Some properties of Rational numbers and Real numbers

1 Preliminaries

Definition 1.1. A binary relation ρ on a nonempty set S is called a *partial order relation* if

1. $a\rho a$ for all $a \in S$ (reflexivity),
2. $a, b \in S$, $a\rho b$ and $b\rho a$ imply $a = b$ (antisymmetry),
3. $a, b, c \in S$, $a\rho b$ and $b\rho c$ imply $a\rho c$ (transitivity).

Definition 1.2. A nonempty set S together with a partial ordering ρ is called a *partially ordered* set. A partially ordered set (S, ρ) is called *totally ordered* if for any $a, b \in S$, either $a\rho b$ or $b\rho a$ or both (in which case $a = b$). A totally ordered set (S, ρ) is called *well-ordered* if every nonempty subset of S has a smallest element.

Definition 1.3. Let $(S, \leq)$ be a partially ordered set (please do not confuse $\leq$ with its usual meaning in $\mathbb{R}$, here $\leq$ is just a replacement for ρ above for convenience). Let A be a nonempty subset of S. An element $x \in S$ is called an *upper bound* of A if $a \leq x$ for all $a \in A$. An upper bound s of A is called the *least upper bound* (in brief, *lub*) or *supremum* (in brief, *sup*) if $s \leq x$ for any upper bound x of A. An element $y \in S$ is called a *lower bound* of A if $y \leq a$ for all $a \in A$. A lower bound t of A is called the *greatest lower bound* (in brief, *glb*) or *infimum* (in brief, *inf*) if $y \leq t$ for any lower bound y of A. An lub (or glb), if it exists, is unique by definition. In this context, we just mention, a partially ordered set is called a *lattice* if there exist lub and glb for every pair of elements in the set. Let A be a nonempty subset of S. If A has an upper bound in S, then A called *bounded above* and if A has a lower bound in S, then A is called *bounded below.* A is called *bounded* if it is both bounded above and bounded below. A totally ordered set $(S, \leq)$ is said to have the *lub property* if every subset A of S which is bounded above has an lub (in S). Similarly, a totally ordered set $(S, \leq)$ is said to have the *glb property* if every subset A of S which is bounded below has a glb (in S).

Definition 1.4. A nonempty set F together with two binary operation $+$ and $\cdot$ is called a *field* if the following conditions are satisfied ($+$ and $\cdot$ are binary operations on F imply that $a+b, a\cdot b \in F$ for all $a, b \in F$):

1. $a + b = b + a$ for all $a, b \in F$,
2. $a + (b + c) = (a + b) + c$ for all $a, b, c \in F$,
3. there exists $0 \in F$ such that $a + 0 = a$ for all $a \in F$,

4. for each $a \in F$, there exists $-a \in F$ such that $a + (-a) = 0$,
5. $a \cdot b = b \cdot a$ for all $a, b \in F$,
6. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in F$,
7. there exists $1 \in F$ such that $a \cdot 1 = a$ for all $a \in F$,
8. for each $0 \neq a \in F$, there exists $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$,
9. $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$.

Notations: We denote the sets of natural numbers, integers, rational numbers and real numbers by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively.

Here we assume the constructions of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ and the following properties:

1. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are totally ordered with usual $\leq$.
2. $\mathbb{N}$ is well-ordered.
3. $(\mathbb{N}, +, \cdot)$ is an additively and multiplicatively commutative *semiring* with multiplicative identity (satisfying 1,2,5,6,7,9 of Definition 1.4).
4. $(\mathbb{Z}, +, \cdot)$ is a commutative *ring* with identity (satisfying 1,2,3,4,5,6,7,9 of Definition 1.4).
5. $(\mathbb{Q}, +, \cdot)$ and $(\mathbb{R}, +, \cdot)$ are fields.
6. $\mathbb{R}$ is a totally ordered field with lub property.

2 Properties of rational numbers

Theorem 2.1. $\mathbb{Q}$ *is* densely ordered*, i.e., between any two rational numbers, there exist infinite number of rational numbers.*

Hint: $a < \frac{a+b}{2} < b$, $a < \frac{a+\frac{a+b}{2}}{2} < \frac{a+b}{2}$ and so on.

Theorem 2.2. $\mathbb{Q}$ *is* archimedian ordered*, i.e., for any $a, b \in \mathbb{Q}$ with $a, b > 0$, there exists $n \in \mathbb{N}$ such that $nb > a$.*

Proof. Suppose $nb \leq a$ for all $n \in \mathbb{N}$. Then $\frac{nb}{a} \leq 1 \leq m$ for all $m, n \in \mathbb{N} \implies \frac{n}{m} \leq \frac{a}{b}$ for all $m, n \in \mathbb{N}$ $\implies x \leq \frac{a}{b}$ for all $x \in \mathbb{Q}^+$, which is a contradiction (as $\frac{a}{b} + 1 > \frac{a}{b}$). $\square$

Proposition 2.3. $\mathbb{Q}$ *does not have the lub property.*

Proof. Let $A = \{x \in \mathbb{Q} \mid x > 0,\ x^2 < 2\}$. Now A is bounded above as $x < 2$ for all $x \in A$. Let $x \in A$. Define $y = \frac{2x+2}{x+2} = x + \frac{2-x^2}{x+2}$. Since $x^2 < 2$, $y > x$. Also $2 - y^2 = \frac{2(2-x^2)}{(x+2)^2} > 0$. Thus $y^2 < 2$ and so $y \in A$. Therefore for each $x \in A$, there is a y in A such that $y > x$, i.e., A has no greatest element. Again let $B = \{x \in \mathbb{Q} \mid x > 0,\ x^2 > 2\}$. Let $x \in B$. Define $y = x - \frac{x^2-2}{x+2} = \frac{2x+2}{x+2}$. Then $0 < y < x$ and $y^2 > 2$. So B has no least element. Finally, there is no rational number y such that $y^2 = 2$ (surely, its proof is known to you). Thus A has no lub. $\square$

3 Properties of $\mathbb{R}$

Theorem 3.1. *$\mathbb{R}$ is archimedean ordered, i.e., for any $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof. Let $A = \{nx \mid n \in \mathbb{N}\}$. If the result is not true, then y is an upper bound of A. But then A has a lub in $\mathbb{R}$. Let $\alpha = \sup(A)$. Since $x > 0$, $\alpha - x < \alpha$ and $\alpha - x$ is not an upper bound of A. Then $\alpha - x < mx$ for some $m \in \mathbb{N}$. But then $\alpha < (m+1)x \in A$ which is impossible as $\alpha = \sup(A)$. $\square$

Theorem 3.2. *$\mathbb{Q}$ is dense in $\mathbb{R}$, i.e., if $x, y \in \mathbb{R}$ with $x < y$, then there exists $p \in \mathbb{Q}$ such that $x < p < y$.*

Proof. We have $y - x > 0$. So there exists $n \in \mathbb{N}$ such that $n(y-x) > 1$, i.e., $ny > 1 + nx$ Also $m_1, m_2 \in \mathbb{N}$ such that $m_1 \cdot 1 > nx$ and $m_2 \cdot 1 > -nx$. Then $-m_2 < nx < m_1$. This implies there exists $m \in \mathbb{N}$ such that $m - 1 \leq nx < m$. Therefore $nx < m \leq 1 + nx < ny$ which imples $x < \frac{m}{n} < y$. This proves the result for $p = \frac{m}{n}$. $\square$

Definition 3.3. (Limit points of a subset of $\mathbb{R}$): Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then x is called a *limit point* of S if for every $\delta > 0$, $(x-\delta, x+\delta) \cap (S \setminus \{x\}) \neq \emptyset$. In other words, every *neighborhood* of x contains at least one point of S different from x itself, where $N \subseteq \mathbb{R}$ is a neighborhood of x if there exists $\delta > 0$, $\delta \in \mathbb{R}$ such that $(x-\delta, x+\delta) \subseteq N$. It is important to understand that δ in the definition of a limit point is arbitrary, i.e., the condition holds for ANY δ (i.e., δ can be as small as you please. But do not romanticize δ to be the length of a very small nano particle! It is simply, arbitrary). Another important thing to note in the definition of a limit point is that $(x-\delta, x+\delta) \cap (S \setminus \{x\}) \neq \emptyset \iff (x-\delta, x+\delta) \cap S$ is infinite (prove!). Finally, note that the limit of a sequence of real numbers is not the same as limit points. For example, the limit of the constant sequence $\{1, 1, 1, \ldots\}$ is 1. But as a set, $\{1, 1, 1, \ldots\} = \{1\}$ and it has no limit points (prove!).

Example 3.4. For example, 0 and 1 are limit points of the set $\{x \in \mathbb{R} \mid 0 < x < 1\}$, i.e., the open interval $(0, 1)$. Note that for any $\delta > 0$, $(-\delta, \delta) \cap (0, 1) \neq \emptyset$ as $0 < \frac{\delta}{2} < \delta$. Similarly, $(1-\delta, 1+\delta) \cap (0, 1) \neq \emptyset$ as $1 - \delta < 1 - \frac{\delta}{2} < 1$. Another example is that 0 is a limit point of set $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ as for any $\delta > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \delta$.

Now which subsets of $\mathbb{R}$ always have limit points? Note that the set $\{1, 2, 3\}$ is finite and bounded, but it has no limits points (prove!). Also $\mathbb{N} = \{1, 2, 3, \ldots\}$ is infinite and not bounded. It has also no limit points (prove!).

Theorem 3.5. *(Bolzano-Weirstrass Theorem) Every bounded infinite subset of $\mathbb{R}$ has a limit point.*

Proof. Let S be a bounded infinite subset of $\mathbb{R}$. Since S is bounded, we have, $S \subseteq [a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$. Let $A = \{x \in \mathbb{R} \mid x \text{ exceeds only a finite number of elements of } S\}$. Now $A \neq \emptyset$ as $a \in A$. Also A is bounded above. In fact, for all $x \in A$, $x < b$ as for any $y \in \mathbb{R}$ with $y \geq b$ exceeds infinite number of elements of S (in fact, all elements of S and S is infinite). Thus by lub property, A has an lub, say, $r \in \mathbb{R}$.

We show that r is a limit point of S. Let $\delta > 0$. Then $r - \delta \in A$ and $r + \delta \notin A$. Thus $(r - \delta, r + \delta) \cap S$ contains infinite number of elements of S. This implies r is a limit point of S. $\square$

3.1 Decimal expansion of Real numbers

Let $x > 0$, $x \in \mathbb{R}$. Let n_0 be the largest integer such that $n_0 \leq x$.

Having chosen $n_0, n_1, \ldots, n_{k-1}$, let n_k be the largest integer such that $n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \leq x$.

Let $E = \left\{n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \mid k = 0, 1, 2, 3, \ldots\right\}$. Then $x = \sup(E)$. The decimal expansion of x is given by $n_0.n_1 n_2 \ldots n_k \ldots$.

$\Diamond$ **Exercise 3.1.** Prove that $\mathbb{R}$ has the glb property.

$\Diamond$ **Exercise 3.2.** Find lub and glb of the following subsets of $\mathbb{R}$:

1. $S = \left\{1 + \frac{1}{n} \mid n \in \mathbb{N}\right\}$,
2. $S = \left\{\frac{n}{n+2} \mid n \in \mathbb{N}\right\}$.

$\Diamond$ **Exercise 3.3.** Let $\emptyset \neq S, T \subseteq \mathbb{R}$ be two bounded sets. Prove the following:

1. $S \subseteq T \Longrightarrow \inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T)$.
2. If $M = \{x \in \mathbb{R} \mid -x \in S\}$, then $\sup(M) = -\inf(S)$ and $\inf(M) = -\sup(S)$.
3. If $A = \{x + y \mid x \in S,\ y \in T\}$, then $\sup(A) = \sup(S) + \sup(T)$ and $\inf(A) = \inf(S) + \inf(T)$.
4. If $B = \{|x - y| \mid x, y \in S\}$, then $\sup(B) = \sup(S) - \inf(S)$ and $\inf(B) = 0$.