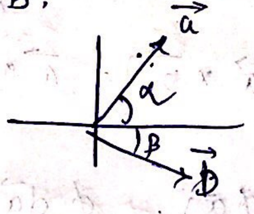
Prove by vector. Mathematica

1) $\boxed{\sin(A+B)} = \sin A\cos B + \cos A\sin B$

2) $\boxed{\cos(A+B)} = \cos A\cos B + \sin A\sin B.$

$\hat{a} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$

$\hat{b} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j}$



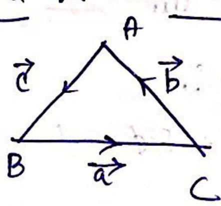
$$\hat{a}\times\hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\alpha & -\sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix}$$

$$= \hat{k}\left(\cos\alpha\sin\beta + \sin\alpha\cos\beta\right).$$

3) $\boxed{\sin(A-B)}$

4) $\boxed{\cos(A-B)}$

5) Show by vector's method
In a ΔABC $\frac{\sin a}{a} = \frac{\sin b}{b} = \frac{\sin c}{c}.$



$\vec{a}+\vec{b}+\vec{c}=0$

$\vec{a}\times(\vec{a}+\vec{b}+\vec{c}) = 0,$

$\vec{a}\times\vec{b} = ab\sin A$

$\vec{a}\times\vec{c} = ac\sin C$

~~$b\sin A = c\sin C$~~
~~$\sin A$~~

$\vec{a}\times\vec{b} = -(\vec{a}\times\vec{c}) = \vec{c}\times\vec{a}$

6) P.T. $[\vec{a}\times\vec{b}\quad \vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

7) Show that $[a\,b\,c]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{a}.\vec{c} & \vec{a}.\vec{c} & \vec{c}.\vec{c} \end{vmatrix}$

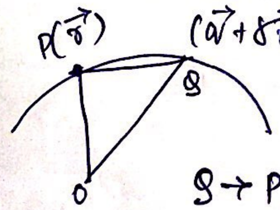
8) If $\vec{a}$ & $\vec{b}$ are 2 non-colinear vectors s.T.
$\vec{a} = \vec{c}+\vec{d}$ where $\vec{c}$ is $\parallel$ to $\vec{b}$ & $\vec{d}$ is $\perp$ to $\vec{b}$
then show that $\vec{c} = \frac{\vec{a}.\vec{b}}{b^2} = \vec{b}$ $\quad \vec{d} = \vec{a} - \left[\left(\frac{\vec{a}.\vec{b}}{b^2}\right).\vec{b}\right].$

a) Then show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\,\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\,\vec{d}$

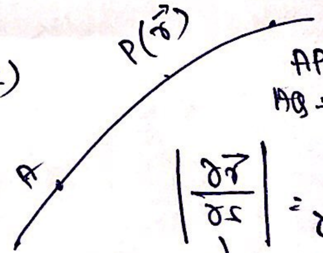
Derivative of vector

$\vec{r} = \vec{f}(t)$

$\frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0}\frac{\bar{f}(t+\delta t) - \bar{f}(t)}{\delta t}.$



$Q\to P,\ \delta t\to 0$



AP = s
AQ = s + δs

$\left|\frac{\delta\vec{r}}{\delta s}\right| = \lim_{\delta s\to 0}\frac{\delta r}{\delta s} = 1$

represent unit tangent vector at point P with pos. vector r & pos. s.

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$$\frac{d}{dt}(\vec{a} \pm \vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$$

$$\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\frac{db}{dt} + \frac{da}{dt}\cdot\vec{B}.$$

$$\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \left(\frac{d\vec{b}}{dt}\right) + \frac{d\vec{a}}{dt} \times \vec{B}.$$

$$\frac{d}{dt}(\phi\vec{a}) = \phi\frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\cdot a.$$

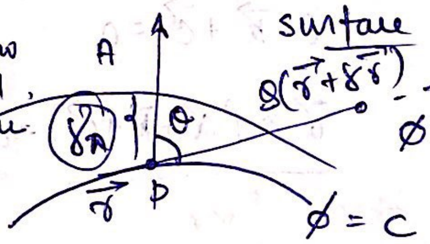
$\phi(x,y,z)$ Grad. ~~Drag~~ rate of a vector in

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}.$$

$\perp$ to the surface

Normal diff b/w 2 nbd surface.



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\partial\vec{r} = \partial x\hat{i} + \partial y\hat{j} + \partial z\hat{k}$.

find $\boxed{\nabla\phi \cdot \delta\vec{r}}$ $\cancel{\frac{d\phi}{dx}}$

$$\nabla\phi \cdot d\vec{r}$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot \left(\partial x\hat{i} + \partial y\hat{j} + \partial z\hat{k}\right)$$

$$= \delta\phi.$$

$Q \to P$

$\delta\vec{r} \to 0$

$\nabla\phi\,\delta\vec{r} = 0$

$\Rightarrow \delta\phi = 0$

$\vec{\nabla}\phi = |\vec{\nabla}\phi|\,\hat{N}$

Let the $\perp$ dist b/w 2 nbd surface be δn.

$$\cancel{\frac{\delta\phi}{\delta n} = \lim_{\delta n\to 0}\frac{\delta\phi}{d} = \lim_{\delta n\to 0}\frac{\vec{\nabla}\cdot\delta\vec{r}}{\delta}}$$

$$\cancel{\nabla^2\phi}\quad \frac{\partial\phi}{\partial n} = \lim_{\partial n\to 0}\frac{\partial\phi}{\partial n} = \lim_{\delta n\to 0}\frac{\vec{\nabla}\phi\,\delta\vec{r}}{\partial n}.$$

$$= \lim_{\partial n\to 0}\ |\vec{\nabla}\phi|\,\hat{N}\cdot\delta\vec{r}$$

$\hat{N}\cdot d\vec{r} = \cos\theta$.

$$= \lim_{\delta n\to 0}\frac{|\nabla\phi|\cdot dn}{dn}$$

<u>Directional derivative</u>

$$= |\nabla\phi|.$$

rate of change of ϕ

$$\frac{\partial\phi}{\partial r} = \lim_{\partial r\to 0}\frac{\partial\phi}{\partial r}$$

r represent the dir^n of line PQ at P.

$$= \lim_{\partial r\to 0}\left[\frac{\partial\phi}{\partial n}\cdot\frac{\partial n}{\partial r}\right]$$

$$\partial\vec{r} = \frac{\partial n}{\cos\theta} = \frac{\partial n}{\hat{N}\cdot\hat{N}'}$$

$$\therefore \lim_{\partial r\to 0}\left[\frac{\partial\phi}{\partial n}\cdot\ \hat{N}\cdot\hat{N}'\right]$$

$$\text{i.e. } \lim_{\partial r\to 0}\ \hat{N}\cdot\hat{N}'\,|\vec{\nabla}\cdot\phi|.$$

$$= \hat{N}'\cdot(\vec{\nabla}\cdot\phi)$$

if dir^n cosine of $\hat{N}'$ is l, m, n.

$$l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z}.$$

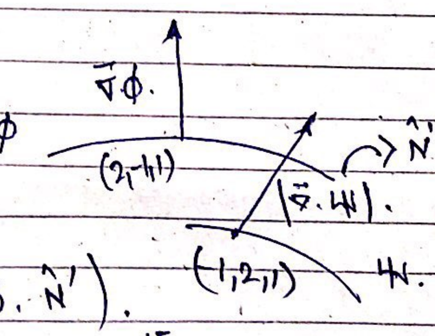
What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the dir^n of normal to the surface $x\log z - y^2 = -4$ at the point $(-1,2,1)$

<u>Sol^n.</u>

$\cancel{x\log z - y^2 = -4}$

$\cancel{(-1,2,1)}$

$\cancel{(2,-1,1)}$ $\cancel{\phi = xy^2+yz^3}$



$\cancel{\vec{\nabla}\cdot\phi}$

$= \cancel{\frac{d\phi}{dx}\hat{i} +}$

$$\left(\vec{\nabla}\cdot\phi\cdot\hat{N}'\right).$$

$$= \frac{15}{\sqrt{17}}$$

$=$

24/9/2019 Vector Prof. Lahiri

1

Prove that ~~divergence of $(\vec{A}\times\vec{B})$~~.

1 $$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = (\vec{\nabla}\times\vec{A})\cdot\vec{B} - (\vec{\nabla}\times\vec{B})\cdot\vec{A}$$

$$\vec{\nabla}(\vec{A}\times\vec{B}) = \sum \hat{i}\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$
$$= \sum \hat{i}\left[\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) + \left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right]$$
$$= \left(\sum \hat{i}\times\frac{\partial A}{\partial x}\right)\cdot\vec{B} - \left(\sum\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)\cdot\vec{A}$$
$$= (\vec{\nabla}\times\vec{A})\cdot\vec{B} - (\vec{\nabla}\times\vec{B})\cdot\vec{A}.$$

2 ~~$\vec{\nabla}\cdot(\vec{A}\times\vec{B})$~~

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}).$$

$$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}.$$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = \sum\hat{i}\times\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$
$$= \sum\hat{i}\times\left[\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right) + \left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right]$$
$$= \sum\hat{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right) + \sum\hat{i}\times\left(\frac{\partial A}{\partial x}\times\vec{B}\right)$$
$$= \sum\left(\hat{i}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{A} - (\hat{i}\cdot\vec{A})\frac{\partial\vec{B}}{\partial x} + \sum(\hat{i}\cdot\vec{B})\frac{\partial\vec{A}}{\partial x} - \left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B}$$
$$= A\sum\hat{i}\frac{\partial\vec{B}}{\partial x} - \vec{B}\sum\hat{i}\frac{\partial\vec{A}}{\partial x} + \sum(\hat{i}\cdot\vec{B})\cdot\frac{\partial\vec{A}}{\partial x} - \sum(\hat{i}\cdot\vec{A})\cdot\frac{\partial\vec{A}}{\partial x}$$
$$= (\vec{B}\cdot\vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}).$$

Note. $\vec{\nabla} \times \vec{F} = 0$

$\vec{F} = \vec{\nabla}(\phi)$

Scalar potential

②

A vector field is given by $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$ show that the field is irrotational.

irrotational if $(\vec{\nabla} \times \vec{A}) = 0$.

solenoidal if $(\vec{\nabla} \cdot \vec{A}) = 0$.

Also find the scalar potential.

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= 0 + 2xy - 2xy = 0.$$

hence $\vec{\nabla} \times \vec{A} = 0.$

$\Rightarrow \vec{\nabla} \times \vec{A} = \vec{\nabla}\phi.$

$$= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\frac{\partial \phi}{\partial x} = x^2 + xy^2$$

$$\frac{\partial \phi}{\partial y} = x^2y + y^2$$

$$\frac{\partial \phi}{\partial z} = 0.$$

$$\phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + \cancel{f_1(y,z)} + C_1$$

$$\phi = \frac{y^3}{3} + \frac{x^2y^2}{2} + \cancel{f_2(x,z)} + C_2$$

$$\phi = \cancel{f_3(x,y)} + C_3.$$

$$\boxed{\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{x^2y^2}{2} + C}$$

Prove that $\vec{f} = (2x + yz)\hat{i} + (4y + zx)\hat{j} - (6z - xy)\hat{k}$ (3)

is irrotational.

Also find the scalar potential.

Soln. $\vec{\nabla} \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+yz & 4y+zx & -(6z-xy) \end{vmatrix}$

$$= i\left(\frac{\partial}{\partial y}\left(\cancel{4y+zx}\ 6z-xy\right) - \frac{\partial}{\partial z}\left(\cancel{2x+yz}\ 4y+zx\right)\right)$$

$$- j\left(\frac{\partial}{\partial x}(6z-xy) - \frac{\partial}{\partial z}(2x+yz)\right)$$

$$+ k\left(\frac{\partial}{\partial x}(4y+zx) - \frac{\partial}{\partial y}(2x+yz)\right)$$

$$= \hat{i}(x - x) - \hat{j}(+y - y)$$

$$+ k(z - z) = 0.$$

$\frac{\partial \phi}{\partial x} = 2x + yz$ | $\phi = x^2 + xyz + f(y,z) + c_1$

$\frac{\partial \phi}{\partial y} = 4y + zx$ | $\phi = 2y^2 + xyz + f(x,z) + c_2$

$\frac{\partial \phi}{\partial z} = xy - 6z$ | $\phi = xyz - 3z^2 + f(x,y) + c_3$

$$\boxed{\phi = x^2 + xyz + 2y^2 - 3z^2 + c}$$

~~What is the directio~~

(5) Show that $\vec{\nabla} \times \left(\frac{\vec{a} \times \vec{r}}{r^3} \right) = -\frac{\vec{a}}{r^3} + \frac{3\vec{r}}{r^5} (\vec{a} \cdot \vec{r})$.
where $\vec{a}$ is a constant vector

Soln.

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r^2 = x^2 + y^2 + z^2$

$$\sum \left(\hat{i} \times \frac{\partial}{\partial x} \left(\frac{\vec{a} \times \vec{r}}{r^3} \right) \right)$$

$$= \sum \hat{i} \times \left[-\frac{3}{r^4} \frac{\partial r}{\partial x} (\vec{a} \times \vec{r}) + \frac{1}{r^3} \left(\vec{a} \times \frac{\partial \vec{r}}{\partial x} \right) \right]$$

$$\Rightarrow \frac{\partial}{\partial x} \left(\frac{\vec{a} \times \vec{r}}{r^3} \right)$$

$$= -\frac{3x}{r^5} (\vec{a} \times \vec{r})$$

$$+ \frac{1}{r^3} (\vec{a} \times \hat{a}).$$

$$\vec{b} \cdot \vec{\nabla} \left(\vec{a} \cdot \vec{\nabla} \frac{1}{r} \right) = 3 \left(\frac{\vec{a} \cdot \vec{r}}{r^5} \right) (\vec{b} \cdot \vec{r}) - \frac{\vec{a} \cdot \vec{b}}{r^3}$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{i}$

a & b ~~are~~ are constant

Q. Find the values of the constant abc. So that the direction derivative of $\phi = axy^2 + byz + cz^2x^3$ (1, 2, -1) in a dirn ∥ to z-axis. (has maxm magnitude 64)

Soln. Grad of ϕ.

$$\vec{\nabla}.\phi = \left(\frac{\partial}{\partial x}\phi\right)i + \left(\frac{\partial}{\partial y}\phi\right)j + \left(\frac{\partial}{\partial z}\phi k\right)$$

$$\Rightarrow \vec{\nabla}.\phi = (ay^2 + 3cz^2x^2)\hat{i}$$
$$+ (2axy + bz)\hat{j}$$
$$+ (0 + by + 2czx^3)\hat{k}.$$

$\vec{\nabla}.\phi\,(1,2,-1)$.

$$= (4a + 3c(1)(1))\hat{i}$$
$$+ (2a1.(2) + \cancel{2(-1)}\, b(-1))\hat{i}$$
$$+ (2\cancel{b} + 2.c.(-1).(1))\hat{k}$$
$$= (4a + 3c)\hat{i} + (4a - b)\hat{i}$$
$$+ (2b - 2c)\hat{k}.$$

$\parallel$ to z-axis.

$4a + 3c = 0$

$4a - b = 0. \Rightarrow b = 4a.$

$2b - 2c = 64.$

~~$8a - 2c = 64$~~

~~$4a - c = 32$~~

~~$4a + 3c = 0$~~

~~$2c = 32$~~

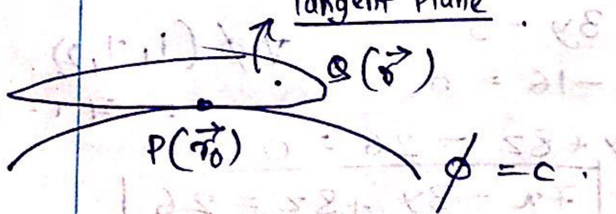
~~$c =$~~

$b = 24$

$a = 6$

$c = -8.$

Eqn of Tangent & Normal Line



$$\vec{r_0} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$$
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{a}.$$

$$\vec{PQ} \perp \vec{\nabla}\phi.$$

(4)
(6)

$$(\vec{r} - r_0)\cdot \vec{\nabla}\phi = 0.$$

cartesian eqn of form

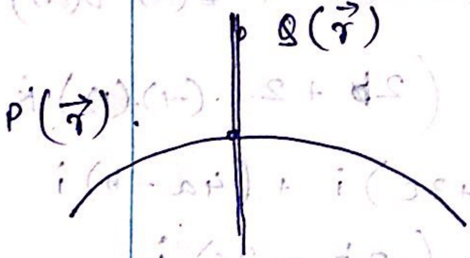
$$= (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}$$

Speigel - Vector Analysis

$$\cdot \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).$$

$$\Rightarrow (x - x_0)\frac{\partial\phi}{\partial x} + (y - y_0)\frac{\partial\phi}{\partial y} + (z - z_0)\frac{\partial\phi}{\partial z} = 0$$

eqn of the normal.



$$(\vec{r} - \vec{r_0}) \parallel \vec{\nabla}\phi.$$

$$(\vec{r} - \vec{r_0}) = p\,\vec{\nabla}.\phi.$$

$$= \frac{x - x_0}{\left(\frac{\partial\phi}{\partial x}\right)} = \frac{y - y_0}{\left(\frac{\partial\phi}{\partial y}\right)} = \frac{z - z_0}{\left(\frac{\partial\phi}{\partial z}\right)} = \phi.$$

Find the eqn of the tangent plane & the normal line to the surface

$$\phi = 2xz^2 - 3xy - 4x - z = 0.$$

$(+1, -1, 2).$

Soln Normal.

$$x_0 = +1, \quad y_0 = -1, \quad z_0 = 2.$$

$$\boxed{\frac{x - 1}{7} = \frac{y + 1}{-3} = \frac{z - 2}{8}}$$

$$\frac{\partial\phi}{\partial x} = 2z^2 - 3y - 4. = 8 + 3 - 4 = 7$$

Tangent $7(x - 1) + -3(y + 1) + 8(z - 2) = 0$

$$\Rightarrow 7x - 7 - 3y - 3 + 8z - 16 = 0$$

$$\nabla\phi(1, -1, 2) = 7\hat{i} - 3\hat{j} + 8\hat{k}$$

$$\Rightarrow 7x - 3y + 8z - 26 = 0$$

$$\Rightarrow \cancel{7x}\ \boxed{7x - 3y + 8z = 26}$$

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Find the equations of the tangent plane & normal line to surface.

$$z = x^2 + y^2 \text{ at } (2,-1,5).$$

Q. Show that the necessary & sufficient cond. that $u(x,y,z)$, $v(x,y,z)$ & $w(x,y,z)$ be functionally related through the eqn

Far Pg. 92 Ex. 4.69B

$$F(u,v,w) = 0 \quad \text{——— (1)}$$

if $[\vec{\nabla} u \ \ \vec{\nabla} v \ \ \vec{\nabla} w] = 0 \Rightarrow$ co-planar.

Soln. $w = f(u,v)$. $\qquad au + bv + cw = 0$

$$w = -\left(\frac{a}{c}u + \frac{b}{c}v\right)$$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} \quad \text{——— (2)}$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} \quad \text{——— (3)}$$

$$\frac{\partial w}{\partial z} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} \quad \text{——— (4)}$$

$$[\vec{\nabla} u \ \ \vec{\nabla} v \ \ \vec{\nabla} w] = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \underbrace{\frac{\partial w}{\partial x}}_{(2)} & \underbrace{\frac{\partial w}{\partial y}}_{(3)} & \underbrace{\frac{\partial w}{\partial z}}_{(4)} \end{vmatrix}$$

$$= 0.$$

if $[\vec{\nabla} u \ \vec{\nabla} v \ \ \vec{\nabla} w] = 0$

then show that u, v, w are functionally related.

co-planar. $\vec{\nabla} u = \lambda \vec{\nabla} v + \mu \vec{\nabla} w$

hence,

$$\therefore \sum i \frac{\partial u}{\partial x} = \lambda \sum i \frac{\partial v}{\partial x} + \mu \sum i \frac{\partial w}{\partial x}.$$

Similar for $\frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ $\qquad \therefore \frac{\partial u}{\partial x} = \lambda \frac{\partial v}{\partial x} + \mu \frac{\partial w}{\partial x}$

$$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}.dy + \frac{\partial u}{\partial z}.dz.$$

$$\Rightarrow \cancel{du} \quad du = \lambda dv + \mu dw$$

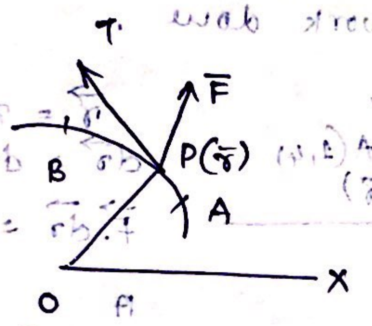
$$\Rightarrow u = \lambda v + \mu w + c.$$

$$\cancel{f(u)} \quad u = f(v, w).$$

$$\Rightarrow \boxed{F(u, v, w) = 0}$$

22/10/2019 — Vector Integration — Prof. Lahiri — ①

Vector Line Integration



Let $\vec{F}$ be the vector fn of AB be a curve. The component of $\vec{F}$ along tangent PT is given by

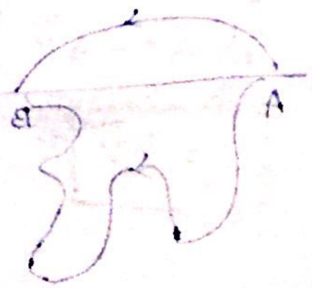
$\vec{F} \cdot \frac{d\vec{r}}{ds}$ $\left(\frac{d\vec{r}}{ds}\right)$. where is the unit tangent vector at P whose position is r

then the Line Integral of the vector fn along curve AB is defined as

$$\int_A^B \left(F \cdot \frac{d\vec{r}}{ds}\right) \cdot ds = \int_a \vec{F} \cdot d\vec{r}$$

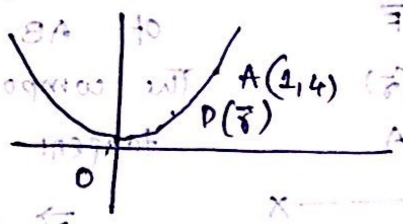
1) If $\vec{F}$ represents a variable force acting on a particle displaces from ~~the particle~~ pt $A \rightarrow B$, the total work done is $\int_A^B \vec{F} \cdot d\vec{r}$.

2) If $\vec{F}$ represents velocity of liquid then $\oint_C \vec{F} \cdot d\vec{r}$ is called circulation of $\vec{F}$ along curve. If $\oint \vec{F} \cdot d\vec{r} = 0$, then $\vec{F}$ is called ~~irratio~~ irrotational.



Q. if $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$
displaces a particle in xy plane
from $(0,0)$ to $(1,4)$ along the
curve $y = 4x^2$. find the
work done.

Soln.



$\vec{r} = x\hat{i} + y\hat{j}$

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$\vec{F}\cdot d\vec{r} = 2x^2y\cdot dx + 3xy\,dy$

$$\int_O^A \vec{F}\cdot d\vec{r} = \int_O^A 2x^2y\,dx + \int_O^A 3xy\cdot dy$$

$dy = 8x\cdot dx$.

$$= \int_O^A 8x^4\cdot dx + \int_O^A 3x(4x^2)\cdot 8x\cdot dx$$

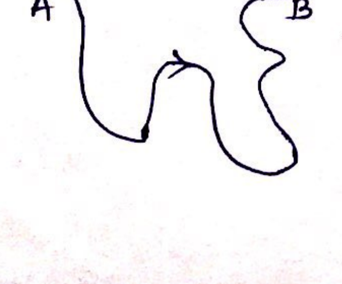
$$= \int_O^1 \left(8x^4 + 12x^4\right)\cdot dx$$

$$= \int_O^1 16\,x^4\,dx$$

$$= 16\cdot\frac{x^5}{5}\Big|_0^1$$

$$= \frac{16}{5}$$

Q. If $\vec{F} = \vec{\nabla}\phi$. Show that the work done
in moving a particle in the force field $\vec{F}$
from ~~A(x1,y~~ $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$
is independent of the path taken.



Soln $WD = \int \vec{F}\cdot d\vec{r} = \int_A^B \vec{\nabla}\phi \cdot d\vec{r}$ ③

$\vec{r} = x\hat{i} + y\hat{j}$

$$= \int_A^B \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_A^B \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz$$

$$= \int_A^B d\phi = \phi\Big]_A^B$$

$$= \phi_B - \phi_A.$$

Converse of this problem

If line integral is independent of the path then $\vec{F}$ is conservative. i.e $\vec{F}$ can be expressed as grad of some fn.

Show the converse

The line integral joining $A(x_1, y_1, z_1)$ & $B(x, y, z)$

Soln.

$$\int_C \vec{F}\cdot d\vec{r} = \int_A^P F\cdot d\vec{r} = \int_A^P \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right) ds$$

$$= \phi(xyz).$$

Diff. under space variable x.

$\frac{d\phi}{ds} = \vec{F}\cdot\frac{d\vec{r}}{ds}$. where $\frac{d\phi}{ds}$ is the directional derivative.

$$\frac{d\phi}{ds} = \vec{\nabla}\phi \cdot \frac{d\vec{r}}{ds}$$

$$\therefore \left(\vec{\nabla}\phi - \vec{F}\right)\cdot\frac{d\vec{r}}{ds} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\phi = F}$$

④

Q. Show that the integral $\int_{(1,2)}^{(3,4)} (xy^2 + y^3)\,dx + (x^2y + 3xy^2)\,dy$ is independent of the path joining the pts $(1,2)$ & $(3,4)$. Hence evaluate the line integral.

Soln

$$\int_{(1,2)}^{(3,4)} \left[(xy^2 + y^3)\hat{i} + (x^2y + 3xy^2)\hat{j}\right] \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_{(1,2)}^{(3,4)} \vec{F} \cdot d\vec{r}.$$

$$\vec{F} = (xy^2 + y^3)\hat{i} + (x^2y + 3xy^2)\hat{j}$$

$\vec{\nabla} \times \vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \\ x^2+y^3 & x^2y+3xy^2 & \end{vmatrix}$$

$$= \hat{k}\left[\frac{\partial}{\partial x}(x^2y + 3xy^2) - \frac{\partial}{\partial y}(x^2 + y^3)\right]$$

$$= \hat{k}\left[2xy + 3y^2 - 3y^2\right]$$

$\vec{\nabla} \times \vec{F} = 0$

then work done is independent of path.

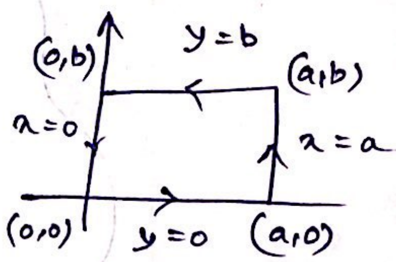
$$\int d\left(\frac{x^2y^2}{2}\right) + d(xy^3)$$

$$= \frac{x^2y^2}{2} + xy^3 \Big|_{(1,2)}^{(3,4)}$$

$= 254$

Q. Evaluate the line integral $\oint \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x^2+y^2)\hat{i} - 2xy\,\hat{j}$, where $x=0,\ y=0$ & $x=a,\ y=b$ are the boundaries.

Soln. $\oint \vec{F}\cdot d\vec{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$



$\vec{F}\cdot d\vec{r} = (x^2+y^2)\,dx - 2xy\,dy$

C_1: $y=0$, $dy=0$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx = \frac{a^3}{3}.$$

C_2: $x=a$

$$\int_0^b -2xy\,dy = -2a\frac{y^2}{2}\Big|_0^b = -ab^2$$

C_3: $y=b$. $dy=0$.

$$\int_a^0 b^2\,dx + \int_a^0 x^2\,dx$$

$$= -b^2a + \cancel{\frac{-a^3}{3}} + \frac{x^3}{3}\Big|_{+a}^{0}$$

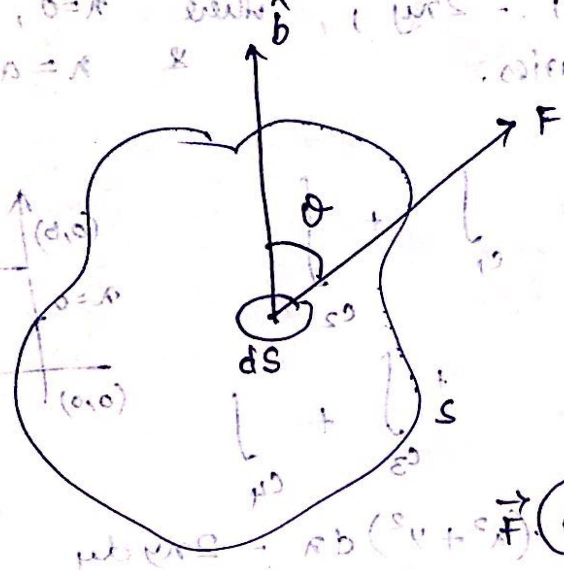
$$= -b^2a + -\frac{a^3}{3}$$

C_4: $x=0$ $dx=0$.

$= 0$.

Total :- $-2ab^2$

Surface Integration



Let $\vec{F}$ be a vector f^n & S be any given surface.
The surface integral of a vector f^n F over the surface S is defined as the integral of component of $\vec{F}$ along the normal to the surface.

$\vec{F} \cdot \hat{n} \rightarrow$ unit outward drawn normal to the surface.

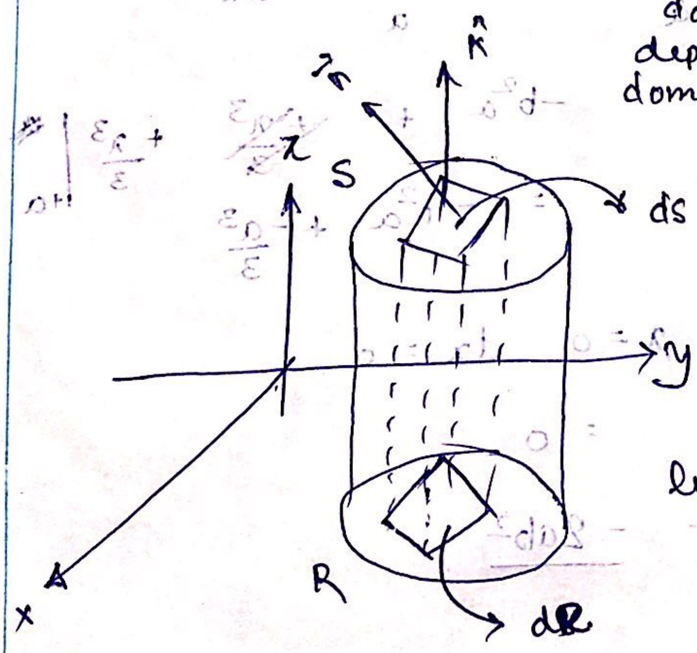
$f = 0 \quad \hat{n} = \dfrac{\vec{\nabla} f}{|\vec{\nabla} f|}$

Total surface integral is defined as:-

$$\iint_S \vec{F}\cdot\vec{n}\,.ds.$$

$$= \iint_{S} \vec{F}\cdot\vec{ds} \rightarrow$$

double/single depends on the domain of integration



$dR = dx.dy.$

Let R be the orthogonal projection of S on x-y plane.

let ~~$\hat{n} =$~~ $\boxed{\hat{n} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}}$

$dx\,dy$ = proj of ds on xy plane. (7)

$= ds.\cos\delta$

$$ds = \frac{dx\,dy}{\cos\delta}$$

$$\iint_S \vec{F}.\vec{n}.ds$$

$$\| $$

$$\iint_R \vec{F}.\vec{n}\,\frac{dx\,dy}{\cos\delta} = \underbrace{\iint_R \vec{F}.\vec{n}.\frac{dx\,dy}{|\vec{n}.\hat{k}|}}_{x\text{-}y \text{ plane}}$$

$= \cancel{\vec{F}.\vec{n}}$

for eg, if x-z plane

$$\iint_{R_1} \vec{F}.\vec{n}.\frac{dx.dz}{|\vec{n}.\hat{j}|}.$$

Read the statement

* Green's theorem

$\phi(x,y)$ $\psi(x,y)$ $\frac{\partial\phi}{\partial y}$ $\frac{\partial\psi}{\partial x}$ be a continuous fn over region R bounded by a closed surface C in xy plane then

$$\oint_{C_1}\left[\phi(x,y)dx + \psi(x,y).dy\right]_{C_2} = \iint_R\left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dx.dy.$$

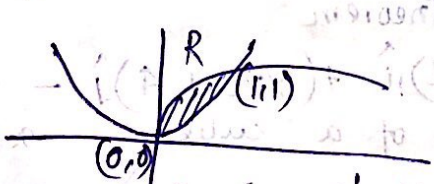
Q. Verify Green's theorem in the plane for

$\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$ where C is the boundary of the region, defined by

1. $y^2 = x$ & $x^2 = y$
2. $x=0,\ y=0,\ x+y=1$.

$C_1 \Rightarrow \cancel{y = x^2}$ $x^2 = y$

$C_2 \Rightarrow y^2 = x$



$$\int_{C_1} + \int_{C_2}$$

C_1 : $\int_0^1 (3x^2 - 8x^4)dx + (4x^2 - 6x^3)\,dx.2x$

$= \cancel{\int 3}\ x^3 - \frac{8x^5}{5} + \frac{4x^3}{3} - \frac{12x^5}{5}\Big|_0^1$

$= \cancel{1 - \frac{8}{5} + \frac{4}{3} - \frac{12}{5}}$

$= \cancel{1 - \frac{4}{5} - \frac{12}{5}}\ \cancel{\frac{5}{3}}\ \cancel{\frac{-11}{5}}$ $\cancel{= \frac{-5}{3}}$ = -1

$\cancel{x^3 - 4x^4 \ldots}$

$\cancel{x^3 - 4\frac{x^5}{5} + \frac{4x^3}{3}}$

$\cancel{1 - \frac{20}{5} + \frac{4}{3}}$ $\cancel{= \frac{-5}{5}}$

$\cancel{= 1 - 4 + \frac{4}{3}}$

$\cancel{= -3 + \frac{4}{3}}$

$C_2 : - \; 5/2.$

$$\boxed{C_1 + C_2 = 3/2}$$

Using Green's theorem $\quad \psi = 4y - 6xy \quad \phi = 3x^2 - 8y^2$

we find $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}.$

$$\iint_R 10\, dy\, dx = 10\int_0^1 \int_{x^2}^{\sqrt{x}} y\, dy . dx .$$

$$= 10\int_0^1 dx . \left(\frac{y^2}{2}\Big|_{x^2}^{\sqrt{x}}\right)$$

$$= 10\int_0^1 dx . \left(\frac{x}{2} - \frac{x^4}{2}\right)$$

$$= \left(\frac{x^2}{4}\Big|_0^1 - \frac{x^5}{10}\Big|_0^1\right)$$

$$= \left(\frac{1}{4} - \frac{1}{10}\right)10$$

$$\left(\frac{5-2}{20}\right).10$$

$$= \frac{3}{2}$$

Stoke's Theorem

Let S be an open surface bounded by a close curve C and if F be a vector fn having cont. 1st order derivative then

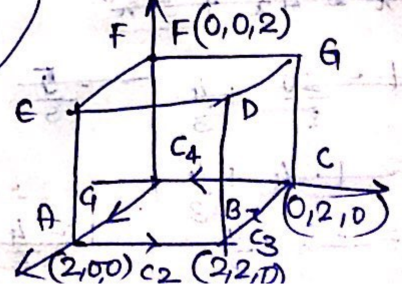
$$\oint_C \vec{F}.\vec{dr} = \iint_S (\vec{\nabla} \times \vec{F})\hat{n}.ds$$

where $\hat{n}$ is the outward drawn normal vector.

Verify Stoke's theorem for $\vec{F} = (y - z + 2)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}$ over the surface of a cube. $x = 0, y = 0, z = 0$ and $x = 2, y = 2, z = 2$

$$\oint \vec{F}.\vec{dr} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

(above x-y-plane the bottom surface is not considered.



(9)

$$\oint \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds.$$

$$\vec{F}\cdot d\vec{r} = (y-z+2)\,dx + (yz+4)\,dy - (xz)\,dz$$

C_1: $y=0,\ z=0$
$\Rightarrow dy=0,\ dz=0$ $\quad \cancel{\vec{F}\cdot 2x}\quad \int_{C_1}\vec{F}\cdot d\vec{r} = \int_0^2 2\,dx = 4.$

C_2: $\cancel{y=}\ x=2,\ z=0$
$dx=0.\quad dz=0$

$$\int_0^2 4\,dy = 8.$$

C_3: $z=0,\ y=2$
$dz=0\quad dy=0$

$$\int_2^0 4\,dx = 4(-2) = -8$$

C_4: $x=0,\ z=0$
$dz=0\quad dx=0$

$$\int_2^0 4\,dy = -8.$$

$\therefore -4$ Ans.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}.$$

$$\vec{\nabla}\times\vec{F} = \hat{i}(-y) + (z-1)\hat{j} - \hat{k}$$

5 surface (the bottom is not considered).

S_1: ABDE $(x=2) \Rightarrow y\text{-}z$ plane
$dS = dy\,dz.$

$$\vec{\nabla}\times\vec{F}\cdot\hat{i} = -y$$

$$\iint_{S_1}(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds = \int_0^2\int_0^2 -\frac{y}{2}\cdot dy\cdot dz$$

$$= \int_0 -y\,dy\,(2)\cdot \cancel{dz}$$

$$= -\frac{2y^2}{2}\Big|_0^2 = -4.$$

S_2 :- OCGF $\cancel{y=2}$ $x=0$

$$\int_0^2\int_0^2 y \cdot dy\,dz = 4.$$

$S_1 + S_2 = 0$

$S_3 + S_4 = 0$

S_5 : EFGD.

$z=2$ $\hat{k}$.

$$\int_0^2\int_0^2 -1 \cdot dx \cdot dy$$

$= -4.$

Total $= -4$

Thus, Stoke's theorem is verified.

~~05/10~~ 05/11/2019 Vector Prof. Lahiri

(Grewal)

Gauss Divergence Theorem

If $\vec{F}$ be a vector having a continuous 1st order derivative then

$$\iiint_V \vec{\nabla}\cdot F\,dv = \iint_S \vec{F}\cdot\vec{n}\,ds$$

$\vec{n}$ is the unit outward drawn vector

Q. Verify divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by cylinder $x^2+y^2=4$, $z=0$, $z=3$

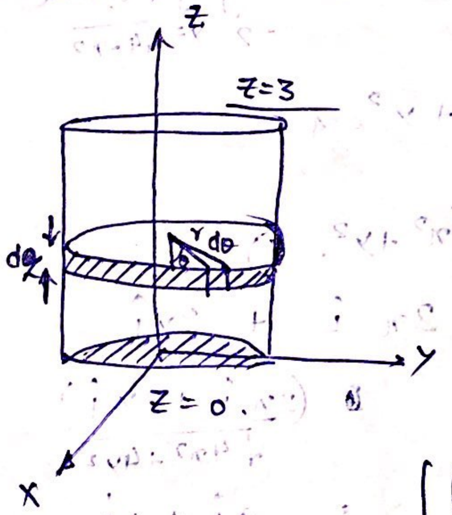
$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(z^2)$

$= \cancel{4 - 4+2}$

$4 - 4y + 2z$

Soln.

considering an elementary strip



$\iiint \vec{\nabla}\cdot\vec{F}\,dv$

$$= \int\int\int_{z=0}^{z=3} (4-4y+2z)\,dz.dy.dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4z - 4zy + z^2\right)\Big|_0^3 dy.dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4\cdot3 - 4\cdot3y + 3^2)\cdot dy\,dx$$

$$\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\,dy\,dx$$

odd

$$\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\,dy\,dx$$

$$= 4\int_0^2\int_0^{\sqrt{4-x^2}} 21\,dy\,dx$$

$\cancel{x^2}$

$x = r\cos\theta$.

$$= \cancel{4\int_0^2 21(\sqrt{4-x^2})\cdot dx}$$

$$= 4\int_0^2 21\left(\sqrt{4-x^2}\right)\cdot dx$$

$= 84\pi$

②

$S_1 : z = 0$

$S_2 : z = 3$

$S_3 : x^2 + y^2 = 4.$

$$\hat{n}_{S_1} = -\hat{k} \qquad \vec{F}\cdot\hat{n} = -z^2$$

$$\iint_{S_1} \vec{F}\cdot\hat{n} = \iint -z^2\big|_{z=0} dS_1 = 0$$

$$\hat{n}_{S_2} = \hat{k} \qquad \vec{F}\cdot\hat{n} = z^2$$

$$\iint_{S_2} \vec{F}\cdot\hat{n} = \iint z^2\big|_{z=3} dS_2$$

$$= \int_{-2}^{2}\int_{y=\sqrt{4-x^2}}^{\sqrt{4-x^2}} 9\, dy\,dx = \pi(2)^2 \cdot 9$$

$$= 36\pi$$

$S_3 : x^2 + y^2 = 4 ;$

$$\vec{\nabla}\cdot(x^2+y^2-4)$$

$$\vec{n} = 2x\,\hat{i} + 2y\,\hat{j}$$

$x = 2\cos\theta$

$y = 2\sin\theta$

$$\hat{n} = \frac{(2x\,\hat{i} + 2y\,\hat{i})}{\sqrt{4x^2+4y^2}}$$

$dS = r\,d\theta\,dz$

$$= \frac{x\,\hat{i}}{\sqrt{x^2+y^2}} + \frac{y\,\hat{j}}{\sqrt{x^2+y^2}}$$

$$= \frac{x\hat{i} + y\hat{j}}{2}$$

$$\iint\left[(4x\,\hat{i} - 2y^2\,\hat{j} + z^2\hat{k})\cdot\frac{(x\hat{i} + y\hat{j})}{2}\right] \cdot \cancel{dx\,dy\,dz}\ dS_3$$

$$\iint\left(\frac{4x^2}{2} - y^3\right)\cdot\cancel{dx\,dy\,dz}\ dS_3$$

$$= \iint (2x^2 - y^3)\ r\,d\theta\,dz$$

$$= \int^{2\pi}\int^{3}\left[2(2\cos\theta)^2 - (2\sin\theta)^3\right] 2.\,d\theta\,dz \quad = 48\pi.$$

$\therefore$ $48\pi + 36\pi$

$= 84\pi.$

Q. Verify divergence theorem given that $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ (3)

and S is the surface of the cube bounded by the planes $x=0,\ x=1\ ;\ y=0,\ y=1\ ;\ z=0,\ z=1$.

Q. Prove that $\iint_S \frac{\vec{r}\cdot\vec{n}}{r^2}\,.dS = \iiint_V \frac{dV}{r^2}$

Soln.

$$\iint_S \frac{\vec{r}\cdot\vec{n}}{r^2}\,dS = \iint_S \left(\frac{\vec{r}}{r^2}\right)\cdot\vec{n}\,.dS$$

$$= \iiint \vec{\nabla}\left(\frac{\vec{r}}{r^2}\right) d\theta.$$

$$= \iiint_{dV} \frac{dV}{r^2}$$

$\cancel{\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}}$

15/11/2019

Vector Calculus

[Courtesy: Titir Adhikary]

Prof. Abhijit Lahiri

①

Fundamental Triads of Lines & Planes associated with any point on a curve.

At every point of curve we associate 3 mutually perpendicular lines known as tangents, normal, binormal and 3 mutually perpendicular planes known as osciliating plane, normal plane & rectifying plane.

Principle normal

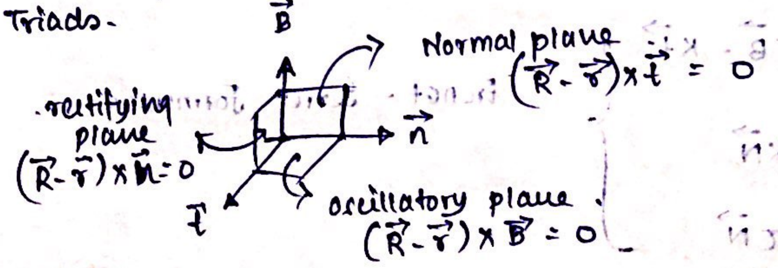
If $\vec{t}$ is a vector having constant magnitude then $\vec{t}\cdot\frac{d\vec{t}}{ds}=0$ i.e. $\frac{d\vec{t}}{ds}$ is $\perp$ to t or

otherwise $\frac{d\vec{t}}{ds}=0$ which is $\vec{t}$ is a constant vector wrt arc length s.

$\frac{d\vec{t}}{ds}$ is $\perp$ to $\vec{t}$ and hence normal to the curve. If we denote a unit normal vector to the curve at the point P by $\vec{N}$ then the direction of $\frac{d\vec{t}}{ds}$ is along $\vec{N}$ which is known as principle normal to the curve at point $\vec{P}$. The plane of $\vec{t}$ & $\vec{N}$ is known as osciliating plane. And hence $\frac{d\vec{t}}{ds}=K\vec{N}$ (K is the constant curvature of curve).

Binormal

The vector $\vec{t}\times\vec{N}$ $\perp$ to both $\vec{t}$ & $\vec{N}$ and thus $\vec{B}=\vec{t}\times\vec{N}$. The 3 vectors $\vec{t},\vec{N}$ and $\vec{B}$ defined at a point on the curve are called Fundamental Triads.



Radius of Curvature

(1) $K = \frac{d\theta}{ds}$ (the rate of change of angle)

$\frac{1}{K}$ = radius of curvature $= \frac{ds}{d\theta}$

Torsion

Since $\vec{B}$ is a unit vector then we have

$$\vec{B} \cdot \frac{d\vec{B}}{ds} = 0 \quad \text{i.e.} \quad \frac{d\vec{B}}{ds} \text{ is } \perp \text{ to } \vec{B}$$

$$\vec{B} \cdot \vec{t} = 0$$

$$\Rightarrow B \cdot \frac{dt}{ds} + \frac{dB}{ds} \cdot \vec{t} = 0$$

$$\Rightarrow \underbrace{\boxed{B(K \cdot \vec{N})}}_{=0} + \frac{d\vec{B}}{ds} \cdot t = 0$$

As $\frac{d\vec{B}}{ds}$ is $\perp$ to $T, \vec{B}$

$\frac{d\vec{B}}{ds}$ is $\parallel$ to $\vec{N}$

$\frac{d\vec{B}}{ds} = -\tau \vec{N}$ (where $\tau \to 0$ ↳ torsion of the curve at point P)

$$\vec{n} = \vec{b} \times \vec{t} \Rightarrow \frac{dn}{ds} = \frac{d\vec{b}}{ds} \times t + b \times \frac{dt}{ds}$$

$$\Rightarrow \frac{d\vec{n}}{ds} = -\tau(\vec{N} \times t) + b \times K\vec{N}$$

$$= +\vec{b}\tau - K\vec{t}.$$

$$\left.\begin{aligned} \frac{d\vec{N}}{ds} &= \tau\vec{B} - K\vec{t} \\ \frac{d\vec{T}}{ds} &= K\vec{N} \\ \frac{d\vec{B}}{ds} &= -\tau\vec{N} \end{aligned}\right\} \text{Frenet - Serret Formulae}$$

Q. PT. Frenet Serret Formula can be written in the form

$$\frac{d\vec{t}}{ds} = (\vec{\omega} \times \vec{t}) \qquad \frac{d\vec{N}}{ds} = (\vec{\omega} \times \vec{N})$$

$$\frac{d\vec{B}}{ds} = (\vec{\omega} \times \vec{B}).$$

$\vec{\omega}$ = Darboux vector.