CHAPTER V

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

5.1. Complementary function and particular integral.

An ordinary linear differential equation of the n-th order has the form
$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_{n-1} \frac{dy}{dx} + P_n y = X, \quad \ldots \quad (1)$$

where X and the coefficients $P_1, P_2, \ldots\ldots, P_n$ are constants or functions of x only. The dependent variable and its derivatives appear only in the first degree and are not multiplied together. If the coefficient of the derivative of the highest order $\frac{d^n y}{dx^n}$ be not unity, then all the terms of the equation can be divided by that coefficient, so that (1) is the most general form of such equations.

In this chapter, we shall consider only the ordinary linear differential equations in which $P_1, P_2, \ldots\ldots, P_n$ are constants and X is a function of x only or a constant. We shall consider two forms of the equation (1). First we consider the form in which the right hand member, that is, X is zero and then we consider the form in which X is a function of x only or a constant. Equations of the first form (that is, when X = 0) are said to be *homogeneous.*

Theorem 1. *If $y = f(x)$ be the general solution of the equation*
$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_{n-1} \frac{dy}{dx} + P_n y = 0 \quad \ldots \quad (1)$$
and $y = \phi(x)$ be a solution of the equation
$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_{n-1} \frac{dy}{dx} + P_n y = X, \quad \ldots \quad (2)$$
then
$$y = f(x) + \phi(x) \quad \ldots \quad (3)$$
is the general solution of the equation (2).

Substituting the value of y from (3) in (2), we get the left hand side of (2) equal to
$$\left(\frac{d^n f}{dx^n} + P_1 \frac{d^{n-1} f}{dx^{n-1}} + \cdots\cdots + P_n f\right) + \left(\frac{d^n \phi}{dx^n} + P_1 \frac{d^{n-1} \phi}{dx^{n-1}} + \cdots\cdots + P_n \phi\right).$$

Now, $y = f(x)$ being a solution of (1), the expression within the former bracket reduces to zero. Similarly $y = \phi(x)$ being a solution of (2), the second group of terms is equal to X.

Hence (3) is a solution of the equation (2).

Theorem 2. *If* $y = y_1\ ,\ y = y_2, \ldots\ldots, y = y_n$ *be integrals of the equation*

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_{n-1} \frac{dy}{dx} + P_n y = 0, \qquad \ldots \quad (1)$$

then $y = C_1 y_1 + C_2 y_2 + \cdots\cdots + C_n y_n$,
where $C_1, C_2, \ldots\ldots, C_n$ *are arbitrary constants, is also an integral of the equation* (1).

Substituting $y = C_1 y_1 + C_2 y_2 + \cdots\cdots + C_n y_n$ in the left hand side of the equation (1), we get

$$C_1\left(\frac{d^n y_1}{dx^n} + P_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots\cdots + P_n y_1\right)$$
$$+ C_2\left(\frac{d^n y_2}{dx^n} + P_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots\cdots + P_n y_2\right)$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ C_n\left(\frac{d^n y_n}{dx^n} + P_1 \frac{d^{n-1} y_n}{dx^{n-1}} + \cdots\cdots + P_n y_n\right).$$

Now, since $y = y_1,\ y = y_2, \ldots\ldots, y = y_n$ are solutions of the given equation, each group of terms within the brackets is zero. This shows that

$$y = C_1 y_1 + C_2 y_2 + \cdots\cdots + C_n y_n$$

is a solution of (1).

Since this solution contains n arbitrary constants, it is the general solution of the equation (1).

From the above two theorems, we see that the general or complete solution of the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = X$$

consists of two parts. The first part is the general solution of the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = 0$$

say, $\quad y = f_1(C_1, C_2, \ldots\ldots, C_n, x)$

containing n arbitrary constants, and the second part

say, $\quad y = f_2(x)$

is a solution of the equation under consideration and does not contain an arbitrary constant.

The first part, that is, the expression $f_1(C_1, C_2, \ldots\ldots, C_n, x)$ is called the *complementary function* (C. F.) and the second part, that is, $f_2(x)$ is called the *particular integral* (P. I).

The *complete* or *general solution* of the equation is thus

$$y = f_1(C_1, C_2, \ldots\ldots, C_n, x) + f_2(x) = C.F. + P.I.$$

Cor. 1. If a function $y_1(x)$ be a solution of a linear differential equation, then the function $Cy_1(x)$, where C is an arbitrary constant, is also a solution of that equation.

Cor. 2. If the functions $y_1(x)$ and $y_2(x)$ be solutions of a linear differential equation, then the sum function $\{y_1(x) + y_2(x)\}$ is also a solution of that equation.

Cor. 3. If a linear differential equation with real coefficients has a complex solution

$$y(x) = u(x) + iv(x),$$

then each of the real part $u(x)$ of this solution and the imaginary part $v(x)$ is also a solution of that equation.

Note 1. Some authors use the symbols y_c and y_p to denote the complementary solution and the particular solution of the equation, so that the general solution is written as

$$y = y_c + y_p.$$

Note 2. $y = C_1 y_1 + C_2 y_2 + \cdots\cdots + C_n y_n$ will be the general solution of

the equation $\dfrac{d^n y}{dx^n} + P_1 \dfrac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = 0$

provided $y_1, y_2, \ldots\ldots, y_n$ are linearly independent*, that is, there does not exist a set of constants $a_1, a_2, \ldots\ldots, a_n$, at least one of which is non-zero, such that

$$a_1 y_1 + a_2 y_2 + \cdots\cdots + a_n y_n \equiv 0.$$

* See Appendix B.

If these functions be linearly dependent, (say $a_1 \neq 0$), then y_1 can be written in terms of the others as

$$y_1 = \frac{-(a_2 y_2 + a_3 y_3 + \cdots\cdots + a_n y_n)}{a_1}.$$

Hence it is clear that this solution can be put as

$$y = \left(C_2 - \frac{a_2 C_1}{a_1} \right) y_2 + \cdots\cdots + \left(C_n - \frac{a_n C_1}{a_1} \right) y_n$$

which contains $(n-1)$ constants and hence is not the general solution.

A linear dependence of a pair of functions means that one of the functions can be obtained from the other by multiplying with a constant.

5.2. Differential operator D.

We use the symbol D for the differential operator $\frac{d}{dx}$ so that for $\frac{d^r y}{dx^r}$ we write $D^r y$.

If m_1 be a constant, then $(D - m_1)\, y \equiv \frac{dy}{dx} - m_1 y$.

The notation $(D - m_1)(D - m_2) y$ is defined to mean that y is operated first with $(D - m_2)$ and then the result is operated with $(D - m_1)$.

Thus, if m_1, m_2 be constants, then

$$(D - m_1)(D - m_2) y = (D - m_1) \left(\frac{dy}{dx} - m_2 y \right)$$

$$= \frac{d^2 y}{dx^2} - m_1 \frac{dy}{dx} - m_2 \frac{dy}{dx} + m_1 m_2 y$$

$$= \frac{d^2 y}{dx^2} - (m_1 + m_2) \frac{dy}{dx} + m_1 m_2 y.$$

It can be easily verified that

$$(D - m_2)(D - m_1) y = \{ D^2 - (m_1 + m_2) D + m_1 m_2 \} y$$

$$= \frac{d^2 y}{dx^2} - (m_1 + m_2) \frac{dy}{dx} + m_1 m_2 y.$$

Thus we see that if m_1 and m_2 be constants, then

$$(D - m_1)(D - m_2) y = (D - m_2)(D - m_1) y,$$

that is, operation is independent of the order in which the factors are used.

5.3. Solution of linear equations with constant coefficients.

Using the symbol $D\left(\equiv \dfrac{d}{dx}\right)$, the equation

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+P_n y=0,$$

where $P_1, P_2, \ldots\ldots, P_n$ are constants, can be written as

$$(D^n+P_1 D^{n-1}+\cdots\cdots+P_n)y=0,$$

that is, $\quad f(D)y=0, \qquad \ldots \quad (1)$

where $\quad f(D)\equiv D^n+P_1 D^{n-1}+\cdots\cdots+P_n.$

Let $P_1, P_2, \ldots\ldots, P_n$ be real so that the roots of the equation $f(m)=0$ are either real or conjugate complex.

Here the degree of the equation $f(m)=0$ is n. Let us assume that the polynomial equation $f(m)=0$ has n real and distinct roots $m_1, m_2, \ldots\ldots, m_n$ so that (1) may be written as

$$(D-m_1)(D-m_2)\cdots\cdots(D-m_n)y=0. \qquad \ldots \quad (2)$$

The solution of any one of the equations

$$(D-m_1)y=0,\ (D-m_2)y=0,\ \cdots\cdots,\ (D-m_n)y=0 \quad \ldots \quad (3)$$

is also a solution of the equation (2). For, if $\phi_2(x)$ be a solution of $(D-m_2)y=0$, then putting $\phi_2(x)$ for y on the left hand expression of (2), we get

$$f(D)\,\phi_2=(D-m_1)(D-m_3)\cdots\cdots(D-m_n)(D-m_2)\phi_2$$

$$=(D-m_1)(D-m_3)\cdots\cdots(D-m_n)(0)$$

$=0$, since the operational factors are independent of the order in which they are used.

Thus $\phi_2(x)$ is a solution of the equation (1) and similar consideration can be made for the other equations in (3).

Now, if we integrate $(D-m)y=0$,

that is, $\quad \dfrac{dy}{dx}-my=0,$

we get $\quad y=Ce^{mx}$, where C is an arbitrary constant.

Hence the solutions of equations (3) are

$$y = C_1 e^{m_1 x},\ y = C_2 e^{m_2 x},\ \ldots\ldots,\ y = C_n e^{m_n x}, \quad \ldots \quad (4)$$

where $C_1, C_2, \ldots\ldots, C_n$ are arbitrary constants.

Each of these solutions being a solution of the equation (1), the general solution of the equation (1) is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots\ldots + C_n e^{m_n x}. \quad \ldots \quad (5)$$

Since the constants $m_1, m_2, \ldots\ldots, m_n$ are distinct, the solutions (4) are linearly independent and hence (5) is the general solution of (1).

It should be noted that the n distinct numbers $m_1, m_2, \ldots\ldots, m_n$ can be found by solving, for m, the equation

$$m^n + P_1 m^{n-1} + \ldots\ldots + P_n = 0$$

which is obtained by substituting e^{mx} for y in (1), since $e^{mx} \neq 0$.

This equation is called the *auxiliary equation.*

In this case the roots of the auxiliary equation are real and distinct.

5.4. Case of the auxiliary equation having equal roots.

When two roots of the auxiliary equation are equal, that is, $m_1 = m_2 = m$ (say), then the solution obtained in the previous article becomes

$$y = (C_1 + C_2)\, e^{mx} + C_3 e^{m_3 x} + \ldots\ldots + C_n e^{m_n x}$$
$$= C e^{mx} + C_3 e^{m_3 x} + \ldots\ldots + C_n e^{m_n x}, \text{ where } C_1 + C_2 = C.$$

This is no longer a general solution, since the number of arbitrary constants is now $(n - 1)$ and not n.

The corresponding part of the solution is, in fact, the solution of

$$(D - m)^2 y = 0$$

or, $\quad (D - m)(D - m)\, y = 0$

or, $\quad (D - m)\, u = 0,$

where u is put for $(D - m)\, y$.

Solution of this equation is $u = C_2 e^{mx}$.

Putting this value of u, we get

$$(D - m) y = u = C_2 e^{mx}, \text{ that is, } \frac{dy}{dx} - my = C_2 e^{mx}$$

which is a linear equation of first order whose integrating factor is e^{-mx} and the solution is

$$y e^{-mx} = \int C_2 e^{mx} . e^{-mx} dx = C_1 + C_2 x .$$

Therefore $$y = (C_1 + C_2 x) e^{mx},$$

in which there are two constants C_1 and C_2.

Thus the general solution in this case is

$$y = (C_1 + C_2 x) e^{mx} + C_3 e^{m_3 x} + \cdots\cdots + C_n e^{m_n x} .$$

Cor. If the auxiliary equation has r equal roots m, then the general solution is

$$y = \left(C_1 + C_2 x + \cdots + C_r x^{r-1} \right) e^{mx} + C_{r+1} e^{m_{r+1} x} + \cdots + C_n e^{m_n x} .$$

5.5. Case of the auxiliary equation having complex roots.

If the auxiliary equation has a pair of complex roots, say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the corresponding part of the solution is

$$C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

$$= e^{\alpha x} \left(C_1 e^{i\beta x} + C_2 e^{-i\beta x} \right)$$

$$= e^{\alpha x} \{ C_1 (\cos \beta x + i \sin \beta x) + C_2 (\cos \beta x - i \sin \beta x) \}$$

$$= e^{\alpha x} \{ (C_1 + C_2) \cos \beta x + i (C_1 - C_2) \sin \beta x \}$$

$$= e^{\alpha x} (A \cos \beta x + B \sin \beta x),$$

in which $A = C_1 + C_2$ and $B = i(C_1 - C_2)$ are arbitrary constants.

If the above pair of complex roots occurs twice in the auxiliary equation, then the corresponding part of the solution is

$$(C_1 + C_2 x) e^{(\alpha + i\beta)x} + (C_3 + C_4 x) e^{(\alpha - i\beta)x}$$

which reduces to

$$e^{\alpha x} \{ (A_1 + A_2 x) \cos \beta x + (B_1 + B_2 x) \sin \beta x \},$$

A_1, A_2, B_1, B_2 being arbitrary constants.

5.6. Illustrative Examples.

Ex. 1. *Solve the equation* $2\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 2y = 0$.

Let $y = e^{mx}$ be a solution of the above equation.

Then the equation becomes

$$(2m^3 - 7m^2 + 7m - 2)e^{mx} = 0.$$

Since $e^{mx} \neq 0$, we get

$$2m^3 - 7m^2 + 7m - 2 = 0$$

or, $(m-1)(m-2)(2m-1) = 0$.

Therefore $m = 1, 2, \frac{1}{2}$.

Hence the general solution is

$$y = C_1e^x + C_2e^{2x} + C_3e^{\frac{1}{2}x}, \text{ where } C_1, C_2, C_3 \text{ are arbitrary constants.}$$

Ex. 2. *Find the general solution of the equation*

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0.$$

Let $y = e^{mx}$ be a solution of the above differential equation ;
then we have $(m^4 - m^3 - 9m^2 - 11m - 4)e^{mx} = 0$.

Therefore $m^4 - m^3 - 9m^2 - 11m - 4 = 0$, since $e^{mx} \neq 0$

or, $(m+1)^3(m-4) = 0$,

giving $m = -1, -1, -1, 4$.

Therefore the general solution is

$$y = (C_1 + C_2x + C_3x^2)e^{-x} + C_4e^{4x},$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

Ex. 3. *Solve the equation* $\frac{d^4y}{dx^4} + a^4y = 0$.

Introducing the differential operator $D\left(\equiv \frac{d}{dx}\right)$, the given equation can be written as

$$(D^4 + a^4)y = 0$$

so that the auxiliary equation is $m^4 + a^4 = 0$,

giving $m = -\frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$ and $m = \frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$.

Hence the general solution is

$$y = \left(C_1 \cos \frac{a}{\sqrt{2}} x + C_2 \sin \frac{a}{\sqrt{2}} x \right) e^{\frac{a}{\sqrt{2}} x} + \left(C_3 \cos \frac{a}{\sqrt{2}} x + C_4 \sin \frac{a}{\sqrt{2}} x \right) e^{-\frac{a}{\sqrt{2}} x},$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

Ex. 4. *Solve the equation* $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0$, *where* $D \equiv \dfrac{d}{dx}$.

Here the auxiliary equation is $(m^2 + 1)^3 (m^2 + m + 1)^2 = 0$.

Therefore $m = \pm i, \pm i, \pm i$ and $m = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}, -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$.

Hence the general solution is

$$y = (C_1 + C_2 x + C_3 x^2) \cos x + (C_4 + C_5 x + C_6 x^2) \sin x$$
$$+ \left\{ (C_7 + C_8 x) \cos \frac{\sqrt{3}}{2} x + (C_9 + C_{10} x) \sin \frac{\sqrt{3}}{2} x \right\} e^{-\frac{1}{2} x},$$

in which $C_1, C_2, \cdots\cdots, C_{10}$ are arbitrary constants.

Note that the general solution contains ten arbitrary constants which is the same as the order of the given equation.

Note. The values of the arbitrary constants and hence the particular solution of the equation can be determined from given conditions.

Examples V(A)

Solve the following differential equations (**1 – 9**) :

1. $4 \dfrac{d^2 y}{dx^2} + 4 \dfrac{dy}{dx} - 3y = 0$. **2.** $\{ D^2 + (a + b) D + ab \} y = 0$.

3. $\dfrac{d^2 y}{dx^2} - 24 \dfrac{dy}{dx} + 144y = 0$. **4.** $\dfrac{d^2 y}{dx^2} + 8 \dfrac{dy}{dx} + 25y = 0$.

5. $\dfrac{d^3 y}{dx^3} + 2 \dfrac{d^2 y}{dx^2} - 5 \dfrac{dy}{dx} - 6y = 0$. **6.** $(D^2 + 4)(D^2 + 1) y = 0$.

7. $\dfrac{d^4 y}{dx^4} - \dfrac{d^3 y}{dx^3} - 9 \dfrac{d^2 y}{dx^2} - 11 \dfrac{dy}{dx} - 4y = 0$.

8. $\dfrac{d^4 y}{dx^4} - 4 \dfrac{d^3 y}{dx^3} + 8 \dfrac{d^2 y}{dx^2} - 8 \dfrac{dy}{dx} + 4y = 0$.

9. $(D - 1)^3 (D^2 - 4)(D + 2) y = 0$.

Answers

1. $y = Ae^{\frac{1}{2}x} + Be^{-\frac{3}{2}x}$.
2. $y = Ae^{-ax} + Be^{-bx}$.
3. $y = (C_1 + C_2 x) e^{12x}$.
4. $y = e^{-4x} (A \cos 3x + B \sin 3x)$.
5. $y = Ae^{-x} + Be^{2x} + Ce^{-3x}$.
6. $y = C_1 \cos x + C_2 \sin x + C_3 \cos 2x + C_4 \sin 2x$.
7. $y = (C_1 + C_2 x + C_3 x^2) e^{-x} + C_4 e^{4x}$.
8. $y = e^{x} \{ (A + Bx) \cos x + (C + Dx) \sin x \}$.
9. $y = (C_1 + C_2 x + C_3 x^2) e^{x} + (C_4 + C_5 x) e^{-2x} + C_6 e^{2x}$.
10. $y = Ae^{2x} + Be^{3x}$; $y = 4e^{2x} - e^{3x}$.
11. $y = (A + Bx) e^{-2x}$; $y = (1 + 2x) e^{-2x}$.
12. $y = 3e^{2x} + 2e^{-2x}$.
13. $x = 4 \cos 2t + \frac{3}{2} \sin 2t$.
14. $y = 4xe^{-x}$.
16. $\theta = \alpha \cos \sqrt{\frac{g}{l}}\, t$.
17. $y = e^{-x} - e^{-2x}$.
20. $s = (s_0 - l) \cos \sqrt{\frac{g}{e}}\, t + l,\ \frac{ds}{dt} = (l - s_0) \sqrt{\frac{g}{e}} \sin \sqrt{\frac{g}{e}}\, t$.

5.7. Symbolic operator $\frac{1}{f(D)}$.

We use the expression $\frac{1}{f(D)} X$ to denote a function of x which does not contain any arbitrary constant and which gives X when operated with $f(D)$. Thus, since

$$(D^2 - D)(x^2 - x) = 3 - 2x,$$

we have

$$\frac{1}{D^2 - D}(3 - 2x) = x^2 - x.$$

The operator $\dfrac{1}{f(D)}$, according to this definition, is the inverse of the operator $f(D)$. If $f(D) = D$, then we have

$$\frac{1}{f(D)} X = \frac{1}{D} X = \int X \, dx .$$

For our future use, we attempt to find v which is obtained by inversely operating on X with the factor $(D - a)$, that is,

$$v = \frac{1}{D-a} X ,$$

in which X is a function of x only and a is a constant. This is, according to the definition,

$$(D - a) v = X$$

or,
$$\frac{dv}{dx} - av = X,$$

which is a linear equation of first order in v and whose integrating factor is e^{-ax}. Therefore its solution is given by

$$v = Ae^{ax} + e^{ax} \int X e^{-ax} \, dx .$$

Now, as v, by definition, will remain free from any arbitrary constant, we have

$$v = e^{ax} \int X e^{-ax} \, dx .$$

This result will be found useful in the discussion of the general method of finding the particular integral of an equation.

Note. If, in particular, $X = e^{ax}$, then $v = xe^{ax}$.

*5.8. General method of finding the particular integral.

Consider the linear equation with constant coefficients

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = X,$$

where X is a function of x only. In symbolic notation, this can be written as

$$f(D) y = X,$$

where
$$f(D) \equiv D^n + P_1 D^{n-1} + \cdots\cdots + P_n$$

and the particular integral is thus $\dfrac{1}{f(D)} X$.

We can now evaluate $\frac{1}{f(D)} X$ by any one of the following methods:

(*i*) Let $f(D)$ can be broken up into linear factors, say,

$$f(D) = (D - m_1)(D - m_2) \ldots\ldots\ldots\ldots (D - m_n),$$

the factors being placed in any order.

Then the particular integral is

$$\frac{1}{D - m_1} \frac{1}{D - m_2} \ldots\ldots\ldots\ldots \frac{1}{D - m_n} X.$$

This expression is defined to mean that X is first inversely operated upon with $(D - m_n)$, then the result is inversely operated upon with $(D - m_{n-1})$ and so on until all the factors are similarly utilised.

After the first operation, it becomes

$$\frac{1}{D - m_1} \frac{1}{D - m_2} \ldots\ldots\ldots\ldots e^{m_n x} \int X e^{-m_n x} dx.$$

Then operating with the second and remaining factors in succession, we get the particular integral as

$$e^{m_1 x} \int e^{(m_2 - m_1)x} \int \ldots\ldots\ldots \int X e^{-m_n x} (dx)^n.$$

(*ii*) Let $\frac{1}{f(D)}$ can be resolved into partial fractions, say,

$$\frac{1}{f(D)} = \frac{N_1}{D - m_1} + \frac{N_2}{D - m_2} + \ldots\ldots\ldots + \frac{N_n}{D - m_n},$$

$N_1, N_2, \ldots\ldots\ldots, N_n$ being constants.

Therefore

$$\frac{1}{f(D)} X = \frac{N_1}{D - m_1} X + \frac{N_2}{D - m_2} X + \ldots\ldots\ldots + \frac{N_n}{D - m_n} X$$

$$= N_1 e^{m_1 x} \int X e^{-m_1 x} dx + \ldots\ldots\ldots + N_n e^{m_n x} \int X e^{-m_n x} dx.$$

5.9. Particular integral by short methods.

The general method of finding the particular integral is a laborious calculation. There are short methods for finding them for some functions which we shall explain now.

(i) *Particular integral for $X = e^{ax}$, a being a constant.*

The equation here is $f(D)\,y = e^{ax}$

so that the particular integral is $\dfrac{1}{f(D)}\,e^{ax}$.

We have

$De^{ax} = ae^{ax},\ D^2e^{ax} = a^2e^{ax}, \ldots, D^{n-1}e^{ax} = a^{n-1}e^{ax},\ D^ne^{ax} = a^ne^{ax}.$

Therefore

$$\begin{aligned} f(D)\,e^{ax} &= (D^n + P_1 D^{n-1} + \cdots\cdots + P_{n-1}D + P_n)\,e^{ax} \\ &= (a^n + P_1 a^{n-1} + \cdots\cdots + P_{n-1}a + P_n)\,e^{ax} \\ &= f(a)\,e^{ax}. \end{aligned}$$

Operating both sides with $\dfrac{1}{f(D)}$, we get

$$\frac{1}{f(D)}\left\{f(D)\,e^{ax}\right\} = \frac{1}{f(D)}\left\{f(a)\,e^{ax}\right\}.$$

Now, since $f(D)$ and $\dfrac{1}{f(D)}$ are operators inverse to one another and $f(a)$ is only an algebraic multiplier, it reduces to

$$e^{ax} = f(a)\cdot\frac{1}{f(D)}\,e^{ax},$$

whence we have $\quad \dfrac{1}{f(D)}\,e^{ax} = \dfrac{1}{f(a)}\,e^{ax}$, provided $f(a) \neq 0$.

If $f(D)$ contains a factor $(D - a)$, then this method fails and we proceed in the following way :

Since $(D - a)$ is a factor of $f(D)$, let $f(D) = (D - a)\,\phi(D)$.

Then $\dfrac{1}{f(D)}\,e^{ax} = \dfrac{1}{D-a}\,\dfrac{1}{\phi(D)}\,e^{ax} = \dfrac{1}{D-a}\,\dfrac{1}{\phi(a)}\,e^{ax}$, provided $\phi(a) \neq 0$

$$= \frac{xe^{ax}}{\phi(a)} \qquad \text{[cf. Note Art. 5.7]}$$

If $(D - a)^2$ be a factor of $f(D)$, let $f(D) = (D - a)^2\,\psi(D)$.

Then $\dfrac{1}{f(D)}\,e^{ax} = \dfrac{1}{(D-a)^2}\,\dfrac{1}{\psi(D)}\,e^{ax} = \dfrac{1}{(D-a)^2}\,\dfrac{1}{\psi(a)}\,e^{ax}$,

provided $\psi(a) \neq 0$

$$= \frac{x^2e^{ax}}{2\,\psi(a)}.$$

Same procedure will be followed when $(D - a)^r$ is a factor of $f(D)$, r being a positive integer.

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(ii) *Particular integral for $X = x^m$, m being a positive integer.*

In order to evaluate $\dfrac{1}{f(D)} x^m$,

expand $\{f(D)\}^{-1}$ and arrange the terms in ascending powers of D and operate on x^m. The result will be the particular integral corresponding to x^m.

It should be noticed that terms of the expansion beyond the m-th power of D need not be written, since $D^{m+1} x^m = 0$.

(iii) *Particular integral for $X = \sin ax$ or $\cos ax$.*

Let us evaluate $\dfrac{1}{f(D)} \sin ax$.

We have
$$D \sin ax = a \cos ax,$$
$$D^2 \sin ax = -a^2 \sin ax,$$
$$D^3 \sin ax = -a^3 \cos ax,$$
$$D^4 \sin ax = a^4 \sin ax.$$

In general, $(D^2)^n \sin ax = (-a^2)^n \sin ax$.

Now, if $f(D)$ contains only even powers of D and we denote it by $\phi(D^2)$, then it is obvious that

$$\phi(D^2) \sin ax = \phi(-a^2) \sin ax.$$

Operating on both sides with $\dfrac{1}{\phi(D^2)}$, we get

$$\sin ax = \frac{1}{\phi(D^2)} \{\phi(-a^2) \sin ax\}.$$

Since $\phi(-a^2)$ is an algebraic multiplier, we get

$$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax, \text{ provided } \phi(-a^2) \neq 0.$$

Similarly we get

$$\frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax, \quad \text{provided } \phi(-a^2) \neq 0.$$

More generally, we have

$$\frac{1}{\phi(D^2)} \sin(ax+b) = \frac{1}{\phi(-a^2)} \sin(ax+b),\ \phi(-a^2) \neq 0$$

and $$\frac{1}{\phi(D^2)} \cos(ax+b) = \frac{1}{\phi(-a^2)} \cos(ax+b),\ \phi(-a^2) \neq 0.$$

The above results do not hold in case $\phi(-a^2) = 0$. This will happen, if $f(D)$ contains a factor (D^2+a^2). In such cases the general method is applied to find the particular integral.

We present a special method here for such cases.

In such cases, instead of computing the particular integral for $\sin ax$ or $\cos ax$, we calculate the particular integral for $(\cos ax + i \sin ax)$, that is, for e^{iax}. Thus

$$\frac{1}{D^2+a^2}(\cos ax + i \sin ax) = \frac{1}{D^2+a^2} e^{iax}$$

$$= \frac{1}{(D+ia)(D-ia)} e^{iax}$$

$$= \frac{e^{iax}}{2ai} \frac{1}{D+ia-ia} 1$$

$$= \frac{e^{iax}}{2ai} \frac{1}{D} 1$$

$$= \frac{x\,e^{iax}}{2ai} = \frac{x}{2ai}(\cos ax + i \sin ax)$$

$$= \frac{x \sin ax}{2a} - i\,\frac{x \cos ax}{2a}.$$

Equating the real and the imaginary parts from both sides, we get

$$\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$$

and $$\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax.$$

(iv) Particular integral for $X = e^{ax} V$, *V being any function of x.*

We are to evaluate $\dfrac{1}{f(D)}(e^{ax} V)$.

Let V_1 be a function of x defined by $V_1 = \dfrac{1}{f(D+a)} V$. ... (1)

We have $D(e^{ax} V_1) = e^{ax} DV_1 + ae^{ax} V_1 = e^{ax}(D+a) V_1$,

$$D^2(e^{ax} V_1) = a e^{ax} (D+a) V_1 + e^{ax} D(D+a) V_1$$
$$= e^{ax}(D+a)^2 V_1.$$

In general, by successive differentiation, we get

$$D^n(e^{ax} V_1) = e^{ax}(D+a)^n V_1.$$

Therefore $f(D)(e^{ax} V_1) = e^{ax} f(D+a) V_1$. $\quad\ldots\quad$ (2)

Putting (1) in (2), we get

$$f(D)\left\{e^{ax}\frac{1}{f(D+a)}V\right\} = e^{ax} V.$$

Now operating both sides of this equation with $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)}(e^{ax} V) = e^{ax}\frac{1}{f(D+a)}V.$$

(v) *Particular integral for $X = xV$, V being any function of x.*

We are to evaluate $\frac{1}{f(D)}(xV)$.

Let V_1 be a function of x defined by $V_1 = \frac{1}{f(D)}V$. $\quad\ldots\quad$ (1)

We have $D(xV_1) = x DV_1 + V_1$,

$$D^2(xV_1) = x D^2 V_1 + 2DV_1 = x D^2 V_1 + \left(\frac{d}{dD}D^2\right)V_1,$$

. . . .

$D^n(xV_1) = x D^n V_1 + n D^{n-1} V_1$, by Leibnitz's theorem

$$= x D^n V_1 + \left(\frac{d}{dD}D^n\right)V_1.$$

Hence $f(D)(xV_1) = x f(D) V_1 + f'(D) V_1$, $\quad\ldots\quad$ (2)

where $f'(D) \equiv \frac{d}{dD}\{f(D)\}$.

Putting (1) in (2), we get

$$f(D)\left\{x\frac{1}{f(D)}V\right\} = xV + f'(D)\frac{1}{f(D)}V.$$

Operating all the terms of this equation with $\frac{1}{f(D)}$, we obtain

$$x\frac{1}{f(D)}V = \frac{1}{f(D)}(xV) + \frac{1}{f(D)}\left\{f'(D)\frac{1}{f(D)}V\right\}$$
$$= \frac{1}{f(D)}(xV) + f'(D)\frac{1}{\{f(D)\}^2}V.$$

Transposing, we get

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - f'(D)\frac{1}{\{f(D)\}^2}V$$

$$= x\frac{1}{f(D)}V + \left[\frac{d}{dD}\left\{\frac{1}{f(D)}\right\}\right]V.$$

The particular integral corresponding to $X = x^m V$, where m is a positive integer can be obtained by repeated application of this method.

5.10. Illustrative Examples.

Ex. 1. *Solve :* $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = x^2.$

Introducing the differential operator $D\left(\equiv \dfrac{d}{dx}\right)$, the given equation can be written as $(D^3 + 3D^2 + 2D)y = x^2.$

The auxiliary equation is $m^3 + 3m^2 + 2m = 0$

or, $m(m+1)(m+2) = 0.$

Therefore $m = 0, -1, -2.$

Hence the complementary function (y_c) is

$$A + Be^{-x} + Ce^{-2x},$$

where A, B, C are arbitrary constants.

The particular integral (y_p) is

$$\frac{1}{D^3 + 3D^2 + 2D}x^2 = \frac{1}{2D}\left(1 + \frac{3D + D^2}{2}\right)^{-1}x^2$$

$$= \frac{1}{2D}\left\{1 - \frac{3D + D^2}{2} + \left(\frac{3D + D^2}{2}\right)^2 - \cdots\cdots\right\}x^2$$

$$= \frac{1}{2D}\left(1 - \frac{3D}{2} - \frac{D^2}{2} + \frac{9D^2 + 6D^3 + D^4}{4} - \cdots\cdots\right)x^2$$

$$= \frac{1}{2D}\left(1 - \frac{3D}{2} + \frac{7}{4}D^2 - \cdots\cdots\cdots\right)x^2$$

$$= \frac{1}{2}\left(\frac{1}{D} - \frac{3}{2} + \frac{7}{4}D\right)x^2 = \frac{1}{2}\left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{7}{4}\cdot 2x\right)$$

$$= \frac{1}{12}(2x^3 - 9x^2 + 21x).$$

Hence the complete solution is

$$y = y_c + y_p = A + Be^{-x} + Ce^{-2x} + \frac{1}{12}(2x^3 - 9x^2 + 21x).$$

Ex. 2. *Solve* : $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x + \cos x.$

The given equation, in terms of the operator $D\left(\equiv \dfrac{d}{dx}\right)$, is

$$(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x.$$

The auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0, \text{ so that } m = 1, 1 \pm i.$$

The complementary function is

$$y_c = Ae^x + (B\cos x + C\sin x)e^x,$$

where A, B, C are arbitrary constants.

The particular integral is

$$y_p = \frac{1}{(D^3 - 3D^2 + 4D - 2)}(e^x + \cos x)$$

$$= \frac{1}{(D-1)(D^2 - 2D + 2)}e^x + \frac{1}{(D-1)(D^2 - 2D + 2)}\cos x$$

$$= \frac{1}{(D-1)(1-2+2)}e^x + \frac{1}{(D-1)(-1-2D+2)}\cos x$$

$$= e^x \frac{1}{(D+1-1)} 1 + \frac{1}{-2D^2 + 3D - 1}\cos x$$

$$= e^x \cdot x + \frac{1}{-2(-1) + 3D - 1}\cos x$$

$$= xe^x + \frac{1}{3D+1}\cos x = xe^x + \frac{3D-1}{9D^2 - 1}\cos x$$

$$= xe^x + \frac{3D-1}{9(-1)-1}\cos x = xe^x - \frac{1}{10}(3D-1)\cos x$$

$$= xe^x + \frac{1}{10}(3\sin x + \cos x).$$

Hence the complete solution is

$$y = y_c + y_p$$

$$= Ae^x + (B\cos x + C\sin x)e^x + xe^x + \frac{1}{10}(3\sin x + \cos x).$$

Ex. 3. *Solve :* $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2 e^{3x}.$

Introducing the differential operator $D\left(\equiv \dfrac{d}{dx}\right)$, the given equation can be written as

$$(D^2 - 2D + 1)y = x^2 e^{3x}.$$

The auxiliary equation is $m^2 - 2m + 1 = 0$.

The roots of this equation are $m = 1, 1$.

Hence the complementary function is $(A + Bx)e^x$, where A, B are arbitrary constants.

The particular integral is

$$\frac{1}{(D-1)^2} x^2 e^{3x} = e^{3x} \frac{1}{(D+3-1)^2} x^2$$

$$= \frac{e^{3x}}{2^2}\left(1 + \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{e^{3x}}{4}\left(1 - D + 3\frac{D^2}{4} - \cdots\cdots\right)x^2$$

$$= \frac{e^{3x}}{4}\left(x^2 - 2x + \frac{3}{4}\cdot 2\right) = \frac{e^{3x}}{8}(2x^2 - 4x + 3).$$

Hence the complete solution is

$$y = (A + Bx)e^x + \frac{e^{3x}}{8}(2x^2 - 4x + 3).$$

Ex. 4. *Solve :* $(D^2 + 1)y = 3\cos^2 x + 2\sin^3 x.$ [*B. H. 1990*]

The given equation can be written as

$$(D^2 + 1)y = \frac{3}{2}(1 + \cos 2x) + \frac{1}{2}(3\sin x - \sin 3x),$$

that is, $$(D^2 + 1)y = \frac{3}{2} + \frac{3}{2}\sin x + \frac{3}{2}\cos 2x - \frac{1}{2}\sin 3x.$$

The auxiliary equation is $m^2 + 1 = 0$, giving $m = \pm i$.

Hence the complementary function is $(A\cos x + B\sin x)$, where A, B are arbitrary constants.

The particular integral for $\dfrac{3}{2}$ is $\dfrac{3}{2}\dfrac{1}{D^2+1}1 = \dfrac{3}{2}$.

The particular integral for $\frac{3}{2}\sin x$ is

$$\frac{3}{2}\frac{1}{D^2+1}\sin x = -\frac{3}{2}\cdot\frac{x}{2}\cos x = -\frac{3}{4}x\cos x,$$

using Art 5.9 (*iii*).

The particular integral for $\frac{3}{2}\cos 2x$ is

$$\frac{3}{2}\frac{1}{D^2+1}\cos 2x = \frac{3}{2}\frac{1}{-2^2+1}\cos 2x = -\frac{1}{2}\cos 2x.$$

The particular integral for $(-\frac{1}{2}\sin 3x)$ is

$$-\frac{1}{2}\frac{1}{D^2+1}\sin 3x = -\frac{1}{2}\frac{1}{-3^2+1}\sin 3x = \frac{1}{16}\sin 3x.$$

Hence the complete solution is

$$y = A\cos x + B\sin x - \frac{3}{4}x\cos x - \frac{1}{2}\cos 2x + \frac{1}{16}\sin 3x + \frac{3}{2}.$$

Ex. 5. *Solve* : $\dfrac{d^3y}{dx^3} + y = e^{2x}\sin x + e^{\frac{x}{2}}\sin\dfrac{\sqrt{3}}{2}x.$ [*C. H. 1992*]

The given equation can be written as

$$(D^3+1)y = e^{2x}\sin x + e^{\frac{x}{2}}\sin\frac{\sqrt{3}}{2}x.$$

The auxiliary equation is $m^3+1=0$,

which gives $\quad m = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$

The complementary function is $C_1e^{-x} + \left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right)e^{\frac{x}{2}}$,

where C_1, C_2, C_3 are arbitrary constants.

The particular integral for $e^{2x}\sin x$ is

$$\frac{1}{D^3+1}e^{2x}\sin x = e^{2x}\frac{1}{(D+2)^3+1}\sin x$$

$$= e^{2x}\frac{1}{D^3+6D^2+12D+9}\sin x$$

$$= e^{2x}\frac{1}{D(D^2+12)+6D^2+9}\sin x$$

$$= e^{2x}\frac{1}{6(-1)+11D+9}\sin x = e^{2x}\frac{3-11D}{9-121D^2}\sin x$$

$$= e^{2x}\frac{3-11D}{9-121(-1)}\sin x = e^{2x}\frac{3\sin x - 11\cos x}{130}.$$

The particular integral for $e^{\frac{x}{2}} \sin \frac{\sqrt{3}}{2} x$ is

$$\frac{1}{D^3+1} e^{\frac{x}{2}} \sin \frac{\sqrt{3}}{2} x = e^{\frac{x}{2}} \frac{1}{\left(D+\frac{1}{2}\right)^3+1} \sin \frac{\sqrt{3}}{2} x$$

$$= e^{\frac{x}{2}} \frac{1}{D^3+\frac{3}{2}D^2+\frac{3}{4}D+\frac{9}{8}} \sin \frac{\sqrt{3}}{2} x$$

$$= e^{\frac{x}{2}} \frac{1}{\left(D^2+\frac{3}{4}\right)\left(D+\frac{3}{2}\right)} \sin \frac{\sqrt{3}}{2} x$$

$$= e^{\frac{x}{2}} \frac{D-\frac{3}{2}}{\left(D^2+\frac{3}{4}\right)\left(D^2-\frac{9}{4}\right)} \sin \frac{\sqrt{3}}{2} x$$

$$= e^{\frac{x}{2}}\left(-\frac{1}{3}\right)\frac{1}{D^2+\frac{3}{4}}\left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}x-\frac{3}{2}\sin\frac{\sqrt{3}}{2}x\right)$$

$$= -\frac{1}{3}e^{\frac{x}{2}}\left\{\frac{\sqrt{3}}{2}\frac{x}{2.\frac{\sqrt{3}}{2}}\sin\frac{\sqrt{3}}{2}x-\frac{3}{2}\left(\frac{-x}{2.\frac{\sqrt{3}}{2}}\cos\frac{\sqrt{3}}{2}x\right)\right\},$$

using Art. 5.9 *(iii)*

$$= -\frac{1}{6}e^{\frac{x}{2}}.x\left(\sin\frac{\sqrt{3}}{2}x+\sqrt{3}\cos\frac{\sqrt{3}}{2}x\right).$$

Thus the complete solution of the given equation is

$$y = C_1 e^{-x} + \left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right)e^{\frac{x}{2}} + \frac{e^{2x}}{130}(3\sin x - 11\cos x)$$
$$-\frac{1}{6}xe^{\frac{x}{2}}\left(\sin\frac{\sqrt{3}}{2}x+\sqrt{3}\cos\frac{\sqrt{3}}{2}x\right).$$

Ex. 6. *Solve :* $\dfrac{d^4y}{dx^4}+2\dfrac{d^2y}{dx^2}+y=x^2\cos x.$

The given equation is $(D^2+1)^2 y = x^2 \cos x$.

The auxiliary equation is $(m^2+1)^2=0$, giving $x = \pm i, \pm i$.

The complementary function is $(A+Bx)\cos x + (C+Dx)\sin x$, where A, B, C, D are arbitrary constants.

The particular integral is $\dfrac{1}{(D^2+1)^2} x^2 \cos x$.

Let $\quad Y = \dfrac{1}{(D^2+1)^2} x^2 \cos x$ and $Z = \dfrac{1}{(D^2+1)^2} x^2 \sin x$.

Therefore $Y + iZ = \dfrac{1}{(D^2+1)^2} x^2 e^{ix} = e^{ix} \dfrac{1}{\left\{(D+i)^2+1\right\}^2} x^2$

$$= e^{ix} \frac{1}{(D^2+2iD)^2} x^2$$

$$= -e^{ix} \frac{1}{4D^2}\left(1+\frac{D}{2i}\right)^{-2} x^2$$

$$= -e^{ix} \frac{1}{4D^2}\left(1-\frac{D}{i}-\frac{3}{4}D^2\right)x^2$$

$$= -\frac{e^{ix}}{4}\left(\frac{1}{D^2}-\frac{1}{iD}-\frac{3}{4}\right)x^2$$

$$= -\frac{e^{ix}}{4}\left(\frac{x^4}{12}-\frac{x^3}{3i}-\frac{3}{4}x^2\right)$$

$$= -\frac{1}{4}(\cos x + i\sin x)\left(\frac{x^4}{12}+\frac{x^3 i}{3}-\frac{3}{4}x^2\right).$$

Equating real parts from both sides, we get

$$Y = \frac{1}{(D^2+1)^2} x^2 \cos x = \frac{1}{12} x^3 \sin x + \left(\frac{3}{16}x^2 - \frac{1}{48}x^4\right)\cos x$$

$$= \frac{1}{12} x^3 \sin x + \frac{1}{48}\left(9x^2 - x^4\right)\cos x.$$

Therefore the complete solution is

$$y = (A+Bx)\cos x + (C+Dx)\sin x + \frac{1}{12}x^3 \sin x + \frac{1}{48}\left(9x^2-x^4\right)\cos x.$$

Ex. 7. *Solve* : $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$.

The auxiliary equation is $m^2 - 4m + 4 = 0$, giving $m = 2, 2$.

Therefore the complementary function is $(A+Bx)e^{2x}$,

where A, B are arbitrary constants.

The particular integral is

$$\frac{1}{(D-2)^2} 8e^{2x} V, \text{ where } V = x^2 \sin 2x$$

$$= 8e^{2x} \frac{1}{(D+2-2)^2} V = 8e^{2x} \frac{1}{D^2} x^2 \sin 2x = 8e^{2x} . S \text{ (say)},$$

where $\quad S = \dfrac{1}{D^2} x^2 \sin 2x$.

Let $P = \dfrac{1}{D^2} x^2 \cos 2x$.

Therefore $P + iS = \dfrac{1}{D^2} x^2 e^{2ix} = e^{2ix} \dfrac{1}{(D+2i)^2} x^2$

$$= \frac{e^{2ix}}{4i^2}\left(1 + \frac{D}{2i}\right)^{-2} x^2$$

$$= -\frac{1}{4} e^{2ix}\left(1 - \frac{iD}{2}\right)^{-2} x^2$$

$$= -\frac{1}{4} e^{2ix}\left\{1 + 2\left(\frac{iD}{2}\right) + 3\left(\frac{iD}{2}\right)^2 + \cdots\cdots\right\} x^2$$

$$= -\frac{1}{4} e^{2ix}\left(1 + iD - \frac{3}{4} D^2 - \cdots\cdots\right) x^2$$

$$= -\frac{1}{4} e^{2ix}\left(x^2 + 2ix - \frac{3}{2}\right)$$

$$= -\frac{1}{4}(\cos 2x + i\sin 2x)\left(x^2 + 2ix - \frac{3}{2}\right).$$

Equating imaginary parts from both sides, we get S.

Thus $S = -\dfrac{1}{4}\left(2x\cos 2x + x^2 \sin 2x - \dfrac{3}{2}\sin 2x\right)$

$$= -\frac{1}{8}\left\{4x\cos 2x + \left(2x^2 - 3\right)\sin 2x\right\}.$$

Therefore the particular integral is

$$-e^{2x}\left\{4x\cos 2x + \left(2x^2 - 3\right)\sin 2x\right\}.$$

Hence the complete solution is

$$y = (A + Bx)e^{2x} - e^{2x}\left\{4x\cos 2x + \left(2x^2 - 3\right)\sin 2x\right\}.$$

Ex. 8. *Solve* : $\dfrac{d^2y}{dx^2} + n^2 y = \sec nx$. [*N. B. H. 1987 ; C. H. 1993*]

The auxiliary equation of the given equation $(D^2 + n^2)y = \sec nx$ is $m^2 + n^2 = 0$, giving $m = \pm in$.

Hence the complementary function is $(A\cos nx + B\sin nx)$, where A, B are arbitrary constants.

The particular integral is

$$\frac{1}{D^2 + n^2}\sec nx = \frac{1}{(D+in)(D-in)}\sec nx$$

$$= \frac{1}{2in}\left(\frac{1}{D-in} - \frac{1}{D+in}\right)\sec nx.$$

Now, $\dfrac{1}{D-in}\sec nx = e^{inx}\displaystyle\int \frac{e^{-inx}}{\cos nx}\,dx$

$$= e^{inx}\int \frac{\cos nx - i\sin nx}{\cos nx}\,dx$$

$$= e^{inx}\left(x + i\cdot\frac{1}{n}\log\cos nx\right).$$

Similarly, $\dfrac{1}{D+in}\sec nx = e^{-inx}\left(x - i\cdot\dfrac{1}{n}\log\cos nx\right).$

Therefore the particular integral is

$$\frac{1}{2in}\left\{e^{inx}\left(x + i\cdot\frac{1}{n}\log\cos nx\right) - e^{-inx}\left(x - i\cdot\frac{1}{n}\log\cos nx\right)\right\}$$

$$= \frac{1}{n}\left(x\sin nx + \frac{1}{n}\cos nx\log\cos nx\right).$$

Hence the complete solution is

$$y = A\cos nx + B\sin nx + \frac{x}{n}\sin nx + \frac{\cos nx}{n^2}\log\cos nx.$$

Examples V (B)

Solve the following equations (1 – 33) :

1. $(D^2 - 4D + 4)y = x^3$.

2. $(D^3 - D^2 - 6D)y = 1 + x^2$. [C. H. 1980]

3. $D^2(D^2 + D + 1)y = x^2$.

4. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{4x}$.

5. $\dfrac{d^2y}{dx^2} + 2a\dfrac{dy}{dx} + (a^2 + b^2)y = e^{px}$.

6. $\dfrac{d^3y}{dx^3} + y = \left(e^x + 1\right)^2$. [C. H. 1982]

7. $(D^3 + 4D^2 + 4D)y = 8e^{-2x}$.

8. $(D^3 + 3D^2 + 3D + 1)y = e^{-x}$.

9. $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x$.

10. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x(x + e^x)$.

11. $(D^3 - 3D^2 + 3D - 1)y = xe^x + e^x$. [B. H. 1981]

12. $(D^3 - 7D - 6)y = e^{2x}(1 + x)$. [V. H. 1987]

13. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x^2 e^{3x}$.

14. $(D-1)^2(D^2+1)^2 y = e^x + x$. [C. H. 1983]

15. $(D^2+a^2)y = \cos ax + \cos bx$.

16. $(D-1)^2(D^2+1)^2 y = \sin^2 \frac{1}{2}x + e^x$.

17. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x \sin x$.

18. $(D^2-1)y = x \sin x + (1+x^2)e^x$.

19. $\dfrac{d^2y}{dx^2} - y = xe^x \sin x$.

20. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x$. [N. B. H. 1986]

21. $(D^3-1)y = x \sin x$. 22. $(D^2-1)y = x \sin x + (1+x^2)e^x$.

23. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^x \cos x$. [V. H. 1988]

24. $\dfrac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$. [V. H. 1992]

25. $\dfrac{d^4y}{dx^4} - y = e^x \cos x$. [B. H. 1991]

26. $\dfrac{d^4y}{dx^4} - y = x \sin x$.

27. $(D^5 - D)y = e^x + \sin x - x$. [C. H. 1981]

28. $(D^5 - D^4 + 2D^3 - 2D^2 + D - 1)y = \cos x$.

29. $(D^4 + 10D^2 + 9)y = 96 \sin 2x \cos x$.

at

$x = -1$.

Answers

1. $y = (C_1 + C_2 x) e^{2x} + \frac{1}{8}(2x^3 + 6x^2 + 9x + 6)$.

2. $y = A + Be^{-x} + Ce^{3x} - \frac{1}{18}x^3 + \frac{1}{36}x^2 - \frac{25}{108}x$.

3. $y = C_1 + C_2 x + \left(C_3 \cos\frac{\sqrt{3}}{2}x + C_4 \sin\frac{\sqrt{3}}{2}x\right)e^{-\frac{1}{2}x} + \frac{1}{12}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2$.

4. $y = Ae^{3x} + Be^{2x} + \frac{1}{2}e^{4x}$.

5. $y = e^{-ax}(A\cos bx + B\sin bx) + \dfrac{e^{px}}{(a+p)^2 + b^2}$.

6. $y = Ae^{-x} + \left(B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right)e^{\frac{1}{2}x} + \frac{1}{9}e^{2x} + e^x + 1$.

7. $y = C_1 + e^{-2x}(C_2 + C_3 x) - 2x^2 e^{-2x}$.

8. $y = (C_1 + C_2 x + C_3 x^2)e^{-x} + \frac{1}{6}e^{-x}x^3$.

9. $y = A + (B + Cx)e^{-x} + \frac{x}{6}(2x^2 - 9x + 24) + \frac{1}{18}e^{2x}$.

10. $y = C_1 e^{2x} + C_2 e^{3x} + \dfrac{18x^2 + 30x + 19}{108} + \dfrac{e^x}{4}(2x + 3)$.

11. $y = (C_1 + C_2 x + C_3 x^2) e^x + \dfrac{1}{24} e^x (x + 1)^4$.

12. $y = Ae^{-x} + Be^{-2x} + Ce^{3x} - \dfrac{1}{12} e^{2x}\left(x + \dfrac{17}{12}\right)$.

13. $y = C_1 e^{2x} + C_2 e^{3x} + e^{3x}\left(\dfrac{1}{3}x^3 - x^2 + 2x\right)$.

14. $y = (C_1 + C_2 x) e^x + (C_3 + C_4 x)\cos x + (C_5 + C_6 x)\sin x$

$$+ \frac{1}{8}x^2 e^x + x + 2.$$

15. $y = A\cos(ax + \beta) + \dfrac{x}{2a}\sin ax + \dfrac{\cos bx}{a^2 - b^2}$.

16. $y = \left(A + Bx + \dfrac{1}{8}x^2\right)e^x + (C + Dx)\sin x + (E + Fx)\cos x$

$$- \frac{1}{32}x^2 \sin x + \frac{1}{2}.$$

17. $y = (C_1 + C_2 x) e^x + \dfrac{1}{2}(\cos x - \sin x + x\cos x)$.

18. $y = C_1 e^x + C_2 e^{-x} + \dfrac{1}{12} x e^x (2x^2 - 3x + 9) - \dfrac{1}{2}(x \sin x + \cos x)$.

19. $y = C_1 e^x + C_2 e^{-x} + \dfrac{1}{25} e^x \{(14 - 5x)\sin x - 2(5x + 1)\cos x\}$.

20. $y = (C_1 + C_2 x) e^x - (x \sin x + 2\cos x) e^x$.

21. $y = C_1 e^x + \left(C_2 \cos \dfrac{\sqrt{3}}{2} x + C_3 \sin \dfrac{\sqrt{3}}{2} x\right) e^{-\frac{1}{2}x}$

$$+ \frac{1}{2}(x\cos x - x\sin x - 3\cos x).$$

22. $y = C_1 e^x + C_2 e^{-x} - \dfrac{1}{2}(x \sin x + \cos x) + \dfrac{1}{12} x e^x (2x^2 - 3x + 9)$.

23. $y = Ae^x \cos(\sqrt{3}\, x + a) + \dfrac{1}{2} e^x \cos x$.

24. $y = A\cos(\sqrt{2}\, x + a) + \dfrac{e^{3x}}{121}\left(11x^2 - 12x + \dfrac{50}{11}\right) + \dfrac{e^x}{17}(4\sin 2x - \cos 2x)$.

25. $y = C_1 e^x + C_2 e^{-x} + C_3 \sin(x + a) - \dfrac{1}{5} e^x \cos x$.

26. $y = Ae^{x} + Be^{-x} + (C\cos x + D\sin x) + \frac{1}{8}(x^2\cos x - 3x\sin x)$.

27. $y = C_1 + C_2 e^{x} + C_3 e^{-x} + C_4\cos x + C_5\sin x + \frac{1}{4}xe^{x} + \frac{1}{4}x\sin x + \frac{1}{2}x^2$.

28. $y = C_1 e^{x} + (C_2 + C_3 x)\cos x + (C_4 + C_5 x)\sin x + \frac{x^2}{16}(\cos x - \sin x)$.

29. $y = A\cos(x - \alpha) + B\cos(3x - \beta) - 3x\cos x + x\cos 3x$.