B. Pharm

Elementary Prob. theory

The word 'Probability' literally denotes 'chance' & the theory of Probability deals with laws governing the chances of occurence of phenomena which are unpredictable in nature.

Defn.

① **Random Experiment.**

The word experiment is used to describe an act which can be repeated under some given conditions. Random experiments are those experiments whose results depend on chance

e.g. tossing a coin, throwing a (several) dice.

② **Outcome** The result of a random experiment will be called an outcome.

e.g. In a random experiment of tossing a coin, there are 2 possibilities. H, T.

③ **Event.** Event is used to denote any phenomenon which occurs in a random experiment.

e.g. When we toss a coin, we may speak of the events 'Head' & 'Tail'.

Events

→ ① Mutually exclusive:

when two or more of them can't occur simultaneously

e.g. Tossing a coin, H & T are mutually excl.

② Exhaustive: If in a set of events, one of them must necessarily occur.

e.g. Two events H & T form an exhaustive set, because one of them necessarily occurs

Cases favourable to an event:

Among all the possible outcomes, of a random exp., those cases which entail occurence of an event A are called 'cases favourable to A'.

Equally likely:

The outcomes of a random experiment are 'equally likely' if after taking into consideration all relevant evidence, none of them can be expected in preference to another.

POCO
SHOT ON POCO F1

Problems.

① When two unbiased coins are tossed, what is the prob. of obtaining

(i) 3 heads

(ii) Not more than 3 heads.

⇒ 4 possible outcomes HH, HT, TH, TT

These are mutually excl. & eq. likely

$\therefore n = 4, \quad m = 3 \text{ heads} = 0$

$\therefore P(A) = 0/4 = 0$

Not more than 3 heads,

$m = 4, \ n = 4$

$\therefore P = 1$

② A bag contains 6 white & 4 black balls. One ball is drawn. What is the prob. that it is white?

③ If 2 balls are drawn one after another from a bag containing 3 white & 5 black balls. What is the prob. -

(i) The 1st ball is white & 2nd is black?

(ii) 1 ball is white & other is black?

Classical defn. of Probability

If a random experiment has n possible outcomes, which are mutually exclusive, exhaustive & equally likely & m of these are favourable to an event A, then the prob. of the event is

$$P(A) = \frac{m}{n}$$

i.e prob. of an event = $\dfrac{\text{No. of outcomes fav. to event}}{\text{Total no. of mutally excl., exhaustive \& eq. likely outcomes of random exp}}$

Class (2)

Jan (1)

Defects:

(1) It is based on the feasibility of subdividing the possible outcomes of the experiments into 'mutually excl.', 'exhaustive', & 'equally likely' cases. Unless this can be done, the formula is inapplicable.

(2) The defn. fails when no. of possible outcomes is infinitely large.

(3) The defn. is only limited to coin tossing, dice throwing etc. We can't find that an Indian aged 25 will die before reaching the age 50.

$\Rightarrow$ (i) $P(A) = \frac{3.5}{56}$ $\rightarrow$ 2 balls may be drawn in $8 \times 7 = 56$ ways, since balls are identical in all respects except in colour, these 56 cases are mutually excl., exhaustive & equally likely.

(ii) $\frac{15+15}{56} = P(A)$

(3) What is the prob. that all 3 children in a family ~~of~~ have different birthdays?

$\Rightarrow$ $p = \frac{365 \times 364 \times 363}{365 \times 365 \times 365} = 0.992$

(4) If 10 persons are arranged at random
 (i) In a line
 (ii) In a ring.
find the prob. that 2 particular persons will be next to each other

$\rightarrow$

(i) 10 persons can be arranged in $10!$ ways which are mutually excl., exh. & eq. likely

$p_1 = \frac{9! \times 2!}{10!} = \frac{1}{5}$

POCO
SHOT ON POCO F1

$\therefore m = 18+16+14+ \cdots +2 = 90$

$\therefore p = \frac{m}{n} = \frac{3}{38} = \frac{90}{1140}$

(6) A group of 2n boys & 2n girls is divided at random into two equal batches. Find the probability that each ~~boys~~ batch will be equally divided into boys & girls.

$\Rightarrow$ The group of 4n boys & girls will be divided into two equal batches, if 2n out of them are selected to form one batch. This selection can be done in ${}^{4n}C_{2n}$ ways. In order that each batch consists of equal numbers of boys & girls, the 1st batch of 2n selected persons should contain n boys & n girls. So, the number of favourable cases is ${}^{2n}C_n \cdot {}^{2n}C_n = ({}^{2n}C_n)^2$. Hence the required probability is,

$$({}^{2n}C_n)^2 / ({}^{4n}C_{2n}).$$

(7) Find the probability p_N that a natural no. chosen at random from the set $\{1, \ldots, N\}$ is divisible by a fixed natural no. k.

$\Rightarrow$ event points are $1, 2, \ldots, N$

dividing N by k, $N = q_N k + r_N$ where $0 \le r_N \le k-1$

The event in the question contains pt's

$k, 2k, \ldots, q_N k$ which are q_N in number.

hence, $p_N = q_N / N$. ◻

(ii) 10 persons can arrange themselves in a ring in 9! ways.

Applying as before,

$$p_2 = \frac{8! \cdot 2!}{9!} = \frac{2}{9}$$

⑤ A box contains 20 tickets of identical appearance, tickets numbered as 1, .., 20.

If 3 tickets are drawn at random, find the prob. that the no's on drawn tickets are in A.P

$\Rightarrow n = {}^{20}C_3 = 1140$

common diff. 1 $\Rightarrow$ 18 sets

(1, 2, 3) (4, 5, 6) - - - (18, 19, 20)

" " 2 $\Rightarrow$ 16 sets

(1, 3, 5), (2, 4, 6), (3, 5, 7), - - -, (16, 18, 20)

" " 3 $\Rightarrow$ 14 sets

(1, 4, 7), (2, 5, 8), - - , (14, 17, 20)

& so on.

Proceeding, with common diff 9,

(1, 10, 19), (2, 11, 20)

(7) From an urn containing N_1 white & N_2 black balls, balls are successively drawn without replacement. What is the prob that i black balls will precede the 1st white ball?

⇒ Suppose all the balls are drawn one by one without replacement & arranged in N different rooms.

The total no. of event points is no. of ways in which N distinguishable balls can be arranged in N rooms, i.e $N!$.

Now, reqd. event means the 1st i rooms are occupied by black balls, the $(i+1)$th room by a white ball, & the last $(N-i-1)$ rooms are filled by remaining $(N-i-1)$ balls in any manner.

∴ reqd. event contains

$$N_2(N_2-1)\cdots(N_2-i+1)\,N_1\,(N-i-1)!$$

event points, hence prob ⇒

$$\frac{N_1 N_2(N_2-1)\cdots(N_2-i+1)}{N(N-1)\cdots(N-i)}$$

∴ Note that,

$$1 = p_N \cdot k + r_N/N$$

& as $N \to \infty$, $r_N/N \to 0$

Since $\{r_N\}$ is bdd,

hence, $\lim p_N = 1/k$.

(8) ~~Prob~~ An urn contains $N = N_1 + N_2$ balls, of which N_1 are white & N_2 black.

(a) A ball is chosen at random from urn, what is the prob. that it is white?

(b) If n balls are drawn, find the prob that exactly i balls are white?

⇒ (a) N_1/N.

(b) In this case, an event point will be group of n balls, the total no. of event points is the no. of different groups of n balls, that can be formed out of N different balls, which is $\binom{N}{n}$

$$\frac{\binom{N_1}{i}\binom{N_2}{n-i}}{\binom{N}{n}}$$

soln. There are two hypotheses:

B_1 = The transferred ball was white

B_2 = " " " " black.

The event A which is stated to have actually happened after the occurence of B_1 or B_2 is:

A $\longrightarrow$ the ball drawn from 2nd box is black.

we have to find $P(B_i/A)$

$P(B_1) = 4/6 = 2/3$

$P(B_2) = 2/6 = 1/3$

also, $P(A/B_1)$ = The prob. that ball drawn from 2nd box is black, assuming that the transferred ball was white

$= 3/5$

Similarly, $P(A/B_2) = 4/5$

using Bayes' formula

$$P(B_1/A) = \frac{P(B_1)\, P(A/B_1)}{P(B_1)\, P(A/B_1) + P(B_2)\, P(A/B_2)}$$

$$= \frac{\frac{2}{5}\times\frac{3}{5}}{\left(\frac{2}{5}\times\frac{3}{5}\right) + \left(\frac{1}{3}\times\frac{4}{5}\right)}$$

$= 3/5$

Baye's theorem

An event A can occur if one of the mutually exclusive & exhaustive set of events $B_1, \ldots, B_n$ occurs.

Suppose, the unconditional probabilities

$P(B_1), \ldots, P(B_n)$ &

conditional probabilities $P(A/B_1), \ldots, P(A/B_n)$ are known.

Then the conditional probability $P(B_i/A)$ is given by,

$$P(B_i/A) = \frac{P(B_i)\, P(A/B_i)}{\sum_i P(B_i)\, P(A/B_i)}$$

This is known as Baye's Theorem

Conditional Probability

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{provided, } P(A) \neq 0$$

Example

① Two boxes contain respectively 4 white, 2 black & 1 white & 3 black balls. One ball is transferred from first box into second & then one ball is drawn from the later. It turns out to be black. What is the prob. that the transferred ball was white?

Random Variable

Let S be a sample space of some given experiment. It has been observed that the outcomes are not always numbers. We may however assign a real no to each sample point according to some definite rule. Such an assignment gives us a "func. defined on the sample space S". This func. is called random variable.

i.e Random variable X may be defined as a func. which assigns a real number X(e) to each sample point e of a given sample space S.

e.g

In the random experiment of tossing 2 coins, let the sample space be $S = \{HH, HT, TH, TT\}$. If X is the r.v denoting no. of heads, then we have assigned a no. to each sample pt. as follows

$$X(HH) = 2, \quad X(HT) = 1, X(TH) = 1, \quad X(TT) = 0$$

Discrete Probability Distr.

Let X be a discrete random variable which can assume the values $x_1, x_2, x_3, \ldots$ (arranged in an increasing order of magntitude) with prob. $p_1, p_2, p_3, \ldots$ respectively, s.t $\sum_i p_i = 1$

The ~~set of~~ specification of the set of values x_i together with their prob. p_i defines the discrete prob. distr. of X.

Let us write $f(x)$ to denote the prob that X takes a specified value x.

i.e $f(x) = P(X=x)$

The func. $f(x)$ is called Prob Mass Func (p.m.F) or simply probability func of the discrete r.v X.

It satisfies two conditions —

(i) $f(x) \geqslant 0$

(ii) $\sum f(x) = 1$

Q Find the prob. distr. of the no. of tails when a coin is thrown repeatedly until the 1st head appears.

⇒ Here $S = \{H, TH, TTH, TTTH, \ldots\}$

Sample pt.	H	TH	TTH	TTTH
X	0	1	2	3

Assuming the coin is unbiased,

$P(X=0) = P(\{H\}) = \frac{1}{2}$

$P(X=1) = P(\{TH\}) = \left(\frac{1}{2}\right)^2$

$P(X=2) = P(\{TTH\}) = \left(\frac{1}{2}\right)^3$

The prob. distr. of X is

x	0	1	2	3 ----	Total
$f(x)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$ --- etc.	1

Mathematical expectation

Suppose a random experiment has n mutually excl. & exh. outcomes corr. to which a variable x takes the values

$$x_1, \ldots, x_n$$

w. p

$$p_1, \ldots, p_n$$

Then mathematical expectation (expected value) of the variable, denoted by $E(x)$ is defined as,

$$E(x) = p_1 x_1 + \cdots + p_n x_n$$
$$= \sum_i p_i x_i$$

where, $\sum_i p_i = 1$

If $E(x) = m$, then mathematical expectation of $(x-m)^2$ is called variance of x, usually written as, $Var(x)$ & σ^2

$$\therefore Var(x) = E(x-m)^2$$
$$= \sum_i p_i (x-m)^2$$

& $\sigma^2 = \sum p_i x_i^2 - m^2$

S.D of x is $\sigma(x)$, where, $\sigma(x) = \sqrt{Var(x)}$

(b) Given,

$$f(x) = \begin{cases} \frac{1}{4} & -2 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

obtain $P\{(2x+3) > 5\}$

$\Rightarrow$ $= P\{(2x+3) > 5\}$

$= P(2x > 2)$

$= P(x > 1)$

$= \int_1^2 f(x)\,dx = \int_1^2 \frac{1}{4}\,dx = \frac{1}{4}$.

Moment:

Let k be a positive integer. The moment of order k or the kth moment about X about a fixed pt. 'a' is defined to be mean value of $E\{(X-a)^k\}$.

$E\{|X-a|^k\}$ will be called the kth absolute moment of X about a.

The kth moment about the origin, which is often simply called the kth moment of X will be denoted by $\alpha_k(X)$

i.e $\alpha_k(X) = E(X^k)$

~~Cont random vari~~

Cont Prob. Distr.

If X is a cont. random variable, the no. of possible values which it can assume is uncountably infinite, and hence the prob func. can't be defined in the same manner as for discrete case.

We introduce, $f(x)$ s.t

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

The prob. that X lies btn. two specified values c&d,

$$P(c \leq X \leq d) = \int_c^d f(x)dx$$

The func. $f(x)$ is called prob. density func. (p.d.f) of cont. random variable X.

e.g

① check if the following a prob density func.?

$$f(x) = \begin{cases} 2x & 0 < x \leq 1 \\ 4-2x & 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ $f(x)$ must satisfy (i) & (ii).

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{\infty}$$

$$= 0 + \int_0^1 2x\,dx + \int_1^2 (4-2x)dx + 0$$

$$= 2$$

as (ii) is not satisfied, $f(x)$ is not p.d.f

E.g 3 coins are tossed. Find the prob. of

(i) 0 head
(ii) 1 head, 2 heads, 3 heads
(iii) More than 1 head
(iii) At least 1 head.

⇒ Denote the occurence of head as success.

$p =$ prob. of success i.e prob. of head in a single toss $= \frac{1}{2}$

$n =$ No. of trials $= 3$

Now, Prob of x successes is,

$$f(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$$= {}^{3}C_{x} \left(\tfrac{1}{2}\right)^{x} \left(\tfrac{1}{2}\right)^{3-x}$$

$$= {}^{3}C_{x} \left(\tfrac{1}{2}\right)^{3}$$

Now, putting values of 0, 1, 2

$f(0) \rightarrow$ prob. of 0 success
$f(1) \rightarrow$ " " 1 "

(ii) Prob. of more than 1 success

$= f(2) + f(3)$

(iii) Prob. of at least 1 success

$= 1 -$ Prob. 0 success
$= 1 - f(0)$

Q. Find mean & ~~variance~~ s.d of binomial distr. with parameters n & p.

⇒ P.M.F $f(x) = {}^nC_x \, p^x q^{n-x}$

mean $= E(x) = \sum_x x f(x)$

$= \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$

$= 0 * ({}^nC_0 \, p^0 q^n) + 1({}^nC_1 \, p q^{n-1}) + 2({}^nC_2 \, p^2 q^{n-2}) + \cdots$

$= npq^{n-1} + 2 \cdot \frac{n(n-1)}{1\times 2} p^2 q^{n-2} + \cdots$

$= npq^{n-1} + n(n-1) p^2 q^{n-2} + \cdots$

$= np\left[q^{n-1} + (n-1) p q^{n-2} + \frac{n(n-1)(n-2)}{1\times 2} p^2 q^{n-3} + \cdots + p^{n-1}\right]$

$= np\left[q^{n-1} + {}^{n-1}C_1 \, p q^{n-2} + {}^{n-1}C_2 \, p q^{n-3} + \cdots + {}^{n-1}C_{n-1} \, p^{n-1}\right]$

$= np(p+q)^{n-1}$

$= np \cdot 1 = np$

Again, $\sigma^2 = E(x^2) - \mu^2$

$E(x^2) = \sum x^2 f(x)$

$= \sum x(x-1) f(x) + \sum x f(x)$

$= \sum x(x-1) f(x) + \mu$.

The kth central moment $\mu_k(X)$ is given by

$$\mu_k = E\{(X-m)^k\}$$

we have,
$$\mu_0 = 1$$
$$\mu_1 = 0$$

If, $Y = ax+b$, $\mu_k(ax+b) = a^k \mu_k(X)$.

Binomial Distr.

Binomial distr. is a discrete prob. distr. & is defined by the p.m.f

$$f(x) = {}^nC_x\, p^x q^{n-x}$$

where p, q are +ve fractions $(p+q=1)$

Suppose that we have a series of n ind. trials in each of which the prob. of occurence of an event is fixed & constantly p. Then the prob. that the event occurs exactly r times in n trials is ${}^nC_r\, p^r q^{n-r}$, where $q = 1-p$ & r may assume any of the values $0, 1, --, n$.

In a series of n ind. trials, if prob. of succe in each trial is a constant p, & the prob. of failure is q.

Properties

1. Binomial distr. is a discrete distr.
2. Mean $= np$, variance $= npq$
   s.d. $\sigma = \sqrt{npq}$

Now, $\sum x(x-1) f(x)$

$= \sum_{x=0}^{n} x(x-1)\, {}^{n}C_{x}\, p^{x} q^{n-x}$

$= 0(-1)\{{}^{n}C_{0}\, p^{0} q^{n}\} + 1\times 0\{{}^{n}C_{1}\, p q^{n-1}\}$

$+ 2\times 1\{{}^{n}C_{2}\, p^{2} q^{n-2}\}$

$+ 3\times 2\{{}^{n}C_{3}\, p^{3} q^{n-3}\} + \cdots$

$= 0 + 0 + 2\, \frac{n(n-1)}{2}\, p^{2} q^{n-2} + (3\times 2)\, \frac{n(n-1)(n-2)}{1\times 2\times 3}\, p^{2} q^{n-2} + \cdots + n(n-1)\, 1\cdot p^{n}$

$= n(n-1)p^{2} q^{n-2} + n(n-1)(n-2)p^{3} q^{n-3} + \cdots + n(n-1)p^{n}$

$= n(n-1)p^{2}\{q^{n-2} + {}^{n-2}C_{1}\, p q^{n-3} + {}^{n-2}C_{r-2}\, p^{n-2}\}$

$= n(n-1)p^{2}(p+q)^{n-2}$

$= n(n-1)p^{2}$

$\therefore E(x^{2}) = n(n-1)p^{2} + \mu$

$\sigma^{2} = E(x^{2}) - \mu^{2}$

$= n(n-1)p^{2} + \mu - \mu^{2}$

$= n(n-1)p^{2} + np - (np)^{2}$

$= n^{2}p^{2} - np^{2} + np - n^{2}p^{2}$

$= np(1-p)$

$= npq$