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# Graph Theory & Combinatorics

Prof. Chintan Kumar Mondol



## Books

1) Combinatorial Theory – Brogn & Lint
2) Graph Theory – Harary
3) . . . – N. Deo
4) Combinatorial theory (Discrete Maths) – Rogen.

## Combinatorics

### Fundamental Counting Techniques/Principles

1) Addition Problem :- If there are $r_1$ different objects in ~~the~~ first set, $r_2$ different objects in second set & $r_m$ objects in mth set & if different sets are disjoint, the no. of ways to select an object from one of the sets is

$$r_1 + r_2 + \dots + r_m = \sum_{i=1}^{m-1} r_i$$

2) Multiplication Principle :- If a procedure can be broken up in to m-successive (ordered) stages, with $r_2$ different outcomes in the 1st stage, $r_2$ different outcomes in ~~diff~~ second stage, & . . . $r_m$ different outcomes in mth stage, the no. of outcomes at each stage is independent of the choices ~~in~~ in previous stages & if the composite outcomes are all distinct, total no. of procedures :- $r_1 \times r_2 \times r_3 \times \dots \times r_m$.

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**Note.** The addn principle requires sets of objects & the multiplication requires procedures broken into ordered sets and that the composite outcomes be distinct.

* The sum $(a+b)$ is the no. of items, resulting when a set $\underline{a}$ items is added to a set of $\underline{b}$ items.

* The product $a \times b$ is the no. of sequences $(A \times B)$ when A can be any of $\underline{a}$ item & B can be any of $\underline{b}$ item.

**Example.** There are 5 different S books,
6 different F books,
8 different T books.
How many ways are there to pick an pair of two books both of them from the same set of books

**Ans.**

SF, ST, FT $\rightarrow$ $6 \times 8 = 48$.

$5 \times 6 = 30$

$5 \times 8 = 40$

$30 + 40 + 48 = \underline{118}$

* "Unordered" means that there is not a first book in the pair & so on. Outcomes cannot be broken into the first stage (first book) & a second stage — i.e. multiplication principle does not apply.

* The question is how to restrict the problem, so that the multiplication/addition principle can be used.

* The problem can be broken up into smaller parts to which the principles apply, the question arises of what parts are into which we should break the problem.

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Possible case :- One possible case when

2 books consist of 1S & 1F book.

This has 2 stages:-

① pick a S book in 5 ways &
F books 6 ways.
total no. of ways $= 6 \times 5 = 30$.

② 1S & 1T $= 5 \times 8 = 40$ ways

③ 1F & 1T $= 6 \times 8 = 48$ ways

The outcomes in these case are clearly disjoint :- $(6 \cdot 5 + 5 \cdot 8 + 6 \cdot 8) = 118$ ways.

## Simple arrangements & selection

Defn :- A permutation of n-distinct objects is an arrangement or ordering of n-objects.

① An r-permutation of n-distinct objects is an arrangement using r of the n-objects $= P(n,r) = \frac{n!}{(n-r)!}$

② An r-combination of n-distinct objects is an ordered selection or subject of r- out of the n-objects.

$$= C(n,r) = \frac{n!}{r!(n-r)!}$$

③ All r-permutations of n-objects can be generated by first picking any r-combination of the n-objects & then rearranging the r-objects in any order.

$$P(n,r) = C(n,r) \times P(r,r).$$

$$\Rightarrow C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!(n-r)!}$$

Note

$C(n,r) \mid \binom{n}{r}$ → these are the binomial coefficient of the binomial expression $(x+y)^n$.

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## Arrangements of repeated letters

(A) How many ways are there to arrange the 7 letters of the word SYSTEMS.

Soln. Pick the posn that the 4 letters E, M, T, Y will occupy in the seven letter arrangement. 3 S will fill the remaining 3 posn.

There are 7 possible posn for E, 6 for M, 5 for T & 4 for Y,

then are :- $P(7,4) = \frac{7!}{3!}$ arrangements.

(B) In how many of the arrangements do the ~~3's~~ 3 S's appear consecutively.

Soln. This can be done by grouping the 3 S's as one S. This can be done by 5! ways.

(1) Another way to look at this problem is to think of temporarily setting aside two of the S's & arranging ~~them~~ the 5 remaining letters i.e. Y, T, C, M, S in 5! ways

## Arrangements & Selections with ~~repe~~ repetition

**Theorem :-** If there are n-objects with $r_1$ of type 1, $r_2$ of type 2, ...., $r_m$ of type m where $\sum_{i=1}^{m} r_i = n$, then the no. of arrangement of n objects is denoted by $P(n: r_1, r_2, \ldots, r_m)$.

i.e. $$P(n: r_1, r_2, \ldots, r_m) = \binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2\ldots r_m}{r_m}$$

$$= \frac{n!}{r_1!\, r_2!\, r_3! \ldots r_m!}$$

where $\left[\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}\right]$

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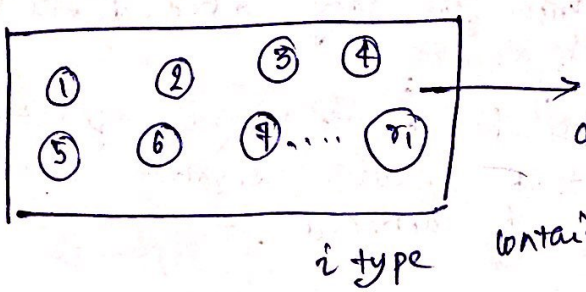
**Proof:-** Suppose that for each type, the $r_i$ ~~it~~ objects of type are given subscripts numbered $1, 2, 3, \ldots, r_i$ to make each object distinct




Then there are $n!$ arrangements of the $n$-distinct objects by enumerating all $P(n: r_1, r_2, \ldots r_m)$ patterns (without subscripts) of the objects & then placing the objects in all possible ways.

eg. the pattern baannna can have subscripts on a's as placed in $3!$ ways.

**Soln** For each of the $3!$ ways to subscript the a's & there are $2!$ ways to subscript the n's.

$ba_1a_2nna_3$
$ba_2a_1nna_3$
$ba_3a_1nna_2$
$ba_3a_2nna_1$
$ba_1a_3nna_2$
$ba_2a_3nna_1$

Thus in general, a pattern will have $r_1!$ ways to subscript the $r_1$ objects of type 1, ....

$$\cancel{n! = P(n: r_1! r_2! \ldots r_m}$$

$$n! = P(n: r_1, r_2 \ldots r_m)\, r_1!\, r_2!\, r_3! \ldots$$

$$\Rightarrow P(n: r_1, r_2, \ldots r_m) = \frac{n!}{r_1!\, r_2! \ldots r_m!}$$

**Theorem:-** The no. of selections with repetitions of $r$ objects chosen from $n$ types of objects is $C(r+n-1, r) = \binom{n+r-1}{r}$.

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eg. How many ways are there to form a sequence of 10 letters, from 4a, 4b, 4c & 4d of each letter appear atleast time.

Ans. To apply the theorem, we must exactly know how many a, b, c, d will be in the arrangement.

Thus, the problem has to be broken up into sub-problems. that each involves sequence with given no. of a, b, c, d.

There are two categories of letter frequencies that sum to 10 with each letter 2 or more times.

Cat 1. There are 4 appearances of 1 letter & 2 appearances of each other letter.

There are 4 cases ($C(4,1)$) for choosing which ~~the~~ letter occurs 4 time & $P(10: 4,2,2,2)$ arrangement of 4 of one letter & 2 of the other.

Cat 2.

3 appearances of 2 letters & 2 appearances of 2 other letters.

∴ there are $\binom{4}{2}$ ways of choosing which two of the 4 letters occur 3 times.

& $P(10; 3,3,2,2)$ ways to arrange 3 of two letters & 2 of the 2 other.

∴ total no. of way is

$$\binom{4}{1} \times P(10; 4,2,2,2) + \binom{4}{2} P(10: 3,3,2,2)$$

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**Distribution :-** This is equivalent to an arrangement or selection problem with repetation.

Modeling distribution problem $\doteq$ { Distribution of distinct object are equivalent to arrangements
& Distribution of ~~actual~~ identical object are equivalent to selection.

**Basic Models :-**

**Distinct objects :-** The process of distributing r-distinct objects into n-different boxes is equivalent to putting the distinct objects in a row & stamping one of the n-different box names on each object.

- The resulting sequence of box names is an arrangement of length r formed from n-items (box names) with repetation.
  There are $n \times n \times \ldots n = n^r$

- If $r_i$ objects must go into, $1 \leq i \leq n$, there there are $P(r; r_1, \ldots r_n)$ distribution.

**Identical Objects**

The process of distributing r-objects into n-different boxes is equivalent to choosing an (unordered) subset of r box names with repeatitions from the n-choice of boxes.

$\therefore$ the total no. of distributing the r-identical objects are.

$$C\binom{n+r-1}{r} = \frac{(r+n-1)!}{r!\,(n-1)!}$$

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(*) How many ways are there to distribute 4 identical & 6 distinct apples (each of different variety) into 5 distinct boxes.

Ans.

① $C(4+5-1|4)$ ways to distribute 4 identical apples into 5 distinct boxes.

② $5^6$ → Putting 6 distinct apples in 5 distinct boxes.

The total no. of ways to distribute 4 identical & 6 distinct apples is

$C(4+5-1,4) \cdot 5^6$ (as both ~~case~~ process are distinct).

There are additional constraints of the two objects in each box.

Case - 1

2 (identical) oranges in each of 2 boxes | $C(5,2)$

& 0 oranges in other 3 boxes

6 distinct apples can be distributed in | $P(6;2,2,2)$

3 other boxes

∴ total no. of distributions $= C(5,2) \times P(6;2,2,2)$.

Case 2

2 oranges in 1 box → $C(5,1)$

1 orange in each of 2 boxes → $C(4,2)$

Similarly :- 2, 1, 1, 0, 0 (thinking of arranging the no. 2, 1, 1, 0, 0 among 5 boxes).

$\equiv P(5;1,2,2)$.

2 boxes still empty with get 2 apples.

2 boxes with 1 orange will get one apple

$\equiv P(6;2,2,1,1)$.

∴ Total no. of distributions $= C(5,1) \times C(4,2) \times P(6;2,2,1,1)$

OR, $P(5;1,2,2) \times P(6;2,2,1,1)$.

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Ex 2 . Integer Soln .

How many integer soln are there to the question :-

$$x_1 + x_2 + x_3 + x_4 = 12$$

with (1) $x_i \geq 0$

(2) $x_i \geq 1$

(3) $x_1 \geq 2, \quad x_2 \geq 2, \quad x_3 \geq 4, \quad x_4 \geq 0.$

Note:- An integer soln to an eqn mean
→ ordered set of integer values for $x_i$'s summary to 12.

$$\langle x_1, x_2, x_3, x_4 \rangle = \langle 2, 3, 3, 4 \rangle$$

One can make the system as ____
(1) distribution of identical objects
(2) selection with repetition .

Let $x_i \longrightarrow$ no. of (identical) objects in box i
or no. of objects of type i chose.

∴ The total no. of integer soln :

$$\binom{12+4-1}{12}$$

Solution with $x_i \geq 1$ ≡ Putting at least one object in each ~~box~~ box at least one object of one type .

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# Diophantine Equation

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Equivalent forms for selection with repeatation.

1) The no. of ways to select r object with ~~repetition~~ repeatition from n-different types of objects.

2) The no. of ways to distribute r object into n distinct object.

3) The no. of non-negative integer sol$^n$

$$x_1 + x_2 + \dots + x_n = r.$$

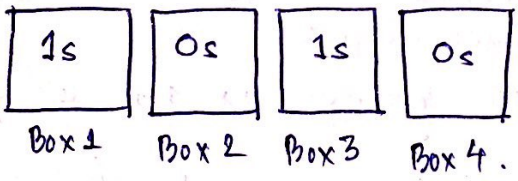
**Example.**

What fraction of binary sequences of length 10. consist of a +ve no. of 1s, followed by 0s followed by 1s, followed by no. of 0s.

eg. 1110111000

Total no. of binary sequences. $2^{10}$.

**Soln.** Modelling the problem:-




* There are 10 identical markings

Each box must have atleast 1 marker, since subsequence of 1s & 0s must be non-empty.

This is similar to putting:-

1 ball in each box with a false bottom ~~to conceal~~ conceal the ball in each box & then count the count the ways to distribute ~~whether~~ without returning the remaining (r-n) balls into the n boxes.

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$$C((r-n)+n-1,\ r-n)$$

$$= \frac{[(r-n)+(n-1)]!}{(r-n)!\,(n-1)!}$$

$$= C(r-1,\ n-1) \text{ ways}$$

for each 4 (n) boxes, distribution of 10's is given by $C(10-1,\ 4-1) = 84$ ways

there are 84 binary sequence.

## Pigeonhole Principle

### Simple form

**Theorem:-** If $(n+1)$ ~~objects pigeons~~ objects are put into $n$ boxes, ~~then~~ then atleast one box contains two or more of ~~the~~ objects.

**Proof:-** If each of the boxes contain atleast one of the objects, then the total no. of objects is at most $n$. Since we start with $n+1$ objects, some boxes contain atleast two of the objects.

**Note**

* The principle/proof does not help in finding a box that contains two or more objects.
* The principle merely generates the outcome of the boxes
* The principle cannot guarantee if there are only $n$ or fewer objects.

### Analogical principle

* If $n$ objects are put into $n$ boxes, and no box is empty, then each box contains only one object.

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* If n-objects are put into n-boxes & no box gets more than one object, then each box has an object in it.

eg. 1 If there are n married couples, how many of the 2n couple must be selected in order to guarantee that one has selected a married couple.

2 Given m integers $a_1, a_2, \ldots, a_m$, there exists integer K & L with $0 \le K \le L \le m$ such that

$$a_{K+1} + a_{K+2} + \ldots + a_L \text{ is divisable by } m.$$

Let formally, there exists consecutive a's in the sequence $a_1, a_2, a_3, a_4, \ldots, a_n$ where sum is divisable by n.

Solution

Let us consider m-sums

$$a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \ldots,\ \sum_{i=1}^{m} a_i$$

If any of the sums is divisable by m, then the conclusion holds.

* We may suppose that each of the sums has a non-zero remainder when divided by m & n. There are remainders $0, 1, 2, 3, \ldots, (m-1)$ ∴ there are ~~remainder~~ integer K & L with $K < L$ such that.

$$a_1 + a_2 + \ldots + a_K = b_m + r$$
$$a_1 + \ldots + a_L = c_m + r.$$
$$(a_{K+1} + \ldots + a_L) = (c-b)m.$$

↳ divisable by m.

eg. Let $m = 7$ & let integers be

2, 4, 6, 3, 5, 5, 6
$a_1$ $a_2$ $a_3$ $a_4$ $a_5$ $a_6$ $a_7$

with $K = 2$, $L = 5$

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Computing ~~the~~ sums.

2, 6, 12, 15, 20, 25, 31

$a_1 + a_2$

$a_1 + a_2 + a_3$.

Remainder when divided by 7 are,

2, 6, 5, 1, 6, 4, 3.

## Chinese Remainder Theorem.

Let m & n be relatively prime +ve numbers. & let a & b be integers where.

$0 \le a \le (m-1)$ & $0 \le b \le (n-1)$.

Then ~~there~~ is a +ve integer x, such that the remainder.

x is divided by m is a

x divided by n is b.

x can be written in the form :-

$x = pm + a$

$x = qn + b$.

for some integer p & q.

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Graph Theory & Combinatorics

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## Chinese Remainder Theorem

~~$x = pm + a$~~ $x = pm + a$

~~$x = qn + b$~~ ~~$y =$~~ $x = qn + b$.

Let us consider the integers :-

$a, m+a, 2m+a, \ldots (n-1)m + a$.

Each of the remainder has $\underline{a}$ when divided by $m$,

Suppose that 2 of them has the same remainder $r$ when divided by $n$

$im + a$ $\quad 0 \le i < j \le (n-1)$

$jm + a$

Then there are 2 integer $q_i$ & $q_j$ such that

$im + a = q_i n + r$ — (1)

$jm + a = q_j n + r$ — (2)

both are divided by $n$ & remainder is $r$.

(2) − (1) −

$(j-i)m = (q_j - q_i)n$

Eq. (3) states that $n$ is a factor of $j-i$

(*) Such that $\underline{n}$ has common factor other than (1) with $m$ it follows that $n$ is a factor of $(j-i)$.

(*) Since $0 \le i \le j \le (n-1)$ implies that

$0 \le (j-i) \le (n-1)$

& hence $n$ cannot be a factor of $(j-i)$.

Note :- This contradiction arises from one supposition that the nos

$a, m+a, \ldots, (n-1)m + a$

has the same remainder when divided by $n$.

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(*) Each of these n-numbers has a different remainder when divided by n

By the PP (pigeon hole principle) each of the n-numbers $0, 1, \ldots, (n-1)$ occurs as a remainder in particular.

Let p be an integer with $0 \le p \le (n-1)$ such that the no. $x = pm + a$ has remainder $\underline{b}$ when divided by $\underline{n}$.

Then for some integer q,

$$x = qn + b.$$

So, $x = pm + a$ & $x = qn + b$.

& i has the required properties

<u>Strong form of PP</u>

<u>Theorem</u> :- Let $a_1, a_2, a_3, \ldots, a_n$ be +ve integer

If $(a_1 + a_2 + \ldots + a_n - n + 1)$ objects are put into n-boxes, then ~~other~~ either the first box contains atleast $a_1$ objects or the second box contains $a_2$ objects or .... $n^{th}$ box contains atleast $a_n$ objects

<u>Proof</u> :- Suppose that we distribute $(a_1 + \ldots + a_n + 1)$ objects among n-boxes.

If for each ~~no~~ $i = 0, 1, \ldots, n$ the $i^{th}$ box contains fewer than $a_i$ objects, then the total no. of objects in all boxes does not exceed :- ~~$(a_1 - 1)$~~ ~~$(a_1 - 1$~~

$$(a_1 - 1) + (a_2 - 1) + \ldots + (a_n - 1)$$

$$= \sum_{i=1}^{n} a_i - n$$

$\because$ this number is less than the no. of objects distributed.

We conclude that for sum i $i = 1, 2, \ldots, n$ the $i^{th}$ box contains at least $a_i$ objects.

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⊛ It is possible to distribute $(\sum a_i - n)$ objects among n-boxes in such a way that for no $i = 1, 2, \ldots n$, the $i$th box contains at least $a_i$ objects.

This can be done by putting $(a_i - 1)$ objects in $i$th box.

Note :- The simple form of PP from string form of PP by taking $a_1 + \ldots + a_n = 2$.

$$\sum a_i - n + 1 = 2n - n + 1 = n + 1.$$

eg.

A basket of fruit n being arranged out of apples, banana & orange. What is the smallest no. of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 8a/6b/9o).

Soln. By strong form of PP,

$$(8 + 6 + 9) - 3 + 1 = 21.$$

∴ there should be atleast 21 fruits such that no other how selected, ~~will~~ will guarantee a basket of fruit with desired properties.

eg. How many ordered pairs of $(a, b)$ are needed to guarantee that there are $(a_1, b_1)$ & $(a_2, b_2)$ pair such that $a_1 \bmod 5 = a_2 \bmod 5$ & $b_1 \bmod 5 = b_2 \bmod 5$.

Ans. There are 5 options for a & 5 options for b.

∴ There are 25 options $(5^2)$ for $(a, b)$. If one has 26, then at least there will be in the same "congruence pair class". However, with 25, it would be each class has one pair, so, if, one has 26 pairs, then at least one class will have 2.

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One needs to consider 25 different cases that both a, b will form. Then, from PP (sample) -

$$\left\lceil n/25 \right\rceil = 2$$ where n = total solutions space

By solving the equality :-

$$\left\lceil \frac{n}{25} \right\rceil - 1 < \frac{n}{25} \leq \frac{n}{25}$$

$$\Rightarrow (2-1) < \frac{n}{25} \leq 2$$

$$\Rightarrow \cancel{25 \leq} \; 25 < n \leq 50$$

∴ there has to be 25+1 = 26 by PP.

Ex:2. Let $x$ be an irrational number. Show that for some +ve integer $j$ not exceeding the +ve integer $n$, the absolute value of the difference $jx$ & the nearest integer to $jx$ is less than $(1/n)$.

Ans. Consider $\{x\} = x - [x]$ be the fractional part of $x$, $0 \leq \{x\} < 1$. Consider the $(n+1)$ numbers $\{ax\}$ where $1 \leq a \leq n+1$. If the irrational nos are put into $n$ pigeonholes,

$$[0, 1/n), [1/n, 2/n), \ldots, [1-1/n, 1).$$

Suppose $\{ax\}$ & $\{bx\}$ belong to the same pigeonhole, where $a > b$ which mean

$$|\{ax\} - \{bx\}| < \frac{1}{n}$$

with $(a-b) < n$

Setting $j = (a-b)$ &

$$k = [ax] - [bx]$$

given $|jx - k| < 1/n$.

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## Averaging Principle

(*) If $n(r-1)+1$ objects are put into $n$ boxes, then at least one of the boxes contains $r$ or more of the objects.

(*) If the avg. of the $n$ - nonnegative integers $m_1, m_2, \ldots, m_n$ is greater than $(r-1)$ i.e.

$$\frac{\sum m_i}{n} > (r-1).$$

then atleast one of the integers is greater than or equal to $r$.

**Proof:-** For $i = 1, 2, \ldots, n$ let $m_i$ be the no. of objects in the $i$th box

$$\frac{\sum m_i}{n} = \frac{n(r-1)+1}{n} = (r-1) + 1/n$$

$\therefore$ the avg is greater than $(r-1)$ of the integers is at least $r$.

One of the boxes contain atleast $r$ objects.

(II) If the avg. of $n$-non negative integers $m_1, m_2, \ldots m_n$ is less than $(r+1)$ i.e

$$\frac{\sum m_i}{n} < (r+1).$$

then atleast one of the integers is less than $(r+1)$.

(III) If the avg of $n$-nonnegative integers $m_1, m_2, \ldots, m_n$ is atleast equal for $r$, then for atleast one of integers $m_1, m_2, \ldots, m_n$

satisfies $m_i \geq r$.

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## Recurrence Relation

The parameter n, denote size of ~~the~~ some underlying set/multiset of a problem, the size of ~~set~~ subset, the no. of positions in the permutation.

∴ The counting problem is often not one individual problem but a sequence of individual problem.

eg. Let $h_n$ denote the permutation $\{1, 2, \ldots, n\}$
i.e. $h_n = n!$
& hence, a sequence of no. can be found $h_0, h_1, h_2, \ldots, h_n$.

eg. number sequence

arithmetic sequence : $h_0, h_0+q, h_0+2q, \ldots, h_0+nq$

geometric sequence :- $h_0, h_0q, h_0q^2, \ldots, h_0q^n$.

**Arithmetic Sequence**

$h_n = h_{n-1} + q$.

**Geometric Sequence**

$h_n = h_{n-1} \cdot q$

**Fibonacci Series :-** $f_n = f_{n-1} + f_{n-2}$ $n \geq 3$

(Rabbit problem) $f_0 = 0, f_1 = 1, f_2 = 1$.

**Property :-** The partial sum of the FS are:-

$$S_n = f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$$

**Proof** by induction

For $n=0$, eq ① reduces to —

$S_0 = f_0 = f_2 - 1 = 1 - 1 = 0$.

Let $n \geq 1$,
Assume then ① holds true for $n$.

For $n = n+1$,

~~$f_0 = 1$~~ $f_0 + f_1 + f_2 + \ldots + f_{n+1} =$

$(f_0 + f_1 + \ldots + f_n) + f_{n+1}$

$= (f_{n+2} - 1) + f_{n+1}$ (By induction assumption)

$= f_{n+1} + f_{n+2} - 1 = f_{n+3} - 1$.

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Let us consider the FS

$$f_n = f_{n-1} + f_{n-2}$$

$$\Rightarrow f_n - f_{n-1} - f_{n-2} = 0$$

Let $f_n = q^n$ where $q \geq 0$

$$\therefore q^n - q^{n-1} - q^{n-2} = 0$$

$$\Rightarrow q^{n-2}(q^2 - q - 1) = 0 \quad n = 2, 3,$$

As $q^{n-2} \neq 0$ ①

then by induction

① holds for $n \geq 0$.

$$\therefore q^2 - q - 1 = 0$$

$$\therefore q = ??$$

$$q_1 = \frac{1+\sqrt{5}}{2} \qquad q_2 = \frac{1-\sqrt{5}}{2}$$

$$f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \text{ or } f_n = \left(\frac{1-\sqrt{5}}{2}\right)$$

$\therefore$ FS relation is linear & homogeneous.
it follows that

$$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$\underline{n=0}$

$$\cancel{f_n =} \; c_1 + c_2 = 0$$

$\underline{n=1}$

$$c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$c_1 = \frac{1}{\sqrt{5}} \qquad c_2 = \frac{-1}{\sqrt{5}}$$

$\therefore$ the fibonacci number satisfies the formula.

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$n \geq 0$

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## Generating fn

* G.F. can be regarded as algebraic object whose formal manipulation allow us to count the no. of possibilities for a problem by means of algebra.

* GF are taylor series (power series expressions) by infinitely differentiable form. If one can find the fn & its taylor series, then the coefficient of Taylor series give soln to problem.

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# Graph Theory & Combinatorics

Prof. Chintan Kumar Mondol

Let $h_0, h_1, h_2, \ldots h_n, \ldots$ be an infinite sequence of number. If generating fn (for any coefficient of $h_0, h_1, h_2, \ldots, h_n$ & corresponding Taylor series expansion) to the infinite series is →

$$g(x) = h_0 + h_1 x + h_2 x^2 + \ldots + h_n x^n + \ldots$$

(*) A finite fn $h_0, h_1, h_2, \ldots, h_n$ can be written as → $h_0, h_1, h_2, \ldots, h_m, 0, 0$.

$$\Rightarrow g(x) = h_0 + h_1 x + h_2 x^2 + \ldots + h_n x^n.$$

Eg. The generating fn of infinite sequence

$1, 1, 1, \ldots 1, \ldots$

$$g(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}.$$

Eg. Let $k$ be an integer & let the sequence $h_0, h_1, \ldots, h_n \ldots$ be defined by letting $h_n$ equal to the non-negative integral soln of $e_1 + e_2 + \ldots + e_k = n$.

$$h_n = \binom{n+k-1}{k-1}, \; n \geq 0.$$

The generating fn for $x \to \infty$.

$$g(x) = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n = \frac{1}{(1-x)^k}$$

(4) $$\frac{1}{(1-x)^k} = \frac{1}{1-x} \cdot \frac{1}{1-x}, \ldots k \text{ times}$$

$$= (1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots) \; k \text{ times}$$

$$= \left(\sum_{e_1=0}^{\infty} x^{e_1}\right)\left(\sum_{e_2=0}^{\infty} x^{e_2}\right) \ldots k \text{ times}.$$

considering, $e_1 + e_2 + e_3 + \ldots + e_k = n$.

$$= x^{e_1} \cdot x^{e_2} \ldots x^{e_k} = x^n.$$

$$= \binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Eg. For what sequence of $(1 + x + x^2 + x^3 + \ldots + x^5)(1 + x + x^2)(1 + x + \ldots x^4)$ the generating fn.

Let $x^{e_1} (0 \leq e_1 \leq 5)$, $x^{e_2} (0 \leq e_2 \leq 2)$ & $x^{e_3} (0 \leq e_3 \leq 4)$ denote typical terms in 1st, 2nd & 3rd factors.

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Multiplying & computing the coeff$^n$ →

$x^{e_1} \cdot x^{e_2} \cdots x^{e_3} = x^n \rightarrow$ provided $e_1 + e_2 + e_3 = n$.

<u>Note</u>. $h_n = 0$ if $n > (5+2+4) = 11$.

## Counting Problem by GF

Find the number $h_n$ of bags of fruit that can be made out of apples (a), bananas (b), oranges (o), & pears (p) where in each bag the no. of a is even,

" b is multiple of 5

" o is at most 4

" p is 0 / 1.

$$g(x) = (1 + x^2 + \cdots + x^{16} + \cdots)(1 + x^5 + x^{10} + \cdots)$$
$$(1 + x + x^2 + x^3 + x^4)(1 + x).$$

$$= \left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^5}\right)\frac{1-x^5}{1-x} \cdot (1+x)$$

$$= \frac{1}{(1-x)^2} \sum_{n=0}^{\infty} \binom{n+1}{n} x^n$$

$$= (n+1)x^n.$$

## Linear Homogeneous Recurrence Relation

Let $h_0, h_1, h_2, \ldots, h_n$ be a sequence of numbers which satisfies a linear Recurrence Relation of order $k$ provided that their exists quantities $a_1, a_2, \ldots, a_k$ with $a_k \neq 0$ & a quantity $b_n$ such that

~~$h_n = a_1 + a_2 + \cdots +$~~

$$\boxed{h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b_n}$$

non-homogeneous Recurrence Relation

where $a_1, a_2, \ldots, a_k, b_n$ may be dependent on $n$.

eg. Let $D_0, D_1, \ldots D_n$ be ~~a num~~ a number such that :→ $D_n = (n-1) D_{n-2} + (n-1) D_{n-1}$ $n \geq 2$

→ order 2

→ $a_1 = (n-1)$ $a_2 = n-1$ $b_n = 0$.

(4) $D_n = n D_{n-1} + (-1)^n$ $n \geq 1$

$a_1 = 1$
$a_2 = 1$
$b_n = 0$

→ order = 1

$a_1 = n$ $b_n = (-1)^n$.

$\boxed{f_n = f_{n-1} + f_{n-2}}$ → order = 2

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## For Linear Homogeneous Eqn

**Relation :-**

$$h_n = a_1 h_{n-1} + \dots + a_k h_{n-k} \quad (n \geq k)$$

$$a_1, a_2, \dots a_k = \text{constant} \ \& \ a_k \neq 0.$$

The sequence of numbers $h_0, h_1, h_2, \dots h_n$ satisfying the above relation is uniquely determined on the values of $h_0, h_1, h_2, \dots h_{k-1}$ i.e. the initial values are preserved.

**Theorem**

Let $q$ be a non-zero number, then $h_n = q^n$ is a solⁿ of the linear homogeneous recurrence relation

$$h_n - a_1 h_{n-1} + a_2 h_{n-2} + \dots + -a_k h_{n-k} = 0$$

$$(a_k \neq 0, \ n \geq k).$$

with constant coefficient iff $q$ is a root of the polynomial eqn :-

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} + \dots - a_k = 0 \quad - (3)$$

If the polynomial eqn has $k$ discrete roots :- $q_1, q_2, q_3, q_k$ then :-

$$h_n = c_1 q_1^n + c_2 q_2^n + \dots + c_k q_k^n \quad - (4)$$

is a general solⁿ of eq in the following sequence.

No matter what initial values for $h_0, h_1, h_2, \dots, h_{k-1}$ are given there are constants $c_1, c_2, \dots, c_k$ so that eq 4 is the unique sequence which satisfies both recurrence relation & the initial value :-

**Proof :-** $h_n = q^n$ is a solⁿ of (2)

iff $q^n - a_1 q^{n-1} - a_2 q^{n-2} + \dots + -a_k q^{n-k} = 0$

$\forall \ n \geq k.$

Since we assume $q \neq 0$, we may can $q^{n-k}$.

$$q^k - a_1 q^{k-1} + \dots + -a_k = 0$$

$\because a_k \neq 0.$

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The linearity & homogeneity imply that :-

$$\boxed{h_n = c_1 q_1^n + c_2 q_2^n + \dots + c_k q_k^n}$$

Suppose for initial values :-

$h_0 = b_0,\ h_1 = b_1,\ \dots,\ h_{k-1} = b_{k-1}$.

$$\Rightarrow \quad c_1 + c_2 + \dots + c_k = h_0$$

$$\vdots$$

$$c_1 q_1^k + \dots + c_k q_k^k = b_k.$$

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ q_1 & q_2 & q_3 & \dots & q_k \\ \vdots & & & & \\ q_1^k & \dots & & & q_k^k \end{bmatrix} \rightarrow \text{vandermode Matrix (VM)}.$$

The VM is an invertible matrix iff $q_1, q_2, \dots q_k$ are distinct in determinant $\prod_{1 \le i \le j \le k} (q_j - q_i) \neq 0$.

$\phi$ = The system of eqn has a unique sol for each choice of $b_0, b_1, b_2, \dots b_{n-1}$

eq (4) ~~has~~ is a general soln of eq (2).

Ex :- Solve the recurrence relation

$$h_n = 2h_{n-1} + h_{n-2} + 2h_{n-3} \quad (n \geq 3).$$

$$h_0 = 1,\ h_1 = 2,\ h_2 = 0$$

$$x^3 - 2x^2 - x + 2 = 0,$$

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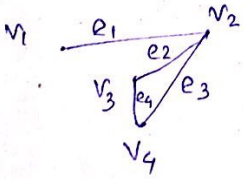
# Graph Theory

13/02/2020
ChM

$G = \{V, E, r\}$

$r$ → set of relations



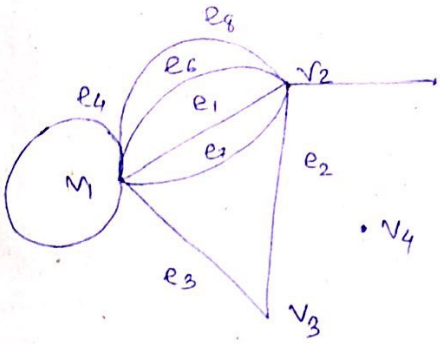

$V = \{v_1, v_2, v_3, v_4\}$

$E = \{e_1, e_2, e_3, e_4\}$

$r = \{ e_1 \mapsto \{v_1, v_2\},\ e_2 \mapsto \{v_2, v_3\},$
$e_3 \mapsto \{v_2, v_4\},\ e_4 \mapsto \{v_3, v_4\} \}$

$E = \{ e_1\{v_1, v_2\},\ e_2\{v_2, v_3\}, \ldots \}$

$E = \{ \{v_1, v_2\}, \{v_2, v_3\} \ldots \}$




$v_3$ → pendent vertex if degree 1.

Degree$(v)$ = no. of edges incident on it.

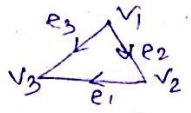
$\deg(v_3) = 1$ (pendant vertex)

$e_1$ is incident to $v_1$ & $v_2$

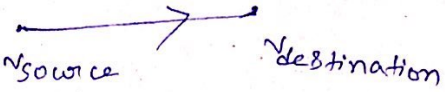
$e_1, e_6, e_7$ → parallel edges.

$\deg(v_4) = 0$

Undirected graph and directed graph




$e_3 = (v_1, v_3)$
$e_2 = (v_1, v_2)$.




$\deg(v)$ = total no of incoming − total no of outgoing edges. ~~vertex~~.

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Simple graph → does not have loop and parallel edges.

|| edge  not || edge

**Defination**

A binary relation from a set S to a set T, is a subset of the cartesian product $(S \times T)$

→ a relation on S is a subset of $(S \times S)$

→ A multirelation on S is a multisubset of $(S \times S)$.

eg. set $S = \{1, 3, 5, 7\}$  multiset $M = \{1, 1, 1, 2, 2, 3\}$

$R = \{(1,3), (3,5), (3,7)\}$ relation.  $T = \{(1,3), (1,3), (1,1), (3,3)\}$ multirelation

**Defination**

A graph is a finite non-empty set, V together with a symmetric & irreflexive binary relation A on V.

The elements of the set V are called vertexes of graph.

The relation R is called the adjacency relation & if K is related to v by R we say that u is adjacent/neighbour to v & write u R v.

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(*) Some R are symmetric relation, if ~~def~~ defines a set E of $V_{(2)}$, the set of two subset of V, ~~narely~~ namely

$$r = \{ v, v \} \mid u, v \in r ; uRv \}$$

The elements of E are called edges of the graph.

$V_i = \{ v_1, v_2, v_3, \ldots v_n \}$

$V_{(2)} = V \times V = u R v,$

$u, v \in V$

~~two subset of~~

**Defination 2**

A graph G is pair (V, E) where V is a non empty set, where elements are called vertices of G & E is a subset of $V_{(2)}$, whose elements are called edges of G.

(*) There is an incidence relation I between the vertex set V & the edge set E of a graph.

if $e \in E$, then there is a pair of distinct vertex u & v such that $e = \{u, v\}$

→ u & v are the end vertex of only of e. & say that u & v are incident on e. with e ( u I e and v I e)

**Defination 3**

A graph G is a pair of disjoint sets of V (non-empty set) & E with a 1-1 incidence function $f : E \to V_{(2)}$

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Note

All 3 defination are equivalent.

* Vertices are ~~refres~~ referred to as nodes / p

* Edges ---- lines

* $|V| \rightarrow$ order of graph

* $|E| \rightarrow$ size of graph

* Since the vertex set of every graph is non-empty, the order of every graph is at least 1.

* A graph with only one vertex, i.e. $|V| = 1$ is a trivial graph

* The order of a non trivial graph is at least 2.

eg. Let, $S = \{2, 3, 5, 8, 13, 21\}$

Let R be the relation of pairs of distinct ~~disjoint~~ integers. belonging to S. where sum/dif $\in S$. eg. $\{2,3\}, \{2,5\}$

$\therefore H = (V, E)$ where $V(H) = S$.

$E(H) = \{\{2,3\}, \{2,5\}, \{3,5\}, \{3,8\}, \{5,8\}, \{5,13\}, \{8,13\}, \{8,21\}, \{13,21\}\}$

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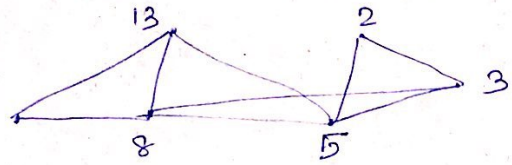
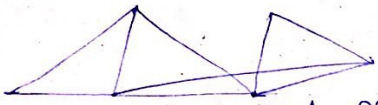
H:




Labelled graph



UnLabelled graph.

Defination of other types of graphs →

**Defination** If such pair $(u,v)$, $[u\,v \in V(G)]$ of a vertex is joined by at most one edge → Simple graph / Graph.

**Defination** A multigraph is a pair $(V,E)$ where $V$ is a non-empty set of vertex & $E$ is a multiset of edges, being a multi subset of $V_{(2)}$.

→ The no. of times an edge $e \to \{u,v\}$ ~~over~~ occurs ~~a~~ is called multiplicity of $e$.

→ The graph obtained by removing the multiple ~~graphs~~ edges is called the underlying graph of $G$.

**Defination** A general graph is a pair $(V,E)$ where $V$ is a set of vertices & $E$ is a multiset of edges, being a ~~multiset of edges~~, multisubset of $V_{(2)}$, the set of

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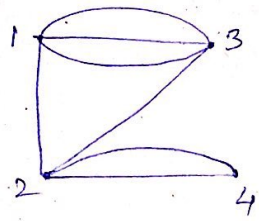
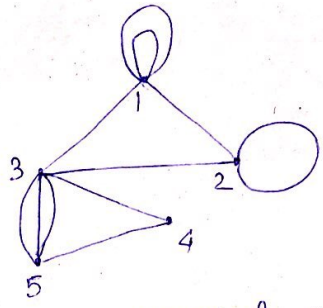
unordered pair, of elements of V, not neccessarily distinct.




multigraph




general graph

## Digraphs / Directed graph

**Defination** A digraph D is a pair (V, E) where V is a non-empty set whose elements are called vertices & E is a subset of $V^{(2)}$ (i.e. the set of ordered pairs of distinct elements of V) whose elements are called arcs of D.

**Defination** A multigraph D is a pair (V, E) whose V is set of vertex & E is a multiset of arcs, being a multisubset of $V^{(2)}$.

→ The no of times an arc occurs in E is called its multiplicity & arcs with multiplicity greater than 1 are the multiple arcs of D.

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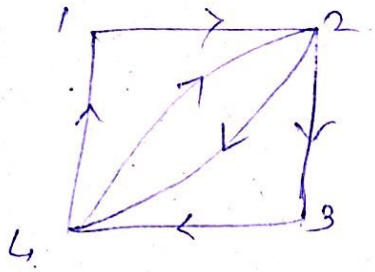
* An arc $(u, v) \in E$ will be denoted as $uv / \overrightarrow{uv}$, where $u \rightarrow$ initial vertex / tail
$v \rightarrow$ terminated vertex / head

* A Digraph is an irreflexive binary relation on V.

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14/2/2020.

# Graph Theory.



Prof. Chintan Kumar Mondal

**Defn.** A subgraph of $G(V,E)$ is a graph $H(U,F)$ with $U \subseteq V$ & $F \subseteq E$.

(#) When $U = V$, $H$ is a spanning subgraph of $G$.

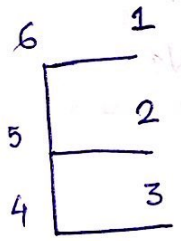
$\rightarrow H \leq G$ ($H$ ...)

$\rightarrow H < G$ ($H$ is a subgraph of $G$).

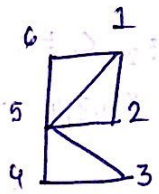
(#) When $F$ consists of all the edges of $G$, joining pair of vertices of $U$, we say that $H$ is <u>vertex induced</u> subgraph of $G$.

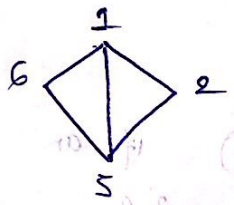
(#) Given a subset $F$ of edges of $G$. Let $U$ be the set of end vertices of edges of $F \rightarrow F$ is edge induced graph of $G$.

eg.




spanning subgraph.

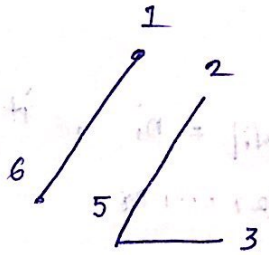



$V = \{1,2,3,4,5,6\}$

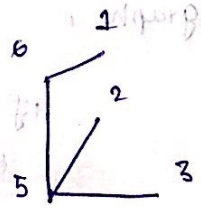
$E = \{12, 15, 16, 25, 65, 35, 54, 34\}$.

$U = \{1,2,5,6\}$ (vertex induced subgraph)




$F = \{16, 25, 35\}$

edge induced subgraph.




subgraph

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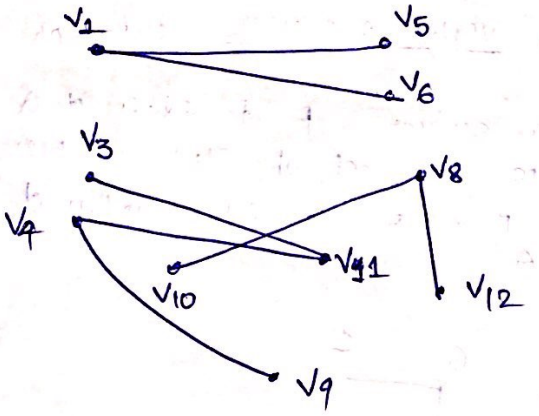
# Partite Graph

**Defn.** A graph $G(V,E)$ is said to be r-partite graph $(r>0)$ if its vertex set can be partitioned as $V = V_1 \cup V_2 \cup V_3 \cup V_4 \ldots \cup V_r$ such that if $uv$ be an edge of $G$, then $u \in V_i$ & $v \in V_j$ i.e. every pair of the induced subgraph $\{V_i\}$ is an empty graph.

If $r=2$, then it is a bi-partite graph.




$V_1 = \{v_1, v_2, v_3, v_4\}$
$V_2 = \{v_5, v_6, v_7, v_8\}$
$V_3 = \{v_9, v_{10}, v_{11}, v_{12}\}$

$G_1 = \{V_1, E_1\}$
$G_2 = \{V_2, E_2\}$
$G_3 = \{V_1, E_3\}$

$E = E_2 \cup E_3 = \{\phi\}$

$K_{4,3,4} \rightarrow$ Complete partial graph

(*) If an r-partite graph as all possible edges, i.e. $u.v \in E$. for every $u \in V_i$ & $v \in V_j$, then it is a complete partite graph.

If $|V_i| = n_i$, it is denoted by $K_{n_1, n_2, \ldots, n_r}$.

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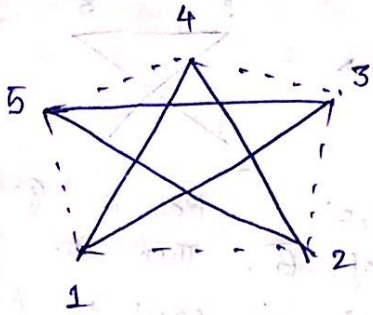
* A complete bi-partite graph is called an n-star or n-claw.

## Complement of graph

**Definition.**

The ~~complete~~ complement of a graph $G(V,E)$ is $\overline{G}(V,E)$, where it has the same set complement $\overline{E}$ of $E$ is $V(2)$ i.e. $uv$ is an edge of $\overline{G}$, iff $uv$ is not an edge of $G$.




A graph is self ~~complement~~ complementary if $G = \overline{G}$.

For a complete graph $K_n$, $\overline{K_n} = \{V, \{\phi\}\} \longrightarrow$ null graph of $|V|$ vertices.

$G = \{V\{v_1, v_2, v_3, v_4, v_5\}, E\{13, 35, 52, 24, 41\}\}$

## Operations on graphs

**Defn:** Let $G(V,E)$ be a graph & $F$ a subset of the edge of $E$. Then the graph $H(V, E-F)$ with the same vertex set as $G$ & edge set $(E-F)$ is said to be obtained, by removing the edge set $F$.

* This is denoted as $G-F$.
* If $F$ consist of a single edge $e$, the ~~group~~ graph obtained is $G-\{e\}$.

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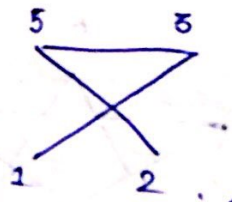
**Defn.** If vertex $v$ of $G$, which is not adjacent with any other vertex of $G$, then $v$ is an isolated vertex.

**Defn.** Let $G(V,E)$ be a graph & $v$ be a vertex of $G$. Let $E_v$ be the set of all edges incident to $v$ then the graph $H = \{V - \{v\}, E - E_v\}$ is said to be obtained from $G$ by removal of the vertex $v$.

(A) $H = G - v$

$G - H \longrightarrow$

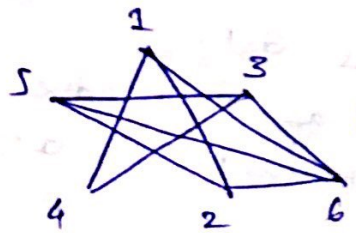



if $v' \subset v$,
$H = G - v'$
$\Rightarrow H = G/v'$

**Defn.** Let $G(V,E)$ be a graph and $f$ an edge of $\overline{G}$. Then the graph $H = \{V, E \cup \{f\}\}$ is said to be obtained from $G$ by addn of the edge $f$, and is denoted by $G + \{f\}$. If $F$ is a subset of edge of $\overline{G}$, the graph obtained from $G$ by adding the edges of $F$ in any order $\rightarrow$ denoted by $G + F$.

**Defn.** Let $G(V,E)$ be a graph so $v \notin V$. Then the graph $H$ with vertex set $V \cup \{v\}$ and edge set $E \cup \{x \mid x \in V\}$ is said to be obtained from $G$ by adding a vertex $v$.

$\rightarrow G + v$.




$G + V_G$

$G + \overline{G} = K_5$.

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**Defn.** Let $G(V,E)$ & $H(U,F)$ be two graphs with disjoint vertex sets ie. $V \cap U = \phi$. Then the join of & it is denoted as $G \cup F$, has vertex set $V \cup U$. & edge set $E \cup F \cup \{uv \mid u \in U, v \in V\}$.

## Degree of Graphs.

**Defn.**

The degree of a vertex $v$ of a graph $G$ is the no. of edges incident to $v$.

$$d(v) = |\{e \in E \mid e = uv \text{ for some } u \in V\}|$$

- The minimum degree of a graph $\rightarrow \delta(G)$,
  The maximum degree of a graph $\rightarrow \Delta(G)$.

For any graph $G$ of order $n$

$$0 \leq \delta(G) \leq \deg(v) \leq \Delta(G) \leq (n-1).$$

**Defn.** A graph is said to be regular / k-regular if all the vertices have the same degree $k$.

eg. - A 3-degree graph is a cubic graph.

- Isolated vertex $\longrightarrow d(\text{Visolated}) = 0$
- $d(v) = 1 \rightarrow$ pendent
- $d(v) = \text{odd} \rightarrow$ odd vertex
- $d(v) = \text{even} \rightarrow$ even vertex
- A loop incident on $v$ is counted as 2

$d(2) = 5$.

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**Defn.** For any $v \in V$, the no. of arcs adjacent to $v$ is the inner clan degree/ in-degree $[d^-(v)]$.

* The no. of arcs adjacent from $v$ is the out-degree/ outer demi degree $[d^+(v)]$.

* The total degree of $v$ is
$$d(v) = d^+(v) + d^-(v)$$

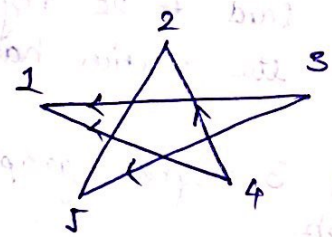
* If $d(v) = k$ for every $v \in V$ the $D$ is a $v$ regular digraph.

* $d^+(v) - d^-(v) = 0$ <u>isolated vertex</u>

* If $d^+(v) > 0$ & $d^-(v) = 0$ <u>transmitter vertex</u>

  if $d^+(v) = d^-(v) = 1$ <u>carrier vertex</u>.

* Any other vertex is ordinary.




$d^-(1) = 2$ $\quad d^+(1) = 0$

$d^-(2) = 2$ $\quad d^+(2) = 0$

$d^-(3) = 0$ $\quad d^+(3) = 2$

$d^-(4) = 0$ $\quad d^+(4) = 2$

$d^-(5) = 1$ $\quad d^+(5) = 1$

$d(5) = 0$

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**Theorem 1**

If $G$ is a graph of size $m$ then $\sum_{v \in V(G)} d(v) = 2m$.

**Proof.**

When summing the degree of the vertex of $G$, we count each edge $e$ of $G$ twice, one for each of the two vertex incident with $e$.

eg ① A certain graph has order 14, & size 27. The degree of each vertex of $G$ is 3, 4, 5 & 6, vertices degree of 4. How many vertices of 4 has degree 3, how many has degree 5 ?

eg ② Suppose that $G$ is a bipartite graph graph of size $m$ with partite set & $u = \{u_1, u_2, u_3, \dots, \}$ & $w = \{w_1, w_2, \dots, w_n\}$. Since every edge of $G$ joins a vertex $u$ & a vertex of $w$.

$$\sum_{i=1}^{5} \deg(v_i) = \sum_{j=1}^{5} \deg(w_j) = m.$$

**Corollary.** Every graph has an even number of odd degree.

**proof.** Let $G$ be a graph of size $m$. Divide $V(G)$ into two subsets $V_1$ & $V_2$, where $V_1 \rightarrow$ consist of all odd vertices, $V_2 \rightarrow$ consish of all even vertices.

By first theorem of graph theory $\rightarrow$

$$\sum_{v \in V(G)} d(v) = \sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v)$$
$$= 2m.$$

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Thus,

$$\sum_{v \in V_1} d(v) = 2m - \sum_{v \in V_2} d(v).$$

which implies that

$$\sum_{v \in V_1} d(v) \text{ is even.}$$

<u>Proposition</u>.

For any graph of order $n$,

$\Delta G \geq n-1$.

<u>Proof</u>.

This is simply true because a vertex can be joined at most $(n-1)$ other ~~vertices~~ vertex & multiple loops are not allowed.

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**Proposition**

For any graph of order $n$, $\Delta(G) \leq n-1$

**Proof:** This is simply true because a vertex can be joined to at most $(n-1)$ other vertices & multiple/loops are not allowed.

Since each of the nos. $d(v)$, $v \in V$, is odd, the no. of odd vertices of $G$ is even

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**Theorem:-** If $G$ is simple graph & $|G| \geq 2$, then there are 2 vertices of the same degree.

**Proof:-** In a simple graph, the maximum degree $\Delta \leq |G| - 1$.

If all the degrees be different, then there would be a sequence of $0, 1, 2, \ldots, |G|-1$. But degree $0$ & $|G|-1$ are mutually exclusive there must be two vertices of the same degree.

Let $V = \{v_1, v_2, v_3 \ldots v_4\}$.

The degree sequence of $G$ is given as.

$Deg(G) = \{d(v_1), d(v_2), \ldots d(v_n)\}$

where the degree are ordered such that,

$$d(v_1) \geq d(v_2) \geq d(v_3) \geq \ldots \geq d(v_n).$$

We say that a sequence $D = \{d_1, d_2 \ldots d_n\}$ is <u>graphic</u> if $d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n$.

& there is a simple graph with $Deg(G) = D$.

Whether graphic/or not $\rightarrow$ Y/N.

$D(4,3,3,2,1) \rightarrow$ N $\quad (\sum d(v_i) \neq 2m)$

$D(7,6,5,4,3,3,2) \rightarrow$ N $\quad$ (Seven vertices, max degree can't be seven)

$D(6,6,5,4,3,3,1) \rightarrow$ N

$\hookrightarrow$ Two nodes have degree 6, & node can't have degree one.

**Problem:** Given a graphic degree sequence, $\longrightarrow$ <u>Greedy</u>.

produce a graph with degree sequence $Deg(G) = D$.

in general, $D = (d_1, d_2 \ldots, d_n)$ where $d_1 \geq d_2 \geq d_3 \ldots \geq d_n$.

S1) The vertex of degree $d_1$ is joined to the $d_1$ vertices of largest degree.

S2) These leaves will then have:-

$d_2 - 1, d_3 - 1, \ldots d_{d_1+2}, \ldots d_n$ in some order

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If the degrees are rearranged into descending order, we get a reduced sequence: $D' = (d_2', d_3', \dots d_n')$ where the first vertex has been deleted.

<u>Theorem.</u> (Havel-Halini Theorem) $\therefore$ $D = (d_1, d_2 \dots d_n)$ is graphic. If $D' = (d_2' \dots d_n')$ is graph.

<u>Theorem</u> Erdos-Gallai Theorem

Let $D = (d_1, d_2 \dots d_n)$ where $d_1 \geq d_2 \geq \dots d_n$.

Then $D$ is graphic if

1) $\sum_{i=1}^{n} d_i$ is even &

2) $\sum_{i=1}^{n} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(k, d_i)$ for $k = 1 \dots n$.

<u>Walks, Trails, Paths & Connectedness.</u>

<u>Def:-</u> A sequence of edges such as $(e_1, e_2 \dots e_L)$ is a walk. if there exists a corresponding sequence of vertices —

$(v_0, v_1, \dots v_L) \dots (1)$ * A walk for which $v_0 = v_L$ is a closed walk.

such that $e_j = (v_{j-1}, v_j)$.

Note :- The vertices doesn't have to be distinct.

<u>Def.</u> The length of a walk is the no. of edges in the sequence i.e. in eq(1), length is 2.

<u>Def:-</u> A ~~trait~~ trail is a walk in which all the edges $e_i$ are distinct & a closed ~~trait~~ trail is a closed walk that is also a trail.

<u>Note</u> :- Closed trail $\equiv$ circuit.

<u>Def:-</u> A path is trail when all the vertices in the sequence is distinct.

* A closed path is a closed trail in which all the vertices are distinct except the first and last vertices.

Def:- A cycle is a closed path that includes three or more edges.

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In set theoretic form ——

walks $\supseteq$ Trail $\supseteq$ Paths

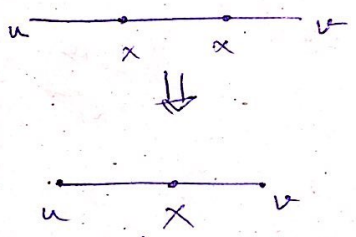
Def:- The distance $d(u,v)$ between two vertices $(u,v)$ in a graph $G$, is the length of the shortest-$uv$-walk in $G$.

Def:- The Diameter $diam(G)$ of $G$ is the min $dist(u,v)$, over all pairs $u$ & $v$ vertices of $G$.

<u>Proposition</u> :- If there is a path from vertex $u$ to $v$, then there is also a path from $u$ to $v$ of length $(|V|-1)$.

<u>Proof</u> :- If all the vertices are distinct, there is nothing to prove.

Otherwise :- suppose some vertex $X$ is repeated in the path. We can remove the segment of the walk between these two $X$'s & obtained another shorten walk from $u$ to $v$.




If this is not a path, one can repeat the above procedure in the walks.

Clearly, after a ~~path~~ finite no. of steps, one can obtain a path.

Def:- In a graph $G(V,E)$, two vertices are set to be connected if there is a walk is given by a vertex sequence $(v_0, \ldots, v_n)$.

Additionally one can say that a vertex is connected to itself.

Def:- A graph in which each pair of vertices is connected is a connected graph.

The connection relation is an equivalence relation i.e. it is.

transitive :- If $G$ has a $u$-$v$ pair & $v$-$w$ path, the $G$ has $u$-$w$ path.

reflexive :- It has paths of length (1) as a vertex is connected to itself.

<u>symmetric</u> :- paths are reversible.

Def:- In an undirected graph $G(V,E)$, a connected component is an equivalence relation under the relation.

"is-connected-to" on $V$.

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~~Def :- The distance $d(u,v)$ between 2 vertices $(u,v)$ is~~

~~Contd~~ Connectedness in Digraph

In digraphs, "is-connected-to" is <u>not an</u> equivalence relation because it is not symmetric..

Def :- In a directed graph $G(V,E)$, vertex $v$ is said to be accessible or reachable from another vertex $u$, if $G$ contains a walk from $u$ to $v$.

(*) Additionally, one can say that all vertices are accessible (reachable) from themselves.

Def :- Two vertices $u$ & $v$ are strongly-connected if $v$ is accessible from $u$ & $u$ is accessible from $v$.

Additionally, a vertex is strongly-connected to itself.

Def :- A directed graph $G(V,E)$ is strongly connected if every pair of vertices is strongly connected.

Equivalently, a digraph is strongly connected if it contains exactly one strongly connected component.

Def :- A directed graph $G(V,E)$ is weakly-connected if when one connects all edges to undirected ones, it becomes a connected undirected graph.

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27/02/2020

# Graph Theory & Combinatorics

Prof. Chintan Kumar Mondal

## Traversibility

**Defn (1).** A circuit/trail (an edge traversal trail) in a graph called an Eulerian circuit, if it contains every edge of G.

(*) Same no. edge is repeated twice in a circuit/trail, every edge appears once in an Eulerian trail/circuit.

**Defn (2) :-** A connected graph is Eulerian if it has a closed trail of all edges. or A connected graph that contains an Eulerian ckt is called an Eulerian graph.

**Defn (3) ..** For a connected graph G, an open trail which contains every edge of G is an Eulerian trail.

## Characterisation Theorem.

**Theorem 1 :-** If every vertex of G has degree at least 2. then G contains a cycle.

**Proof :-** Let P be a minimal path in G, & let u be an endpoint of P. Since P cannot be extended, every neighbor of u must already be a vertex of P.

Since ~~u~~ u has a degree at least 2, it has a neighbor v in V(P) then, is an edge uv in P.

The edge uv completes a cycle with the portion from u to v.

**Note :-** If $V(G) = Z$ & $E(G) = \{ij : |i-j| = 1\}$ every vertex of G has degree 2, but ~~G~~ G has not cycle in no extendable path.

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**Theorem 2 :-** A graph G is Eulerian, iff has at most one trivial component & its vertices all have even degree.

**Proof :-** **Necessity :-** Suppose that G has an Eulerian ckt C. Each passage of C through a vertex uses two incident edges, and first edge is parallel with the last at the first vertex. Hence, every vertex has even degree. Also, the edges can be in same trail only when they are the same component, so there is one non-trivial component.

**Sufficiency :-** Assuming that the condition, we obtain an Eulerian ckt. using induction on the no. of edges m.

**Basis step :-** $m = 0 \longrightarrow$ a closed trail consisting of one vertex suffices.

**Induction step :** $m > 0 \longrightarrow$

With even degrees, each vertex in the non-trivial component of G, has degree of at least 2. This means that it has a cycle (previous theorem). Let G' be the graph obtained from G by deleting an edge in E(C).

Since C has 0 or 2 edges at each vertex, each component of G' is also an even graph.

Since each component is also connected & has fewer than m-edges, we can apply the induction hypothesis to conclude that each component of G' has an Eulerian circuit. To combine these into an Eulerian circuit of G', we traverse C, but when a component of G' is entered for the first time, we detour along an Eulerian ckt of that component. This circuit ends at the vertex, where we began the detour.

When we complete the traversal of G, one has completed an Eulerian ckt of G.

**Proposition 1 :-** Every even graph decomposes into cycle.

**Corollary :-** A connected graph contains an Eulerian trail iff exactly two vertices of G have odd degree. Furthermore, each Eulerian ~~circuit~~ trail begins with one of the odd vertices & ends at the other.

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**Proposition:-** If $G$ is a simple graph on which every vertex has degree at least $k$, then $G$ contains a path of length at least $k$. If $k \geq 2$, then $G$ also contains a cycle of at least $k$.

**Theorem:** For a connected non-trivial graph with exactly $2k$ odd vertices, the min number of trails that decompose it is $\max\{k, 1\}$.

**Proposition:-** Let $G$ & $H$ be non-trivial connected graphs. Then $G$ & $H$ are Eulerian or every vertex of $G$ & $H$ is odd.

## Finding the connectivity of a graph

**Theorem:-** Let $G$ be a graph with vertices labelled $v_1, v_2, v_3, v_4, \ldots, v_n$ & let $A$ be the corresponding adjacent matrix.

For any +ve integer $k$, the $(i,j)$ entry of $A^k$ is equal to the no. of walks from $v_i$ to $v_j$ that are exactly $k$ edges.

**Proof:** (By induction) For $k=1$, the result is true as $[A]=1$ when there is exactly one edge walk between $v_i$ & $v_j$.

Now suppose that for every $i$ & $j$, the $(i,j)$ edge entry for $A^{k-1}$ is the no of walks from $v_i$ to $v_j$ that are exactly $(k-1)$ edges. For each edge walk from $v_i$ to $v_j$, there exist a vertex $v_n$ such that the walk can be thought of as a $(k-1)$ edge walk from $v_i$ to $v_n$ combined with edge from $v_n$ to $v_j$.

The total no. of $k$ edge walks —

$$\sum_{v \in V} \#(k-1) \text{ edge walks from } v_i \text{ to } v_n.$$

By induction hypothesis this can be rewritten as

$$\sum_{v \in V} [A^{k-1}]_{i,n} = \sum_{n=1}^{|V|} [A^{n-1}]_{i,n} [A]_{n,j} = [A^k]_{i,j}$$

**Note:-** ① If a graph with $n$-vertices is a connected graph, then the length of the longest possible path in $G$ is $(n-1)$, that can be generated by $A^{n-1}$.

② For every entry $(i,j)$ of $B = A + A^2 + A^3 + \ldots + A_k$ $(k>1)$ gives the no. of walks less than or equal to $k$.

③ $S = M + M^2 + M^3 + M^4 + \ldots + M^{n-1}$.
If there is a zero in the matrix $S$, then it is impossible to connect the pair of vertices in the $n^{th}$ step or more. Thus, graph is not connected.

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