Proofs 05/11/2021
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dyadic fairness

→ link prediction are stat independent of the sensitive attributes from vertices

→ demographic parity

$P(C \mid A) = P(C)$

disparate impact

→ Gender Shades

Fairwalk

independence / dem

$P(C \mid A) = P(C)$

separation

$P(C \mid A, Y) = P(C \mid Y)$

sufficiency

$P(Y \mid C, A) = P(Y \mid C)$

$G = (V, E)$

$X \in \mathbb{R}^{N \times M}$

$A \in \mathbb{R}^{N \times N}$

$\mapsto a_{uv}: (u,v)$

$S(v)$ → sensitive attribute $: \in \{0, 1\}$

$T(v)$: 1-hop neighborhood of v

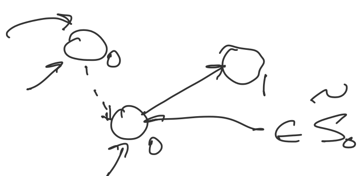
intra: $S(v) = S(u) \leftarrow (u,v)$

inter: $S(v) \neq S(u)$

$|S_i|$: # of nodes for which $\forall n \in V,\ S(n) = i$

$|S_0|$ $|S_1|$

female male



$\tilde{S}_0 = \{ v \in S_0 \mid T(v) \cap S_1 \neq \emptyset \}$

$U \sim \text{Disc Unif } V$

$g(\cdot, \cdot) : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}$

* predict links ind of if they have the same sensitive attribute

have the same ...

→ vertex reprs v and u

satisfy dyadic fairness

→ $Pr(g(u,v) \mid S(u) = S(v))$

$= Pr(g(u,v) \mid S(u) \neq S(v))$

fair vertex reprs

→ fair link prediction

Prop 4.1

$\underline{g(v,u) = v^T \Sigma u}$, $\underline{\Sigma} \in S_{++}^{M}$

$\exists Q > 0, \forall v \sim V, \|v\|_2 \le Q$

$\mathbb{E}_{v \sim U}[v] \in \mathbb{R}^M$

$\|\mathbb{E}_{v \sim U}[v \mid v \in S_0] - \mathbb{E}_{v \sim U}[v \mid v \in S_1]\|_2 \le \delta$

$$\Rightarrow$$

$$\underbrace{\Delta_{DP}} \leq Q \|\Sigma\|_2 \cdot \delta$$

$$\left| \mathbb{E}[g(v,u) \mid S(v) = S(u)] - \mathbb{E}[g(v,u) \mid S(v) \neq S(u)] \right|$$

$$p := \mathbb{E}[v \mid v \in S_0]$$

$$q := \mathbb{E}[v \mid v \in S_1]$$

$$|\mathbb{E}_{intra} - \mathbb{E}_{inter}|$$

interof

$$= \left| \mathbb{E}[v^T \Sigma u \mid v \in S_0, u \in S_1] - \mathbb{E}[v^T \Sigma u \mid v \in S_0, u \in S_0 \lor v \in S_1, u \in S_1] \right|$$

$$= \left| p^T \Sigma q - \left(\frac{|s_0|^2}{|s_0|^2+|s_1|^2} p^T \Sigma p + \frac{|s_1|^2}{|s_0|^2+|s_1|^2} q^T \Sigma q \right) \right|$$

$$= \left| (q-p)^T \left(\frac{|s_0|^2}{|s_0|^2+|s_1|^2} \Sigma p - \frac{|s_1|^2}{|s_0|^2+|s_1|^2} \Sigma q \right) \right|$$

$$\alpha = \frac{|s_0|^2}{|s_0|^2+|s_1|^2}, \quad \beta = \frac{|s_1|^2}{|s_0|^2+|s_1|^2}$$

$$\left| (q-p)^T (\alpha \Sigma p - \beta \Sigma q) \right|$$

$$\leq \|q-p\|_2 \cdot \|\alpha \Sigma p - \beta \Sigma q\|_2$$

$$\leq \delta \cdot \|\Sigma\|_2 \cdot (\alpha \|p\|_2 + \beta \|q\|_2)$$

$\|A-B\|_2$ $\qquad\qquad$ $\|v\|_2 \leq Q$

$$\leq \|A + (-B)\|_2$$
$$\leq \|A\|_2 + \|B\|_2$$

$$\|\mathbb{E}[v]\|_2 \leq \mathbb{E}[\|v\|_2] \leq Q$$

$$\|p\|_2 \leq Q, \quad \|q\|_2 \leq Q$$

$$\leq \delta \cdot \|\Sigma\|_2 \cdot (\alpha Q + \beta Q)$$
$$= \underline{Q \|\Sigma\|_2 \delta}$$

$$\mathrm{GNN}_\theta(X, \tilde{A}) := \rho(\underbrace{\tilde{A} X}_{\text{neighborhood aggr}} \overbrace{W_\theta}^{\text{embedding}})$$

$$\tilde{A} = D^{-1} A$$

$$\tilde{A} X$$

$$\mathrm{Agg}(v) := \underline{[\deg(v)]^{-1}} \sum_{u \in \Gamma(v)} a_{vu} u$$

$\mu_0 := \mathbb{E}[v \mid v \in S_0]$

$\mu_1 := \mathbb{E}[v \mid v \in S_1]$

σ, $\forall v \in S_0$, $\|\underline{v} - \underline{\mu_0}\|_\infty \leq \sigma$

$\forall v \in S_1$, $\|v - \mu_1\|_\infty \leq \sigma$

$D_{max} := \max_{v \in V} \deg(v)$

$m := \sum_{S(v) \neq S(u)} a_{vu}$

<u>Theorem 4.1</u>

$$\Delta_{DP}^{Aggr} := \| \mathbb{E}[Agg(v) \mid v \in S_0] - \mathbb{E}[Agg(v) \mid v \in S_1] \|_2$$

① $\geq \max\{\alpha_{min} \|\mu_0 - \mu_1\|_\infty - 2\sigma, 0\}$

② $\leq \alpha_{max} \|\mu_0 - \mu_1\|_2 + 2\sqrt{M}\,\sigma$

$$Agg(v) = \frac{1}{\deg(v)} \sum_{u \in \Gamma(v)} a_{vu} u$$

$$= \frac{1}{\deg(v)} \underbrace{\left(\sum_{u \in \Gamma(v) \cap S_0} a_{vu} u + \sum_{u \in \Gamma(v) \cap S_1} a_{vu} u \right)}$$

$\boxed{u}$ $\mu - \sigma \leq u \leq \mu + \sigma$

$$\Rightarrow u \in [\mu \pm \sigma]$$

$$Agg(v) \in \left[\frac{\sum_{u \in \Gamma(v) \cap S_0} a_{vu}}{\deg(v)} \mu_0 + \frac{\sum_{u \in \Gamma(v) \cap S_1} a_{vu}}{\deg(v)} \frac{\mu_1}{\pm \sigma \cdot 1} \right]$$

$$\in \left[\left(\mu_0 + \frac{\sum_{u \in \Gamma(v)} a_{vu \cap S_1}}{\deg(v)} (\mu_1 - \mu_0) \right) \frac{\pm}{\sigma \cdot 1} \right]$$

$$\deg(v) = \sum_{u \in \Gamma(v) \cap S_0} a_{vu} + \sum_{u \in \Gamma(v) \cap S_1} a_{vu}$$

$$\beta_v = \frac{\sum_{u \in \Gamma(v) \cap S_{opp}(v)} a_{vu}}{\deg(v)}$$

① $\rightarrow \mathbb{E}[Agg(v) \mid v \in S_0]$

$$\in \left[\left(\frac{1}{|S_0|} \sum_{v \in S_0} (\mu_0 + \beta_v (\mu_1 - \mu_0)) \right) \pm \sigma \cdot \mathbb{1} \right]$$

$$\in \left[\left(\mu_0 + \frac{1}{|S_0|} \sum_{v \in S_0} (\beta_v (\mu_1 - \mu_0)) \pm \sigma \cdot \mathbb{1} \right] \right.$$

② $\rightarrow \mathbb{E}[Agg(v) \mid v \in S_1]$

$$\in \left[\left(\mu_1 + \frac{1}{|S_1|} \sum_{v \in S_1} \beta_v (\mu_0 - \mu_1) \right) \pm \sigma \cdot \mathbb{1} \right]$$

$$\text{diff} = \left[\left(1 - \left(\frac{1}{|S_0|} \sum_{v \in S_0} \beta_v + \frac{1}{|S_1|} \sum_{v \in S_1} \beta_v \right) \right) \right.$$

$$v \in S_0 \qquad v \in S_1$$

$$\cdot\,(\mu_0 - \mu_1) \pm 2\sigma \cdot 1 \,\Big]$$

$$\alpha := 1 - \left(\frac{1}{|S_0|} \sum_{v \in S_0} \beta_v + \frac{1}{|S_1|} \sum_{v \in S_1} \beta_v \right)$$

S_0

$$\sum_{v \in S_0} \beta_v = \sum_{v \in S_0} \left[\frac{\sum_{u \in \Gamma(v) \cap S_1} a_{vu}}{\deg(v)} \right]$$

$$\geq \frac{1}{D_{max}} \sum_{v \in S_0} \sum_{u \in \Gamma(v) \cap S_1} a_{vu} = \frac{m}{D_{max}}$$

$$\deg(v) \geq \sum_{u \in \Gamma(v) \cap S_1} a_{vu}$$

$$\beta_v = \frac{\sum_{u \in \Gamma(v) \cap S_1} a_{vu}}{\deg(v)} \leq 1$$

$$\sum_{v \in S_0} \beta_v = \sum_{v \in \tilde{S}_0} \beta_v \leq \sum_{v \in \tilde{S}_0} 1$$

$$= |\tilde{S}_0|$$

$$\frac{m}{D_{max}} \le \sum_{v \in S_0} B_v \le |\tilde{S}_0|$$

$$\frac{m}{D_{max}} \le \sum_{v \in S_1} B_v \le |\tilde{S}_1|$$

$$\underline{\alpha'} \in \Big[\underbrace{1 - \left(\frac{|\tilde{S}_0|}{|S_0|} + \frac{|\tilde{S}_1|}{|S_1|} \right)}, \underline{\underline{1 - \frac{m}{D_{max}} \left(\frac{1}{|S_0|} + \frac{1}{|S_1|} \right)}} \Big]$$

$$|\alpha'|$$

$$\alpha_{max} = \max \left\{ 1 - \left(\frac{|\tilde{S}_0|}{|S_0|} + \frac{|\tilde{S}_1|}{|S_1|} \right), 1 - \frac{m}{D_{max}} \left(\frac{1}{|S_0|} + \frac{1}{|S_1|} \right) \right\}$$

$$\alpha_{min} = \cancel{\max} \min$$

$$\Delta_{DP}^{Aggr} \leq \alpha_{max} \underline{\|\mu_0 - \mu_1\|_2} + \underline{2\sqrt{M}\sigma}$$

$$\| 2\sigma \cdot \mathbb{1} \|_2 = 2\sqrt{M}\sigma$$
$$\in \mathbb{R}^M$$

μ_0^i, μ_1^i

$$\left| \left(1 - \left(\frac{1}{|S_0|} \sum_{v \in S_0} \beta_v + \frac{1}{|S_1|} \sum_{v \in S_1} \beta_v\right)\right) \cdot (\mu_0^i - \mu_1^i) \right| \leq 2\sigma$$

$$\Delta_{DP}^{Aggr} \geq \max\{\underline{\alpha_{min}} \|\mu_0 - \mu_1\|_\infty - 2\sigma, 0\}$$

$$GNN_\theta := \mathcal{P}(\tilde{A} X W_\theta)$$

ONE-LAYER

Δ post-aggr

$$\leq Q^{(1)} \|\Sigma\|_2 \cdot \underline{\|\mu_0' - \mu_1'\|_2}$$

$$\leq \underline{Q L^2 H \|\Sigma\|_2 \|W_\theta\|_2^2} .$$

$$\left(\alpha \|\mu_0 - \mu_1\|_2 + 2\sqrt{M\theta}\right)$$

$$Q' \leq L \|W_\theta\|_2 Q$$