following very useful but rather technical theorem, we shall continue in Secs. 6 and 7 with an exploration of the implications of these ideas.

The theorem we have in mind—the *Schroeder-Bernstein theorem*—is the following: *if X and Y are two sets each of which is numerically equivalent to a subset of the other, then all of X is numerically equivalent to all of Y.* There are several proofs of this classic theorem, some of which are quite difficult. The very elegant proof we give is essentially due to Birkhoff and MacLane.

Now for the proof. We assume that $f:X \to Y$ is a one-to-one mapping of X into Y, and that $g:Y \to X$ is a one-to-one mapping of Y into X. Our task is to produce a mapping $F:X \to Y$ which is one-to-one *onto*. We may assume that neither f nor g is onto, since if f is, we can define F to be f, and if g is, we can define F to be g^{-1}. Since both f and g are one-to-one, it is permissible to use the mappings f^{-1} and g^{-1} as long as we clearly understand that f^{-1} is defined only on $f(X)$ and g^{-1} only on $g(Y)$. We obtain the mapping F by splitting both X and Y into subsets which we characterize in terms of the ancestry of their elements. Let x be an element of X. We apply g^{-1} to it (if we can) to get the element $g^{-1}(x)$ in Y. If $g^{-1}(x)$ exists, we call it the first ancestor of x. The element x itself we call the zeroth ancestor of x. We now apply f^{-1} to $g^{-1}(x)$ if we can, and if $(f^{-1}g^{-1})(x)$ exists, we call it the second ancestor of x. We now apply g^{-1} to $(f^{-1}g^{-1})(x)$ if we can, and if $(g^{-1}f^{-1}g^{-1})(x)$ exists, we call it the third ancestor of x. As we continue this process of tracing back the ancestry of x, it becomes apparent that there are three possibilities. (1) x has infinitely many ancestors. We denote by X_i the subset of X which consists of all elements with infinitely many ancestors. (2) x has an even number of ancestors; this means that x has a last ancestor (that is, one which itself has no first ancestor) in X. We denote by X_e the subset of X consisting of all elements with an even number of ancestors. (3) x has an odd number of ancestors; this means that x has a last ancestor in Y. We denote by X_o the subset of X which consists of all elements with an odd number of ancestors. The three sets X_i, X_e, X_o form a disjoint class whose union is X. We decompose Y in just the same way into three subsets Y_i, Y_e, Y_o. It is easy to see that f maps X_i onto Y_i and X_e onto Y_o, and that g^{-1} maps X_o onto Y_e; and we complete the proof by defining F in the following piecemeal manner:

$$F(x) = \begin{cases} f(x) & \text{if } x \in X_i \cup X_e, \\ g^{-1}(x) & \text{if } x \in X_o. \end{cases}$$

We attempt to illustrate these ideas in Fig. 12. Here we present two replicas of the situation: on the left, X and Y are represented by the vertical lines, and f and g by the lines slanting down to the right and