CHAPTER-6

Solutions of Non-homogeneous Linear Partial Differential Equations with Constant Coefficients

6.1 Introduction

In the previous chapter, we have discussed the methods of solution of homogeneous linear partial differential equations with constant coefficients. In this chapter, we discuss the methods of solution of non-homogeneous linear partial differential equations with constant coefficients because such type of equations are frequently encountered in our study.

6.2 Non-homogeneous Linear Partial Differential Equations with Constant Coefficients

A linear partial differential equation with constant coefficients is known as **non-homogeneous linear partial differential equation with constant coefficients** if the orders of all the partial derivatives involved in the equation are not equal, e.g

The equations $(D^2 - D'^2 + D - D')z = 0$, $(D^2 + DD' + D' - 1)z = 0$, $r + 2s + t + 2p + 2q + z = 0$ and $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} + \frac{\partial z}{\partial x} + z = xy$ are non-homogenous linear partial differential equations with constant coefficients.

6.3 Reducible and Irreducible Linear Differential Operators

A differentiable operator $f(D, D')$ is known as a **reducible linear differentiable operator**, if it can be written as the product of linear factors of the form $aD + bD' + c$ with a, b and c as constants.

A differentiable operator $f(D, D')$ is known as **irreducible linear differentiable operator**, if it is not reducible.

For example, the differential operator $D^2 - D'^2$ which can be written in the form $(D + D')(D - D')$ is reducible, whereas the operator $D^2 - D'^3$ which cannot be decomposed into linear factors is irreducible.

6.4 Reducible and Irreducible Linear Partial Differential Equations with Constant Coefficients

A linear partial differential equation with constant coefficients $f(D, D')z = F(x, y)$ is called as the **reducible linear partial differential equation,** if $f(D, D')$ is reducible.

Again, a linear partial differential equation with constant coefficients $f(D, D')z = F(x, y)$ is called as the **irreducible linear partial differential equation** if $f(D, D')$ is irreducible.

For example, $(D^2 - D'^2)z = x^2y^3$ is a reducible partial differential equation with constant coefficients, since $D^2 - D'^2 = (D + D')(D - D')$, whereas $(D^2 - D'^3)z = x^2y^3$ is an irreducible partial differential equation with constant coefficients, since $D^2 - D'^3$ cannot be decomposed into linear factors.

Further, the equation $\left(D^2 - {D'}^2 + 2D + 1\right)z = x^2 + xy$ is a reducible partial differential equation, since
$D^2 - {D'}^2 + 2D + 1 = (D + D')\,(D - D') + (D + D')(D - D' + 1)$
$= (D + D')\ (D - D' + 1) + (D - D' + 1) = (D + D' + 1)(D - D' + 1)$, while the equation $(DD' + D'^3)z = \cos(x + 2y)$ is irreducible, since $DD' + D'^2 = D'(D + D'^2)$, where $D + D'^2$ can not be decomposed into linear factors.

6.5 Important Theorem on Linear Factors of $f(D, D')$

If $f(D, D')$ is a reducible differential operator, then the order in which the linear factors of $f(D, D')$ occur is unimportant

Proof. In order to prove the theorem, we must show that

$$(a_r D + b_r D' + c_r)(a_s D + b_s D' + c_s)$$
$$= (a_s D + b_s D' + c_s)(a_r D + b_r D' + c_r) \qquad \ldots(1)$$

where $a_r D + b_r D' + c_r$ and $a_s D + b_s D' + c_s$ are linear factors of $f(D, D')$.

In general, the reducible differential operator $f(D, D')$ can be decomposed in n linear factors and hence can be written as

$$f(D, D') = \prod_{r=1}^{n}(a_r D + b_r D' + c_r) \qquad \ldots(2)$$

Therefore, the proof of (1) is immediate, since both sides of it are equal to $a_r a_s D^2 + (a_s b_r + a_r b_s)DD' + b_r b_s {D'}^2$

$$+(c_s a_r + c_r a_s)D + (c_s b_r + c_r b_s)D' + c_s c_r$$

Note: We are interested only in finding C.F. of reducible and irreducible non-homogeneous linear partial differential equations, because the methods of finding P.I. in both cases, are same as the methods of finding P.I. of homogeneous linear partial differential equations.

6.6 Complementary Function (C.F.) of Reducible Non-homogeneous Linear Partial Differential Equation with Constant Coefficients Given by $f(D, D')z = 0$

Let $$f(D, D')z = 0 \qquad \ldots(1)$$

be the reducible non-homogeneous linear partial differential equation with constant coefficients.

Also, let $f(D, D') = (b_1D - a_1D' - c_1)(b_2D - a_2D' - c_2)$

$$\ldots (b_nD - a_nD' - c_n) \qquad \ldots(2)$$

where a's, b's and c's are constants. Then (1) becomes

$$(b_1D - a_1D' - c_1)(b_2D - a_2D' - c_2) \ldots (b_nD - a_nD' - c_n)z = 0 \ldots(3)$$

It shows that any solution of the equation of the form

$(b_rD - a_rD' - c_r)z = 0, r = 1,2, \ldots n$ is a solution of (3) i.e.,

$$b_rD_z - a_rD'z - c_rz = 0 \quad \text{or} \quad b_rp - a_rq = c_rz \qquad \ldots(4)$$

which is of Lagrange's form $\quad Pp + Qq = R.$

∴ Lagrange's auxiliary equations for (4) are

$$\frac{dx}{b_r} = \frac{dy}{-a_r} = \frac{dz}{c_rz} \qquad \ldots(5)$$

Proceeding as usual, two independent integrals of (5) are

$$b_ry + a_rx = c_1$$

and
$$z = \begin{cases} c_2 e^{(c_r/b_r)x}, & \text{if} \quad b_r \neq 0 \\ c'_2 e^{-(c_r/a_r)y}, & \text{if} \quad a_r \neq 0 \end{cases}$$

∴ The general solution of equation (4) is given by

$$z = \begin{cases} e^{(c_r/b_r)x}\phi_r(b_r y + a_r x), & \text{if } b_r \neq 0 \\ e^{-(c_r/a_r)y}\psi_r(b_r y + a_r x), & \text{if } a_r \neq 0 \end{cases} \qquad \ldots(6)$$

where ϕ_r and ψ_r are arbitrary functions.

The general solution of (3) is the sum of the solutions of the equations of the form (4) corresponding to each factor in (2).

Case of Repeated Factors: While factorizing $f(D, D')$ into linear factors, it may possible that a number of its factors are repeated. Let the repeated factors of $f(D, D')$, repeated two times be $bD - aD' - c$. Now, let us we consider the partial differential equation

$$(bD - aD' - c)(bD - aD' - c)z = 0 \qquad \ldots(8)$$

Again, let us take $$(bD - aD' - c)z = v \qquad \ldots(9)$$

Then (8) reduces to $$(bD - aD' - c)v = 0 \qquad \ldots(10)$$

As before, the general solution of equation (10) is given by

$$v = e^{(c/b)x}\phi(by + ax), \text{ if } \quad b \neq 0 \qquad \ldots(11)$$

or
$$v = e^{-(c/a)y}\psi(by + ax), \text{ if } \quad a \neq 0 \qquad \ldots(12)$$

where ϕ and ψ are arbitrary functions.

Substituting the value of v from (11) in (9), we have

$$(bD - aD' - c)z = e^{(c/b)x}\phi(by + ax)$$

or $$bp - aq = cz + e^{(c/b)x}\phi(by + ax) \qquad \ldots(13)$$

which is of Lagrange's form $Pp + Qq = R$.

$\therefore$ Lagrange's auxiliary equations for (13) are

$$\frac{dx}{b} = \frac{dy}{-a} = \frac{dz}{cz + e^{(c/b)x}\phi(by + ax)} \qquad \ldots(14)$$

Taking the first two fractions of (14), we get

$adx + bdy = 0$ so that $by + ax = \lambda$, say $\qquad \ldots(15)$

where λ is an arbitrary constant.

Taking the first and the third fractions of (14), we get

$$\frac{dz}{dx} - \frac{c}{b}z = \frac{1}{b}e^{(c/b)x}\phi(by + ax)$$

or $$\frac{dz}{dx} - \frac{c}{b}z = \frac{1}{b}e^{(c/b)x}\phi(\lambda), \text{ on using (15)}$$

This is a linear ordinary differential equation, whose integrating factor (I.F.) is given by I.F. $= e^{-\int (c/b)dx} = e^{-(c/b)x}$

and therefore, its solution is given by

$$ze^{-(c/b)x} = \int \frac{1}{b}e^{(c/b)x}\phi(\lambda)e^{-(c/b)x}dx$$

i.e., $$ze^{-(c/b)x} - (x/b)\phi(\lambda) = \mu$$

which on using (15), can be written as

$$ze^{-(c/b)x} - (x/b)\phi(by + ax) = \mu \qquad \ldots(16)$$

where μ is an arbitrary constant.

From (15) and (16), the general solution of (13) or (8) is

$$ze^{-(c/b)x} - (x/b)\phi(by + ax) = \phi_1(by + ax)$$

or $z = e^{-(c/b)x}[\phi_1(by + ax) + x\phi_2(by + ax)]$, if $b \neq 0$ ...(17)

where ϕ_1 and ϕ_2 are arbitrary functions.

Again, substituting the value of v from (12) in (9), we obtain

$$z = e^{-(c/a)x}[\psi_1(by + ax) + y\psi_2(by + ax)], \text{ if } a \neq 0 \quad \text{...(18)}$$

where ψ_1 and ψ_2 are arbitrary functions.

In general, if $(bD - aD' - c)$ is repeated r times, then we get

$$z = e^{-(c/b)x} \sum_{i=1}^{r} x^{i-1}\, \phi_i(by + ax), \text{ if } b = 0 \quad \text{...(19)}$$

or $$z = e^{-(c/a)y} \sum_{i=1}^{r} y^{i-1}\, \psi_i(by + ax), \text{ if } a \neq 0 \quad \text{...(20)}$$

6.7 Working Rule for Finding C.F. of Reducible Non-homogeneous Linear Partial Differential Equation with Constant Coefficients

Let the given reducible non-homogeneous linear partial differential equation be $f(D, D') = F(x, y)$.

First of all, we factorize $f(D, D')$ into linear factors. Then, we use the following rules to have the complementary function (C.F.) of $f(D, D')z = F(x, y)$.

Rule I: Corresponding to each non-repeated factor$(bD - aD' - c)$ the part of C.F. is taken as $e^{(cx/b)}\,\phi(by + ax)$, if $b \neq 0$.

We now have the following three particular cases of Rule I:

Rule I(A): Take $c = 0$ in rule I. Then, corresponding to each linear factor $(bD - aD')$, the part of C.F. is $\phi(by + ax)$, $b \neq 0$.

Rule I(B): Take $a = 0$ in rule I. Then, corresponding to each linear factor $(bD - c)$, the part of C.F. is $e^{(cx/b)}\phi(by), b \neq 0$.

Rule I(C): Take $a = c = 0$ and $b = 1$ in rule I. Then, corresponding to linear factor D, the part of C.F. is $\phi(y)$.

Rule II: Corresponding to a repeated factor $(bD - aD' - c)^r$, the part of C.F. is taken as $e^{(cx/b)}[\phi_1(by + ax) + x\phi_2(by + ax)$

$$+x^2\phi_3(by + ax) + \ldots + x^{r-1}\phi_r(by + ax)], \text{ if } b \neq 0$$

We now have the following three particular cases of Rule II.

Rule II(A): Take $c = 0$ in rule II. Then, corresponding to each repeated factor $(bD - aD')^r$, the part of C.F. is

$$\phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + \cdots$$
$$+x^{r-1}\phi_r(by + ax), \; b \neq 0$$

Rule II(B): Take $a = 0$ in rule II. Then, corresponding to a repeated factor $(bD - c)^r$, the part of C.F. is

$$e^{(cx/b)}[\phi_1(by) + x\phi_2(by) + x^2\phi_3(by) + \ldots + x^{r-1}\phi_r(by)], b \neq 0$$

Rule II(C): Take $a = c = 0$ and $b = 1$ in rule II. Then, correspon-ding to a repeated factor D^r, the part of C.F. is

$$\phi_1(y) + x\phi_2(y) + x^2\phi_3(y) + \ldots + x^{r-1}\phi_r(y)$$

Rule III: Corresponding to each non-repeated linear factor $(bD - aD' - c)$, the part of C.F. is taken as

$$e^{-(cy/a)}\phi(by + ax), \text{ if } a \neq 0$$

We now have the following three particular cases of Rule III:

Rule III(A). Take $c = 0$ in rule III. Then, corresponding to each linear factor $(bD - aD')$, the part of C.F. is $\phi(by + ax), a \neq 0$.

Rule III(B). Take $b = 0$ in rule III. Then, corresponding to each linear factor $(aD' + c)$, the part of C.F. is $e^{-(cy/a)}\phi(ax), a \neq 0$.

Rule III(C). Take $b = c = 0$ and $a = 1$ in rule III. Then, corresp-onding to linear factor D', the part of C.F. is $\phi(x)$.

Rule IV. Corresponding to a repeated factor $(bD - aD' - c)^r$, the part of C.F. is taken as $e^{-(cy/a)}[\phi_1(by + ax) + y\phi_2(by + ax)$

$$+y^2\phi_3(by + ax) + \ldots + y^{r-1}\phi_r(by + ax)], \text{ if } a \neq 0$$

We now have the following three particular cases of Rule IV.

Rule IV(A). Take $c = 0$ in rule IV. Then, corresponding to repeated factor $(bD - aD')^r$, the part of C.F. is

$$\phi_1(by + ax) + y\phi_2(by + ax) + y^2\phi_3(by + ax) + \cdots$$

$$+y^{r-1}\phi_r(by+ax),\ a \neq 0.$$

Rule IV(B). Take $b = 0$ in rule IV. Then, corresponding to a repeated factor $(aD' + c)^r$, the part of C.F. is

$$e^{-(cy/a)}[\phi_1(ax) + y\phi_2(ax) + y^2\phi_3(ax) + \ldots + y^{r-1}\phi_r(ax)],\ a \neq 0$$

Rule IV(C). Take $b = c = 0$ and $a = 1$ in rule IV. Then, corresponding to repeated factor D'^r, the part of C.F. is

$$\phi_1(x) + y\phi_2(x) + y^2\phi_3(x) + \ldots + y^{r-1}\phi_r(x)$$

SOLVED EXAMPLES

Example 1. Solve $(D^2 - D'^2 + D - D')z = 0$.

Solution. The given partial differential equation can be written as

$$(D - D')(D + D' + 1)z = 0 \qquad \ldots(1)$$

Here, we see that equation (1) is a **reducible non-homogeneous linear partial differential equation**.

The part of complementary function C.F. corresponding to the factor $(D - D')$ is $\phi_1(y + x)$.

Again, the part of complementary function C.F. corresponding to the factor $(D + D' + 1)$ is $e^{-x}\phi_2(y - x)$.

Therefore, the general solution of (1) is given by

$$z = \phi_1(y + x) + e^{-x}\phi_2(y - x) \qquad \ldots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Example 2. Solve $(D^2 - a^2D'^2 + 2abD + 2a^2bD')z = 0.$

Solution. The given partial differential equation can be written as

$$[(D + aD')(D - aD') + 2ab(D + aD')]z = 0$$

or $$(D + aD')(D - aD' + 2ab)z = 0 \qquad \ldots(1)$$

Here, we see that equation (1) and hence the given partial differential equation is a **reducible non-homogeneous linear partial differential equation.**

The part complementary function of C.F. corresponding to the factor $(D + aD')$ is $\phi_1(y - ax)$.

Again, the part of complementary function C.F. corresponding to the factor $(D - aD' + 2ab)$ is $e^{-2abx}\phi_2(y + ax)$

Hence, the general solution of (1) is given by

$$z = \phi_1(y - ax) + e^{-2abx}\phi_2(y + ax) \qquad \ldots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Example 3. Solve $r + 2s + t + 2p + 2q + z = 0.$

Solution. The given partial differential equation can be written as

$$\left(\frac{\partial^2 z}{\partial x^2}\right) + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\frac{\partial^2 z}{\partial y^2}\right) + 2\left(\frac{\partial z}{\partial x}\right) + 2\left(\frac{\partial z}{\partial y}\right) + z = 0$$

or $$(D^2 + 2DD' + D'^2 + 2D + 2D' + 1)z = 0$$

or $$[(D + D')^2 + 2(D + D') + 1]z = 0$$

or $$(D + D' + 1)^2 z = 0 \qquad \dots(1)$$

Here, we see that equation (1) is a **reducible non-homogeneous linear partial differential equation** and there are two repeated linear factors $(D + D' + 1)$.

$\therefore$ The required general solution is given by

$$z = e^{-x}[\phi_1(y - x) + x\phi_2(y - x)] \qquad \dots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

EXERCISE 6(A)

Solve the following partial differential equations:

1. $(D - D' + 1)(D + 2D' - 3)z = 0$
2. $(DD' + aD + bD' + ab)z = 0$
3. $r + 2s + t + 2p + 2q + z = 0$
4. $(D + 1)(D + D' - 1)z = 0$
5. $(D^2 - D'^2 + D - D')z = 0$
6. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$
7. $(D^2 - DD' + D' - 1)z = 0$
8. $(D^2 + DD' + D' - 1)z = 0$

ANSWERS

1. $z = e^{-x}\phi_1(y + x) + e^{2x}\phi_2(y - 2x)$
2. $z = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x)$
3. $z = e^{-x}\{\phi_1(y - x) + x\phi_2(y - x)\}$
4. $z = e^{-x}\phi_1(y) + e^{x}\phi_2(y - x)$
5. $z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$

6. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$
7. $z = e^{x}\phi_1(y) + e^{-x}\phi_2(y + x)$
8. $z = e^{-x}\phi_1(y) + e^{x}\phi_2(y - x)$

6.8 C.F. of Irreducible Non-homogeneous Linear Partial Differential Equation with Constant Coefficients Given by $\boldsymbol{f(D, D')z = F(x, y)}$

Let $$f(D, D')z = F(x, y) \qquad \ldots(1)$$

be the irreducible non-homogeneous linear partial differential equation.

When the operator $f(D, D')$ in (1) is irreducible, it is not always possible to find a solution with the full number of arbitrary functions, but it is possible to construct solutions which contain as many arbitrary constants as we wish. We now state and prove a theorem which will be used to find C.F. of (1).

Theorem. If $f(D, D')$ is an irreducible differential operator, then

$$f(D, D')e^{ax+by} = f(a, b)e^{ax+by}$$

Proof. Since $f(D, D')$ is an irreducible differential operator, therefore, it consists of terms of the form $c_{rs}D^{r}D'^{s}$.

Also $D^{r}(e^{ax+by}) = a^{r}e^{ax+by}$, $D'^{s}(e^{ax+by}) = b^{s}e^{ax+by}$

$\therefore$ We have $(c_{rs}D^{r}D'^{s})(e^{ax+by}) = c_{rs}a^{r}b^{s}e^{ax+by}$

The theorem follows by combining the terms of the differential operator $f(D, D')$.

We now discuss the method of finding C.F. of (1).

Let us consider the PDE $\qquad f(D, D')z = 0 \qquad$ ...(2)

To find C.F. of above, we assume a trail solution of the form

$$z = ce^{ax+by}$$

$\therefore \qquad Dz = cae^{ax+by}, D'z = cb^{ax+by}, DD'z = cabe^{ax+by} \qquad$ ...(3)

Putting the values from (2) & (3) in (1) , we get

$cf(a, b)e^{ax+by} = 0 \Rightarrow f(a, b) = 0$, since c and $e^{ax+by} \neq 0$

This means that a and b are connected by the relation

$$f(a, b) = 0 \qquad \text{...(4)}$$

Let $f(a, b)$ is of degree r in b. Then, we solve $f(a, b) = 0$ for various values of b. Let $f_1(a), f_2(b), \ldots, f_r(a)$ be the roots of $f(a, b) = 0$.

$\therefore\ z = c_i e^{ax+by} + f_i(a)y$, $1 \leq i \ \leq r$ (5) are the solutions of (2).

Thus, we can construct the solution of (2) containing as many arbitrary constants as we need. The series (5) may not be finite, but if it is infinite, it is necessary that it should be uniformly convergent, if it has to be a solution of (2).

The general solution of an irreducible non-homogeneous partial differential equation (1) can be put in the following form:

$$C.F. = \sum_{i=1}^{r} c_i e^{ax+f_i(a)y}$$

where $c_i, a\ and\ f_i(a)$ are arbitrary constants such that $f(a, b) = 0$.

6.9 Working Rules for C.F. of Irreducible Non-homogeneous Linear Partial Differential Equation with Constant Coefficients of the Form $\boldsymbol{f(D, D')z = 0}$

The following steps are to be kept in mind, while finding the C.F. of an irreducible non-homogeneous linear partial differential equation. $f(D, D')\ z = 0$:

Step 1. If necessary, factorize $f(D, D')$ into factors $f_1(D, D')$ and $f_2(D, D')$, where $f_1(D, D')$ consists of a product of linear factors in D and D' and $f_2(D, D')$ consists of a product of irreducible factors in D and D'.

Step 2. Write down the part of C.F. corresponding to the factor of $f_1(D, D')$.

Step 3. Write down the part of C.F. corresponding to the factor of $f_2(D, D')$.

Step 4. Adding the C.F. corresponding to $f_1(D, D')$ obtained in step 2 and the C.F. corresponding to $f_2(D, D')$ obtained in step 3, we obtain the C.F. of the given partial differential equation $f(D, D')z = 0$ i.e., $f_1(D, D')f_2(D, D')z = 0$.

SOLVED EXAMPLES

Example 1. Solve $(D - D'^2)z = 0$.

Solution. Here, the differential operator $D - D'^2$ is not a linear factor in D and D'. Therefore, the given equation is an **irreducible non-homogeneous linear partial differential equation.**

Let $z = ce^{ax+by}$ be a trial solution of the given PDE.

Then, we have $Dz = ca\, e^{ax+by}$ and $D'^2 z = cb^2 e^{ax+by}$.

Putting these values in the given differential equation, we get

$$c(a - b^2)e^{ax+by} = 0 \quad \text{so that} \quad a - b^2 = 0 \ \text{ or } \ a = b^2$$

Replacing a by b^2 , the most general solution of the given partial differential equation is given by

$$z = \sum c_i\, e^{b_i{}^2 x + b_i y}$$

where c_i and b_i are arbitrary constants.

Example 2. Solve $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0.$

Solution. Since the factor $(D - 2D' - 1)$ is linear in D and D', therefore, the part of C.F. corresponding to it is $e^x \phi(y + 2x)$, where ϕ is an arbitrary function.

Again, to find C.F. corresponding to the non-linear factor $(D - 2D'^2 - 1)$, we proceed as follows:

Let $z = ce^{ax+by}$ ...(1)

be a trial solution of $(D - 2D'^2 - 1)z = 0$ ...(2)

$\therefore$ $Dz = cae^{ax+by}, D'z = cbe^{ax+by}$ and $D'^2 z = cb^2 e^{ax+by}$.

Thus, from the partial differential equation (2), we get

$$c(a - 2b^2 - 1)e^{ax+by} = 0$$

This will be true if and only if, we have

$$a - 2b^2 - 1 = 0 \qquad \text{or} \qquad a = 2b^2 + 1 \qquad \text{...(3)}$$

Replacing a by $2b^2 + 1$ in (2), the solution of (1) i.e. the part of C.F. corresponding to $\left(D - 2{D'}^2 - 1\right)$ is given by

$$\sum c_i e^{(2b_i{}^2 + 1)x + b_i y}$$

where c_i and b_i are arbitrary constants.

∴ The required solution of the given PDE is

$$z = e^x \phi(y + 2x) + \sum c_i e^{(2b_i{}^2+1)x+b_i y} \qquad \ldots(4)$$

Example 3. Solve $(2D^4 - 3D^2D' + D'^2)z = 0$**.**

Solution. The given partial differential equation can be written as

$$(2D^2 - D')(D^2 - D')z = 0 \qquad \ldots(1)$$

Here, we see that the equation (1) is an **irreducible non-homogeneous linear partial differential equation with constant coefficients.**

Let $z = ce^{ax+by}$ be a trail solution of $(2D^2 - D')z = 0$.

∴ $(2a^2 - b)ce^{ax+by} = 0$ so that $2a^2 - b = 0$ or $b = 2a^2$

Hence, C.F. corresponding to the factor $(2D^2 - D')$ is

$$\text{C.F.} = \sum c_i\, e^{a_i x + 2a_i{}^2 y} \qquad \ldots(2)$$

Again, let $z = e^{a'x+b'y}$ be a solution of $(D^2 - D')z = 0$.

∴ $(a'^2 - b')c'e^{a'x+b'y} = 0$ so that $a'^2 - b' = 0$ or $b' = a'^2$

Hence, C.F. corresponding to the factor $(D^2 - D')$ is

$$\text{C.F.} = \sum c_i\, e^{a_i{}'x + a_i{}'^2 y} \qquad \ldots(3)$$

From (2) and (3), the general solution of (1) is given by

$$z = \sum c_i e^{a_i x + 2a_i^2 y} + \sum_i c'_i e^{a_i x + 2a_i^2 y} \qquad \ldots(4)$$

where c_i, a_i, c'_i and a'_i are arbitrary constants.

Example 4. Solve $(D + 2D' - 3)(D^2 + D')z = 0$.

Solution. We see that the given partial differential equation is an irreducible non-homogeneous linear partial differential equation

C.F. corresponding to the linear factor $(D + 2D' - 3)$ is

$$\text{C.F.} = e^{3x}\phi(y - 2x) \qquad \ldots(1)$$

We now find C.F. corresponding to the irreducible factor $(D^2 + D')$ as follows:

Let $z = ce^{ax+by}$ be a solution of $(D^2 + D')z = 0$.

$\therefore \quad (a^2 + b)ce^{ax+by} = 0$ so that $a^2 + b = 0$ or $b = -a^2$

Hence, C.F. corresponding to the factor $(D^2 + D')$ is

$$\text{C.F.} = \sum c_i \, e^{a_i x - a_i{}^2 y} \qquad \ldots(2)$$

Therefore, the general solution of given equation is

$$z = e^{3x}\phi(y - 2x) + \sum c_i \, e^{a_i x - a_i^2 y} \qquad \ldots(3)$$

where ϕ is an arbitrary function and c_i, a_i are arbitrary constants.

EXERCISE 6(B)

Solve the following partial differential equations:

1. $(D^2 + D + D')z = 0.$

2. $(2D^2 - D'^2 + D)z = 0.$

3. $(D' + 3D)^2(D^2 + 5D + D')z = 0.$

4. $(2D - 3D' + 7)^2(D^2 + 3D')z = 0.$

ANSWERS

1. $z = \sum c_i e^{a_i - (a_i^2 + a_i)y}$, where c_i and a_i are arbitrary constants.

2. $z = \sum c_i e^{a_i x + b_i y}$, where $2a_i^2 - b_i^2 + a_i = 0$ and c_i, a_i, b_i are arbitrary constants.

3. $z = \phi_1(3y - x) + x\phi_2(3y - x) + \sum c_i e^{a_i x - (a_i^2 + 5a_i)y}$, where ϕ_1, ϕ_2 are arbitrary function and c_i, a_i are arbitrary constants.

4. $z = e^{-(7x/2)}\{\phi_1(2y + 3x) + x\phi_2(2y + x)\} + \sum c_i e^{a_i x - (a_i^2 y)/3}$, where ϕ_1, ϕ_2 are arbitrary functions and c_i, a_i are arbitrary constants.

6.10 Particular Integral (P.I.) of Non-homogeneous Linear Partial Differential Equation $f(D, D')z = F(x, y)$ whether It is Reducible or Irreducible

Consider the following non-homogeneous linear partial differential equation $$f(D, D')z = F(x, y) \qquad \ldots(1)$$

The methods of finding particular integrals of the non-homogeneous partial differential equations are very similar to those of ordinary linear differential equations with constant coefficients.

We give below a list of some cases of finding P.I. of (1).

Case I. When $\boldsymbol{F(x,y) = e^{ax+by}}$ and $\boldsymbol{f(a,b) \neq 0}$.

In this case, the particular integral (P.I.) is given by

$$\text{P.I.} = \frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$$

Thus, in this case, replacing D by a and D' by b, we get the required particular integral (P.I.).

SOLVED EXAMPLES

Example 1. Solve $(DD' + aD + bD')z = e^{mx+ny}$.

Solution. The given partial differential equation can be written as

$$(D + b)(D' + a)z = e^{mx+ny} \qquad \dots(1)$$

$$\therefore \qquad \text{C.F.} = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x) \qquad \dots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

and $$\text{P.I.} = \frac{1}{(D+b)(D'+a)} e^{mx+ny} = \frac{1}{(m+b)(n+a)} e^{mx+ny} \qquad \dots(3)$$

Hence, the required general solution of the given equation is

$$z = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x) + [(m+b)(n+a)]^{-1} e^{mx+ny}$$

Example 2. Solve $(D^2 - D'^2 + D - D')z = e^{2x+3y}$.

Solution. The given partial differential equation can be written as

$$[(D - D')(D + D') + (D - D')]z = e^{2x+3y}$$

or $$(D - D')(D + D' + 1)z = e^{2x+3y} \quad \ldots(1)$$

$\therefore$ $$\text{C.F.} = \phi_1(y + x) + e^{-x}\phi_2(y - x) \quad \ldots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

and $$\text{P.I.} = \frac{1}{(D-D')(D+D'+1)} e^{2x+3y}$$

or $$\text{P.I.} = \frac{1}{(2-3)(2+3+1)} e^{2x+3y} = -\frac{1}{6} e^{2x+3y} \quad \ldots(3)$$

Hence, the required general solution of the given equation is

$$z = \phi_1(y + x) + e^{-x}\phi_2(y - x) - \frac{1}{6} e^{2x+3y} \quad \ldots(4)$$

Example 3. Solve $\left(\frac{\partial^2 y}{\partial x^2}\right) - \left(\frac{\partial^2 y}{\partial z^2}\right) = y + e^{x+z}$.

Solution. The given partial differential equation can be written as

$$(D^2 - D'^2 - 1)y = e^{x+z} \quad \ldots(1)$$

$\therefore$ $$\text{C.F.} = \sum c_i\, e^{a_i x + b_i z}\ , \quad \text{where } a_i^2 - b_i^2 - 1 = 0 \quad \ldots(2)$$

and $$\text{P.I.} = \frac{1}{D^2 - D'^2 - 1} e^{x+z} = \frac{1}{1-1-1} e^{x+z} = -e^{x+z} \quad \ldots(3)$$

$\therefore$ The required general solution of (1) is given by

$$y = \sum c_i\, e^{a_i x + b_i z} - e^{x+z} \quad \ldots(4)$$

where c_i, a_i and b_i are arbitrary constants and a_i and b_i are connected by relation $a_i^2 - b_i^2 - 1 = 0$.

EXERCISE 6(C)

Solve the following partial differential equations:

1. $(D^2 - DD' - 2D)z = e^{2x+y}$
2. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y}$
3. $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y}$
4. $(D^2 - D' + 4)z = e^{4x-y}$
5. $(D^2 + D'^2 - 3D')z = e^{x+2y}$

ANSWERS

1. $z = \phi_1(y) + e^{2x}\phi_2(y + x) - \frac{1}{2}e^{2x+y}$
2. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - \frac{1}{10}e^{2x+3y}$
3. $z = e^{-2x}\phi_1(y + x) + e^{x}\phi_2(y - x) - \frac{1}{4}e^{x-y}$
4. $z = \sum c_i\, e^{a_i x - (a_i^2+4)y} + \frac{1}{19}e^{4x-y}$, where c_i and a_i are arbitrary constants.
5. $z = \sum c_i\, e^{(b_i^2+3b_i)^{1/2}x + ky} - \frac{1}{19}e^{x+2y}$ where c_i and a_i are arbitrary constants.

Case II. When $F(x, y) = \sin(ax + by)$ or $\cos(ax + by)$

In this case, the particular integral (P.I.) is given by

$$\text{P.I.} = \frac{1}{f(D,D')}\sin(ax + by) \text{ or } \text{P.I.} = \frac{1}{f(D,D')}\cos(ax + by)$$

which is evaluated by putting $D^2 = -a^2, D'^2 = -b^2, DD' = -ab$, provided the denominator is non-zero.

SOLVED EXAMPLES

Example 1. Solve $(D^2 + DD' + D' - 1)z = \sin(x + 2y)$.

Solution. The given partial differential equation can be written as

$$(D + 1)(D + D' - 1)z = \sin(x + 2y) \qquad \ldots(1)$$

$$\therefore \qquad \text{C.F.} = e^{-x}\phi_1(y) + e^{x}\phi_2(y - x) \qquad \ldots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2+DD'+D'-1}\sin(x + 2y) = \frac{1}{-1^2-(1.2)+D'-1}\sin(x + 2y)$$

$$= \frac{1}{D'-4}\sin(x + 2y) = (D' + 4)\frac{1}{D'^2-16}\sin(x + 2y)$$

$$= (D' + 4)\frac{1}{-2^2-16}\sin(x + 2y) = -\frac{1}{20}(D' + 4)\sin(x + 2y)$$

$$= -\frac{1}{20}[D'\sin(x + 2y) + 4\sin(x + 2y)]$$

$$= -\frac{1}{20}[2\cos(x + 2y) + 4\sin(x + 2y)] \qquad \ldots(3)$$

$\therefore$ The required solution of the given PDE is

$$z = e^{-x}\phi_1(y) + e^{x}\phi_2(y - x)$$

$$-\frac{1}{10}[\cos(x + 2y) + 2\sin(x + 2y)]$$

Example 2. Solve $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$.

Solution. Here, $C.F. = e^{x}\phi_1(y + x) + e^{2x}\phi_2(y + x) \qquad \ldots(1)$

where ϕ_1 and ϕ_2 are arbitrary functions.

and P.I. $= \frac{1}{(D-D'-1)(D-D'-2)}\sin(2x+3y)$

$$= \frac{1}{D^2-2DD'+D'^2-3D+3D'+2}\sin(2x+3y)$$

$$= \frac{1}{-2^2+2\times(2\times3)-3^2-3D+3D'+2}\sin(2x+3y)$$

$$= \frac{1}{-3D+3D'+1}\sin(2x+3y) = D\frac{1}{-3D^2+3DD'+D}\sin(2x+3y)$$

$$= D\frac{1}{-3\times(-2^2)+3\times(2\times3)+D}\sin(2x+3y) = D\frac{1}{D-6}\sin(2x+3y)$$

$$= D(D+6)\frac{1}{D^2-36}\sin(2x+3y) = (D^2+6D)\frac{1}{-2^2-36}\sin(3x+2y)$$

$$= -\frac{1}{40}[D^2\sin(2x+3y) + 6D\sin(2x+3y)]$$

$$= -\frac{1}{40}[-4\sin(2x+3y) + 12\cos(2x+3y)] \qquad \ldots(2)$$

∴ The general solution of the given partial differential equation is

$$z = e^x\phi_1(y+x) + e^{2x}\phi_2(y+x)$$

$$+\frac{1}{10}[\sin(2x+3y) - 3\cos(2x+3y)]$$

Example 3. Solve $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x+y)$.

Solution. The given partial differential equation can be written as

$$(D+D')(D-2D'+2)z = \sin(2x+y) \qquad \ldots(1)$$

$\therefore \qquad \text{C.F.} = \phi_1(y+x) + e^{-2x}\phi_2(y+2x) \qquad \ldots(2)$

where ϕ_1 and ϕ_2 are arbitrary functions.

and P.I. $= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x+y)$

$= \frac{1}{-2^2 + (2\times1) - 2\times(-1^2) + 2D + 2D'} \sin(2x+y)$

$= \frac{1}{2(D+D')} \sin(2x+y) = \frac{D-D'}{2} \frac{1}{D^2 - D'^2} \sin(2x+y)$

$= \frac{D-D'}{2} \frac{1}{-2^2 - (-1^2)} \sin(2x+y) = -\frac{1}{6}(D-D') \sin(2x+y)$

$= -\frac{1}{6}\{2\cos(2x+y) - \cos(2x+y)\} \qquad \ldots(3)$

$\therefore$ The general solution of the given partial differential equation is

$$z = \phi_1(y-x) + e^{-2x}\phi_2(y+2x) - \frac{1}{6}\cos(2x+y)$$

EXERCISE 6(D)

Solve the following partial differential equations:

1. $(2DD' + D'^2 - 3D')z = 3\cos(3x - 2y)$

2. $(D^2 - D')(D - D - D'^2)z = \sin(2x+y)$

3. $2\left(\frac{\partial^2 z}{\partial x^2}\right) + \left(\frac{\partial^2 z}{\partial y^2}\right) - 3\left(\frac{\partial z}{\partial y}\right) = 5\cos(3x - 2y)$

ANSWERS

1. $z = \phi_1(x) + e^{3x/2}\phi_2(2y - x) + \frac{3}{50}\{4\cos(3x - 2y)$

$$+3\sin(3x-2y)\}$$

2. $z = \sum c_i\, e^{a_i x - a_i^2 y} + \sum c'_i\, e^{\left(b'_i + b'^2_i\right)x + b_i y}$

$$-\frac{1}{10}\{5\sin(2x+y) - 3\cos(2x+y)\}$$

where c_i, a_i, c'_i, b'_i are arbitrary constants.

3. $z = \phi_1(x) + e^{3x/2}\phi_2(2y-x)$

$$+\frac{1}{10}[4\cos(3x+2y) + 3\sin(3x-2y)]$$

Case III. When $\boldsymbol{F(x,y) = x^m y^n}$

In this case, the particular integral (P.I.) is given by

$$\text{P.I.} = \frac{1}{f(D,D')}x^m y^n = [f(D,D')]^{-1}x^m y^n$$

which is evaluated by expanding $[fF(D,D')]^{-1}$ in ascending powers of D'/D or D/D' or D or D' as the case may be. In practice, we shall expand in ascending powers of D'/D. However, if we expand in ascending powers of D/D', we shall get a P.I. of apparently different form. In this connection, remember that both forms of P.I. are correct because any one can be transformed into other with the help of C.F. of the given equation.

SOLVED EXAMPLES

Example 1. Solve $s + p - q = z + xy$.

Solution. The given partial differential equation can be written as

$$\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\frac{\partial z}{\partial x}\right) - \left(\frac{\partial z}{\partial y}\right) - z = xy \text{ or } (DD' + D - D' - 1)z = xy$$

or $$(D-1)(D'+1)z = xy \qquad \ldots(1)$$

$\therefore$ $$\text{C.F.} = e^x \phi_1(y) + e^{-y} \phi_2(x) \qquad \ldots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

and $$\text{P.I.} = \frac{1}{(D-1)(D'+1)} xy = -\frac{1}{(1-D)(1+D')} xy$$

$$= -(1-D)^{-1}(1+D')^{-1} xy$$

$$= -(1 + D + \ldots)(1 - D' + \ldots) xy$$

$$= -(1 + D - D' - DD' + \ldots) xy$$

$$= -xy - y + x + 1 \qquad \ldots(3)$$

$\therefore$ The required general solution of (1) is given by

$$z = e^x \phi_1(y) + e^{-y} \phi_2(x) - xy - y + x + 1 \qquad \ldots(4)$$

Example 2. Solve $r - s + p = 1$.

Solution. The given partial differential equation can be written as

$$\left(\frac{\partial^2 z}{\partial x^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right) + \left(\frac{\partial z}{\partial x}\right) = 1 \text{ or } (D^2 - DD' + D)z = 1$$

or $$D(D - D' + 1)z = 1 \qquad \ldots(1)$$

$\therefore$ $$\text{C.F.} = \phi_1(y) + e^{-x} \phi_2(y + x)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

and $\quad$ P.I. $= \frac{1}{D(1+D-D')}1 = \frac{1}{D}(1+D-D')^{-1}1$

$$= \frac{1}{D}[1-(D-D')+\ldots]1 = \frac{1}{D}1 = x \qquad \ldots(3)$$

$\therefore$ The required general solution of the given PDE is

$$z = \phi_1(y) + e^{-x}\phi_2(y+x) + x \qquad \ldots(4)$$

Example 3. Solve $(D^2 - D'^2 - 3D + 3D')z = xy$.

Solution. The given partial differential equation can be written as

$$(D-D')(D+D'-3)z = xy \qquad \ldots(1)$$

$\therefore$ $\quad$ C.F. = $\phi_1(y+x) + e^{3x}\phi_2(x-y)$ $\qquad \ldots(2)$

where ϕ_1 and ϕ_2 are arbitrary functions.

and $\quad$ P.I. $= \frac{1}{(D-D')(D+D'-3)} = -\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(1-\frac{D+D'}{3}\right)xy$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+\ldots\right)\left\{1+\frac{D+D'}{3}+\frac{(D+D')^2}{9}+\ldots\right\}xy$$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+\ldots\right)\left(1+\frac{1}{3}D+\frac{1}{3}D'+\frac{2}{9}DD'+\ldots\right)xy$$

$$= -\frac{1}{3D}\left(xy+\frac{1}{3}y+\frac{2}{3}x+\frac{2}{9}+\frac{x^2}{2}\right)$$

$$= -\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{2x}{9}+\frac{x^3}{6}\right) \qquad \ldots(3)$$

$\therefore$ The general solution of the given PDE is

$$z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - \frac{1}{3}\left(\frac{x^2y}{2} + \frac{xy}{3} + \frac{x^2}{3} + \frac{2x}{9} + \frac{x^3}{6}\right)$$

EXERCISE 6(E)

Solve the following partial differential equations:

1. $(D - D' - 1)(D - D' - 2)z = x.$
2. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = xy.$
3. $(D^2 - D'^2 + D + 3D' - 2)z = x^2y.$

ANSWERS

1. $z = e^x\phi_1(y+x) + e^{2x}\phi_2(y+x) + \frac{1}{4}(2x+3)$
2. $z = \phi_1(y-x) + e^{-2x}\phi_2(y+2x) + \frac{1}{4}(x^2y - xy - 2x) + \frac{3x^2}{8} - \frac{1}{12}x^2$, where ϕ_1 and ϕ_2 are arbitrary functions.
3. $z = e^{-2x}\phi_1(y+x) + e^x\phi_2(y-x) - \frac{1}{8}(4x^2y + 4xy + 6x^2 + 6y + 12x + 21)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Case IV. When $\boldsymbol{F(x,y) = Ve^{ax+by}}$, where V is some Function of $\boldsymbol{x}$ and $\boldsymbol{y}$.

In this case, the particular integral (P.I.) is given by

$$\text{P.I.} = \frac{1}{f(D,D')}Ve^{ax+by} = e^{ax+by}\frac{1}{f(D+a.D'+b)}e^{ax+by}$$

Important Note. If $f(a,b) = 0$ and $F(x,y) = e^{ax+by}$. Then, P.I. $\frac{1}{f(D,D')}e^{ax+by}$ in which, case I fails, can be found by treating

e^{ax+by} as product of e^{ax+by} with '1' and applying the result of case IV. Thus, we can evaluate P.I. in this case, as follows:

$$\text{P.I.} = \frac{1}{f(D,D')}e^{ax+by}(1) = e^{ax+by}\frac{1}{F(D+a,D'+b)}(1)$$

which can be evaluated as explained in case III by treating $1 = x^0y^0$.

SOLVED EXAMPLES

Example 1. Solve $(D^2 - D')z = xe^{ax+a^2y}$.

Solution. Since $(D^2 - D')$ cannot be resolved into linear factors in D and D', therefore, C.F. is obtained by considering the equation

$$(D^2 - D')z = 0 \qquad \ldots(1)$$

Let a trial solution of (1) be $z = ce^{ax+by}$ …(2)

$\therefore$ $D^2z = ca^2e^{ax+by}$ and $D'z = cbe^{ax+by}$.

Then, equation (1) becomes $c(a^2 - b)e^{ax+by} = 0$

so that $a^2 - b = 0$ or $b = a^2$

$\therefore$ From (2), we have C.F. $= \sum c_i\, e^{a_i^x + a_i^2 y}$ …(3)

where c_i and a_i are arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2-D'}xe^{ax+a^2y} = e^{ax+a^2y}\frac{1}{(D+a)^2(D'+a^2)}x$$

$$= e^{ax+a^2y}\frac{1}{D^2+2aD-D'}x = e^{ax+a^2y}\frac{1}{2aD}\left(1+\frac{D}{2a}-\frac{D'}{2aD}\right)^{-1}x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left\{1-\left(\frac{D}{2a}-\frac{D'}{2aD}\right)+\ldots\right\}x$$

$$= e^{ax+a^2y}\frac{1}{2aD}\left(x-\frac{1}{2a}\right) = e^{ax+a^2y}\left(\frac{x^2}{4a}-\frac{x}{4a^2}\right) \qquad \ldots(4)$$

∴ The general solution of the given PDE is

$$z = \sum c_i\, e^{a_i^x+a_i^2y} + \left(\frac{x^2}{4a}-\frac{x}{4a^2}\right)e^{ax+a^2y} \qquad \ldots(5)$$

Example 2. Solve $(D-3D'-2)^2z = 2e^{2x}\sin(y+3x)$.

Solution. C.F. $= e^{2x}[\phi_1(y+3x) + x\phi_1(y+3x)]$ $\qquad \ldots(1)$

where ϕ_1 and ϕ_2 are arbitrary functions.

and P.I. $= \frac{1}{(D-3D'-2)^2}2e^{2x+0.y}\sin(y+3x)$

$$= 2e^{2x+0.y}\frac{1}{\{(D+2)-3(D'+0)-2\}^2}\sin(y+3x)$$

$$= 2e^{2x}\frac{1}{(D'-3D')^2}\sin(y+3x)$$

$$= 2e^{2x}\frac{x^2}{1^2 2!}\sin(y+3x) = x^2e^{2x}\sin(y+3x) \qquad \ldots(2)$$

∴ The required solution of the given equation is

$$z = e^{2x}[\phi_1(y+3x) + x\phi_2(y+3x)] + x^2e^{2x}\sin(y+3x)$$

Example 3. Solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x\partial y} + 4\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial z}{\partial z} - 2\frac{\partial z}{\partial y} = e^{x+y}$.

Solution. Given that $(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$

or $$(D-2D')(D-2D'+1)z = e^{x+y} \qquad \ldots(1)$$

$\therefore$ $$\text{C.F.} = \phi_1(y+2x) + e^{-x}\phi_2(y+2x) \qquad \ldots(2)$$

where $\phi_1$and ϕ_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-2D'+1)}\left[\frac{1}{D-2D'}e^{x+y}\right] = \frac{1}{D-2D'-1}\left(\frac{1}{1-2}\right)e^{x+y}$$

$$= -e^{x+y}\frac{1}{(D+1)-2(D'+1)+1}(1) = -e^{x+y}\frac{1}{D-2D'}(1)$$

$$= -e^{x+y}\frac{1}{D}\left(1-\frac{2D'}{D}\right)^{-1}(1) = -e^{x+y}\frac{1}{D}(1) = -xe^{x+y} \qquad \ldots(3)$$

$\therefore$ The required solution of the given equation is

$$z = \phi_1(y+2x) + e^{-x}\phi_2(y+2x) - xe^{x+y} \qquad \ldots(4)$$

Example 4. Solve $(D^2 - D'^2 - 3D + 3D')z = e^{x+2y}$.

Solution. The given partial differential equation can be written as

$$(D-D')(D+D'-3)z = e^{x+2y} \qquad \ldots(1)$$

$\therefore$ $$\text{C.F.} = \phi_1(y+x) + e^{3x}\phi_2(y-x) \qquad \ldots(2)$$

where $\phi_1$and ϕ_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{D+D'-3}\left\{\frac{1}{D-D'}e^{x+2y}\right\} = \frac{1}{D+D'-3}\frac{1}{(1-2)}e^{x+2y}.1$$

$$= -e^{x+2y}\frac{1}{D+1+D'+2-3}1 = -e^{x+2y}\frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1}(1)$$

$$= -e^{x+2y}\left(\frac{1}{D}\right)\left(1-\frac{D'}{D}+\ldots\right)1 = -xe^{x+2y} \qquad \ldots(3)$$

∴ The general solution of the given equation is

$$z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - xe^{x+2y} \qquad \ldots(4)$$

Example 5. Solve $r - 3s + 2t - p + 2q = (2+4x)e^{-y}$.

Solution. The given partial differential equation can be written as

$$(D^2 - 3DD' + 2D'^2 - D + 2D')z = (2+4x)e^{-y}$$

or
$$(D-2D')(D-D'-1)z = (2+4x)e^{-y} \qquad \ldots(1)$$

∴
$$\text{C.F.} = \phi_1(y+2x) + e^x\phi_2(y+x) \qquad \ldots(2)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-2D')(D-D'-1)}2\,e^{0x-y}(1+2x)$$

$$= 2\,e^{0x-y}\frac{1}{\{D+0-2(D'-1)\}\{D+0-(D'-1)-1\}}(1+2x)$$

$$= 2e^{-y}\frac{1}{(D-2D'+2)(D-D')}(1+2x)$$

$$= 2e^{-y}\frac{1}{D}\left(1+\frac{D-2D'}{2}\right)^{-1}\left(1-\frac{D'}{D}\right)^{-1}(1+2x)$$

$$= e^{-y}\frac{1}{D}\left\{1-\frac{1}{2}(D-2D')+\ldots\right\}\left\{1+\frac{D\prime}{D}+\ldots\right\}(1+2x)$$

$$= e^{-y}\frac{1}{D}\left(1-\frac{D}{2}+\ldots\right)(1+2x)$$

$$= e^{-y}\left(\frac{1}{D}\right)(1+2x-1) = x^2 e^{-y} \qquad \ldots(3)$$

∴ The required general solution of the given PDE is

$$z = \phi_1(y+2x) + e^x\phi_2(y+x) + x^2e^{-y} \qquad \ldots(4)$$

EXERCISE 6(F)

Solve the following partial differential equations:

1. $D(D-2D')(D+D')z = e^{x+2y}(x^2+4y^2)$.
2. $(D^2+DD'+D+D'-1)z = e^{-2x}(x^2+y^2)$.
3. $(D^2D'+D'^2-2)z = e^{2y}\sin 3x - e^x\cos y$.
4. $(D^2-DD'+D'-1)z = e^y$.
5. $(D^2-DD'+D'-1)z = e^x$.

ANSWERS

1. $z = \phi_1(y) + \phi_2(y+2x) + \phi_3(y-x) - \frac{1}{81}(9x^2+36y^2-18x-72y+76)e^{x-2y}$, where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.
2. $z = \sum A\, e^{hx+ky} + \frac{1}{27}e^{-2x}(9x^2+9y^2+18x+6y+14)$, where $h^2+hk+h+k+1=0$.
3. $z = \sum A\, e^{hx+ky} - \frac{1}{16}e^{2y}\sin 3x + \frac{1}{20}e^x(3\cos 2y - \sin 2y)$, where h and k are related by $h^2k+k^2-2=0$.
4. $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) - xe^y$
5. $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + \frac{1}{2}xe^x$

MISCELANEOUS SOLVED EXAMPLES

Example 1. Solve $(D^2-DD'+D'-1)z = \cos(x+2y) + e^y$

Solution. The given equation can be written as

$$(D-1)(D-D'+1)z=\cos(x+2y)+e^y \qquad \ldots(1)$$

$\therefore$ $$\text{C.F.}=e^x\phi_1(y)+e^{-x}\phi_2(y+x)$$

where ϕ_1 and ϕ_2 being arbitrary functions.

P.I. corresponding to $\cos(x+2y)$is $=\frac{1}{D^2-DD'+D'-1}\cos(x+2y)$

$$=\frac{1}{-1^2+(1\times 2)+D'-1}\cos(x+2y)=\left(\frac{1}{D'}\right)\cos(x+2y)$$

$$=\frac{1}{2}\sin(x+2y) \qquad \ldots(3)$$

P.I. corresponding to e^y is $=\frac{1}{(D-1)(D-D'+1)}e^y=\frac{1}{D-D'+1}\frac{1}{D-1}e^{0.x+1.y}$

$$=\frac{1}{D-D'-1}\frac{1}{0-1}e^{0.x+1.y}=-e^{0.x+1.y}\frac{1}{(D+0)-(D'-1)+1}(1)$$

$$=-e^y\frac{1}{D(1-D'/D)}(1)=-e^y\frac{1}{D}\left(1-\frac{D'}{D}+\ldots\right)^{-1}(1)$$

$$=-e^y\left(\frac{1}{D}\right)(1+\ldots)(1)=-xe^y \qquad \ldots(4)$$

$\therefore$ The general solution of the given PDE is

$$z=e^x\phi_1(y)+e^{-x}\phi_2(y+x)+\frac{1}{2}\sin(x+2y)-xe^y \qquad \ldots(5)$$

Example 2. Find a particular integral of the differential equation: $(D^2-D')z=e^{x+y}+5\cos(x+2y)$.

Solution. P.I. corresponding to e^{x+y} is $=\frac{1}{D^2-D'}e^{x+y}$

$= e^{x+y} \frac{1}{(D+1)^2-(D'+1)}(1) = e^{x+y} \frac{1}{D^2+2D-D'}(1)$

$= e^{x+y} \frac{1}{2D}\left[1+\left(\frac{D}{2}-\frac{D'}{2D}\right)\right]^{-1}(1) = e^{x+y} \frac{1}{2D}\{1+\ldots\}(1) = \frac{1}{2} x e^{x+y}$

P.I. corresponding to $5\cos(x+2y)$ is $= 5\frac{1}{D^2-D'}\cos(x+2y)$

$= 5\frac{1}{-1^2-D'}\cos(x+2y) = -\frac{5}{D'+1}\cos(x+2y)$

$= -5(D'-1)\frac{1}{D'^2-1}\cos(x+2y) = -5\frac{1}{-2^2-1}(D'-1)\cos(x+2y)$

$= (D'-1)\cos(x+2y) = D'\cos(x+2y) - \cos(x+2y)$

$= -2\sin(x+2y) - \cos(x+2y)$

$\therefore$ Required P.I. $= \frac{1}{2} x e^{x+y} - 2\sin(x+2y) - \cos(x+2y)$

Example 3. Solve $(D-D'-1)(D-D'-2)z = e^{2x-y} + x$

Solution. Here, C.F. $= e^{x}\phi_1(y+x) + e^{2x}\phi_2(y+x)$,

where ϕ_1 and ϕ_2 are arbitrary functions.

Now, P.I. corresponding to e^{2x-y} is $= \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y}$

$= \frac{1}{\{2-(-1)-1\}\{2-(-1)-2\}} e^{2x-y} = \frac{1}{2} e^{2x-y}$

and P.I. corresponding to x is $= \frac{1}{(D-D'-1)(D-D'-2)} x$

$= \frac{1}{2\{1-(D-D')\}\{1-(D-D')/2\}} x = \frac{1}{2}[1-(D-D')]^{-1}\left\{1-\frac{D-D'}{2}\right\}^{-1} x$

$= \frac{1}{2}[1+(D-D')+\ldots]\left\{1+\frac{D-D\prime}{2}+\ldots\right\}x$

$= \frac{1}{2}\left\{1+(D-D')+\frac{D-D\prime}{2}+\ldots\right\}x = \frac{1}{2}\left\{1+\frac{3}{2}D+\ldots\right\}x = \frac{1}{2}\left(x+\frac{3}{2}\right)$

$\therefore$ The general solution of the given PDE is

$$z = e^{x}\phi_1(y+x) + e^{2x}\phi_2(y+x) + \frac{1}{2}e^{2x-y} + \frac{1}{2}\left(x+\frac{3}{2}\right)$$

Example 4. Solve $(D^2-D')(D-2D')z = e^{2x+y} + xy$.

Solution. C.F. corresponding to linear factor $(D-2D')$ is $\phi(y+2x)$. Now, (D^2-D') cannot be resolved into linear factors in D and D'. To find C.F. corresponding to it, we consider

$$(D^2-D')z = 0 \qquad \ldots(1)$$

Let a trial solution of (1) be $\qquad z = ce^{ax+by} \qquad \ldots(2)$

$\therefore$ $D^2z = ca^2e^{ax+by}$ and $D'z = cbe^{ax+by}$. Then, (1) becomes

$c(a^2-b)e^{ax+by} = 0$ so that $\qquad a^2-b=0$ or $b=a^2$.

$\therefore$ From (2), C.F. corresponding to (D^2-D') is $\sum e^{c_i a_i^x + a_i^2 y}$

Now, P.I. corresponding to e^{2x+y} is $= \frac{1}{D-2D'}\frac{1}{D^2-D'}e^{2x+y}$

$= \frac{1}{D-2D'}\frac{1}{2^2-1}e^{2x+y} = \frac{1}{3}\frac{1}{D-2D'}e^{2x+y}.1 = \frac{1}{3}e^{2x+y}\frac{1}{(D+2)-2(D'+1)}(1)$

$= \frac{1}{3}e^{2x+y}\frac{1}{D(1-2D'/D)}(1) = \frac{1}{3}e^{2x+y}\frac{1}{D}\left(1-\frac{2D\prime}{D}\right)^{-1}(1)$

$$= \frac{1}{3} e^{2x+y} \left(\frac{1}{D}\right)(1+\ldots)1 = \frac{1}{3} x\, e^{2x+y}$$

and P.I. corresponding to xy is $= \frac{1}{(D-2D')(D^2-D')} xy$

$$= \frac{1}{(-2D')(1-D/2D')(-D')(1-D^2/D')} xy = \frac{1}{2D'^2}\left(1-\frac{D}{2D'}\right)^{-1}\left(1-\frac{D^2}{D'}\right)^{-1} xy$$

$$= \frac{1}{2D'^2}\left(1+\frac{D}{2D'}+\ldots\right)\left(1+\frac{D^2}{D'}+\ldots\right) xy$$

$$= \frac{1}{2D'^2}\left(1+\frac{D}{2D'}+\cdots\right) xy = \frac{1}{2D'^2}\left(xy+\frac{1}{2D'}y\right)$$

$$= \frac{1}{2D'^2}\left(xy+\frac{y^2}{4}\right) = \frac{1}{2}\left(\frac{xy^2}{6}+\frac{y^4}{3\times4\times4}\right) = \frac{1}{12}\left(xy^2+\frac{1}{4}y^4\right)$$

$\therefore$ The general solution of the given PDE is

$$z = \phi(y+2x) + \sum c_i e^{c_i a_i^x + a_i^2 y} + \frac{1}{3} x e^{2x+y} + \frac{1}{12}\left(xy^2+\frac{1}{4}y^4\right)$$

where ϕ is an arbitrary function and c_i and a_i are arbitrary constants.

Example 5. Solve the following partial differential equation: $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y + xy + 1.$

Solution. The given PDE can be written as

$$(D-1)(D-D'+1)z = \cos(x+2y) + e^y + xy + 1 \quad \ldots(1)$$

$\therefore$ C.F. of (1) is $= e^x\phi_1(y) + e^{-x}\phi_2(y+x)$ ...(2)

where ϕ_1 and ϕ_2 are arbitrary functions.

Now, P.I. corresponding to $\cos(x+2y)$ is

$$= \frac{1}{D^2-DD'+D'-1}\cos(x+y) = \frac{1}{-1^2+(1\times 2)D'-1}\cos(x+2y)$$

$$= (1/D')\cos(x+2y) = \left(\frac{1}{2}\right)\sin(x+2y) \qquad \ldots(3)$$

P.I. corresponding to e^y is $= \frac{1}{D^2-DD'+D'-1} e^{0.x+1.y}(1)$

$$= e^{0.x+1.y}\frac{1}{(D+0)^2(D+0)(D'+1)+(D'+1)-1}(1) = e^y \frac{1}{D^2-DD'+D\prime}(1)$$

$$= e^y \frac{1}{D\prime}\left\{1+\left(\frac{D^2}{D\prime}-D\right)\right\}^{-1}(1) = e^y \frac{1}{D\prime}\{1+\ldots\}(1)y = e^y \qquad \ldots(4)$$

and P.I. corresponding to $(xy+1)$ is $= \frac{1}{(D-1)(D-D'+1)}(xy+1)$

$$= -(1-D)^{-1}\{1+(D-D')\}^{-1}(xy+1)$$

$$= -\{1+D+\ldots\}\{1-(D-D')+(D-D')^2-\ldots\}(xy+1)$$

$$= -(1+D+\ldots)(1-D+D'-2DD'+\ldots)(xy+1)$$

$$= -(1+D+\cdots)(xy+1-y+x-2)$$

$$= -(1+D+\cdots)(xy-y+x-1) \qquad \ldots(5)$$

$$= -(xy-y+x-1+y+1) = -(xy+x)$$

$\therefore$ The general solution of the given PDE is

$$z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + \frac{1}{2}\sin(x+2y) + ye^y - (xy+x)$$

EXERCISE 6(G)

Solve the following partial differential equations:

1. $(D^2 - DD' - 2D^2 + 2D + 2D')z = e^{2x+3y} + \sin(2x + y)$.
2. $(D^2 - DD' + D' - 1)z = e^y + xy$.
3. $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^x$.

ANSWERS

1. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - \frac{1}{10}e^{2x+3y} - \frac{1}{6}\cos(2x + y)$
2. $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x)ye^y - xy - x + 1$
3. $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + \frac{1}{2}\sin(x + 2y) + \frac{1}{2}e^x$.

6.11 General Method of Finding Particular Integral (P.I.) of Reducible Non-homogeneous Linear Partial Differential Equation $f(D, D')z = F(x, y)$

Consider the following reducible non-homogeneous linear partial differential equation $f(D, D')z = F(x, y)$ …(1)

Let $F(D, D') = (a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2)$

$$... (a_nD + b_nD' + c_n) \qquad ...(2)$$

$\therefore$ Particular of (1) is $= \frac{1}{f(D,D')}F(x, y)$

or P.I. $= \frac{1}{(a_1D+b_1D'+c_1)(a_2D+b_2D'+c_2)...(a_nD+b_nD'+c_2)}F(x, y)$ …(3)

In order to evaluate P.I. given by (3), we consider a solution of the following equation (assuming that $a \neq 0$):

$$(aD + bD + c)z = F(x, y)$$

or $$ap + bq = F(x, y) - cz \qquad \ldots(4)$$

which is of the form $Pp + Qq = R$.

∴ Lagrange's auxiliary equations for (4) are

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{F(x,y) - cz} \qquad \ldots(5)$$

Taking the first two fractions of (5), we get

$$ady - bdx = 0 \qquad \ldots(6)$$

Integrating (6), we get $ay - bx = d$ $\ldots(7)$

where d is an arbitrary constant.

or $$y = (d + bx)/a, \text{ if } a \neq 0 \qquad \ldots(8)$$

Taking first and last fractions of (5) and using (8), we get

$$\frac{dz}{dx} = \frac{F(x,y) - cz}{a} = -\frac{cz}{a} + \frac{1}{a} F\left(x, \frac{d + bx}{a}\right)$$

or $$\frac{dz}{dx} + \frac{c}{a} z = \frac{1}{a} F\left(x, \frac{d + bx}{a}\right) \qquad \ldots(9)$$

which is a linear differential equation whose I.F. is given by

$$\text{I.F.} = e^{\int (c/a) dx} = e^{cx/a}$$

and its solution is given by $ze^{cx/a} = \frac{1}{a}\int F\left(x, \frac{d+bx}{a}\right) dx$ ...(10)

so that $z = \frac{e^{-(cx/a)}}{a}\int F\left(x, \frac{d+bx}{a}\right) dx,\ \ a \neq 0$ and $d = ay - bx$

∴ From (4), we get

$$\frac{1}{(aD+bD'+c)}F(x,y) = \frac{e^{-(cx/a)}}{a}\int F\left(x, \frac{d+bx}{a}\right) dx \qquad ...(11)$$

where $ay - bx = d$ and $a \neq 0$.

Similarly, if $b \neq 0$, we can show that

$$\frac{1}{(aD+bD'+c)}F(x,y) = \frac{e^{-(cy/b)}}{b}\int e^{cy/b}\, F\left(\frac{d+ay}{b}\right) dy \qquad ...(12)$$

where $bx - ay = d$ and $b \neq 0$.

Results (11) and (12) will be used to evaluate P.I. given by (3).

6.12 Working Rule for Finding P.I. of Any Reducible Linear Partial Differential Equation (Homogeneous or Non-homogeneous) Given by $f(D, D')z = F(x, y)$

Let us consider the linear partial differential equation

$$f(D, D')z = F(x, y) \qquad ...(1)$$

Then, following rules are important while finding its P.I.:

Rule I. $\frac{1}{aD+bD'+c}F(x,y) = \frac{e^{-(cx/a)}}{a}\int e^{cx/a}\, F\left(x, \frac{d+bx}{a}\right) dx, a \neq 0$
where $ay - bx = d$

Note that constant d must be replaced by $ay - bx$ after integration is performed.

Rule II. $\frac{1}{aD+bD'+c}F(x,y) = \frac{e^{-(cx/b)}}{b}\int e^{cx/b}\, F\left(\frac{d+ax}{a}\right)dy, \quad a \neq 0$ where $bx - ay = d$.

Note that constant d must be replaced by $bx - ay$ after integration is performed.

We now consider some special cases of the above rules:

Rule III. $\frac{1}{aD+c}F(x,y) = \frac{e^{-(cx/a)}}{a}\int e^{cx/a}\, F(x, d/a)dy$, where $ay = d$.

Rule IV. $\frac{1}{D-mD'}F(x,y) = \int F\,(x, d - mx)dx$,where $y + mx = d$.

Rule V. $\frac{1}{bD'+c}f(x,y) = \frac{e^{-(cy/b)}}{b}\int e^{cy/b}\, f(d/b, y)dy$,where$bx = d$.

Rule VI. $\frac{1}{D'-mD}F(x,y) = \int F(d - my, y)\, dy$,where $x - my = d$.

Note. If $f(D, D')$ can be factored as $\prod_{r=1}^{n}(a_r D + b_r D' + c_r)$, then

$$\text{P.I. for (1)} = \frac{1}{f(D,D')}F(x,y) = \frac{1}{\prod_{r=1}^{n}(a_r D+b_r D'+c_r)}F(x,y)$$

$$= \frac{1}{(a_1 D+b_1 D'+c_1)(a_2 D+b_2 D'+c_2)\ldots(a_n D+b_n D'+c_n)}F(x,y)$$

which is evaluated by using the above six rules for each factor, in succession, from right to the left.

SOLVED EXAMPLES

Example 1. Solve $(D + D')(D + D' - 2)z = \sin(x + 2y)$.

Solution. C.F. $= \phi_1(y - x) + e^{2x}\phi_2(y - x)$ ...(1)

where ϕ_1 and ϕ_2 are arbitrary function.

P.I. $= \frac{1}{D+D'-2}\left[\frac{1}{D+D'}\sin(x + 2y)\right] = \frac{1}{D+D'-2}\left[\frac{1}{D+D'}\sin(3x + 2y - 2x)\right]$

$= \frac{1}{D+D'-2}\int \sin(3x + 2d)\, dx$, where $y - x = d$ [using rule IV]

$= \frac{1}{D+D'-2}\left[-\frac{\cos(3x+2d)}{3}\right] = -\frac{1}{3}\frac{1}{D+D'-2}\cos(2x + y)$

$= -\frac{1}{3}e^{2x}\int e^{-2x}\cos(3x + 2d)\, dx$, where $y - x = d$ [using rule I]

$= -\frac{1}{3}e^{2x}\frac{1}{(-2)^2+3^2}e^{-2x}\{-2\cos(3x + 2d) + 3\sin(3x + 2d)\}$

$= \frac{2}{39}\cos(x + 2y) - \frac{1}{13}\sin(x + 2y)$ ...(2)

∴ The general solution of the given PDE is

$$z = \phi_1(y - x) + e^{2x}\phi_2(y - x) + \frac{2}{39}\cos(x + 2y) - \frac{1}{13}\sin(x + 2y)$$

Example 2. Solve $(D^3 - DD'^2 - D^2 + DD')z = (x + 2)/x^3$.

Solution. The given partial differential equation can be written as

$$D(D - D')(D + D' - 1)z = \frac{1}{x^2} + \frac{2}{x^3} \qquad \text{...(1)}$$

∴ C.F. of (1) $= \phi_1(y) + \phi_2(y + x)$ ...(2)

P.I. $= \frac{1}{(D-D')(D+D'-1)}\frac{1}{D}\left(\frac{1}{x^2}+\frac{2}{x^3}\right) = \frac{1}{(D+D'-1)(D-D)}\left(-\frac{1}{x}-\frac{1}{x^2}\right)$

$= \frac{1}{D+D'-1}\int\left(-\frac{1}{x}-\frac{1}{x^2}\right)dx = \frac{1}{D+D'-1}\left(-\log x+\frac{1}{x}\right)$

$= e^x\int e^{-x}\left(-\log x+\frac{1}{x}\right)dx$

$= -e^x\int e^{-x}\log x\,dx + e^x\int(e^{-x})\left(\frac{1}{x}\right)dx$

$= -e^x\left[(-e^{-x})\log x - \int e^{-x}\left(\frac{1}{x}\right)dx\right] - e^x\int e^{-x}\left(\frac{1}{x}\right)dx$

$= \log x$ $\quad\ldots(3)$

$\therefore$ The general solution of the given PDE is

$$z = \phi_1(y) + \phi_2(y+x) + e^x\phi_3(y-x) + \log x \qquad \ldots(4)$$

Example 3. Solve $(D^2 + DD' + D' - 1)z = 4\sinh x$.

Solution. The given partial differential equation can be written as

$$(D+1)(D+D'-1)z = 2(e^x - e^{-x}) \qquad \ldots(1)$$

$\therefore$ C.F. of (1) $= e^{-x}\phi_1(y) + e^x\phi_2(y-x)$ $\quad\ldots(2)$

where ϕ_1 and ϕ_2 are arbitrary functions.

P.I. $= \frac{1}{(D+1)(D+D'-1)}2(e^x-e^{-x}) = \frac{1}{(D+1)}2e^x\int e^{-x}(e^x-e^{-x})dx$

$= \frac{1}{(D+1)}2e^x\left(1+\frac{1}{2}e^{-2x}\right) = \frac{1}{D+1}(2xe^x+e^{-x})$

$= e^{-x}\int e^x(2x\,e^x+e^{-x})dx$, using rule III

$$= 2e^{-x}\int x\,e^{2x}dx + e^{-x}x = 2e^{-x}\left[x\left(\frac{e^{2x}}{2}\right) - \int 1.\left(\frac{e^{2x}}{2}\right)dx\right]$$

$$= \left(x - \frac{1}{2}\right)e^{x} + xe^{-x} \qquad \ldots(3)$$

∴ The general solution of the given PDE is

$$z = e^{-x}\phi_1(y) + e^{x}\phi_2(y - x) + \left(x - \frac{1}{2}\right)e^{x} + xe^{-x} \qquad \ldots(4)$$

EXERCISE 6(H)

1. Solve $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^{y}$.

2. Solve the PDE $2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial y} = 5\cos(3x - 2)$.

ANSWERS

1. $z = e^{x}\phi_1(y) + e^{-x}\phi_2(y + x) + \frac{1}{2}\sin(x + 2y) - xe^{y}$

2. $z = \phi_1(x) + e^{3x/2}\phi_2(2y - x) + \frac{4}{10}\cos(3x - 2y)$

$$+ \frac{3}{10}\sin(3x - 2y)$$

OBJECTIVE TYPE QUESTIONS

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^3 z}{\partial y^3} + \frac{\partial z}{\partial x} + z = x + y$ is an example of

(a) homogeneous PDE (b) non-homogeneous PDE

(c) cubic equation (d) quadratic equation

2. Solution of $r + 2s + t + 2p + 2q + z = 0$ is

(a) $z = e^{-x}[\phi_1(y-x) + x\phi_2(y-x)]$

(b) $z = e^{x}[\phi_1(y-x) + x\phi_2(y-x)]$

(c) $z = e^{-x}[\phi_1(y-x) + \phi_2(y+x)]$

(d) $z = e^{x}[\phi_1(y-x) + x\phi_2(y+x)]$

3. C.F. of $(D - D'^2)z = 0$ is

(a) $z = \sum Ae^{k^2x+ky}$ (b) $z = \sum Ae^{k(x+y)}$

(c) $z = \sum A(x+ky)$ (d) $z = \sum A(x-ky)$

4. C.F. of $(D^2 + D'^2 + D - D')z = e^{2x+3y}$ is

(a) $\phi_2(y-x) + \phi_2(y+x)$ (b)$\phi_1(y+x) + e^{x}\phi_2(y-x)$

(c) $\phi_1(y+x) + e^{-x}\phi_2(y-x)$ (d)$\phi_1(y-x) - e^{x}\phi_2(y+x)$

ANSWERS

1. (b) **2.** (a) **3.** (a) **4.** (c)