CHAPTER VI

HOMOGENEOUS LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

6.1. Homogeneous linear equations.

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + P_n y = X, \qquad \ldots \quad (1)$$

where $P_1, P_2, \cdots\cdots, P_n$ are constants and X is either a constant or a function of x only is called a *homogeneous linear differential equation.* This is also known as *Euler-Cauchy type of equations.*

Equations of this type are solved by transforming them to equations with constant coefficients through a change of the independent variable x to z by the relation

$$x = e^z, \qquad \text{that is,} \quad z = \log x.$$

When this change is effected, we have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right),$$

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^ny}{dx^n} = \frac{1}{x^n}\left\{\frac{d^ny}{dz^n} - \frac{n(n-1)}{2}\frac{d^{n-1}y}{dz^{n-1}} + \cdots \right.$$

$$\left. \cdots + (-1)^{n-1}(n-1)!\,\frac{dy}{dz}\right\}.$$

We use the symbol D' for the differential operator $\frac{d}{dz}$.

Thus $\quad D' \equiv \frac{d}{dz}$ and $D'' = \frac{d^2}{dz^2}$. $\quad$ Also $\quad D' \equiv x\frac{d}{dx}$.

Putting this differential operator D' for $\frac{d}{dz}$, we get

$$x\frac{dy}{dx} = D'y,$$

$$x^2\frac{d^2y}{dx^2} = D'(D'-1)\,y,$$

$$x^3\frac{d^3y}{dx^3} = D'(D'-1)(D'-2)\,y,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x^n\frac{d^ny}{dx^n} = D'(D'-1)(D'-2)\ldots\ldots(D'-n+1)\,y.$$

Substituting these relations in (1), we get the transformed equation as

$$\{D'(D'-1)\ldots\ldots(D'-n+1)$$
$$+ P_1 D'(D'-1)\ldots\ldots(D'-n+2)+\cdots\cdots+P_n\}\,y = Z, \quad \ldots \quad (2)$$

where Z is a function of z into which X is transformed by the substitution $x = e^z$.

This is an equation with constant coefficients and can be easily solved. If this equation (2) be written in the form

$$f(D')\,y = Z, \quad \ldots \quad (3)$$

where $f(D') \equiv D'(D'-1)\ldots\ldots(D'-n+1)+$

$$P_1 D'(D'-1)\ldots\ldots(D'-n+2)+\cdots\cdots+P_n,$$

then the complementary function will be given by different functions as determined by the roots of the auxiliary equation $f(m) = 0$, as in the previous chapter.

The particular integral will be given by

$$\frac{1}{f(D')}Z$$

and can be evaluated by applying the methods discussed in the previous chapter.

6.5. Equations reducible to homogeneous linear form.

The equations of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + P_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots$$

$$\cdots + P_{n-1} (a+bx) \frac{dy}{dx} + P_n y = X, \qquad \ldots \quad (1)$$

where $P_1, P_2, \ldots\ldots, P_n$ are constants and X is either a constant or a function of x only, can easily be reduced to the homogeneous linear form and hence also to the form of linear equations with constant coefficients. For this purpose, we write $a + bx = z$, so that

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = b \frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(b\frac{dy}{dz}\right) = \frac{d}{dz}\left(b\frac{dy}{dz}\right)\cdot\frac{dz}{dx} = b^2\frac{d^2y}{dz^2},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^ny}{dx^n} = b^n\frac{d^ny}{dz^n}.$$

Substituting these in (1), we get the reduced equation as

$$z^n\frac{d^ny}{dz^n} + \frac{P_1}{b}z^{n-1}\frac{d^{n-1}y}{dz^{n-1}} + \frac{P_2}{b^2}z^{n-2}\frac{d^{n-2}y}{dz^{n-2}} + \cdots$$

$$\cdots + \frac{P_{n-1}}{b^{n-1}}z\frac{dy}{dz} + \frac{P_n}{b^n}y = \frac{1}{b^n}Z, \qquad \ldots \quad (2)$$

where Z is a function of z into which X is transformed by the substitution $x = \frac{z-a}{b}$.

This is an equation of homogeneous form and can be easily solved.

If $y = G(z)$ be the solution of the equation (2), then

$y = G(a+bx)$ is the solution of the equation (1).

If e^t had been substituted for $(a+bx)$, the independent variable thus being changed to t from x, we would get a linear equation with constant coefficients.

6.6. Illustrative Examples.

Ex. 1. *Solve* : $\quad x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = x^3 + 3x.$

We first change the independent variable x to z by the substitution $x = e^z$, that is, $z = \log x$ so that $x\frac{d}{dx} \equiv \frac{d}{dz} \equiv D'$, say.

The equation is then reduced to

$$\{D'(D'-1)(D'-2) - D'(D'-1) + 2D' - 2\}\,y = e^{3z} + 3e^z$$

or, $\quad (D'^3 - 4D'^2 + 5D' - 2)\,y = e^{3z} + 3e^z$

or, $\quad (D'-1)^2(D'-2)\,y = e^{3z} + 3e^z.$

Here the auxiliary equation $(m-1)^2(m-2) = 0$ has the roots 1, 1, 2 .

Thus the complementary function is

$$(C_1 + C_2z)e^z + C_3e^{2z} = (C_1 + C_2\log x)\,x + C_3x^2.$$

The particular integral is

$$\frac{1}{(D'-1)^2(D'-2)}(e^{3z}+3e^z)$$

$$=\frac{1}{(D'-1)^2(D'-2)}e^{3z}+3\frac{1}{(D'-1)^2(D'-2)}e^z$$

$$=\frac{1}{4}e^{3z}-3\frac{1}{(D'-1)^2}e^z$$

$$=\frac{1}{4}e^{3z}-3e^z\frac{1}{(D'+1-1)^2}1$$

$$=\frac{1}{4}e^{3z}-3e^z\frac{1}{D'^2}1$$

$$=\frac{1}{4}e^{3z}-3e^z\frac{z^2}{2}=\frac{1}{4}x^3-\frac{3}{2}x(\log x)^2.$$

Hence the complete solution is

$$y=(C_1+C_2\log x)x+C_3x^2+\frac{1}{4}x^3-\frac{3}{2}x(\log x)^2.$$

Ex. 2. *Solve* : $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$. [C. H. 1991, 1993]

Let us put $x=e^z$, that is, $z=\log x$ so that $x\frac{d}{dx}\equiv\frac{d}{dz}\equiv D'$, say.

Then the given equation becomes

$$\{D'(D'-1)(D'-2)+2D'(D'-1)+2\}y=10(e^z+e^{-z})$$

or, $(D'+1)(D'^2-2D'+2)y=10(e^z+e^{-z})$.

The roots of the auxiliary equation $(m+1)(m^2-2m+2)=0$ are $-1,\ 1\pm i$.

Thus the complementary function is

$$C_1e^{-z}+(C_2\cos z+C_3\sin z)e^z$$

$$=C_1x^{-1}+\{C_2\cos(\log x)+C_3\sin(\log x)\}x.$$

The particular integral is

$$\frac{1}{(D'+1)(D'^2-2D'+2)}10(e^z+e^{-z})$$

$$=\frac{1}{(D'+1)(D'^2-2D'+2)}10e^z+\frac{1}{(D'+1)(D'^2-2D'+2)}10e^{-z}$$

$$=5e^z+\frac{1}{D'+1}2e^{-z}=5e^z+e^{-z}\frac{1}{D'-1+1}2$$

$$=5e^z+e^{-z}2z=5x+2x^{-1}\log x.$$

Hence the complete solution is

$$y = x\{C_2 \cos(\log x) + C_3 \sin(\log x) + 5\} + x^{-1}(C_1 + 2\log x).$$

Ex. 3. *Solve* : $(x^2D^2 - 3xD + 5)y = x^2 \sin(\log x)$, *where* $D \equiv \dfrac{d}{dx}$.

Let us put $x = e^z$ so that $z = \log x$.

Then the given equation reduces to

$$\{D'(D'-1) - 3D' + 5\}y = e^{2z}\sin z, \text{ where } D' \equiv x\frac{d}{dx} \equiv \frac{d}{dz}$$

or, $(D'^2 - 4D' + 5)y = e^{2z}\sin z$.

The roots of the auxiliary equation $m^2 - 4m + 5 = 0$ are $2 \pm i$.

Thus the complementary function is

$e^{2z}(A\cos z + B\sin z) = x^2\{A\cos(\log x) + B\sin(\log x)\}$.

The particular integral is

$$\frac{1}{D'^2 - 4D' + 5}e^{2z}\sin z = e^{2z}\frac{1}{(D'+2)^2 - 4(D'+2) + 5}\sin z$$

$$= e^{2z}\frac{1}{D'^2 + 1}\sin z$$

$$= e^{2z}\left(-\frac{z}{2}\cos z\right)$$

$$= -\frac{1}{2}x^2\log x\cos(\log x).$$

Hence the complete solution is

$$y = x^2\{A\cos(\log x) + B\sin(\log x) - \tfrac{1}{2}\log x\cos(\log x)\}.$$

Ex. 4. *Solve* : $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1-x)^2}$, *where* $D \equiv \dfrac{d}{dx}$.

[*N.B.H.1988*]

Let us put $x = e^z$ so that $z = \log x$.

The given equation then reduces to

$$\{D'(D'-1) + 3D' + 1\}y = \frac{1}{(1-x)^2}, \text{ where } D' \equiv \frac{d}{dz}$$

or, $(D'+1)^2 y = \dfrac{1}{(1-x)^2}$.

Now, the roots of the auxiliary equation are $-1, -1$.

Thus the complementary function is

$$(C_1 + C_2 z)e^{-z} = (C_1 + C_2\log x)x^{-1}.$$

The particular integral is

$$\frac{1}{(D'+1)^2}(1-x)^{-2} = \frac{1}{D'+1} x^{-1}\int x^{1-1}(1-x)^{-2}\,dx$$

$$= \frac{1}{D'+1} x^{-1}(1-x)^{-1}$$

$$= x^{-1}\int x^{1-1}x^{-1}(1-x)^{-1}\,dx$$

$$= x^{-1}\int \frac{1}{x(1-x)}\,dx$$

$$= x^{-1}\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx$$

$$= x^{-1}\log\frac{x}{x-1}.$$

Hence the complete solution is

$$y = x^{-1}\left(C_1 + C_2\log x + \log\frac{x}{x-1}\right).$$

Ex. 5. *Solve* :

$$(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)\,y = (1+\log x)^2,$$

where $D \equiv \dfrac{d}{dx}$.

Let us put $x = e^z$ so that $z = \log x$.

The given equation then reduces to

$$\{D'(D'-1)(D'-2)(D'-3) + 6D'(D'-1)(D'-2) + 9D'(D'-1) + 3D' + 1\}\,y = (1+z)^2.$$

Simplifying, we get

$$(D'^2+1)^2\,y = (1+z)^2.$$

Now, the roots of the auxiliary equation are $\pm i, \pm i$.

Thus the complementary function is

$$(C_1 + C_2 z)\cos z + (C_3 + C_4 z)\sin z$$

$$= (C_1 + C_2\log x)\cos(\log x) + (C_3 + C_4\log x)\sin(\log x).$$

The particular integral is

$$\frac{1}{(D'^2+1)^2}(1+z)^2 = (1+D'^2)^{-2}(1+2z+z^2)$$

$$= (1-2D'^2 - \cdots\cdots)(1+2z+z^2)$$

$$= 1+2z+z^2-4 = z^2+2z-3 = (\log x)^2 + 2\log x - 3.$$

Hence the complete solution is

$$y = (C_1 + C_2 \log x)\cos(\log x) + (C_3 + C_4 \log x)\sin(\log x) + (\log x)^2 + 2\log x - 3.$$

Ex. 6. *Solve* : $(1+2x)^2 \dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$.

[*V. H. 1987*]

Let us put $1+2x = e^z$ in the given equation so that $z = \log(1+2x)$.

Then we have $\dfrac{dy}{dx} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dx} = \dfrac{1}{1+2x}\cdot 2\dfrac{dy}{dz}$.

and $$\frac{d^2y}{dx^2} = \frac{2}{1+2x}\frac{d^2y}{dz^2}\cdot\frac{2}{1+2x} - \frac{4}{(1+2x)^2}\frac{dy}{dz}$$

$$= \frac{4}{(1+2x)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right).$$

Therefore $(1+2x)\dfrac{dy}{dx} = 2\dfrac{dy}{dz}$ and $(1+2x)^2\dfrac{d^2y}{dx^2} = 4\left(\dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}\right)$

Substituting these in the given equation, we get

$$4\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) - 12\frac{dy}{dz} + 16y = 8e^{2z}$$

or, $$\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 4y = 2e^{2z}$$

or, $$(D'-2)^2 y = 2e^{2z}.$$

The roots of the auxiliary equation are 2, 2.

Thus the complementary function is

$$(C_1 + C_2 z)e^{2z} = \{C_1 + C_2 \log(1+2x)\}(1+2x)^2.$$

The particular integral is

$$\frac{1}{(D'-2)^2}2e^{2z} = e^{2z}\frac{1}{(D'+2-2)^2}2 = e^{2z}\frac{1}{D'^2}2$$

$$= e^{2z}z^2 = (1+2x)^2\{\log(1+2x)\}^2.$$

Hence the complete solution is

$$y = \left[C_1 + C_2\log(1+2x) + \{\log(1+2x)\}^2\right](1+2x)^2.$$

Examples VI

Solve the following equations :

1. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 2\log x$.

2. $x^2 \dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} = 0$.

3. $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \dfrac{1}{x}$.

4. $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$.

5. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2x^2$. [*C. H. 1989*

6. $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (x+1)^2$.

7. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x\log x$.

8. $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$.

9. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \log x$.

10. $(x^2D^2 - xD + 4)\,y = \cos(\log x) + x\sin(\log x)$, $D \equiv \dfrac{d}{dx}$.

11. $\{x^2D^2 - (2m-1)xD + (m^2+n^2)\}\,y = n^2x^m \log x$.

12. $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x + \sin x$.

13. $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$.

14. $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^2e^{2x}$.

15. (*a*) $(x^2D^2 + xD - 1)\,y = x^m$.

(*b*) $(x^2D^2 - 3xD + 4)\,y = x^m$.

16. $(x^3D^3 + xD - 1)y = x^2$. [V. H. 1997]

17. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\log x \,.\, \sin(\log x) + 1}{x}$.

18. $x^3\dfrac{d^3y}{dx^3} - x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 2y = x^3 + 3x$.

19. $(5 + 2x)^2\dfrac{d^2y}{dx^2} - 6(5 + 2x)\dfrac{dy}{dx} + 8y = 0$.

20. $(x + a)^2\dfrac{d^2y}{dx^2} - 4(x + a)\dfrac{dy}{dx} + 6y = x$. [C. H. 1995]

21. $(1 + x)^2\dfrac{d^2y}{dx^2} + (1 + x)\dfrac{dy}{dx} + y = 4\cos\log(1 + x)$.

22. $(3x + 2)^2\dfrac{d^2y}{dx^2} + 5(3x + 2)\dfrac{dy}{dx} - 3y = x^2 + x + 1$.

23. $(2x - 1)^3\dfrac{d^3y}{dx^3} + (2x - 1)\dfrac{dy}{dx} - 2y = 0$.

24. $2x^2y\dfrac{d^2y}{dx^2} + 4y^2 = x^2\left(\dfrac{dy}{dx}\right)^2 + 2xy\dfrac{dy}{dx}$ after making it homogeneous by the substitution $y = z^2$.

25. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 12y = 12$, satisfying $y(1) = 0$, $y'(1) = 0$.

Answers

1. $y = x(C_1 + C_2\log x) + 2\log x + 4$.
2. $y = C_1x^3 + C_2 + C_3\log x$.
3. $y = (C_1 + C_2\log x)x + C_3x^{-1} + \frac{1}{4}x^{-1}\log x$.
4. $y = C_1x^{-1} + C_2x^4 + \frac{1}{5}x^4\log x$.
5. $y = (C_1 + C_2\log x)x^2 + x^2(\log x)^2$.
6. $y = C_1x^{-5} + C_2x^4 - \frac{1}{14}x^2 - \frac{1}{9}x - \frac{1}{20}$.
7. $y = x\{C_1\cos(\log x) + C_2\sin(\log x)\} + x\log x$.

8. $y = C_1 \cos(\log x) + C_2 \sin(\log x) + \dfrac{\log x}{4}\{\sin(\log x) - \log x \cos(\log x)\}.$

9. $y = C_1 x^3 + C_2 x^{-1} - \frac{1}{3}x^2\left(\log x + \frac{2}{3}\right).$

10. $y = x\{C_1 \cos(\sqrt{3}\log x) + C_2 \sin(\sqrt{3}\log x)\} + \frac{1}{13}\{3\cos(\log x) - 2\sin(\log x)\} + \frac{1}{2}x\sin(\log x).$

11. $y = x^m\{C_1 \cos(n\log x) + C_2 \sin(n\log x)\} + x^m \log x.$

12. $y = C_1 x^{-1} + C_2 x^{-2} + \frac{1}{6}x - x^{-2}\sin x.$

13. $y = C_1 x^{-1} + C_2 x^{-2} + \dfrac{1}{x^2}e^x.$

14. $y = C_1 x + C_2 x^{-1} + \frac{1}{8}(2 - x^{-1})e^{2x}.$

15. (*a*) $y = C_1 x + C_2 x^{-1} + \dfrac{x^m}{m^2 - 1}.$

(*b*) $y = x^2(C_1 + C_2 \log x) + \dfrac{x^m}{(m-2)^2}.$

16. $y = \{C_1 + C_2 \log x + C_3(\log x)^2\}x + x^2.$

17. $y = x^2(C_1 x^{\sqrt{3}} + C_2 x^{-\sqrt{3}}) + \dfrac{1}{6x} + \dfrac{\log x}{61x}\{5\sin(\log x) + 6\cos(\log x)\} + \dfrac{2}{3721x}\{27\sin(\log x) + 191\cos(\log x)\}.$

18. $y = (A + B\log x)x + Cx^2 + \frac{1}{4}x^3 - \frac{3}{2}x(\log x)^2.$

19. $y = (5 + 2x)^2\{A(5 + 2x)^{\sqrt{2}} + B(5 + 2x)^{-\sqrt{2}}\}.$

20. $y = C_1(x + a)^3 + C_2(x + a)^2 + \dfrac{1}{2}(x + a) - \dfrac{1}{6}a.$

21. $y = C_1 \cos\{\log(1 + x)\} + C_2 \sin\{\log(1 + x)\} + 2\log(1 + x)\sin\{\log(1 + x)\}.$

22. $y = C_1(3x + 2)^{-1} + C_2(3x + 2)^{\frac{1}{3}} + \frac{1}{405}(3x + 2)^2 - \frac{1}{108}(3x + 2) - \frac{7}{27}.$

23. $y = A(2x - 1) + B(2x - 1)\cosh\left\{\dfrac{\sqrt{3}}{2}\log(2x - 1) + C\right\}.$

24. $y = (A + B\log x)^2 . x^2.$

25. $y = -1 + \frac{1}{7}(3x^{-4} + 4x^3).$