Vector Calculus

Vector function:
Let P be a variable point on a curve in space and the position vector of P relative to a fixed origin O be $\vec{r}$. If there exists an independent scalar variable t such that corresponding to each value of t in a definite domain, we get a definite position of P, that is, a unique vector $\vec{r}$, then $\vec{r}$ is called a single-valued vector function of the scalar variable t in that domain .It is usually denoted by $\vec{r}=\overrightarrow{f(t)}$.

$\overrightarrow{f(c)}$ denotes the particular vector for some fixed value c of t.

If $\vec{i},\vec{j},\vec{k}$ denote a fixed triad of mutually orthogonal unit vectors, then the vector function $\overrightarrow{f(t)}$ of the scalar parameter t can be decomposed to express if as in the form $\vec{r}=\overrightarrow{f(t)}=f_1(t)\,\vec{i}+f_2(t)\,\vec{j}+f_3(t)\,\vec{k}$ in which $f_1(t)$, $f_2(t)$, $f_3(t)$ are three scalar function of t.

The point P, whose Cartesian co-ordinates are $(f_1,\ f_2, f_3)$, describes a certain curve as t varies and hence the function $\vec{f}$ represents a curve.

For example $\vec{r}=\overrightarrow{f(t)}=at\,\vec{i}+b(1-t)\,\vec{j}$ is the vector equation of the straight line $\dfrac{x}{a}+\dfrac{y}{b}=1$, $\vec{r}=\overrightarrow{f(\alpha)}=a\cos\alpha\,\vec{i}+b\sin\alpha\,\vec{i}+0\,\vec{k}$, α being a scalar variable, is the vector equation of an ellipse with $2a$ and $2b$ as the major and minor axes respectively.

Limit and continuity of Vector function:

A vector function $\overrightarrow{f(t)}$ of the scalar parameter t is said to tend to a limit $\vec{l}$ as t tends to t_0, if corresponding to any pre-assigned positive quantity ε, however small, we can find out another positive quantity δ, such that

$\left|\overrightarrow{f(t)}-\vec{l}\right|<\in$, when $0<\left|t-t_0\right|<\delta$.

This is expressed by writing $\lim_{t\to t_0}\overrightarrow{f(t)}=\vec{l}$.

A vector function $\overrightarrow{f(t)}$ is said to be continuous at $t=t_0$, if $\lim_{t\to t_0}\overrightarrow{f(t)}$ exists, is finite and is equal to $\overrightarrow{f(t_0)}$

If $\overrightarrow{f(t)}$ be continuous for every value of t in a domain, then it is said to be continuous in that domain.

Derivative of a vector:

The derivative of a vector function $\vec{a}=\overrightarrow{f(t)}$ is denoted by

$$\overrightarrow{f'(t)}=\frac{d\vec{a}}{dt}=\lim_{\Delta t\to 0}\frac{\overrightarrow{f(t+\Delta t)}-\overrightarrow{f(t)}}{\Delta t}$$

When this limit exists, $\vec{a}$ is said to be derivable or differentiable.

Space Curve :

If, in particular, $\overrightarrow{f(t)}$ be the position vector $\overrightarrow{r(t)}$ of any point (x, y, z) relative to a set of rectangular axes with the origin O, then we have

$\overrightarrow{r(t)} = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$.

As t changes, the terminal point $\vec{r}$ describes a space curve, having parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$.

Then $\dfrac{\Delta\vec{r}}{\Delta t} = \dfrac{\overrightarrow{r(t+\Delta t)} - \overrightarrow{r(t)}}{\Delta t}$

is a vector in the direction of $\Delta\vec{r}$. If the limit of $\dfrac{\Delta\vec{r}}{\Delta t}$ exists as $\Delta t \to 0$ and is equal to $\dfrac{d\vec{r}}{dt}$, then this limit will be a vector in the direction of the tangent to the space curve at (x, y, z) and will be given by

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

Note: The derivative of a constant vector is the zero vector. If t denotes the time, $\dfrac{d\vec{r}}{dt}$ represents the velocity $\vec{v}$ with which the terminal point of $\vec{r}$ describes the curve. Similarly, $\dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$ represents its acceleration $\vec{a}$ along the curve.

Differentiation formulae:

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ differentiable vector functions of a scalar u, and ϕ is a differentiable scalar function of u, then

1. $\dfrac{d}{du}(\vec{A} + \vec{B}) = \dfrac{d\vec{A}}{du} + \dfrac{d\vec{B}}{du}$
2. $\dfrac{d}{du}(\vec{A}.\vec{B}) = \vec{A}.\dfrac{d\vec{B}}{du} + \dfrac{d\vec{A}}{du}.\vec{B}$
3. $\dfrac{d}{du}(\vec{A} \times \vec{B}) = \vec{A} \times \dfrac{d\vec{B}}{du} + \dfrac{d\vec{A}}{du} \times \vec{B}$
4. $\dfrac{d}{du}(\phi\vec{A}) = \phi\,\dfrac{d\vec{A}}{du} + \dfrac{d\phi}{du}\vec{A}$
5. $\dfrac{d}{du}(\vec{A} \times \vec{B} \times \vec{C}) = \vec{A}.\vec{B} \times \dfrac{d\vec{c}}{du} + \vec{A}.\dfrac{d\vec{B}}{du} \times \vec{C} + \dfrac{d\vec{A}}{du}.\vec{B} \times \vec{C}$
6. $\dfrac{d}{du}\{\vec{A} \times (\vec{B} \times \vec{C})\} = \vec{A} \times (\vec{B} \times \dfrac{d\vec{c}}{du}) + \vec{A} \times (\dfrac{d\vec{B}}{du} \times \vec{C}) + \dfrac{d\vec{A}}{du} \times (\vec{B} \times \vec{C})$

Theorem: If $\overrightarrow{F'(t)}$ exists at $t = t_0$, then $\overrightarrow{F(t)}$ is continuous at $t = t_0$

Proof: Let $\overrightarrow{F'(t)}$ exists at $t = t_0$. Then $\overrightarrow{F'(t_0)} = \lim\limits_{\Delta t \to 0} \dfrac{\overrightarrow{F(t_0 + \Delta t)} - \overrightarrow{F(t_0)}}{\Delta t}$ exists

Now, $\lim\limits_{\Delta t \to 0} \left| \overrightarrow{F(t_0 + \Delta t) - F(t)} \right| = \lim\limits_{\Delta t \to 0} \left[\Delta t \left\{ \dfrac{\overrightarrow{F(t_0 + \Delta t)} - \overrightarrow{F(t_0)}}{\Delta t} \right\} \right]$

$$= \lim_{\Delta t \to 0} (\Delta t) \lim_{\Delta t \to 0} \left[\frac{\overrightarrow{F(t_0 + \Delta t)} - \overrightarrow{F(t_0)}}{\Delta t} \right]$$

$$= 0\,\overrightarrow{F'(t_0)} = \vec{0}$$

Therefore, $\lim\limits_{\Delta t \to 0} \overrightarrow{F(t_0 + \Delta t)} = \overrightarrow{F(t_0)}$ and this shows that $\overrightarrow{F(t)}$ is continuous at $t = t_0$.

Converse: The converse of the above theorem is not always true.

e.g., $\overrightarrow{F(t)} = |t|\vec{i}$, is continuous at $t = 0$ but not derivable there.

For, $\left| \overrightarrow{F(t)} - \overrightarrow{F(0)} \right| = \left| |t|\vec{i} - \vec{0} \right| = |t|$

whence , $\lim\limits_{t \to 0} \overrightarrow{F(t)} = \vec{0} = \overrightarrow{F(0)}$.

So that $\overrightarrow{F(t)}$ is continuous at $t = 0$.

But $\dfrac{\overrightarrow{F(t)} - \overrightarrow{F(0)}}{t - 0} = \dfrac{|t|\vec{i}}{t}$

So that the limit is $\vec{i}$ and $-\vec{i}$ according as t tends to zero through positive or through negative values. Hence $\overrightarrow{F'(0)}$ does not exist, since the limit is not unique.

Theorem: The necessary and sufficient condition for a vector function $\overrightarrow{f(t)}$ to be a constant is that $\dfrac{d}{dt}(\overrightarrow{f(t)}) = \vec{0}$

Proof: If $\overrightarrow{f(t)}$ be a constant vector, then for every change h of the scalar variable t, $\overrightarrow{f(t+h)} - \overrightarrow{f(t)} = \vec{0}$

Hence $\dfrac{d\vec{f}}{dt} = \lim\limits_{h \to 0} \dfrac{\overrightarrow{f(t+h)} - \overrightarrow{f(t)}}{h} = \vec{0}$

Thus the condition is necessary.

To prove that this condition is also sufficient, we assume that the derivatives of $\overrightarrow{f(t)}$ is zero vector.

Let us express $\overrightarrow{f(t)}$ as $\overrightarrow{f(t)} = f_1(t)\,\vec{i} + f_2(t)\,\vec{j} + f_3(t)\,\vec{k}$, in which $f_1(t), f_2(t), f_3(t)$ are three scalar functions of t.

Then $\dfrac{d\vec{f}}{dt} = \vec{0} = \dfrac{df_1}{dt}\,\vec{i} + \dfrac{df_2}{dt}\,\vec{j} + \dfrac{df_3}{dt}\,\vec{k}$

This implies $\dfrac{df_1}{dt} = \dfrac{df_2}{dt} = \dfrac{df_3}{dt} = 0$ and hence the scalar functions $f_1(t), f_2(t), f_3(t)$ are constants. Hence $\overrightarrow{f(t)}$ is a constant vector.

Theorem: The necessary and sufficient condition for a vector function $\vec{r}=\overrightarrow{f(t)}$ to have a constant magnitude is that $\vec{f}\,.\dfrac{d\vec{f}}{dt}=0$

Proof: Let $\vec{r}=\overrightarrow{f(t)}$ be a vector function of a scalar variable t.

Let $\left|\overrightarrow{f(t)}\right|$=constant. Then $\overrightarrow{f(t)}\,.\,\overrightarrow{f(t)}=\left|\overrightarrow{f(t)}\right|^2$=constant.

$\therefore\ \dfrac{d}{dt}\left(\overrightarrow{f(t)}.\overrightarrow{f(t)}\right)=0$ or $\overrightarrow{f(t)}.\dfrac{d}{dt}\left(\overrightarrow{f(t)}\right)+\dfrac{d}{dt}\left(\overrightarrow{f(t)}\right)\overrightarrow{f(t)}=0$

or $2\,\overrightarrow{f(t)}.\dfrac{d}{dt}\left(\overrightarrow{f(t)}\right)=0$ or $\overrightarrow{f(t)}.\dfrac{d}{dt}\left(\overrightarrow{f(t)}\right)=0$

Therefore, the condition is necessary.

To prove that this condition is also sufficient, let $\overrightarrow{f(t)}$ be a vector function such that the condition $\vec{f}.\dfrac{d\vec{f}}{dt}=0$ holds.

Then we have $2\vec{f}.\dfrac{d\vec{f}}{dt}=0$ or $\vec{f}.\dfrac{d\vec{f}}{dt}+\dfrac{d\vec{f}}{dt}.\vec{f}=0$ or, $\dfrac{d}{dt}\left(\overrightarrow{f(t)}.\overrightarrow{f(t)}\right)=0$.

Therefore, $\left|\overrightarrow{f(t)}\right|^2=$ constant or, $\left|\overrightarrow{f(t)}\right|=$constant.

Note: If a vector function $\overrightarrow{f(t)}$ has a constant length, then $\overrightarrow{f(t)}$ and $\dfrac{d\vec{f}}{dt}$ are perpendicular.

Theorem : The necessary and sufficient condition for a vector $\vec{r}=\overrightarrow{f(t)}$ to have a constant direction is that $\vec{f}\times\dfrac{d\vec{f}}{dt}=\vec{0}$

Proof: Let $g(t)$ be the magnitude of $\overrightarrow{f(t)}$ and $\overrightarrow{F(t)}$ be a vector function in the direction of $\overrightarrow{f(t)}$ whose modulus is unity for all values of t, so that $\overrightarrow{f(t)}=g(t)\,\vec{F}$ and therefore $\dfrac{d\vec{f}}{dt}=g(t)\,\dfrac{d\vec{F}}{dt}+\dfrac{dg}{dt}\,\vec{F}$.

Thus we have, $\vec{f}\times\dfrac{d\vec{f}}{dt}=\vec{f}\times(\text{g}\dfrac{d\vec{F}}{dt}+\dfrac{dg}{dt}\,\vec{F}\,)$

$=\text{g}\vec{F}\times(\text{g}\dfrac{d\vec{F}}{dt}+\dfrac{dg}{dt}\,\vec{F}\,)$

$=g^2\,\vec{F}\times\dfrac{d\vec{F}}{dt}$, since $\vec{F}\times\vec{F}=\vec{0}$ .... (1)

Now, if the direction of $\overrightarrow{f(t)}$ be constant, then $\vec{F}$ is a constant vector. So we have $\dfrac{d\vec{F}}{dt}=\vec{0}$

Hence, from (1), in this case $\vec{f}\times\dfrac{d\vec{f}}{dt}=\vec{0}$

Thus the condition is necessary.

To prove that this condition is also sufficient, we assume that $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$

Then, from (1), we have $g^2 \vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}$ ……………… (2)

Since $g(t)$ is not always zero,

we have from (2), $\vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}$ ………………… (3)

Now, $\vec{F}$ being the vector with unit (constant) modulus,

so, we have, $\vec{F} \cdot \frac{d\vec{F}}{dt} = 0$ …………………… (4)

From (3) and (4), we have $\frac{d\vec{F}}{dt} = \vec{0}$

This implies that $\vec{F}$ is a constant vector.

Hence $\overrightarrow{f(t)}$ has a constant direction.

Differential geometry: Differential geometry involves a study of space curves and surfaces. If c is a space curve defined by the function $\overrightarrow{r(u)}$, then we have seen that $\frac{d\vec{r}}{du}$ is a vector in the direction of the tangent to c.

If the scalar u is taken as the arc length s measured from some fixed point on c, then $\frac{d\vec{r}}{ds}$ is a unit tangent vector to c and is denoted by $\vec{T}$. The rate at which $\vec{T}$ changes with respect to s is a measure of the curvature of c and is given by $\frac{d\vec{T}}{ds}$. The direction of $\frac{d\vec{T}}{ds}$ at any given point on c is normal to the curve at that point. If $\vec{N}$ is a unit vector in this normal direction, it is called the principal normal to the curve. Then $\frac{d\vec{T}}{ds} = k\,\vec{N}$, where k is called the curvature of c at the specified point. The quantity $\rho = \frac{1}{k}$ is called the radius of curvature.

A unit vector $\vec{B}$ perpendicular to the plane of $\vec{T}$ and $\vec{N}$ and such that $\vec{B} = \vec{T} \times \vec{N}$, is called the binormal to the curve. It follows that directions $\vec{T}, \vec{N}, \vec{B}$ from a localized right-handed rectangular coordinate system at any specified point of c. This coordinate system is called the trihedral or triad at the point. As s changes, the coordinate system moves and is known as the moving trihedral.

A set of relations involving derivatives of the fundamental vectors $\vec{T}, \vec{N}, \vec{B}$ is known as collectively as the Frenet-Serret formulae given by $\frac{d\vec{T}}{ds} = k\overrightarrow{N}$, $\frac{d\vec{N}}{ds} = \tau\,\vec{B} - \kappa\,\vec{T}$, $\frac{d\vec{B}}{ds} = -\tau\,\vec{N}$

where τ is a scalar called the torsion. The quantity $\sigma = \frac{1}{\tau}$ is called the radius of torsion.

Surfaces, normal and tangent plane :

Consider the equation $\vec{r} = \overrightarrow{r(u,v)}$ …………….. (1)
in which u and v are two parameters. If u takes a constant value u_0, then $\vec{r} = \overrightarrow{r(u_0,v)}$ represents a curve, which we denote by $u = u_0$.

Thus as u varies, that is, takes up different fixed values, $\vec{r} = \overrightarrow{r(u,v)}$ represents a moving curve in space which generates a surface S.

Hence we see that (1) represents a surface, on which $u = u_0$, $u = u_1$, …….. are curves.

Similarly, $v = v_0$, $v = v_1$,…….. are curves on the surface (1) and are given by constant values of v.

The curves $u = u_0$ and $v = v_0$ intersect at the point (u_0, v_0) on the surface given by the equation (1). These parameters u and v are called the curvilinear coordinates on the surface.

Let us consider a point P on the surface S whose curvilinear coordinates are (u_0, v_0). If $v = v_0$, then we know that $\frac{\partial \vec{r}}{\partial u}$ represents a vector tangent to this curve at the point. Similarly, $\frac{\partial \vec{r}}{\partial v}$ represents a vector tangent to the curve $u = u_0$ at the point

But $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ represent vectors at the point $P(u_0, v_0)$ tangent to the curves at P lying on the surface given by (1). Thus we get that the vector normal to the surface S at the point P is given by $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ ………(2)

Unit normal vector is obtained by dividing $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ by $\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|$.

Worked out exercises:

Exercise1: If $\hat{a}$ is a unit vector in the direction of the vector $\vec{b}$ then show that $\hat{a} \times \frac{d\hat{a}}{dt} = \frac{\left(\vec{b} \times \frac{d\vec{b}}{dt}\right)}{\vec{b}.\vec{b}}$.

☺. Since $\hat{a}$ is a unit vector in the direction of the vector $\vec{b}$, therefore we have $\hat{a} = \frac{\vec{b}}{|\vec{b}|}$.

Now, $\frac{d\hat{a}}{dt} = \frac{1}{|\vec{b}|}\frac{d\vec{b}}{dt} - \frac{1}{|\vec{b}|^2}\frac{d|\vec{b}|}{dt}\vec{b}$

$$\hat{a}\times\frac{d\hat{a}}{dt}=\left(\vec{b}\times\frac{d\vec{b}}{dt}\right)\frac{1}{\left|\vec{b}\right|^2}-\frac{1}{\left|\vec{b}\right|^3}\frac{d\left|\vec{b}\right|}{dt}\left(\vec{b}\times\vec{b}\right)=\frac{\left(\vec{b}\times\frac{d\vec{b}}{dt}\right)}{\vec{b}.\vec{b}}$$, since $\left(\vec{b}\times\vec{b}\right)=\vec{0}$.

Exercise2: If $\vec{\omega}$ is a constant vector, $\vec{r}$ and $\vec{s}$ are vector functions of a scalar variable t and if $\frac{d\vec{r}}{dt}=\vec{\omega}\times\vec{r},\frac{d\vec{s}}{dt}=\vec{\omega}\times\vec{s}$ then show that $\frac{d}{dt}\left(\vec{r}\times\vec{s}\right)=\vec{\omega}\times\left(\vec{r}\times\vec{s}\right)$

☺. $\frac{d}{dt}\left(\vec{r}\times\vec{s}\right)=\frac{d\vec{r}}{dt}\times\vec{s}+\vec{r}\times\frac{d\vec{s}}{dt}=\left(\vec{\omega}\times\vec{r}\right)\times\vec{s}+\vec{r}\times\left(\vec{\omega}\times\vec{s}\right)$

$=\left(\vec{\omega}.\vec{s}\right)\vec{r}-\left(\vec{r}.\vec{s}\right)\vec{\omega}+\left(\vec{r}.\vec{s}\right)\vec{\omega}-\left(\vec{r}.\vec{\omega}\right)\vec{s}=\left(\vec{\omega}.\vec{s}\right)\vec{r}-\left(\vec{\omega}.\vec{r}\right)\vec{s}=\vec{\omega}\times\left(\vec{r}\times\vec{s}\right)$

Exercise3: A particle moves along a curve whose parametric equations are $x=e^{-t}$, $y=2\cos 3t$, $z=2\sin 3t$, where t is the time.

(a) Determine its velocity and acceleration at any time.

(b) Find the magnitudes of the velocity and acceleration at $t=0$.

☺. (a) The position vector $\vec{r}$ of the particle is

$$\vec{r}=x\,\vec{i}+y\,\vec{j}+z\,\vec{k}=e^{-t}\,\vec{i}\,+2\cos 3t\,\vec{j}+2\sin 3t\,\vec{k}$$

Then the velocity $\vec{v}=\frac{d\vec{r}}{dt}=-e^{-t}\,\vec{i}-6\sin 3t\,\vec{j}+6\cos 3t\,\vec{k}$ and the acceleration is

$\vec{a}=\frac{d^2\vec{r}}{dt^2}=e^{-t}\vec{i}-18\cos 3t\vec{j}-18\sin 3t\vec{k}$

(b) At $t=0$, $\frac{d\vec{r}}{dt}=-\vec{i}+6\vec{k}$ and $\frac{d^2\vec{r}}{dt^2}=\vec{i}-18\vec{j}$

Then the magnitude of velocity at t=0 is $\sqrt{(-1)^2+6^2}=\sqrt{37}$

magnitude of acceleration at t=0 is $\sqrt{(1)^2+(-18)^2}=\sqrt{325}$

Exercise4: A particle moves along a curve $x=2t^2$, $y=t^2-4t$, $z=3t-5$, where t is the time. Find the components of its velocity and acceleration at time t=1 in the direction $\vec{i}-3\vec{j}+2\vec{k}$.

☺. The position vector $\vec{r}$ of the particle is

$$\vec{r}=x\,\vec{i}+y\,\vec{j}+z\,\vec{k}=2t^2\,\vec{i}+(t^2-4t)\,\vec{j}+(3t-5)\,\vec{k}$$

Then the velocity $\vec{v}=\frac{d\vec{r}}{dt}=4t\,\vec{i}+(2t-4)\,\vec{j}+3\,\vec{k}$

and the acceleration is $\vec{a}=\frac{d^2\vec{r}}{dt^2}=4\,\vec{i}+2\,\vec{j}$

At t=1, $\frac{d\vec{r}}{dt}=4\,\vec{i}-2\,\vec{j}+3\,\vec{k}$, $\frac{d^2\vec{r}}{dt^2}=4\,\vec{i}+2\,\vec{j}$

Unit vector in the direction of $\vec{i}-3\,\vec{j}+2\,\vec{k}$ is

$$\frac{\vec{i}-3\vec{j}+2\vec{k}}{\sqrt{1^2+(-3)^2+2^2}}=\frac{\vec{i}-3\vec{j}+2\vec{k}}{\sqrt{14}}$$

Then the component of the velocity in the given direction is

$$\frac{(4\vec{i}-2\vec{j}+3\vec{k}).(\vec{i}-3\vec{j}+2\vec{k})}{\sqrt{14}}=\frac{4+6+6}{\sqrt{14}}=\frac{16}{\sqrt{14}}=\frac{8\sqrt{14}}{7}$$

and the component of the acceleration in the given direction is

$$\frac{(4\vec{i}+2\vec{j}).(\vec{i}-3\vec{j}+2\vec{k})}{\sqrt{14}}=\frac{4-6}{\sqrt{14}}=\frac{-2}{\sqrt{14}}=\frac{-\sqrt{14}}{7}$$

Exercise5: A particle moves so that its position vector is given by $\vec{r}=\cos\omega t\ \vec{i}+\sin\omega t\ \vec{j}$ where ω is a constant. Show that

(a) the velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$.

(b) the acceleration $\vec{a}$ is directed towards to the origin and has magnitude proportional to the distance from the origin,

(c) $\vec{r}\times\vec{v}$=a constant vector.

☺. (a) $\vec{v}=\frac{d\vec{r}}{dt}=-\omega\sin\omega t\ \vec{i}+\omega\cos\omega t\ \vec{j}$

Then $\vec{r}.\vec{v}$ =($\cos\omega t\ \vec{i}+\sin\omega t\ \vec{j}$).($-\omega\sin\omega t\ \vec{i}+\omega\cos\omega t\ \vec{j}$)

=($\cos\omega t$)($-\omega\sin\omega t$)+($\sin\omega t$)($\omega\cos\omega t$)=0

Therefore, $\vec{r}$ and $\vec{v}$ are perpendicular.

(b) $\frac{d^2\vec{r}}{dt^2}=\frac{d\vec{v}}{dt}=-\omega^2\cos\omega t\ \vec{i}-\omega^2\sin\omega t\ \vec{j}$

$=-\omega^2(\cos\omega t\ \vec{i}+\sin\omega t\ \vec{j})=-\omega^2\ \vec{r}$

Then the acceleration is opposite to the direction of $\vec{r}$, i.e. it is directed toward the origin. Its magnitude is proportional to $|\vec{r}|$ which is the distance from the origin.

(c) $\vec{r}\times\vec{v}$=($\cos\omega t\ \vec{i}+\sin\omega t\ \vec{j}$)×($-\omega\sin\omega t\ \vec{i}+\omega\cos\omega t\ \vec{j}$)

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \cos wt & \sin wt & 0\\ -w\sin wt & w\cos wt & 0\end{vmatrix}$$

$=w(\cos^2\omega t+\sin^2\omega t)\vec{k}=\omega\ \vec{k}$, a constant vector.

Note: Physically, the motion is that of a particle moving on the circumference of a circle with constant angular speed ω. The acceleration, directed toward the centre of the circle, is the centripetal acceleration.

Exercise6: Prove the Frenet-Serret formulae

(a) $\frac{d\vec{T}}{ds}=k\vec{N}$ (b) $\frac{d\vec{B}}{ds}=-\tau\ \vec{N}$ (c) $\frac{d\vec{N}}{ds}=\tau\ \vec{B}-\kappa\ \vec{T}$

☺. (a) Since $\vec{T}$ is a unit vector.

Therefore, $\vec{T}.\vec{T}=|\vec{T}|^2=1$

Differentiating w.r.t. the arc length s we have,

$$\vec{T}.\frac{d\vec{T}}{ds}+\frac{d\vec{T}}{ds}.\vec{T}=0$$

or, $2\vec{T}.\frac{d\vec{T}}{ds}=0$, $\quad(\because \vec{T}.\frac{d\vec{T}}{ds}=\frac{d\vec{T}}{ds}.\vec{T})$

or, $\vec{T}.\frac{d\vec{T}}{ds}=0$, i.e. $\frac{d\vec{T}}{ds}$ is perpendicular to $\vec{T}$.

If $\vec{N}$ is a unit vector in the direction $\frac{d\vec{T}}{ds}$, then $\frac{d\vec{T}}{ds} = k\vec{N}$. We call $\vec{N}$ the principal normal, κ the curvature and $\rho = \frac{1}{k}$ =the radius of curvature.

(b) Let $\vec{B} = \vec{T} \times \vec{N}$, so that $\frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds} + \frac{d\vec{T}}{ds} \times \vec{N}$

$$= \vec{T} \times \frac{d\vec{N}}{ds} + \text{k}\vec{N} \times \vec{N}, \left(\because \frac{d\vec{T}}{ds} = \text{k}\vec{N}\right)$$

$$= \vec{T} \times \frac{d\vec{N}}{ds}, \left(\because \vec{N} \times \vec{N} = \vec{0}\right)$$

Then $\vec{T} . \frac{d\vec{B}}{ds} = \vec{T} . \vec{T} \times \frac{d\vec{N}}{ds} = 0$

So that $\vec{T}$ is perpendicular to $\frac{d\vec{B}}{ds}$.

But from $\vec{B} . \vec{B} = 1$ it follows that $\vec{B} . \frac{d\vec{B}}{ds} = 0$

So that $\frac{d\vec{B}}{ds}$ is perpendicular to $\vec{B}$ and is thus in the plane of $\vec{T}$ and $\vec{N}$.

Since $\frac{d\vec{B}}{ds}$ is in the plane of $\vec{T}$ and $\vec{N}$ and is perpendicular to $\vec{T}$, it must be parallel to $\vec{N}$; then $\frac{d\vec{B}}{ds} = -\tau\vec{N}$.

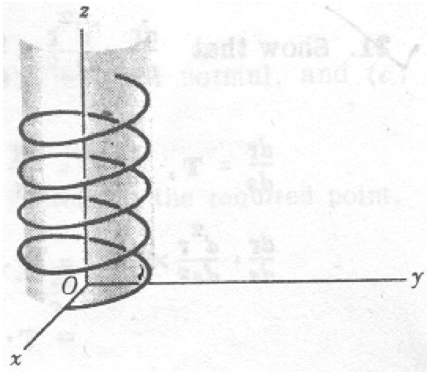
We call $\vec{B}$ the binormal, τ the torsion and $\sigma = \frac{1}{\tau}$ the radius of torsion.

(c) Since $\vec{T}, \vec{N}, \vec{B}$ form a right-handed system, so do $\vec{N}, \vec{B}, \vec{T}$, i.e., $\vec{N} = \vec{B} \times \vec{T}$

Then $\frac{d\vec{N}}{ds} = \frac{d\vec{B}}{ds} \times \vec{T} + \vec{B} \times \frac{d\vec{T}}{ds} = -\tau\vec{N} \times \vec{T} + \vec{B} \times \kappa\vec{N} = \tau\vec{B} - \kappa\vec{T}$

Exercise7: Sketch the space curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$ and find (a) the unit tangent $\vec{T}$, (b) the principal normal $\vec{N}$, curvature k and radius of curvature ρ, (c) the binormal $\vec{B}$, torsion τ and radius of torsion σ.

☺**.** The space curve is a circular helix. Since t=z/4, the curve has equations $x = 3\cos(z/4)$, $y=3\sin(z/4)$ and therefore lies on the cylinder $x^2 + y^2 = 9$



(a) The position vector for any point on the curve is $\vec{r} = 3\cos t\vec{i} + 3\sin t\vec{j} + 4t\vec{k}$.

Then $\dfrac{d\vec{r}}{dt} = -3\sin t\vec{i} + 3\cos t\vec{j} + 4\vec{k}$

$\dfrac{ds}{dt} = \left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{(\text{-3sint})^2 + (\text{3cost})^2 + (4)^2} = 5$

Then $\vec{T} = \dfrac{d\vec{r}}{ds} = \dfrac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = -\dfrac{3}{5}\sin t\,\vec{i} + \dfrac{3}{5}\cos t\,\vec{j} + \dfrac{4}{5}\vec{k}$

(b) $\dfrac{d\vec{T}}{dt} = \dfrac{d}{dt}\left(-\dfrac{3}{5}\sin t\,\vec{i} + \dfrac{3}{5}\cos t\,\vec{j} + \dfrac{4}{5}\vec{k}\right) = -\dfrac{3}{5}\cos t\,\vec{i} - \dfrac{3}{5}\sin t\,\vec{j}$

$\dfrac{d\vec{T}}{ds} = \dfrac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} = -\dfrac{3}{25}\cos t\,\vec{i} - \dfrac{3}{25}\sin t\,\vec{j}$

since $\dfrac{d\vec{T}}{ds} = k\,\vec{N}$, $\left|\dfrac{d\vec{T}}{ds}\right| = |k||\vec{N}| = k$ as $\kappa \geq 0$

Then $\kappa = \left|\dfrac{d\vec{T}}{ds}\right| = \sqrt{\left(-\dfrac{3}{25}\cos t\right)^2 + \left(-\dfrac{3}{25}\sin t\right)^2} = \dfrac{3}{25}$ and $\rho = \dfrac{1}{k} = \dfrac{25}{3}$

From $\dfrac{d\vec{T}}{ds} = \text{k}\vec{N}$, we obtain $\vec{N} = \dfrac{1}{k}\dfrac{d\vec{T}}{ds} = -\cos t\vec{i} - \sin t\vec{j}$,

(c) $\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$

$= \dfrac{4}{5}\sin t\,\vec{i} - \dfrac{4}{5}\cos t\,\vec{j} + \dfrac{3}{5}\vec{k}$

$\dfrac{d\vec{B}}{dt} = \dfrac{4}{5}\cos t\,\vec{i} + \dfrac{4}{5}\sin t\,\vec{j}$,

$$\frac{d\vec{B}}{ds}=\frac{\frac{d\vec{B}}{dt}}{\frac{ds}{dt}}=\frac{4}{25}\cos t\ \vec{i}+\frac{4}{25}\sin t\ \vec{j}$$

$$-\tau\ \vec{N}=\tau\,(-\cos t\ \vec{i}\ -\sin t\ \vec{j}\ )=\frac{4}{25}\cos t\ \vec{i}+\frac{4}{25}\sin t\ \vec{j}\ ,\quad (\because \frac{d\vec{B}}{ds}=-\tau\ \vec{N}\ )$$

$$\therefore\ \tau=\frac{4}{25}\ \text{ and }\ \sigma=\frac{1}{\tau}=\frac{25}{4}.$$

Exercise8: Show that $\frac{d\vec{r}}{ds}\cdot\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}=\frac{\tau}{\rho^2}$

☺. $\frac{d\vec{r}}{ds}=\vec{T}$, $\frac{d^2\vec{r}}{ds^2}=\frac{d\vec{T}}{ds}=\kappa\ \vec{N}$

$$\frac{d^3\vec{r}}{ds^3}=k\ \frac{d\vec{N}}{ds}+\frac{dk}{ds}\ \vec{N}=\kappa\,(\ \tau\ \vec{B}\ -\kappa\ \vec{T}\ )+\frac{dk}{ds}\ \vec{N}$$

$$=\kappa\ \tau\ \vec{B}\ -k^2\ \vec{T}+\frac{dk}{ds}\ \vec{N}$$

$$\frac{d\vec{r}}{ds}\cdot\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}=\left(\frac{d\vec{r}}{ds}\times\frac{d^2\vec{r}}{dt^2}\right)\cdot\left(\frac{d^3\vec{r}}{ds^3}\right)$$

$$=\kappa\,(\vec{T}\times\vec{N}\ ).\frac{d^3\vec{r}}{ds^3}=\kappa\ \vec{B}\ .\frac{d^3\vec{r}}{ds^3}=k^2\ \tau=\frac{\tau}{\rho^2}$$

Exercise9: Given the space curve $x=t$, $y=t^2$, $z=\frac{2}{3}t^3$. Find (a) the curvature κ, (b) the torsion τ.

☺. (a) The position vector is $\vec{r}=t\ \vec{i}+\ t^2\ \vec{j}\ +\frac{2}{3}\ t^3\ \vec{k}$

Then $\frac{d\vec{r}}{dt}=\vec{i}\ +2\,t\ \vec{j}\ +2\,t^2\ \vec{k}$

$$\frac{ds}{dt}=\left|\frac{d\vec{r}}{dt}\right|=\sqrt{(1)^2+(2\mathrm{t})^2+(2\mathrm{t}^2)^2}\ =\sqrt{(1+2t^2)^2}=(1+2\mathrm{t}^2)$$

and $\vec{T}=\frac{d\vec{r}}{ds}=\frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}}=\frac{\vec{\mathrm{i}}+2\mathrm{t}\vec{\mathrm{j}}+2\mathrm{t}^2\vec{k}}{(1+2t^2)}$

$$\frac{d\vec{T}}{dt}=\frac{(1+2t^2)(2\vec{j}+4t\vec{k})-(\vec{i}+2t\vec{j}+2t^2\vec{k})(4t)}{(1+2t^2)^2}$$

$$=\frac{-4t\vec{i}+(2-4t^2)\vec{j}+4t\vec{k}}{(1+2t^2)^2}$$

Then $\dfrac{d\vec{T}}{ds}=\dfrac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}=\dfrac{-4t\vec{i}+(2-4t^2)\vec{j}+4t\vec{k}}{(1+2t^2)^2}$

Since $\dfrac{d\vec{T}}{ds}=\kappa\ \vec{N}$, $\kappa=\left|\dfrac{d\vec{T}}{ds}\right|=\dfrac{\sqrt{(-4t)^2+(2-4t^2)^2+(4t)^2}}{(1+2t^2)^3}$

$=\dfrac{\sqrt{16t^2+4-16t^2+16t^4+16t^2}}{(1+2t^2)^3}=\dfrac{\sqrt{4(1+4t^2+4t^4)}}{(1+2t^2)^3}=\dfrac{2}{(1+2t^2)^2}$

(b) From (a), $\vec{N}=\dfrac{1}{k}\dfrac{d\vec{T}}{ds}=\dfrac{-2t\vec{i}+(1-2t^2)\vec{j}+2t\vec{k}}{1+2t^2}$

Then $\vec{B}=\vec{T}\times\vec{N}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{1}{1+2t^2} & \dfrac{2t}{1+2t^2} & \dfrac{2t^2}{1+2t^2}\\ -\dfrac{2t}{1+2t^2} & \dfrac{1-2t^2}{1+2t^2} & \dfrac{2t}{1+2t^2}\end{vmatrix}$

$=\dfrac{(4t^2-2t^2+4t^4)\vec{i}+(-4t^3-2t)\vec{j}+(1-2t^2+4t^2)\vec{k}}{(1+2t^2)^2}$

$=\dfrac{2t^2\vec{i}-2t\vec{j}+\vec{k}}{1+2t^2}$

Now, $\dfrac{d\vec{B}}{dt}=\dfrac{(1+2t^2)(4t\vec{i}-2\vec{j})-(2t^2\vec{i}-2t\vec{j}+\vec{k})4t}{(1+2t^2)^2}=\dfrac{4t\vec{i}+(4t^2-2)\vec{j}-4t\vec{k}}{(1+2t^2)^2}$

and $\dfrac{d\vec{B}}{ds}=\dfrac{\frac{d\vec{B}}{dt}}{\frac{ds}{dt}}=\dfrac{4t\vec{i}+(4t^2-2)\vec{j}-4t\vec{k}}{(1+2t^2)^3}=\dfrac{2(2t\vec{i}+(2t^2-1)\vec{j}-2t\vec{k})}{(1+2t^2)^3}$

Also, $-\tau\ \vec{N}=-\tau\left[\dfrac{-2t\vec{i}+(1-2t^2)\vec{j}+2t\vec{k}}{1+2t^2}\right]$

$=\tau\ \dfrac{2t\vec{i}+(2t^2-1)\vec{j}-2t\vec{k}}{1+2t^2}$

Since $\dfrac{d\vec{B}}{ds}=-\tau\ \vec{N}$, we find $\tau=\dfrac{2}{(1+2t^2)^2}$

Note: Note that $\kappa=\tau$ for this curve.

Exercise10: Find the unit normal to the surface ($a>0$) $\vec{r}=a\cos u\sin v\ \vec{i}+a\sin u\sin v\ \vec{j}+a\cos v\ \vec{k}$

☺. $\vec{r}=a\cos u\sin v\ \vec{i}+\ a\sin u\sin v\ \vec{j}+a\cos v\ \vec{k}$

$\dfrac{\partial\vec{r}}{\partial u}=-a\sin u\sin v\ \vec{i}+\ a\cos u\sin v\ \vec{j}$

$\dfrac{\partial\vec{r}}{\partial v}=\ a\cos u\cos v\ \vec{i}+\ a\sin u\cos v\ \vec{j}\ -a\sin v\ \vec{k}$

Then $\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin u \sin v & a\cos u \sin v & 0 \\ a\cos u \cos v & a\sin u \cos v & -a\sin v \end{vmatrix}$

$$= -a^2 \cos u \sin^2 v\, \vec{i} - a^2 \sin u \sin^2 v\, \vec{j} - a^2 \sin v \cos v\, \vec{k}$$

represents a vector normal to the surface at any point (u,v).

A unit normal is obtained by dividing $\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}$ by its magnitude, $\left|\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}\right|$, given by $\sqrt{a^4 \cos^2 u \sin^4 v + a^4 \sin^2 u \sin^4 v + a^4 \sin^2 v \cos^2 v}$

$= \sqrt{a^4 (\cos^2 u + \sin^2 u)\sin^4 v + a^4 \sin^2 v \cos^2 v}$

$= \sqrt{a^4 \sin^4 v + a^4 \sin^2 v \cos^2 v}$

$= \sqrt{a^4 \sin^2 v (\sin^2 v + \cos^2 v)} = \begin{cases} \text{a}^2\text{sinv} & \text{if } \sin \text{v} > 0 \\ \text{-a}^2\text{sinv} & \text{if } \sin \text{v} < 0 \end{cases}$

Then there are two unit normals given by $\pm(\cos u \sin v\, \vec{i} + \sin u \sin v\, \vec{j} + \cos v\, \vec{k})$

Exercise11: If $\vec{\alpha} = t^2\, \vec{i} - t\, \vec{j} + (2t+1)\, \vec{k}$ and $\vec{\beta} = (2t-3)\, \vec{i} + \vec{j} - t\, \vec{k}$, where $\vec{i}, \vec{j}, \vec{k}$ have their usual meanings, then $\dfrac{d}{dt}(\vec{\alpha} \times \dfrac{d\vec{\beta}}{dt})$ at $t = 2$.

☺. We have $\dfrac{d}{dt}(\vec{\alpha} \times \dfrac{d\vec{\beta}}{dt}) = \vec{\alpha} \times \dfrac{d^2\vec{\beta}}{dt^2} + \dfrac{d\vec{\alpha}}{dt} \times \dfrac{d\vec{\beta}}{dt}$

Now $\vec{\alpha} = t^2\, \vec{i} - t\, \vec{j} + (2t+1)\, \vec{k}$, $\dfrac{d\vec{\alpha}}{dt} = 2t\, \vec{i} - \vec{j} + 2\vec{k}$

$\vec{\beta} = (2t-3)\, \vec{i} + \vec{j} - t\, \vec{k}$, $\dfrac{d\vec{\beta}}{dt} = 2\vec{i} - 2\vec{k}$, $\dfrac{d^2\vec{\beta}}{dt^2} = \vec{0}$

$\therefore$ At $t=2$, $\vec{\alpha} = 4\vec{i} - 2\vec{j} + 5\vec{k}$, $\dfrac{d\vec{\alpha}}{dt} = 4\vec{i} - \vec{j} + 2\vec{k}$

$\vec{\beta} = \vec{i} + \vec{j} - 2\vec{k}$, $\dfrac{d\vec{\beta}}{dt} = 2\vec{i} - \vec{k}$, $\dfrac{d^2\vec{\beta}}{dt^2} = \vec{0}$

$$\frac{d}{dt}(\vec{\alpha} \times \frac{d\vec{\beta}}{dt}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 5 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} = \vec{i} + 8\vec{j} + 2\vec{k}.$$

Exercise12: Find $\vec{T}, \vec{N}, \vec{B}$ for the circular helix $\vec{r} = a(\cos\theta, \sin\theta, \theta\cot\beta)$. Find also expressions for curvature and torsion at a point on the curve.

☺. Now, $\vec{r} = a\cos\theta\, \vec{i} + a\sin\theta\, \vec{j} + a\theta\cot\beta\, \vec{k}$

$\therefore \dfrac{d\vec{r}}{d\theta} = -a\sin\theta\vec{i} + a\cos\theta\vec{j} + a\cot\beta\vec{k}$

so that $\dfrac{ds}{d\theta} = \sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2 + (a\cot\beta)^2} = a\,\mathrm{cosec}\beta$

$$\vec{T}=\frac{d\vec{r}}{ds}=\frac{\frac{d\vec{r}}{d\theta}}{\frac{ds}{d\theta}}=\sin\beta(-\sin\theta\vec{i}+\cos\theta\vec{j}+\cot\beta\vec{k})$$

$$\frac{d\vec{T}}{d\theta}=\sin\beta(-\cos\theta\vec{i}-\sin\theta\vec{j})$$

$$\frac{d\vec{T}}{ds}=\frac{\frac{d\vec{T}}{d\theta}}{\frac{ds}{d\theta}}=\frac{1}{a}\sin^2\beta(-\cos\theta\vec{i}-\sin\theta\vec{j})$$

Since, $\frac{d\vec{T}}{ds}=\kappa\vec{N}$, therefore, $\left|\frac{d\vec{T}}{ds}\right|=\kappa\left|\vec{N}\right|$

So that $\kappa=\frac{1}{a}\sin^2\beta$ ……………………………………(2)

Now, $\vec{N}=\frac{1}{\kappa}\frac{d\vec{T}}{ds}=-\cos\theta\vec{i}-\sin\theta\vec{j}$

Also $\vec{B}=\vec{T}\times\vec{N}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -\sin\beta\sin\theta & \sin\beta\cos\theta & \cos\beta\\ -\cos\theta & -\sin\theta & 0\end{vmatrix}$

$$=\cos\beta\sin\theta\ \vec{i}\ -\cos\beta\cos\theta\ \vec{j}+\sin\beta\ \vec{k}$$

$$\frac{d\vec{B}}{d\theta}=\cos\beta\cos\theta\vec{i}+\cos\beta\sin\theta\vec{j}$$

$$\frac{d\vec{B}}{ds}=\frac{\frac{d\vec{B}}{d\theta}}{\frac{ds}{d\theta}}=\frac{1}{a}\sin\beta\cos\beta(\cos\theta\vec{i}+\sin\theta\vec{j})$$

Since, $\frac{d\vec{B}}{ds}=-\tau\vec{N}$,therefore, $\tau=\left|\frac{d\vec{B}}{ds}\right|=\frac{1}{a}\sin\beta\cos\beta$ ………(3)

(2) and (3) gives the expressions for curvature and torsion.

Exercise13: Show that the curve $\vec{r}=\vec{r}\,(s)$ is a helix ($\frac{k}{\tau}$=constant) if $\frac{d^2\vec{r}}{ds^2}.(\frac{d^3\vec{r}}{ds^3}\times\frac{d^4\vec{r}}{ds^4})$=0, where s is the arc of the curve.

☺. $\frac{d\vec{r}}{ds}=\vec{T}$, $\frac{d^2\vec{r}}{ds^2}=\frac{d\vec{T}}{ds}=k\ \vec{N}$

$$\frac{d^3\vec{r}}{ds^3}=\kappa\ \frac{d\vec{N}}{ds}+\frac{dk}{ds}\ \vec{N}=\kappa\,(\tau\ \vec{B}-k\vec{T})+\frac{dk}{ds}\ \vec{N}=\kappa\ \tau\ \vec{B}-\kappa^2\ \vec{T}+\frac{dk}{ds}\ \vec{N}$$

$$\frac{d^4\vec{r}}{ds^4}=\frac{dk}{ds}\ \tau\ \vec{B}+\kappa\ \frac{d\tau}{ds}\ \vec{B}+\ k\ \tau\ \frac{d\vec{B}}{ds}-2k\ \frac{dk}{ds}\ \vec{T}-k^2\ \frac{d\vec{T}}{ds}+\frac{d^2k}{ds^2}\ \vec{N}+\frac{dk}{ds}\ \frac{d\vec{N}}{ds}$$

$$=\frac{dk}{ds}\tau\,\vec{B}+k\,\frac{d\tau}{ds}\,\vec{B}+\kappa\,\tau\,(-\tau\,\vec{N})-2\kappa\,\frac{dk}{ds}\,\vec{T}-\kappa^2\kappa\,\vec{N}+\frac{d^2k}{ds^2}\,\vec{N}+\frac{dk}{ds}(\tau\,\vec{B}-k\vec{T})$$

$$=(2\,\tau\,\frac{dk}{ds}+\kappa\,\frac{d\tau}{ds})\,\vec{B}-3\kappa\,\frac{dk}{ds}\,\vec{T}+(\frac{d^2k}{ds^2}-\kappa\,\tau^2-\kappa^3)\,\vec{N}$$

$$\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}=\kappa\,\vec{N}\times(\kappa\,\tau\,\vec{B}-\kappa^2\,\vec{T}+\frac{dk}{ds}\,\vec{N})\;=\;\kappa^2\,\tau\,\vec{T}+\kappa^3\,\vec{B}$$

$$\therefore\ \frac{d^2\vec{r}}{ds^2}.(\frac{d^3\vec{r}}{ds^3}\times\frac{d^4\vec{r}}{ds^4})=(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}).\frac{d^4\vec{r}}{ds^4}$$

$$=2\,\kappa^3\,\tau\,\frac{dk}{ds}+\kappa^4\,\frac{d\tau}{ds}-3\kappa^3\,\tau\,\frac{dk}{ds}$$

$$=\kappa^4\,\frac{d\tau}{ds}-k^3\tau\,\frac{dk}{ds}$$

$$=\kappa^3\,(\kappa\,\frac{d\tau}{ds}-\tau\,\frac{dk}{ds})$$

$$=-\kappa^3\,\tau^2\,\frac{\left(\tau\frac{dk}{ds}-\kappa\frac{d\tau}{ds}\right)}{\tau^2}$$

$$=-\kappa^3\,\tau^2\,\frac{d}{ds}(\frac{k}{\tau})\ \ldots\ldots\ldots\ldots.\ (1)$$

If the curve be a helix, then $\frac{k}{\tau}$ is constant.

$$\therefore\ \frac{d}{ds}(\frac{k}{\tau})=0$$

So we have from (1), $\frac{d^2\vec{r}}{ds^2}.(\frac{d^3\vec{r}}{ds^3}\times\frac{d^4\vec{r}}{ds^4})=0$

Thus the condition is necessary.

To prove that this condition is also sufficient, we assume that

$$\frac{d^2\vec{r}}{ds^2}.(\frac{d^3\vec{r}}{ds^3}\times\frac{d^4\vec{r}}{ds^4})=0$$

Then from (1), we have, $-\kappa^3\,\tau^2\,\frac{d}{ds}(\frac{k}{\tau})=0$

or, $\frac{d}{ds}(\frac{k}{\tau})=0$ ($\because\ \kappa\,,\tau\neq 0$, for a curve)

This implies that $\frac{k}{\tau}$ is constant and hence the curve is a helix.

Exercise14: Prove that the curvature of the space curve $\vec{r}=\vec{r}\,(t)$ is given numerically by $\kappa=\frac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$, where dots denote differentiation with respect to t.

☺**.** The given curve is $\vec{r}=\vec{r}\,(t)$.

Now, $\dot{\vec{r}}=\frac{d\vec{r}}{ds}\,\frac{ds}{dt}=\vec{T}\,\dot{s}$, so that $|\dot{\vec{r}}|=|\vec{T}|\,\dot{s}=\dot{s}$

$$\ddot{\vec{r}} = \frac{d\vec{T}}{ds}\dot{s}^2 + \vec{T}\ \ddot{s} \ = \kappa\ \dot{s}^2\ \vec{N} + \ddot{s}\ \vec{T}$$

$\therefore\ \dot{\vec{r}} \times \ddot{\vec{r}} = k\dot{s}^3\ \vec{B}$

or, $\left|\dot{\mathrm{r}} \times \ddot{\vec{r}}\right| = k\dot{s}^3\ \left|\vec{B}\right|$, since $k \geq 0$

or, $k = \dfrac{\left|\dot{\mathrm{r}} \times \ddot{\vec{r}}\right|}{\dot{s}^3} = \dfrac{\left|\dot{\mathrm{r}} \times \ddot{\vec{r}}\right|}{\left|\dot{\vec{r}}\right|^3}$

Exercise15: Show that the Frenet-Serret formulae can be written in the form $\dfrac{d\vec{T}}{ds} = \vec{W} \times \vec{T}$, $\dfrac{d\vec{N}}{ds} = \vec{W} \times \vec{N}$, $\dfrac{d\vec{B}}{ds} = \vec{W} \times \vec{B}$ and determine $\vec{W}$.

☺**.** The Frenet-Serret formulae are $\dfrac{d\vec{T}}{ds} = \kappa\vec{N}$, $\dfrac{d\vec{N}}{ds} = \tau\vec{B} - \kappa\vec{T}$ and $\dfrac{d\vec{B}}{ds} = -\tau\vec{N}$

Now, $\dfrac{d\vec{N}}{ds} = \tau\vec{B} - \kappa\vec{T} = \tau(\vec{T} \times \vec{N}) - \kappa(\vec{N} \times \vec{B}) = (\tau\vec{T} + \kappa\vec{B}) \times \vec{N} = \vec{W} \times \vec{N}$, where

$\vec{W} = \tau\vec{T} + \kappa\vec{B}$ (say).

$$\frac{d\vec{T}}{ds} = \kappa\vec{N} = \kappa(\vec{B} \times \vec{T}) + \tau(\vec{T} \times \vec{T}) = (\tau\vec{T} + \kappa\vec{B}) \times \vec{T} = \vec{W} \times \vec{T}$$

$$\frac{d\vec{B}}{ds} = -\tau\vec{N} = \ -\tau(\vec{B} \times \vec{T}) + \kappa(\vec{B} \times \vec{B}) = (\tau\vec{T} + \kappa\vec{B}) \times \vec{B} = \vec{W} \times \vec{B}$$

Therefore, the Frenet-Serret formulae can be written as

$$\frac{d\vec{T}}{ds} = \vec{W} \times \vec{T}, \frac{d\vec{N}}{ds} = \vec{W} \times \vec{N}, \frac{d\vec{B}}{ds} = \vec{W} \times \vec{B}, \text{ where } \vec{W} = \tau\vec{T} + \kappa\vec{B}$$

Note: $\tau\vec{T} + \kappa\vec{B}$ is called the **Darboux vector**.

Exercise16: Show that the acceleration $\vec{a}$ of a particle which travels along a space curve with velocity $\vec{v}$ is given by $\vec{a} = \dfrac{dv}{dt}\vec{T} + \dfrac{v^2}{\rho}\vec{N}$ where $\vec{T}$ is the unit tangent vector to the space curve, $\vec{N}$ is its units principal normal, and ρ is the radius of curvature.

☺**.** Velocity $\vec{v}$ =magnitude of $\vec{v}$ multiplied by unit tangent vector $\vec{T}$.

Therefore, $\vec{v} = v\vec{T}$

Differentiating, $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}\left(v\,\vec{T}\right) = \dfrac{dv}{dt}\vec{T} + v\,\dfrac{d\vec{T}}{dt}$

Now, $\dfrac{d\vec{T}}{dt} = \dfrac{d\vec{T}}{ds}\,\dfrac{ds}{dt} = \kappa\ \vec{N}\ \dfrac{ds}{dt} = \kappa\ v\ \vec{N} = \dfrac{v\vec{N}}{\rho}$

Then $\vec{a} = \dfrac{dv}{dt}\vec{T} + v\left(\dfrac{v\vec{N}}{\rho}\right) = \dfrac{dv}{dt}\vec{T} + \dfrac{v^2}{\rho}\vec{N}$

This shows that the component of the acceleration is $\frac{dv}{dt}$ in a direction tangent to the path and $\frac{v^2}{\rho}$ in a direction of the principal normal to the path. The later acceleration is often called the centripetal acceleration.

Exercise17: Establish the following formula for the derivatives of the cross products of two vectors $\vec{\alpha}$ and $\vec{\beta}$ which depend on a parameter t :

$$\frac{d}{dt}(\vec{\alpha}\times\vec{\beta})=\vec{\alpha}\times\frac{d\vec{B}}{dt}+\frac{d\vec{\alpha}}{dt}\times\vec{\beta}$$

☺. Let $\Delta\vec{\alpha},\Delta\vec{\beta}$ be the increments of $\vec{\alpha}$ and $\vec{\beta}$ corresponding to the increment Δt of t.

Then the increment of the vector product $\vec{\alpha}\times\vec{\beta}$ is $\Delta(\vec{\alpha}\times\vec{\beta})=(\vec{\alpha}+\Delta\vec{\alpha})\times(\vec{\beta}+\Delta\vec{\beta})-\vec{\alpha}\times\vec{\beta}$

Since vector product obeys distributive law, we find

$$\Delta(\vec{\alpha}\times\vec{\beta})=\vec{\alpha}\times\Delta\vec{\beta}+\Delta\vec{\alpha}\times\vec{\beta}+\Delta\vec{\alpha}\times\Delta\vec{\beta}$$

$$\text{Or, } \frac{\Delta(\vec{\alpha}\times\vec{\beta})}{\Delta t}=\vec{\alpha}\times\frac{\Delta\vec{\beta}}{\Delta t}+\frac{\Delta\vec{\alpha}}{\Delta t}\times\vec{\beta}+\frac{\Delta\vec{\alpha}}{\Delta\vec{\beta}}\times\Delta\vec{\beta}$$

Taking limit as $\Delta t\to 0$, when each of $\Delta\vec{\alpha}$ and $\Delta\vec{\beta}$ tends to zero, we get

$$\frac{d}{dt}(\vec{\alpha}\times\vec{\beta})=\vec{\alpha}\times\frac{d\vec{B}}{dt}+\frac{d\vec{\alpha}}{dt}\times\vec{\beta}$$

Exercise18: Solve the vector equation for $\vec{r}=p\,\vec{r}+(\vec{r}.\vec{b})\vec{a}=\vec{c}$, p is the scalar($\neq 0$)

☺. The vector equation is $p\,\vec{r}+(\vec{r}.\vec{b})\vec{a}=\vec{c}$, $p\neq 0$

$$\therefore p(\vec{r}.\vec{b})+(\vec{r}.\vec{b})(\vec{a}.\vec{b})=\vec{c}.\vec{b}$$

$$\text{Or, } (p-\vec{a}.\vec{b})\vec{r}.\vec{b}=\vec{b}.\vec{c}$$

$$\text{Or, } \vec{r}.\vec{b}=\frac{\vec{b}.\vec{c}}{p+\vec{a}.\vec{b}}, \text{ assuming } p+\vec{a}.\vec{b}\neq 0$$

$$\therefore p\vec{r}+\frac{\vec{b}.\vec{c}}{p+\vec{a}.\vec{b}}=\vec{c}$$

$$\text{or, } p\vec{r}=\vec{c}-\frac{\vec{b}.\vec{c}}{p+\vec{a}.\vec{b}}=\frac{p\vec{c}+(\vec{a}.\vec{b})\vec{c}-(\vec{b}.\vec{c})\vec{a}}{p+\vec{a}.\vec{b}}=\frac{p\vec{c}+(\vec{a}\times\vec{c})\times\vec{b}}{p+\vec{a}.\vec{b}}$$

or, $\vec{r}=\frac{p\vec{c}+(\vec{a}\times\vec{c})\times\vec{b}}{p(p+\vec{a}.\vec{b})}$, this is the required solution for $\vec{r}$.

Exercise19: A particle moves along the curve $\vec{r}=(t^3-4t)\vec{i}+(t^2+4t)\vec{j}+(8t^2-3t^3)\vec{k}$, where t is the time. Find the magnitude of the tangential and normal acceleration when $t=2$

☺. The position vector of ant point on the curve is

$$\vec{r}=(t^3-4t)\vec{i}+(t^2+4t)\vec{j}+(8t^2-3t^3)\vec{k}$$

$$\therefore \quad \frac{d\vec{r}}{dt} = \left(3t^2-4\right)\vec{i} + \left(2t+4\right)\vec{j} + \left(16t-9t^2\right)\vec{k}$$

and $\dfrac{d^2\vec{r}}{dt^2} = 6t\vec{i} + 2\vec{j} + \left(16-18t\right)\vec{k}$

At $t=2$, $\dfrac{d\vec{r}}{dt} = 8\vec{i} + 8\vec{j} - 4\vec{k}$ and $\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{8^2+8^2+\left(-4\right)^2} = 12$

$$\vec{T} = \frac{\dfrac{d\vec{r}}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{3}\vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = 12\vec{i} + 2\vec{j} - 20\vec{k}\,, \quad \left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{12^2+2^2+\left(-20\right)^2} = \sqrt{548}$$

Therefore the magnitude of the tangential components of acceleration of the particle is $\left(12\vec{i} + 2\vec{j} - 20\vec{k}\right).\left(\dfrac{2}{3}\vec{i} + \dfrac{2}{3}\vec{j} - \dfrac{1}{3}\vec{k}\right) = \dfrac{24+4+20}{3} = 16$

The magnitude of the normal components of acceleration of the particle is $\sqrt{548-16^2} = \sqrt{292} = 2\sqrt{73}$

Exercise20: Show that for a plane curve $\tau = 0$.

☺**.** If the curve lies in a plane and we select origin in this plane, then we see that both the vectors $\vec{T} = \dfrac{d\vec{r}}{ds}$ and $\dfrac{\vec{N}}{\rho} = \dfrac{d^2\vec{r}}{ds^2}$ lie on this plane. This implies that $\vec{T}$ and $\vec{N}$ lie on this plane and $\vec{B}$ is a constant vector, its direction being orthogonal to the plane.

Accordingly, $\dfrac{d\vec{B}}{ds} = \vec{0}$ and this implies $\tau = 0$

Exercises:

1. If $\vec{r} = a\cos t\hat{i} + a\sin t\hat{j} + bt\hat{k}$ then find the value of $\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}$.

2. If $\vec{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$ then find the value of $\left[\dfrac{d\vec{r}}{dt}\,\dfrac{d^2\vec{r}}{dt^2}\,\dfrac{d^3\vec{r}}{dt^3}\right]$

3. Show the curvature is the same at every point of the curve $\vec{r}$=(a$\cos\theta$)$\vec{i}$ +(a$\sin\theta$)$\vec{j}$ +$\theta\,\vec{k}$.

4. Show that every differentiable function is continuous.

5. For the circular helix $\vec{r}$ =a$\cos\theta\,\vec{i}$ +a$\sin\theta\,\vec{j}$ +b$\theta\,\vec{k}$, show that $\dfrac{1}{\rho} = a(a^2+b^2)^{-1}$ and $\tau = b(a^2+b^2)^{-1}$

6. Find the curvature and radius of curvature of the curve with position vector $\vec{r} = a\cos u\,\vec{i} + b\sin u\,\vec{j}$, where a and b are positive constants. Interpret the case where $a = b$.

☺. $k=\dfrac{ab}{(a^2\sin^2 u+b^2\cos^2 u)^{3/2}}=\dfrac{1}{\rho}$; If $a=b$, given curve which is an ellipse, becomes a circle of radius a and its radius of curvature $\rho=a$.

7. Prove that $\tau=\dfrac{\dot{\vec r}.\ddot{\vec r}\times\dddot{\vec r}}{\left|\dot{\vec r}\times\ddot{\vec r}\right|^2}$ for the space curve $\vec r=\vec r$ (t).

8. A particle moves along the curve $x=t^3+1$, $y=t^2$, $z=2t+5$ where t is the time. Obtain the components of its velocity and acceleration at $t=1$ in the direction $\vec i+\vec j+3\vec k$.

9. If the vector equation of a curve in E^3 (the Euclidean three-dimensional space) is $\vec r=(4\cos t)\,\vec i+(4\sin t)\,\vec j+(2t)\,\vec k$. Show that its curvature is same for all points.

10. A particle moves according to the law $\vec r=\cos t\vec i+\sin t\vec j+t^2\vec k$ **.** Find the tangential and normal acceleration of the particle.

11. If a particle has velocity $\vec v$ and $\vec a$ along a space curve, prove that the radius of curvature of its path is given numerically by $\rho=\dfrac{v^3}{\left|\vec v\times\vec a\right|}$