Second-Order Equations in Two Independent Variables

The general linear second-order partial differential equation in one dependent variable u may be written as

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Fu = G,$$

in which we assume $A_{ij} = A_{ji}$ and A_{ij}, B_i, F, and G are real-valued functions defined in some region of the space $(x_1, x_2, \ldots, x_n)$.

Here we shall be concerned with second-order equations in the dependent variable u and the independent variables x, y.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where the coefficients are functions of x and y and do not vanish simultaneously. We shall assume that the function u and the coefficients are twice continuously differentiable in some domain in $\mathbb{R}^2$.

The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry. The equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

represents hyperbola, parabola, or ellipse accordingly as $B^2 - 4AC$ is positive, zero, or negative.

The classification of second-order equations is based upon the possibility of reducing equation by coordinate transformation to *canonical* or *standard* form at a point. An equation is said to be *hyperbolic*, *parabolic*, or *elliptic* at a point (x_0, y_0) accordingly as

$$B^2(x_0, y_0) - 4A(x_0, y_0)\,C(x_0, y_0)$$

is positive, zero, or negative. If this is true at all points, then the equation is said to be hyperbolic, parabolic, or elliptic in a domain. In the case of two independent variables, a transformation can always be found to reduce the given equation to canonical form in a given domain. However, in the case of several independent variables, it is not, in general, possible to find such a transformation.

Examples:

Wave equation

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = f(x, t), \qquad \text{(Hyperbolic)}$$

Laplace or Poisson's equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = f(x, y), \qquad \text{(Elliptic)}$$

or Fourier's heat equation

$$\frac{\partial^2 \varphi}{\partial x^2} - \kappa\frac{\partial \varphi}{\partial t} = f(x, t). \qquad \text{(Parabolic)}$$

Separation of Variables

In this section we introduce the technique, called the method of separations of variables, for solving initial boundary value-problems.

Heat Equation

We consider the heat equation satisfying the initial conditions

$$\begin{cases} u_t = ku_{xx}, & x \in [0, L],\ t > 0 \\ u(x, 0) = \phi(x), & x \in [0, L] \end{cases}$$

We seek a solution u satisfying certain boundary conditions. The boundary conditions could be as follows:

(a) *Dirichlet* $u(0, t) = u(L, t) = 0$.

(b) *Neumann* $u_x(0, t) = u_x(L, t)$.

(c) *Periodic* $u(-L, t) = u(L, t)$ and $u_x(-L, t) = u_x(L, t)$.

We look for solutions of the form

$$u(x, t) = X(x)T(t)$$

where X and T are function which have to be determined. Substituting $u(x, t) = X(x)T(t)$ into the equation, we obtain

$$X(x)T'(t) = kX''(x)T(t)$$

from which, after dividing by $kX(x)T(t)$, we get

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$

The left side depends only on t whereas the right hand side depends only on x. Since they are equal, they must be equal to some constant $-\lambda$. Thus

$$T' + \lambda k T = 0$$

$$X'' + \lambda X = 0.$$

The general solution of the first equation is given

$$T(t) = Be^{-\lambda k t}$$

for an arbitrary constant B. The general solutions of the second equation are as follows.

(1) If $\lambda < 0$, then $X(x) = \alpha \cosh \sqrt{-\lambda} x + \beta \sinh \sqrt{-\lambda} x$.

(2) If $\lambda = 0$, then $X(x) = \alpha x + \beta$.

(3) If $\lambda > 0$, then $X(x) = \alpha \cos \sqrt{\lambda} x + \beta \sin \sqrt{\lambda} x$.

In addition, the function X which solves the second equation will satisfy boundary conditions depending on the boundary condition imposed on u. The problem

$$\begin{cases} X'' + \lambda X = 0 \\ X \text{ satisfies boundary conditions} \end{cases}$$

is called the *eigenvalue problem*, a nontrivial solution is called an *eigenfunction* associated with the *eigenvalue* λ.

Heat equation with Dirichlet boundary conditions

We consider the Dirichlet condition

$$u(0,t) = u(L,t) = 0 \quad \text{for all } t \geq 0.$$

In this case the eigenvalue problem becomes

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0. \end{cases}$$

We have to find nontrivial solutions X of the eigenvalue problem If $\lambda = 0$, then $X(x) = \alpha x + \beta$ and $0 = X(0) = \alpha \cdot 0 + \beta$ implies that $\beta = 0$ and $0 = X(L) = \alpha L$ implies that $\alpha = 0$. If $\lambda < 0$. Then $0 = X(0) = \alpha \cosh 0 + \beta \sinh 0 = \alpha$ and $0 = X(L) = \beta \sinh L$ shows that also $\beta = 0$. We conclude that $\lambda \leq 0$ is not an eigenvalue of the problem Finally, consider $\lambda > 0$.Then $0 = X(0) = \alpha \cos \sqrt{\lambda} \cdot 0 = \alpha$ and $0 = X(L) = \beta \sin \sqrt{\lambda} L$. Since X is nontrivial solution, $\beta \neq 0$ and hence $\sin \sqrt{\lambda} L = 0$. Consequently,

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n \geq 1$$

and the corresponding eigenfunction is given by

$$X_n(x) = \sin \frac{n\pi x}{L} x$$

After substituting $\lambda = (n\pi/L)^2$ we get the family of solutions

$$T_n(t) = B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

Thus we have obtained the following sequence of solutions

$$u_n(x,t) = X_n(x)T_n(x) = B_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

We obtain more solutions by taking linear combinations of the u_n's (recall the superposition principle)

$$u(x,t) = \sum_{n=1}^{N} u_n(x,t) = \sum_{n=1}^{N} B_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t},$$

and then by passing to the limit $N \to \infty$,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

Finally, we consider the initial condition. At $t = 0$, we must have

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = \phi(x).$$

The coefficients, B_n can be computed as follows. Fix $m \in \mathbb{N}$. Multiplying the above equality by $\sin\frac{m\pi x}{L}$ and then integrating over $[0, L]$, we get

$$\begin{aligned}\int_0^L \phi(x)\sin\frac{m\pi x}{L}dx &= \int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx \\ &= \sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx.\end{aligned}$$

Since $\displaystyle\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m, \end{cases}$

$$B_m = \frac{2}{L}\int_0^L \phi(x)\sin\frac{m\pi x}{L}\,dx.$$

Example *Consider the problem,*

$$
\begin{aligned}
&u_t - u_{xx} = 0, \quad 0 < x < \pi,\ t > 0 \\
&u(0,t) = u(\pi,t) = 0, \quad t \geq 0 \\
&u(x,0) = \phi(x) = \begin{cases} x & 0 \leq x \leq \pi/2 \\ \pi - x & \pi/2 \leq x \leq \pi. \end{cases}
\end{aligned}
$$

Solution Here $k = 1$ and $[0, L] = [0, \pi]$.

Thus
$$u(x,t) = \sum_{n=1}^{\infty} B_n (\sin nx) e^{-n^2 t}.$$

where
$$
\begin{aligned}
B_n &= \frac{2}{\pi} \int_0^{\pi} \phi(x) \sin nx \, dx \\
&= \frac{2}{\pi} \int_0^{\pi/2} x \sin nx \, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx.
\end{aligned}
$$

Integrating by parts we find that the right-hand side is equal to

$$\frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin x}{n^2}\right]_0^{\pi/2} + \frac{2}{\pi}\left[\frac{-(\pi - x)\cos nx}{n} - \frac{\sin x}{n^2}\right]_{\pi/2}^{\pi} = \frac{4}{\pi n^2} \sin \frac{n\pi}{2}.$$

Since
$$\sin \frac{n\pi}{2} = \begin{cases} 0 & n = 2k \\ (-1)^{k+1} & n = 2k - 1 \end{cases}$$

for $k \geq 1$, we get
$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} B_n (\sin nx) e^{-n^2 t} \\
&= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin[(2n-1)x] e^{-(2n-1)^2 t}.
\end{aligned}
$$

Heat equation with Neumann boundary conditions

We consider the heat equation but with Neumann boundary conditions

$$u_x(0,t) = u_x(0,t) = 0 \qquad \text{for all } t \geq 0.$$

In this case the eigenvalue problem becomes

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases}$$

As before the problem doesn't have negative eigenvalues. If $\lambda = 0$, the general solution is $X(x) = \alpha x + \beta$ so that $0 = X'(0) = \beta$, implies that $\lambda_0 = 0$ is an eigenvalue with the unique (up to multiplication by a constant) eigenfunction $X_0(x) \equiv 1$. If $\lambda > 0$, then the general solution of the problem is $X(x) = \alpha \cos\sqrt{\lambda}x + \beta \sin\sqrt{\lambda}x$ form which we conclude that $0 = X'(0) = \beta$ and $0 = X'(L) = -\sqrt{\lambda}\alpha \sin\sqrt{\lambda}L$ implies that $\lambda > 0$ is an eigenvalue if and only if

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \geq 1$$

and the corresponding eigenfunction $X_n(x)$ is given by

$$X_n(x) = \cos\frac{n\pi x}{L}.$$

Then the corresponding solutions of $T' + \lambda k T = 0$ are

$$\begin{aligned} T_0(t) &= B_0 \\ T_n(t) &= B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad n \geq 1. \end{aligned}$$

Thus we obtain a sequence of solutions

$$u_n(x,t) = B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\frac{n\pi x}{L}, \qquad n \geq 0$$

which we combine to form the series

$$u(x,t) = \sum_{n\geq 0} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\frac{n\pi}{L}x.$$

At $t = 0$, we have

$$\phi(x) = u(x,0) = \sum_{n\geq 0} B_n \cos\frac{n\pi x}{L}.$$

To compute B_m's, multiply this equality by $\cos\frac{m\pi x}{L}$ and integrate over $[0, L]$. Then

$$\int_0^L \phi(x)\cos\frac{m\pi x}{L}\,dx = \sum_{n\geq 0} B_n \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx.$$

Since $\displaystyle\int_0^L \cos\frac{m\pi x}{L}\,dx = \begin{cases} L & m = 0 \\ 0 & m \geq 1, \end{cases}$

and $\displaystyle\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m, \end{cases}$

it follows that

$$B_0 = \frac{1}{L}\int_0^L \phi(x)\,dx \quad \text{and} \quad B_m = \frac{2}{L}\int_0^L \phi(x)\cos\frac{m\pi x}{L}\,dx, \;\; m \geq 1.$$

Heat equation with periodic boundary conditions

Next we consider the periodic boundary conditions

$$u(-L,t) = u(L,t) \quad \text{and} \quad u_x(-L,t) = u_x(L,t) \quad \text{for all } t \geq 0.$$

In this case the eigenvalue problem takes the form

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L),\ X'(0) = X'(L) \end{cases}$$

This follows from $X(-L)T(t) = X(L)T(t)$ and $X'(-L)T(t) = X(L)T(t)$.

To find eigenvalues, we first consider $\lambda < 0$. since sinh is

odd and cosh is even, the condition $X(-L) = X(L)$ implies that

$$\beta \sinh \sqrt{-\lambda} L = 0$$

so that $\beta = 0$. The condition $X'(-L) = X'(L)$ implies that

$$\alpha \sinh \sqrt{-\lambda} L = 0$$

so that $\alpha = 0$. If $\lambda = 0$, then $X(-L) = -\alpha L + \beta = \alpha L + \beta = X(L)$ so that $\alpha = 0$. So $\lambda_0 = 0$ is an eigenvalue with the corresponding eigenfunction $X_0(x) \equiv 1$. Finally, let $\lambda > 0$. Then $X(-L) = X(L)$ gives either $\beta = 0$ or $\sqrt{\lambda} = \frac{n\pi}{L}$ and the condition $X'(-L) = X'(L)$ gives either $\alpha = 0$ or $\sqrt{\lambda} = \frac{n\pi}{L}$. Hence the positive eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \geq 1$$

and the corresponding eigenfunctions are

$$X_n(x) = B_n \cos\left(\frac{n\pi x}{L}\right) + C_n \sin\left(\frac{n\pi x}{L}\right).$$

Thus the product solutions of the periodic boundary problem are

$$\begin{aligned} u_0(x,t) &= A_0 \\ u_n(x,t) &= \left(B_n \cos\left(\frac{n\pi x}{L}\right) + C_n \sin\left(\frac{n\pi x}{L}\right)\right) e^{-\left(\frac{n\pi}{L}\right)^2 t} \end{aligned}$$

which can be combined to form the series

$$u(x,t) = A_0 + \sum_{n\geq 1} \left(B_n \cos\left(\frac{n\pi x}{L}\right) + C_n \sin\left(\frac{n\pi x}{L}\right)\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}.$$

At $t = 0$, we have

$$\phi(x) = u(x,0) = A_0 + \sum_{n\geq 1} \left(B_n \cos\left(\frac{n\pi x}{L}\right) + C_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

Integrating over $[-L, L]$, we get

$$A_0 = \frac{1}{2L} \int_{-L}^{L} \phi(x)$$

since $\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\ dx = \int_{-L}^{L} \sin\left(\frac{n\pi}{L}\right)\ dx = 0.$

Next multiplying both sides

by $\cos\left(\frac{m\pi x}{L}\right)$ and integrating over $[-L, L]$ leads to

$$B_m = \frac{1}{L}\int_{-L}^{L} \phi(x)\cos\left(\frac{m\pi x}{L}\right)\,dx$$

since

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0, \quad \text{all } n, m \geq 1.$$

Finally, multiplying both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating over $[-L, L]$ leads to

$$C_m = \frac{1}{L}\int_{-L}^{L} \phi(x)\sin\left(\frac{m\pi x}{L}\right)\,dx$$

since

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & n \neq m \\ L & n = m. \end{cases}$$

1(a) $u_t = 4\,u_{xx}, \qquad 0 < x < 1, \qquad t > 0,$

$u\,(x, 0) = x^2\,(1 - x), \qquad 0 \le x \le 1,$

$u\,(0, t) = 0, \qquad u\,(l, t) = 0, \qquad t \ge 0.$

(b) $u_t = k\,u_{xx}, \qquad 0 < x < \pi, \qquad t > 0,$

$u\,(x, 0) = \sin^2 x, \qquad 0 \le x \le \pi,$

$u\,(0, t) = 0, \qquad u\,(\pi, t) = 0, \qquad t \ge 0.$

(c) $u_t = u_{xx}, \qquad 0 < x < 2, \qquad t > 0,$

$u\,(x, 0) = x, \qquad 0 \le x \le 2,$

$u\,(0, t) = 0, \qquad u_x\,(2, t) = 1, \qquad t \ge 0.$

(d) $u_t = k\,u_{xx}, \qquad 0 < x < l, \qquad t > 0,$

$u\,(x, 0) = \sin\,(\pi x/2l), \qquad 0 \le x \le l,$

$u\,(0, t) = 0, \qquad u\,(l, t) = 1, \qquad t \ge 0.$

Answer:

1. (a) $u\,(x, t) = \sum\limits_{n=1}^{\infty} \frac{4}{n^3\pi^3} \left[2\,(-1)^{n+1} - 1\right] e^{-4n^2\pi^2 t} \sin\,(n\pi x).$

(b) $u\,(x, t) = \sum\limits_{n=1,3,4,\ldots}^{\infty} \left[(-1)^n - 1\right] \left[\frac{n}{\pi(4-n^2)} - \frac{1}{n\pi}\right] e^{-n^2 kt} \sin\,(nx).$

Separation of variable for the wave equation

Dirichlet boundary conditions

$$
\begin{aligned}
& u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < L,\ t > 0 \\
& u(0,t) = u(L,t) = 0, \qquad t \geq 0 \\
& u(x,0) = \phi, \qquad 0 < x < L \\
& u_t(x,0) = \psi, \qquad 0 < x < L.
\end{aligned}
$$

Solution If we take $u(x,t) = X(x)T(t)$,

We get $$X(x)T''(t) = c^2 X''(x)T(t).$$

Dividing by $c^2 X(x)T(t)$, we get

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}$$

from which we conclude that both sides of this equality must be equal to a constant $-\lambda$. Thus, we obtain two second order differential equations

$$X'' + \lambda X = 0$$

$$T'' + \lambda c^2 T.$$

The boundary conditions imply that

$$X(0) = X(L).$$

so that the function X should be a solution of the eigenvalue problem,

$$
\begin{aligned}
& X'' + \lambda X = 0, \qquad 0 < x < L \\
& X(0) = X(L) = 0.
\end{aligned}
$$

Just as in the case of the heat equation the eigenvalues are and the associated eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad X_n(x) = \sin\frac{n\pi x}{L} \quad n \geq 1.$$

Then the solution with $\lambda = \lambda_n$ is of the form

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)$$

where A_n and B_n are constants. The product solutions of the boundary value problem are given by

$$u_n(x,t) = \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\frac{n\pi x}{L}$$

which can be combined in the series

$$u(x,t) = \sum_{n\geq 1} \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\frac{n\pi x}{L}.$$

-----------(1)

Setting $t = 0$, we get

$$\phi(x) = u(x,0) = \sum_{n\geq 1} A_n \cos\left(\frac{n\pi ct}{L}\right) \sin\frac{n\pi x}{L}.$$

Multiplying by $\sin\frac{m\pi x}{L}$ and integrating over $[0, L]$ we find that

$$A_n = \frac{2}{L}\int_0^L \phi(x) \sin\frac{n\pi x}{L}\, dx.$$

Next, differentiate (1) with respect to t at $t = 0$ to get

$$\psi(x) = u_t(x,0) = \frac{n\pi c}{L} \sum_{n\geq 1} B_n \sin\frac{n\pi x}{L}.$$

To compute B_n's, multiply both sides by $\sin\frac{m\pi x}{L}$ and integrate over $[0, L]$ to get

$$B_m = \frac{2}{L}\frac{L}{m\pi c}\int_0^L \phi(x)\sin\frac{m\pi x}{L}\,dx = \frac{2}{m\pi c}\int_0^L \phi(x)\sin\frac{m\pi x}{L}\,dx.$$

Example

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & (x,t) \in (0,\pi)\times(0,\infty) \\ u(0,t) = u(0,t), & t \geq 0 \\ u(x,0) = \sin 2x,\ u_t(x,0) = 0, & 0 \leq x \leq \pi \end{cases}$$

Here $c = 2$, $[0, L] = [0, \pi]$. The formal solution is of the form

$$u(x,t) = \sum_{n\geq 1}(A_n \cos 2nt + B_n \sin 2nt)\cdot \sin nx.$$

From the above formulae, $B_n = 0$ since $\psi \equiv 0$ and

$$A_n = \frac{2}{L}\int_0^L \sin 2x \sin nx dx = \begin{cases} 1 & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Consequently,

$$u(x,t) = \cos 4t \sin 2x.$$

Exercises

A lightly stretched string with fixed ends $x = 0$ and $x = l$ is initially in a position given by the deflection $f(x)$ as follows and then released. Find the displacement of any point x of the string at any time $t > 0$

(i) $f(x) = u_0 \sin^3(\pi x/l),\ 0 \le x \le l.$

(ii) $f(x) = 10 \sin(\pi x/l),\ 0 \le x \le l.$

(iii) $f(x) = u_0 \sin(2\pi x/l),\ 0 \le x \le l.$

(iv) $f(x) = u_0 x(l - x),\ 0 \le x \le l.$

(v) $f(x) = u_0 \sin^2(\pi x/l),\ 0 \le x \le l.$

(vi) $f(x) = \begin{cases} \frac{2\lambda x}{l}, & 0 < x < l/2 \\ \frac{2\lambda(l-x)}{l}, & l/2 < x < l \end{cases}$

Answers

(i) $u(x,t) = \frac{1}{4} u_0 \{3 \sin(\pi x/l) \cos(\pi c t/l) - \sin(3\pi x/l) \cos(3\pi c t/l)\}.$

(ii) $u(x,t) = 10 \cos(\pi c t/l) \sin(\pi x/l).$

(iii) $u(x,t) = u_0 \sin(2\pi x/l) \cos(2\pi c t/l).$

(iv) $u(x,t) = \frac{8u_0 l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin\left\{\frac{(2n-1)\pi x}{l}\right\} \cos\left\{\frac{(2n-1)\pi c t}{l}\right\}.$

(v) $u(x,t) = \frac{u_0 l}{12 c \pi} \{9 \sin(\pi x/l) \cos(\pi c t/l) - \sin(3\pi x/l) \cos(3\pi c t/l)\}.$

(vi) $u(x,t) = \frac{8\lambda}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi c t}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$

Separation of variables for Laplace Equation

The two dimensional Laplace's equation is

$$u_{xx} + u_{yy} = 0 \quad (1)$$

$0 < x < 1,\ 0 < y < 1$

As an example consider the boundary conditions

$$u(0,y) = 0, \ \ u(1,y) = 0, \ \ u(x,0) = 0, \ \ u(x,1) = x - x^2$$

If we assume separable solutions of the form

$$u(x,y) = X(x)Y(y),$$

then substituting this into (1) We get

$$X''Y + XY'' = 0.$$

Dividing by XY and expanding gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0,$$

and since each term is only a function of x or y, then each must be constant giving

$$\frac{X''}{X} = \lambda, \qquad \frac{Y''}{Y} = -\lambda.$$

The boundary conditions give

$$X(0) = 0, \quad X(1) = 0, \quad Y(0) = 0.$$

In order to obtain non-trivial solution for x,

it is necessary to set $\lambda = -k^2$.

Solving X equation, $X = c_1 \sin kx + c_2 \cos kx$

X(0) = 0 gives $c_2 = 0$

X(1) = 0 implies $k = n\pi, \;\; k \in \mathbb{Z}^+$

So $X(x) = c_1 \sin n\pi x.$

we obtain the solution to the Y equation

$$Y(y) = c_3 \sinh n\pi y + c_4 \cosh n\pi y$$

Since $Y(0) = 0$ this implies $c_4 = 0$ so

$$X(x)Y(y) = a_n \sin n\pi x \sinh n\pi y$$

where we have chosen $a_n = c_1 c_3$.

Using the principle of superposition

$$u = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi y.$$

The remaining boundary condition gives

$$u(x,1) = x - x^2 = \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi.$$

This looks like a Fourier sine series and if we let $A_n = a_n \sinh n\pi$, this becomes

$$\sum_{n=1}^{\infty} A_n \sin n\pi x = x - x^2.$$

which is precisely a Fourier sine series. The coefficients A_n are given by

$$\begin{aligned} A_n &= \frac{2}{1}\int_0^2 \left(x - x^2\right) \sin n\pi x \, dx \\ &= \frac{16}{n^3\pi^3}(1 - \cos n\pi), \end{aligned}$$

and since $A_n = a_n \sinh n\pi$, this gives

$$a_n = \frac{4(1 - (-1)^n)}{n^3\pi^3 \sinh n\pi}.$$

The required solution is

$$u(x,y) = \frac{4}{\pi^3}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin n\pi x \frac{\sinh n\pi y}{\sinh n\pi}.$$

Example

Solve

$$u_{xx} + u_{yy} = 0,$$

0 < x < 1, 0 < y < 1

subject to

$$u(x,0) = 0, \quad u(x,1) = 0,$$

$$u(0,y) = 0, \quad u(1,y) = y - y^2.$$

Answer $$u = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \frac{\sinh n\pi x}{\sinh n\pi} \sin n\pi y.$$

Example

Solve

$$u_{xx} + u_{yy} = 0$$

0 < x < 1, 0 < y < 1

subject to

$$u(x,0) = x - x^2, \quad u(x,1) = 0$$

$$u(0,y) = 0, \quad u(1,y) = 0.$$

Answer $$u(x,y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin n\pi x \frac{\sinh n\pi(1-y)}{\sinh n\pi}.$$

Example

Solve

$$u_{xx} + u_{yy} = 0$$

subject to

$$u(x,0) = 0, \;\; u(x,1) = 0$$
$$u(0,y) = y - y^2, \;\; u(1,y) = 0.$$

Answer.

$$u(x,y) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \frac{\sinh n\pi(1-x)}{\sinh n\pi} \sin n\pi y.$$