

# Graph Theory - Lecture 7

## Planar Graphs

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### 1 Basics

Case Study (GAS-WATER-ELECTRICITY PROBLEM) ;

Three sworn enemies **A-B-C** want to share Gas(G), Water(W) and Electricity(E). To avoid confrontations, the paths of the supplies should not be crossed.

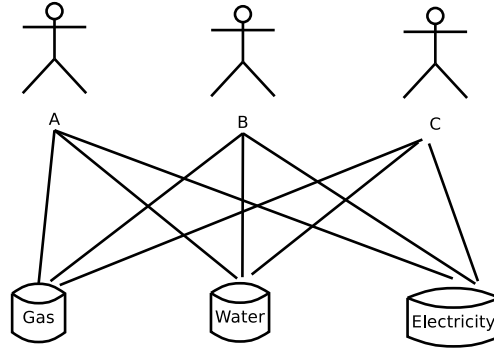


Figure 1: Gas-Water-Electricity Problem

Let  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  be a graph and let  $\mathbf{S}$  be any surface (e.g. plane, sphere).

Let  $P = \{p_1, p_2, \dots, p_{|V(G)|}\}$  be a set of  $|V(G)|$ -distinct points of  $\mathbf{S}$ ,  $p_i$  corresponding to  $v_i \in V(G)$ . If  $e_i = v_j v_k$ , draw an *arc*,  $\mathbf{J}_i$  on  $\mathbf{S}$  from  $p_i$  to  $p_k$  such that  $\mathbf{J}_i$  does not pass through any other  $p_i$ . Then  $P \cup \{\mathbf{J}_1, \mathbf{J}_2, \dots, \mathbf{J}_m\}$  is called a *drawing* of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  on  $\mathbf{S}$  or a diagram representing  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  on  $\mathbf{S}$ .

**Definition 1.1.** A *drawing* of a graph  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is a function defined on  $V(G) \cup E(G)$  that assigns each vertex  $v_i$  a point  $f(v_i)$  in a plane and assigns each edge with endpoints  $v_i, v_j$  a polygonal  $f(v_i)f(v_j)$ -curve; the images of the vertices being distinct.

A point in  $f(e_{ij}) \cap f(e_{kl})$  that is not a common-point is called a crossing

**Definition 1.2.** A graph  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is **planar** if it has a drawing without crossings.

Such a drawing is a **planar embedding** of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ . A *plane graph* is particular planar embedding of planar graph.

A curve is closed if its first and last points are same. It is simple if it has no repeated points except first and last points.

**Definition 1.3.** A **open set** in the plane is a set  $U \subset \mathbb{R}^2$  such that for every  $p_i \in U$ , all points within some small distance from  $p_i$  belong to  $U$ .

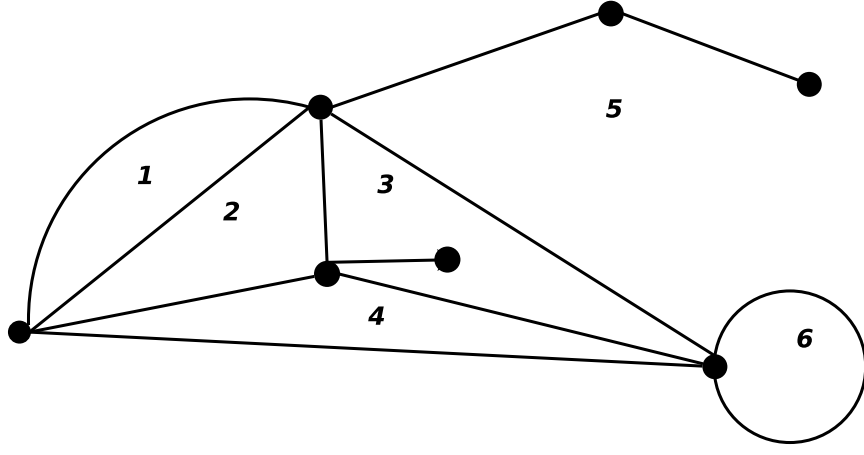


Figure 2: Planar representation of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  , where  $\#$  represents the regions

**Definition 1.4.** A **region** is an open set  $U$  that contains a polygonal  $v_i v_j$ -curve for every pair  $v_i, v_j \in U$ .

**Definition 1.5.** The **face (f)** of a plane graph are the maximal regions of the plane that contain no other point used in the embedding.

- A **plane** representation of a graph divides the plane into *regions* also called *windows*, *faces* and *meshes*.
- A *region* is characterised by the set of edges (or the set of vertices) forming its boundary.
- A *region* **is not** defined in a non-planar graph, or even in a planar graph not embedded in a plane.
- ★★ A **region** is a property of the specific plane representation of a graph and not of an abstract graph
- A finite plane graph,  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  has **one unbounded face** (also called outer face).
- The faces are pairwise disjoint.
- Points  $p_i, p_j \in \mathbb{R}^2$  lying in no edge of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  are in the same face iff there is a polygonal  $p_i p_j$ -curve that crosses no edge.

## 2 Euler's Theorem

Euler's Theorem gives the relation between the number of vertices  $|V(G)|$ , number of edges  $|E(G)|$ , the number of faces of the planar graph,  $|F(G)|$  and the number of components,  $K$

**Theorem 2.1.** If a graph  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is a planar graph with  $|V(G)|$  vertices,  $|E(G)|$  edges,  $|F(G)|$  faces and  $K$  components, then

$$\boxed{|V(G)| - |E(G)| + |F(G)| = K + 1} \quad (1)$$

**Theorem 2.2.** In a connected planar graph with  $|V(G)|$  vertices,  $|E(G)|$  edges,  $|F(G)|$  faces

$$\boxed{|V(G)| - |E(G)| + |F(G)| = 2} \quad (2)$$

**Proof 1 of Theorem 2.2 .** Without loss of generality, assume that the planar graph is simple. Since, any simple planar graph can have a plane representation such that each edge is a straight line, any planar graph can be drawn such that each region is a polygon (a polygon net). Let the polygon net representing the given graph consists of  $|F(G)|$  regions. Let  $k_p$  be the number of  $p$ -sided regions. Since, each edge is on the boundary of exactly two regions,

$$3k_3 + 4k_4 + 5k_5 + \dots + rk_r = 2|E(G)| \quad (3)$$

where  $k_r$  is the number of polygons with  $r$  edges.

Also,

$$k_3 + k_4 + k_5 + \dots + k_r = |F(G)| \quad (4)$$

The sum of all angles subtended at each vertex in the polygon net is

$$2|V(G)|\pi \quad (5)$$

Now, the sum of all interior angles of a  $p$ -sided polygon is  $\pi(p-2)$  and the sum of the exterior angles is  $\pi(p+2)$ .

Equation (5) is the total sum of all interior angles of  $|F(G)| - 1$  finite regions plus the sum of the exterior angles of the polygon defining the infinite regions. This sum is

$$\begin{aligned} & \pi(3-2)k_3 + \pi(4-2)k_4 + \dots + \pi(r-2)k_r + 4\pi \\ &= \pi[3k_3 + 4k_4 + \dots + rk_r + 2(k_3 + k_4 + \dots + k_r)] + 4\pi \\ &= \pi(2|E(G)| - 2|F(G)|) + 4\pi \\ &= 2\pi(|E(G)| - |F(G)| + 2) \end{aligned} \quad (6)$$

Equating equations (5), (6), we get

$$2\pi[|E(G)| - |F(G)| + 2] = 2\pi|V(G)| \quad (7)$$

so that  $|F(G)| = |E(G)| - |V(G)| + 2$

□

**Definition 2.1** (Region of a Planar Graph). Let  $\phi$  be a region of a planar graph  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ . We define the degree of  $\phi$ , denoted by  $d(\phi)$ , as the number of edges of the boundary of  $\phi$ .

**Proof 2 of Theorem 2.2.** We use *induction* on the number of edges of the planar graph,  $|E(G)|$ .

If  $|E(G)| = 0$ , then  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is  $K_1$ , a complete graph with 1 vertex and 1 region. So,  $|V(G)| - |E(G)| + |F(G)| = 1 - 0 + 1 = 2$ , and the result is *TRUE*.

If  $|E(G)| = 1$ , then the number of vertices in  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is either *one* or *two*, the first possibility occurring when the edge is a *loop*. These two possibilities give rise respectively to two regions and one region.

Therefore,

$$|V(G)| - |E(G)| + |F(G)| = \begin{cases} 1 - 1 + 2 & \text{in the loop case} \\ 2 - 1 + 1 & \text{in the non-loop case} \end{cases}$$

Thus,  $|V(G)| - |E(G)| + |F(G)| = 2$  is *TRUE*.

Assume that the result is true for all connected planar graphs with fewer than  $|E(G)|$  edges. Let  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  have  $|E(G)|$  edges.

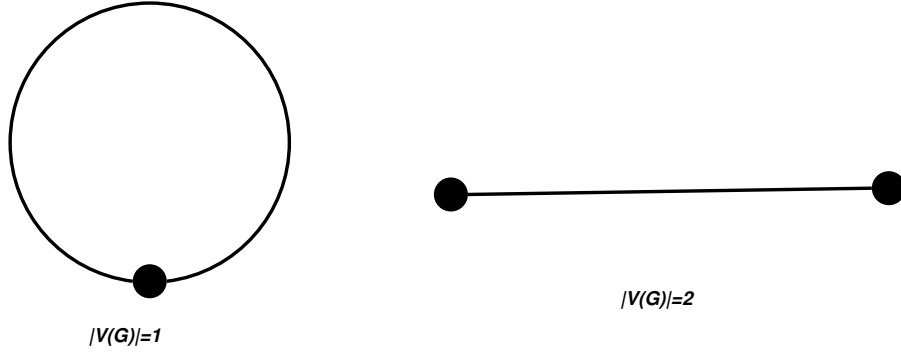


Figure 3: Possibilities for one or two regions

**Case 1 :** Suppose  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is a *tree*. Then  $|E(G)| = |V(G)| - 1$  and  $|F(G)| = 1$ , because a planar representation of a tree has one region. Thus,  $|V(G)| - |E(G)| + |F(G)| = |V(G)| - (|V(G)| - 1) + 1 = 2$ , and the result holds.

**Case 2 :** Suppose  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is not a *tree*. Then  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  has cycles. Let  $C(G)$  be a cycle in  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ . Let  $e \in C(G)$  be an edge of the cycle  $C(G)$ . The graph  $\mathbf{G}(\mathbf{V}, \mathbf{E}) - e$  has one edge less than the graph  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ . Also the number of vertices in  $V(G - e)$  and  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  are same. Since, removing  $e$  coalesces two regions in  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  into one in  $\mathbf{G}(\mathbf{V}, \mathbf{E}) - e$ , therefore  $\mathbf{G}(\mathbf{V}, \mathbf{E}) - e$  has one region less than in  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ .

Thus, by induction hypothesis in  $\mathbf{G}(\mathbf{V}, \mathbf{E}) - e$ , we have  $|V(G)| - (E(G) - 1) + (F(G) - 1) = 2$ , so that  $|V(G)| - |E(G)| + |F(G)| = 2$

Hence, the result.

□

### 3 Upper Bound for the Edges of a Simple Planar Graph

**Theorem 3.1.** Let graph  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  be a simple planar graph with  $|\mathbf{V}(\mathbf{G})|$  vertices and  $|\mathbf{E}(\mathbf{G})|$  edges. where  $|\mathbf{V}(\mathbf{G})| \geq 3$ . Then  $|\mathbf{E}(\mathbf{G})| \leq 3|\mathbf{V}(\mathbf{G})| - 6$

*Proof.* First, assume that the planar graph  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is connected.

If  $|\mathbf{V}(\mathbf{G})| = 3$ , then since  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is simple. therefore  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  has at most three edges. Thus,  $|\mathbf{E}(\mathbf{G})| \leq 3$ , i.e.  $|\mathbf{E}(\mathbf{G})| \leq 3 \times 3 - 6$ , and the result is *TRUE*.

Now, let  $|\mathbf{V}(\mathbf{G})| = 4$ .

First, let  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  be a tree so that  $|\mathbf{E}(\mathbf{G})| = |\mathbf{V}(\mathbf{G})| - 1$ . Since,  $|\mathbf{V}(\mathbf{G})| \geq 4$ , obviously we have  $|\mathbf{V}(\mathbf{G})| - 1 \leq 3|\mathbf{V}(\mathbf{G})| - 6$ .

Now, let  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  be not a tree. Then,  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  has cycles. Clearly, there is a cycle in  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ , all of whose edges lie on the boundary of the exterior region of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ . Then, since  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is simple, we have  $d(\phi) \geq 3$ , for each region  $\phi$  of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ .

Let

$$b = \sum_{\phi \in F(G)} d(\phi)$$

, where  $F(G)$  denotes the set of all regions of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ .

Since, each region has at least three edges on its boundary, therefore we have  $b \geq 3|F(G)|$ , where  $|F(G)|$  is the number of regions of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ . However, when we sum to get  $b$ , each edge of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is counted either once or twice (twice when it occurs as a boundary edge for two regions).

Therefore,  $b \leq 2|\mathbf{V}(\mathbf{G})|$ , so that  $3|F(G)| \leq b \leq 2|\mathbf{V}(\mathbf{G})|$ . In particular, we have  $3|F(G)| \leq 2|\mathbf{E}(\mathbf{G})|$ .

Using Euler's formula,  $|\mathbf{V}(\mathbf{G})| - |\mathbf{E}(\mathbf{G})| + |F(G)|$ , we get  $|\mathbf{E}(\mathbf{G})| \leq 3|\mathbf{V}(\mathbf{G})| - 6$ .

Let graph  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  be not connected and let  $G_1, G_2, \dots, G_t$  be its connected components. For each  $i, 1 \leq i \leq t$ , let  $|V(G_i)|$  and  $|E(G_i)|$  denote the number of vertices and edges in  $G_i$ . Since each  $G_i$  is a planar simple graph, by the above argument, we have  $|E(G_i)| \leq 3|V(G_i)| - 6$ , for each  $i, 1 \leq i \leq t$ .

$$\text{Also, } |\mathbf{V}(\mathbf{G})| = \sum_{i=1}^t |V(G_i)| \text{ and } |\mathbf{E}(\mathbf{G})| = \sum_{i=1}^t |E(G_i)|.$$

Hence,

$$|\mathbf{E}(\mathbf{G})| = \sum_{i=1}^t |V(G_i)| \leq \sum_{i=1}^t (3|V(G_i)| - 6) = 3 \sum_{i=1}^t |V(G_i)| - 6t \leq 3|\mathbf{V}(\mathbf{G})| - 6$$

□

## 4 Existence of a vertex of degree less than six in a simple planar graph

**Theorem 4.1.** *If graph  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is a simple planar graph, then  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  has a vertex  $v$  of degree less than 6*

*Proof.* If  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  has only one vertex, then this vertex has degree zero. If  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  has only two vertices, then both vertices have degree at most one.

Let  $|\mathbf{V}(\mathbf{G})| \geq 3$ . Assume degree of every vertex in  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is at least six.

$$\text{Then, } \sum_{v \in V(G)} d(v) \geq 6|\mathbf{V}(\mathbf{G})|$$

$$\text{We know, } \sum_{v \in V(G)} d(v) \geq 2|\mathbf{E}(\mathbf{G})|$$

$$\text{Thus, } 2|\mathbf{E}(\mathbf{G})| \geq 6|\mathbf{V}(\mathbf{G})|, \text{ so that } |\mathbf{E}(\mathbf{G})| \geq 3|\mathbf{V}(\mathbf{G})|.$$

But, this is not possible, because by Theorem 3.1, we have  $|\mathbf{E}(\mathbf{G})| \leq 3|\mathbf{V}(\mathbf{G})| - 6$ . Thus, we get a contradiction. Hence,  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  has at least one vertex of degree less than 6. □

**Corollary 4.2.**  $K_5$  is non-planar.

*Proof.* Here,  $|\mathbf{V}(\mathbf{G})| = 5$  and  $|\mathbf{E}(\mathbf{G})| = 10$ .

$$\text{So, } 3|\mathbf{V}(\mathbf{G})| - 6 = 15 - 6 = 9.$$

$$\text{Thus, } |\mathbf{E}(\mathbf{G})| > 3|\mathbf{V}(\mathbf{G})| - 6.$$

Therefore,  $K_5$  is non-planar. □

**Corollary 4.3.**  $K_{3,3}$  is non-planar

*Proof.* Since,  $K - 3, 3$  is bipartite, it contains no odd-cycles., and so cycle of length **three** exists. It follows that every region of a plane drawing of  $K_{3,3}$  if it exists, has at least **four** boundary edges. We have, thus,  $d(\phi) \geq 4$ . for each region  $\phi$  of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ .

Let  $b = \sum_{\phi \in F(G)} d(\phi)$ , where  $F(G)$  denotes the set of all regions of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ . Since, each region has at least **four** edges on its boundary, we have  $b \geq 4|F(G)|$ .

Now, when we sum up to get  $b$ , each edge of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  is counted either once or twice, and so,  $b \leq 2|\mathbf{E}(\mathbf{G})|$ . Thus,  $4|F(G)| \leq b \leq 2|\mathbf{E}(\mathbf{G})|$ , so that  $2|F(G)| \leq |\mathbf{E}(\mathbf{G})|$ .

For,  $K_{3,3}$ , we have  $|\mathbf{E}(\mathbf{G})| = 9$ , and so  $2|F(G)| \leq 9$  giving  $|F(G)| \leq \frac{9}{2}$ . But, by Euler's formula,  $|F(G)| = |\mathbf{E}(\mathbf{G})| - |\mathbf{V}(\mathbf{G})| + 2 = 9 - 6 + 2 = 5$ , a contradiction.

Hence,  $K_{3,3}$  is non-planar. □

## 5 Kuratowski's Graphs

**Definition 5.1.** The complete graphs  $K_5$  and  $K_{3,3}$  are non-planar graphs, which are referred to as Kuratowski's Graphs.

### Observations

1. Both are *regular*
2. Both are non-planar
3.  $K_5$  is a non-planar complete graph with the **smallest number of vertices** and  $K_{3,3}$  is the non-planar bipartite graph with **smallest number of edges**

The following result is used in proving Kuratowski's theorem.

### Theorem 5.1.

1. If  $\mathbf{G}(\mathbf{V}, \mathbf{E}) \setminus e$  contains a subdivision of  $K_5$ , then  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  contains a subdivision of  $K_5$  or  $K_{3,3}$ .
2. If  $\mathbf{G}(\mathbf{V}, \mathbf{E}) \setminus e$  contains a subdivision of  $K_{3,3}$ , then  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  contains a subdivision of  $K_{3,3}$ .

*Proof.* Let  $\mathbf{G}(\mathbf{V}, \mathbf{E})' = \mathbf{G}(\mathbf{V}, \mathbf{E}) \setminus e$  be a graph obtained by contracting the edge  $e = v_x v_y$  of  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ . Let  $v_w$  be the vertex of  $\mathbf{G}(\mathbf{V}, \mathbf{E})'$  obtained by contracting  $e = v_x v_y$ .

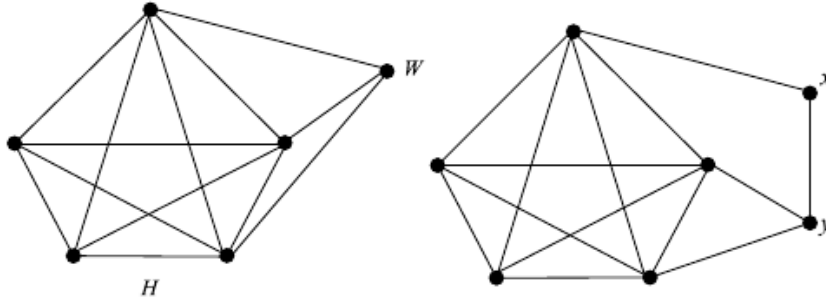


Figure 4:

1. Let  $\mathbf{G}(\mathbf{V}, \mathbf{E}) \setminus e$  contains a subdivision of  $K_5$ , say  $H(V, E)$ . If  $v_w$  is not a branch vertex of  $H(V, E)$ , then  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  also contains a subdivision of  $K_5$ , obtained by expanding  $v_w$  back into the edge  $v_x v_y$ , if necessary (Figure 4).

Assume  $v_w$  is a branch vertex (inner vertex of a tree) of  $H(V, E)$  and each of  $v_x, v_y$  is incident in  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  to two of the four edges incident to  $v_w$  in  $H(V, E)$ . Let  $u_1$  and  $u_2$  be the branch vertices of  $H(V, E)$  that are at the other ends of the paths leaving  $v_w$  on edges incident to  $v_x$  in  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ . Let  $v_1, v_2$  be the branch vertices of  $H(V, E)$  that are at the other ends of the paths leaving  $v_w$  on edges incident to  $v_y$  in  $\mathbf{G}(\mathbf{V}, \mathbf{E})$  (Figure 5)

By deleting the  $u_1 - u_2$  path and  $v_1 - v_2$  path from  $H(V, E)$ , we obtain a subdivision of  $K_{3,3}$  in  $\mathbf{G}(\mathbf{V}, \mathbf{E})$ , in which  $v_y, u_1, u_2$  are branch vertices for one partite set, and  $v_x, v_1, v_2$  are branch vertices of the other.

□

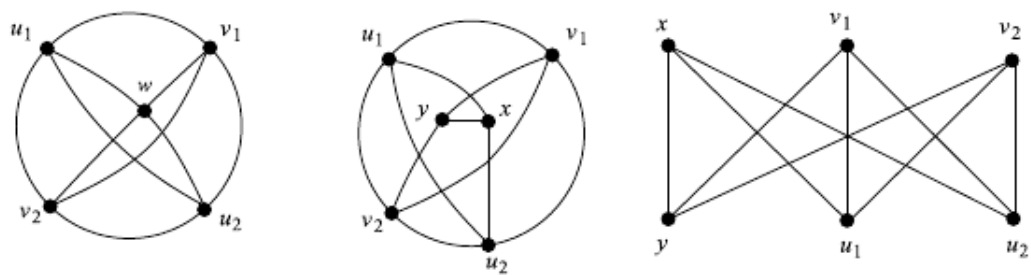


Figure 5: