Graph Theory - Lecture 4
Connectedness

Chintan Kr Mandal
Department of Computer Science and Engineering
Jadavpur University, India

Equivalence Relation

Definition 0.1 (Equivalence Relation). *A relation $\sim$ on a set S is an* **equivalence relation** *if is:*

1. **Reflexive :** $a \sim a, \quad \forall a \in S$
2. **Symmetric :** $a \sim b \Rightarrow b \sim a, \quad \forall a, b \in S$
3. **Transitive :** $a \sim b$ *and* $b \sim c \Rightarrow a \sim c \quad \forall a, b, c \in S$

The main use of an equivalence relation on S is that it decomposes S into a collection of disjoint *equivalence classes*. That is, we can write

$$S = \bigcup_j S_j$$

, where $S_j \bigcap S_k = \phi$ if $j \neq k$ and $a \sim b$ iff $a, b \in S_j$ for some j.

1 Connectedness

Two vertices are said to be connected if one can get from one to the other by moving along the edges of the graph.

Definition 1.1 (Connectedness). *In a graph* $\mathbf{G(V, E)}$ *, two vertices,* v_0 *and* v_l *are said to be* **connected** *if there is a walk given by a sequence* $(v_0, v_1, \ldots, v_l)$*.*

Additionally, we say that a vertex is **connected** *to each other.*

(a) Connected Graph from a-b (b) Disconnected Graph

Figure 1: Connectedness of Graphs

Definition 1.2. *A graph in which each pair of vertices is connnected is a* **connected graph**

1.1 Connectedness in Undirected Graphs

The relation "is-connected-to" is an equivalence relation on the vertex set of a graph $\mathbf{G}(\mathbf{V},\mathbf{E})$.
Based on the three properties of the Equivalence relation :

1. **Reflexive :** This is *TRUE* by definition, as a vertex is **connected** to itself.
2. **Symmetric :** If there is walk from $v_x \in V(G)$ to $v_y \in V(G)$, one can simply reverse the corresponding sequence of edges to get a walk from v_y to v_x
3. **Transitive :** Suppose
   - $v_x \in V(G)$ is connected to $v_z \in V(G)$ through a walk corresponding to some vertex sequence $W(G, \{v_x, v_z\}) = (v_x = v_0, v_1, \ldots, v_{L_1-1}, v_{L_1} = v_z)$.
   - $v_z \in V(G)$ is connected to $v_y \in V(G)$ through a walk corresponding to some vertex sequence $W(G, \{v_z, v_y\}) = (v_z = v'_0, v'_1, \ldots, v'_{L_2-1}, v'_{L_2} = v_y)$.

   Thus, if wants to take a walk from v_x to v_y, one can do so taking a walk through $W(G, \{v_x, v_z\})$ and $W(G, \{v_z, v_y\})$ one after the other, whereas the sequence becomes

   $$W(G, \{v_x, v_y\}) = (v_x = v_0, v_1, \ldots, v_{L_1-1}, v_z, v'_1, \ldots, v'_{L_2-1}, v'_{L_2} = v_y)$$

   The above is also referred to as *concatenation of walks*

Definition 1.3. *In an undirected graph* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *, a* **connected component** *is an equivalence class under the relation "is-connected-to" on* $V(G)$*.*

Theorem 1.1. *A graph* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *is connected if and only if for each possible subdivision of its set of vertices into two groups (not necessarily equal), there is at least one edge whose two extremities are in the two different groups*

Proof. Suppose the graph, $\mathbf{G}(\mathbf{V},\mathbf{E})$ is connected and V_1, V_2 denote the division of set of vertices in the two groups. The number of ways in divinding the set for $|V(G)| = n$

$\lvert V_1\rvert$	$\lvert V_2\rvert$	No. of ways
1	$n-1$	C_1^n
2	$n-2$	C_2^n
$\vdots$	$\vdots$	$\vdots$

Therefore, the total number of ways to group them are $C_1^n + C_2^n + \ldots = 2^n - 1$.

Pick any arbitrary vertex $u \in V_1$ and another element $v \in V_2$. The graph, $\mathbf{G}(\mathbf{V},\mathbf{E})$ being connected, there must exist a path from u to v. On this path, let w be the first vertex that is from V_2 after u.

Fact 1. If there is a path from u to w and w to v, then there is a walk from u to v . . . . . . Transitive Relation

Fact 2. Let $\mathbf{G}(\mathbf{V},\mathbf{E})$ be a connected graph. Introduce a new vertex w and arbitrarily join w to some vertex $v \in V(G)$ by an edge. Then, $G \bigcup \{w\}$ is also a connected graph.

Conversely, suppose $\mathbf{G}(\mathbf{V},\mathbf{E})$ satisfies the given facts, we have to show that $\mathbf{G}(\mathbf{V},\mathbf{E})$ is connected. In other words, given any two vertices u, v, we have to find a path joining them.

Each step in Algorithm 1 increases the number of elements of $V(G_1)$ by 1 and decreases that of $G(V_2)$ by 1.

Algorithm 1 Connectivity of a $\mathbf{G}(\mathbf{V},\mathbf{E})$

Require: $\mathbf{G}(\mathbf{V},\mathbf{E})$ is divided into two groups, $V(G_1) = \{u\}$, $V(G_2) = V(G) - \{u\}$, $v \in V(G_2)$
Ensure: A path in $\mathbf{G}(\mathbf{V},\mathbf{E})$

while $w \neq v$ **do** ▷ The algorithm terminates until $v \in V(G_2)$ is not found
  Select an $e = uw$ such that $u \in V(G_1)$ and $w \in V(G_2)$
  $V(G_1) = V(G_1) \bigcup \{w\}$
  $V(G_2) = V(G_2) - \{w\}$
end while

Analysis of Algorithm 1

1. At the beginning, $V(G_2)$ has $(n-1)$ vertices. In the worst case, after $(n-2)$ iterations, $V(G_2)$ will contain just one element, which must be v.

   So, in the next iteration, i.e. in the $(n-1)^{th}$ iteration, an edge must be found which joins an element of $V(G_1)$ with u.

2. The algorithm collects at most $(n-1)$ edges but all these edges may not be part of the path.

3. All the edges selected *form* a connected subgraph. Hence in this subgraph, there is a path from u to v.

□

1.2 Connectedness in Directed Graphs

In a directed graph, $\mathbf{D}(\mathbf{V},\mathbf{E})$, the "is-connected-to" isn't an equivalence relation because it's not *symmetric*. This implies that even if there exists a path from some vertex $v_x \in V(D)$ to $v_y \in V(D)$, there is no guarantee of a walk from $v_y \in V(D)$ to $v_x \in V(D)$.

Definition 1.4. *In a directed graph,* $\mathbf{D}(\mathbf{V},\mathbf{E})$ *, a vertex* v_y *is said to be* **accessible** *or* **reachable** *from another vertex* v_x*, if* $\mathbf{D}(\mathbf{V},\mathbf{E})$ *contains a walk from* v_x *to* v_y*.*
Additionally, all vertices are accessible (or reachable) from themselves.

Definition 1.5 (Strongly Connected)**.** *Two vertices* $v_x \in D(V,E)$ *and* $v_y \in D(V,E)$*, if* v_x *is accessible from* v_y *and* v_y *is accessible from* v_x*.*
Additionally, a vertex is strongly connected to itself.

Note : It is easy to show that "is-strongly-connected-to" is an equivalence relation and so the vertex set decomposes into a disjoint union of *strongly connected components*.

Definition 1.6. *A directed graph* $\mathbf{D}(\mathbf{V},\mathbf{E})$ *is* **strongly connected** *if every pair of its vertices is strongly connected. Equivalently, a digraph is strongly connected if it contains exactly one strongly connected component.*

Definition 1.7 (Weakly Connected)**.** *A directed graph* $\mathbf{D}(\mathbf{V},\mathbf{E})$ *is* **weakly connected** *if, when one converts all its edges to undirected ones, it becomes a connected, undirected graph.*

Definition 1.8. *If* $\mathbf{D}(\mathbf{V},\mathbf{E})$ *is a directed multigraph, then* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *is the undirected multigraph produced by ignoring the directedness of the edges. If both the directed edges* (v_x, v_y) *and* (v_y, v_x) *are present in digraph,* $\mathbf{D}(\mathbf{V},\mathbf{E})$ *, then two parallel copies of the undirected edge* $\{v_x, v_y\}$ *appear in* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *.*

Note : In a digraph, a **king** is a vertex from which every vertex is reachable by a path of length at most 2.

Theorem 1.2 (Landau 1953)**.** *Every tournament graph has a king.*

Proof. Let v_x be a vertex with maximum out-degree in a tournament $T(G)$. We claim v_x is a king.

We will prove this claim by contradiction.

Otherwise, there is a vertex v_y can not reached by v_x in at most 2 steps. So v_y must reach v_x. If v_z can be reached by v_x, v_y must reach v_z as well. In particular, we have $d^+(y) > d^+(x)$. Contradiction to the choice of v_x. □

1.3 Finding Connectivity of Graphs

Proposition 1.1. *If a graph* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *having* $|V(G)|$ *vertices has more than* C_2^{n-1} *edges, then it must be connected.*

Proof. Suppose, we have divided the $|V(G)|$ vertices of $\mathbf{G}(\mathbf{V},\mathbf{E})$ into **two** groups of vertices V_p and V_q, such that $|V_p| + |V_q| = |V(G)|$. Clearly, any edge, $e \in E(G)$ of this graph must have either

1. both extremities of e is in V_p : Maximum number of edges are $C_2^{|V_p|}$
2. both extremities of e is in V_q : Maximum number of edges are $C_2^{|V_q|}$
3. one extremity of e lies in V_p and the other lies in V_q : The total number of edges is **greater than** : $C_2^{|V_p|} + C_2^{|V_q|}$

The exact values of $|V_p|, |V_q|$ are not known **but** $|V_p| + |V_q| = |V(G)|$ **holds**.

Thus, the maximum value of the expression $C_2^{|V_p|} + C_2^{|V_q|}$ is subject to the constraint $|V_p| + |V_q| = |V(G)|$.

Stating: $|V(G)| = n, |V_p| = p, |V_q| = q$

$$\begin{aligned} C_2^p + C_2^q &= \tfrac{p(p-1)+q(q-1)}{2} \\ &= \tfrac{(p+q)^2-2pq-(p+q)}{2} \\ &= \tfrac{n^2-n-2pq}{2} \\ &= \tfrac{n(n-1)-2p(n-p)}{2} \end{aligned} \tag{1}$$

Expression (1) is **maximum** when the quantity $2p(n-p)$ is **minimum**, i.e. when $p = 1$ or $p = n - 1$.

When $p = 1$

$$\begin{aligned} C_2^p + C_2^q &= \tfrac{n(n-1)-2p(n-p)}{2} \\ &\leq \tfrac{n^2-n-2(n-1)}{2} \ldots\ldots p = 1 \\ &= \tfrac{(n-2)(n-1)}{2} \end{aligned} \tag{2}$$

When $1 < p < (n-1)$

$$\begin{aligned} (n-p)p > &\ (n-1).1 = n-1 \\ \Rightarrow np - p^2 > &\ p^2 - 1 \\ \Rightarrow\ np - n > &\ p^2 - 1 \\ \Rightarrow\ n(p-1) > &\ (p-1)(p+1) \\ \Rightarrow\ n > &\ p+1 \\ \Rightarrow\ (n-1) > &\ p \ldots\ldots (p > 1) \end{aligned} \tag{3}$$

Suppose $n = 5$

p	$n-p$	*product*
1	4	4
2	3	6
3	2	6

Thus, the minimum value of the quantity pq subject to the restriction $P = q = n$ is $(n-1)$.

$$\begin{aligned} C_2^p + C_2^q &= \tfrac{p^2+q^2-(p+q)}{2} \\ &= \tfrac{n^2-n-2pq}{2} \\ \therefore \qquad &\leq \tfrac{n^2-n-2(n-1)}{2} \\ &= \tfrac{(n-1)(n-2)}{2} \\ &= C_2^{n-1} \end{aligned} \tag{4}$$

So, if the total number of edges of the graph is $C_2^{n-1}+1$ are more for every possible subdivision of its vertices into two groups, there is at least one edge whose two extremities are in **two** extremities are in the two different groups and hence the graph is connected.

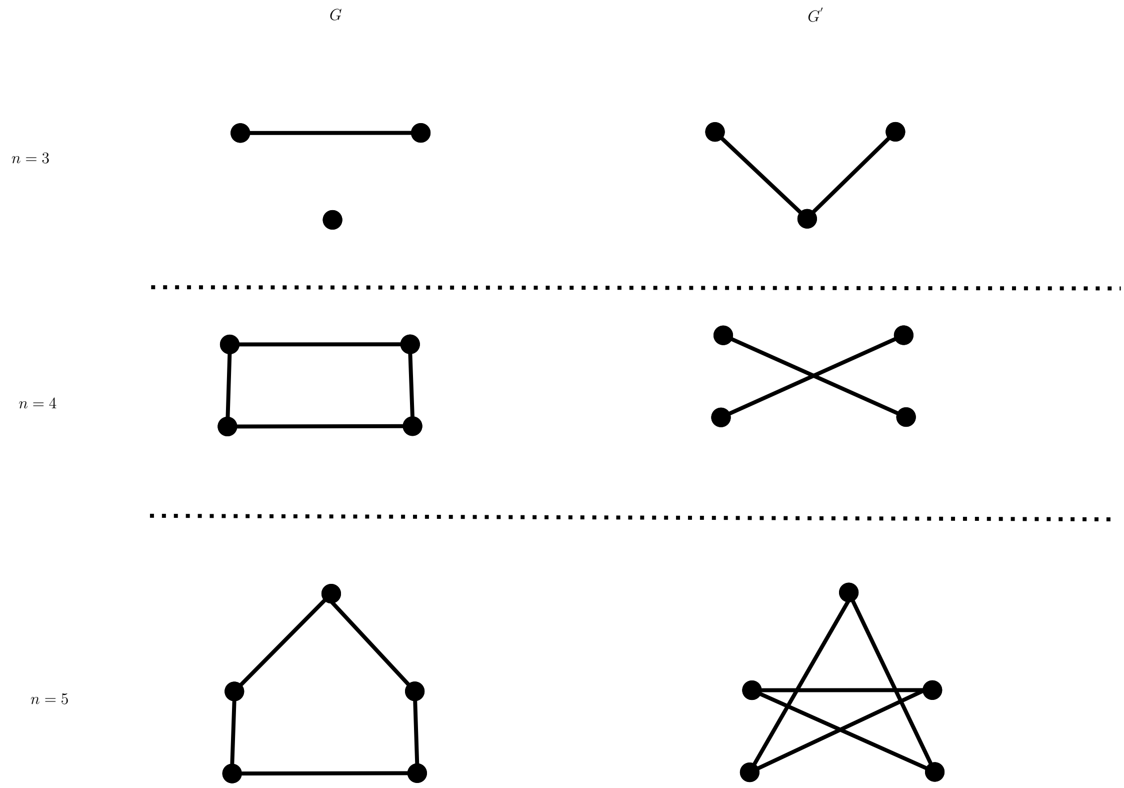


Figure 2: Counter Example

∎

Theorem 1.3. *Let* $\mathbf{G(V, E)}$ *be a graph with vertices labelled as* $v_1, v_2, \ldots, v_{|V(G)|}$ *and let* $[A]$ *be its corresponding adjacency matrix. For any positive integer* k*, the* $(i,j)^{th}$ *entry of* A^k *represents the number of* **walks** *from* v_i *to* v_j *that use exactly* $k-$*edges.*

Proof. For $k = 1$, the result is **true** as $[A] = 1$, when then is a **single edge walk** between v_i to v_j.

Now, suppose that, for every i and j, the $(i,j)^{th}$ entry for A^{k-1} is the number of walks from v_i to v_j that use exactly $(k-1)$ edges.

For each $k-$edge walk from v_i to v_j, there exists a vertex v_h such that the **walk** can be thought of as a $(k-1)$ edge walk from v_i to v_h combined with an edge from v_h to v_j.

The total number of these $k-$edge walks then is $\sum_{v \in V(G)} \#(k-1)$ edge from v_i to v_h.

By induction hypothesis, this sum can be sum can be rewritten as

$$\sum_{v_h \in V(G} [A^{k-1}]_{i.k} = \sum_{h=1}^{|V(G)|} [A^{k-1}]_{i,h}[A]_{h,j} = [A^k]_{i,j}$$

□

Note :

1. If a $\mathbf{G(V,E)}$ with $|V(G)|$ vertices is a connected graph, then the length of the longest possible path in $\mathbf{G(V,E)}$ is $|V(G)|-1$ that can be generated by $[A]^{|V(G)|-1}$.
2. Any entry (i,j) of $B[] = A + A^2 + \ldots + A^k \quad (k \geq 1)$ gives the number of walks less than equal to k
3. If any entry $S[i,j] \equiv 0$ of $S[] = A + A^2 + \ldots + A^{|V(G)|-1}$, then it is impossible to connect the pair (v_i, v_j) in the $|V(G)|^{th}$ step or more. **Thus, the graph is not connected.**

2 Components of a Graph

Definition 2.1 (Maximal subgraph)**.** *A* **Maximal subgraph** *is the largest possible subgraph such that one could not find another node anywhere in the graph such that it could be added to the subgraph and all the nodes in the subgraph would still be connected.*

Definition 2.2 (Connected Component)**.** *A* **connected component** *is a maximal subgraph in which all nodes are reachable from every other.*

A component (or graph) is trivial if it has *no edges*, otherwise it is non-trivial.
For directed graphs, there **strong components** and **weak components**.

- A **strong component** is a maximal subgraph in which there is a path from every point to every point following all the arcs in the direction they are pointing.
- A **weak component** is a maximal subgraph which would be connected if one ignores the direction of the arcs.

Remarks

- Components are pairwise disjoint i.e. no two components share a vertex.
- Adding an edge with endpoints in distinct components combines them into one component, thus decreasing the number ofcomponents by 1
- Deleting an edge increases the number of components by 0 or 1, considering that there exists no other path between any pair of vertices in the two components

Proposition 2.1. *Each graph,* $\mathbf{G(V,E)}$ *may be uniquely decomposed into its connected components*

Proof.

1. Select an arbitary vertex v_x

2. Select all vertices, $\forall\, v \in V(G)$ that are reachable from v_x.

3. Collect all edges whose both extremities are in $V(v_x)$ and let these edges be $E(v_x)$

Thus, $G(V(v_x), E(v_x))$ becomes a connected component.

It is maximal as vertex addition is not possible preseving the connectivity. Similarly, introduction of new edges is not possible preserving the connectivity.

If $V(G) - V(v_x)$ is not empty, select arbitrarily another element $v_y \in V(G) - V(v_x)$ and form the next connected component $G(V'(v_y), E'(v_y))$ just as above.

Clearly, this process must terminate after a finite number of steps, as each step reduces the number of available vertices for the next component by a finite amount. □

Proposition 2.2. *Every graph,* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *with* $|V(G)|$ *vertices and* $|E(G)|$ *edges has at least* $|V(G)| - |E(G)|$ *components.*

Proof. An $|V(G)|$-vertex graph with no edges has $|V(G)|$ components. As each addition of an edge reduces the number of components by at most 1, so when $|E(G)|$ edges have been added, the number of components is still at least $|V(G)| - |E(G)|$ □

Theorem 2.1. *If a graph* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *is disconnected, then its complement is connected*

Proof. Let us pick any two vertices $u, v \in V(G)$ from a disconnected graph $\mathbf{G}(\mathbf{V}, \mathbf{E})$. We need to show that there is a path between them in $\overline{G(V, E)}$.

First, if u, v are in different components of $\mathbf{G}(\mathbf{V}, \mathbf{E})$, then $uv \notin E(G)$, which means $uv \in E(\overline{G})$ and hence we have a path from u to v in $\overline{G(V, E)}$.

On the other hand, if u and v are in the same component of $\mathbf{G}(\mathbf{V}, \mathbf{E})$, then pick some other vertex w in a different component. Then, $uw, wv \notin E(G)$, meaning they are in $E(\overline{G})$. Thus, the path from u to w to v is a path in $E(\overline{G})$ from u to v. □

2.1 Cut-Edge and Cut vertex

Definition 2.3. *A* **cut-edge/cut-vertex** *of a graph* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *is an edge/vertex whose deletion increases the number of components.*

Definition 2.4 (Bridge)**.**

Re-defining "Induced Subgraphs"

An **Induced subgraph,** $H(U, F)$ is a subgraph obtained by deleting a set of vertices.

- When $H \subseteq V(G)$, the induced subgraph $H(U, F)$ consists of vertices $U(H) \subseteq V(G)$ and all edges whose endpoints are contained in $H(U, F)$.
- The full graph is itself an induced subgraph, as are individual vertices.
- A set S of vertices is an independent set iff the subgraph induced by it has no edges.

3 Vertex Connectivity and Edge Connectivity

3.1 Vertex Cut and Vertex connectivity

Definition 3.1 (Vertex Cut)**.** *A* **vertex cut** *in a graph* $\mathbf{G}(\mathbf{V}, \mathbf{E})$ *is a set of vertices* $S \subset V(G)$*, such that* $G - S$ *is disconnected*

A vertex cut of minimum cardinality in $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is called a **minimum vertex cut**

Definition 3.2 (Vertex Connectivity)**.** *For a graph,* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *, that is not complete, the vertex-connectivity (or simply connectivity),* $\boldsymbol{\kappa}(G)$ *is defined as the cardinality of a minimum vertex-cut of* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *.*

- If $\mathbf{G}(\mathbf{V},\mathbf{E})$ is not a complete graph, K_n, then $\mathbf{G}(\mathbf{V},\mathbf{E})$ contains two non-adjacent vertices.
- The removal of all vertices of $\mathbf{G}(\mathbf{V},\mathbf{E})$ except these two non-adjacent vertices producesa disconnected graph – every graph that is not *complete* has a vertex cut.
- The removal of any proper subset of a vertices of a complete graph results in another complete graph.
- If $S(V(G))$ is a minimum vertex cut in a non-complete graph $\mathbf{G}(\mathbf{V},\mathbf{E})$, then $G-S$ is disconnected and contains components $G_1, G_2, \ldots, G_k \quad k \geq 2$.
- Every vertex $v \in S(V(G))$ is adjacent to at least one vertex in G_i for each $i, i \leq k$; for otherwise $S - \{v\}$ is also a vertex-cut, which is impossible.
- If $G \cong K_n$, for some positive integer $|V(G)| = n$, then $\kappa(G) = n$
- In general, the connectivity $\kappa(G)$ of a graph is the minimum value of $|S(V(G))|$ among all subsets $S(V(G))$ such that $G-S$ is either disconnected or trivial

$$0 \leq \kappa(G) \leq n-1$$

- A graph is said to be k-connected if $\kappa(G) \geq k$. Thus, a k-connected graph is also l-connected for every integer l with $0 \leq l \leq k$. In general, a graph is k-connected iff the removal of fewer tha k vertices does not result in a disconnected or trivial graph.

e.g. In Figure 3, G_4, G_5, G_6, G_7 : 2 connected; G_6, G_7 : 3 connected

Note : $\mathbf{G}(\mathbf{V},\mathbf{E})$ is 1-connected iff $\mathbf{G}(\mathbf{V},\mathbf{E})$ is non-trivial and connected

Note : $\mathbf{G}(\mathbf{V},\mathbf{E})$ is 2-connected iff $\mathbf{G}(\mathbf{V},\mathbf{E})$ is non-seperable and hs order at least 3.

3.2 Edge Cut and Edge connectivity

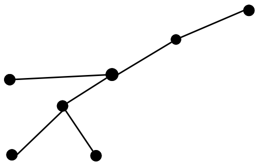
Definition 3.3 (Edge Cut)**.** *An* **Edge Cut** *in a non-trivial graph* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *is a set of* X *edges of* $\mathbf{G}(\mathbf{V},\mathbf{E})$ *, such that* $G-X$ *is disconnected.*

1. An **edge cut** X of a connected graph $\mathbf{G}(\mathbf{V},\mathbf{E})$ is *minimal* if no proper subset of X is an edge cut of $\mathbf{G}(\mathbf{V},\mathbf{E})$ such thta $G-X$ is disconnected.
2. An **edge cut** of minimum cardinality is called a minimum edge cut

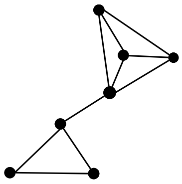
Note : While every edge cut is a minimal edge cut, the converse is **NOT TRUE**.

Example 3.1. *Consider the edge sets in Figure 4*

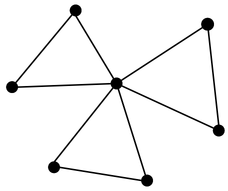
1. $X_1 = \{e_3, e_4, e_5\}$
2. $X_2 = \{e_1, e_2, e_6\}$
3. $X_3 = \{e_1, e_6\}$



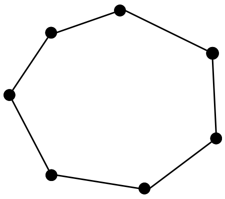
(a) $G_1 : \kappa(G_1) = 1$



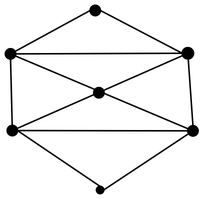
(b) $G_2 : \kappa(G_2) = 1$



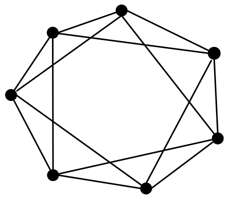
(c) $G_3 : \kappa(G_3) = 1$



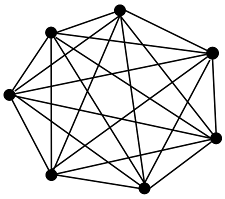
(d) $G_4 : \kappa(G_4) = 2$



(e) $G_5 : \kappa(G_5) = 2$



(f) $G_6 : \kappa(G_6) = 4$



(g) $G_7 : \kappa(G_7) = 6$

Figure 3: Connectivity of Graphs

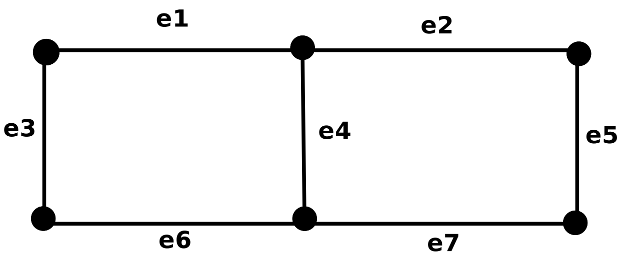


Figure 4: Edge Connectivity

X_1, X_3 *: Minimal Edge Cuts*

X_2 *: Not "Minimal Edge Cut" as* X_3 *is a proper subset of* X_2

X_3 *: Minimum Edge Cut*

Definition 3.4 (Edge Connectivity)**.** *The* **Edge Connectivity** *(*$\boldsymbol{\lambda}(G)$*) of a non-trivial graph* $\mathbf{G(V, E)}$ *is the cardinality of a minimum edge cut of* $\mathbf{G(V, E)}$ *where* $\lambda(K_1) = 0$

OR

The **Edge Connectivity** *(*$\boldsymbol{\lambda}(G)$*) is the minimum value of* $|X|$*, i.e the cardinality of the edge cut, among all subsets* X *of* $E(G)$ *such that* $G - X$ *is either a disconnected or trivial graph.*

For every graph $\mathbf{G(V, E)}$, $0 \leq \lambda(G) \leq |V(G)|$.

3.3 Relationship between $\kappa(G)$ and $\lambda(G)$

Theorem 3.1 (Hassler-Whitney)**.** *For every graph,* $\mathbf{G(V, E)}$

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

, where $\delta(G)$ *is the minimum degree of* $\mathbf{G(V, E)}$

Proof. If graph $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is disconnected or trivial, then $\kappa(G) = \lambda(G) = 0$ and the inequalities **holds**.

If $\mathbf{G}(\mathbf{V}, \mathbf{E}) \equiv K_n$ for some integer $n \geq 2$, then $\kappa(G) = \lambda(G) = \delta(G) = |V(K_n)| - 1 = n - 1$.

Case $\lambda(\mathbf{G}) \leq \delta(\mathbf{G})$: If the graph $\mathbf{G}(\mathbf{V}, \mathbf{E})$ is a connected graph of order $|V(G)| \geq 3$ and is also not connected. Hence $\lambda(G) \leq n - 2$.

Let $v \in V(G)$ with $deg(v) = \delta(G)$.

Since the set of $\delta(G)$ edges incident on v is an edge-cut of $\mathbf{G}(\mathbf{V}, \mathbf{E})$, it follows that

$$\lambda(G) \leq \delta(G) \leq n - 2$$

Case $\kappa(\mathbf{G}) \leq \lambda(\mathbf{G})$: Let $X \subset E(G)$ be the minimum edge cut of $\mathbf{G}(\mathbf{V}, \mathbf{E})$.

Then, $|X| = \lambda(G) \leq |V(G)| - 2$. Necessarily, $G - X$ contains exactly two components G_1 and G_2.

Let the order of $G_1 = k, \quad k \geq 1$. Therefore, the order of $G_2 = |V(G)| - k, \quad |V(G)| - k \geq 1$.

Consequently, every edge in X joins a vertex $v \in V(G_1)$ and a vertex $u \in V(G_2)$. We consider two cases :

Case 1 : Every vertex of G_1 is adjacent in $\mathbf{G}(\mathbf{V}, \mathbf{E})$ to every vertex of $\mathbf{G}(\mathbf{V}, \mathbf{E})$. Thus, $|X| = k(n - k)$.

Since, $(k - 1)(n - k - 1) \geq 0$, it follows that

$$(k - 1)(n - k - 1) = k(n - k) - n + 1 \geq 0$$

. Thus, $\lambda(G) = |X| = k(n - k) \geq n - 1$.

However, $\lambda(G) \leq n - 1$.

Therefore, this case does not exist.

Case 2 : There exists $u \in V(G_1)$ and $v \in V(G_2)$ such that u and v are not adjacent in $\mathbf{G}(\mathbf{V}, \mathbf{E})$.

We define a set $U \subset V(G)$. For each $e \in E(X)$, we select a vertex for U in the following way

If u is incident with e, then choose the other vertex in G_2 that is incident with e as an elemnt of U; otherwise select the vertex that is incident with e and belongs to G_1 as an element of U.

Then, $|U| \leq |X|$.

Since, $u, v \notin U$ and there is no $u - v$ path in $G - U$, it follows that $G - U$ is disconnected, and so U is a vertex cut.

Hence,

$$\kappa(G) \leq |U| \leq |X| \leq \lambda(G)$$

□

Note : We have introduced two measures of good connection : invulnerability to deletions and multiplicity of alternate paths

4 Connectivity using Matrices

Theorem 4.1. *Let* $\mathbf{G(V, E)}$ *be a graph with vertices labelled as* $V(G) : v_1, v_2, \ldots, v_n$ *and let* $[A]$ *be its corresponding adjancecy matrix.*

For any positive integer, k*, the* $(i, j)^{th}$ *entry of* $[A]^k$ *is equal to the number of walks from* $v_i - v_j$ *that uses exactly* <u>*k-edges.*</u>

By Induction. For $k = 1$, the result is true as $[A]_{ij} = 1$ when there is <u>**one edge**</u> walk between $v_i - v_j$.

Now, suppose that, for every edge $E_{ij} : \{v_i, v_j\}$, the $(i, j)^{th}$ entry for $[A]^{k-1}$ is the number of walks from $v_i - v_j$ that use exactly $(k - 1)$ edges.

For each k-edge walk from $v_i - v_j$, there exists a vertex v_h such that the walk can be thought of as a $(k - 1)$ edge walk from $v_i - v_h$ combined with an edge from $v_h - v_j$

The total number of these k-edge walks then is

$$\sum_{v \in V(G)} \#(k-1) \text{ edge walks from } v_i - v_h$$

Therefore, by induction hypothesis, this summmation can be re-written as

$$\sum_{v_h \in V(G)} [A]_{i,j}^{k-1} = \sum_{k=1}^{|V(G)|} [A]_{i,h}^{k-1} [A]_{h,j} = [A]_{i,j}^{k}$$

□

Notes :

1. If a graph $\mathbf{G(V, E)}$ with $|V(G)|$ vertices is a connected graph, then the length of the largest possible path in $\mathbf{G(V, E)}$ is $(|V(G) - 1|)$ that can be generated by $[A]^{|V(G)|-1}$.

2. Any entry $[B]_{i,j}$ of $B = A + A^2 + A^3 \ldots A^k$, $K > 1$ gives the number of walks less than or equal to k

3. If there is a zero(0) in the matrix $S = A + A^2 + A^3 \ldots A^{|V(G)|-1}$, then it is impossible to connect the pair of vertices in the $|V(G)|^{th}$ step or more.

   Thus, the graph $\mathbf{G(V, E)}$ is **NOT connected**

Note : Can we infer about anything on Euler graph from S ?