Vector Integration

Ordinary integrals of vectors:

Let $\vec{R}(u)=R_1(u)\vec{i}+R_2(u)\vec{j}+R_3(u)\vec{k}$ be a vector on a scalar variable u, where $R_1(u), R_2(u), R_3(u)$ are supposed continuous in a specified interval. Then

$$\int \vec{R}(u)du=\vec{i}\int R_1(u)du+\vec{j}\int R_2(u)du+\vec{k}\int R_3(u)du$$

is called an indefinite integral of $\vec{R}(u)$. If there exists a vector $\vec{S}(u)$ such that $\vec{R}(u)=\frac{d}{du}\left(\vec{S}(u)\right)$, then

$$\int \vec{R}(u)du=\int \frac{d}{du}\left(\vec{S}(u)\right)du=\vec{S}(u)+\vec{c}$$

where $\vec{c}$ is an arbitrary constant vector independent of u. The definite integral between limits $u=a$ and $u=b$ can in such case be written

$$\int_a^b \vec{R}(u)du=\int_a^b \frac{d}{du}\left(\vec{S}(u)\right)du=\left[\vec{S}(u)+\vec{c}\right]_a^b=\vec{S}(b)-\vec{S}(a)$$

Line integrals:

Let $\vec{r}(u)=x(u)\vec{i}+y(u)\vec{j}+z(u)\vec{k}$, where $\vec{r}(u)$ is the position vector of (x,y,z) define a curve C joining points P_1 and P_2, where $u=u_1$ and $u=u_2$ respectively.

We assume that C is composed of a finite number of curves for each of which $\vec{r}(u)$ has a continuous derivative.

Let $\vec{A}(x,y,z)=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\vec{A}$ along C from P_1 to P_2 written as

$$\int_{P_2}^{P_2} \vec{A}.d\vec{r}=\int_c \vec{A}.d\vec{r}=\int_c \left(A_1dx+A_2dy+A_3dz\right)$$

is an example of a line integral. If $\vec{A}$ is the force $\vec{F}$ on a particle moving along C, this line integral represents the work done by the force. If C is a closed curve(which we shall suppose is a simple closed curve, i.e., a curve which does not intersect itself anywhere) the integral around C is often denoted by $\oint \vec{A}.d\vec{r}=\oint\left(A_1dx+A_2dy+A_3dz\right)$.

In aerodynamics and fluid mechanics this integral is called the **circulation** of $\vec{A}$ about C, where $\vec{A}$ represents the velocity of a fluid.

Theorem: If $\vec{A}=\vec{\nabla}\phi$ everywhere in a region R of space, defined by $a_1\le x\le a_2$, $b_1\le y\le b_2$, $c_1\le z\le c_2$, where $\phi(x,y,z)$ is single-valued and has continuous derivatives in R, then

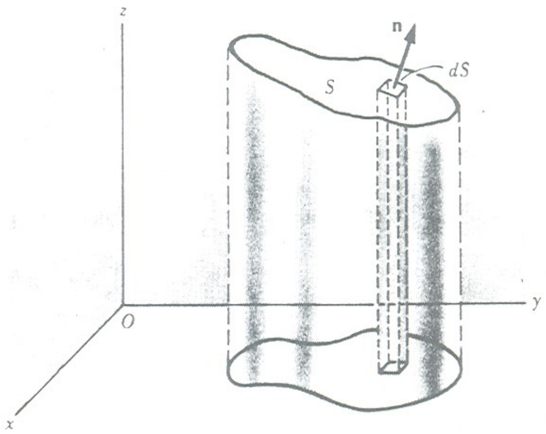
1. $\int_{P_1}^{P_2}\vec{A}.d\vec{r}$ is independent of the path C in R joining P_1 and P_2.

2. $\oint_C \vec{A}.d\vec{r}=0$ around any closed curve C in R.

In such cases $\vec{A}$ is called a **conservative vector field** and ϕ is its **scalar potential**.

Surface integrals:

Associate with the differential of surface area dS a vector $d\vec{S}$ whose magnitude is dS and whose direction is that of $\vec{n}$. Then $d\vec{S}=\vec{n}dS$, where $\vec{n}$ is the outward drawn unit normal vector to the surface S. The integral $\iint \vec{A}.d\vec{S}$ is an example of a surface integral called flux of $\vec{A}$ over S. Other surface integrals are $\iint\limits_S \phi\, dS,\ \iint\limits_S \phi\vec{n}\, dS,\ \iint\limits_S \vec{A}\times d\vec{S}$



Volume integrals:

Consider a closed surface in space enclosing a volume V. Then

$\iiint\limits_V \vec{A}\, dV$ and $\iiint\limits_V \phi\, dV$ are examples of volume integral.

Note: The notation $\oiint\limits_S$ is sometimes used to indicate integration over the closed surface S, where no confusion can arise the notation $\oint\limits_S$ may also be used.

Exercise1: The acceleration of a particle at any time $t \geq 0$ is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = 12\cos 2t\vec{i} - 8\sin 2t\vec{j} + 16t\vec{k}$$

If the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at $t=0$, find $\vec{v}$ and $\vec{r}$ at any time.

☺**.** Now, $\dfrac{d\vec{v}}{dt} = 12\cos 2t\vec{i} - 8\sin 2t\vec{j} + 16t\vec{k}$

Integrating, $\vec{v} = 6\sin 2t\vec{i} + 4\cos 2t\vec{j} + 8t^2\vec{k} + \vec{c}_1$

Putting $\vec{v} = \vec{0}$ when $t=0$,

we find $\vec{0} = 0\vec{i} + 4\vec{j} + 0\vec{k} + \vec{c}_1$ and $\vec{c}_1 = -4\vec{j}$

Then $\vec{v} = 6\sin 2t\vec{i} + (4\cos 2t - 4)\vec{j} + 8t^2\vec{k}$

so that $\dfrac{d\vec{r}}{dt} = 6\sin 2t\vec{i} + (4\cos 2t - 4)\vec{j} + 8t^2\vec{k}$

Integrating $\vec{r} = -3\cos 2t\vec{i} + (2\sin 2t - 4t)\vec{j} + \dfrac{8t^3}{3}\vec{k} + \vec{c}_2$

Putting $\vec{r} = \vec{0}$ when $t=0$ we find,

$$\vec{0} = -3\vec{i} + \vec{c}_2 \text{ and } \vec{c}_2 = 3\vec{i}$$

Then $\vec{r} = (3 - 3\cos 2t)\vec{i} + (2\sin 2t - 4t)\vec{j} + \dfrac{8t^3}{3}\vec{k}$

Exercise2: The equation of motion of a particle P of mass m is given by

$m\frac{d^2\vec{r}}{dt^2}=f(r)\vec{r}_1$ where $\vec{r}$ is the position vector of P measured from an origin O, $\vec{r}_1$ is a unit vector in the direction $\vec{r}$, and $f(r)$ is a function of the distance of P from O.

(a) Show that $\vec{r}\times\frac{d\vec{r}}{dt}=\vec{c}$ where $\vec{c}$ is a constant vector.
(b) Interpret physically the cases $f(r)<0$ and $f(r)>0$.
(c) Interpret the result in (a) geometrically.
(d) Describe how the results obtained relate to the motion of the planets in our solar system.

☺. (a) Multiply both sides of $m\frac{d^2\vec{r}}{dt^2}=f(r)\vec{r}_1$ by $\vec{r}\times$.

Then $m\vec{r}\times\frac{d^2\vec{r}}{dt^2}=f(r)\vec{r}\times\vec{r}_1=\vec{0}$, since $\vec{r}$ and $\vec{r}_1$ are collinear and so $\vec{r}\times\vec{r}_1=\vec{0}$.

Thus $\vec{r}\times\frac{d^2\vec{r}}{dt^2}=\vec{0}$ and $\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)=\vec{0}$.

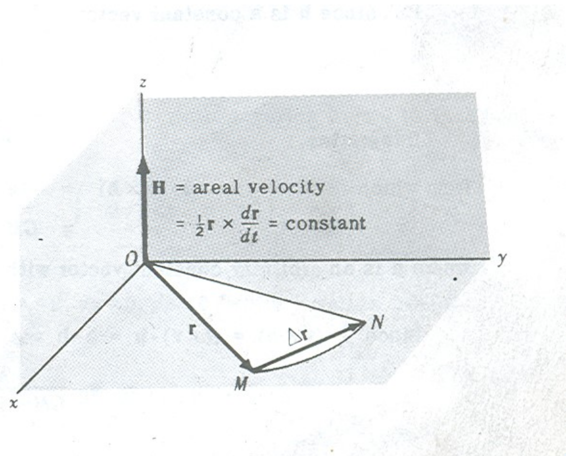
Integrating, $\vec{r}\times\frac{d\vec{r}}{dt}=\vec{c}$, where $\vec{c}$ is a constant vector.

(b) If $f(r)<0$ the acceleration $\frac{d^2\vec{r}}{dt^2}$ has direction opposite to $\vec{r}_1$; hence the force is directed toward O and the particle is always attracted toward O.

If $f(r)>0$ the force is directed away from O and the particle is under the influence of a repulsive force at O.

A force directed toward or away from a fixed point O and having magnitude depending only on the distance r from O is called a central force.

(c)



In time Δt the particle moves from M to N. The area swept out by the position vector in this time is approximately half the area of a parallelogram with sides $\vec{r}$ and $\Delta\vec{r}$, or $\frac{1}{2}\vec{r}\times\Delta\vec{r}$. Then the approximate area swept out by the radius vector per unit time is $\frac{1}{2}\vec{r}\times\frac{\Delta\vec{r}}{\Delta t}$; hence the instantaneous time rate of change in area is $\lim_{\Delta t\to 0}\frac{1}{2}\vec{r}\times\frac{\Delta\vec{r}}{\Delta t}=\frac{1}{2}\vec{r}\times\frac{d\vec{r}}{dt}=\frac{1}{2}\vec{r}\times\vec{v}$ where $\vec{v}$ is the instantaneous velocity of the particle. The quantity $\vec{H}=\frac{1}{2}\vec{r}\times\frac{d\vec{r}}{dt}=\frac{1}{2}\vec{r}\times\vec{v}$ is called the areal velocity. From part (a), Areal velocity$=\vec{H}=\frac{1}{2}\vec{r}\times\frac{d\vec{r}}{dt}$=constant.

Since $\vec{r}.\vec{H}=0$, the motion takes place in a plane, which we take as the xy plane in the figure above.

(d) A planet (such as the earth) is attracted toward the sun according to Newton's universal law of gravitation which states that any two objects of mass m and M respectively are attracted toward each other with a force of magnitude $F=\dfrac{GmM}{r^2}$, where r is the distance between objects and G is a universal constant. Let m and M be the masses of the planet and the sun respectively and choose a set of coordinate axes with the origin O at the sun. Then the equation of motion of the planet is

$$m\frac{d^2\vec{r}}{dt^2}=-\frac{GMm}{r^2}\vec{r}_1 \quad \text{or,} \quad \frac{d^2\vec{r}}{dt^2}=-\frac{GM}{r^2}\vec{r}_1$$

Assuming the influence of the other planets to be negligible. According to part (c), a planet moves around the sun so that its position vector sweeps out equal areas in equal times.

Exercise3: If $\vec{A}=\left(3x^2+6y\right)\vec{i}-14yz\vec{j}+20xz^2\vec{k}$, evaluate $\oint_C \vec{A}.d\vec{r}$ from (0,0,0) to (1,1,1) along the following paths C**:**

(a) $x=t, y=t^2, z=t^3$

(b) the straight lines from (0,0,0) to (1,0,0), then (1,0,0) to (1,1,0) and then to (1,1,1).

(c) the straight line joining (0,0,0) and (1,1,1).

☺. $$\int_C \vec{A}.d\vec{r}=\int_C\left[(3x^2+6y)\vec{i}-14yz\vec{j}+20xz^2\vec{k}\right].\left(dx\vec{i}+dy\vec{j}+dz\vec{k}\right)$$
$$=\int_C\left\{\left(3x^2+6y\right)dx-14yzdy+20xz^2dz\right\}$$

(a) If $x=t, y=t^2, z=t^3$, points (0,0,0) and (1,1,1) correspond to t=0 and t=1 respectively. Then

$$\int_C \vec{A}.d\vec{r}=\int_0^1\left\{\left(3t^2+6t^2\right)dt-14\left(t^2\right)\left(t^3\right)2tdt+20\left(t\right)\left(t^3\right)^2 3t^2dt\right\}$$
$$=\int_0^1\left(9t^2-28t^6+60t^9\right)dt=\left[3t^3-4t^7+6t^{10}\right]_0^1=5$$

(b) Along the straight line from (0,0,0) to (1,0,0) $y=0, z=0, dy=0$, dz=0, while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_0^1\left(3x^2+6(0)\right)dx-14(0)(0)(0)+20x(0)^2=\int_{x=0}^1 3x^2dx=1$$

Along the straight line from $(1,0,0)$ to $(1,1,0)$ $x=1, z=0, dx=0, dz=0$, while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_0^1\left(3(1)^2+6y\right)0-14y(0)dy+20(1)(0)^2 0=0$$

Along the straight line from $(1,1,0)$ to $(1,1,1)$ $x=1, y=1, dx=0, dy=0$ while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_0^1\left(3(1)^2+6(1)\right)0-14(1)z(0)+20(1)z^2dz=\int_{z=0}^1 20z^2dz=\left[\frac{20z^3}{3}\right]_0^1=\frac{20}{3}$$

Adding, $\int_C \vec{A}.d\vec{r}=1+0+\dfrac{20}{3}=\dfrac{23}{3}$

(c) Along the straight line joining $(0,0,0)$ and $(1,1,1)$ is given in parametric from by $x=t, y=t, z=t$.

Then $\int_C \vec{A}.d\vec{r} = \int_0^1 \left\{ \left(3t^2+6t\right)dt - 14(t)(t)dt + 20(t)(t)^2 dt \right\}$

$$= \int_0^1 \left(3t^2+6t-14t^2+20t^3\right)dt = \int_0^1 \left(6t-11t^2+20t^3\right)dt = \frac{13}{3}$$

Exercise4: Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

☺. Total work $= \int_C \vec{F}.d\vec{r}$. Here C is the curve $x = t^2+1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$

$$= \int_C \left(3xy\vec{i} - 5z\vec{j} + 10x\vec{k}\right).\left(dx\vec{i} + dy\vec{j} + dz\vec{k}\right)$$

$$= \int_C \left(3xydx - 5zdy + 10xdz\right)$$

$$= \int_1^2 \left\{3\left(t^2+1\right)\left(2t^2\right)2tdt - 5\left(t^3\right)4tdt + 10\left(t^2+1\right)3t^2dt\right\}$$

$$= \int_1^2 \left(12t^5+10t^4+12t^3+30t^2\right)dt = 303$$

Exercise5: If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$, evaluate $\int_C \vec{F}.d\vec{r}$ where C is the curve in the xy plane, $y = 2x^2$, from $(0,0)$ to $(1,2)$.

☺. Since the integration is performed in the xy plane $(z = 0)$, we can take $\vec{r} = x\vec{i} + y\vec{j}$. Then

$$\int_C \vec{F}.d\vec{r} = \int_C \left(3xy\vec{i} - y^2\vec{j}\right).\left(dx\vec{i} + dy\vec{j}\right) = 3xydx - y^2dy$$

Let $x = t$ in $y = 2x^2$. Then the parametric equations of C are $x = t$, $y = 2t^2$. Points $(0,0)$ and $(1,2)$ correspond to $t = 0$ and $t = 1$ respectively. Then

$$\int_C \vec{F}.d\vec{r} = \int_0^1 \left\{3(t)\left(2t^2\right)dt - \left(2t^2\right)4tdt\right\} = -\frac{7}{6}$$

Exercise6: In a plane field $\vec{a} = xy^2\vec{i} + 2x\vec{j}$, find the line integral of $\vec{a}$ along the curve $y = x^2$ from the point $O(0,0)$ to the point $P(1,1)$

☺. In the xy-plane $d\vec{r} = dx\vec{i} + dy\vec{j}$. Therefore $\vec{a}.d\vec{r} = \left(xy^2\vec{i} + 2x\vec{j}\right).\left(dx\vec{i} + dy\vec{j}\right) = xy^2dx + 2xdy$

Let $x = t$ in $y = x^2$. Then the parametric equations of the curve $y = x^2$ are $x = t, y = t^2$. Points $O(0,0)$ and $P(1,1)$ correspond to $t = 0$ and $t = 1$ respectively. Then the line integral equals

$$\int_0^1 \left[t\left(t^2\right)^2 dt + 2t\left(2tdt\right)\right] = \int_0^1 \left(t^5 + 4t^2\right)dt = \left[\frac{t^6}{6} + 4\frac{t^3}{3}\right]_0^1 = \frac{1}{6} + \frac{4}{3} = \frac{1+8}{6} = \frac{3}{2}$$

Exercise7: Evaluate $\int_C \left(xy\vec{i} - z\vec{j} + x^2\vec{k}\right) \times d\vec{r}$ along the curve C given by $x = t^2$, $y = 2t$, $z = t^3$ from $(0,0,0)$ to $(1,2,1)$.

☺. Let $\vec{F} = xy\vec{i} - z\vec{j} + x^2\vec{k}$.

Along C, $\vec{F} = xy\vec{i} - z\vec{j} + x^2\vec{k} = 2t^3\vec{i} - t^3\vec{j} + t^4\vec{k}$.

Then $\vec{F} \times d\vec{r} = \left(2t^3\vec{i} - t^3\vec{j} + t^4\vec{k}\right) \times \left(2t\vec{i} + 2\vec{j} + 3t^2\vec{k}\right)dt$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} dt = \left[\left(-3t^5 - 2t^4\right)\vec{i} + \left(2t^5 - 6t^5\right)\vec{j} + \left(4t^3 + 2t^4\right)\vec{k} \right] dt$$

Therefore the given integral equals $\int\limits_C \vec{F} \times d\vec{r} = \vec{i}\int\limits_0^1 \left(-3t^5 - 2t^4\right)dt + \vec{j}\int\limits_0^1 \left(-4t^5\right)dt + \vec{k}\int\limits_0^1 \left(4t^3 + 2t^4\right)dt$

$$= -\frac{9}{10}\vec{i} - \frac{2}{3}\vec{j} + \frac{7}{5}\vec{k}$$

Exercise8: Find the work done in moving a particle once around a circle C in the xy plane, if the circle has centre at the origin and radius 3 and if the force field is given by

$$\vec{F} = (2x - y + z)\vec{i} + \left(x + y - z^2\right)\vec{j} + (3x - 2y + 4z)\vec{k}$$

☺. In the plane $z = 0$, $\vec{F} = (2x - y)\vec{i} + (x + y)\vec{j} + (3x - 2y)\vec{k}$ and $d\vec{r} = dx\vec{i} + dy\vec{j}$ so that the work done is $\int\limits_C \vec{F}.d\vec{r} = \int\limits_C \left[(2x - y)\vec{i} + (x + y)\vec{j} + (3x - 2y)\vec{k}\right].\left(dx\vec{i} + dy\vec{j} + dz\vec{k}\right)$

$$= \int\limits_C \left\{(2x - y)dx + (x + y)dy\right\}$$

Choose the parametric equations of the circle as $x = 3\cos t$, $y = 3\sin t$, where t varies from 0 to 2π.

Then the integral equals $\int\limits_0^{2\pi} \left[\left\{2(3\cos t) - 3\sin t\right\}(-3\sin t)dt + \left\{3\cos t + 3\sin t\right\}3\cos t dt\right]$

$$= \int\limits_0^{2\pi} (9 - 9\sin t\cos t)dt = \left[9t - \frac{9}{2}\sin^2 t\right]_0^{2\pi} = 18\pi$$

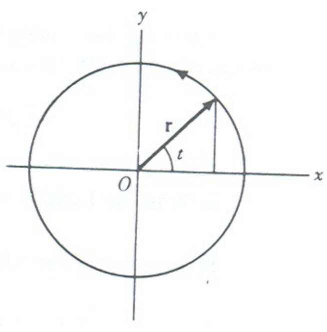
Exercise9: If $\vec{A} = (y - 2x)\vec{i} + (3x + 2y)\vec{j}$ compute the circulation of $\vec{A}$ about a circle C in the xy plane with centre at the origin and radius 2, if C is traversed in the positive direction.

☺. In the plane $z = 0$, $d\vec{r} = dx\vec{i} + dy\vec{j}$ so that the circulation of $\vec{A}$ around the circle C is

$\oint\limits_C \vec{A}.d\vec{r} = \oint\limits_C \left[(y - 2x)\vec{i} + (3x + 2y)\vec{j}\right].\left(dx\vec{i} + dy\vec{j}\right)$ where C is the circle in the xy plane with centre at the origin and radius 2.

$$= \oint\limits_C \left[(y - 2x)dx + (3x + 2y)dy\right]$$

Choose the parametric equation of the circle C as $x = 2\cos t, y = 2\sin t$ where C varies from 0 to 2π.



$$\vec{r} = x\vec{i} + y\vec{j} = 2\cos t\vec{i} + 2\sin t\vec{j}$$

Then the circulation equals $\int\limits_0^{2\pi} \left[(2\sin t - 4\cos t)(-2\sin t dt) + (6\cos t + 4\sin t)(2\cos t dt)\right]$

$$=\int_0^{2\pi}\left(-4\sin^2 t+8\sin 2t+12\cos 2t\right)dt$$

$$=\int_0^{2\pi}\left[-2\left(1-\cos 2t\right)+8\sin 2t+6\left(1+\cos 2t\right)\right]dt$$

$$=\int_0^{2\pi}\left(8\cos 2t+4-4\cos 2t\right)dt$$

$$=\left[4\sin 2t+4t-4\cos 2t\right]_0^{2\pi}=8\pi$$

Exercise10: If $\vec{F}=\left(yz+2x\right)\vec{i}+xz\vec{j}+\left(xy+2z\right)\vec{k}$ evaluate $\int_C \vec{F}.d\vec{r}$ along the curve $C: x^2+y^2=1, z=1$ in the positive direction from $(0,1,1)$ to $(1,0,1)$.

☺. In the plane $z=1$, $dz=0$. So that $\vec{F}.d\vec{r}=\left\{\left(y+2x\right)\vec{i}+x\vec{j}+\left(xy+2\right)\vec{k}\right\}.\left(dx\vec{i}+dy\vec{j}\right)=\left(y+2x\right)dx+xdy$
Choose the parametric equations of the curve C as $x=\cos t, y=\sin t, z=1$. In the positive direction points $(0,1,1)$ and $(1,0,1)$ corresponds to $t=\frac{\pi}{2}$ to $t=2\pi$. Thus the required integral is

$$\int_{\frac{\pi}{2}}^{2\pi}\left[\left(\sin t+2\cos t\right)\left(-\sin t dt\right)+\cos t\left(\cos t dt\right)\right]=\int_{\frac{\pi}{2}}^{2\pi}\left(\cos 2t-\sin 2t\right)dt=\left[\frac{\sin 2t}{2}+\frac{\cos 2t}{2}\right]_{\frac{\pi}{2}}^{2\pi}=1.$$

Exercise11: (a) If $\vec{F}=\vec{\nabla}\phi$, where ϕ is a single-valued and has continuous partial derivatives, show that the work done in moving a particle from one point $P_1\left(x_1,y_1,z_1\right)$ in this field to another point $P_2\left(x_2,y_2,z_2\right)$ is independent of the path joining the two points.

(b) Conversely if $\int_C \vec{F}.d\vec{r}$ is independent of the path C joining any two points, show that there exists a function such that $\vec{F}=\vec{\nabla}\phi$

☺. (a) Work done $=\int_{P_1}^{P_2}\vec{F}.d\vec{r}=\int_{P_1}^{P_2}\vec{\nabla}\phi.d\vec{r}$, (Since $\vec{F}=\vec{\nabla}\phi$)

$$=\int_{P_1}^{P_2}\left(\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}\right).\left(dx\vec{i}+dy\vec{j}+dz\vec{k}\right)$$

$$=\int_{P_1}^{P_2}\left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right)$$

$$=\int_{P_1}^{P_2}d\phi=\phi\left(P_2\right)-\phi\left(P_1\right)=\phi\left(x_2,y_2,z_2\right)-\phi\left(x_1,y_1,z_1\right)$$

Then the integrals depends only on points P_1 and P_2 and not on the path joining them.

(b) Let $\vec{F}=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$. By hypothesis, $\int_C \vec{F}.d\vec{r}$ is independent of the path C joining any two points which we take as $\left(x_1,y_1,z_1\right)$ and $\left(x,y,z\right)$ respectively. Then

$$\phi\left(x,y,z\right)=\int_{\left(x_1,y_1,z_1\right)}^{\left(x,y,z\right)}\vec{F}.d\vec{r}=\int_{\left(x_1,y_1,z_1\right)}^{\left(x,y,z\right)}\vec{F}.\frac{d\vec{r}}{ds}ds$$

By differentiation, $\frac{d\phi}{ds}=\vec{F}.\frac{d\vec{r}}{ds}$.

But $\frac{d\phi}{ds}=\frac{\partial\phi}{\partial x}\frac{dx}{ds}+\frac{\partial\phi}{\partial y}\frac{dy}{ds}+\frac{\partial\phi}{\partial z}\frac{dz}{ds}$

$$=\left(\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}\right).\left(\frac{dx}{ds}\vec{i}+\frac{dy}{ds}\vec{j}+\frac{dz}{ds}\vec{k}\right)$$

$$=\vec{\nabla}\phi.\frac{d\vec{r}}{ds}$$

or, $\left(\vec{\nabla}\phi-\vec{F}\right).\frac{d\vec{r}}{ds}=0$

Since this must hold irrespective of $\frac{d\vec{r}}{ds}$, we have $\vec{F}=\vec{\nabla}\phi$.

Exercise12: If $\vec{F}$ is a conservative field, prove that curl $\vec{F}=\vec{\nabla}\times\vec{F}=\vec{0}$ (i.e. $\vec{F}$ is irrotational).

☺**.** Let $\vec{F}$ be a conservative field, then we have, $\vec{F}=\vec{\nabla}\phi$, where ϕ is a scalar .

Thus curl $\vec{F}=\vec{\nabla}\times\vec{\nabla}\phi=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}$

$$=\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)\vec{i}+\left(\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right)\vec{j}+\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)\vec{k}=\vec{0}$$

Exercise13: Evaluate $\int_{(0,0)}^{(2,1)}\left\{\left(10x^4-2xy^3\right)dx-3x^2y^2dy\right\}$ along the path $x^4-6xy^3=4y^2$

☺**.** $\int_{(0,0)}^{(2,1)}\left\{\left(10x^4-2xy^3\right)dx-3x^2y^2dy\right\}$

$$=\int_{(0,0)}^{(2,1)}\left\{\left(10x^4-2xy^3\right)\vec{i}-3x^2y^2\vec{j}\right\}.\left(dx\vec{i}+dy\vec{j}+dz\vec{k}\right)$$

$=\int_{(0,0)}^{(2,1)}\vec{A}.d\vec{r}$ …… (1) where $\vec{A}=\left(10x^4-2xy^3\right)\vec{i}-3x^2y^2\vec{j}$

Now, $\vec{\nabla}\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 10x^4-2xy^3 & -3x^2y^2 & 0\end{vmatrix}=\vec{0}$

So the vector $\vec{A}$ represents a conservative force field.
Thus the integral (1) is independent of path.

$$\vec{A}.d\vec{r}=\left(10x^4-2xy^3\right)dx-3x^2y^2dy$$

$$=10x^4dx-\left(2xy^3dx+3x^2y^2dy\right)$$

$$=10x^4dx-d\left(x^2y^3\right)$$

$$=d\left(2x^5-x^2y^3\right)\ \ldots\ldots\ldots\ldots(2)$$

$$\therefore\int_{(0,0)}^{(2,1)}\left\{\left(10x^4-2xy^3\right)dx-3x^2y^2dy\right\}=\left[2x^5-3x^2y^3\right]_{(0,0)}^{(2,1)}=60$$

Exercise14: (a) Show that $\vec{F}=\left(2xy+z^3\right)\vec{i}+x^2\vec{j}+3xz^2\vec{k}$ is a conservative force field.

(b) Find the scalar potential.

(c) Find the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$

☺. (a) A necessary and sufficient condition that a force will be conservative is that curl $\vec{F}=\vec{\nabla}\times\vec{F}=\vec{0}$

$$\text{Now, } \vec{\nabla}\times\vec{F}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2 & 3xz^2 \end{vmatrix}=(0-0)\vec{i}+\left(3z^2-3z^2\right)\vec{j}+(2x-2x)\vec{k}=\vec{0}$$

Thus $\vec{F}$ is a conservative force field.

(b) Since $\vec{F}$ is a conservative force field. So we can write

$$\vec{F}=\vec{\nabla}\phi$$

$$\text{or, } \frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}=\left(2xy+z^3\right)\vec{i}+x^2\vec{j}+3xz^2\vec{k}$$

$$\text{Then } \frac{\partial\phi}{\partial x}=2xy+z^3,\ \frac{\partial\phi}{\partial y}=x^2,\ \frac{\partial\phi}{\partial z}=3xz^2$$

$$\begin{aligned}\text{Now, } d\phi &= \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz \\ &=(2xy+z^3)dx+x^2dy+3xz^2dz \\ &=\left(2xydx+x^2dy\right)+\left(z^3dx+3xz^2dz\right) \\ &= d\left(x^2y\right)+d\left(xz^3\right)=d\left(x^2y+xz^3\right)\end{aligned}$$

Therefore, $\phi=x^2y+xz^3$ +constant

(c) $\vec{F}.d\vec{r}=\vec{\nabla}\phi.d\vec{r}$ ($\because$ $\vec{F}$ is conservative force)

$$=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=d\phi$$

Thus the total work done$=\int_{(1-2,1)}^{(3,1,4)} d\phi=\phi(3,1,4)-\phi(1,-2,1)=202$.

Exercise15: (a) Show that a necessary and sufficient condition that $F_1dx+F_2dy+F_3dz$ be an exact differential is that $\vec{\nabla}\times\vec{F}=\vec{0}$ where $\vec{F}=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$

(b)Show that $\left(y^2z^3\cos x-4x^3z\right)dx+2z^3y\sin x dy+\left(3y^2z^2\sin x-x^4\right)dz$ is an exact differential of a function ϕ and find ϕ.

☺. (a) Suppose $F_1dx+F_2dy+F_3dz=d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$, an exact differential.

Then since x,y,z are independent variables, $F_1=\frac{\partial\phi}{\partial x}, F_2=\frac{\partial\phi}{\partial y}, F_3=\frac{\partial\phi}{\partial z}$.

Therefore, $\vec{F}=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}=\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}=\vec{\nabla}\phi$

$$\text{Thus } \vec{\nabla}\times\vec{F}=\vec{\nabla}\times\vec{\nabla}\phi=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}=\vec{0}$$

Conversely, if $\vec{\nabla}\times\vec{F}=\vec{0}$, then we have $\vec{F}=\vec{\nabla}\phi$ where ϕ is a scalar function.

Now, $F_1dx+F_2dy+F_3dz=\left(F_1\vec{i}+F_2\vec{j}+F_3\vec{k}\right).\left(dx\vec{i}+dy\vec{j}+dz\vec{k}\right)=\vec{F}.d\vec{r}$

$$=\vec{\nabla}\phi.d\vec{r}=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$$

$=d\phi$, an exact differential

(b) Let $\vec{F}=\left(y^2z^3\cos x-4x^3z\right)\vec{i}+2z^3y\sin x\vec{j}+\left(3y^2z^2\sin x-x^4\right)\vec{k}$

$$\therefore\ \vec{\nabla}\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2z^3\cos x-4x^3z & 2z^3y\sin x & 3y^2z^2\sin x-x^4\end{vmatrix}$$

$$=\left(6yz^2\sin x-6yz^2\sin x\right)\vec{i}+\left(3y^2z^2\cos x-4x^3-3y^2z^2\cos x+4x^3\right)\vec{j}$$
$$+\left(2z^3y\cos x-2yz^3\cos x\right)\vec{k}=\vec{0}$$

Therefore, $\left(y^2z^3\cos x-4x^3z\right)dx+2z^3y\sin xdy+\left(3y^2z^2\sin x-x^4\right)dz$ is an exact differential.

Let $\left(y^2z^3\cos x-4x^3z\right)dx+2z^3y\sin xdy+\left(3y^2z^2\sin x-x^4\right)dz=d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$

$$\therefore\ \frac{\partial\phi}{\partial x}=y^2z^3\cos x-4x^3z\,,\frac{\partial\phi}{\partial y}=2z^3y\sin x\,,\frac{\partial\phi}{\partial z}=3y^2z^2\sin x-x^4$$

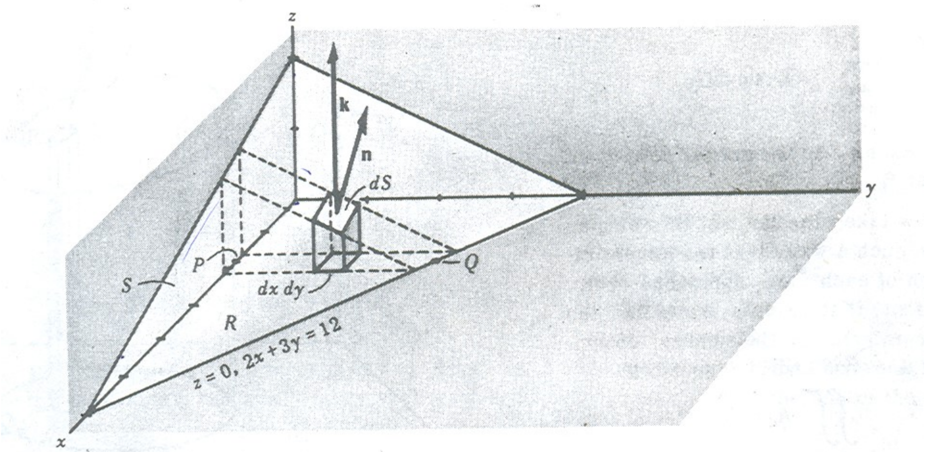
Now $d\phi=\left(y^2z^3\cos x-4x^3z\right)dx+2z^3y\sin xdy+\left(3y^2z^2\sin x-x^4\right)dz$

$$=\left(y^2z^3\cos xdx+2z^3y\sin xdy+3y^2z^2\sin xdz\right)-\left(4x^3zdx+x^4dz\right)$$
$$=d\left(y^2z^3\sin x-x^4z\right)$$

$\therefore\ \phi=y^2z^3\sin x-x^4z$ +constant

Exercise16: Evaluate $\iint_S\vec{A}.\vec{n}dS$ where $\vec{A}=18z\vec{i}-12\vec{j}+3y\vec{k}$ and S is that part of the plane $2x+3y+6z=12$ which is located in the first octant.

☺.



Let R be the projection of S on the xy plane. Then we have ,

$$\iint_S\vec{A}.\vec{n}dS=\iint_R\vec{A}.\vec{n}\frac{dxdy}{\left|\vec{n}.\vec{k}\right|}$$

To obtain $\vec{n}$ note that a vector perpendicular to the surface $2x+3y+6z=12$ is given by

$\vec{\nabla}\left(2x+3y+6z\right)=2\vec{i}+3\vec{j}+6\vec{k}$

Then a unit normal to any point of S is $\vec{n}=\dfrac{2\vec{i}+3\vec{j}+6\vec{k}}{\sqrt{2^2+3^2+6^2}}=\dfrac{2}{7}\vec{i}+\dfrac{3}{7}\vec{j}+\dfrac{6}{7}\vec{k}$

Thus $\vec{n}.\vec{k}=\left(\dfrac{2}{7}\vec{i}+\dfrac{3}{7}\vec{j}+\dfrac{6}{7}\vec{k}\right).\vec{k}=\dfrac{6}{7}$ and so $\dfrac{dxdy}{\left|\vec{n}.\vec{k}\right|}=\dfrac{7}{6}\,dxdy$

Also $\vec{A}.\vec{n}=\left(18z\vec{i}-12\vec{j}+3y\vec{k}\right).\left(\dfrac{2}{7}\vec{i}+\dfrac{3}{7}\vec{j}+\dfrac{6}{7}\vec{k}\right)$

$$=\frac{36z-36+18y}{7}=\frac{6(12-2x-3y)-36+18y}{7}=\frac{36-12x}{7}$$

Then $\displaystyle\iint_S \vec{A}.\vec{n}dS=\iint_S \vec{A}.\vec{n}\frac{dxdy}{\left|\vec{n}.\vec{k}\right|}=\iint_R \frac{36-12x}{7}\,\frac{7}{6}\,dxdy=\iint_R (6-2x)dxdy$

$$=\int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}}(6-2x)dydx$$

$$=\int_{x=0}^{6}(6-2x)\left(\frac{12-2x}{3}\right)dx$$

$$=\int_{x=0}^{6}\left(24-12x+\frac{4x^2}{3}\right)dx$$

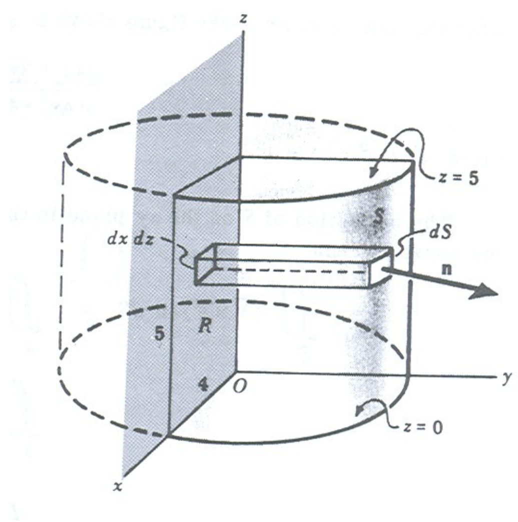
$$=\left[24x-6x^2+\frac{4x^3}{9}\right]_0^6$$

$=24$

Note: $d\vec{S}=\vec{n}dS=dydz\vec{i}+dzdx\vec{j}+dxdy\vec{k}$. So that $dS=\dfrac{dydz}{\left|\vec{n}.\vec{i}\right|}=\dfrac{dzdx}{\left|\vec{n}.\vec{j}\right|}=\dfrac{dxdy}{\left|\vec{n}.\vec{k}\right|}$.

Exrecise17: Evaluate $\displaystyle\iint_S \vec{A}.\vec{n}dS$, where $\vec{A}=z\vec{i}+x\vec{j}-3y^2\vec{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

☺.



Let R be the projection of the surface S on the xz plane.

Then we have $\displaystyle\iint_S \vec{A}.\vec{n}dS=\iint_R \vec{A}.\vec{n}\frac{dxdz}{\left|\vec{n}.\vec{j}\right|}$

A normal to $x^2+y^2=16$ is $\vec{\nabla}\left(x^2+y^2\right)=2x\vec{i}+2y\vec{j}$

Thus the unit normal to S is $\vec{n}=\dfrac{2x\vec{i}+2y\vec{j}}{\sqrt{(2x)^2+(2y)^2}}=\dfrac{x\vec{i}+y\vec{j}}{4}$, since $x^2+y^2=16$ on S.

$$\vec{A}.\vec{n}=\left(z\vec{i}+x\vec{j}-3y^2z\vec{k}\right).\left(\frac{x\vec{i}+y\vec{j}}{4}\right)=\frac{1}{4}\left(xz+xy\right)$$

$$\vec{n}.\vec{j}=\left(\frac{x\vec{i}+y\vec{j}}{4}\right).\vec{j}=\frac{y}{4}$$

Then the surface integral equals

$$\iint_R \frac{xz+xy}{y}\,dxdz=\iint_R\left(\frac{xz}{\sqrt{16-x^2}}+x\right)dxdz$$

$$=\int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}}+x\right)dxdz$$

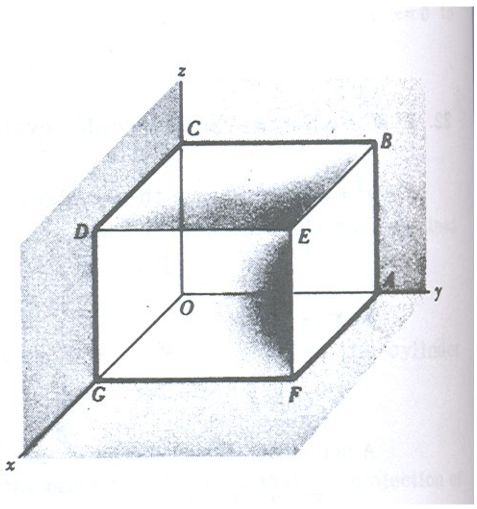
$$=\int_{z=0}^{5}\left[-\sqrt{(16-x^2)}z+\frac{x^2}{2}\right]_0^4 dz$$

$$=\int_{z=0}^{5}\left(4z+8\right)dz=\left[2z^2+8z\right]_0^5=50+40=90$$

Exercise18: If $\vec{F}=4xz\vec{i}-y^2\vec{j}+yz\vec{k}$,evaluate $\iint_S \vec{F}.\vec{n}dS$ where S is the surface of the cube bounded by

$$x=0,x=1,y=0,y=1,z=0,z=1$$

☺.



On the face DEFG: $\vec{n}=\vec{i}$, $x=1$

Then $\displaystyle\iint_{DEFG}\vec{F}.\vec{n}dS=\int_0^1\int_0^1\left(4z\vec{i}-y^2\vec{j}+yz\vec{k}\right).\vec{i}dydz$

$$=\int_0^1\int_0^1 4zdydz=2$$

On the face ABCO: $\vec{n}=-\vec{i}$, $x=0$

Then $\displaystyle\iint_{ABCO}\vec{F}.\vec{n}dS=\int_0^1\int_0^1\left(-y^2\vec{j}+yz\vec{k}\right).\left(-\vec{i}\right)dydz=0$

On the face ABEF: $\vec{n}=\vec{j}$, $y=1$

Then $\iint\limits_{ABEF} \vec{F}.\vec{n}dS = \int_0^1\int_0^1 \left(4xz\vec{i} - \vec{j} + z\vec{k}\right).\vec{j}dxdz$

$= \int_0^1\int_0^1 (-dxdz) = -1$

<u>On the face OGDC</u>: $\vec{n} = -\vec{j}$, $y = 0$

Then $\iint\limits_{OGDC} \vec{F}.\vec{n}dS = \int_0^1\int_0^1 \left(4xz\vec{i}\right).\left(-\vec{j}\right)dxdz = 0$

<u>On the face BCDE</u>: $\vec{n} = \vec{k}$, $z = 1$

Then $\iint\limits_{BCDE} \vec{F}.\vec{n}dS = \int_0^1\int_0^1 \left(4x\vec{i} - y^2\vec{j} + y\vec{k}\right).\vec{k}dxdy$

$= \int_0^1\int_0^1 ydxdy = \frac{1}{2}$

<u>On the face AFGO</u>: $\vec{n} = -\vec{k}$, $z = 0$

Then $\iint\limits_{AFGO} \vec{F}.\vec{n}dS = \int_0^1\int_0^1 \left(-y^2\vec{j}\right).\left(-\vec{k}\right)dxdy = 0$

Adding $\iint\limits_{S} \vec{F}.\vec{n}dS = 2 + 0 + (-1) + 0 + \frac{1}{2} + 0 = \frac{3}{2}$

Exercise19: If $\vec{F} = 2y\vec{i} - z\vec{j} + x^2\vec{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, evaluate $\iint\limits_{S} \vec{F}.\vec{n}dS$

☺**.** Let R be the projection of the surface S on the yz-plane, then we have

$$\iint\limits_{S} \vec{F}.\vec{n}dS = \iint\limits_{R} \vec{F}.\vec{n}\frac{dydz}{\left|\vec{n}.\vec{i}\right|}$$

A normal to the surface S is $\nabla\left(8x - y^2\right) = 8\vec{i} - 2y\vec{j}$.

A unit normal to S is $\dfrac{8\vec{i} - 2y\vec{j}}{\sqrt{64 + 4y^2}} = \dfrac{4\vec{i} - y\vec{j}}{\sqrt{16 + y^2}}$

So that $\vec{n}.\vec{i} = \dfrac{4}{\sqrt{16 + y^2}}$ and $\vec{F}.\vec{n} = \dfrac{8y + yz}{\sqrt{16 + y^2}}$

Thus the given surface integral equals $\iint\limits_{R} \dfrac{8y + yz}{\sqrt{16 + y^2}} \dfrac{dydz}{\dfrac{4}{\sqrt{16 + y^2}}} = \iint\limits_{R} \dfrac{8y + yz}{4}dydz = \int_{z=0}^{6}\left\{\int_{y=0}^{4} \frac{y}{4}(8 + z)dy\right\}dz$

$$= \int_{z=0}^{6} 2(8 + z)dz = \left[(8 + z)^2\right]_0^6 = 132$$

Exercises:

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1. The acceleration $\vec{a}$ of a particle at any time $t \ge 0$ is given by $\vec{a} = e^{-t}\vec{i} - 6(t + 1)\vec{j} + 3\sin t\vec{k}$. If the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at $t = 0$, find $\vec{v}$ and $\vec{r}$ at any time.

2. If $\vec{A} = (2y + 3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$, evaluate $\int\limits_{C} \vec{A}.d\vec{r}$ along the following path C:

(a) $x = 2t^2, y = t, z = t^3$ from $t = 0$ to $t = 1$,

(b) the straight lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$, and then to $(2,1,1)$.

(c) the straight line joining $(0,0,0)$ to $(2,1,1)$

3. If $\vec{F}=(5xy-6x^2)\vec{i}+(2y-4x)\vec{j}$, evaluate $\int_C \vec{F}.d\vec{r}$ along the curve in the xy plane, $y=x^3$ from the point $(1,1)$ to $(2,8)$

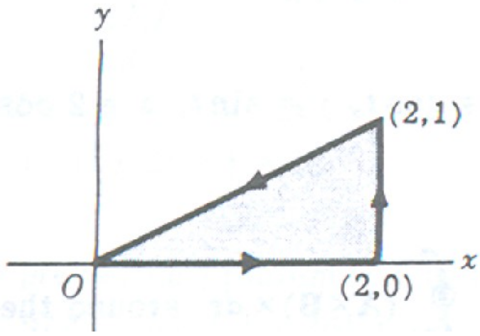
4. Find the work done in moving particle in the force field $\vec{F}=3x^2\vec{i}+(2xz-y)\vec{j}+z\vec{k}$ along

(a) the straight line from $(0,0,0)$ to $(2,1,3)$.

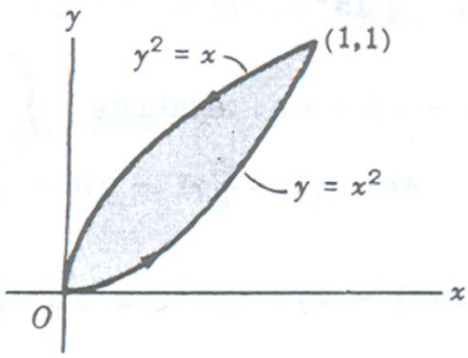
(b) the space curve $x=2t^2, y=t, z=4t^2-t$ from t=0 to t=1

(c) the curve defined by $x^2=4y, 3x^3=8z$ from x=0 to x=2

5. If $\vec{F}=(2x+y^2)\vec{i}+(3y-4x)\vec{j}$, evaluate $\oint_C \vec{F}.d\vec{r}$ around the triangle C of the figure (a) in the indicated direction, (b) opposite to the indicated direction.



6. Evaluate $\oint_C \vec{A}.d\vec{r}$ around the closed curve C of the figure if $\vec{A}=(x-y)\vec{i}+(x+y)\vec{j}$



7. (a) Prove that $\vec{F}=(y^2\cos x+z^3)\vec{i}+(2y\sin x-4)\vec{j}+(3xz^2+2)\vec{k}$ is a conservative force field.

(b) Find the scalar potential .

(c) Find the work done in moving an object in this field from $(0,1,-1)$ to $\left(\frac{\pi}{2},-1,2\right)$

8. A particle moves in a field of force $\vec{F}$ given by $\vec{F}=yz(1-2xyz)\vec{i}+zx(1-2xyz)\vec{j}+xy(1-2xyz)\vec{k}$, verify that the force is conservative and find the potential function from which it is derivable.

9. Prove that $\vec{F}=r^2\vec{r}$ is conservative and find the scalar potential.