# Graph Theory - Lecture 3
# Walks, Trials, Paths

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## 1 Walks

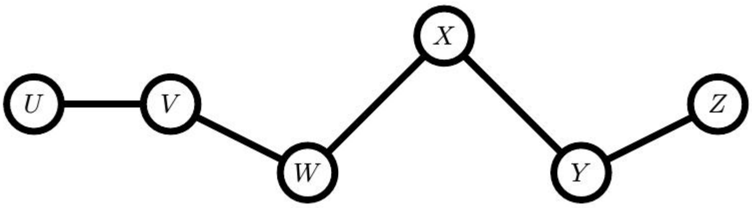



Figure 1: Simple Walk : $UVWXYZ$

Let us consider a graph $\mathbf{G}(\mathbf{V},\mathbf{E})$ , where $E(G) = \{e_i \in E(G) | i = 1, 2, \ldots, n = |E(G)|\}$ and $V(G) = \{v_i \in V(G) | i = 1, 2, \ldots, k = |V(G)|\}$

**Definition 1.1** (Walk)**.** *An ordered sequence of edges such as*

$$W(G) = (e_1, e_2, \ldots, e_l); e_j \in E(G) \tag{1}$$

*where $e_i \in E(G)$ is a* **walk** *if there exists a corresponding sequence of vertices $(v_0, v_1, \ldots v_l)$ such that $e_j = (v_{j-1}, v_j)$; the vertices in the* **walk** *may not be distinct.*

Note : A **walk** for which $v_0 = v_l$ is a **closed walk**.

**Definition 1.2.** *The* **length** *of a walk is the number of edges in the sequence.*

E.g. in [Eq 1], the length of the walk is $l$.

**Definition 1.3** (Distance)**.** *The* **distance** *$d(u, v)$ between two vertices $u$ and $v$ in a graph* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *is the length of the shortest uv path* [1] *in $G$ and for any three vertices $u, v, w$,*

$$d(u, w) + d(w, v) \geq d(u, v). \tag{2}$$

Equation 2 is referred to as the **Tiangular Inequality**

**Definition 1.4** (Eccentricity)**.** *The maximum distance from a vertex $u \in V(G)$ to any other vertex, $v$ in* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *is called the* **eccentricity,** *$e(u)$.*

If $w$ is a vertex such that $e(v) = d(v, w)$, then it is said that the eccentricity of the vertex $v$ is realized by the vertex $w$.

**Definition 1.5** (Radius)**.** *The smallest eccentricity, i.e.* $Rad(G) = \min_{v \in V(G)} e(v)$ *is the* **radius** *of the* $\mathbf{G(V,E)}$ *.*

**Definition 1.6** (Diameter)**.** *The* **diameter** $D(G)/Diam(G)$ *of* $\mathbf{G(V,E)}$ *is the* <u>*minimum distance,* $d(u,v)$</u> *over all pairs of* $\{u,v\}$ *pairs of vertices of* $V(G)$*, i.e.*

$$Diam(G) = \min_{\forall \{u,v\} \in V(G)} d(u,v)$$

.

**OR**

*The maximum eccentricity, i.e.* $Diam(G) = \max_{v \in V(G)} e(v)$ *is the* **diameter** *of the* $\mathbf{G(V,E)}$

**Definition 1.7** (Central Vertex)**.** *The* **Central vertex** *is the vertex, for which* $e(v) = Rad(G)$*.*

**Theorem 1.1.** *For every non-trivial connected graph* $\mathbf{G(V,E)}$ *,*

$$Rad(G) \leq Diam(G) \leq 2 \times Rad(G)$$

*Proof.* The inequality $Rad(G) \leq Diam(G)$ is immediate since the smallest eccentricity cannot exceed the largest eccentricity.

Let $u$ and $v$ be two vertices such that $d(u,v) = Diam(G)$ and let $w$ be a **Central vertex** of $\mathbf{G(V,E)}$ .

Therefore, $e(w) = Rad(G)$. Hence, the distance between $w$ and any other vertex of $\mathbf{G(V,E)}$ is at most $Rad(G)$.

By *triangle inequality*,

$$\begin{aligned} Diam(G) = &\ d(u,v) \\ \leq &\ d(u,w) + d(w,v) \\ \leq &\ Rad(G) + Rad(G) = 2 \times Rad(G) \end{aligned}$$

□

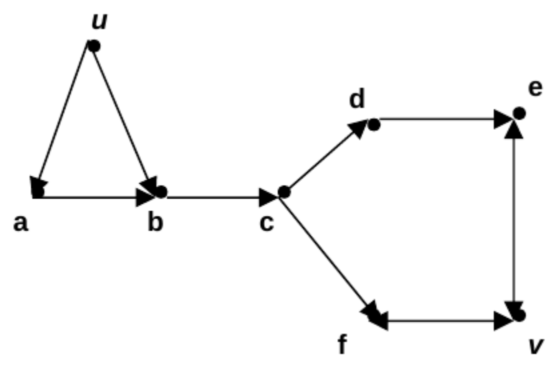



Figure 2: Diameter = $\infty$; Radius = 3

**Proposition 1.1.** *If there is a* **walk** *from vertex* $u$ *to* $v$*, then there is also a path from* $u$ *to* $v$ *of length* $(|V(G)| - 1)$*.*

*Proof.* If all the vertices are distinct, then there is nothing to prove.

Otherwise, suppose some vertex $v_x$ is repeated in the path. We can remove the segment of the walk between these two $v_x$'s and obtain another shorter walk from $u$ to $v$. If this is not a path, we may again repeat the above procedure on this walk.

Clearly, after a finite number of steps, one can obtain a path. □

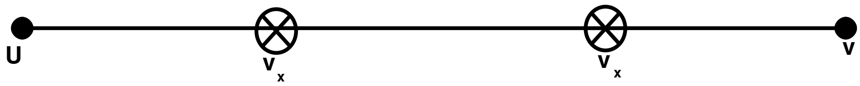



Figure 3: Shorter Path

# 2 Trial

**Definition 2.1** (Trial)**.** *A* **trial** *is a walk in which all the edges $e_j$ are distinct and a* **closed trial** *is a closed walk, that is also a trial.*

A **closed trial** is also referred to as a **Circuit**.

# 3 Path

**Definition 3.1** (Path)**.** *A* **path** *is a trial in which all the vertices in the sequence, Eq 1 are* **distinct**

A path is referred to as a $P_n$, where $n = |P_n|$ is the number of vertices of the path OR order of the path.

Note : A **closed path** is a closed trial in which all the vertices are distinct except $v_0 = v_l$.

# 4 Cycle

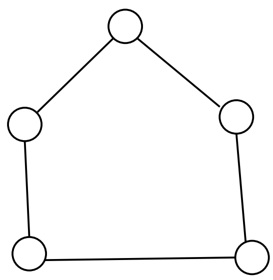


Figure 4: Cycle $C_5$

**Definition 4.1** (Cycle)**.** *A* **cycle** *($C_n$, where $n = |C_n|$ is the order of $C_n$) is a closed path that includes three or more edges.*

Note :

- Equivalently, a cycle is a subgraph isomorphic to one of the cycle graphs $C_n$, e.g. Figure 4
- In set theoretic notations

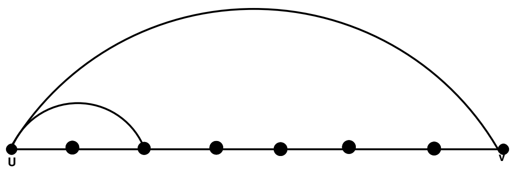
$$Walks \supseteq Trials \supseteq Paths$$

**Theorem 4.1.** *If every vertex of a graph* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *has degree at least 2, then* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *contains a cycle*

*Proof.* Let $P(G)$ be a maximal path in $\mathbf{G}(\mathbf{V}, \mathbf{E})$ , and let $u \in V(G)$ be an endpoint of $P(G)$. Since $P(G)$ cannot be extended, every neighbour of $u$ must already be a vertex of $P(G)$. Since, $u$ has degree at least 2, it has a neighbour $v \in V(P)$ via an edge *not in* $P(G)$

The edge $uv$ completes a cycle with the portion of $P(G)$ from $v$ to $u$

**Note :** If $|V(G)| = Z$ and $E(G) = \{ij : |i - j| = 1\}$, then every vertex has degree 2, but $\mathbf{G}(\mathbf{V}, \mathbf{E})$ has no cycle, i.e. no non-extendible path. $\square$




**Lemma 4.2.** *If every vertex of a (finite) graph* $\mathbf{D}(\mathbf{V},\mathbf{E})$ *has out-degree/*$d^{+}(v)$ *(or in-degree/*$d^{-}(v)$*) at least 1, then* $\mathbf{D}(\mathbf{V},\mathbf{E})$ *contains a cycle.*

*Proof.* Let $P(D)$ be a maximal path in $\mathbf{D}(\mathbf{V},\mathbf{E})$ , and $u$ be the last vertex on $P(D)$. Since $P(D)$ can not be extended, every successor of $u$ is in $V(D)$. There is at least one successor of $u$, say $v$. This edge $uv$ and the path from $v$ to $u$ form a cycle. □

**Abstract**

**Walk :** Vertices may REPEAT, edges may REPEAT

**Trail :** Vertices may REPEAT, edges CANNOT REPEAT

**Circuit :** Vertices may REPEAT (Closed Trial)

**Path :** Vertices CANNOT REPEAT, **thus** edges also CANNOT REPEAT.

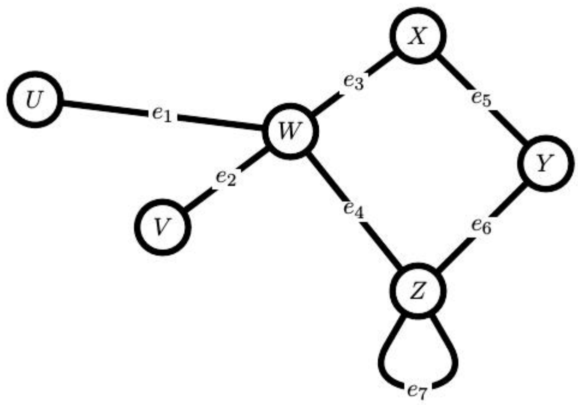



Figure 5: Graph with examples

With respect to Figure 5

1. **Direct walk** between $U$ and $Y$ graph is: $UWXY$
2. A more **roundabout walk** between $U$ and $Y$ could be: $UWVWZZY$
3. Alternative Notation : $Ue_1We_2Ve_2We_4Ze_7Ze_6Y$
4. The walk $Ue_1We_4Ze_7Ze_6Y$ is an **open walk**
5. The walk $Ue_1We_3Xe_5Ye_6Ze_4We_1U$ is **closed walk**
6. The walk $Ue_1We_3Xe_5Ye_6Z$ is a **path**
7. The walk $We_4Ze_7Ze_6Ye_5Xe_3W$ is a **closed trail**
8. The walk $We_3Xe_5Ye_6Ze_4W$ is a **cycle**
9. The walk $Ue_1We_2Ve_2We_4Ze_7Ze_6Ye_7$ is **not a trail or a path**
10. The walk $Ue_1We_4Ze_7Ze_6Y$ **is a trail but not path**