Theorem 6.2.2 (Chebyshev's inequality) If X is a random variable with mean μ and finite variance σ^2, then for any $k > 0$,

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Proof Assume that X is a continuous random variable. Then

$$E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$$= \int_{|x-\mu| \geq k} (x-\mu)^2 f_X(x)\,dx + \int_{|x-\mu| < k} (x-\mu)^2 f_X(x)\,dx$$

$$\geq \int_{|x-\mu| \geq k} (x-\mu)^2 f_X(x)\,dx \geq k^2 \int_{|x-\mu| \geq k} f_X(x)\,dx = k^2 P(|X-\mu| \geq k)$$

i.e.

$$P(|X-\mu| \geq k) \leq \frac{1}{k^2} E[(X-\mu)^2]$$

i.e.

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

When X is a discrete random variable,

$$E[(X-\mu)^2] = \sum_i (x_i-\mu)^2 P(X = x_i)$$

$$\geq \sum_{|x_i-\mu| \geq k} (x_i-\mu)^2 P(X = x_i)$$

$$\geq k^2 \sum_{|x_i-\mu| \geq k} P(X = x_i) = k^2 P(|X-\mu| \geq k)$$

So,

$$P(|X-\mu| \geq k) \leq \frac{1}{k^2} E[(X-\mu)^2]$$

i.e.

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Note Chebyshev's inequality can also be written in the equivalent form of

$$P(|X-\mu| < k) \geq 1 - \frac{\sigma^2}{k^2}$$

Remark 6.2.1 Ordinarily, when we obtain the probability of an event described by a random variable, its distribution or density is required. Chebyshev's inequality gives a bound for the probability of a particular event which does not depend on the distribution of the random variable, except for its mean and variance.

The following example clarifies the situation.

Example 6.2.2 Let X be a random variable with the distribution given by

$$P(X = k) = 2^{-k} \quad k = 1, 2, \ldots$$

$$E(X) = \sum_{k=1}^{\infty} k2^{-k} = 1 \times \frac{1}{2} + 2 \times \frac{1}{2^2} + 3 \times \frac{1}{2^3} + \ldots = 2$$

$$E(X^2) = \sum_{k=1}^{\infty} k^2 2^{-k} = 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{2^2} + \ldots = 6$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 2$$

Hence by Chebyshev's inequality,

$$P(|X-2| > 2) \leq \frac{\text{Var}(X)}{2^2} = \frac{1}{2}$$

Thus Chebyshev's inequality only gives upper bound 1/2 for this probability. Also, we see that

$$P(|X-2| > 2) = 1 - P(0 \leq X \leq 4)$$

$$= 1 - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right) = \frac{1}{16} = 2^{-4}$$

Since the actual probability is 2^{-4}, which is far less than 1/2, we conclude that Chebyshev's inequality gives a rough bound for the probability of the event in question.

possible parameters.

Example 6.2.4 If a die is thrown 3,600 times, show that the probability that the number of sixes lies between 550 and 650 is at least 4/5.

Solution Let X denote the number of sixes. Clearly X is a binomial $b(n, p)$ random variable with $n = 3{,}600$ and $p = 1/6$. So

$$E(X) = np = 600$$

$$\text{Var}(X) = npq = 3600 \times \frac{1}{6} \times \frac{5}{6} = 500$$

Hence, by Chebyshev's inequality

$$P(|X - 600| < 50) \geq 1 - \frac{\text{Var}(X)}{50^2} = 1 - \frac{1}{5} = \frac{4}{5}$$

i.e.

$$P(550 < X < 650) \geq \frac{4}{5}$$