(6) $A(a) \to H(a)$ (5) and Specification

(7) $H(a)$ (4), (6), and Inference Rule (1)

3. (b), (f), (g), and (h) are invalid.

1.10 MATHEMATICAL INDUCTION

In mathematics, as in science there are two main aspects of inquiry whereby we can discover new results: deductive and inductive. As we have said the deductive aspect involves accepting certain statements as premises and axioms and then deducing other statements on the basis of valid inferences. The inductive aspect, on the other hand, is concerned with the search for facts by observation and experimentation—we arrive at a conjecture for a general rule by inductive reasoning. Frequently we may arrive at a conjecture that we believe to be true *for all positive integers n*. But then before we can put any confidence in our conjecture we need to verify the truth of the conjecture. There is a proof technique that is useful in verifying such conjectures. Let us describe that technique now.

The Principle of Mathematical Induction. Let $P(n)$ be a statement which, for each integer n, may be either true or false. To prove $P(n)$ is true for all integers $n \geq 1$, it suffices to prove:

1. $P(1)$ is true.
2. For all $k \geq 1$, $P(k)$ implies $P(k+1)$.

If one replaces (1) and (2) by (1′) $P(n_0)$ is true, and (2′) For all $k \geq n_0$, $P(k)$ implies $P(k+1)$, then we can prove $P(n)$ is true for all $n \geq n_0$, and the starting point n_0, or *basis of induction*, may be any integer—positive, negative, or zero. Normally we expect to prove $P(k) \to P(k+1)$ directly so there are 3 steps to a proof using the principle of mathematical induction:

(i) (Basis of induction) Show $P(n_0)$ is true.
(ii) (Inductive hypothesis) Assume $P(k)$ is true for $k \geq n_0$.
(iii) (Inductive step) Show that $P(k+1)$ is true on the basis of the inductive hypothesis.

We emphasize that the inductive hypothesis is not tantamount to assuming what is to be proved; it is just part of proving the implication

$$P(k) \to P(k+1).$$

Now the principle of mathematical induction is a reasonable method of proof for part (1) tells us that $P(1)$ is true. Then using (2) and the fact that part (1) tells us that $P(1)$ is true, we conclude $P(2)$ is true. But then (2) implies that $P(2+1) = P(3)$ is true, and so on. Continuing in this way we would ultimately reach the conclusion that $P(n)$ is true for any fixed positive integer n. The principle of mathematical induction is much like the game we played as children where we would stand up dominos so that if one fell over it would collide with the next domino in line. This is like part (2) of the principle. Then we would tip over the first domino (this is like part (1) of the principle). Then what would happen? All the dominos would fall down—like the conclusion that $P(n)$ is true for all positive integers n.

Example 1.10.1. Let us use this approach on the problem of determining a formula for the sum of the first n positive integers. Let $S(n) = 1 + 2 + 3 + \cdots + n$. Let us examine a few values for $S(n)$ and list them in the following table:

n	1	2	3	4	5	6	7
$S(n)$	1	3	6	10	15	21	28

The task of guessing a formula for $S(n)$ may not be an easy one and there is no sure-fire approach for obtaining a formula. Nevertheless, one might observe the following pattern:

$$\begin{aligned}
2\,S(1) &= 2 = 1 \cdot 2 \\
2\,S(2) &= 6 = 2 \cdot 3 \\
2\,S(3) &= 12 = 3 \cdot 4 \\
2\,S(4) &= 20 = 4 \cdot 5 \\
2\,S(5) &= 30 = 5 \cdot 6 \\
2\,S(6) &= 42 = 6 \cdot 7
\end{aligned}$$

This leads us to conjecture that

$$2\,S(n) = n(n+1) \text{ or that } S(n) = \frac{n(n+1)}{2}.$$

Now let us use mathematical induction to prove this formula. Let $P(n)$ be the statement: the sum $S(n)$ of the first n positive integers is equal to $n(n+1)/2$.

1. **Basis of Induction.** Since $S(1) = 1 = 1(1 + 1)/2$, the formula is true for $n = 1$.

2. **Inductive Hypothesis.** Assume the statement $P(n)$ is true for $n = k$, that is, that $S(k) = 1 + 2 + \cdots + k = k(k + 1)/2$.

3. **Inductive Step.** Now show that the formula is true for $n = k + 1$, that is, show that $S(k + 1) = (k + 1)(k + 2)/2$ follows from the inductive hypothesis. To do this, we observe that $S(k + 1) = 1 + 2 + \cdots + (k + 1) = S(k) + (k + 1)$.

Since $S(k) = k(k + 1)/2$ by the inductive hypothesis, we have

$$S(k+1) = S(k) + (k+1) = \frac{k}{2}(k+1) + (k+1) = (k+1)\left(\frac{k}{2} + 1\right)$$
$$= \frac{(k+1)(k+2)}{2},$$

and the formula holds for $k + 1$. So, by assuming the formula was true for k, we have been able to prove the formula holds for $k + 1$, and the proof is complete by the principle of mathematical induction.

The principle of mathematical induction is based on a result that may be considered one of the axioms for the set of positive integers. This axiom is called the *well-ordered property* of the positive integers; its statement is the following: *Any nonempty set of positive integers contains a least positive integer.*

Example 1.10.2. Find and prove a formula for the sum of the first n cubes, that is, $1^3 + 2^3 + \cdots + n^3$.

We consider the first few cases:

$$1^3 = 1 = 1^2$$
$$1^3 + 2^3 = 9 = 3^2$$
$$1^3 + 2^3 + 3^3 = 36 = 6^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = 100 = 10^2$$

From this meager information we expect that $1^3 + 2^3 + 3^3 + 4^3 + 5^3$ to be a perfect square. But the square of what integer? After computing we find that it is 15^2. Still we may not see the pattern at first, but by comparing the table for $S(n)$ in Example 1.11.1 we see that we have obtained thus far,

$$[S(1)]^2 = 1^2, [S(2)]^2 = 3^2, [S(3)]^2 = 6^2, [S(4)]^2 = 10^2, \text{and } [S(5)]^2 = 15^2.$$

We conjecture then that $1^3 + 2^3 + \cdots + n^3 = [n(n + 1)/2]^2$. Let us verify this formula by mathematical induction:

1. **Basis of Induction.** Since $1^3 = [1\,(1 + 1)/2]^2$ the formula holds for $n = 1$.

2. **Inductive Hypothesis.** Suppose the formula holds for $n = k$. Thus, suppose $1^3 + 2^3 + \cdots + k^3 = [k(k + 1)/2]^2$.

3. **Inductive Step.** Show the formula holds for $n = k + 1$; that is, show $1^3 + 2^3 + \cdots + k^3 + (k + 1)^3 = [(k + 1)\,(k + 2)/2]^2$.

Now $1^3 + 2^3 + \cdots + k^3 + (k + 1)^3$ is nothing more that the sum of $1^3 + 2^3 + \cdots + k^3$ and $(k + 1)^3$, so we use the inductive hypothesis to replace $1^3 + 2^3 + \cdots + k^3$ by $[k\,(k + 1)/2]^2$. Thus,

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k + 1)^3 &= \left[\frac{k(k + 1)}{2}\right]^2 + (k + 1)^3 \\
&= (k + 1)^2 \left[\left(\frac{k}{2}\right)^2 + k + 1\right] = (k + 1)^2 \left[\frac{k^2}{4} + k + 1\right] \\
&= (k + 1)^2 \left[\frac{k^2 + 4k + 4}{4}\right] \\
&= (k + 1)^2 \left[\frac{k + 2}{2}\right]^2 \\
&= \left[\frac{(k + 1)\,(k + 2)}{2}\right]^2
\end{aligned}$$

Hence, the formula holds for $k + 1$ and thus by the principle of mathematical induction for all positive integers n.

Example 1.10.3. Prove by mathematical induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n.

1. First we show that $6^{1+2} + 7^{2+1} = 6^3 + 7^3$ is divisible by 43. But this follows because $6^3 + 7^3 = 559 = 43(13)$.

2. Next we suppose that $6^{k+2} + 7^{2k+1} = 43x$ for some integer x.

3. Then we show that, on the basis of the inductive hypothesis, $6^{k+3} + 7^{2(k+1)+1} = 6^{k+3} + 7^{2k+3}$ is divisible by 43.

We showed in Section 1.4 that $6^{k+3} + 7^{2k+3} = 6(6^{k+2} + 7^{2k+1}) + 43(7^{2k+1})$. Thus, $6^{k+3} + 7^{2k+3} = 6(43x) + 43(7^{2k+1}) = 43[6x + 7^{2k+1}]$ or $6^{k+3} + 7^{2k+3} = 43(y)$ where y is an integer.

Hence, $6^{k+3} + 7^{2k+3}$ is divisible by 43, and by the principle of mathematical induction $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n.

Example 1.10.4. For each positive integer n, there are more than n prime integers.

Let $P(n)$ be the proposition: there are more than n prime integers.

1. $P(1)$ is true since 2 and 3 are primes.
2. Assume $P(k)$ is true.
3. Let $a_1, a_2, \ldots, a_k, a_{k+1}$ be $k + 1$ distinct prime integers whose existence is guaranteed since $P(k)$ is true. Form the integer

$$N = a_1 a_2 \ldots a_k a_{k+1} + 1 = \prod_{i=1}^{k+1} a_i + 1.$$

Now N is not divisible by any of the primes a_i. But N is either a prime or is divisible by a new prime a_{k+2}. In either case there are more than $k + 1$ primes.

Example 1.10.5. Suppose the Postal Department prints only 5- and 9-cent stamps. Prove that it is possible to make up any postage of n-cents using only 5- and 9-cent stamps for $n \geq 35$.

1. First, we see that postage of exactly 35 cents can be made up with seven 5-cent stamps.
2. Assume that n-cents postage can be made up with 5- and 9-cent stamps where $n \geq 35$.
3. Now consider postage of $n + 1$ cents. There are two possibilities to consider:
   (a) The n cents postage is made up with only 5-cent stamps, or
   (b) there is at least one 9-cent stamp involved in the makeup of n-cents postage.

n = 5|5|-|5 | 5|5|-|
1 2 7
n+1 = 9|9|9|9|5|5|
4

In case (a), the number of 5-cent stamps is at least seven since $n \geq 35$. Thus, we can replace those seven 5-cent stamps by four 9-cent stamps and make up $n + 1$ cents postage.

In case (b), the n cents postage includes at least one 9-cent stamp. Therefore, if we replace that one 9-cent stamp by two 5-cent stamps we can make up $n + 1$ cents postage.

n = 9| . 5|5|
n+1 = 5|5 |5|5
2

Therefore, in either case we have shown how to make up $n + 1$ cents postage in terms of only 5- and 9-cent stamps.

Example 1.10.6. Prove that for all integers $n \geq 4$, $3^n > n^3$.

Let $P(n)$ be the statement: $3^n > n^3$.

1. First, $P(4)$ is true because $3^4 = 81 > 4^3 = 64$.
2. Assume $P(n)$ is true for $n \geq 4$, that is, assume $3^n > n^3$ for $n \geq 4$.
3. We prove $P(n+1)$ is true on the basis of our assumption. Thus, we must prove $3^{n+1} > (n+1)^3$. Let us rewrite $(n+1)^3 = n^3 + 3n^2 + 3n + 1 = n^3(1 + 3/n + 3/n^2 + 1/n^3)$. Since the inductive hypothesis gives us that $3^n > n^3$, we would be done if we could also prove that $3 > 1 + 3/n + 3/n^2 + 1/n^3$ for $n \geq 4$. We now prove this. Observe that the function $f(n) = 1 + 3/n + 3/n^2 + 1/n^3$ decreases as n increases, so then $f(n)$ is largest when n is smallest. In other words $f(4)$ is the largest value of $f(n)$ when n ranges over the integers greater than or equal to 4. Since

$$f(4) = 1 + 3/4 + 3/4^2 + 1/4^3 = 125/64$$

is obviously less than 3, we have for any integer $n \geq 4$, $3 > 1 + 3/n + 3/n^2 + 1/n^3$. Thus, combining the two facts: $3 > 1 + 3/n + 3/n^2 + 1/n^2$ and $3^n > n^3$ for $n \geq 4$, we can multiply and obtain

$$3^{n+1} > 3n^3 > n^3(1 + 3/n + 3/n^2 + 1/n^3)$$
$$= (n+1)^3, \text{ and the proof is complete.}$$

$(n+1)^3 = n^3 + 3n^2 + 3n + 1$
$= n^3(1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3})$
$3^n > n^3$
$3 > 1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}$
$3^{n+1} > (n+1)^3$

$n \to \infty$, $f(n) \to 1$
$f(4) < 3$, $n \geq 4$

Recursion

In computer programming the evaluation of a function or the execution of a procedure is usually achieved at machine-language level by the use of a subroutine. The idea of a subroutine which itself calls another subroutine is common; however, it is frequently beneficial to have a subroutine that contains a call to itself. Such a routine is called a *recursive subroutine*. Informally speaking, we give the name *recursion* to the technique of defining a function, a set, or an algorithm *in terms of itself* where it is generally understood that the definition will be in terms of "previous" values. Thus, a recursive subroutine applied to a list of objects would be defined in terms of applying the subroutine to proper sublists.

Moreover, a function f from the set N of nonnegative integers is *defined recursively* if the value of f at 0 is given and for each positive integer n the value of f at n is defined in terms of the values of f at k where $0 \leq k < n$.