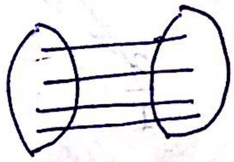
Chapter 4 .
Chapter 14 . } Discrete Mathematics
Swarmu Outline . Shamik .



$A \underset{\text{onto}}{\overset{1-1}{\longleftrightarrow}} B$.

George Cantor $\Rightarrow$ $\mathbb{N}$, $\mathbb{R} \Rightarrow$ Set of Real Nos
$\Downarrow$
Set of natural nos .

$\mathbb{Z} \to$ Set of Integer
$\mathbb{Q} \Rightarrow$ Set of Rational Nos
$\mathbb{C} \to$ Set of Complex Nos

$\mathbb{N}$:- 1 2 3 4 5 6 }
$2 \cdot \mathbb{N}$:- 2 4 6 8 . . . } One-to-one mapping con be established .
$2\mathbb{N}-1$:- 1 3 5 7 9 . . .

$f : \mathbb{N} \to 2\mathbb{N}$.
$f(n) = 2n$.

$f(m) = f(n)$
$\Rightarrow 2m = 2n$
$\Rightarrow m = n$.
$\therefore f$ is one-one
f is onto by defn .
f is bijective .

$|\mathbb{N}| = |2 \cdot \mathbb{N}|$

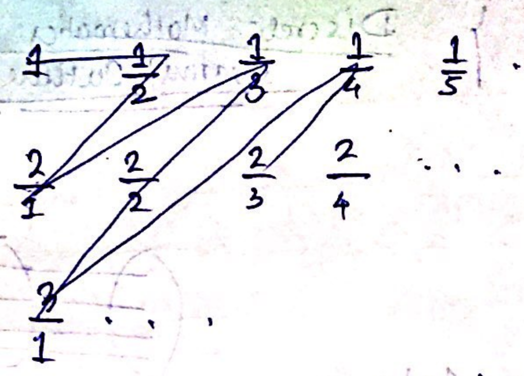
$\mathbb{N} \longleftrightarrow \mathbb{Z}$

$$f(N) = \begin{cases} \frac{N}{2} & \text{if } n \in \text{Even No.} \\ (-1) \cdot \left(\frac{N-1}{2}\right) & \text{if } n \in \text{odd No.} \end{cases}$$

$\Rightarrow |\mathbb{N}| = |\mathbb{Z}|$

$f(n) = f(m)$,
& f is clearly onto .

$\mathbb{N} \leftrightarrow \mathbb{Q}$



~~Repet~~
Repetitions are discarded.

$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$.

~~Ex~~ Ex. suppose $\frac{m}{n} = a_k$. Find k.

Prove: $|\mathbb{R}| > |\mathbb{N}|$

Cantor's diagonalization theorem

$|\mathbb{R}| = |2^{\mathbb{N}}| = |\wp(\mathbb{N})|$

$= |\text{set of all subsets of } \mathbb{N}|$

$c = |\mathbb{R}|$ $\qquad \aleph_0 = |\mathbb{N}|$.

$c = 2^{\aleph_0}$

07/02/2020 . **Mathematics** . Prof. Shamik Ghosh ①

$\mathbb{N} \xleftrightarrow{bij} \mathbb{N}\cdot 2$

$\mathbb{N} \xleftrightarrow{bij} 2\mathbb{N}-1$.

$\mathbb{N} \xleftrightarrow{bij} \mathbb{Z}$

$\mathbb{N} \xleftrightarrow{bij} \mathbb{Q}$

$\mathbb{Q}^+ \xleftrightarrow{bij} \mathbb{N}$.

$f: \mathbb{Q}^+ \xleftrightarrow{bij} \mathbb{N}$.

$g: \mathbb{Q}^+ \longleftrightarrow \mathbb{Z}$

$\mathbb{Q}^+ \longleftrightarrow \mathbb{N} \leftrightarrow \mathbb{Z}^+$

$\mathbb{Q}^- \leftrightarrow \mathbb{Z}^-$

$0 \longleftrightarrow 0$.

$$g(x) = \begin{cases} f(x), & \text{if } x>0 \\ -f(x), & \text{if } x<0 \\ 0, & \text{if } x=0. \end{cases}$$

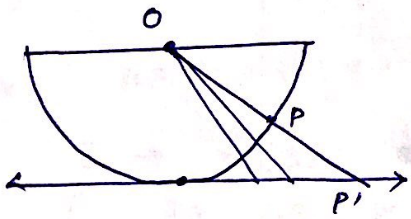
$|\mathbb{N}| < |\mathbb{R}|$

Let $\aleph_0 = |\mathbb{N}|$ — Verify g is bijective.

$c = |\mathbb{R}|$

$\mathcal{P}(\mathbb{N}) \xleftrightarrow{bij} \mathbb{R}$.

$$\boxed{2^{\aleph_0} = c}$$



$(0,1) \xleftrightarrow{bij} (a,b)$.

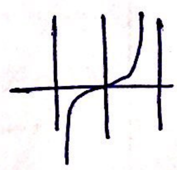
$f(x) = (b-a)x + a$.

for every point on the semi-circle, $\exists$ a unique point on the tangent line; (1-1 correspondence),

$f(0,1) \to \mathbb{R}$.

$f(x) = \tan \pi(x - \frac{1}{2})$.

Eg. P.T. f is bijective.



Suppose.

$A \xleftrightarrow{bij} \mathbb{N}$.,

B = finite set ; $= \{b_1, b_2, \dots, b_n\}$

P.T. $\exists$ a bijection from

$A \cup B \longrightarrow \mathbb{N}$.

$A = \{a_1, a_2, a_3, \dots\}$

$\mathbb{N} : 1, 2, 3, 4, 5, \dots\dots, n_1, \dots$

$A\cup B : b_1, b_2, \dots\dots, b_n, a_1, a_2, \dots$

$b_i \longleftrightarrow i$

$a_j \longleftrightarrow n+j$

Suppose A is an infinite set. $A \neq \phi$ (2)

$\Rightarrow \exists a_1 \in A$.

$A \setminus \{a_1\} \neq \phi$

~~$\Rightarrow \exists a_2 \in A \setminus f$~~ $\Rightarrow \exists a_2 \in A - \{a_1\}$

if we have $B = \{a_1, a_2, a_3, \dots\} \subseteq A$.

$\Rightarrow B \xleftrightarrow{bij} \mathbb{N}$.

Let C = finite set.

$B \cup C \xleftrightarrow{bij} B$.

$A = (A \setminus B) \cup B$

$g : A \cup C \longleftrightarrow A$.

$$g(x) = \begin{cases} x & \text{if } x \in A - B \\ f(x) & \text{if } x \in (B \cup C). \end{cases}$$

$\mathbb{N} \cup B \longleftrightarrow \mathbb{N}$.

Thm. $|(0,1)| = |[0,1]| = |\mathbb{R}|$

$= |(a,b)|$

$= |[a,b]|$.

$2^{\aleph_0} = c$.

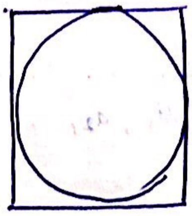
We will show there exist a bijection from the set of all subset of $\mathbb{N}$ onto $[0,1]$

$f : P(\mathbb{N}) \longrightarrow [0,1]$

$A \subseteq \mathbb{N}$ (Binary)

$A \longrightarrow 0.d_1 d_2 d_3 d_4 \dots d_n \dots$?

$d_i = 1$ iff $i \in A$.



0 1

$[0,1] \times [0,1] \xleftrightarrow[?]{bij} (0,1)$.

$(0.d_1 d_2 \dots d_n, 0.e_1 e_2 \dots) \rightarrow 0.d_1 e_1 d_2 e_2 d_3 e_3 \dots$

$$|\mathbb{R}\times\mathbb{R}| = |\mathbb{R}|.$$

$$|\mathbb{R}^3| = |\mathbb{R}\times\mathbb{R}\times\mathbb{R}| = |\mathbb{R}\times\mathbb{R}| = |\mathbb{R}|$$

For any set

$|A| < |\mathcal{P}(A)|$

$f: A \to \mathcal{P}(A)$
$f(a) = \{a\}$
f is 1-1.
$|A| \leq |\mathcal{P}(A)|$

Suppose g is a bijective map from A onto $\mathcal{P}(A)$.

$g: A \xrightarrow{\text{bij}} \mathcal{P}(A)$.

$B = \{x\in A \mid x \notin g(x)\}$
$g(x) \subseteq A$.

eg. $A = \{1,2,3,4,5,\ldots 19\}$
$3 \to \{13,5,19\}$
$3 \in \{13,3,19\}$ ✓
$3 \notin \{5,7,9\}$

let $B = \{x\in A \mid x\notin g(x)\} \subseteq A$
g is onto $\exists\, b\in A$
s.t. $g(b) = B$.

Let $b \in B$
$\Rightarrow b \notin g(b) = B$
$\Rightarrow b \notin B \to$ Contradiction.

if $b \notin B$. by defn $b \in g(b)$
$\Rightarrow b\in B$
Contradiction.

hence no-bijection exists.

$$\aleph_0 < 2^{\aleph_0} < 2^{c} < 2^{2^{c}} < 2^{2^{2^{c}}} < \cdots < \cdots$$

Introduction to Topology & Modern Analysis
— G.F. Simmons

Topics in Algebra
— I.N. Herstein

12/2/2020.

Set

$$S = \{ \underset{P(x)}{\overset{n}{\text{objects}}} \}$$

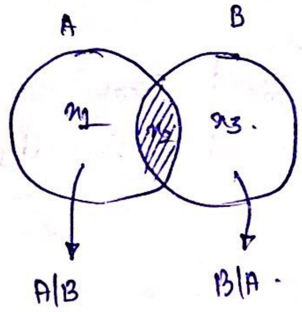
well defined:

Eg. Barber.

→ Those people who can't share their hair ~~are~~ come to Barber.

→ Those people who can share their hair do not come to Barber.

Question :- Where does Barber belong?



$A \Delta B = (A|B) \cup (B|A)$.

$A = \{a_1, a_2, \ldots, a_n\}$ $\qquad |A \times B| = mn$.

$B = \{b_1, b_2, \ldots, b_n\}$

$A \times B$
$= \{(a_i, b_j) \mid a_i \in A, b_j \in B\}$.

Relation on a set

$A, B \neq \phi$.

$\mathcal{R} \subseteq A \times B$

$(x, y) \in \mathbb{N} \times \mathbb{N}$

$x+y =$ even.) → subset of $\mathbb{N} \times \mathbb{N}$.

Properties . $\rho \subseteq A \times A$

1) Reflexive. $(a,a) \in \rho . \quad \forall\, a \in A$

2) Symmetric $(a,b) \in \rho \Rightarrow (b,a) \in \rho, \quad \forall\, a,b \in A.$

3) Transitive $\forall\, a,b,c \in A.$

$a \rho b \ \& \ b \rho c \Rightarrow a \rho c.$

$(a,b) \ \& \ (b,c) \in \rho . \Rightarrow (a,c) \in \rho .$

~~Relati~~

4) Anti-symmetric $a \rho b \ \& \ b \rho a$ hold $\Rightarrow a = b$.

→ if $A = \mathbb{Z}$

→ $a \rho b \iff a - b \rightarrow$ even, → Not anti-symmetric

→ $a \rho b \iff a / b \longrightarrow$ Anti-symmetric.

→ $a \rho b \iff a \equiv b \pmod{m} \longrightarrow$ Anti-symmetric does not hold.

$(A, \rho) \xrightarrow{\text{satisfies}}$ ① Reflexive. ② Antisymmetric ③ Transitive | Poset. Partially Ordered Set.

$A = \{1,2,3\}$

$P(A) = \{ \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\} \}.$

$x \rho y$ when $x \subseteq y$

$A = \{1,2,3\}$

Poset. $\{P(A), \rho\}$

if we have (A, ρ).

$a,b \in A$.

$a \rho b$, $b \rho a$

$(\mathbb{N}, \rho)$ $3 \nmid 5$, $5 \nmid 3$.

hence. for not all $a,b \in A$

$a \rho b \ \& \ b \rho a$.

(A, ρ) with $a \rho b$ or $b \rho a$
is said to be totally ordered set.

eg. (N, ρ) $a \rho b \iff a \le b$.
$\longrightarrow$ totally ordered set.

We can arrange the elements in a chain.
$a_1 \rho a_2 \rho a_3 \rho a_4 \dots \rho a_n$.

Equivalence Relation.

$\rho \to$ equivalence relation.
1) Reflexive 2) Symmetric 3) Transitive.

Equivalence Class.

$A \to$ set

$[x] = \{ y \mid x \rho y \}$
$\parallel$
x
equivalence $(x, y) \in \rho$
class of A.

$A = \{1, 2, 3\}$ $\rho = \{ (1,1), (2,2), (3,3), (1,2), (2,1) \}$.
~~$\rho = \frac{x}{x}$~~
$[1] = \{1, 2\}$
$[2] = \{2, 1\}$
$[3] = \{3\}$.

$A = \mathbb{R}$. $\rho = \{ (x, y) \mid x^2 = y^2 \}$.
$[x] = \{ x, -x \}$.

(A, ρ)

$\hookrightarrow$ equivalence relation.

Please complete the proof

1) $\forall x \in A$, $x \in [x]$

2) if $y \in [x]$, then $x \in [y]$ & $[x] = [y]$

3) For any $x \& y \in A$, either $[x] = [y]$

& $[x] \cap [y] = \phi$,

2. $\forall y_i \in [x]$

we know $x \rho y_i$ holds

$\Rightarrow$ because of symmetric relation $y_1 \in x$ holds.

$\Rightarrow$ so, $x \in [y_i]$

Let z be element $z \in [y_i]$

... (complete the proof)

3. ~~$z \in [x]$ from (2) $z \in [y]$~~

~~$\Rightarrow z \in \rho$. $\Rightarrow x \in [z]$~~

$x \in [y] \Rightarrow (x, y) \in \rho$

Let z be like $(x, z) \in \rho$.

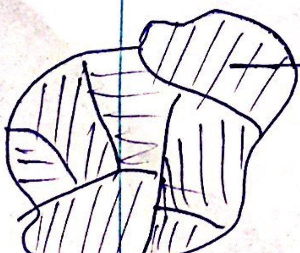
$(x, y) \in \rho \Rightarrow (y, x) \in \rho$.

$(y, x) \in \rho \wedge (x, z) \in \rho$

$\Rightarrow (y, z) \in \rho$

$z \in [y]$.

$(A, \rho) \longrightarrow$ we can partition the set which are disjoint subsets & are individually equivalence classes.



$\rightarrow$ Partition.

A $\rightarrow$ set

& partitions are given to you.

Can you define the equivalence relation which will yield those equivalence classes / partitions?

Predecessor

$A \to$ set

$\rho \to$ partial order.

$a, b \in A$

$a \neq b$

$a \rho b$ holds & $\nexists$ $c \in A$ for which $a \rho c$ & $c \rho b$ holds simultaneously.

eg. 3, 9 ✓

3, 6, 12 ✗

$A = \{1, 2, 3, 4, 5, \ldots 12\}$

Hasse Diagram.

Supremum.

$(A, \rho) \to$ partial order.

Defn:- $B \subset A$ $\quad a \in A \to$ upper bound of B.

1) $b \rho a \quad \forall\, b \in B$.

2) $a \rho x \quad \forall$ upperbound x for B.

$A = \mathbb{R}$ $\quad B = \{x \mid 5 < x < 7\}$

$B \subset A$. $\quad x \geq 7 \to$ upper bound.

$B = A$.

Then upperbound $x = 7$.

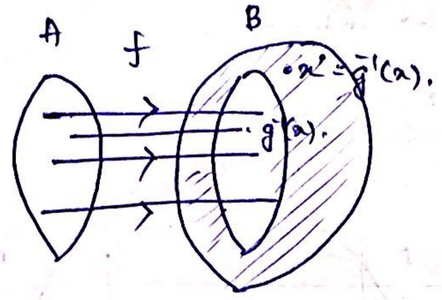
Proof:- Supremum should be unique.

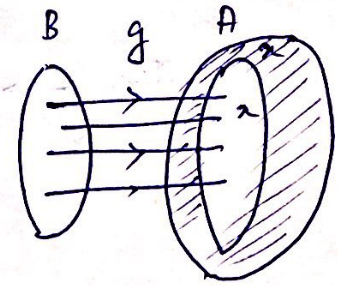
14/02/2020 Mathematics

① Prof. Sharmila Ghosh

Schröder - Bernstein Theorem

Let A & B be two sets such that there exist injective maps (one to one) $f: A \to B$ and $g: B \to A$. Then there is a bijection b/w A & B.





$x \in A \setminus g(B)$

Ⓐ		Ⓑ	
x	$\longrightarrow$	$g^{-1}(x)$.	predecessor of x.
$f^{-1}(g^{-1}(x))$	$\longrightarrow$	$g^{-1} \cdot f^{-1} g^{-1}(x)$.	3rd predecessor of x.

even/odd.

$A = A_o \cup A_e \cup A_i \Rightarrow$ infinite no. of predecessor.

A_o: set of x in A which has odd no. of ~~th~~ predecessor.

A_e: ... even no. of predecessor

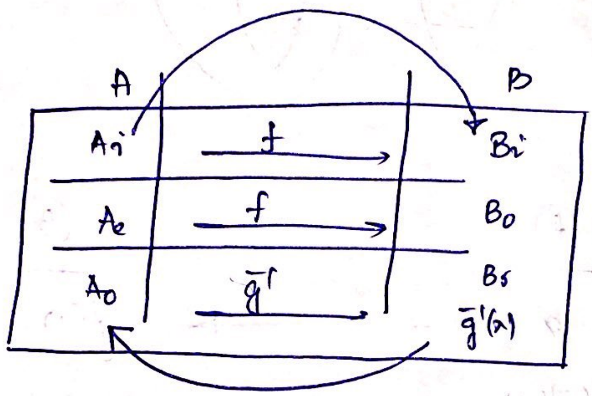
Ⓐ		Ⓑ
1st predecessor of Y. $f^{-1}(y)$	$\xleftarrow{\text{if } y \in f(A).}$	y.
	if $f^{-1}(y) \in g(B)$.	2nd pred. of y. $g^{-1} \cdot f^{-1}(y)$.

$B = B_0 \cup B_e \cup B_i \Rightarrow$ infinite no. of predecessor.

$B_0 \Downarrow$ set of y in B with odd no. of predecessor.

$B_e \Downarrow$ even ..

$f^{-1}(x) \longrightarrow x \longrightarrow g^{-1}(x) \longrightarrow f^{-1} \cdot g^{-1}(x) \rightarrow g^{-1}f^{-1}g^{-1}(x) \rightarrow \ldots$
$\rightarrow \ldots \rightarrow$ infinite.



$x \in A_0$.
x has atleast one predecessor.
$\therefore g^{-1}(x)$ exists.

Define $h : A \to B$. by

$$h(x) = \begin{cases} f(x). & \text{if } x \in A_i \cup A_e. \\ g^{-1}(x). & \text{if } x \in A_0. \end{cases}$$

We state that h is bijective.

Now as. f, g are 1-1, h is so.
As g^{-1} is defined on A_0 i.e. on the element of A ~~other~~ which has at best one predecessor.
g^{-1} is well defined.

If $y \in B_i$
$\Rightarrow$ y has infinite predecessor.
$\Rightarrow$ y has atleast one predecessor.
$f^{-1}(y)$, exist.

if $y \in B_o$.

$\Rightarrow y$ has odd predecessor.

$\Rightarrow y$ has at least 1 predecessor.

$f^{-1}(y)$ exists $\quad f(f^{-1}(y)) = y$.

$y \in B_e$.

$g^{-1}(g(y)) = y$ as g is 1-1.

Cardinal Number

Let $\mathcal{C}$ be a collection of sets. Define a binary relation $\sim$ on $\mathcal{C}$ by

$A \sim B$ iff $\exists$ bijective map $f : A \to B$.

Show that $\sim$ is an equivalence relation.

ρ equivalence relation

reflexive.

$x \rho x \quad \forall x \in A$.

Symmetric.

$x \rho y \Rightarrow y \rho x$.

Trans.

$x \rho y, \ y \rho z \Rightarrow x \rho z$.

$i_A : A \to A$

$i_A(a) = a$

i_A is bijective.

$A \sim A \quad \forall A \in \mathcal{C}$.

1-1.

$f : A \to B$

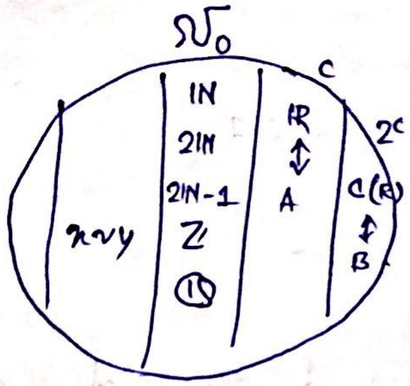
onto.

$f^{-1} : B^{-1} \xrightarrow{1\text{-}1} A^{-1}$

onto.

$$A \xrightarrow[\text{onto}]{1-1} B \xrightarrow[\text{onto}]{1-1} C.$$

$$g \circ f : \quad A \xrightarrow[\text{onto}]{1-1} C$$



Defn.

→ for each $A \in \mathcal{C}$, the equivalence class of A defined by $\{B \in \mathcal{C} \mid A \sim B\}$ is called the cardinal number of A and is denoted by $|A|$.

A partial ordering of Cardinal number.

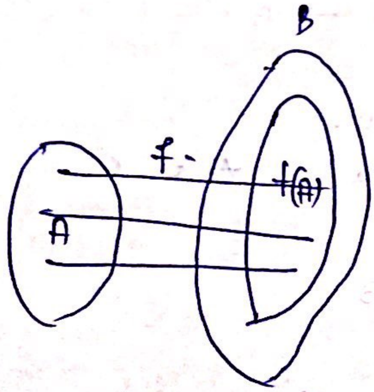
$x \rho y$ on A.

$x \rho x \; \forall x \in A$.

eg. $x \mid y$. $\quad x \rho y$ & $y \rho x \Rightarrow x = y$,
$\leq \qquad \leq$

$x \rho y$, $y \rho x \Rightarrow x \rho z$.
$\leq \qquad \leq \qquad \leq$

• Define. $\quad |A| \leq |B|$ iff $\exists$ injective (1-1) map from A into B
$$F : A \xrightarrow{1-1} B.$$



Verify that $\leq$ is a partial order relation.

0 $n = |\{1, 2, 3, \ldots, n\}|$.

1. $\aleph_0 = |\mathbb{N}|$.

2. $n < \aleph_0 \quad \forall \; n \in \mathbb{N}$.

We denote $|A| < |B|$ iff $\exists$ an injective map from A into B but $\nexists$ any bijective map between A & B. i.e. $\nexists$ any injective map from B into A.

$A = \{1, 2, 3, 4, 5, 6, 7, \ldots, n\}$.

$f : A \longrightarrow \mathbb{N}$,

$f(a) = a$.

$\therefore f$ is 1-1.

if $\exists$ a bijection map, g from A onto $\mathbb{N}$,

then $|g(A)| = \{a_1, a_2, \ldots, a_n\}$.

Take an element $b \in \mathbb{N}$,

s.t. $b \notin \{a_1, a_2, \ldots, a_n\}$.

then b has no pre-image.

b exist as $\mathbb{N}$ is infinite.

3. $|$Every infinite subset of $\mathbb{N}| = \aleph_0$.

4. Let $|A_i| = \aleph_0 \quad \forall \; i \in I \leq \mathbb{N}$.

then $\sum_{i \in I} |A_i| = \aleph_0$.

(In. Countable union of countable sets is countable),

A partial ordering of cardinal nos.

if ~~$|A_i| \leq$~~ $|A| \leq \aleph_0$.

countably infinite

if $|A| = \aleph_0$.

$A_1 \cup A_2 \cup A_3 \cup \ldots \cup$

5. $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ i.e. $\aleph_0^2 = \aleph_0$.

<u>By induction</u>.

$\aleph_0^n = \aleph_0 \quad \forall\ n \in \mathbb{N}$.

6. $|\mathbb{Q}| = |\mathbb{N}| = \aleph_0$.

7. $|$Any subset of $\mathbb{Q}| \leq \aleph_0$.

$|$Any infinite subset of $\mathbb{Q}| = \aleph_0$.

8. $|\mathbb{Q}^n| = \aleph_0$.

$\left|\bigcup_{i \in I} A_i\right| \leq \aleph_0$, $A_i \in \mathbb{Q} \quad \forall\ i \in I \subseteq \mathbb{N}$.

9. Let $c = |\mathbb{R}|$ Then $c > \aleph_0$.

10. $2^{\aleph_0} = c$.

11. For any set A, $|2^A| = |\mathcal{P}(A)| > |A|$.

12. $\aleph_0 < 2^{\aleph_0}.\ c < 2^c < 2^{2^c} < \ldots$

<u>Defn</u>. A set A is called uncountable if $|A| > \aleph_0$.