(a) Proportion of days on which neither car is used

$= P(X = 0) = e^{-1.5} = 0.223.$

(b) Proportion of days on which some demands is refused

$= P(X > 2) = 1 - P(X \leq 2)$
$= 1 - \{P(X = 0) + P(X = 1) + P(X = 2)\}$
$= 1 - e^{-1.5}\left\{1 + 1.5 + \frac{(1.5)^2}{2}\right\}$
$= 0.1916.$

Normal Distribution The most commonly encountered physical phenomena provide plenty of normal distributions. Carl Fredrich Gauss (1777-1855) while studying the nature of errors made in any scientific measurement came out with a peculiar distribution, presently, known as *Gaussian distribution or normal distribution.* This distribution plays a vital role in statistical decision theory and estimation.
The normal distribution of a random variable X with parameter μ or m and $\sigma(> 0)$ is defined by the probability function $f(x)$ given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty.$$

A random variable with the above probability function is called normal variate. The fact that X follows normal distribution with parameter μ and σ is expressed symbolically by $X \sim N(\mu, \sigma^2)$.
In particular, the random variable Z with $\mu = 0$ and $\sigma = 1$ is called the *standard normal variate.* Thus, if Z is the standard normal variate then $Z \sim N(0, 1)$.

Property If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$ and $Var(X) = \sigma^2$.
To see the above, we note that

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx && (1.6) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx && (1.7) \end{aligned}$$

Putting $\frac{x-\mu}{\sqrt{2}\sigma} = t$, we see $dx = \sqrt{2}\,\sigma dt$ and $t \to \infty$ as $x \to \infty$, also $t \to -\infty$ as $x \to -\infty$.

$$\begin{aligned} \therefore E(X) &= \frac{1}{\sqrt{2\pi}\sigma}\sqrt{2}\sigma \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma t)e^{-t^2} dt \\ &= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt + \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} te^{-t^2} dt \\ &= \frac{\mu}{\sqrt{\pi}} \times \sqrt{\pi} + \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \times 0 = \mu \end{aligned}$$

since te^{-t^2} is and odd function and $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$.

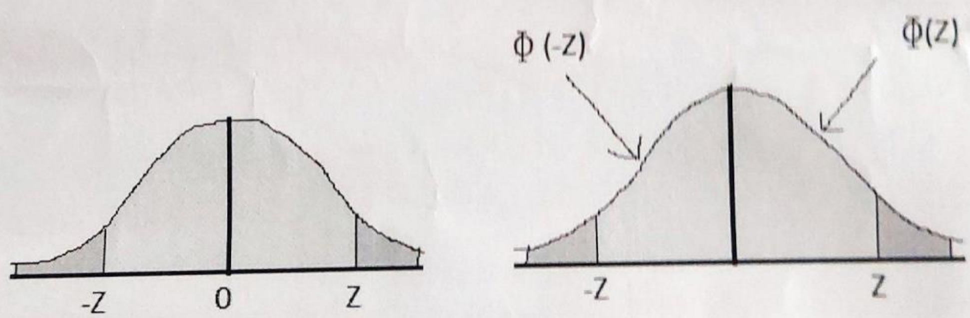
Now

$$
\begin{aligned}
E(X^2) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx && (1.8)\\
&= \frac{1}{\sqrt{2\pi}\sigma}\sqrt{2}\sigma\int_{-\infty}^{\infty}(\mu+\sqrt{2}\sigma t)^2 e^{-t^2}\,dt\\
&= \frac{\mu^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\,dt + \frac{2\sqrt{2}\mu\sigma}{\sqrt{\pi}}\int_{-\infty}^{\infty} te^{-t^2}\,dt + \frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^2e^{-t^2}\,dt\\
&= \mu^2+0+\sigma^2,\ \text{since} \int_{-\infty}^{\infty} t^2e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}. && (1.9)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Var(X) &= E(X^2) - \{E(X)\}^2\\
&= \mu^2+\sigma^2-\mu^2\\
&= \sigma^2.
\end{aligned}
$$

The Normal Probability Table A table showing the probability that Z is less than or equal to a particular value of z is known as the probability table. The value is usually denoted by $\Phi(z)$. Thus,



$$\phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{Z} e^{-\frac{t^2}{2}}\,dt$$

Clearly,

$$\Phi(0) = 0.5,\ \Phi(-z) = 1-\Phi(z) \text{ by symmetry.}$$

Also

$$P(a \le x \le b) = \Phi(b) - \Phi(a) \text{ since } P(X=a)=0.$$

Example. If $X \sim N(30, 25)$, then find

(i)$P(X \geq 45)$, (ii) $P(26 \leq X \leq 40)$, (iii) $P(|X-30| > 5)$.

Solution: Since $X \sim N(30, 25)$, $Z = \frac{X-30}{5} \sim N(0, 1)$.

(9)

Now,

$$\begin{aligned}
P(X \geq 45) &= P\left(\frac{X-30}{5} \geq \frac{45-30}{5}\right) \\
&= P(Z \geq 3) = 1 - P(Z < 3) = 1 - P(Z \leq 3), \text{ since } P(Z = 3) = 0 \\
&= 1 - \Phi(3) = 1 - 0.99865 \text{ from normal probability table} \\
&= 0.00135.
\end{aligned}$$

Again,

$$\begin{aligned}
P(26 \leq X \leq 40) &= P\left(\frac{26-30}{5} \leq \frac{X-30}{5} \leq \frac{40-30}{5}\right) \\
&= P(-0.8 \leq Z \leq 2) \\
&= \Phi(2) - \Phi(-0.8) \\
&= 0.7653
\end{aligned}$$

Finally,

$$\begin{aligned}
P(|X-30| > 5) &= P\left(\left|\frac{X-30}{5}\right| > 1\right) \\
&= P(Z > 1) \\
&= 1 - P(|Z| \leq 1) = 1 - P(-1 \leq Z \leq 1) \\
&= 1 - \{\Phi(1) - \Phi(-1)\} \\
&= 0.3174.
\end{aligned}$$

Example. Assuming that the lifespan of a type of transistor is normal, find the mean and standard deviation if 84 % of the transistors have lifespan less than 65.2 months and 68 % have lifespan lying between 65.2 and 62.8 months.

Solution: Let X denote lifespan of the said type of transistor and let its mean be μ and variance be σ^2. By the assumption $X \sim N(\mu, \sigma^2)$.

$$P(X \leq 65.2) = 0.84 \text{ and } P(62.8 \leq X \leq 65.2) = 0.68.$$

Then,

$$p\left(\frac{X-\mu}{\sigma} \leq \frac{65.2-\mu}{\sigma}\right) = 0.84 \text{ and } p\left(\frac{62.8-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{65.2-\mu}{\sigma}\right) = 0.68$$

or,

$$P\left(Z \leq \frac{65.2-\mu}{\sigma}\right) = 0.84 \text{ and } p\left(Z \leq \frac{62.8-\mu}{\sigma}\right) = 0.16$$

But from the normal probability table,

$$P(Z \leq 0.9) = 0.84 \text{ and } P(Z \leq -0.9) = 0.16.$$

Hence,

$$\frac{65.2-\mu}{\sigma} = 0.9 \text{ and } \frac{62.8-\mu}{\sigma} = -0.9$$

or,

$$\mu + 0.9\sigma = 65.2 \text{ and } \mu - 0.9\sigma = 62.8.$$

$$\therefore \mu = 64 \text{ months}, \ \sigma = 1.33 \text{ months}.$$

Example. The marks obtained by 1000 students in a final examination are found to be approximately normally distributed with mean 70 and standard deviation 5. Estimate the number of students whose marks will be between 60 and 75, both inclusive, given that the area under the normal curve $f(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ between$z = 0$ and $z = 2$ is 0.4772 and between $z = 0$ and $z = 1$ is 0.3413.

Solution: Let X denote the marks of a student in the examination. Then X is normal variate with mean 70 and standard deviation 5.

$$\therefore Z = \frac{X-70}{5} \text{ is the standard normal variate.}$$

$$\begin{aligned}
\therefore P(60 \le X \le 75) &= P\left(\frac{60-70}{5} \le \frac{X-70}{5} \le \frac{75-70}{5}\right) \\
&= P(-2 \le Z \le 1) \\
&= \Phi(1) - \Phi(-2) \\
&= 0.8413 - 0.0228 = 0.8185, \text{ since the normal curve is symmetrical.}
\end{aligned}$$

Exercise

1. It is known that one in every 10 villagers of a certain village contract leprosy. If 7 people are selected at random from the village, find the probability that 3 of them will have leprosy in future. *Ans* : $\mathbf{\frac{9^4.35}{10^7}}$.
2. Two fair dice are rolled 100 times. Find the probability of getting at least once a double six. *Ans* : $\mathbf{1 - \left(\frac{35}{36}\right)^{100}}$.
3. Suppose the probability of a new born baby being a boy is 0.51. In a family of 8 children, calculate the probability that there are 4 or 5 boys. *Ans* : **0.5003**.
4. A random variable X follows binomial distribution with mean $\frac{5}{3}$ and $P(X = 2) = P(X = 1)$. Find variance, $P(X \ge 1)$ and $P(X \le 1)$. *Ans* : $\mathbf{\frac{10}{9}, \frac{211}{243}, \frac{112}{243}}$.
5. A discrete random variable X has the mean 6 and variance 2. Assuming the distribution to be binomial, find the probability that $5 \le X \le 7$. *Ans* : $\mathbf{\frac{2^6 \times 73}{3^8}}$.
6. If X is Poisson variate such that $P(X = 1) = 0.2$ and $P(X = 2) = 0.2$, find $P(X = 0)$. *Ans* : **0.1**.