Graph Theory - Lecture 2
Degrees and Degree Sequences

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1 Degrees

Definition 1.1 (Degree). *The* **degree** *of a vertex* $v \in V$ *of a graph* $\mathbf{G(V,E)}$*is the number of edgs of* G *which are incident with* v

$$d(v) = |\{e \in E | e = uv \text{ for some } u \in V\}|$$

Note : When the graph has to be specified, the notation used is : $d(v|G)$

Minimum Degree for $\mathbf{G(V,E)}$**:** $\delta(G)$

Maximum Degree for $\mathbf{G(V,E)}$**:** $\Delta(G)$

Indegree for $\mathbf{D(V,E)}$**:** Minimum Indegree : $\delta^-(G(V,E))$; Maximum Indegree : $\Delta^-(G(V,E))$

Outdegree for $\mathbf{D(V,E)}$**:** Minimum Outdegree : $\delta^+(G(V,E))$; Maximum Outdegree : $\Delta^+(G(V,E))$

For any graph $\mathbf{G(V,E)}$of order $|\boldsymbol{V}|$:

$$0 \leq \delta(G) \leq d(v) \leq \Delta(G) \leq (|\boldsymbol{V}| - 1)$$

Definition 1.2 (Regular Graph). *A graph* $\mathbf{G(V,E)}$*is said to be* **regular / k-regular** *if all the vertices have the same degree,* $\boldsymbol{k}$*.*

- A 3-regular graph is **cubic graph**
- A vertex with degree=**zero(0)** is called an **isolated vertex**
- A vertex with degree=**one(1)** is called a **pendant vertex**
- A vertex with odd degree : **odd vertex**
- A vertex with even degree : **even vertex**
- A *loop* incident on v is counted as **two(2) edges incident with** v i.e. $d(v) = 2$

Definition 1.3 (Degree of a Digraph). *For any* $v \in V$ *in graph* $\mathbf{D(V,E)}$,

- *the number of arcs/edges adjacent* **to v** *is the* **in-degree of v/inner-demi degree :** $d^-(v)$ *and*

- *the number of arcs/edges adjacent* **from v** *is the* **out-degree of v/outer-demi degree :** $d^+(v)$

and the total degree of v i.e

$$d(v) = d^-(v) + d^+(v)$$

Properties and Some special Graphs

Regular Digraph : $\{d(v) = k | \forall v \in V(D)\}$ i.e. all vertices $v \in V$ has the same degree

Isograph : $\forall v \in V$, $d^-(v) = d^+(v)$ i.e all vertices $v \in V$ has the same out-degree and in-degree

Isolated Vertex : A vertex with $d^-(v) = d^+(v) = 0$

Transmitter Vertex : If $d^+(v) > 0,\ d^-(v) = 0$

Reciever Vertex : If $d^+(v) = 0,\ d^-(v) > 0$

Carrier Vertex : If $d^-(v) = d^+(v) = 1$

Ordinary Vertex : Any other vertex is an **Ordinary Vertex**

Theorem 1.1 (First Theorem of Graph Theory)**.** *If* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *is a regular graph of size* $|V|$, *then*

$$\sum_{v \in V(G)} d(v) = 2|V| \tag{1}$$

Proof. When summing the degrees of the vertices of G, one counts each edge $e \in E(G)$ **twice**, once for each of the two vertices incident with $e \in E(G)$ $\square$

Proposition 1.1. *Suppose G is a bi-partite graph of size m with partite sets $U = \{u_1, u_2, \ldots, u_s\}$ and $W = \{w_1, w_2, \ldots, w_t\}$. Since every edge of G joins a vertex of U and a vertex of W,*

$$\sum_{i=1}^{s} d(u_i) = \sum_{j=1}^{t} d(w_i) = m \tag{2}$$

Corollary 1.2. *Every graph* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *has an* **even** *number of* <u>*odd vertices*</u>.

Proof. Let $\mathbf{G}(\mathbf{V}, \mathbf{E})$ be a graph of size $|E|$.
Divide $V(G)$ into two subsets V_1 consisting of odd vertices and V_2 consisting of even vertices..
By the First Theorem [Theorem 1.1]

$$\sum_{v \in V(G)} d(v) = \sum_{v \in V_1} d(v) + \sum_{v \in V_2} = 2m. \tag{3}$$

Thus,

$$\sum_{v \in V_1} d(v) = 2m - \sum_{v \in V_2} \tag{4}$$

which implies that $\sum_{v \in V_1} d(v)$ is **even**.
Since, each of the numbers $d(v)_{v \in V_1}$ is **odd**, the number of **odd vertices** of G is **even**. $\square$

Proposition 1.2. *For any graph* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *of order* $|V|$, $\Delta(G) \leq |V| - 1$

Proof. This is simply because a vertex can be joined to at most $(|V|-1)$ other vertices, multiple edges not being allowed. $\square$

Theorem 1.3 (Handshaking Dilemma)**.** *In any digraph, the sum of all the out-degrees and the sum of the in-degrees are both equal to the number of arcs.*

Proof. In any digraph, each arc has 2 ends, so it contributes exactly 1 to the sum of the out-degrees and 1 to the sum of the in-degrees. $\square$

(Prove) Use the Handshaking Dilemma to prove that, in any digraph, if the number of vertices with odd out-degree is odd then the number of vertices with odd in- degree is odd.

There are a few intuitive implications of the handshaking lemma:

- For a graph, the sum of degrees of all its nodes is even.
- In any graph, the sum of all the vertex-degrees is an even number.
- In any graph, the number of vertices of odd degree is even.
- If $\mathbf{G}(\mathbf{V},\mathbf{E})$has $|V(G)|$ vertices and is regular of degree r, then $\mathbf{G}(\mathbf{V},\mathbf{E})$has exactly $\frac{|V(G)|\times r}{2}$ edges.

Example 1.1. *A certain graph G has order 14 and size 27. The degree of each vertex of G is 3, 4, 5 respectively and 6 vertices of degree 4. How many vertices have degree 3 and how many have degee 5.*

Answer. *Let x be the number of vertices of G having degree 3.*
$\therefore \quad 3\times x + 4\times 6 + 5\times((14-6)-x) = 2\times 27. \therefore \quad x = 5$

2 Degree Sequences

Theorem 2.1. *If $\mathbf{G}(\mathbf{V},\mathbf{E})$is simple and order of graph $|V(G)| \geq 2$, then there are two vertices of the same degree*

Proof. In a simple graph, the maximum degree is $\Delta \leq |V(G)|-1$. If all the degrees were different, then there would be $0, 1, 2, \ldots, |V(G)|-1$. But degree 0 and $|V(G)|-1$ are **mutually exclusive**. Therefore, they must be two vertices of the same degree. $\square$

Note. *A graph, $\mathbf{G}(\mathbf{V},\mathbf{E})$cannot have a node with degree $d(v) = 0$ and another node with $d(v) = |V(G)-1|$, which means the node is connected to all other nodes.*

Let $V(G) = \{v_1, v_2, \ldots, n_n\}$, the degree sequence of which is given by $Deg(G) = (d(v_1), d(v_2), \ldots, d(v_n))$, where they are ordered as $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. We say a sequence $D = (d(v_1), d(v_2), \ldots, d(v_n))$ is **graphic** if $d(v_1) \geq d(v_2) \geq \ldots, \geq d(v_n)$ and there exists a simple graph with $D = Deg(G)$

Example 2.1.

- $D = (4,3,3,2,1)$ *: Not graphic as number of odd vertices is odd*
- $D = (7,6,5,4,3,3,2)$ *: Not graphic as $\Delta(G) = |V(G)| = 7$, which is the same as the order of the graph.*
- $D = (6,6,5,4,3,3,1)$ *: Not graphic; $\Delta(G) = 6$, order of graph $V(G) = 7$, $\delta(G) = 1$*

  But two(2) vertices have degree $V(G)-1 = 6$, it is not possible to have **only one(1)** *vertex to have degree, $d(v) = 1$ with this degree sequence.*

Problem. *Given a* **graphic** *sequence, produce a graph* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *with a given degree sequence* $Deg(G) = D$, *i.e given the sequence* $D = (d(v_1), d(v_2), \ldots, d(v_n))$, *where* $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$

Step (1) The vertex of degree $d(v_1)$ is joined to the $d(v_1)$ vertices of the *largest degree*

Step (2) These leaves the degrees of the vertices as $d(v_2) - 1, d(v_3) - 1, \ldots, d(d(v_1) + 1) - 1, d(d(v_1) + 2), \ldots, d(v_n)$ in some order

Step (3) Rearrange the above into a descending order getting a new sequence $D' = (d'(v_2), d'(v_3) \ldots, d'(v_n)$ wher the first vertex is deleted

Step (4) Repeat from Step (1), replacing using D'

Example 2.2. *Let the given sequence be* $D = (3, 3, 3, 3, 3, 3)$

1. *The first vertex will be joined to the 3 vertices of the largest degree.*

   The reduced sequence becomes $(*, 3, 3, 2, 2, 2) \Rightarrow D' = (*, 3, 3, 2, 2, 2)$

2. $(*, *, 2, 1, 1, 2) \Rightarrow D'' = (*, *, 2, 2, 1, 1)$

3. $(*, *, *, 1, 0, 1) \Rightarrow D''' = (*, *, *, 1, 1, 0)$

4. $(*, *, *, 1, 1, 0) \Rightarrow D'''' = (*, *, *, *, 0, 0)$

   which happens to be **graphic**

Algorithm 1 Check Graphic Sequence

Require: An ordered sequence of $D = (d(v_1), d(v_2), \ldots, d(v_n))$
Ensure: **TRUE** if D is **Graphic** ELSE **FALSE**

```
procedure GrapgGen(D)                              ▷ To check if a degree sequence is graphic or not.
    graphic = FALSE
    i ⟵ 1
    while D[i] > 0 do
        k ⟵ D[i]
        if there are at least k vertices with d(v_i) > 0 then
            Join v_i to the k vertices of the largest degrees
            Decrease each of the k vertex degrees by 1
            D[i] ⟵ 0                                ▷ Vertex v_i is now completely joined
        else
            EXIT                                    ▷ v_i cannot be "joined" i ⟵ i + 1
        end if
    end while
    graphic = TRUE
end procedure
```

Theorem 2.2 (Havel-Hakimi Theorem). $D = (d(v_1), d(v_2), \ldots, d(v_n))$ *is graphic if and only if* $D' = (d'(v_2), d'(v_3) \ldots, d'(v_n))$ *is graphic*

Theorem 2.3 (Erdos-Gallai Theorem). *Let* $D = (d(v_1), d(v_2), \ldots, d(v_n))$, *where* $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. *Then* D *is graphic iff*

1. $\sum_{i=1}^{n} d(v_i)$ *is* **even** *and*

2. $\sum_{i=1}^{k} d(v_i) \leq k(k-1) + \sum_{i=k+1}^{m} \min(k, d(v_i))$ *for* $k = 1, 2, \ldots n$