Graph Theory - Lecture 6
Trees

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1 Basics

Definition 1.1 (Trees)**.** *A connected acyclic graph,* $\mathbf{G(V,E)}$ *is a* **Tree***.*

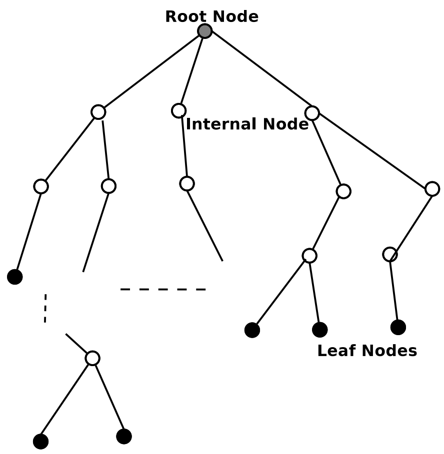


Figure 1: Tree

Properties :

1. Between any two vertices, $u, v \in \mathbf{G(V,E)}$, there exists a unique *path.*
2. The total number of edges is $|V(G)| - 1$
3. There are three types of vertices (also referred to as **Nodes**) in a tree
   (a) **Root Node :** This is the main node through which the tree can be accessed. For any given tree, there is only one single **Root Node**
   (b) **Leaf Node(s) :** These are the nodes beyond which the tree cannot be anymore accessed. They have degree 1.
   (c) **Internal Node(s) :** All the other nodes of the tree other than the above are the **internal nodes**

Theorem 1.1. *Every edge is a tree is a cut-edge/bridge i.e if any is removed, the tree becomes disconnected*

Proof. Consider any edge uv. If this edge is removed, vertex v is no longer reachable from u.

For if v is still reachable from u in the resulting graph, it must be a path of length greater than 1. The edge uv creates a cycle with this path, contradicting the fact that the graph is *acyclic*. In fact, the graph $\mathbf{G(V,E)}$ is disconnected in two components : $Comp_1$ = all vertices reachable from u; $Comp_2$ = all vertices reachable from v in the graph with the edge uv removed.

$Comp_1$, $Comp_2$ must be disjoint. For if there is a vertex which is reachable from both u, v, then u is reachable from v. A vertex must belong to either $Comp_1$, or $Comp_2$. In the original tree, there was a path from u to every other vertex $v_x \in V(G)$.

This path may or may not use the path using uv. □

Theorem 1.2. *If any two non-adjacent vertices, u, v of a tree are joined by an edge uv, exactly one cycle is connected*

Proof. As there was already a path between u and v, introduction of this edge creates a cycle.

If more than one cycle is created, then this new edge uv must be a member of all such cycles, the graph being acyclic. As these two cycles are different, there must be some difference among the edges other than uv between these two cycles.

Thus, there exists two different paths between u and v consisting of these two sets of remaining edges of the two cycles. □

Theorem 1.3. *Every tree must contain at least two vertices of degree 1*

Proof. Let $d_1, d_2, \ldots, d_{|V(G)|}$ denote the respective degrees of the $|V(G)|$ vertices of the tree $\mathbf{G(V,E)}$.

$$\therefore,\ d_1 + d_2 + \ldots + d_{|V(G)|} = 2 \times |E(G)| = 2(|V(G)| - 1)$$

.

Clearly, if every $d_i \geq 2$, then

$$d_1 + d_2 + \ldots + d_{|V(G)|} \geq 2|V(G)|$$

which is a contradiction.

Let k be the number of nodes whose degree is 1 and the remaining be greater than 2. Then,

$$\begin{aligned} d_1 + d_2 + \ldots + d_{|V(G)|} &\geq k + 2 \times (|V(G)| - k) \\ &= 2|V(G)| - k \\ &\leq 2(|V(G)| - 1) \end{aligned}$$

Hence, $\mathbf{k \geq 2}$

□

Lemma 1.4. *Every tree, $\mathbf{G(V,E)}$ with at least two vertices has at least two leaves. Deleting a leaf from from an $|V(G)|$-vertex tree produces a tree with $|V(G)| - 1$ vertices.*

Theorem 1.5. *For an $|V(G)|$-vertex graph, $\mathbf{G(V,E)}$ (with $|V(G)| \geq 1$, the following are equivalent (and characterize the trees with $|V(G)|$ vertices).*

(A) $\mathbf{G(V,E)}$ *is connected and has no cycles*

(B) $\mathbf{G(V,E)}$ *is connected and has $|V(G)| - 1$ edges.*

(C) $\mathbf{G(V,E)}$ *has $|V(G)| - 1$ edges and no cycles*

(D) For $u, v \in V(G)$, $\mathbf{G(V,E)}$ *has exactly one uv-path*

Proof. We first demonstrate A, B and C by proving that any two of {connected, acyclic, $|V(G)|-1$ edges} together imply the third.

Case 1. $A \Rightarrow \{BC\}$. We use induction for $|V(G)|$. For $|V(G)| = 1$, an acyclic 1-vertex graph has no edge. For $|V(G)| > 1$, we suppose that the implication holds gor graphs with fewer than $|V(G)|$. Given an acylic connected graph $\mathbf{G}(\mathbf{V}, \mathbf{E})$, Lemma 1.4 provides a leaf v and states that $G' = G - v$ also is acyclic and connected. Applying the induction hypothesis to G' yields $E(G') = |V(G)| - 2$. Since only one edge is incident to v, we have $E(G) = |V(G)| - 1$

Connected and No cycles/acyclic

Case 2. $B \Rightarrow \{AC\}$. Delete edges from cycles of $\mathbf{G}(\mathbf{V}, \mathbf{E})$ one by one until the rsulting grapg G' is acyclic. Since no edge of a cycle is a cut-edge, G' is connected. Case 1 implies that $E(G') = |V(G)|-1$. Since we are given $E(G) = |V(G)|-1$, no edges were deleted. Thus, $G' = G$, and G is acyclic.

Acyclic and $|V(G)| - 1$ edges

Case 3. $C \Rightarrow \{AB\}$. Let $G_1, G_2, \ldots, G_k$ be the components of $\mathbf{G}(\mathbf{V}, \mathbf{E})$. Since every vertex appears in one component, $\sum_i |V(G_i)| = |V(G)|$. Since $\mathbf{G}(\mathbf{V}, \mathbf{E})$ has no cycles, each component satisfies Property A. Thus, $|E(G_i)| = |V(G_i)| - 1$. Summing over i, $|E(G)| = \sum_i |V(G_i)| - 1 = |V(G)| - k$. We are given $E(G) = |V(G)| - 1$. so $k = 1$, and $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is connected.

Connected and acyclic

Case 4. $C \Rightarrow D$. Since $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is conncted, each pair of vertices is connected by a path. If some pair is connected by more than one, we choose a shorted (total length) pair P, Q of distinct paths with the same endpoints. By this extremal choice, no internal vertex of P or Q can belong to the other path (Figure 2). This implies that $P \bigcup Q$ is a cycle, which contradicts hypothesis A.

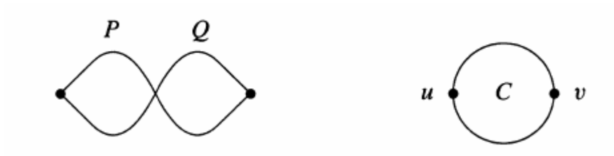


Figure 2:

Case 5. $D \Rightarrow A$. If there is u, v-path for every $u, v \in V(G)$, then $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is connected. If $\mathbf{G}(\mathbf{V}, \mathbf{E})$ has a cycle C_G, then $\mathbf{G}(\mathbf{V}, \mathbf{E})$ has two u, v-paths for two $u, v \in V(C_G)$; hence C_G is acyclic (this also forbids loops).

$\square$

Corollary 1.6.

(a) *Every edge of a tree is a cut-edge.*

(b) *Adding one edge to aree forms exactly one loop.*

(c) *Every connected graph contains a spanning tree.*

2 Spanning Trees in Graphs

Definition 2.1 (Spaning Tree)**.** *A spanning tree of a graph* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *is a tree* $T(V, E')$ *which is a spanning subgraph of* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *i.e. ,* $T(V, E')$ *has the same vertex set as* $\mathbf{G}(\mathbf{V},\mathbf{E})$

Proposition 2.1. *Let* $\tau(G)$ *denote the number of spanning trees of a graph* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *. If* $e \in E(G)$ *is not a loop, then* $\tau(G) = \tau(G-e) + \tau(G/e)$

Proof. The spanning trees of $\mathbf{G}(\mathbf{V},\mathbf{E})$ that omit e are precisely the spanning trees of $G - e$. To show that $\mathbf{G}(\mathbf{V},\mathbf{E})$ has $\tau(G/e)$ spanning trees containing e, we show that contraction of e, i.e. G/e [Note : See First Lecture] defiesn a bijection from the set of spanning trees of $\mathbf{G}(\mathbf{V},\mathbf{E})$ containing e tothe set of spanning trees of G/e.

When we contact e in a spanning tree that contains e, we obtain a spanning tree of G/e, because the resulting subgraph of G/e is spanning and connected and has the right number of edges. The other edges maintain their identity under contraction, so no two trees are mapped to the same spanning tree of G/e by this operation. Also, each spanning tree of G/e rises in this way, since expanding the new vertex back into e yields a spanning tree of $\mathbf{G}(\mathbf{V},\mathbf{E})$. Since each spanning tree of G/e arises exactly once, the function is bijection. $\square$

Remark :

- If $\mathbf{G}(\mathbf{V},\mathbf{E})$ is a connected loopless with no cycle of length at least 3, then $\tau(G)$ is the product of edge multiplicities.
- Since loops do not affect the number of spanning trees, we can delete loops as they arise.
- A computation by deleting or contracting every edge in a loopless graph $\mathbf{G}(\mathbf{V},\mathbf{E})$ **computes** as many as $2^{|E(G)|}$ terms. Thus the computation grows exponentially with the size of the graph, which is **impractical**.
- A disconnected graph has no spanning trees.

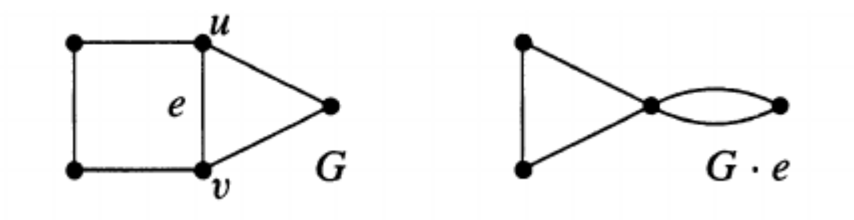


Figure 3: The graphs on the right each have 4 spanning trees, which implies (Proposition 2.1) that the kite has 8 spanning trees. **Without multiple edges, the computation would fail**

2.1 Kirchoff's Matrix-Tree Computation for undirected Graphs

The **Matrix Tree Theorem** computes $\tau(G)$ using a determinant because determinants of n-by-n matrices can be computed using fewer than n^3 operations.

Definition 2.2 (Laplacian Matrix)**.** *If* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *is a graph on* $|V(G)|$ *vertices with* $V = \{v_1, v_2, \ldots, v_{|V(G)|}\}$ *then its* **graph Laplacian** L *is an* $|V(G)| \times |V(G)|$ *matrix whose entries are*

$$L_{ij} = \begin{cases} Deg(v_j) & \ldots \quad \text{If } i = j \\ -1 & \ldots \quad \text{If } i \neq j \text{ and } e : (v_i, v_j) \in E(G) \\ 0 & \ldots \quad \text{Otherwise} \end{cases}$$

Equivalenty, $L = D - A$, where D is a diagonal matrix with $D_{jj} = deg(v_j)$ and A is the **adjacency matrix** *of the graph* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *.*

The Laplacian graph L for Kite graph in Figure 3 is

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

Theorem 2.1 (Kirchoff's Matrix-Tree Theorem, 1847)**.** *If* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *is a connected labelled graph with $|V(G)|$ vertices, then the number of spanning trees of* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *is given by*

$$\tau(G) = \frac{1}{|V(G)|} \lambda_1 \lambda_2 \ldots \lambda_{n-1} \tag{1}$$

where $\lambda_1 \lambda_2 \ldots \lambda_{n-1}$ are non-zero Eigen-values of its Laplacian Matrix

The above theorem can be interpreted as : *The number of spanning trees is equal to any cofactor of the Laplacian matrix of* $\mathbf{G}(\mathbf{V}, \mathbf{E})$

Proof. Let $\mathbf{G}(\mathbf{V}, \mathbf{E})$ has $|V(G)|$ vertices and $|E(G)|$ edges. Since $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is connected, it has at least $|V(G)| - 1$ edges.

Let M be the $|V(G)| \times |E(G)|$ matrix that results from changing the top-most 1 in each column to -1.

To prove the result, we require to prove two more claims

Claim (1) $\mathbf{MM^T = D - A = L}$

Proof. We can prove this by comparing the (i, j) entries of MM^T and L. By rules of Matrix Multiplication,

$$\begin{aligned} MM^T &= [[M]_{i,1}, [M]_{i,2}, \ldots, [M]_{i,|E(G)|}].[[M^T]_{1,j}, [M^T]_{2,j}, \ldots, [M^T]_{|E(G)|,j}] \\ &= [[M]_{i,1}, [M]_{i,2}, \ldots, [M]_{i,|E(G)|}].[[M]_{j,1}, [M]_{j,2}, \ldots, [M]_{j,|E(G)|}] \\ &= \sum_{r=1}^{|E(G)|} [M]_{i,r}[M]_{j,r} \end{aligned} \tag{2}$$

(a) If $i = j$, then this sum counts one for every nonzero entry in row i; that is, it counts the degree of v_i.

(b) If $i \neq j$ and $v_i v_j \notin E(G)$, then there is no column of M in which both the row i and j entries are nonzero. Hence the value of the sum in this case is 0.

(c) If $i = j$ and $v_i v_j \in E(G)$, then the only column in which both the row i and the row j entries are nonzero is the column that represents the edge $v_i v_j$. Since one of these entries is 1 and the other is -1, the value of the sum is -1

We have shown that the (i, j) entry of MM^T is the same as the (i, j) entry of L, and thus Claim 1 is proved. □

Claim (2) `Let` $H(V, E')$ `be a subgraph of` $\mathbf{G}(\mathbf{V}, \mathbf{E})$ `with` $V(G)$ `vertices and` $|V(G)|-1$ `edges. Let` p `be an arbitrary integer between 1 and` $|V(G)|$`, and let` M' `be the` $(|V(G)|-1)\times(|V(G)|-1)$ `submatrix of` M `formed by all rows of` M `except row` p `and the columns that correspond to the edges in` $H(V, E')$`.`

If H is a tree, then $|det(M)| = 1$. Otherwise, $det(M) = 0$.

Proof. First suppose that H is not a tree. Since H has $|V(G)|$ vertices and $|V(G)|1$ edges, $H(V, E')$ must be disconnected.

Let H_1 be a connected component that does not contain the vertex v_p. Let M'' be the $|V(H_1)| \times (|V(H) - 1)$ submatrix of M' formed by eliminating all rows other than the ones corresponding to vertices of H_1. Each column of M contains exactly two nonzero entries: 1 and 1. Therefore, the sum of all of the row vectors of M is the zero vector, so the rows of M'' are linearly dependent. Since these rows are also rows of M', we see that $det(M') = 0$.

Now suppose that H is a tree. Choose some leaf of H that is not v_p and call it u_1. Let us also say that e_1 is the edge of H that is incident with u_1. In the tree $H - u_1$, choose u_2 to be some leaf other than v_p. Let e_2 be the edge of $H - u_1$ incident with u_2. Keep removing leaves in this fashion until v_p is the only vertex left. Having established the list of vertices $u_1, u_2, \ldots, u_{n1}$, we now create a new $(|V(G)|-1) \times (|V(G)|-1)$ matrix M by rearranging the rows of M in the following way: row i of M^* will be the row of M that corresponds to the vertex u_i.

A useful property of the matrix M is that it is lower triangular (we know this because for each i, vertex u_i is not incident with any of $e_{i+1}, e_{i+2}, \ldots, e_{n1}$). Thus, the determinant of M is equal to the product of the main diagonal entries, which are either 1 or -1, since every u_i is incident with e_i. Thus, $|det(M)| = 1$. This proves Claim 2

□

Fact (1) If the row sums and column sums of a matrix are all 0, then the cofactors all have the same value.

MM^T satisfies the above condition. So we need to consider only one of its cofactors. Let $i = 1$ and $j = 1$. So the $(1, 1)$ cofactor of $D - A$ is

Cauchy-Binet Formula : If A is a $m \times n$ matrix and B is an $n \times m$ matrix where $m \leq n$, then

$$Det(AB) = \text{ sum of the principal } m \times m \text{ minors of } B^T A^T$$

Example 2.1. *Taking* $m = 2$ *and* $n = 3$, *and matrices* $A = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & -1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 \\ 3 & 1 \\ 0 & 2 \end{pmatrix}$, *the Cauchy–Binet formula gives the determinant*

$$\det(AB) = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}.$$

Indeed $AB = \begin{pmatrix} 4 & 6 \\ 6 & 2 \end{pmatrix}$ *and its determinant is* -28 *which equals* $-2 \times -2 + -3 \times 6 + -7 \times 2$ *from the right hand side of the formula*

$$\begin{aligned} \tau(G) &= det((DA)(1|1)) \\ &= det(MM^T(1|1)) \\ &= det(M_1 M_1^T) \end{aligned} \tag{3}$$

where M_1 is the matrix obtained by deleting the first row of $D - A$.

We have already seen (in Claim 2) that any $(|V(G)|1) \times (|V(G)|1)$ submatrix that corresponds to a spanning tree of $\mathbf{G}(\mathbf{V}, \mathbf{E})$ will contribute 1 to the sum, while all others contribute 0. This tells us that the value of $\tau(G) = det(D - A) = det(MM^T)$ is precisely the number of spanning trees of $\mathbf{G}(\mathbf{V}, \mathbf{E})$.

Thus we have proved the 2^{nd} statement of the Matrix Tree Theorem.

Fact (2) The product of all non-zero eigenvalues of the Laplacian L is equal to the sum of all principal minors of L. That is,

$$\lambda_1\lambda_2\ldots\lambda_{n-1} = \sum_{k=1}^{|V(G)|}[L]_{k,k} \tag{4}$$

where $[L]_{k,k}$ is the k^{th} principal minor of L.

Fact (3) Diagonal cofactors of an $|V(G)| \times |V(G)|$ square matrix are equal to corresponding principal minors. That is, (k,k) cofactor of

$$L = [L]_{k,k} \text{ for } 1 \leq k \leq n \tag{5}$$

where $[L]_{k,k}$ is the k^{th} principal minor of L.

This Fact shows the equivalence between this and the 1^{st} statement.

□

Theorem 2.2. *If* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *is a loopless graph,* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *and* L *is its graph Laplacian, then the number* $\tau(G)$ *of spanning trees contained in* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *, having distinct vertices is given by the following computation*

1. *Choose a vertex* v_j *and eliminate the* j^{th} *row and column from* L *to get a new matrix* $\hat{L_j}$
2. *Compute*

$$\tau(G) = Det(\hat{L_j}) \tag{6}$$

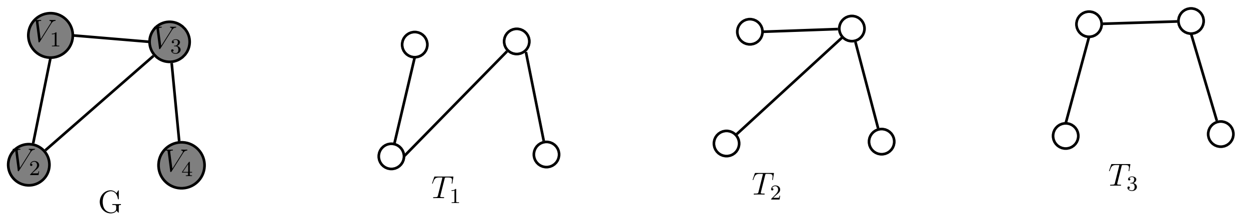


Figure 4: Note that although T_1 and T_3 are isomorphic, it is regarded as **different** spanning trees for the purposes of the Matrix-Tree Theorem

The number $\tau(G)$ in Eq 6 counts the spanning trees that are distinct **as subgraphs** of G. Thus, someof the trees that contibute to $\tau(G)$ may be isomorphic, e.g. in Fig 4.

2.2 Tutte's Matrix Tree Theorem for Directed Graphs (digraphs)

Definition 2.3. *A vertex* $v \in \mathbf{D}(\mathbf{V},\mathbf{E})$ *is a* **root** *if every other vertex is accessible from* v

Definition 2.4. *A* $\mathbf{D}(\mathbf{V},\mathbf{E})$ *is a* **directed tree** *or arborescence if*

(i) A digraph $\mathbf{D}(\mathbf{V},\mathbf{E})$ *contains a* **root**

(ii) The graph, $\mathbf{G}(\mathbf{V},\mathbf{E})$ *that one obtains by ignoring the directedness of the edges of* $\mathbf{D}(\mathbf{V},\mathbf{E})$ *is a* **tree**

An analogue of a spanning tree :

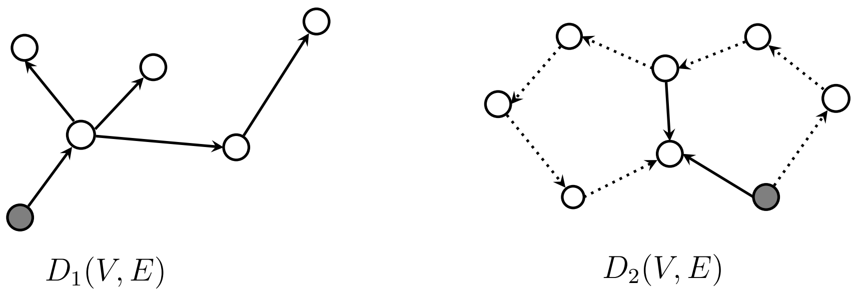


Figure 5: $D_1(V, E)$ is an arborescence with root vertex (*gray shade*).
$D_2(V, E)$ contains a spanning arborescence with root vertex (*gray shade*) and $-- >$ tree edges

Definition 2.5. *A subgraph* $\mathbf{T}(\mathbf{V}, \mathbf{E}')$ *of a digraph* $\mathbf{D}(\mathbf{V}, \mathbf{E})$ *is a* **spanning arborescence** *if* $\mathbf{T}(\mathbf{V}, \mathbf{E}')$ *is an arborescence that contains all the vertices of* $\mathbf{D}(\mathbf{V}, \mathbf{E})$ *.*

Theorem 2.3 (Tutte's Directed Matrix-Tree Theorem, 1948)**.** *If* $\mathbf{D}(\mathbf{V}, \mathbf{E})$ *is a digraph and* $L[n, n]$ *be its equivalent Laplacian Matrix whose entries are given by*

$$L_{ij} = \begin{cases} Deg_{in}(v_j) & \ldots \quad \textit{In-Degree of } v_j \textit{ if } i = j \\ -1 & \ldots \quad \textit{If } i \neq j \textit{ and } e : (v_i, v_j) \in E(D) \\ 0 & \ldots \quad \textit{Otherwise} \end{cases}$$

then the number N_j *of spanning arborescences with root at* v_j *is*

$$N_j = det(\hat{L})$$

where $\hat{L}$ *is the matrix by deleting the* j^{th} *row and column from* L

N_j counts the spanning arborescenes that are distinct as **subgraphs of** $\mathbf{D}(\mathbf{V}, \mathbf{E})$: equivalently, one regards the vertices as distinguishable. Thus, some of the arborescences that contribute to N_j may be isomorphic, but if they involve different edges, one can count them seperately.

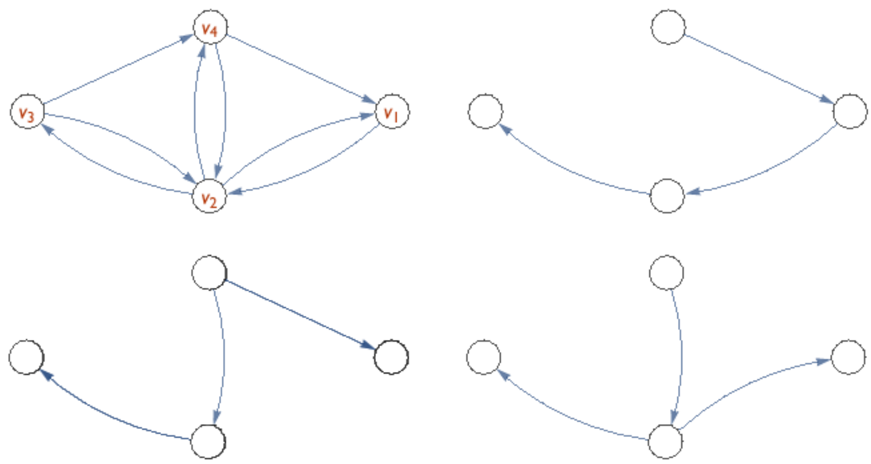


Figure 6: Three (3) spanning aborescences rooted at v_4

Example 2.2. *The Laplacian of the above graph is given as*

$$
\begin{aligned}
L = \;& D_{in} - A \\
= \;& \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \\
= \;& \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & -1 \\ -1 & -1 & 0 & 2 \end{bmatrix}
\end{aligned}
$$

j	$\hat{L_j}$	$det(\hat{L_j})$
1	$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$	2
2	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$	4
3	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$	7
4	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & -1 \end{bmatrix}$	3

Table 1: The number of spanning aborescences for the four possible roots in the graph in Figure 6

2.3 Generic Theory for Counting the Spanning Trees using Matrices

Theorem 2.4. Matrix Tree Theorem *Given a loopless graph,* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *with vertex set* $V = \{v_1, \ldots, v_n\}$*, let* $a_{i,j}$ *be the number of edges with endpoints* v_i*and* v_j*. Let* Q *be the matrix in which entry* (i, j) *is* $-a_{i,j}$ *when* $i \neq j$ *and is* $deg(v_i)$ *when* $i = j$*. If* Q^* *is a matrix obtained by deleting row* s *and column* t *of* Q*, then* $\tau(G) = (-1)^{s+t} det(Q^*)$*, where* $\tau(G)$ *is the number of spanning trees of* $\mathbf{G}(\mathbf{V}, \mathbf{E})$

Step (1) *If* D *is an orientation of* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *, and* M *is the incidence matrix of* D*, then* $Q = MM^T$*.*

With edges $e_i, \ldots, e_m$, the entries of M are $m_{i,j} = 1$ when v_i is the tail of e_j, $m_{i,j} = -1$ when v_i is the head of e_j, and $m_{i,j} = 0$ otherwise.

Entry i, j in MM^T is the dot product of rows i and j of M. When $i \neq j$, the product counts -1 for every edge of $\mathbf{G}(\mathbf{V}, \mathbf{E})$ joining the two vertices; when $i = j$, it counts 1 for every incident edge and yields the degree.

Step (2) *If B is and $(n-1) - by - (n-1)$ submatrix of M, then $|B| = \pm 1$.*

(a) For $n = 1$, by convention a 0×0 matrix has a determinant 1.

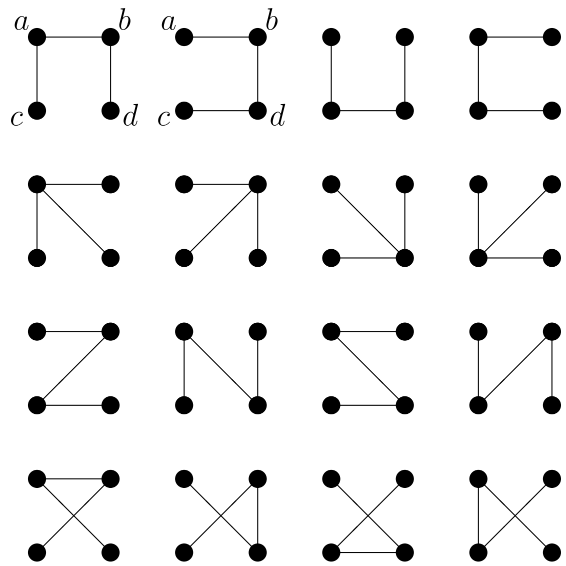
(b) For $n > 1$, let T be the spanning tree whose edges are the columns of B.
Since, T has at least two leaves and only one row is *deleted*, B has a row corresponding to a leaf $x \in T$.

We have shown that a tree must have at least one vertex of degree 1 and that it cannot have exactly one, so it must have at least two. □

3 Cayley's Formula

Using the above definitions, we can now begin to discuss Cayley's formula and its proofs. Cayley's Formula tells us how many different trees we can construct on n vertices. We can think about this process as beginning with n vertices and then placing edges to make a tree. Another way to think about it involves beginning with the complete graph on n vertices, K_n, and then removing edges in order to make a tree. Cayley's formula tells us how many different ways we can do this. These are called *spanning trees* on n vertices, and we will denote the set of these spanning trees by T_n.

The following is a diagram of all of elements of T_4:



Notice that the figures in each row are just rotations of the first one. Each of these graphs is distinct because each has a different set of adjacencies. For example, $a \sim c$ in the first graph above, but $a \nsim c$ in the second graph. Again, these graphs can be obtained by adding edges to 4 vertices or from taking edges away from K_4.

In its simplest form, Cayley's Formula says:

$$|T_n| = n^{n-2} \tag{1}$$

From our above example, we can see that $|T_4| = 16 = 4^2$. It is trivial that there is only one tree on 2 vertices (so $|T_2| = 1 = 2^0$). Also, the only possible tree type on 3 vertices is a 'V' and the 2 other trees are just rotations of that (so $|T_3| = 3 = 3^1$). We can see that Cayley's Formula holds for small n, but how can we prove that it is true for all n? We shall see how we can do this in different ways in the following sections.

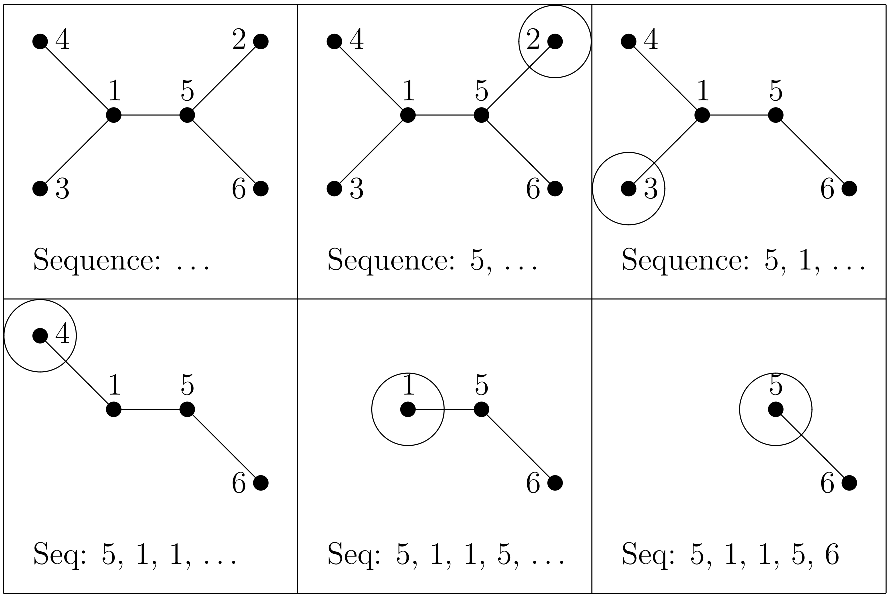
4 Prüfer Encoding

The most straight forward method of showing that a set has a certain number of elements is to find a bijection between that set and some other set with a known number of elements. In this case, we are going to find a bijection between the set of Prüfer sequences and the set of spanning trees.

A Prüfer sequence is a sequence of $n-2$ numbers, each being one of the numbers 1 through n. We should initially note that indeed there are n^{n-2} Prüfer sequences for any given n. The following is an algorithm that can be used to encode any tree into a Prüfer sequence:

1. Take any tree, $T \in T_n$, whose vertices are labeled from 1 to n in any manner.
2. Take the vertex with the smallest label whose degree is equal to 1, delete it from the tree and write down the value of its only neighbor. (Note: above we showed that any tree must have at least two vertices of degree 1.)
3. Repeat this process with the new, smaller tree. Continue until only one vertex remains.

This algorithm will give us a sequence of $n-1$ terms, but we know that the last term will always be the number n because even if initially $d(n) = 1$, there will always be another vertex of degree 1 with a smaller label. Since we already know the number of vertices on our graph by the length of our sequence, we can drop the last term as it is redundant. So now we have a sequence of $n-2$ elements encoded from our tree. Below is an example of encoding a tree on 6 vertices:



After encoding our tree, we end up with the sequence: 5, 1, 1, 5, 6; then we can drop the ending 6 and end with our Prüfer Sequence and denote it by P. $P = 5, 1, 1, 5$. So what should make us think that this is the only tree that gives us this sequence? First, we must notice that all of the vertices of degree 1 do not occur in P. With a little thought we can see that this is true for any tree, as the vertices of degree 1 will never be written down as the neighbors of other degree 1 vertices (except when vertex n is of degree 1, but this will never end up in our sequence). In fact, it follows from this that every vertex has degree equal to $1 + a$, where a is the number of times that vertex appears in our sequence.

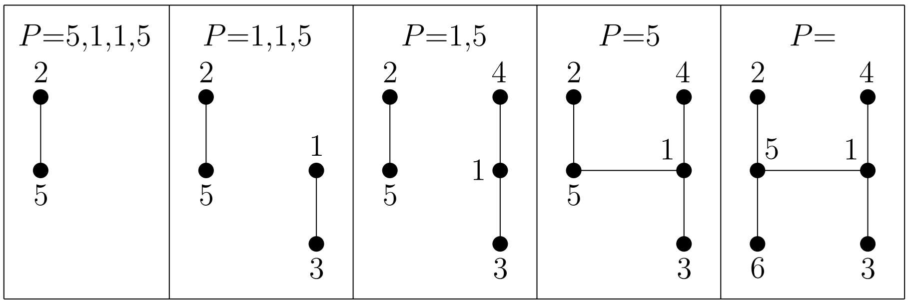
This way of analyzing a Prüfer Sequence provides us with a way of reconstructing an encoded tree. The algorithm goes as follows:

1. Find the smallest number from 1 to n that is not in the sequence P and attach the vertex with that number to the vertex with the first number in P. (We know that $n = 2 +$ number of elements in P.)

2. Remove the first number of P from the sequence. Repeat this process considering only the numbers whose vertices have not yet attained their correct degree.

3. Do this until there are no numbers left in P. Remember to attach the last number in P to vertex n.

Let's reconstruct our original tree from our sequence, $P = 5, 1, 1, 5$:



Following the above steps, we have now reconstructed our original tree on 6 vertices. It may be oriented differently, but all of the vertices are adjacent to their correct neighbors, and so we have the correct tree back. Since there were no ambiguities on how to encode the tree or decode the sequence, we can see that for every tree there is exactly one corresponding Prüfer Sequence, and for each Prüfer Sequence there is exactly one corresponding tree. More formally, the encoding function can be thought of as taking a member of the set of spanning trees on n vertices, T_n, to the set of Prüfer Sequences with $n-2$ terms, P_n. Decoding would then be the inverse of the encoding function, and we have seen that composing these two functions results in the identity map. If we let f be the encoding function, then the above statements can be summarized as follows:

$$f : T_n \longrightarrow P_n, \quad f^{-1} : P_n \longrightarrow T_n, \quad \text{and} \quad f^{-1} \circ f = Id.$$

Since we have found a bijective function between T_n and P_n, we know that they must have the same number of elements. We know that $|P_n| = n^{n-2}$, and so $|T_n| = n^{n-2}$.

5 A Forest of Trees

Another common way of proving something in mathematics is to prove something more general of which what you want to prove is a specific case. We can use this method to prove Cayley's formula as well. First, we must define what a forest is. A *forest* on n vertices is a graph that contains no cycles, but does not need to be connected like a tree. In fact, a forest can be thought of as a group of smaller trees, hence the name forest.