the code.