Exam Roll No. - CSE 214021

Class Roll No. - 00191 0501061

BCSE 2nd year, 2nd semester, 2020-21, Math-iv, part-2

part-2

1.

a) Chebyshev's inequality :-

If x is a random variable with finite mean μ and variance σ^2, then, for any value $k>0$

$$P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

b) Bayes' theorem -

Bayes's theorem is stated as the following equation:

$$P(A/B) = \frac{P(B/A) \cdot P(A)}{P(B)}$$

where A and B are events and $P(B) \neq 0$

it may be derived from the defination of conditional probability:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \text{ if } P(B) \neq 0$$

$$= \frac{P(B \cap A)}{P(B)}$$

general form :- Let $A_1, A_2, \ldots A_n$ be n pairwise mutually exclusive and exhaustive events connected with the random experiment E.

Let X be an arbitraty event connected with E, where $P(X) \neq 0$, Also, let the probablities $P(X/A_1), P(X/A_2)$, ... $P(X/A_n)$ be all known. Then

$$P(A_i/X) = \frac{P(A_i) \cdot P(X/A_i)}{\sum_{p=1}^{n} P(A_p) \cdot P(X/A_p)}, i = 1, 2, 3$$

c) we say that X is a uniform random variable on the interval (α, β) if the probablity density function of X is given by-

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

2.

a) Number of ways of choosing 3 tickets from 20 tickits (numbered through $1, 2, \ldots, 20$) $= {}^{20}C_3$

Let, x and y be 2 integers

then, arithmatic mean between them (AM) is given by $AM = \frac{x+y}{2}$

$\therefore$ AM must be even. So, summation of 2 integers is even iff both are even or both are odd.

Number of ways of choosing 2 even number or two odd numbers from $\{1, 2, \ldots, 20\}$

$$= {}^{10}C_2 + {}^{10}C_2 = 2 \cdot {}^{10}C_2$$

$\therefore$ Required probablity $= \frac{2 \cdot {}^{10}C_2}{{}^{20}C_3} \approx 0.0789$

b).

Number of white balls = N_1

Number of black balls = N_2

Let, number of black balls preceeding the first white ball be K and the required probablity be $P(K)$

For $K=1$

$$P(1) = \frac{N_2}{N_1+N_2} \times \frac{N_1}{(N_1+N_2)-1}$$

For $K=2$

$P(2) = $ (Black) (Black) (white)

$$= \frac{N_2}{N_1+N_2} \cdot \frac{N_2-1}{N_1+N_2-1} \cdot \frac{N_1}{N_1+N_2-2}$$

For $K = i$

$$p(i) = \frac{N_2}{N_1+N_2} \cdot \frac{N_2-1}{N_1+N_2-1} \cdots \frac{N_2-i+1}{N_1+N_2-i+1} \cdot \frac{N_1}{N_1+N_2-i}$$

$$= \frac{{}^{N_2}C_i}{{}^{(N_1+N_2)}C_i} \cdot \frac{N_1}{N_1+N_2-i}$$

e) Total number of outcomes possible for (a,b) pair (where $1 \le a \le 6 \times 1 \le b \le 6$)

$= 36 \ (6\times 6)$

Now, the required probablity is for getting atleast once a double six $(6,6)$,

It can be found by computing

1 − (No. of times no double six occurs)

probablity of getting a double 6 (6, 6)

$= \frac{1}{36}$

probablity of ~~getting~~ not getting a double

six (6, 6) $= 1 - \frac{1}{36} = \frac{35}{36}$

Total no. of throws $= 100$

probablity of no occurence of a double

6 in 100 throws $= \left(\frac{35}{36}\right)^{100}$

$\therefore$ Required probablity $= 1 - \left(\frac{35}{36}\right)^{100}$

3) a)

Joint density of two random variable X and Y are given by,

$$f(x, y) = \begin{cases} e^{-(x+y)} & 0 < x < \alpha, 0 < y < \alpha \\ 0 & \text{otherwise} \end{cases}$$

Let density function of the random variable $\frac{X}{Y}$; $f_{X/Y}(a)$

$F_{X/Y}(a) = P\{\frac{X}{Y} \leq a\}$

$$= \iint\limits_{(X,Y):\, \frac{x}{y} \leq a} e^{-(x+y)}\, dx\, dy$$

$$= \int_0^{\alpha} \int_0^{ay} e^{-(x+y)}\, dx\, dy$$

$$= \cancel{\int_0^{\alpha} \left[\int_0^{ay} e^{x}/dx\right] e^{y}/dy}$$

$$\cancel{= \int_0^{\alpha} e^{ay}/e^{y}\, dy} = \cancel{\int_0^{\alpha} e^{(a+1)y}\, dy}$$

$$= \int_0^\infty \int_0^{ay} (e^{-x-y})\, dx\, dy$$

$$= \int_0^\infty [\int_0^{ay} e^{-x} dx]\, e^{-y} dy$$

$$= \int_0^\infty [-e^{-x}]_0^{ay}\, e^{-y} dy$$

$$= \int_0^\infty (1-e^{-ay})\, e^{-y} dy$$

$$= \int_0^\infty [e^{-y} - e^{-(a+1)y}]\, dy$$

$$= [-e^{-y} + \frac{e^{-(a+1)y}}{(a+1)}]_0^\infty$$

$$= [-e^{-y}]_0^\infty + [\frac{e^{-(a+1)y}}{a+1}]_0^\infty$$

$$= 1 + \frac{0-1}{a+1} = 1 - \frac{1}{a+1}$$

Now, to calculate the density function of $\frac{X}{Y}$, we differenciate (i), which gives, $f_{X/Y}(a) = \frac{1}{(a+1)^2}$ $0<a<\infty$

b)

(i) from Markov's Inequality we can say,

If x is a random variable that takes only non-negetive values, then, for any value $a>0$,

$$p\{x \geq a\} \leq \frac{E(x)}{a}$$

Here, number of ~~tod~~ item produced during a week is a random variable with mean 50.

So, probablity that this weak's production exceeds 75 =>

$$P(x \geq 75) \leq \frac{E(x)}{75}$$

=> Here $E(x) = 50$

$$\Rightarrow P(x \geq 75) \leq \frac{50}{75} = \frac{2}{3}$$

(ii)

Variance (σ^2) of a week's production is 25,

by chebyshev's Inequality we can say,

If X is a random variable with finite mean μ and variance σ^2, then for any value $k > 0$

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

So, Here $k = 60 - 50 = 50 - 40 = 10$ and $\sigma^2 = 25$ and $\mu = 50$

∴ probablity that this week's production will be between 40 and 60 => $P\{|X - 50| < 10\}$

~~$1 - P\{|X - 50| \geq 10\}$~~

Here, by chebyshev's inequality,

$$P\{|X - 50| \geq 10\} \leq \frac{\sigma^2}{10^2} = \frac{25}{100} = \frac{1}{4}$$

$$\therefore P\{|X - 50| < 10\} \geq 1 - P\{|X - 50| \geq 10\} = 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

4.

a)

Champman- Kolmogorov eauation

$$P_{ij} = \sum_{k=0}^{M} p_{ik}^{(p)} \, p_{kj}^{(n-p)} \quad \text{for all } 0<p<n$$

proof:

$$p_{ij}^{(n)} = p\{X_n=j \mid X_0=i\} = \sum p\{X_n=j, X_p=k \mid X_0=i\}$$

$$= \sum_{k} p\{X_n=j \mid X_p=k, X_0=i\} \cdot p\{X_p=k \mid X_0=i\}$$

$$= \sum_{k} p_{kj}^{n-p} \cdot p_{ki}^{p}$$

For a large number of Markov chains it turns out that $p_{ij}^{(n)}$ converges to a value π_j as $p\to\alpha$. π_j depends only on j. ie; for large 'n', the prob of being in state j, after n transitions is approximately eaual to π_j, no matter what the initial state was

The sufficient condition for a Markov chain to possess the above property is that for some $n>0$, $P_{ij}^{(n)}>0$ for all $i=0,1,\dots M$ → (1)

Markov chains satisfying (1), is said to be 'ergodic'

Now, (1) yields

$$P_{ij}^{(n+1)} = \sum_{k=0}^{M} p_{ik}^{(n)} \, p_{kj} \quad \text{— (2)}$$

it follows by letting $n\to\alpha$ for ergodic chains

$$\pi_j = \sum_{k=0}^{M} \pi_k \, p_{kj} \quad \text{— (3)}$$

Furthermore, since $1 = \sum_{j=0}^{M} P_{ij}^{(n)}$, we also obtain, by letting $n \to \alpha$ $\sum_{j=0}^{M} \pi_j = 1$ — (4)

it can be shown that $\pi_j, 0 \leq j \leq M$ are unique non-negative solutions of eqn (3) and (4)

Theorm :-

~~For an ergodic Markov~~

(i) For an erodic Markov chain,

$\pi_j = \lim_{n \to \alpha} P_{ij}^{(n)}$ exist and π_j, $0 \leq j \leq M$ are unique non-negative solution of

$$\pi_j = \sum_{k=0}^{M} \pi_k P_{kj}$$

$$\sum_{j=0}^{M} \pi_j = 1$$

from this ~~equation~~ theorm, the limiting probablities π_0 and π_1 of pain and no pain are given by,

$$\pi_0 = \alpha \pi_0 + \beta \pi_1$$

$$\pi_1 = (1-\alpha)\pi_0 + (1-\beta)\pi_1$$

$$\pi_0 + \pi_1 = 1$$

which yields,

$$\pi_0 = \frac{\beta}{1+\beta-\alpha}, \quad \pi_1 = \frac{1-\alpha}{1+\beta-\alpha}$$

~~for~~ ~~now~~ no, ~~a~~given $\alpha = 0.6$ and $\beta = 0.3$

then the limiting problality of pain on the nth day is $\pi_0 = \frac{0.3}{1+0.3-0.6} = \frac{3}{7}$

$$\pi_1 = \frac{1-0.6}{1+0.3-0.6} = \frac{4}{7}$$

e)

absorbing and transient states for finite Markov chain :-

A state of a Markov chain is called an absorbing state, if once the Markov chain enters the state, it remains there forever.

i.e., $P_{kk} = 1$ and $P_{kj} = 0$ for $j \neq k$ and $0 \leq k \leq M$

A state is called transient if the system, starts from ~~that~~ that particular state and have zero probablity of returning to the same state.