MATH 239 Spring 2024: Assignment 7 problems

Due: Wednesday July 17, 11:59pm EDT

Preamble.

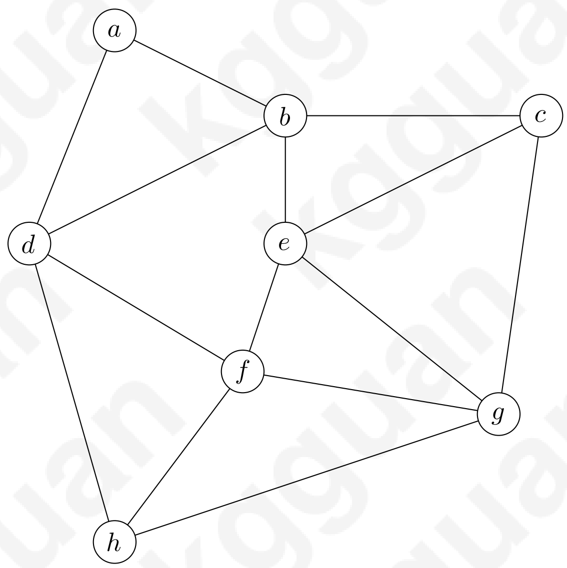
- You will receive a Crowdmark link for the assignment. Submit your solutions on Crowdmark.
- You should show all of your work and justify all of your solutions, unless you are explicitly asked not to.
- You are graded on both your accuracy and your presentation. A correct solution that is poorly presented may not receive full marks. The markers will not spend a lot of time figuring out something you write that is not clear.
- Each of the problems is worth 10 marks.
- When discussing assignment problems on Piazza, use private posts if you think that you might spoil an answer or major thinking point of a problem.
- Read the assignment policies on the course outline for what is allowed and not allowed in working on the assignment.
- Reproduction, sharing or online posting of the document is strictly forbidden.

Assignment problems.

A7-1. **Eulerian Circuits**

Given a graph G, we say that a walk is *Eulerian* if it uses every edge exactly once.

(a) Find an Eulerian walk for the following graph, by labelling the edges from 1 to 14 in the order of the walk.



(b) Let G be a connected graph. Prove that G has an Eulerian walk if and only if G has at most two vertices of odd degree.

A7-2. **Trees and forests**

(a) Let $T = (V, E)$ be a tree with at least 2 vertices. Let k be an integer such that $k \geq 2$. Let x be the number of leaves in T and let y be the number of vertices of degree at least k. Prove that $x \geq 2 + (k-2)y$.

(b) Let $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ be two forests (on the same vertex set). Assume that $|E_1| > |E_2|$. Prove that there exists e in $E_1 \setminus E_2$ such that $(V, E_2 \cup \{e\})$ is a forest.

A7-3. **Cuts, bridges, and spanning trees**

(a) Let $G = (V, E)$ be a connected graph, and let f be an edge of G. Prove that f is a bridge if and only if every spanning tree of G contains f.

(b) Let $k \geq 1$ be an integer. Suppose $G = (V, E)$ is a connected graph such that for every $S \subseteq V$ where $S \neq V$, $S \neq \varnothing$, the cut induced by S has at least k edges. Prove or disprove: for every $k-1$ edges $e_1, \ldots, e_{k-1}$, there is a spanning tree that does not contain any of the edges $e_1, \ldots, e_{k-1}$.

A7-4. **Bipartite characterization**

(a) Let G be a graph. Prove or disprove: G is bipartite if and only if $G - e$ is bipartite for every edge e in G. ($G - e$ is the graph obtained from G by removing the edge e.)

(b) Let $G = (V, E)$ be a connected graph and $e = uw$ be an edge in G. Consider the graph G' with the vertex set $V \cup \{v'_e, v''_e\}$ and edge set $(E \cup \{uv'_e, v'_e v''_e, v''_e w\}) \setminus \{uw\}$, where v'_e, v''_e are new vertices. Prove that G' is bipartite if and only if G is bipartite.