Chapter 4

Logic and Propositional Calculus

4.1 INTRODUCTION

Many proofs in mathematics and many algorithms in computer science use logical expressions such as

"IF p THEN q" or "IF p_1 AND p_2, THEN q_1 OR q_2"

It is therefore necessary to know the cases in which these expressions are either TRUE or FALSE: what we refer to as the truth values of such expressions. We discuss these issues in this section.

We also investigate the truth value of quantified statements, which are statements which use the logical quantifiers "for every" and "there exists".

4.2 PROPOSITIONS AND COMPOUND PROPOSITIONS

A *proposition* (or *statement*) is a declarative sentence which is true or false, but not both. Consider, for example, the following eight sentences:

(i) Paris is in France.
(ii) $1 + 1 = 2$.
(iii) $2 + 2 = 3$.
(iv) London is in Denmark.
(v) $9 < 6$.
(vi) $x = 2$ is a solution of $x^2 = 4$.
(vii) Where are you going?
(viii) Do your homework.

All of them are propositions except (vii) and (viii). Moreover, (i), (ii), and (vi) are true, whereas (iii), (iv), and (v) are false.

Compound Propositions

Many propositions are *composite*, that is, composed of *subpropositions* and various connectives discussed subsequently. Such composite propositions are called *compound propositions*. A proposition is said to be *primitive* if it cannot be broken down into simpler propositions, that is, if it is not composite.

EXAMPLE 4.1

(*a*) "Roses are red and violets are blue" is a compound proposition with subpropositions "Roses are red" and "Violets are blue".

(*b*) "John is intelligent or studies every night" is a compound proposition with subpropositions "John is intelligent" and "John studies every night".

(*c*) The above propositions (i) through (vi) are all primitive propositions; they cannot be broken down into simpler propositions.

The fundamental property of a compound proposition is that its truth value is completely determined by the truth values of its subpropositions together with the way in which they are connected to form the compound propositions. The next section studies some of these connectives.

4.3 BASIC LOGICAL OPERATIONS

This section discusses the three basic logical operations of conjunction, disjunction, and negation which correspond, respectively, to the English words "and", "or", and "not"

Conjunction, $p \wedge q$

Any two propositions can be combined by the word "and" to form a compound proposition called the *conjunction* of the original propositions. Symbolically,

$$p \wedge q$$

read "p and q", denotes the conjunction of p and q. Since $p \wedge q$ is a proposition it has a truth value, and this truth value depends only on the truth values of p and q. Specifically:

Definition 4.1: If p and q are true, then $p \wedge q$ is true; otherwise $p \wedge q$ is false.

The truth value of $p \wedge q$ may be defined equivalently by the table in Fig. 4-1(a). Here, the first line is a short way of saying that if p is true and q is true, then $p \wedge q$ is true. The second line says that if p is true and q is false, then $p \wedge q$ is false. And so on. Observe that there are four lines corresponding to the four possible combinations of T and F for the two subpropositions p and q. Note that $p \wedge q$ is true only when both p and q are true.

EXAMPLE 4.2 Consider the following four statements:

(i) Paris is in France and $2 + 2 = 4$.
(ii) Paris is in France and $2 + 2 = 5$.
(iii) Paris is in England and $2 + 2 = 4$.
(iv) Paris is in England and $2 + 2 = 5$.

Only the first statement is true. Each of the other statements is false, since at least one of its substatements is false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(a) "p and q"

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) "p or q"

p	$\neg p$
T	F
F	T

(c) "not p"

Fig. 4-1

Disjunction, $p \vee q$

Any two propositions can be combined by the word "or" to form a compound proposition called the *disjunction* of the original propositions. Symbolically,

$$p \vee q$$

read "p or q", denotes the disjunction of p and q. The truth value of $p \vee q$ depends only on the truth values of p and q as follows.

Definition 4.2: If p and q are false, then $p \vee q$ is false; otherwise $p \vee q$ is false.

The truth value of $p \vee q$ may be defined equivalently by the table in Fig. 4-1(b). Observe that $p \vee q$ is false only in the fourth case when both p and q are false.

(*b*) Next, additional truth values are entered into the truth table in various steps as shown in Fig. 4-4. That is, first the truth values of the variables are entered under the variables in the proposition, and then there is a column of truth values entered under each logical operation. We also indicate the step in which each column of truth values is entered in the table.

The truth table of the proposition then consists of the original columns under the variables and the last step, that is, the last column entered into the table.

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T			T
T	F		T			F
F	T		F			T
F	F		F			F
Step			1			1

(*a*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T		F	T
T	F		T		T	F
F	T		F		F	T
F	F		F		T	F
Step			1		2	1

(*b*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T	F	F	T
T	F		T	T	T	F
F	T		F	F	F	T
F	F		F	F	T	F
Step			1	3	2	1

(*c*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T	T	T	F	F	T
F	T	F	T	T	T	F
F	F	T	F	F	F	T
F	F	T	F	F	T	F
Step		4	1	3	2	1

(*d*)

Fig. 4-4

4.5 TAUTOLOGIES AND CONTRADICTIONS

Some propositions $P(p, q, \ldots)$ contain only T in the last column of their truth tables or, in other words, they are true for any truth values of their variables. Such propositions are called *tautologies.* Analogously, a proposition $P(p, q, \ldots)$ is called a *contradiction* if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables. For example, the proposition "p or not p", that is, $p \vee \neg p$, is a tautology, and the proposition "p and not p", that is, $p \wedge \neg p$, is a contradiction. This is verified by looking at their truth tables in Fig. 4-5. (The truth tables have only two rows since each proposition has only the one variable p.)

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(*a*) $p \vee \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(*b*) $p \wedge \neg p$

Fig. 4-5

Note that the negation of a tautology is a contradiction since it is always false, and the negation of a contradiction is a tautology since it is always true.

Now let $P(p, q, \ldots)$ be a tautology, and let $P_1(p, q, \ldots), P_2(p, q, \ldots), \ldots$ be any propositions. Since $P(p, q, \ldots)$ does not depend upon the particular truth values of its variables $p, q, \ldots$, we can substitute P_1 for p, P_2 for $q, \ldots$ in the tautology $P(p, q, \ldots)$ and still have a tautology. In other words:

Theorem 4.1 (Principle of Substitution): If $P(p, q, \ldots)$ is a tautology, then $P(P_1, P_2, \ldots)$ is a tautology for any propositions $P_1, P_2, \ldots$.

4.6 LOGICAL EQUIVALENCE

Two propositions $P(p, q, \ldots)$ and $Q(p, q, \ldots)$ are said to be *logically equivalent*, or simply *equivalent* or *equal*, denoted by

$$P(p, q, \ldots) \equiv Q(p, q, \ldots)$$

if they have identical truth tables. Consider, for example, the truth tables of $\neg (p \wedge q)$ and $\neg p \vee \neg q$ appearing in Fig. 4-6. Observe that both truth tables are the same, that is, both propositions are false in the first case and true in the other three cases. Accordingly, we can write

$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$

In other words, the propositions are logically equivalent.

Remark: Consider the statement

"It is not the case that roses are red and violets are blue"

This statement can be written in the form $\neg (p \vee q)$ where:

p is "roses are red" and q is "violets are blue"

However, as noted above, $\neg (p \wedge q) \equiv \neg p \vee \neg q$. Thus the statement

"Roses are not red, or violets are not Blue."

has the same meaning as the given statement.

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(*a*) $\neg(p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(*b*) $\neg p \vee \neg q$

Fig. 4-6

4.7 ALGEBRA OF PROPOSITIONS

Propositions satisfy various laws which are listed in Table 4-1. (In this table, T and F are restricted to the truth values "true" and "false", respectively.) We state this result formally.

Theorem 4.2: Propositions satisfy the laws of Table 4-1.

Table 4-1 Laws of the algebra of propositions

Idempotent laws	
(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws	
(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws	
(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws	
(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws	
(5a) $p \vee \mathrm{T} \equiv p$	(5b) $p \wedge \mathrm{F} \equiv p$
(6a) $p \vee \mathrm{T} \equiv \mathrm{T}$	(6b) $p \wedge \mathrm{F} \equiv \mathrm{F}$
Complement laws	
(7a) $p \vee \neg p \equiv \mathrm{T}$	(7b) $p \wedge \neg p \equiv \mathrm{F}$
(8a) $\neg \mathrm{T} \equiv \mathrm{F}$	(8b) $\neg \mathrm{F} \equiv \mathrm{T}$
Involution law	
(9) $\neg\neg p \equiv p$	
DeMorgan's laws	
(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

4.8 CONDITIONAL AND BICONDITIONAL STATEMENTS

Many statements, particularly in mathematics, are of the form "If p then q". Such statements are called *conditional* statements and are denoted by

$$p \rightarrow q$$

The conditional $p \rightarrow q$ is frequently read "p implies q" or "p only if q".

Another common statement is of the form "p if and only if q". Such statements are called *biconditional* statements and are denoted by

$$p \leftrightarrow q$$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(a) $p \rightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(b) $p \leftrightarrow q$

Fig. 4-7

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

$\neg p \vee q$

Fig. 4-8

The truth values of $p \to q$ and $p \leftrightarrow q$ are defined by the tables in Fig. 4-7. Observe that:

(*a*) The conditional $p \to q$ is false only when the first part p is true and the second part q is false. Accordingly, when p is false, the conditional $p \to q$ is true regardless of the truth value of q.

(*b*) The biconditional $p \leftrightarrow q$ is true whenever p and q have the same truth values and false otherwise.

The truth table of the proposition $\neg p \vee q$ appears in Fig. 4-8. Observe that the truth tables of $\neg p \vee q$ and $p \to q$ are identical, that is, they are both false only in the second case. Accordingly, $p \to q$ is logically equivalent to $\neg p \vee q$; that is,

$$p \to q \equiv \neg p \vee q$$

In other words, the conditional statement "If p then q" is logically equivalent to the statement "Not p or q" which only involves the connectives $\vee$ and $\neg$ and thus was already a part of our language. We may regard $p \to q$ as an abbreviation for an oft-recurring statement.

4.9 ARGUMENTS

An *argument* is an assertion that a given set of propositions $P_1, P_2, \ldots, P_n$, called *premises*, yields (has a consequence) another proposition Q, called the *conclusion*. Such an argument is denoted by

$$P_1, P_2, \ldots, P_n \vdash Q$$

The notion of a "logical argument" or "valid argument" is formalized as follows:

Definition 4.4: An argument $P_1, P_2, \ldots, P_n \vdash Q$ is said to be *valid* if Q is true whenever all the premises $P_1, P_2, \ldots, P_n$ are true.

An argument which is not valid is called a *fallacy*.

EXAMPLE 4.5

(*a*) The following argument is valid:

$$p, p \to q \vdash q \qquad \textbf{(\textit{Law of Detachment})}$$

The proof of this rule follows from the truth table in Fig. 4-9. Specifically, p and $p \to q$ are true simultaneously only in Case (row) 1, and in this case q is true.

(*b*) The following argument is a fallacy:

$$p \to q, q \vdash p$$

For $p \to q$ and q are both true in Case (row) 3 in the truth table in Fig. 4-9, but in this case p is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Fig. 4-9

Now the propositions $P_1, P_2, \ldots, P_n$ are true simultaneously if and only if the proposition $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ is true. Thus the argument $P_1, P_2, \ldots, P_n \vdash Q$ is valid if and only if Q is true whenever $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ is true or, equivalently, if the proposition $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \to Q$ is a tautology. We state this result formally.

Theorem 4.3: The argument $P_1, P_2, \ldots, P_n \vdash Q$ is valid if and only if the proposition $(P_1 \wedge P_2 \ldots \wedge P_n) \rightarrow Q$ is a tautology.

We apply this theorem in the next example.

EXAMPLE 4.6 A fundamental principle of logical reasoning states:

"If p implies q and q implies r, then p implies r."

That is, the following argument is valid:

$$p \rightarrow q,\ q \rightarrow r \vdash p \rightarrow r \quad (\textit{Law of Syllogism})$$

This fact is verified by the truth table in Fig. 4-10 which shows that the following proposition is a tautology:

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

Equivalently, the argument is valid since the premises $p \rightarrow q$ and $q \rightarrow r$ are true simultaneously only in Cases (rows) 1, 5, 7 and 8, and in these cases the conclusion $p \rightarrow r$ is also true. (Observe that the truth table required $2^3 = 8$ lines since there are three variables p, q and r.)

p	q	r	$[(p$	$\rightarrow$	$q)$	$\wedge$	$(q$	$\rightarrow$	$r)]$	$\rightarrow$	$(p$	$\rightarrow$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	F	F	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T	T	T	F	T	T
F	T	F	F	T	T	F	T	F	F	T	F	T	F
F	F	T	F	T	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F
Step			1	2	1	3	1	2	1	4	1	2	1

Fig. 4-10

We now apply the above theory to arguments involving specific statements. We emphasize that the validity of an argument does not depend upon the truth values nor the content of the statements appearing in the argument, but upon the particular form of the argument. This is illustrated in the following example.

EXAMPLE 4.7 Consider the following argument:

S_1: If a man is a bachelor, he is unhappy.
S_2: If a man is unhappy, he dies young.
....
S: Bachelors die young.

Here the statement S below the line denotes the conclusion of the argument, and the statements S_1 and S_2 above the line denote the premises. We claim that the argument $S_1, S_2 \vdash S$ is valid. For the argument is of the form

$$p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$$

where p is "He is a bachelor", q is "He is unhappy" and r is "He dies young"; and by Example 4.6 this argument (Law of Syllogism) is valid.

4.10 LOGICAL IMPLICATION

A proposition $P(p, q, \ldots)$ is said to *logically imply* a proposition $Q(p, q, \ldots)$, written

$$P(p, q, \ldots) \Rightarrow Q(p, q, \ldots)$$

if $Q(p, q, \ldots)$ is true whenever $P(p, q, \ldots)$ is true.

EXAMPLE 4.8 We claim that p logically implies $p \vee q$. For consider the truth table in Fig. 4-11. Observe that p is true in Cases (rows) 1 and 2, and in these cases $p \vee q$ is also true. Thus $p \Rightarrow p \vee q$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Fig. 4-11

Now if $Q(p, q, \ldots)$ is true whenever $P(p, q, \ldots)$ is true, then the argument

$$P(p, q, \ldots) \vdash Q(p, q, \ldots)$$

is valid; and conversely. Furthermore, the argument $P \vdash Q$ is valid if and only if the conditional statement $P \rightarrow Q$ is always true, i.e., a tautology. We state this result formally.

Theorem 4.4: For any propositions $P(p, q, \ldots)$ and $Q(p, q, \ldots)$, the following three statements are equivalent:

(i) $P(p, q, \ldots)$ logically implies $Q(p, q, \ldots)$.

(ii) The argument $P(p, q, \ldots) \vdash Q(p, q, \ldots)$ is valid.

(iii) The proposition $P(p, q, \ldots) \rightarrow Q(p, q, \ldots)$ is a tautology.

We note that some logicians and many texts use the word "implies" in the same sense as we use "logically implies", and so they distinguish between "implies" and "if... then". These two distinct concepts are, of course, intimately related as seen in the above theorem.

4.11 PROPOSITIONAL FUNCTIONS, QUANTIFIERS

Let A be a given set. A *propositional function* (or: an *open sentence* or *condition*) defined on A is an expression

$$p(x)$$

which has the property that $p(a)$ is true or false for each $a \in A$. That is, $p(x)$ becomes a statement (with a truth value) whenever any element $a \in A$ is substituted for the variable x. The set A is called the *domain* of $p(x)$, and the set T_p of all elements of A for which $p(a)$ is true is called the *truth set* of $p(x)$. In other words,

$$T_p = \{x\colon x \in A,\ p(x) \text{ is true}\} \qquad \text{or} \qquad T_p = \{x\colon p(x)\}$$

Frequently, when A is some set of numbers, the condition $p(x)$ has the form of an equation or inequality involving the variable x.

EXAMPLE 4.9 Find the truth set of each propositional function $p(x)$ defined on the set **N** of positive integers.

(*a*) Let $p(x)$ be "$x+2>7$". Its truth set is

$$\{x: x \in \mathbf{N},\ x+2>7\} = \{6, 7, 8, \ldots\}$$

consisting of all integers greater than 5.

(*b*) Let $p(x)$ be "$x+5<3$". Its truth set is

$$\{x: x \in \mathbf{N},\ x+5<3\} = \varnothing$$

the empty set. In other words, $p(x)$ is not true for any positive integer in **N**.

(*c*) Let $p(x)$ be "$x+5>1$". Its truth set is

$$\{x: x \in \mathbf{N},\ x+5>1\} = \mathbf{N}$$

Thus $p(x)$ is true for every element in **N**.

Remark: The above example shows that if $p(x)$ is a propositional function defined on a set A then $p(x)$ could be true for all $x \in A$, for some $x \in A$, or for no $x \in A$. The next two subsections discusses quantifiers related to such propositional functions.

Universal Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\forall x \in A)p(x) \quad \text{or} \quad \forall x\, p(x) \tag{4.1}$$

which reads "For every x in A, $p(x)$ is a true statement" or, simply, "For all x, $p(x)$". The symbol

$$\forall$$

which reads "for all" or "for every" is called the *universal quantifier*. The statement (*4.1*) is equivalent to the statement

$$T_p = \{x: x \in A,\ p(x)\} = A \tag{4.2}$$

that is, that the truth set of $p(x)$ is the entire set A.

The expression $p(x)$ by itself is an open sentence or condition and therefore has no truth value. However, $\forall x\, p(x)$, that is $p(x)$ preceded by the quantifier $\forall$, does have a truth value which follows from the equivalence of (*4.1*) and (*4.2*). Specifically:

Q_1: If $\{x: x \in A, p(x)\} = A$ then $\forall x\, p(x)$ is true; otherwise, $\forall x\, p(x)$ is false.

EXAMPLE 4.10

(*a*) The proposition $(\forall n \in \mathbf{N})(n+4>3)$ is true since

$$\{n: n+4>3\} = \{1, 2, 3, \ldots\} = \mathbf{N}$$

(*b*) The proposition $(\forall n \in \mathbf{N})(n+2>8)$ is false since

$$\{n: n+2>8\} = \{7, 8, \ldots\} \neq \mathbf{N}$$

(*c*) The symbol $\forall$ can be used to define the intersection of an indexed collection $\{A_i: i \in I\}$ of sets A_i as follows:

$$\cap(A_i: i \in I) = \{x: \forall_i \in I,\ x \in A_i\}$$

Existential Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\exists x \in A)p(x) \quad \text{or} \quad \exists X,\ p(x) \tag{4.3}$$

which reads "There exists an x in A such that $p(x)$ is a true statement" or, simply, "For some x, $p(x)$". The symbol

$$\exists$$

which reads "there exists" or "for some" or "for at least one" is called the *existential quantifier*. Statement (*4.3*) is equivalent to the statement

$$T_p = \{x: x \in A,\ p(x)\} \neq \varnothing \tag{4.4}$$

i.e., that the truth set of $p(x)$ is not empty. Accordingly, $\exists x\, p(x)$, that is, $p(x)$ preceded by the quantifier $\exists$, does have a truth value. Specifically:

Q_2: If $\{x: p(x)\} \neq \varnothing$ then $\exists x\ p(x)$ is true; otherwise, $\exists x\, p(x)$ is false.

EXAMPLE 4.11

(*a*) The proposition $(\exists n \in \mathbf{N})(n+4<7)$ is true since $\{n: n+4<7\} = \{1,2\} \neq \varnothing$.

(*b*) The proposition $(\exists n \in \mathbf{N})(n+6<4)$ is false since $\{n: n+6<4\} = \varnothing$.

(*c*) The symbol $\exists$ can be used to define the union of an indexed collection $\{A_i: i \in I\}$ of sets A_i as follows:

$$\cup(A_i: i \in I) = \{x: \exists i \in I,\ x \in A_i\}$$

Notation

Let $A = \{2,3,5\}$ and let $p(x)$ be the sentence "x is a prime number" or, simply "x is prime". Then

"Two is prime and three is prime and five is prime" (*)

can be denoted by

$$p(2) \wedge p(3) \wedge p(5) \quad \text{or} \quad \wedge(a \in A,\ p(a))$$

which is equivalent to the statement

"Every number in A is prime" or $\forall a \in A,\ p(a)$ (**)

Similarly, the proposition

"Two is prime or three is prime or five is prime."

can be denoted by

$$p(2) \vee p(3) \vee p(5) \quad \text{or} \quad \vee(a \in A,\ p(a))$$

which is equivalent to the statement

"At least one number in A is prime" or $\exists a \in A,\ p(a)$

In other words

$$\wedge(a \in A,\ p(a)) \equiv \forall a \in A,\ p(a) \quad \text{and} \quad \vee(a \in A,\ p(a)) \equiv \exists a \in A,\ p(a)$$

Thus the symbols $\wedge$ and $\vee$ are sometimes used instead of $\forall$ and $\exists$.

Remark: If A were an infinite set, then a statement of the form (*) cannot be made since the sentence would not end; but a statement of the form (**) can always be made, even when A is infinite.

4.12 NEGATION OF QUANTIFIED STATEMENTS

Consider the statement: "All math majors are male". Its negation reads:

"It is not the case that all math majors are male" or, equivalently,

"There exists at least one math major who is a female (not male)"

Symbolically, using M to denoted the set of math majors, the above can be written as

$$\neg(\forall x \in M)\,(x \text{ is male}) \equiv (\exists x \in M)\,(x \text{ is not male})$$

or, when $p(x)$ denotes "x is male",

$$\neg(\forall x \in M)p(x) \equiv (\exists x \in M)\neg p(x) \qquad \text{or} \qquad \neg\forall x p(x) \equiv \exists x \neg p(x)$$

The above is true for any proposition p(x). That is:

Theorem 4.5 (DeMorgan): $\neg(\forall x \in A)p(x) \equiv (\exists x \in A)\neg p(x)$.

In other words, the following two statements are equivalent:

(1) It is not true that, for all $a \in A$, $p(a)$ is true.
(2) There exists an $a \in A$ such that $p(a)$ is false.

There is an analogous theorem for the negation of a proposition which contains the existential quantifier.

Theorem 4.6 (DeMorgan): $\neg(\exists x \in A)p(x) \equiv (\forall x \in A)\neg p(x)$.

That is, the following two statements are equivalent:

(1) It is not true that for some $a \in A$, $p(a)$ is true.
(2) For all $a \in A$, $p(a)$ is false.

EXAMPLE 4.12

(*a*) The following statements are negatives of each other:

"For all positive integers n we have $n + 2 > 8$"
"There exists a positive integer n such that $n + 2 \not> 8$"

(*b*) The following statements are also negatives of each other:

"There exists a (living) person who is 150 years old"
"Every living person is not 150 years old"

Remark: The expression $\neg p(x)$ has the obvious meaning; that is:

"The statement $\neg p(a)$ is true when $p(a)$ is false, and vice versa"

Previously, $\neg$ was used as an operation on statements; here $\neg$ is used as an operation on propositional functions. Similarly, $p(x) \wedge q(x)$, read "$p(x)$ and $q(x)$", is defined by:

"The statement $p(a) \wedge q(a)$ is true when $p(a)$ and $q(a)$ are true"

Similarly, $p(x) \vee q(x)$, read "$p(x)$ or $q(x)$", is defined by:

"The statement $p(a) \vee q(a)$ is true when $p(a)$ or $q(a)$ is true"

Thus in terms of truth sets:

(i) $\neg p(x)$ is the complement of $p(x)$.
(ii) $p(x) \wedge q(x)$ is the intersection of $p(x)$ and $q(x)$.
(iii) $p(x) \vee q(x)$ is the union of $p(x)$ and $q(x)$.

One can also show that the laws for propositions also hold for propositional functions. For example, we have DeMorgan's laws;

$$\neg(p(x) \wedge q(x)) \equiv \neg p(x) \vee \neg q(x) \quad \text{and} \quad \neg(p(x) \vee q(x)) \equiv \neg p(x) \wedge \neg q(x)$$

Counterexample

Theorem 4.6 tells us that to show that a statement $\forall x, p(x)$ is false, it is equivalent to show that $\exists x \neg p(x)$ is true or, in other words, that there is an element x_0 with the property that $p(x_0)$ is false. Such an element x_0 is called a *counterexample* to the statement $\forall x, p(x)$.

EXAMPLE 4.13

(*a*) Consider the statement $\forall x \in \mathbf{R}, |x| \neq 0$. The statement is false since 0 is a counterexample, that is, $|0| \neq 0$ is not true.

(*b*) Consider the statement $\forall x \in \mathbf{R}, x^2 \geq x$. The statement is not true since, for example, $\frac{1}{2}$ is a counterexample. Specifically, $(\frac{1}{2})^2 \geq \frac{1}{2}$ is not true, that is, $(\frac{1}{2})^2 < \frac{1}{2}$.

(*c*) Consider the statement $\forall x \in \mathbf{N}, x^2 \geq x$. This statement is true where $\mathbf{N}$ is the set of positive integers. In other words, there does not exist a positive integer n for which $n^2 < n$.

Propositional Functions with More Than One Variable

A propositional function (of n variables) defined over a product set $A = A_1 \times \ldots \times A_n$ is an expression

$$p(x_1, x_2, \ldots, x_n)$$

which has the property that $p(a_1, a_2, \ldots, a_n)$ is true or false for any n-tuple $(a_1, \ldots a_n)$ in A. For example,

$$x + 2y + 3z < 18$$

is a propositional function on $\mathbf{N}^3 = \mathbf{N} \times \mathbf{N} \times \mathbf{N}$. Such a propositional function has no truth value. However, we do have the following:

Basic Principle: A propositional function preceded by a quantifier for each variable, for example,

$$\forall x \exists y, p(x, y) \quad \text{or} \quad \exists x\, \forall y \exists z, p(x, y, z)$$

denotes a statement and has a truth value.

EXAMPLE 4.14 Let $B = \{1, 2, 3, \ldots, 9\}$ and let $p(x, y)$ denote "$x + y = 10$". Then $p(x, y)$ is a propositional function on $A = B^2 = B \times B$.

(*a*) The following is a statement since there is a quantifier for each variable:

$\forall x \exists y, p(x, y)$, that is, "For every x, there exists a y such that $x + y = 10$"

This statement is true. For example, if $x = 1$, let $y = 9$; if $x = 2$, let $y = 8$, and so on.

(*b*) The following is also a statement:

$\exists y \forall x, p(x, y)$, that is, "There exists a y such that, for every x, we have $x + y = 10$"

No such y exists; hence this statement is false.

Note that the only difference between (*a*) and (*b*) is the order of the quantifiers. Thus a different ordering of the quantifiers may yield a different statement. We note that, when translating such quantified statements into English, the expression "such that" frequently follows "there exists".

Negating Quantified Statements with More Than One Variable

Quantified statements with more than one variable may be negated by successively applying Theorems 4.5 and 4.6. Thus each $\forall$ is changed to $\exists$ and each $\exists$ is changed to $\forall$ as the negation symbol $\neg$ passes through the statement from left to right. For example,

$$\neg[\forall x \exists y \exists z, p(x,y,z)] \equiv \exists x \neg[\exists y \exists z, p(x,y,z)] \equiv \exists x \forall y[\neg \exists z, p(x,y,z)$$
$$\equiv \exists x \forall y \forall z, \neg p(x,y,z)$$

Naturally, we do not put in all the steps when negating such quantified statements.

EXAMPLE 4.15

(*a*) . Consider the quantified statement:

"Every student has at least one course where the lecturer is a teaching assistant"

Its negation is the statement:

"There is a student such that in every course the lecturer is not a teaching assistant"

(*b*) The formal definition that L is the limit of a sequence $a_1, a_2, \ldots$ follows:

$$\forall \in > 0, \exists n_0 \in \mathbf{N}, \quad \forall n > n_0, \quad |a_n - L| < \in$$

Thus L is not the limit of the sequence $a_1, a_2, \ldots$ when:

$$\exists \in > 0, \quad \forall n_0 \in \mathbf{N}, \quad \exists n > n_0, \quad |a_n - L| \geq \in$$

Solved Problems

PROPOSITIONS AND LOGICAL OPERATIONS

4.1. Let p be "It is cold" and let q be "It is raining". Give a simple verbal sentence which describes each of the following statements: (*a*) $\neg p$; (*b*) $p \wedge q$; (*c*) $p \vee q$; (*d*) $q \vee \neg p$.

In each case, translate $\wedge$, $\vee$, and $\sim$ to read "and", "or", and "It is false that" or "not", respectively, and then simplify the English sentence.

(*a*) It is not cold.

(*b*) It is cold and raining.

(*c*) It is cold or it is raining.

(*d*) It is raining or it is not cold.

4.2. Let p be "Erik reads *Newsweek*", let q be "Erik reads *The New Yorker*", and let r be "Erik reads *Time*". Write each of the following in symbolic form:

(*a*) Erik reads *Newsweek* or *The New Yorker*, but not *Time*.

(*b*) Erik reads *Newsweek* and *The New Yorker*, or he does not read *Newsweek* and *Time*.

(*c*) It is not true that Erik reads *Newsweek* but not *Time*.

(*d*) It is not true that Erik reads *Time* or *The New Yorker* but not *Newsweek*.

Use $\vee$ for "or", $\wedge$ for "and" (or, its logical equivalent, "but"), and $\neg$ for "not" (negation).

(*a*) $(p \vee q) \wedge \neg r$; (*b*) $(p \wedge q) \vee \neg (p \wedge r)$; (*c*) $\neg (p \wedge \neg r)$; (*d*) $\neg [(r \vee q) \wedge \neg p]$.

TRUTH VALUES AND TRUTH TABLES

4.3. Determine the truth value of each of the following statements:

(*a*) $4+2=5$ and $6+3=9$. (*c*) $4+5=9$ and $1+2=4$.
(*b*) $3+2=5$ and $6+1=7$. (*d*) $3+2=5$ and $4+7=11$.

The statement "*p* and *q*" is true only when both substatements are true. Thus: (*a*) false; (*b*) true; (*c*) false; (*d*) true.

4.4. Find the truth table of $\neg p \wedge q$.

See Fig. 4-12, which gives both methods for constructing the truth table.

p	q	$\neg p$	$\neg p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

(*a*) Method 1

p	q	$\neg$	p	$\wedge$	q
T	T	F	T	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	F	F	F
	Step	2	1	3	1

(*b*) Method 2

Fig. 4-12

4.5. Verify that the proposition $p \vee \neg (p \wedge q)$ is tautology.

Construct the truth table of $p \vee \neg (p \wedge q)$ as shown in Fig. 4-13. Since the truth value of $p \vee \neg (p \wedge q)$ is T for all values of p and q, the proposition is a tautology.

p	q	$p \wedge q$	$\neg (p \wedge q)$	$p \vee \neg (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Fig. 4-13

4.6. Show that the propositions $\neg (p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Construct the truth tables for $\neg (p \wedge q)$ and $\neg p \vee \neg q$ as in Fig. 4-14. Since the truth tables are the same (both propositions are false in the first case and true in the other three cases), the propositions $\neg (p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent and we can write

$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$

p	q	$p \wedge q$	$\neg (p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(*a*) $\neg (p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(*b*) $\neg p \vee \neg q$

Fig. 4-14

4.7. Use the laws in Table 4-1 to show that $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$.

	Statement	Reason
(1)	$\neg(p \vee q) \vee (\neg p \wedge q) \equiv (\neg p \wedge \neg q) \vee (\neg p \wedge q)$	DeMorgan's law
(2)	$\equiv \neg p \wedge (\neg q \vee q)$	Distributive law
(3)	$\equiv \neg p \wedge T$	Complement law
(4)	$\equiv \neg p$	Identity law

CONDITIONAL STATEMENTS

4.8. Rewrite the following statements without using the conditional:

(*a*) If it is cold, he wears a hat.

(*b*) If productivity increases, then wages rise.

Recall that "If p then q" is equivalent to "Not p or q"; that is, $p \rightarrow q \equiv \neg p \vee q$. Hence,

(*a*) It is not cold or he wears a hat.

(*b*) Productivity does not increase or wages rise.

4.9. Determine the contrapositive of each statement:

(*a*) If John is a poet, then he is poor;

(*b*) Only if Marc studies will he pass the test.

(*a*) The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$. Hence the contrapositive of the given statement is

If John is not poor, then he is not a poet.

(*b*) The given statement is equivalent to "If Marc passes the test, then he studied". Hence its contrapositive is

If Marc does not study, then he will not pass the test.

4.10. Consider the conditional proposition $p \rightarrow q$. The simple propositions $q \rightarrow p$, $\neg p \rightarrow \neg q$, and $\neg q \rightarrow \neg p$ are called, respectively, the *converse*, *inverse*, and *contrapositive* of the conditional $p \rightarrow q$. Which if any of these propositions are logically equivalent to $p \rightarrow q$?

Construct their truth tables as in Fig. 4-15. Only the contrapositive $\neg q \rightarrow \neg p$ is logically equivalent to the original conditional proposition $p \rightarrow q$.

				Conditional	Converse	Inverse	Contrapositive
p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Fig. 4-15

4.11. Write the negation of each statement as simply as possible:

(*a*) If she works, she will earn money.
(*b*) He swims if and only if the water is warm.
(*c*) If it snows, then they do not drive the car.

(*a*) Note that $\neg (p \rightarrow q) \equiv p \wedge \neg q$; hence the negation of the statement follows:

She works or she will not earn money.

(*b*) Note that $\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q \equiv \neg p \leftrightarrow q$; hence the negation of the statement is either of the following:

He swims if and only if the water is not warm.
He does not swim if and only if the water is warm.

(*c*) Note that $\neg (p \rightarrow \neg q) \equiv p \wedge \neg \neg q \equiv p \wedge q$. Hence the negation of the statement follows:

It snows and they drive the car.

ARGUMENTS

4.12. Show that the following argument is a fallacy: $p \rightarrow q, \neg p \vdash \neg q$.

Construct the truth table for $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ as in Fig. 4-16. Since the proposition $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ is not a tautology, the argument is a fallacy. Equivalently, the argument is a fallacy since in third line of the truth table $p \rightarrow q$ and $\neg p$ are true but $\neg q$ is false.

p	q	$p \rightarrow q$	$\neg p$	$(p \rightarrow q) \wedge \neg p$	$\neg q$	$[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

Fig. 4-16

4.13. Determine the validity of the following argument: $p \rightarrow q, \neg q \vdash \neg p$.

Construct the truth table for $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$ as in Fig. 4-17. Since the proposition $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$ is a tautology, the argument is valid.

p	q	$[(p$	$\rightarrow$	$q)$	$\wedge$	$\neg$	$q]$	$\rightarrow$	$\neg$	p
T	T	T	T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	F	T	F	T
F	T	F	T	T	F	F	T	T	T	F
F	F	F	T	F	T	T	F	T	T	F
Step		1	2	1	3	2	1	4	2	1

Fig. 4-17

4.14. Prove that the following argument is valid: $p \rightarrow \neg q, r \rightarrow q, r \vdash \neg p$.

Construct the truth tables of the premises and conclusion as in Fig. 4-18. Now, $p \rightarrow \neg q$, $r \rightarrow q$, and r are true simultaneously only in the fifth row of the table, where $\neg p$ is also true. Hence, the argument is valid.

	p	q	r	$p \rightarrow \neg q$	$r \rightarrow q$	$\neg q$
1	T	T	T	F	T	F
2	T	T	F	F	T	F
3	T	F	T	T	F	F
4	T	F	F	T	T	F
5	F	T	T	T	T	T
6	F	T	F	T	T	T
7	F	F	T	T	F	T
8	F	F	F	T	T	T

Fig. 4-18

4.15. Test the validity of the following argument:

If two sides of a triangle are equal, then the opposite angles are equal.
Two sides of a triangle are not equal.

The opposite angles are not equal.

First translate the argument into the symbolic form $p \rightarrow q, \neg p \vdash \neg q$, where p is "Two sides of a triangle are equal" and q is "The opposite angles are equal". By Problem 4.12, this argument is a fallacy.

Remark: Although the conclusion *does* follow from the second premise and axioms of Euclidean geometry, the above argument does not constitute such a proof since the argument is a fallacy.

4.16. Determine the validity of the following argument:

If 7 is less than 4, then 7 is not a prime number.
7 is not less than 4.

---

7 is a prime number.

First translate the argument into symbolic form. Let p be "7 is less than 4" and q be "7 is a prime number". Then the argument is of the form

$$p \rightarrow \neg q, \neg p \vdash q$$

Now, we construct a truth table as shown in Fig. 4-19. The above argument is shown to be a fallacy since, in the fourth line of the truth table, the premises $p \rightarrow \neg q$ and $\neg p$ are true, but the conclusion q is false.

Remark: The fact that the conclusion of the argument happens to be a true statement is irrelevant to the fact that the argument presented is a fallacy.

p	q	$\neg q$	$p \rightarrow \neg q$	$\neg p$
T	T	F	F	F
T	F	T	T	F
F	T	F	T	T
F	F	T	T	T

Fig. 4-19

QUANTIFIERS AND PROPOSITIONAL FUNCTIONS

4.17. Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of each of the following statements:

(a) $(\exists x \in A)(x + 3 = 10)$ (b) $(\forall x \in A)(x + 3 < 10)$

(c) $(\exists x \in A)(x + 3 < 5)$ (d) $(\forall x \in A)(x + 3 \leq 7)$

(a) False. For no number in A is a solution to $x + 3 = 10$.

(b) True. For every number in A satisfies $x + 3 < 10$.

(c) True. For if $x_0 = 1$, then $x_0 + 3 < 5$, i.e., 1 is a solution.

(d) False. For if $x_0 = 5$, then $x_0 + 3$ is not less than or equal 7. In other words, 5 is not a solution to the given condition.

4.18. Determine the truth value of each of the following statements where $\mathbf{U} = \{1, 2, 3\}$ is the universal set: (a) $\exists x\ \forall y, x^2 < y + 1$; (b) $\forall x\ \exists y, x^2 + y^2 < 12$; (c) $\forall x\ \forall y, x^2 + y^2 < 12$.

(a) True. For if $x = 1$, then 1, 2 and 3 are all solutions to $1 < y + 1$.

(b) True. For each x_0, let $y = 1$; then $x_0^2 + 1 < 12$ is a true statement.

(c) False. For if $x_0 = 2$ and $y_0 = 3$, then $x_0^2 + y_0^2 < 12$ is not a true statement.

4.19. Negate each of the following statements:

(a) $\exists x\ \forall y,\ p(x, y)$; (b) $\exists x \forall y,\ p(x, y)$; (c) $\exists y\ \exists x\ \forall z,\ p(x, y, z)$.

Use $\neg\ \forall x\ p(x) \equiv \exists x\ \neg\ p(x)$ and $\neg\ \exists x\ p(x) \equiv \forall x\ \neg\ p(x)$:

(a) $\neg\ (\exists x\ \forall y,\ p(x, y)) \equiv \forall x\ \exists y\ \neg\ p(x, y)$.

(b) $\neg\ (\forall x\ \forall y,\ p(x, y)) \equiv \exists x\ \exists y\ \neg\ p(x, y)$.

(c) $\neg\ (\exists y\ \exists x\ \forall z,\ p(x, y, z)) \equiv \forall y\ \forall x\ \exists z\ \neg\ p(x, y, z)$.

4.20. Let $p(x)$ denote the sentence "$x + 2 > 5$". State whether or not $p(x)$ is a propositional function on each of the following sets: (a) $\mathbf{N}$, the set of positive integers; (b) $M = \{-1, -2, -3, \ldots\}$; (c) $\mathbf{C}$, the set of complex number.

(a) Yes.

(b) Although $p(x)$ is false for every element in M, $p(x)$ is still a propositional function on M.

(c) No. Note that $2i + 2 > 5$ does not have any meaning. In other words, inequalities are not defined for complex numbers.

4.21. Negate each of the following statements: (a) All students live in the dormitories. (b) All mathematics majors are males. (c) Some students are 25 (years) or older.

Use Theorem 4.5 to negate the quantifiers.

(a) At least one student does not live in the dormitories. (Some students do not live in the dormitories.)

(b) At least one mathematics major is female. (Some mathematics majors are female.)

(c) None of the students is 25 or older. (All the students are under 25.)

Supplementary Problems

PROPOSITION AND LOGICAL OPERATIONS

4.22. Let p be "Audrey speaks French" and let q be "Audrey speaks Danish". Give a simple verbal sentence which describes each of the following:

(a) $p \vee q$; (b) $p \wedge q$; (c) $p \wedge \neg q$; (d) $\neg p \vee \neg q$; (e) $\neg\neg p$; (f) $\neg(\neg p \wedge \neg q)$.

4.23. Let p denote "He is rich" and let q denote "He is happy". Write each statement in symbolic form using p and q. Note that "He is poor" and "He is unhappy" are equivalent to $\neg p$ and $\neg q$, respectively.

(a) If he is rich, then he is unhappy.
(b) He is neither rich nor happy.
(c) It is necessary to be poor in order to be happy.
(d) To be poor is to be unhappy.

4.24. Find the truth tables for: (a) $p \vee \neg q$; (b) $\neg p \wedge \neg q$.

4.25. Verify that the proposition $(p \wedge q) \wedge \neg (p \vee q)$ is a contradiction.

ARGUMENTS

4.26. Test the validity of each argument:

(a) If it rains, Erik will be sick.
It did not rain.

Erik was not sick.

(b) If it rains, Erik will be sick.
Erik was not sick.

It did not rain.

4.27. Test the validity of the following argument:

If I study, then I will not fail mathematics.
If I do not play basketball, then I will study.
But I failed mathematics.

Therefore I must have played basketball.

4.28. Show that: (a) $p \wedge q$ logically implies $p \leftrightarrow q$, (b) $p \leftrightarrow \neg q$ does not logically imply $p \rightarrow q$.

QUANTIFIERS

4.29. Let $A = \{1, 2, \ldots, 9, 10\}$. Consider each of the following sentences. If it is a statement, then determine its truth value. If it is a propositional function, determine its truth set.

(a) $(\forall x \in A)(\exists y \in A)(x + y < 14)$.
(b) $(\forall y \in A)(x + y < 14)$.
(c) $(\forall x \in A)(\forall y \in A)(x + y < 14)$.
(d) $(\exists y \in A)(x + y < 14)$.

4.30. Negate each of the following statements:

(*a*) If the teacher is absent, then some students do not complete their homework.

(*b*) All the students completed their homework and the teacher is present.

(*c*) Some of the students did not complete their homework or the teacher is absent.

4.31. Negate each of the statements in Problem 4.17.

4.32. Find a counterexample for each statement where $U = \{3, 5, 7, 9\}$ is the universal set: (*a*) $\forall x, x + 3 \geq 7$; (*b*) $\forall x$, x is odd; (*c*) $\forall x$, x is prime; (*d*) $\forall x, |x| = x$.

Answers to Supplementary Problems

4.22. In each case, translate $\wedge$, $\vee$, and $\neg$ to read "and", "or", and "It is false that" or "not", respectively, and then simplify the English sentence.

(*a*) Audrey speaks French or Danish.

(*b*) Audrey speaks French and Danish.

(*c*) Audrey speaks French but not Danish.

(*d*) Audrey does not speak French or she does not speak Danish.

(*e*) It is not true that Audrey does not speak English.

(*f*) It is not true that Audrey speaks neither French nor Danish.

4.23. (*a*) $p \rightarrow \neg q$; (*b*) $\neg p \wedge \neg q$; (*c*) $q \rightarrow \neg p$; (*d*) $\neg p \leftrightarrow \neg q$.

4.24. The truth tables appear in Fig. 4-20.

p	q	$\neg q$	$p \vee \neg q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(*a*)

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(*b*)

Fig. 4-20

4.25. It is a contradiction since its truth table in Fig. 4-21 is false for all values of p and q.

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Fig. 4-21

4.26. First translate the arguments into symbolic form using, say, p for "It rains" and q for "Erik is sick" as follows:

$$(a)\ p \to q, \neg p \vdash \neg q, \qquad (b)\ p \to q, \neg q \vdash \neg p$$

By Problem 4.12, argument (*a*) is a fallacy. By Problem 4.13, argument (*b*) is valid.

4.27. Letting p be "I study", q be "I fail mathematics", and r be "I play basketball", the given argument has the form:

$$p \to \neg q, \neg r \to p, q \vdash r$$

Construct the truth tables as in Fig. 4-22 where the premises $p \to \neg q$, $\neg r \to p$, and q are true simultaneously only in the fifth line of the table, and in that case the conclusion r is also true. Hence the arguments is valid.

p	q	r	$\neg q$	$p \to \neg q$	$\neg r$	$\neg r \to p$
T	T	T	F	F	F	T
T	T	F	F	F	T	T
T	F	T	T	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	T	F	F	T	T	F
F	F	T	T	T	F	T
F	F	F	T	T	T	F

Fig. 4-22

4.28. (*a*) Construct the truth tables of $p \wedge q$ and $p \leftrightarrow q$ as in Fig. 4-23(*a*). Note that $p \wedge q$ is true only in the first line of the table where $p \leftrightarrow q$ is also true.

(*b*) Construct the truth tables of $p \leftrightarrow \neg q$ and $p \to q$ as in Fig. 4-23(*b*). Note that $p \leftrightarrow \neg q$ is true in the second line of the table where $p \to q$ is false.

p	q	$p \wedge q$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	T

(*a*)

p	q	$\neg q$	$p \leftrightarrow \neg q$	$p \to q$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	T
F	F	T	F	T

(*b*)

Fig. 4-23

4.29. (*a*) The open sentence in two variables is preceded by two quantifiers; hence it is a statement. Moreover, the statement is true.

(*b*) The open sentence is preceded by one quantifier; hence it is a propositional function of the other variable. Note that for every $y \in A$, $x_0 + y < 14$ if and only if $x_0 = 1, 2$, or 3. Hence the truth set is $\{1, 2, 3\}$.

(*c*) It is a statement and it is false: if $x_0 = 8$ and $y_0 = 9$, then $x_0 + y_0 < 14$ is not true.

(*d*) It is an open sentence in x. The truth set is A itself.

4.30. (*a*) The teacher is absent and all the students completed their homework.

(*b*) Some of the students did not complete their homework or the teacher is absent.

(*c*) All the students completed their homework and the teacher is present.

4.31. (*a*) $(\forall x \in A)(x + 3 \neq 10)$.

(*b*) $(\exists x \in A)(x + 3 \geq 10)$.

(*c*) $(\forall x \in A)(x + 3 \geq 5)$.

(*d*) $(\exists x \in A)(x + 3 > 7)$.

4.32. (*a*) Here 5, 7, and 9 are counterexamples.

(*b*) The statement is true; hence no counterexample exists.

(*c*) Here 9 is the only counterexample.

(*d*) The statement is true; hence there is no counterexample.