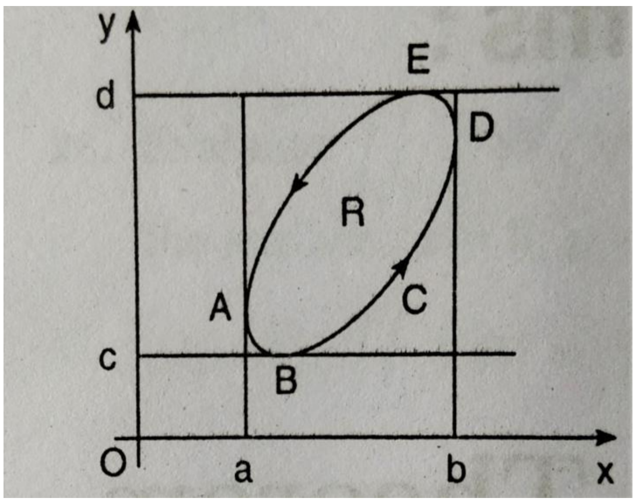
Three Important Theorems

1.Green's theorem in a plane: If R be a closed region of the xy-plane bounded by a closed curve C and if M and N be two functions of x and y, which are continuous and possessing continuous derivatives in S, then

$$\oint_C M(x,y)dx + N(x,y)dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

Where C is traversed in anticlockwise direction (taken positive)

Proof. 

Let R be enclosed by the lines $x = a, x = b, y = c$ and $y = d.$ Let any line parallel to either coordinate axes cuts the curve C in at most two points. Let any point on the portion ABD of C satisfies $y = \varphi_1(x)$ and any point on the portion DEA satisfies $y = \varphi_2(x)$.

$$\text{So } \iint_R \frac{\partial M}{\partial y} dxdy = \int_a^b \{ \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial M}{\partial y}\ dy\}\ dx$$

$$= \int_a^b M\big(x,\varphi_2(x)\big)dx - \int_a^b M\big(x,\varphi_1(x)\big)dx$$

$$= -[\int_b^a M\big(x,\varphi_2(x)\big)dx + \int_a^b M\big(x,\varphi_1(x)\big)dx]$$

$$= -\oint_C M(x,y)dx$$

$$\text{Or, } \oint_c M(x,y)dx = -\iint_R \frac{\partial M}{\partial y} dxdy \qquad (1)$$

Let any on the portion EAB on C satisfies $x = \Psi_1(y)$ and any point BDE on C satisfies $x = \Psi_2(y)$

So $\iint\limits_R \frac{\partial N}{\partial x} dxdy = \int_c^d \{ \int_{\Psi_1(y))}^{\Psi_2(y)} \frac{\partial N}{\partial x} \; dx\} \, dy$

$$= \int_c^d N(\Psi_2(y)\,, y)dy - \int_d^c N(\Psi_1(y)\,, y)dy$$

$$= \int_c^d N(\Psi_2(y)\,, y)dy + \int_d^c N(\Psi_1(y)\,, y)dy$$

$$= \int_{\text{BDE}} N(\Psi_2(y)\,, y)dy + \int_{\text{EAB}} N(\Psi_1(y)\,, y)dy$$

$$= \oint_c N(x, y)dx$$

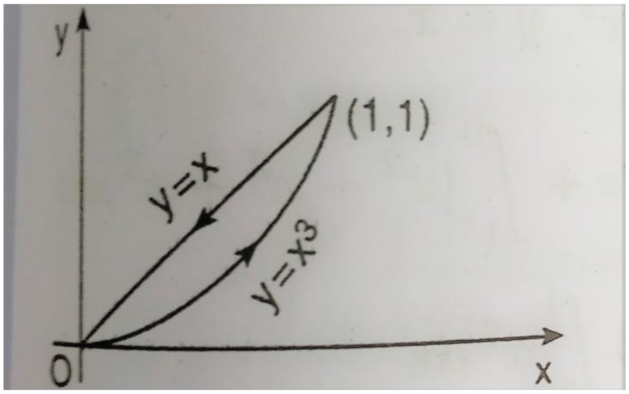
So, $$\oint_{\text{C}} N(x, y)dx = \iint_R \frac{\partial N}{\partial x} dxdy \qquad (2)$$

Adding (1) and (2) we get,

$$\oint_{\text{C}} M(x, y)dx + N(x, y)dy = \iint_{\text{R}} (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy$$

Ex 1. Verify Green's theorem for $\oint_{\text{C}} \{(x^2 - xy) \, dx + (y - x^2)dy\}$ where C is the boundary of the region bounded by the curves $C_1 : y = x^3$ and $C_2 \; : \; y = x$.

Solution:



Points of intersection of $y = x^3$ and $y = x$ are (0,0) and (1,1). Let S be the region whose boundary is the closed curve C.

$$\oint_c (x^2 - xy) \, dx + (y - x^2)dy = \int_{c_1} \{(x^2 - xy) \, dx + (y - x^2)dy\}$$

$$+ \int_{C_2}\{(x^2 - xy)\, dx + (y - x^2)dy\}$$

$$= \int_0^1 \int\{(x^2 - x^4) + (x^3 - x^2).3x^2\}dx$$
$$+ \int_1^0 \int\{(x^2 - x^2) + (x - x^2)\}dx$$

$$= \int_0^1 \int 3x^5 - 4x^4\, dx + \int_1^0 (x - x^2)dx$$

$$= -\frac{2}{15}.$$

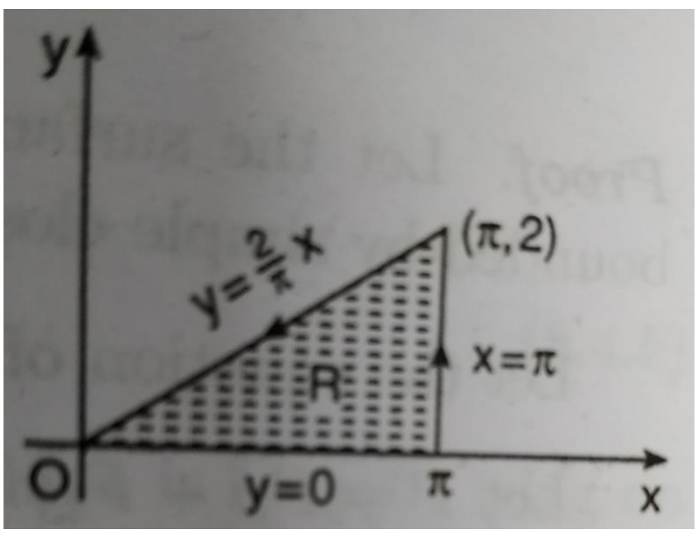
Now, $\iint_S\{\frac{\partial}{\partial x}(y - x^2) - \frac{\partial}{\partial y}(x^2 - y)\}dxdy = \iint_S(-2x + x)dydx$

$$= -\int_0^1 \int_{x^3}^{x} x\, dydx = -\int_0^1 x(x - x^3)dx$$
$$= -\frac{2}{15}.$$

Thus Green's theorem is verified.

Ex 2. Verify Green's theorem for $\oint_C\{(y - \sin x)\, dx + \cos x\, dy\}$ where C is the triangle enclosed by the lines $y = 0, x = \pi,\ y = \frac{2}{\pi}x$.

Solution:



Let S be the triangular region in the xy −plane formed by the lines $C_1: y = 0$, $C_2: x = \pi$ and $C_3: y = \frac{2}{\pi}x$.

$$\oint_C \{(y - \sin x)\, dx + \cos x\, dy\} = \int_{C_1} \{(y - \sin x)\, dx + \cos x\, dy\} + \int_{C_2} \{(y - \sin x)\, dx + \cos x\, dy\} + \int_{C_3} \{(y - \sin x)\, dx + \cos x\, dy\}$$

$C_1 : y = 0,\ dy = 0;\quad \int_{C_1} \{(y - \sin x)\, dx + \cos x\, dy\} = \int_0^{\pi} - \sin x\, dx = -2$

$C_2 : x = \pi,\ dx = 0;\quad \int_{C_2} \{(y - \sin x)\, dx + \cos x\, dy\} = \int_0^{2} -1\, dy = -2$

$C_3 : y = \frac{2}{\pi} x\ ,\ dy = \frac{2}{\pi}\, dx;$

$\int_{C_3} \{(y - \sin x)\, dx + \cos x\, dy\} = \int_{\pi}^{0} (\frac{2}{\pi} x - \sin x + \frac{2}{\pi} \cos x) dx = -(\pi - 2)$

So,

$\oint_C \{(y - \sin x)\, dx + \cos x\, dy\} = -(\pi + 2)$

Now,

$$\oint_C \{(y - \sin x)\, dx + \cos x\, dy\} = \iint_S \{\frac{\partial}{\partial x}(\cos x) - \frac{\partial}{\partial y}(y - \sin x)\}\, dxdy$$

$$= \iint_S (-\sin x - 1) dxdy = -\int_0^{\pi} \int_0^{\frac{2}{\pi}x} (1 + \sin x) dydx$$

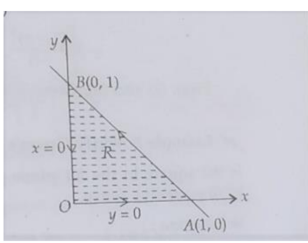
$$= -\int_0^{\pi} \frac{2}{\pi} x(1 + \sin x) dx = -(\pi + 2).$$

Thus Green’s theorem is verified.

Ex 3. Verify Green’s theorem for $\oint_C [(3x - 8y^2)dx + (4y - 6xy)dy$

Where c is the boundary of the region bounded by $x = 0, y = 0,$ and $x + y = 1.$

Solution:



By Green’s theorem, we have

$$\oint_c M(x,y)dx + N(x,y)dy = \iint_s \left(\frac{\partial N}{\partial x} - - \frac{\partial M}{\partial y}\right)dxdy$$

Here, $M(x,y) = 3x - 8y^2$ and $N(x,y) = 4y - 6xy$

$$\oint_c M(x,y)dx + N(x,y)dy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BO} Mdx + Ndy$$

On OA, $y = 0$ ie $dy = 0$ and x varies from 0 to 1 along OA.

On AB, $x + y = 1$ ie $y = 1 - x$ ie $dy = -dx$ and x varies from 1 to 0 along AB.

On BO, $x = 0$ ie $dx = 0$ and y varies from 1 to 0 along BO.

$$\oint_c M(x,y)dx + N(x,y)dy =$$

$$\int_0^1 3x\,dx + \int_1^0 [\{3x - 8(1 - x^2\}dx + \{4(1 - x) - 6x(1 - x)(-d\,x)] + \int_1^0 4y\,dy$$

$$= \frac{5}{3}$$

$$\iint_s \left(\frac{\partial N}{\partial x} - - \frac{\partial M}{\partial y}\right)dxdy = \iint_s \left\{\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x - 8y^2\right\}dxdy = \iint_s 10ydydx$$

$$= \int_{x=0}^1 \left[\int_{y=0}^{1-x} 10ydy\right]dx\,,$$

Since in the triangle region x varies from 0 to 1 and y varies from 0 to $1 - x$.

$$= \int_{x=0}^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = 5\int_0^1 (1 - x)^2\,dx = \frac{5}{3}$$

Thus Green's theorem is verified.

Ex.4 Evaluate by Green's theorem $\oint_C \{(\,Cosx \sin y - xy)\,dx + Sinx \cos y\,dy\}$

Where C is the circle $x^2 + y^2 = 1$.

Solution:

$$\oint_c \{(Cosx \sin y - xy)\,dx + Sinx \cos y\,dy\} = \iint_s \left\{\frac{\partial}{\partial x}(Sinx \cos y) - \frac{\partial}{\partial y}(CosxSiny - xy\right\}dxdy$$

$$= \iint_S \{CosxCosy - CosxCosy + x\}dydx$$

$$= \int_{-1}^1 x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydx$$

$$= 2\int_{-1}^1 x\sqrt{1 - x^2}dx$$

$= 0$ (as the integral is an odd function)

Ex.5 Evaluate $\oint_C [(3x + 4y)dx + (2x - 3y)dy]$ by Green's theorem, where C is the circle in the xy −plane centered at the origin having radius 2 units.

Solution.

The equation of the circle is $x^2 + y^2 = 4$.

By Green's theorem,

$$\oint_C [(3x+4y)dx + (2x-3y)dy] = \iint_S \{\frac{\partial}{\partial x}(2x-3y) - \frac{\partial}{\partial y}(3x+4y\}dxdy$$

$$= -\iint 2\, dydx$$

$$= \quad -\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dydx = -4\int_{-2}^{2}\sqrt{4-x^2}\ dx.$$

$$= \quad -8\int_0^2 \sqrt{4-x^2}dx$$

$$= \quad -8\int_0^{\frac{\pi}{2}} 4Cos^2\theta\ d\theta \qquad \text{where } x = 2Sin\theta, dx = 2Cos\theta d\theta$$

x	0	2
θ	0	$\frac{\pi}{2}$

$$= -8\pi.$$

Alternate: $-\iint 2\, dydx = -2\int_{\theta=0}^{2\pi}\int_{r=0}^{2} d\theta dr = -8\pi.$

2. Stokes' Theorem: If $\vec{F}$ is continuously differentiable vector point function in a surface S bounded by a curve C, then

$$\oint_C \vec{F}.d\vec{r} = \iint_S Curl\ \vec{F}.\hat{n}\, ds$$

Where $\hat{n}$ is the unit normal to S and line integral is taken along the posiyive direction of C.

Ex 6. Evaluate by Stoke's theorem $\oint_c (\sin z\, dx - \cos x\, dy + \sin y dz)$,

Where C is the boundary of the rectangle: $0 \le x \le \pi, 0 \le y \le 1, z = 0$

Solution: Here surface S is a plane surface on the plane $z = 0$. So, $\hat{n} = \hat{k}$.

$$\oint_c (\sin z\, dx - \cos x\, dy + \sin y dz) = \oint_c \vec{F}\,.\,d\vec{r}$$

Where $\vec{F} = \sin z\,\hat{\imath} - \cos x\,\hat{\jmath} + Siny\,\hat{k}$

By Stokes' theorem,

$$\oint_c (\sin z\, dx - \cos x\, dy + \sin y dz) = \iint_s (\bar{\nabla} \times \bar{F}).\hat{k} ds$$

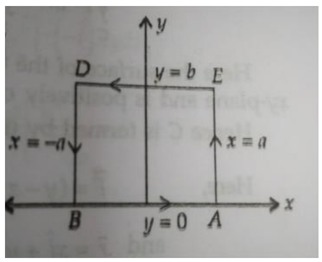
Where S is the region of the rectangle bounded by C.

Now, $\overline{\nabla} \times \bar{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix} = \cos y\, \hat{\imath} + \cos z\, \hat{\jmath} + \sin x\, \hat{k}$

$$\oint_C (\sin z\, dx - \cos x\, dy + \sin y dz) = \int_0^{\pi} \int_0^1 \sin x\, dx\, dy = [\cos x]_{\pi}^{0}\, [y]_0^1 = 2.$$

Ex 7. Verify Stokes' theorem for $\bar{F} = (x^2 + y^2)\hat{\imath} - 2xy\hat{\jmath}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.

Solution:



By Stokes' theorem,

$$\oint_C \vec{F}.d\vec{r} = \iint_S Curl\ \vec{F}.\vec{n}\, ds$$

In this problem surface S is $xy -$ plane.. The curve C is formed by the lines BA, AE,ED and DB.
$\bar{F} = (x^2 + y^2)\hat{\imath} - 2xy\hat{\jmath}$ and $\bar{r} = x\hat{\imath} + y\hat{\jmath}$, so, $\bar{F}.\bar{r} = (x^2 + y^2)dx - 2xydy$.

$$\oint_C \bar{F}\,.\,d\bar{r} = \int_{BA} \bar{F}\,.\,d\bar{r} + \int_{AE} \bar{F}\,.\,d\bar{r} + \int_{ED} \bar{F}\,.\,d\bar{r} + \int_{DB} \bar{F}\,.\,d\bar{r} \qquad (1)$$

Equation of line BA is $y = 0, ie\ dy = 0$

Equation of line AE is $x = a, ie\ dx = 0$

Equation of line ED is $y = b, ie\ dy = 0$ and

Equation of line DB is $x = -a, ie\ dx = 0$

So from equation (1)

$$\oint_C \bar{F}\,.\,d\bar{r} = \int_{-a}^{a} x^2\, dx + \int_{-a}^{a} -2aydy \pm \int_{a}^{a} (x^2 + y^2)\, dx + \int_b^0 2aydy$$

$$= -4ab^2. \qquad (2)$$

Again, $\overline{\nabla} \times \bar{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix} = -4y\hat{k}.$

For the surface S, $\hat{n} = \hat{k}$, as S lies on xy-plane.

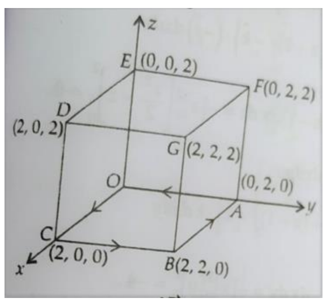
$\overline{\nabla} \times \bar{F}.\hat{n} = -4y$,

$$\iint_S \overline{\nabla} \times \bar{F}.\hat{n}\, ds = \int_{y=0}^{b}\int_{x=-a}^{a} -4y\, dydx = -4ab^2 \qquad (3)$$

Hence, Stoke's theorem is verified.

Ex. 8 Verify Stoke's theorem for $\bar{F} = (y - z + 2)\hat{\imath} + (yz + 4)\hat{\jmath} - xz\hat{k}$ over the surface of the cube $x = y = z = 0$ and $x = y = z = 2$ above xy −plane.

Solution:



By Stoke's theorem,

$$\oint_C \vec{F}.d\vec{r} = \iint_S Curl\ \vec{F}.\bar{n}\, ds$$

The surface S is above xy −plane. So the surface S is open at the xy −plane and is positively oriented at xy −plane. In this case C is formed by the lines OC, CB, BA, AO.

Since, $\bar{F} = (y - z + 2)\hat{\imath} + (yz + 4)\hat{\jmath} - xz\hat{k}$ and $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

So, $\bar{F}.\bar{r} = (y - z + 2)dx + (yz + 4)dy - xz\, dz$.

Therefore,

$$\oint_C \bar{F}\,.\,d\bar{r} = \int_{OC} \bar{F}\,.\,d\bar{r} + \int_{CB} \bar{F}\,.\,d\bar{r} + \int_{BA} \bar{F}\,.\,d\bar{r} + \int_{AO} \bar{F}\,.\,d\bar{r} \qquad (1)$$

Equation of line OC is $y = 0, z = 0$ ie $dy = 0, dz = 0$ and x varies from 0 to 2.

Equation of line CB is $x = 2, z = 0,$ ie $dx = 0, dz = 0$ and y varies from 0 to 2

Equation of line BA is $y = 2, z = 0,$ ie $dy = 0, dz = 0$ and x varies from 2 to 0

Equation of line AO is $x = 0, z = 0$ ie $dx = 0, dz = 0$ and y varies from 2 to 0.

Therefore from (1),

$$\oint_C \bar{F}\,.\,d\bar{r} = \int_0^2 2\,dx + \int_0^2 4\,dy + \int_2^0 4\,dx + \int_2^0 4\,dy = -4 \qquad (2)$$

Now, $\overline{\nabla} \times \bar{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix} = -y\hat{\imath} + (z-1)\hat{\jmath} - \hat{k}.$

Now, $\iint_S Curl\ \vec{F}.\bar{n}\,ds$ is to be evaluated over the five surfaces i.e. CBGA, OAFE,BAFG,OCDE and FGDE.

The equation of the surface CBGA, $x = 2, \hat{n} = \hat{\imath},\ ds = dydz$

So, $\iint_S Curl\ \vec{F}.\bar{n}\,ds = \iint_{CBGA}\{-y\hat{\imath} + (z-1)\hat{\jmath} - \hat{k}\}.\hat{\imath}\,dydz$

$$= \int_{z=0}^{2}\int_{y=0}^{2} -y\,dydz = -[z]_0^2\left[\frac{y^2}{2}\right]_0^2 = -4.$$

The equation of the surface OAFE, $x = 0, \hat{n} = -\hat{\imath},\ ds = dydz$

So, $\iint_S Curl\ \vec{F}.\bar{n}\,ds = \iint_{OAFE}\{-y\hat{\imath} + (z-1)\hat{\jmath} - \hat{k}\}.(-\hat{\imath})\,dydz$

$$= \int_{z=0}^{2}\int_{y=0}^{2} y\,dydz = [z]_0^2\left[\frac{y^2}{2}\right]_0^2 = 4.$$

The equation of the surface BAFG, $y = 2, \hat{n} = \hat{\jmath},\ ds = dxdz$

So, $\iint_S Curl\ \vec{F}.\bar{n}\,ds = \iint_{BAFG}\{-y\hat{\imath} + (z-1)\hat{\jmath} - \hat{k}\}.(j)\,dxdz$

$$= \int_{x=0}^{2}\int_{z=0}^{2}(z-1)\,dxdz = [x]_0^2\left[\frac{z^2}{2} - z\right]_0^2 = 0.$$

The equation of the surface OCDE, $y = 0, \hat{n} = -\hat{\jmath},\ ds = dxdz$

So, $\iint_S Curl\ \vec{F}.\bar{n}\,ds = \iint_{OCDE}\{-y\hat{\imath} + (z-1)\hat{\jmath} - \hat{k}\}.(-j)\,dxdz$

$$= \int_{x=0}^{2}\int_{z=0}^{2}(z-1)\,dxdz = -[x]_0^2\left[\frac{z^2}{2} - z\right]_0^2 = 0.$$

The equation of the surface FGDE, $z = 2, \hat{n} = \hat{k},\ ds = dxdy$

So, $\iint_S Curl\ \vec{F}.\bar{n}\,ds = \iint_{FGDE}\{-y\hat{\imath} + (z-1)\hat{\jmath} - \hat{k}\}.(\hat{k})\,dxdy$

$$= -\int_{x=0}^{2}\int_{y=0}^{2} dxdy = -[x]_0^2[y]_0^2 = -4.$$

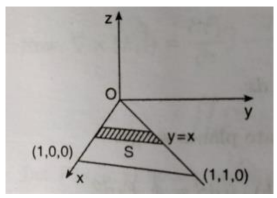
$$\iint_S Curl\ \vec{F}.\bar{n}\, ds = \iint_{CBGD} Curl\ \vec{F}.\bar{n}\, ds + \iint_{OAFE} Curl\ \vec{F}.\bar{n}\, ds + \iint_{BAFG} Curl\ \vec{F}.\bar{n}\, ds + \iint_{OCDE} Curl\ \vec{F}.\bar{n}\, ds + \iint_{FGDE} Curl\ \vec{F}.\bar{n}\, ds$$

$$= -4 + 4 + 0 + 0 - 4 = -4 \qquad (3)$$

Hence from (2) and (3) Stoke's theorem is verified.

EX 9. Evaluate $\oint_C \bar{F}\,.\,d\bar{r}$ by Stoke's theorem, where $\bar{F} = y^2\hat{\imath} + x^2\hat{\jmath} - (x+z)\hat{k}$ and C is the boundary of the triangle with vertices (0,0,0),(1,0,0),(1,1,0).

Solution.



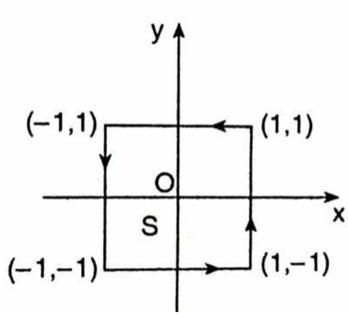
By Stoke's theorem $\oint_C \vec{F}.d\vec{r} = \iint Curl\ \vec{F}.\bar{k}\, ds$

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{\imath} + (x-y)\hat{k} - \hat{k}.$$

$$\oint_C \vec{F}.d\vec{r} = 2\iint_S (x-y)\, dydx = 2\int_0^1 [xy - \frac{y^2}{2}]_0^x\, dx = 2\int_0^2 \frac{x^2}{2}\, dx = \frac{1}{3}$$

Ex.10 Use Stoke's threorem to determine $\oint_C (xydx + xy^2dy)$, where C is a square having vertices $(1,1), (-1,1), (-1,-1), (1,-1)$ in xy-plane

Solution.



Let $\bar{F} = xy\hat{\imath} + xy^2\hat{\jmath}$, $\bar{r} = x\hat{\imath} + y\hat{\jmath}$, so $d\bar{r} = dx\hat{\imath} + dy\hat{\jmath}$

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = (y^2 - x)\hat{k}$$

$\oint_C (xydx + xy^2dy) = \oint_C \bar{F}.d\bar{r} = \iint_S Curl\ \vec{F}.\bar{n}\, ds$ (by Stoke's theorem)

(S is the square region enclosed by C)

$$= \iint_S (y^2 - x)\hat{k}.\hat{n}ds = \iint_S (y^2 - x)\hat{k}.\hat{k}ds$$
$$= \iint_S (y^2 - x)dx\, dy \quad (\text{ as } ds = dx\, dy)$$
$$= \int_{-1}^{1}\int_{-1}^{1}(y^2 - x)\, dxdy = \int_{-1}^{1}[xy^2 - \frac{x^2}{2}]_{-1}^{1}\, dy$$
$$= \int_{-1}^{1} 2y^2\, dy = \frac{4}{3}.$$

Ex.11 By Stoke's theorem show that $\vec{\nabla} \times \vec{\nabla}\varphi = \bar{0}$

Solution.

$$\iint_S \vec{\nabla} \times \vec{\nabla}\varphi\, .\hat{n}ds = \oint_C \vec{\nabla}\varphi\, .d\bar{r} = \oint_C (\frac{\partial\varphi}{\partial x}\hat{\imath} + \frac{\partial\varphi}{\partial y}\hat{\jmath} + \frac{\partial\varphi}{\partial y}\hat{k}).(dx\hat{\imath} + dy\hat{\jmath} + dz\,\hat{k})$$

$$= \oint_C (\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial y}dz) = \oint_C d\varphi = 0.$$

Since $\iint_S \vec{\nabla} \times \vec{\nabla}\varphi\, .\hat{n}ds$ is zero for every surface S, we conclude that $\vec{\nabla} \times \vec{\nabla}\varphi = \bar{0}$.

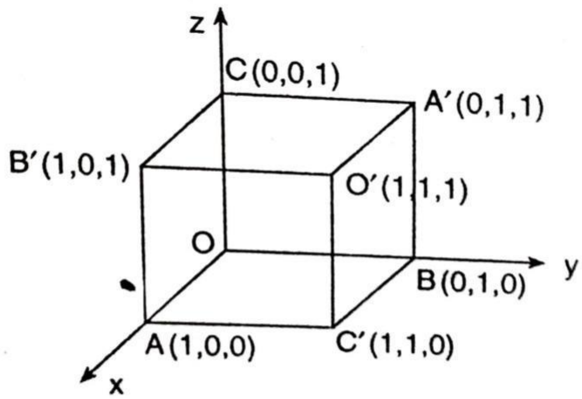
3. Gauss' Divergence Theorem

If $\vec{F}$ is a continuously differentiable vector point function and S be a closed surface enclosing volume V, then

$$\iint_S \vec{F}\, .\hat{n}\, ds = \iiint_V \vec{\nabla}.\vec{F}dv$$

Ex.12 Verify the divergence theorem for the function $\bar{F} = x^2\hat{\imath} + z\hat{\jmath} + yz\hat{k}$ over a unit cube.

Solution:



The divergence theorem states that $\iint_S \vec{F}.\vec{n}\,ds = \iiint_V \vec{\nabla}.\vec{F}dv$

$$\iiint_V \vec{\nabla}.\vec{F}dv = \int_0^1\int_0^1\int_0^1 (2x+y)dzdydx = \int_0^1\int_0^1 [2xz+yz]_0^1\,dydx$$

$$= \int_0^1\int_0^1 (2x+y)dy\,dx = \int_0^1 [2xy+\frac{y^2}{2}]_0^1\,dx = \int_0^1 (2x+\frac{1}{2})dx = \frac{3}{2}.$$

$$\iint_S \vec{F}.\vec{n}\,ds = \iint_{OAC'B} \vec{F}.\vec{n}\,ds + \iint_{CB'O'A'} \vec{F}.\vec{n}\,ds + \iint_{OAB'C} \vec{F}.\vec{n}\,ds + \iint_{BC'O'A'} \vec{F}.\vec{n}\,ds + \iint_{OBA'C} \vec{F}.\vec{n}\,ds + \iint_{AC'O'B'} \vec{F}.\vec{n}\,ds$$

On the face $OAC'B$: $z=0, \hat{n}=-\hat{k}$ On the face $CB'O'A'$: $z=1, \hat{n}=\hat{k}$

On the face $OAB'C$: $y=0, \hat{n}=-\hat{j}$ On the face $BC'O'A'$: $y=1, \hat{n}=\hat{j}$

On the face $OBA'C$: $x=0, \hat{n}=-\hat{\imath}$ On the face $AC'O'B'$: $x=1, \hat{n}=\hat{\imath}$

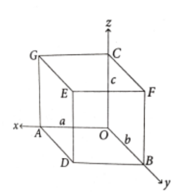
$$\iint_S \vec{F}.\vec{n}\,ds = 0 + \int_0^1\int_0^1 y\,dxdy - \int_0^1\int_0^1 z\,dxdz + \int_0^1\int_0^1 z\,dxdz + 0 + \int_0^1\int_0^1 dydz$$

$$= \frac{1}{2} + 1 = \frac{3}{2}$$

Ex 13. Verify the divergence theorem for the function

$\bar{F} = (x^2 - yz)\hat{\imath} + (y^2 - zx)\hat{\jmath} + (z^2 - xy)\hat{k}$, taken over the rectangular parallelepiped $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$

Solution :



To verify Gauss divergence theorem, we have to show

$$\iint_S \vec{F}.\vec{n}\,ds = \iiint_V \vec{\nabla}.\vec{F}dv$$

Now, $\overline{\nabla}.\overline{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2(x+y+z)$

$$\iiint_V \vec{\nabla}.\vec{F}dv = \int_0^c\int_0^b\int_0^a 2\,(x+y+z)dxdydz = abc(a+b+c). \tag{1}$$

$$\iint_S \vec{F}.\vec{n}\,ds = \iint_{OABD} \vec{F}.\vec{n}\,ds + \iint_{CGEF} \vec{F}.\vec{n}\,ds + \iint_{ADEG} \vec{F}.\vec{n}\,ds + \iint_{OBFC} \vec{F}.\vec{n}\,ds + \iint_{OAGC} \vec{F}.\vec{n}\,ds + \iint_{DBFE} \vec{F}.\vec{n}\,ds \tag{2}$$

Now for the face OADB, $\hat{n} = -\hat{k}\,, z = 0, dz = 0, ds = dxdy$

$$\iint_{OABD} \vec{F}.\vec{n}\,ds = \iint (x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}).(-\hat{k})\,ds = = \int_0^b\int_0^a xydxdy = \frac{a^2b^2}{4}$$

Now for the face CGEF, $\hat{n} = \hat{k}\,, z = c, dz = 0, ds = dxdy$

$$\iint_{CGEF} \vec{F}.\vec{n}\,ds = \iint_{CGEF} (x^2 - cy)\hat{\imath} + (y^2 - cx)\hat{\jmath} + (z^2 - xy)\hat{k}).(\hat{k})\,ds$$

$$= \int_0^b\int_0^a (c^2 - xy)dxdy = abc^2 - \frac{a^2b^2}{4}$$

Now for the face ADEG, $\hat{n} = \hat{\imath}\,, x = a, and\ dx = 0, ds = dydz$

$$\iint_{ADEG} \vec{F}.\vec{n}\,ds = \iint_{ADEG} (a^2 - yz)\hat{\imath} + (y^2 - ca)\hat{\jmath} + (z^2 - ay)\hat{k}).(\hat{\imath})\,ds$$

$$= \int_0^c\int_0^b (a^2 - yz)dydz = a^2bc - \frac{b^2c^2}{4}.$$

Now for the face OBFC, $\hat{n} = -\hat{\imath}\,, x = 0, and\ dx = 0, ds = dydz$

$$\iint_{OBFC} \vec{F}.\vec{n}\,ds = \iint_{OBFC} (a^2 - yz)\hat{\imath} + (y^2 - ca)\hat{\jmath} + (z^2 - ay)\hat{k}).(-\hat{\imath})\,ds$$

$$= \int_0^c\int_0^b yz\,dydz = -\frac{b^2c^2}{4}.$$

Now for the face OAGC, $\hat{n} = -\hat{\jmath}\,, y = 0, and\ dy = 0, ds = dzdx$

$$\iint_{OAGC} \vec{F}.\vec{n}\,ds = \int_0^a\int_0^c zx\,dzdx = \frac{a^2c^2}{4}$$

Now for the face DBFE, $\hat{n} = \hat{j}$, $y = b, and\ dy = 0, ds = dzdx$

$$\iint_{DBFE} \vec{F}.\vec{n}\, ds = \int_0^a \int_0^c (b^2 - zx)\, dzdx = ab^2c\ \frac{a^2c^2}{4}$$

Hence from (2) , we get

$$\iint_S \vec{F}.\vec{n}\, ds = abc(a+b+c)$$

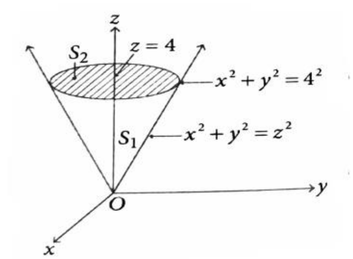
Hence, $\iint_S \vec{F}.\vec{n}\, ds = \iiint_V \vec{\nabla}.\vec{F} dv$

Hence Gauss divergence theorem verified.

Ex.14 Verify Gauss' divergence theorem for $\vec{F} = 4xz\hat{\imath} + xyz^2\hat{\jmath} + 3z\hat{k}$ where S is the surface of the cone bounded by $z^2 = x^2 + y^2$ and the plane $z = 4$.

Solution. To verify Gauss' divergence theorem we are to verify that

$$\iint_S \vec{F}.\hat{n} ds = \iiint_V \vec{\nabla}.\vec{F}$$



Now $\vec{\nabla}.\vec{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z) = 4z + xz^2 + 3$

To find the volume integral of the cone, we consider cylindrical polar coordinates

$x = rcos\theta, y = rsin\theta\ , z = z$ then $dv = dxdydz = dr.rd\theta.dz = rdr\ d\theta\ dz$

Thus,

$$\iiint_v \vec{\nabla}.\vec{F} dv = \int_{r=0}^{4}\int_{\theta=0}^{2\pi}\int_{z=r}^{4}(4z + rcos\theta.z^2 + 3)rdrd\theta dz.$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4}\left[\frac{4z^2}{2} + rcos\theta\frac{z^3}{3} + 3z\right]_r^4 r\, drd\theta$$

$$= \int_0^{2\pi}\int_0^4 [32 + \frac{64}{3}rcos\theta + 12 - 2r^2 - \frac{r^4}{3}cos\theta - 3r]rdrd\theta$$

(Since on any point on the cone, $x^2 + y^2 = z^2\ ie, r^2 = z^2 so, z \to r\ to\ 4$)

$$= \int_0^{2\pi}[32\frac{r^2}{2} + 64\, cos\theta\left(\frac{r^3}{3}\right) + 12\frac{r^2}{2} - 2\ \frac{r^4}{4} - \frac{r^6}{18}cos\theta - r^3]_0^4\, d\theta$$

$$= \int_0^{2\pi}[\ 256 + \frac{64}{3}cos\theta\frac{64}{3} + 96 - 128 - \ \frac{4^6}{18}cos\theta - 64]d\theta$$

$$= \ 320\pi.$$

Again,

$$\iint_s \vec{F}\,.\vec{n}\,ds = \iint_{s_1}\vec{F}\,.\vec{n}\,ds + \iint_{s_2}\vec{F}\,.\vec{n}\,ds$$

Where s_1 is the curved surface of the cone $x^2 + y^2 = z^2$ and s_2 is the plane surface of the circle $x^2 + y^2 = 4$ in z = 4 plane.

Now the surface $s_2 \equiv \ x^2 + y^2 - z^2 = 0$

$$\hat{n} = \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|}\ \text{[where, } \varphi \equiv \ x^2 + y^2 - z^2\ so, \vec{\nabla}\varphi = 2x\hat{\imath} + 2y\hat{\jmath} - 2z\hat{k}]$$

$$= \frac{2x\hat{\imath}+2y\hat{\jmath}-2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{\imath}+y\hat{\jmath}-z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

So, $\vec{F}.\hat{n} = \left(4xz\hat{\imath} + xyz^2\hat{\jmath} + 3z\hat{k}\right).\frac{x\hat{\imath}+y\hat{\jmath}-z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{4x^2z+xy^2z^2-3z^2}{\sqrt{x^2+y^2+z^2}}$

Now, an elementary area ds_2 of the curved surface s_1 on xy −plane is

$$\frac{dx.dy}{|\hat{n}.\hat{k}|} = \frac{dxdy}{\frac{z}{\sqrt{x^2+y^2+z^2}}} = \frac{\sqrt{x^2+y^2+z^2}}{z}dxdy$$

Therefore,

$$\iint_{s_1}\vec{F}\,.\vec{n}\,ds = \iint_{s_1}\frac{4x^2z+xy^2z^2-3z^2}{\sqrt{x^2+y^2+z^2}} \times \frac{\sqrt{x^2+y^2+z^2}}{z}dxdy$$

$$= \iint_{s_1}(4x^2 + xy^2z - 3z)dxdy$$

$$= \iint_{s_1}(4x^2\sqrt{x^2+y^2} + xy^2\sqrt{x^2+y^2} - 3\sqrt{x^2+y^2})dxdy$$

(Since on $s_1 = \sqrt{x^2+y^2}$)

$$= \int_0^{2\pi}\int_0^4[4r^2\, cos^2\theta + rcos\theta.r^2sin^2\theta - 3r]rdrd\theta$$

$$= \int_0^{2\pi}[\ 4cos^2\theta(\frac{r^4}{4})_0^4 + cos\theta sin^2\theta(\frac{r^6}{6})_0^4 - 3\left(\frac{r^3}{3}\right)_0^4]d\theta$$

$$= \ 256\int_0^{2\pi}cos^2\theta d\theta + \frac{(4)^6}{6}\int_0^{2\pi}cos\theta\, sin^2\theta d\theta - 64\int_0^{2\pi}d\theta$$

$$= 256 \times \frac{1}{2}\int_0^{2\pi}(1 + cos\theta)d\theta + \frac{(4)^6}{6}[\frac{sin^3\theta}{3}]_0^{2\pi} - 64.2\pi$$

$$= 128 \times [2\pi + (sin2\theta)_0^{2\pi} + 0 - 128\pi] = 128\pi$$

Also on $S_{2,}\ z = 4\ and\ \overline{F}.\hat{n} = 3z. so, ds_2 = dxdy$

$$\iint_{s_2}\vec{F}\,.\vec{n}\,ds = \iint_{s_2}3zdxdy = \iint_{s_2}12dxdy\ \text{on}\,[\,S_2, z = 4]$$

$$= 12\ \iint_{s_2}dxdy = 12 \times area\ of\ the\ circle\ x^2 + y^2 = 16$$

$$= 12 \times 4^2.\pi = 192\pi$$

So,

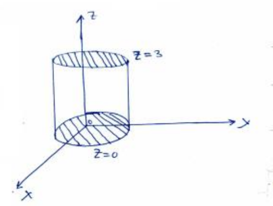
$$\iint_S \vec{F}.\vec{n}\,ds = \iint_{S_1} \vec{F}.\vec{n}\,ds + \iint_{S_2} \vec{F}.\vec{n}\,ds = 128\pi + 192\pi = 320\pi$$

Hence the divergence theorem is verified.

Ex 15. Verify divergence theorem for $\vec{F} = 4x\hat{\imath} - 2y^2\hat{\jmath} + z^2\hat{k}$, taken over the region bounded by the cylinder $x^2 + y^2 = 4\,, z = 0\ and\ z = 3$.

Solution.

$$\vec{\nabla}.\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$
$$= 4 - 4y + 2z.$$



$$\iiint_v \vec{\nabla}.\vec{F}\,dv = \iiint_v (4 - 4y + 2z)dxdydz$$
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4 - 4y + 2z)dzdydx$$
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4z - 4yz + z^2]_0^3\,dydx$$
$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\,dydx = \int_{-2}^{2} 42\sqrt{4 - x^2}dx \qquad (\int_{-a}^{a} 12y\,dy = 0)$$
$$= \int_0^2 84\sqrt{4 - x^2}dx = 2.84\int_0^{\pi/2} 2cos^2\theta\ d\theta \ \text{(where } x = 2sin\theta, dx = 2cos\theta d\theta.\text{)}$$
$$= 168\int_0^{\pi/2}(1 + cos\theta)d\theta = 168\left[\theta + \frac{sin2\theta}{2}\right]_0^{\frac{\pi}{2}} = 168\left[\frac{\pi}{2}\right] = 84\pi.$$

$$\iint_S \vec{F}.\vec{n}\,ds = \iint_{S_1} \vec{F}.\vec{n}\,ds + \iint_{S_2} \vec{F}.\vec{n}\,ds + \iint_{S_3} \vec{F}.\vec{n}\,ds$$

Where,

S_1= the circular base in the plane $z = 0$.
S_2= the circular top in the plane $z = 3$.
S_3= the curved surface of the cylinder given by the equation $x^2 + y^2 = 4$.

On $S_1 (z = 0), \hat{n} = -\hat{k}, \vec{F} = 4x\hat{\imath} - 2y^2\hat{\jmath}$

$$\vec{F}.\hat{n} = (4x\,\hat{\imath} - 2y^2\hat{\jmath}).\hat{k} = 0$$

So, $\iint\limits_{S_1} \vec{F}.\hat{n}\, ds_1 = 0$

On $S_2 (z = 3), \hat{n} = \hat{k}, \vec{F} = 4x\hat{\imath} - 2y^2\hat{\jmath} + 9\hat{k}$

$\vec{F}.\hat{n} = (4x\,\hat{\imath} - 2y^2\hat{\jmath} + 9\hat{k}).\hat{k} = 9$

So, $\iint\limits_{S_2} \vec{F}.\hat{n}\ ds_2 = \iint\limits_{S_2} 9\ ds_2 = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 9\, dy\, dx$

$= 4 \times 9 \int_0^2 \int_0^{\sqrt{4-x^2}} dy\, dx$

$= 36 \int_0^2 \sqrt{4 - x^2}\, dx \qquad x = 2cos\theta, dx = -2sin\theta d\theta$

$= 36 \int_0^{\frac{\pi}{2}} 2\cos\theta\,.2cos\theta\ d\theta$

$= 72 \int_0^{\frac{\pi}{2}} (1 + cos\theta) d\theta = 72 \left[\theta + \frac{sin2\theta}{2}\right]_0^{\frac{\pi}{2}} = 36\pi.$

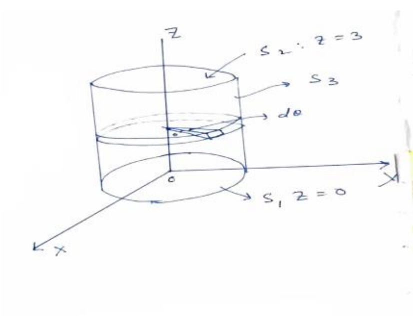
On S_3, $x^2 + y^2 = 4$

A vector normal to the surface S_3 is given by $\vec{\nabla}(x^2 + y^2 - 4) = 2x\hat{\imath} + 2y\hat{\jmath}$

$\hat{n}$= a unit vector normal to the surface S_3.

$= \frac{2x\hat{\imath}+2y\hat{\jmath}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{\imath}+y\hat{\jmath}}{2} \qquad since\ x^2 + y^2 = 4$

$\vec{F}.\hat{n} = (4x\hat{\imath} - 2y^2\hat{\jmath} + z^2\hat{k}).\frac{x\hat{\imath}+y\hat{\jmath}}{2} = 2x^2 - y^3$



From the figure it is clear that $x = 2cos\theta, y = 2sin\theta,\ ds_3 = 2d\theta dz$

So,

$\iint\limits_{S_3} \vec{F}.\hat{n}\ ds_3 = \int_0^{2\pi}\int_0^3 [2(2cos\theta)^2 - (2sin\theta)^3] 2dzd\theta$

$= \int_0^{2\pi} (48\, cos^2\theta - 48sin^3\theta) d\theta = \int_0^{2\pi} (48\, cos^2\theta) d\theta = 48\pi.$

$\therefore \qquad \iint\limits_{S} \vec{F}.\vec{n}\, ds = 0 + 36\pi + 48\pi = 84\pi.$

Hence, $\iint\limits_{s} \vec{F}.\vec{n}\, ds = \iiint\limits_{v} \vec{\nabla}.\vec{F}\, dv$