Definition 1.2.8 (Continuity)

A function $f(x,y)$ is said to be continuous at a point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) \tag{2}$$

i.e., the limit of f as (x,y) tends to (a,b) = the value of f at (a,b).
A function is said to be continuous in a domain if it is continuous at every point of the domain.
Equation 2 can also be written as

$$\lim_{(h,k)\to(0,0)} f(a+h,b+k) = f(a,b) \tag{3}$$

If f is not continuous at (a,b), it is said to be discontinuous at (a,b).

Remark 1.2.9

If $f(x,y)$ and $g(x,y)$ are continuous at (a,b) then $f \pm g, f.g$ and f/g are continuous at (a,b).

Definition 1.2.10 (Test for continuity at a Point (a, b))

Step I: $f(a, b)$ should be well defined

Step II: $\lim\limits_{f(x,y)}$ as $(x, y) \to (a, b)$ should exists (must be unique and same along any path).

Step III: The limit of f = value of f at (a, b).

Example 1.2.11

Discuss the continuity of the function $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ when $(x,y) \neq (0,0)$ and $f(x,y) = 2$ when $(x,y) = (0,0)$.

At first, evaluate the limit

$$\text{Along path I:} \qquad \lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{\sqrt{x^2+y^2}} = 0$$

$$\text{Along path II:} \qquad \lim_{\substack{y\to 0\\ x\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x}{\sqrt{x^2}} = \lim_{x\to 0} 1 = 1$$

Since the limits along paths I and II are different, the limit itself does not exist. Therefore the function is discontinuous at the origin.

Example 1.2.12

If $f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$ when $x \neq 0$, $y \neq 0$ and $f(x, y) = 0$ when $x = 0$, $y = 0$, find out whether the function $f(x, y)$ is continuous at origin.

First calculate the limit of the function:

1. $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim\limits_{y \to 0} \left(\frac{-y^3}{y^2} \right) = \lim\limits_{y \to 0} (-y) = 0$
2. $\lim\limits_{\substack{y \to 0 \\ x \to 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim\limits_{x \to 0} \left(\frac{x^3}{x^2} \right) = \lim\limits_{x \to 0} (x) = 0$
3. $\lim\limits_{\substack{y = mx \\ x \to 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim\limits_{x \to 0} \frac{x^3 - m^3 x^3}{x^2 + m^2 x^2} == \lim\limits_{x \to 0} = \frac{(1 - m^3)}{(1 + m^2)} x = 0$
4. $\lim\limits_{\substack{y = mx^2 \\ x \to 0}} \frac{x^3 - y^3}{x^2 + y^2} = \lim\limits_{x \to 0} \frac{x^3(1 - m^3 x^3)}{x^2(1 + m^2 x^2)} == \lim\limits_{x \to 0} = \frac{(1 - m^3 x^3)}{(1 + m^2 x^2)} x = 0.$

Since the limit along any path is same, the limit exists and equal to zero which is the value of the function $f(x, y)$ at the origin. Hence the function f is continuous at the origin.

Example 1.2.13

Examine for continuity at origin of the function defined by

$$f(x,y) = \frac{x^2}{\sqrt{x^2+y^2}}, \text{ for } (x \neq 0,\ y \neq 0)$$
$$= 3, \text{ for } (x = 0, y = 0)$$

Redefine the function to make it continuous.

Initially, find the limit

1. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{\sqrt{x^2+y^2}} = \lim\limits_{y\to 0} 0 = 0$
2. $\lim\limits_{\substack{y\to 0\\ x\to 0}} \frac{x^2}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0} \frac{x^2}{\sqrt{x^2}} = \lim\limits_{x\to 0} x = 0$
3. $\lim\limits_{\substack{y\to mx\\ x\to 0}} \frac{x^2}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0} \frac{x^2}{x\sqrt{(1+m^2)}} = \frac{1}{\sqrt{1+m^2}} \lim\limits_{x\to 0} x = 0$
4. $\lim\limits_{\substack{y\to mx^n\\ x\to 0}} \frac{x^2}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0} \frac{x^2}{x\sqrt{(1+m^2x^{2n-2})}} = \frac{0}{\sqrt{1+0}} = 0$

Thus the limit along any path exists and is the same and the common value equals to zero i.e.,

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0$$

However the value of the functions at origin is 3, i.e.,

$$f(0,0) = 3$$

Therefore f is discontinuous at origin because

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0 \neq 3 = f(0,0)$$

The function can be 'made' continuous at origin by redefining the function as $f(0,0) = 0$, since in this case

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0 = f(0,0).$$

Example 1.2.14

If $f(x,y) = \frac{(x^2-y^2)}{(x^2+y^2)}$ when $x \neq 0, y \neq 0$ $f(x,y) = 0$ when $(x = 0, y = 0)$, show that f is discontinuous at origin.

Example 1.2.15

1. Is the function $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ when $(x,y) \neq (0,0)$ and $f(0,0) = 4$ continuous at origin.
2. Redefine if necessary to make it continuous at $(0,0)$.

Ans

1. not continuous: $\lim = 0 \neq 4 = f(0,0)$
2. continuous if $f(0,0) = 0$

Example 1.2.16

If $f(x,y) = x^3 + y^2$ determine where the function is continuous.
Ans. continuous for every x and y i.e., everywhere

Example 1.2.17

If $f(x,y) = \frac{x^3-y^3}{x^3+y^3}$, for $(x,y) \neq (0,0)$ $f(x,y) = 15$, at $(0,0)$, show that f is discontinuous at origin.

Example 1.2.18

Find whether $f(x,y) = \frac{x^3y^3}{x^3+y^3}$ is continuous at $(0,0)$ when (a) $f(0,0) = -15$, (b) $f(0,0) = 0$.
Ans. a. not continuous: $\lim = 0 \neq -15 = f(0,0)$
b. continuous since $\lim = 0 = 0 = f(0,0)$

Example 1.2.19

Given $f(x, y) = x^3 + 3y^2 + 2x + y$ for every (x, y) except at $(2, 3)$ where $f(2, 3) = 10$. Examine whether f is continuous at (a) point $(2, 3)$ (b) at any other points (c) can the function be made continuous at $(2, 3)$ by redefining f at $(2, 3)$.

Ans

1. discontinuous at $(2, 3)$
2. continuous for every x and y i.e., everywhere except at $(2, 3)$
3. f becomes continuous by redefining f at $(2, 3)$ as $f(2, 3) = 42$

Example 1.2.20

1. Show that $f(x,y) = \frac{xy}{x^2+y^2}$, $x = y \neq 0$ is discontinuous at origin when $f(0,0) = 0$.
2. Can it be made continuous by defining f ?

Ans

1. since limit along $y = mx$ is $\frac{m}{1+m^2}$, not unique, the limit does not exist, so discontinuous.
2. can not make f continuous by redefining f at $(0,0)$ i.e., for any choice of $f(0,0)$, since the limit at $(0,0)$ does not exist.

Example 1.2.21

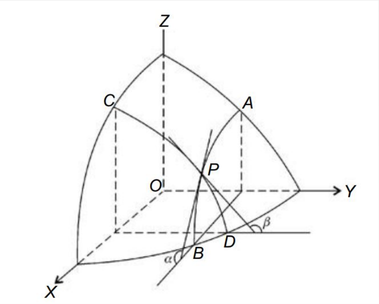
Prove that $f(x,y) = \frac{x^2-y^3}{x^2-y^2}$ when $(x,y) \neq (0,0)$ and $f(0,0) = 0$ is discontinuous at origin.
Hint: $\lim f$ along $y = mx$ is not unique, depends on m. So limit does not exist.

Example 1.2.22

Find out (and give reason) whether $f(x,y)$ is continuous at $(0,0)$ if $f(0,0)=0$ and for $(x,y)\neq(0,0)$ the function f is equal to:

1. $\frac{y}{\sqrt{x^2+y^2}}$ Ans. discontinuous
2. $\frac{x}{\left(1+\sqrt{x^2+y^2}\right)}$ Ans. continuous
3. $\frac{xy}{(x^2+y^2)^{\frac{1}{2}}}$ Ans. continuous
4. $\frac{(x^2-y^2)}{\sqrt{x^2+y^2}}$ Ans. continuous

A partial derivative of a function of several variables is the ordinary derivative with respect to one of the variables when all the remaining variables are held constant. Partial differentiation is the process of finding partial derivatives. All the rules of differentiation applicable to function of a single independent variable are also applicable in partial differentiation with the only difference that while differentiating (partially) with respect to one variable, all the *other* variables are treated (temporarily) as constants.



Consider a function u of three independent variables $x, y, z,$ (refer figure. 39)

$$u = f(x, y, z) \tag{4}$$

Keeping y, z constant and varying only $x,$ the partial derivative of u with respect to x is denoted by $\frac{\partial u}{\partial x}$ and is defined as the limit

$$\frac{\partial u(x, y, z)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}.$$

Partial derivatives of u w.r.t. y and z can be defined similarly and are denoted by $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

Remark 1.3.1 (Partial derivatives notation)

The partial derivative $\frac{\partial u}{\partial x}$ *is also denoted by* $\frac{\partial f}{\partial x}$ *or* $f_x(x,y,z)$ *or* f_x *or* $D_x f$ *or* $f_1(x,y,z)$ *where the subscripts* x *and* 1 *denote the variable w.r.t. which the partial differentiation is carried out. Thus we can have* $\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} = f_y(x,y,z) = f_y = D_y f = f_2(x,y,z)$ *etc. The value of a partial derivative at a point* (a,b,c) *is denoted by*

$$\left.\frac{\partial u}{\partial x}\right|_{x=a,y=b,z=c} = \left.\frac{\partial f}{\partial x}\right|_{(a,b,c)} = f_x(a,b,c)$$

Geometrical interpretation of a partial derivative of a function of two variables: $z = f(x, y)$ represents the equation of a surface in xyz- coordinate system. Let APB the curve, which a plane through any point P on the surface parallel to the xz-plane, cuts. As point P moves along this curve APB, its coordinates z and x vary while y remains constant. The slope of the tangent line at P to APB represents the rate at which z changes w.r.t. x.

Thus

$\frac{\partial z}{\partial x} = \tan\alpha =$ slope of the curve APB at the point P

Similarly,

$\frac{\partial z}{\partial y} = \tan\beta$ = slope of the curve CPD at P.

Partial derivatives of higher order, of a function $f(x,y,z)$ are calculated by successive differentiation. Thus if $u = f(x,y,z)$ then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{x,x} = f_{11}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{y,x} = f_{21}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = f_{12}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} = f_{22}$$

$$\frac{\partial^3 u}{\partial z^2 \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial z \partial y}\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right) = f_{yzz} = f_{233}$$

$$\frac{\partial^4 u}{\partial x \partial y \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^3 f}{\partial y \partial z^2}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z^2}\right)\right) = f_{zzyx} = f_{3321}$$

The partial derivative $\frac{\partial f}{\partial x}$ obtained by differentiating once is known as first order partial derivative, while $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ which are obtained by differentiating twice are known as second-order derivatives. 3rd order, 4th order derivatives involve 3, 4 times differentiation respectively.

Remark 1.4.1

The crossed or mixed partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ *and* $\frac{\partial^2 f}{\partial x \partial y}$ *are in general, equal*

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

i.e., the order of differentiation is immaterial if the derivatives involved are continuous.

Remark 1.4.2

In the subscript notation, the subscripts are written in the same order in which differentiation is carried out, while in the $'\partial'$ *notation the order is opposite, for example,*

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = f_{xy}$$

Remark 1.4.3

A function of 2 *variables has two first order derivatives, four second order derivatives and* 2^n *of* n^{th} *order derivatives. A function of* m *independent variables will have* m^n *derivatives of order* n.

Example 1.4.4

Find the first order partial derivatives $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ when:

1. $w = e^x \cos y$
2. $w = \tan^{-1} \frac{y}{x}$
3. $w = \log \sqrt{x^2 + y^2}$

Ans: a)

$$\frac{\partial w}{\partial x} = \cos y \frac{\partial}{\partial x}(e^x) = e^x \cos y$$
$$\frac{\partial w}{\partial y} = e^x \frac{\partial}{\partial y}(\cos y) = -e^x \sin y$$

Ans: b)

$$\frac{\partial w}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial w}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2+y^2}$$

Ans: c)

$$\frac{\partial w}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\frac{\partial}{\partial x}(x^2) = \frac{x}{x^2+y^2}$$

$$\frac{\partial w}{\partial y} = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\frac{\partial}{\partial y}(y^2) = \frac{y}{x^2+y^2}$$

Example 1.4.5

Find partial derivative of f with respect to each of the independent variable:

1. $f(x,y,z,w) = x^2 e^{2y+3z} \cos(4w)$
2. $f(r,\theta,z) = \frac{r(2-\cos 2\theta)}{r^2+z^2}$

Ans: a

$$f_x = \frac{\partial f}{\partial x} = e^{2y+3z}\cos(4w)\frac{\partial}{\partial x}(x^2)$$

$$f_x = 2xe^{2y+3z}\cos(4w) = \frac{2f}{x}$$

$$f_y = x^2e^{3z}\cos(4w)\frac{\partial}{\partial y}e^{2y} = x^2e^{3z}\cos(4w)(2e^{2y})$$

$$f_y = 2f$$

$$f_z = x^2e^{2y}\cos(4w)\frac{\partial}{\partial z}e^{3z} = x^2e^{2y}\cos(4w)3e^{3z}$$

$$f_z = 3f$$

$$\begin{aligned} f_w &= x^2e^{2y+3z}\frac{\partial}{\partial w}(\cos(4w)) \\ &= x^2e^{2y+3z}(-\sin 4w)4 \end{aligned}$$

$$f_w = -4x^2e^{2y+3z}\sin(4w)$$

Ans: b

$$f(r,\theta,z) = \frac{r(2-\cos 2\theta)}{r^2+z^2}$$
$$\frac{\partial f}{\partial r} = f_r = \frac{(r^2+z^2)(2-\cos 2\theta) - r(2-\cos 2\theta)2r}{(r^2+z^2)^2}$$
$$f_r = \frac{(z^2-r^2)(2-\cos 2\theta)}{(r^2+z^2)^2}$$
$$\frac{\partial f}{\partial \theta} = f_\theta = \frac{r}{r^2+z^2} 2\sin 2\theta$$
$$\frac{\partial f}{\partial z} = f_z = \frac{r(2-\cos 2\theta)}{(r^2+z^2)^2}(-2z).$$