03/09/19 **Mathematics** ①

Differential Eqn

$$\frac{dy}{dx} = f(x,y). = \varphi\left(\frac{y}{x}\right) \longrightarrow \text{homogeneou}$$

$$y = vx.$$

$$f(x,y) = \frac{x+2y+1}{3x+5y+4}$$

$$x' = x+h$$

$$y' = y+k \quad \Big| \quad xdy + ydx = d(xy).$$

$$\frac{dy'}{dx'} = f'(x', y')$$

Exact soln

$$Mdx + Ndy = 0 \longrightarrow du$$

$$u(x,y) = c$$

Cond^n $\rightarrow$ $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

Proof. $du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy$

$\Rightarrow du = Mdx + Ndy$

$M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}.$

$P = \int M.dx$

$\Rightarrow \frac{\partial P}{\partial x} = M$

$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial^2 P}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right). \Big| \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow \boxed{N = \frac{\partial P}{\partial y} + f(y)}$

$$\frac{\partial P}{\partial x}.dx + \left(\frac{\partial P}{\partial y} + f(y)\right) dy = 0$$

$$\Rightarrow dP + df(y). = 0$$

$$d(P + f(y)) = 0$$

$$\Rightarrow du = 0$$

$$\Rightarrow u' = c.$$

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② eg. $(x+y)^2 dx - (y^2 - 2xy - x^2)\,dy = 0$

$M = (x+y)^2 \qquad N = (y^2 - 2xy - x^2)$

$\frac{dM}{dy} = 2(x+y)$, ~~x is taken as zero~~

$\frac{\partial N}{\partial x} = (-2y - 2x) = +2(x+y)$.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$(x+y)^2 dx - y^2 dy = 0$

$\Rightarrow (x^2+y^2+2xy)\,dx - y^2 dy = 0$

$\Rightarrow$ ~~$\frac{x^3}{3} + x^2y + xy^2 - \frac{y^3}{3} =$~~

$\Rightarrow \boxed{\frac{x^3}{3} + x^2y + xy^2 - \frac{y^3}{3} = \text{constant}}$

Integrating factor

$M dx + N dy = 0$

$\frac{dM}{dy} \neq \frac{dN}{dx}$

① $Mx + Ny = f(x,y) \neq 0$, $\frac{1}{Mx+Ny}$ is the integrating factor of eqn $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

② $Mx - Ny = g(x,y)$, $\frac{1}{Mx-Ny}$ is the integrating factor of the eqn $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

③ $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, $e^{\int f(x).dx}$

④ $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \phi(y)$, $e^{\int \phi(y).dy}$

⑤ $x^\alpha y^\beta (mydx + nxdy) = 0 \qquad \alpha, \beta, m, n \in \mathbb{R}$.

$x^{km-1-\alpha}\, y^{kn-1-\beta}$.

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Eg. ① $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.

$M = x^2y - 2xy^2$

$N = -(x^3 - 3x^2y)$.

$\frac{\partial M}{\partial y} = x^2 - 4xy$

$\frac{\partial N}{\partial x} = -3x^2 + 6xy$

$I.F = \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$

Multiplying both sides we get,

$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) \cdot dy = 0$

$\Rightarrow \left(\frac{1}{y}dx - \frac{x}{y^2}dy\right) - \frac{2}{x}dx + \frac{3}{y}dy = 0$

$\Rightarrow \boxed{\frac{x}{y} - 2\ln|x| + 3\ln|y| = c.}$

$\boxed{x^2 = cy^3 e^{\frac{x}{y}}}$

② $y(1 + xy) \cdot dx + x(1 - xy)dy = 0$

$M = y(1 + xy)$ $\quad N = x(1 - xy)$.

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$Mx - Ny = 2x^2y^2$

$I.F = \frac{1}{2x^2y^2}$.

Multiplying both sides by I.F.

we get, $\left|\frac{x}{y}\right| = ce^{\frac{1}{xy}}$.

$\frac{1}{2x^2y}(1 + xy)dx + \frac{1}{2xy^2}(1 - xy)\,dy = 0$

$\frac{\partial M}{\partial y} = -\frac{1}{2x^2y^2}$

$\frac{\partial N}{\partial x} = -\frac{1}{2x^2y^2}$

③ $(x^2 + y^2 + x)\,dx + xy\,dy = 0$

$M = x^2 + y^2 + x$

$N = xy$.

$\frac{\partial M}{\partial y} = +2y$, $\frac{\partial N}{\partial x} = y$

$I.F. = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x}$

$I.F. = e^{\int \frac{1}{x} \cdot dx} = e^{\ln(x)} = x.$

checking again $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$

Ans. $3x^4 + 6x^2y^2 + 4x^3 = \text{const}$

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4) $\left[2xy^4e^y + 2xy^3 + y\right]dx$

$+ \left(x^2y^4e^4 - x^2y^2 - 3x\right)dy = 0.$

Soln. $M = 2xy^4e^y + 2xy^3 + y.$

$N = x^2y^4e^4 - x^2y^2 - 3x.$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ | $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \ldots$

using

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{4}{y}.$

I.F. $e^{\int -\frac{4}{y}dy} = \frac{1}{y^4}$

Multiplying both sides by I.F.

we get. $\left(x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}\right) = \text{constant}$

5) $\underline{xy^3\left(ydx + 2xdy\right)} + \underline{\left(3ydx + 5dy\right)} = 0$

$x^\alpha y^\beta\left(mydx + nxdy\right) = 0$ | $\alpha_2 = 0, \beta_2 = 0$

$\alpha, \beta, x, y \in \mathbb{R}.$

$\alpha_1 = 1, \beta_1 = 3$

$x^{k_1 - 2}y^{2k_1 - 4}$

$x^{3k_2 - 1}y^{5k_2 - 1}.$

$k_1 = 4$

$k_2 = 1.$

$\underline{x^2y^4}.$

Multiplying with I.F.

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Differential Calculus

①

Solve

$$xy^3(ydx + 2xdy) + (3ydx + 5xdy) = 0$$

Ans. $\frac{x^4}{4}y^8 + x^3y^5 = c$

1st order linear differential equation

$$\frac{dy}{dx} + Py = Q$$

P, Q are funtion

$$IF = e^{\int P.dx}.$$

$$y(I.F.) = \int Q(I.F).dx + c$$

eg $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2}$

$$I.F. \quad e^{\int \frac{2x}{1+x^2}.dx} = e^{\ln(1+x^2)} = 1+x^2$$

~~eg. $\frac{dy}{dx} + 2x$~~

$$y(1+x^2) = \int (1+x^2).\frac{1}{(1+x^2)^2}dx + c$$

$$\Rightarrow \boxed{y(1+x^2) = \tan^{-1}(x) + c}$$

Bernoulli's form

$$\frac{dy}{dx} + Py = Qy^n$$

$$\Rightarrow y^{-n}.\frac{dy}{dx} + Py^{1-n} = Q.$$

$$\Rightarrow \frac{1}{1-n}.\frac{dz}{dx} + Pz = Q.$$

$$y^{1-n} = z$$

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}.$$

$$y^{-n}.\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}.$$

$$\frac{dz}{dx} + (1-n)Pz = (1-n)Q.$$

$$I.F = e^{\int (1-n)Pdx}.$$

$$z.(I.F) = \int (1-n)(I.F).dx + c$$

$$y^{1-n}(IF) = \int (1-n)(IF)dx + c.$$

Eg. $x\frac{dy}{dx} + xy^2 = y$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = -y^2$$

$$\Rightarrow -\frac{1}{y^2}.\frac{dy}{dx} + \frac{1}{x}.\frac{1}{y} = 1.$$

$$\frac{dz}{dx} + \frac{z}{x} = 1.$$

$$I.F. \quad e^{\int \frac{dx}{x}} = e^{\ln(x)} = x.$$

$$z.x = \int x\,dx + c$$

$$\Rightarrow \frac{1}{y}.x = \frac{x^2}{2} + c$$

$$y = \left(\frac{x}{2} + \frac{c}{x}\right)^{-1}$$

4

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2nd order differential equation of constant coefficient

$$\boxed{\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = X(x)} = \begin{cases} 0 & \text{homogeneous} \\ X(x) & \text{non-homogeneous} \end{cases}$$

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

e^{mx} as the trial soln

$$(m^2 + am + b)\, e^{mx} = 0$$

$\Rightarrow \quad m^2 + am + b = 0$

- Two distinct real roots $\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$
- Two equal real roots $\boxed{y = (c_1 + c_2 x) e^{mx}}$
- Two complex conjugate roots $a \pm ib$
  $y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$y = e^{mx}$ as the trial soln

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2 e^{mx}$$

$$m^2 e^{mx} - 5m e^{mx} + 6 e^{mx} = 0$$

$$(m^2 - 5m + 6)\, e^{mx} = 0$$

$$\therefore \quad y = c_1 e^{2x} + c_2 e^{3x}.$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

$$m = 1, 1$$

$$y = (c_1 + c_2 x) e^{x}.$$

$$D = \frac{d}{dx} \qquad D^2 = \frac{d}{dx^2}$$

$$(D^2 - 2D + 1)\, y = 0$$

$$\Rightarrow \quad (D-1)^2 y = 0$$

$$\Rightarrow \quad (D-1)(D-1)\, y = 0$$

$$(D-1)\, y = u$$

$$y = \frac{u}{D-1}$$

$$(D-1)\, u = 0$$

$$\frac{du}{dx} - u = 0$$

$$\frac{du}{dx} = u$$

$$\int \frac{du}{u} = \int dx + c$$

$$\Rightarrow \quad u = c e^{x}.$$

$$(D-1)\, y = c e^{x}$$

$\vdots$

$$\boxed{y = (c_1 x + c_2)\, e^{x}}$$

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③

$$\frac{d^2y}{dx^2} + 4y = 0$$

$$\Rightarrow (m^2+4) = 0$$

$$m = \pm 2i$$

$$y = (c_1 \cos 2x + c_2 \sin 2x).$$

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = P(x), \quad \Rightarrow f(D)\cdot y = P(x)$$

$$f(D) = (D^2 + aD + b).$$

$$= (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n).$$

$$y = \underset{\text{Complementary fn}}{\overset{R}{(CF)}} + \underset{\text{Partical Integral}}{\overset{S}{(PI)}}$$

$$\frac{d^2R}{dx^2} + a\cdot\frac{dR}{dx} + bR = 0$$

$$\frac{d^2S}{dx^2} + a\frac{dS}{dx} + bS = P(x). = \frac{1}{f(D)}\cdot P(x)$$

① $P(x) = e^{mx}$

$$P.I = \frac{e^{mx}}{f(m)} \quad \text{If } f(m) \neq 0.$$

$$\left(\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y\right) = e^x$$

$\hookrightarrow$ C.F. $= c_1 e^{2x} + c_2 e^{3x}$

$$P.I = \frac{1}{(D^2-5D+6)}\cdot e^x = \frac{1}{(D-2)(D-3)} e^x.$$

$$= \cancel{\frac{e^x}{(-2)(-3)}} \; \frac{e^x}{(1-2)(1-3)}. \quad m=1$$

$$\boxed{y = c_1 e^{2x} + c_2 e^{3x} + \frac{e^x}{2}}$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x}$$

$\hookrightarrow$ Ans. $\boxed{y = c_1 e^{2x} + c_2 e^{3x} - xe^{2x}}$

$$P(x) = x^n$$

$$P.I = \frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$$

$$= [1 + f(D)] x^n$$

$$\frac{d^2y}{dx^2} - 4y = x^3$$

$$\Rightarrow m^2 - 4 = 0$$

$$\Rightarrow \boxed{m = \pm 2}$$

$$C.F. = c_1 e^{2x} + c_2 e^{-2x}$$

$$P.I. = \frac{1}{D^2-4} x^3$$

$$= \frac{-1}{4} \cdot \frac{1}{1 - \frac{D^2}{4}} \cdot x^3$$

$$= -\frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1} x^3$$

$$\Rightarrow P.I = -1/4\left[x^3 + 3x/2\right]$$

Ans. $c_1 e^{2x} + c_2 e^{2x} - \frac{1}{4}\left(x^3 + \frac{3x}{2}\right).$

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3) $P(x) = \sin mx$ or $\cos mx$.

$$\frac{1}{f(D)} \cdot \sin mx$$
$$= \frac{1}{f(D)}\left(\operatorname{Im}(e^{imx})\right)$$
$$= \operatorname{Im}\left(\frac{1}{f(D)} e^{imx}\right).$$

$$\frac{d^2y}{dx^2} - 4y = 2\sin\left(\frac{x}{2}\right)$$

$m^2 - 4 = 0$

$m = \pm 2$

$C.F = c_1 e^{2x} + c_2 e^{-2x}$

$$P.I = \frac{1}{D^2-4} \cdot 2\sin\left(\frac{x}{2}\right)$$
$$= 2\,\frac{1}{D^2-4} \cdot \sin(x/2).$$
$$2\left(\operatorname{Im}\left(\frac{1}{D^2-4}\right) e^{\frac{ix}{2}}\right)$$
$$= 2\operatorname{Im}\left(\frac{1}{\left(\frac{1}{2}\right)^2-4}\right) e^{\frac{ix}{2}}.$$

$$\frac{1}{f(D)} e^{mx} = \frac{e^{mx}}{f(m)}$$

$$(D^2 - 4D + 4)y = 2\sin\frac{x}{2}$$

$(m-2)^2 = 0$

$\Rightarrow m = 2, 2$

P.I. $\dfrac{1}{D^2-4D+4}\, 2\sin\dfrac{x}{2}$

$$\Rightarrow 2 \cdot \operatorname{Im}\left(\frac{1}{D^2-4D+4} e^{\frac{ix}{2}}\right)$$
$$= 2\operatorname{Im}\left(\frac{e^{\frac{ix}{2}}}{\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right)+4}\right)$$
$$\ldots\ldots = \frac{32}{289}\left(2\cos\frac{x}{2} + \frac{15}{4}\sin\frac{x}{2}\right).$$

4) $P(x) = e^{mx} v(x)$.

~~$f(D) e^{mx} v$~~ $\quad f(D)\, e^{mx} v(x) = e^{mx} \cdot \dfrac{1}{f(D+m)} v(x).$

$$(D^2 + 3D + 2)y = e^{2x}\sin x.$$

$m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2.$

C.F $\quad c_1 e^{-x} + c_2 e^{-2x}$.

P.I $\quad \dfrac{1}{D^2+3D+2}\, e^{2x}\sin x \ldots$

$$= \frac{e^{2x}}{170}(11\sin x - 7\cos x).$$

5) $P(x) = x\,v(x)$

$$\frac{1}{F(D)} \cdot \cancel{x} \cdot x \cdot v(x) = \left[x - \frac{F'(D)}{F(D)}\right] \cdot \frac{1}{F(D)} \cdot v.$$

$(D^2 + 4)y = x\sin x$

$m^2 + 4 = 0 \Rightarrow m = \pm 2i.$

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\left[x\sin x - \frac{2}{3}\cos x\right]$$

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2nd order differential eqn with variable coefficient

Euler Equation

$$x^n \frac{dy^n}{dx^n} + x^{n-1} \frac{dy^{n-1}}{dx^{n-1}} + \dots + y = 0. = X(x).$$

$$z = \log x$$

$$\Rightarrow dz = \frac{1}{x} \cdot dx.$$

$$\frac{d}{dz} = x \cdot \frac{d}{dx}$$

$$\frac{dy}{dz} = x \frac{dy}{dx},$$

$$\therefore \; x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{d^2y}{dz^2}$$

$$\Rightarrow \quad x^2 \frac{d^2y}{dz^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}.$$

$$x^3 \frac{d^3y}{dx^3} - 4x^2 \cdot \frac{d^2y}{dx^2} + 5x \cdot \frac{dy}{dx} - 2y = x^3.$$

$$z = \log x$$

$$\Rightarrow x = e^z.$$

$$\frac{d}{dz} = D$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)z.$$

$$\left[D(D-1)(D-2) - 4D(D-1)\cancel{(D-2)} + 5D - 2\right] z = e^{3z}.$$

$$\Rightarrow \left(D^3 - 7D^2 + 11D - 2\right) z = 0.$$

$$m = 2, \frac{5 \pm \sqrt{21}}{2}.$$

$$C.F = c_1 e^{2} + c_2 e^{\frac{5+\sqrt{21}}{2}} + c_3 e^{\left(\frac{5-\sqrt{21}}{2}\right)}.$$

$$P.I = \frac{1}{D^3 - 7D^2 + 11D - 2} e^{3z}$$

$$= \frac{e^{3z}}{3^3 - 7.3^2 + 11.3 - 2}$$

$$= -\frac{1}{5} x^3$$

$$y = c_1 x^2 + c_2 x^{\frac{5+\sqrt{21}}{2}} + c_3 x^{\frac{5-\sqrt{21}}{2}} - \frac{1}{5} x^3$$

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$$(x+a)^2\frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = 1$$

$\Rightarrow x + a = t$

$\Rightarrow dx = dt$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \Rightarrow \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}$

$$t^2\left(\frac{d^2y}{dt^2}\right) - 4t\cdot\frac{dy}{dt} + 6y = (t-a)$$

$z = \log t \Rightarrow t = e^z \quad D = \frac{d}{dz}$

$$[D(D-1) - 4D + 6]y = e^z - a$$

$$\Rightarrow (D^2 - 5D + 6)y = e^z - a$$

$$CF = c_1 e^{2z} + c_2 e^{3z}$$

$$\Rightarrow \boxed{CF = c_1(x+a)^2 + c_2(x+a)^3}$$

$$P.I. = \frac{1}{D^2-5D+6}(e^z - a).$$

$$\Rightarrow P.I = \frac{1}{D^2-5D+6}e^z - \frac{1}{D^2-5D+6}ae^{0.z}.$$

$$= \frac{x}{2} + \frac{a}{3}.$$

$$\boxed{y = c_1(x+a)^2 + c_2(x+a)^3 + \frac{x}{2} + \frac{a}{3}}$$

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08/11/2019

Differential Equations

Prof. Akash Bose

x, y are independent variable.

z, u, w which are fn of x, y (dependent variables)

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} = 0.$$

$\rightarrow$ 2nd order.

$$\frac{\partial z}{\partial x} = p.$$

$$\frac{\partial z}{\partial y} = q$$

$$\frac{\partial^2 z}{\partial x^2} = r$$

$$\frac{\partial^2 z}{\partial x \cdot \partial y} = s$$

$$\frac{\partial^2 z}{\partial y^2} = t$$

1st order PDE

- Linear
- Semi-Linear
- Quasi-Linear
- Non-Linear

1st order PDE

(*) $f(x, y, z, a, b) = 0$. a, b are arbitrary constant

diff. wrt x & y respectively & eliminating a, b we obtain 1st order PDE

Linear $\rightarrow$ $P(x,y)p + Q(x,y).q = S(x,y) + R(x,y)z$

Semi-Linear $\rightarrow$ $P(x,y)p + Q(x,y)q = R(x,y,z)$

$\downarrow$

z^2/z^3.

Quasi-Linear $\rightarrow$ ~~$P(x,y)$~~ $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$.

Non-Linear $\rightarrow$ $F(x,y,z,p,q) = 0$

$\Rightarrow p^2 + q^2 = z$

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Eg. $(x-b)^2 + (y-a)^2 + z^2 = 0$

Partially diff. wrt x.

$$\cancel{2}(x-b) + \cancel{2}z \cdot \frac{dz}{dx} = 0$$

$$\Rightarrow \boxed{(x-b) = -pz}$$

wrt. y.

$$2(y-a) + 2z \cdot \frac{dz}{dy} = 0 \Rightarrow \boxed{y-a = -qz}$$

→ (please check another example, this is incomplete)

(*) u & v are LHS x, y, z related by $\varphi(u, v) = 0$

then PDE $Pp + Qq = R$

where.

$$P = \frac{\partial(u,v)}{\partial(y,z)} = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}$$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial v}{\partial y} \cdot \frac{\partial v}{\partial x}.$$

u, v are fn of x, y, z

$$u = u(x, y, z)$$
$$v = v(x, y, z)$$

$$\cancel{\varphi(x,y,z) = 0} \quad \varphi(u, v) = 0$$

$$\frac{d\varphi}{du} \cdot \frac{du}{dx} + \frac{\partial \varphi}{\partial v} \cdot \frac{dv}{dx} = 0$$

d or in D, not ∂

$$\Rightarrow \frac{\partial \varphi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \varphi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

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Similarly

$$\frac{\partial \varphi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \varphi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$$

eliminating $\frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v}$

$$\begin{vmatrix} \frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z} \end{vmatrix} = 0$$

eq

$\cancel{\varphi(x+y+z)}$ $\varphi(x+y+z,\ x^2+y^2+z^2) = 0$

$u = x+y+z$

$v = x^2+y^2+z^2$

$$P = \frac{\partial(u,v)}{\partial(y,z)} = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$$

$$= 1.2z - 1.2y = 2(z-y).$$

Similary $Q = \frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}$

$$= 1.2x - 1.2z$$

$$= 2(x-z).$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

$$= 1.2y - \cancel{1.2x}\ 1.2x$$

$$= 2(y-x).$$

PDE

$$Pp + Qq = R$$

$$\Rightarrow \boxed{p(z-y) + q(x-z) = (y-x)}$$

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Lagrange Method

$Pp + Qq = R$ (P, Q, R are fn of x, y, z)

Auxillary eqn

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

and taking suitable combination

$$u(x,y,z) = c_1$$
$$v(x,y,z) = c_2$$

then soln. $\phi(u, v) = 0$

$$(y-z).p + (z-x).q = (x-y).$$

Auxillary eqn

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

$$= \frac{dx + dy + dz}{(y-z) + (z-x) + (x-y)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{(xy - zx) + (yz - xy) + (xz - yz)}$$

$$= \frac{d(x+y+z)}{0} = \frac{\frac{1}{2}\,d(x^2+y^2+z^2)}{0}$$

$$u(x,y,z) = x + y + z = c_1$$
$$v(x,y,z) = x^2 + y^2 + z^2 = c_2$$

$$\phi(u, v) = 0$$

$$\Rightarrow \phi(x+y+z,\ x^2+y^2+z^2) = 0$$

$$\Rightarrow \boxed{x+y+z = f(x^2+y^2+z^2)}$$

$$\Rightarrow \boxed{x^2+y^2+z^2 = g(x+y+z)}$$

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$$z = xy + f(x^2+y^2)$$

<u>p.d.wrt x</u>

$$p = y + f'(x^2+y^2)\,2x$$

$$\Rightarrow f'(x^2+y^2) = \frac{p-y}{2x}$$

<u>p.d.wrt y</u>

$$q = x + f'(x^2+y^2)\,2y$$

$$\Rightarrow f'(x^2+y^2) = \frac{q-x}{2y}$$

$$\frac{p-y}{2x} = \frac{q-x}{2y}$$

$$\Rightarrow y(p-y) = x(q-x)$$

$$\Rightarrow \boxed{py - qx = y^2 - x^2}$$

$$D \equiv \frac{\partial}{\partial x} \qquad D' \equiv \frac{\partial}{\partial y}$$

$$(D^2 + aDD' + bD'^2)\,z = 0$$

Auxillary eqn replace D by m & D' by 1

$$(D^2 + aD + b)\,z = 0$$

$$f(m,1) = m^2 + am + b = 0$$

$$= (m_1)(m_2)$$

$$z = f(y+m_1x) + g(y+m_2x), \qquad m_1 \neq m_2$$

$$z = f(y+mx) + xg(y+mx), \qquad m_1 = m_2$$

m_1, m_2 may be real/complex

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$$\frac{\partial^2 z}{\partial x . \partial y} = \sin x$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x$$

$$\frac{\partial z}{\partial y} = \int \sin x . dx + f(y)$$

$$= -\cos x + f(y).$$

$$z = -y\cos x + \int f(y).dy + h(x)$$

$$\Rightarrow \boxed{z = -y\cos x + g(y) + h(x)}.$$

P.I

① $$\frac{1}{f(D, D')} e^{ax+by} = \frac{e^{ax+by}}{f(a, b)}.$$

② $$\frac{1}{(bd - aD')^n} e^{ax+by} = \frac{x^n}{b^n n!} e^{ax+by}.$$

③ $$\frac{1}{F(D^2, DD', D'^2)} \sin(ax+by) = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax+by).$$

④ $$\frac{1}{F(D, D')} e^{ax+by} Q(x, y) = e^{ax+by} \cdot \frac{1}{F(D+a, D'+b)} Q(a, y)$$

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2nd order

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} + d\frac{\partial z}{\partial x} + e\frac{\partial z}{\partial y} = f(x,y,z).$$

$$\Rightarrow \frac{a\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \cdot \partial y} + c\cdot\frac{\partial^2 z}{\partial y^2} = f(x,y,z,z_x,z_y).$$

if. $ax^2 + by^2 + 2bxy + dx + ey + f = 0$

$$a\frac{\partial^2 z}{\partial x^2} + 2b\frac{\partial^2 z}{\partial x \cdot \partial y} + c\cdot\frac{\partial^2 z}{\partial y^2} + d\cdot\frac{\partial z}{\partial x} + e\frac{\partial z}{\partial y} = f(x,y,z)$$

$$\Rightarrow a\frac{\partial^2 z}{\partial x^2} + \cancel{2b}\ 2b\frac{\partial^2 z}{\partial x \cdot \partial y} + c\frac{\partial^2 z}{\partial y^2} = f(x,y,z,z_x,z_y).$$

$\cancel{\Rightarrow}$

$b^2 - ac > 0 \Rightarrow$ hyperbolic pde

$b^2 = ac \Rightarrow$ parabolic pde

$b^2 - ac < 0 \Rightarrow$ elliptic pde.

$u(x,t) = X(x)T(t)$

$\cancel{X(T}$ $X \cdot T'' - c^2 X'' T = 0$

$$\Rightarrow \frac{1}{c^2}\cdot\frac{T''}{T} = \frac{X''}{X} = \lambda$$

$$f = \begin{cases} +ve \\ 0 \\ -ve \end{cases} \quad X, T$$

Dirichlet problem

$u(0,t) = 0$

$u(a,t) = 0$

Neuman problem

$\frac{\partial u}{\partial x}(0,t) = 0$

$\frac{\partial u}{\partial x}(a,t) = 0$

$u(0,t) + \frac{\partial u}{\partial x}(a,t) = 0$

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$$x = A\cos px + B\sin px$$
$$= \sin(n\pi)$$
$$p = n\pi$$
$$n = 0, 1, 2, \ldots$$

! Variation of parameter, please derive the formula

Ronskien - (determinant).

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