Graph Theory - Lecture 1
Terminologies and Definitions

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1 Sets and Set Theory : Some Definitions

Definition 1.1 *A* **set** *is a collection of distinct objects.*

Object : An *object* occurs only once in a set

Definition 1.2 *A* **multi-set** *is a collection of objects in which an object may occur several times.*

Multiplicity : The number of times an object occurs is the *Multiplicity* of the Sets

Definition 1.3 *A* **binary relation** *from a set* $\boldsymbol{S}$ *to a set* $\boldsymbol{T}$ *is a subset of the Cartesian product of* $(\boldsymbol{S} \times \boldsymbol{T})$

Relation : A *relation* on $\boldsymbol{S}$ is a seubset of $(\boldsymbol{S} \times \boldsymbol{S})$

Multi-relation : A *multi-relation* on $\boldsymbol{S}$ is a *multi-subset* of $(\boldsymbol{S} \times \boldsymbol{S})$

Example :

Set $\boldsymbol{S = \{1, 3, 5, 7\}}$

Multi-Set $\boldsymbol{M = \{1, 1, 1, 2, 2, 3\}}$

Relation $\boldsymbol{\Re = \{(1, 3), (3, 5), (3, 7)\}}$

Multi-Relation $\boldsymbol{T = \{(1, 3), (1, 3), (1, 1), (3, 3), (5, 7)\}}$

1.1 Definition of Graph

Definition 1.4 *A graph* $\boldsymbol{G}$ *is a finite non-empty set* $\boldsymbol{V}$ *together with a symmetric irreflexive binary relation* $\boldsymbol{A}$ *on* $\boldsymbol{V}$

- Definition 1.4 is an *algebraic set theoretic* definition of a graph
- The elements of the set $\boldsymbol{V}$ are called the *vertices* of the graph
- The relation $\boldsymbol{\Re}$ is called the *adjacency relation* and if $\boldsymbol{u}$ is related to $\boldsymbol{v}$ by $\boldsymbol{\Re}$, we say that $\boldsymbol{u}$ *is adjacent to* $\boldsymbol{v}$ and write $\boldsymbol{u \Re v}$

- Since $\Re$ is a *Symmetric Relation*, it defines a set $\boldsymbol{E}$ of $\mathbf{V_{(2)}}$, the set of **two**-subsets of $\boldsymbol{V}$, namely

$$\boldsymbol{E = \{\{u, v\} | u, v \in V; u \Re v\}}$$

$\Rightarrow$ The elements of the set $\boldsymbol{E}$ are called the edges of the graph, where the elements are referred to as $\boldsymbol{e_{uv} \in E(G)}$, where $\boldsymbol{u, v \in V(G)}$.

Sometimes, the elements of $\boldsymbol{E(G)}$ is also represented as $\boldsymbol{e_1, e_2, \ldots, e_n}$, where $\boldsymbol{n = |E(G)|}$.

Definition 1.5 *A graph* $\boldsymbol{G}$ *is a pair* $\boldsymbol{(V, E)}$*, where* $\boldsymbol{V(G)}$ *is as non-empty set whose elements are called of* $\boldsymbol{G}$ *and* $\boldsymbol{E(G)}$ *is a subset of* $\mathbf{V_{(2)}}$*, whose elements are called the edges of* $\boldsymbol{G}$*.*

Note : There is an *incidence relation* $\boldsymbol{I}$ between the vertex set $\boldsymbol{V}$ and the edge set $\boldsymbol{E(G)}$ of a graph.

If $\boldsymbol{e \in E(G)}$, then there is a pair of *distinct vertices* $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\boldsymbol{e = \{u, v\}}$.

$\Rightarrow$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are the <u>end-vertices</u> of ends of $\boldsymbol{e}$ and say that $\boldsymbol{u}$ and $\boldsymbol{v}$ are incident on $\boldsymbol{e}$ with $\boldsymbol{e(uIe \text{ } and \text{ } vIe)}$.

$\Rightarrow$ We can also say $\boldsymbol{e}$ is incident with $\boldsymbol{u}$ and $\boldsymbol{v}$: $\boldsymbol{eI'u}$ and $\boldsymbol{eI'v}$, where $\boldsymbol{I'}$ is a relation converse of $\boldsymbol{I}$.

Definition 1.6 *A graph* $\boldsymbol{G}$ *is a pair of disjoint set of* $\boldsymbol{V(G)}$ *(non-empty) and* $\boldsymbol{E(G)}$ *and a 1-1* <u>*incidence function*</u> $\boldsymbol{f : E(G) \to \mathbf{V_{(2)}}}$

If $\boldsymbol{f(e) = \{u, v\}}$*,* $\boldsymbol{e}$ *is said to be incident on* $\boldsymbol{u}$ *and* $\boldsymbol{v}$*;* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *is said to be incident with* $\boldsymbol{e}$

Note : All the 3 definitions are equivalent : Definition 1.4, 1.5, 1.6

- **Vertices** are sometimes referred to as *nodes/points.*
- **Edges** are sometimes referred to as *lines*

  $\boldsymbol{|V(G)|}$: Order of the Graph is the ***cardinality*** *of the number of vertices*

  $\boldsymbol{|E(G)|}$: Size of the Graph is the ***cardinality*** *of the number of edges*

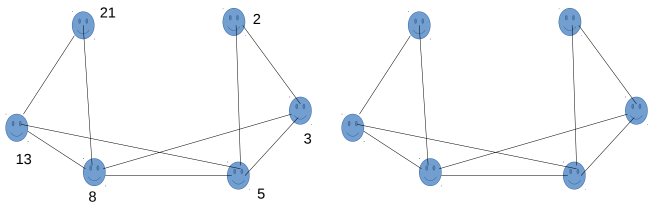
- Since the vertex set of every graph is non-empty, the order of every graph is *at least* $\mathbf{1}$.
- A graph with exactly one vertex is called a *trivial graph* and all other graphs *non-trivial.*
- The *order* of a non-trivial graph is *at least* $\mathbf{1}$.

Example 1.1 *Consider the set* $\boldsymbol{S = \{2, 3, 5, 8, 13, 21\}}$

Let $\Re$ *be the relation of pairs of distinct integers belonging to* $\boldsymbol{S}$ *whose sum/difference* $\in \boldsymbol{S}$*, e.g.* $\boldsymbol{\{2, 3\}, \{2, 5\}, \ldots}$

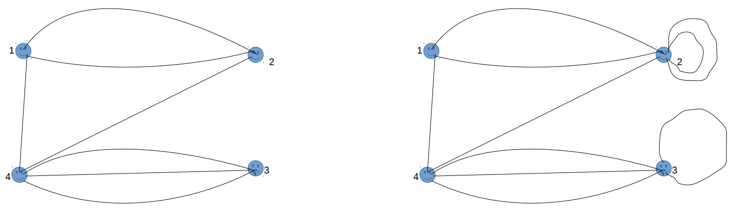
$\therefore$*,* $\boldsymbol{H = (V, E)}$*, where* $\boldsymbol{V(H) = S}$ *and*

$$\boldsymbol{E(H) = \{\{2, 3\}, \{2, 5\}, \{3, 5\}, \{3, 8\}, \{5, 8\}, \{5, 13\}, \{8, 13\}, \{8, 21\}, \{13, 21\}\}}$$



(a) Labeled Graph (b) Unlabeled Graph

Figure 1: Graph Types



(a) Multi-Graph (b) General Graph/Pseudograph

Figure 2: Graph Types

2 Other Definitions of Graphs

Definition 2.1 (Simple Graph) *A* **Simple Graph** *is a graph if each pair of vertices,* $\{u, v\}$ *is joined by at most one edge.*

Parallel Edges : Multiple edges/lines joining a pair of vertices.

Loops : In some cases, we have to allow one or more lines joining a point to itself

Loops with multiplicity greater than 1 are **multiple loops**

Definition 2.2 (Multigraph) *A* **Multigraph** *is a pair* $\mathbf{G(V,E)}$*, where* $\boldsymbol{V(G)}$ *is a non-empty set of vertices and* $\boldsymbol{E(G)}$ *is a multi-set of edges, being a multi-subset of* $\mathbf{V_{(2)}}$*.*
The number of times an edge $\boldsymbol{e = uv}$ *occurs is called* **multiplicity of** $\boldsymbol{e}$

Underlying Graph : The graph obtained by replacing all multiple edges by a single edge is called the **Underlying** graph of $\boldsymbol{G}$.

Definition 2.3 (General Graph/Pseudograph) *A* **General Graph/Pseudograph** *is a pair* $\mathbf{G(V,E)}$*, where* $\boldsymbol{V(G)}$ *is a non-empty set of vertices and* $\boldsymbol{E(G)}$ *is a multiset of edges, being a multi-subset of* $\mathbf{V_{(2)}}$*, the set of unordered pairs of elements of* $\boldsymbol{V(G)}$*, not necessirily distinct.*

3 Complete Graph

A graph $\mathbf{G(V,E)}$for which there exists all incidence relations between all vertices, $\boldsymbol{V(G)}$ is known as the **complete graph**, $\boldsymbol{K_{|V(G)|}}$ or $\boldsymbol{K_n}$, where $\boldsymbol{n = |V(G)|}$. The order of a complete graph is $|\boldsymbol{V(G)}|$.

In Figure 3, the order of the graph is 5.

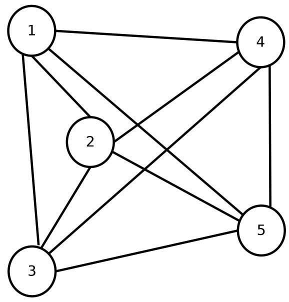


Figure 3: Complete Graph : $\boldsymbol{K_5}$

4 Directed Graphs / Digraphs

Directed Graphs/Digraphs are used for modelling of systems, where is a natural sense of orientation.

Definition 4.1 (Digraph) *A* **Digraph** *is defined as* $\mathbf{D(V,E)}$*, where* $\boldsymbol{V(D)}$ *is a non-empty set whose elements are called the* *vertices* *and* $\boldsymbol{E(D)}$ *is a subset of* $\boldsymbol{V^{(2)}}$ *(i.e. the set of ordered pairs of distinct elements of* $\boldsymbol{V(D)}$*), whose elements are called the* **arcs** *of* $\boldsymbol{D}$*.*

OR

A **Digraph** $\mathbf{D(V,E)}$ *is a quadrapule consisting of a vertex set* $\boldsymbol{V(D)}$*, an edge set* $\boldsymbol{E(D)}$*, a function indicating each edge as an ordered pair of vertices and another funtion assigning the weights to each ordered pair, where in the ordered pair :*

- *the first vertex of the ordered pair is the* *tail* *of the edge and the second is the* *head**, together they being the* *endpoints.*
- *we say that an edge is* *from its tail to its head.*

Definition 4.2 (Multi-Digraph) *A* **multi-digraph** $\mathbf{D(V,E)}$ *is a pair of* $\boldsymbol{(V,E)}$*, where* $\boldsymbol{V(D)}$ *is a non-empty set of vertices and* $\boldsymbol{E(D)}$ *is a multiset of arcs, being a multi-subset of* $\boldsymbol{V^{(2)}}$*.*

- The number of times an arc occurs in $\boldsymbol{E}$ is called its *multiplicity* and arcs with multiplicity greater than $\mathbf{1}$ are called multiple arcs of $\boldsymbol{D}$.
- An arc $\boldsymbol{(u,v)} \in \boldsymbol{E(D)}$ can also denoted as $\boldsymbol{uv}$ or $\overrightarrow{\boldsymbol{uv}}$, where

$\boldsymbol{u}$ initial vertex/tail/source

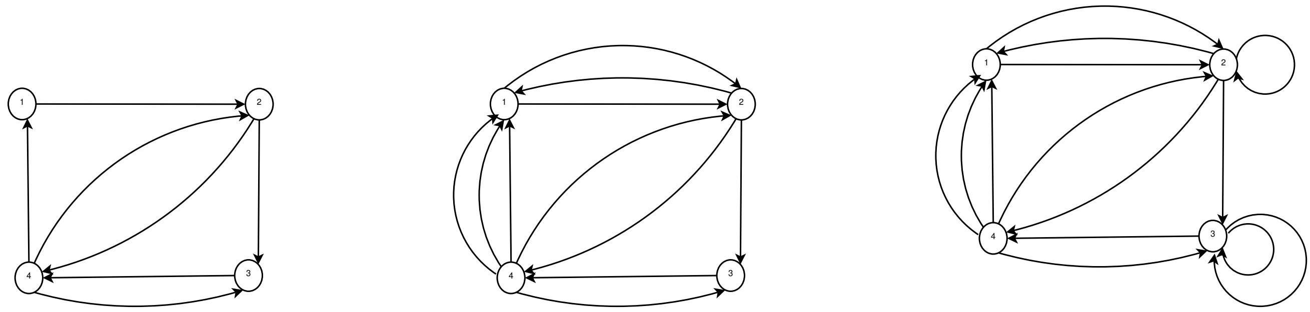
$\boldsymbol{v}$ terminal vertex/head/sink

- A **digraph** is an **irreflexive binary relation** on $\boldsymbol{V}$.
- The **out-neighbourhood** or **successor set** is defined as

$$\boldsymbol{N^+(v \in V(D)) = \{x \in V(D) : v \to x\}}$$

- The **in-neighbourhood** or **predecessor set** is defined as

$$\boldsymbol{N^-(v \in V(D)) = \{x \in V(D) : x \to v\}}$$



(a) Digraph (b) Multi-Digraph (c) General-Digraph

Figure 4: Directed Graphs

Example : Modelling a Coin Toss

- There are Two States Only Ever: Heads (H) or Tails (T)
- Each coin toss is a finite state or event.
- Coin can either stay in same state (say another Head) or change (to Tail)

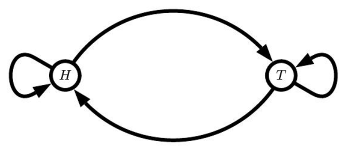


Figure 5: Modelling a Coin Toss

5 Orientations and Tournaments

Definition 5.1 (Orientation of a Graph) *An orientation of a graph* $\mathbf{G(V, E)}$*is a digraph* $\mathbf{D(V, E)}$*obtained from* $\mathbf{G(V, E)}$*by choosing an orientation* $\boldsymbol{x \Rightarrow y}$ *or* $\boldsymbol{y \Rightarrow x}$ *for every edge* $\boldsymbol{xy \in E(G)}$.

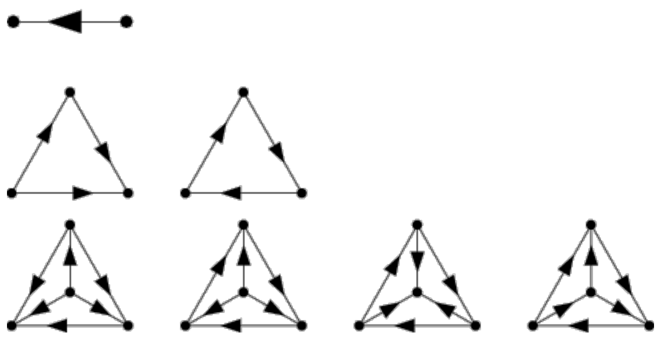


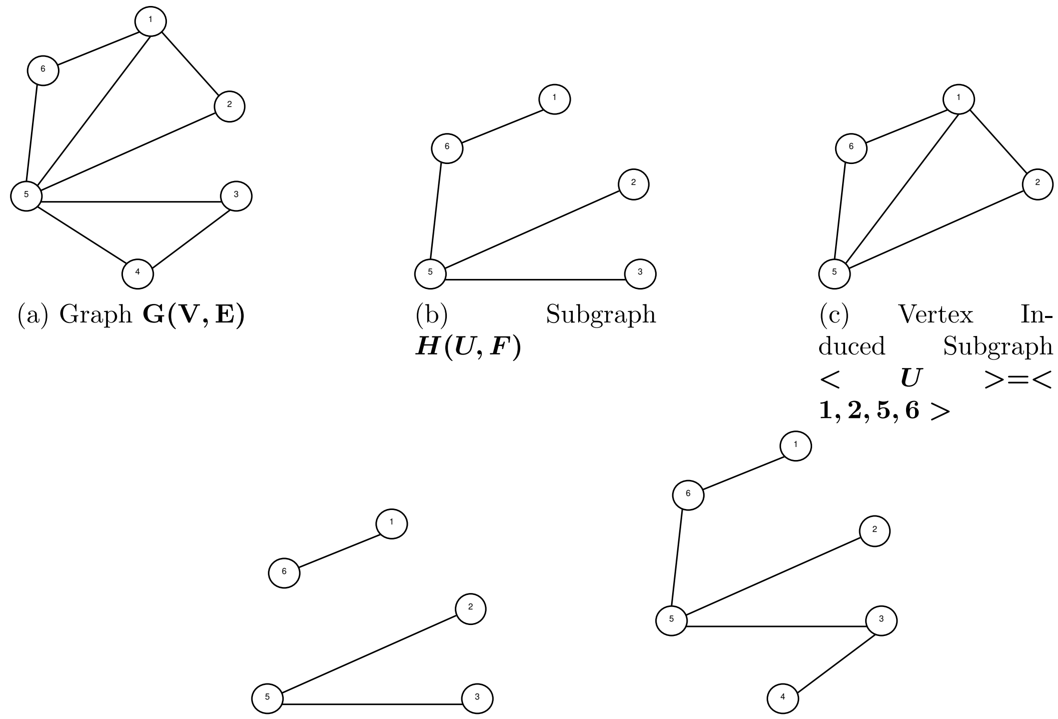
Figure 6: Tournament : The first and second 3-node tournaments shown above are called a transitive triple and cyclic triple, respectively (Harary 1994, p. 204)

Tournaments / tournament graphs is an orientation of a complete graph. They are so named because an $\boldsymbol{n}$-node tournament graph corresponds to a tournament in which each member of a group of $\boldsymbol{n}$ players plays all other $\boldsymbol{n-1}$ players, and each game results in a win for one player and a loss for the other.

Example 5.1 *A so-called score sequence can be associated with every tournament giving the set of scores that would be obtained by the players in the tournament, with each win counting as one point and each loss counting as no points.*

A different scoring system is used to compute a tournament's so-called tournament matrix, with 1 point awarded for a win and -1 points for a loss.

6 Subgraphs



(a) Graph $\mathbf{G(V,E)}$

(b) Subgraph $\boldsymbol{H(U,F)}$

(c) Vertex Induced Subgraph $<\boldsymbol{U}>=<\mathbf{1,2,5,6}>$

(d) Edge Induced Subgraph $<\boldsymbol{F}>=<\mathbf{\{1,6\},\{2,5\},\{3,5\}}>$

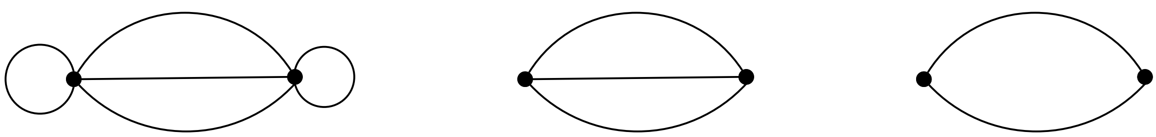
(e) Spanning Subgraph

Figure 7: Sub-Graph Types

Definition 6.1 (Subgraph) *A* **Subgraph** *of a graph* $\mathbf{G(V,E)}$ *is a graph* $\boldsymbol{H(U,F)}$ *with* $\boldsymbol{U(H)} \subseteq \boldsymbol{V(G)}$ *and* $\boldsymbol{F(H)} \subseteq \boldsymbol{E(G)}$

- When $\boldsymbol{U(H)} = \boldsymbol{V(G)}$, $\boldsymbol{H}$ is a spanning subgraph of $\boldsymbol{G}$
- If $\boldsymbol{H}$ is a subgraph of $\boldsymbol{G} \Rightarrow \boldsymbol{G}$ is a **super-graph** of $\boldsymbol{H}$

6.1 Induced Subgraph



(a) Graph $\mathbf{G(V,E)}$ (b) Induced graph (c) Subgraph

Figure 8: Sub-graph and Induced-graph Types

Definition 6.2 (Induced Subgraph) *If a subgraph has every possible edge, it is an induced subgraph.*

- When $F(H)$ consists of all the edges of G joining pairs of vertices of $U(H)$, we say H is **vertex induced** subgraph of G.
- Given a subset F of edges of G, let $U(H)$ be the set of end-vertices of the edge of $F(H)$, then $F(H)$ is the **edge induced** subgraph of G.

7 Partite Graphs

Definition 7.1 (r-Partite) *A graph* $\mathbf{G(V,E)}$*is said to be r-Partite* ($\mathbf{r > 0}$)*, if its vertexx set can be partitioned as* $V = V_1 \cup V_2 \ldots \cup V_r$ *such that if* $uv \in V(G)$*, then* $u \in V_i$ *and* $v \in V_j$*, i.e. everyone of the induced sub-graph* $< V_i >$ *is an empty graph.*

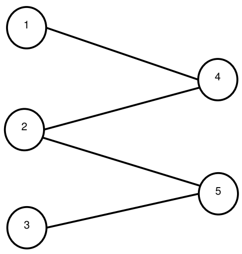


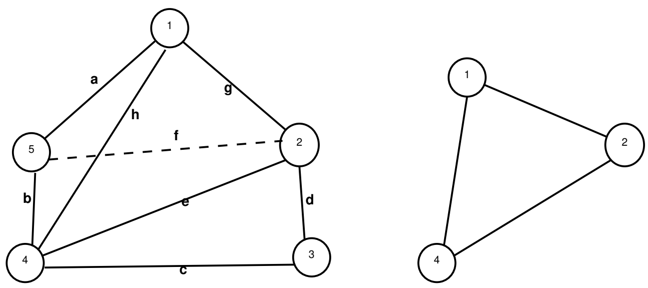
Figure 9: Bipartite Graph : $< V_1 >=< 1, 2, 3 >$,
$< V_2 >=< 4, 5 >$, $E(V_1) = E(V_2) = \phi$

If an r-partite graph has all possible edges, i.e. $uv \in E$ for all pairs $u \in V_i, v \in V_j$, then it is called a *complete r-partite graph.* If $|V_i| = n_i$, then the partite graph is denoted as $K_{n_1,n_2,\ldots,n_k}$.

A 2-partite/*bipartite* graph is defined as a graph $\mathbf{G(V,E)}$, where $V = V_1 \cup V_2$ is the bipartition of the vertices V. If there exists all possible edges beteen V_1 and V_2, then it is referred to as a *complete bipartite graph*, denoted as $K_{|V_1|,|V_2|}$.

The complete bipartite graph $K_{1,n}$ is called an *n-star* or an *n-claw* graph.

8 Operations on Graphs



(a) $\mathbf{G(V,E)}$ (b) $H(U,F)$

Figure 10: Main Operational Graphs

8.1 Complement of a Graph

Definition 8.1 (Complement of a Graph) *The* **complement** $\overline{G} = (V, \overline{E})$ *of a graph* $\mathrm{G(V,E)}$ *has the same vertex set as* G *and its edge set is the set complement* $\overline{E}(\overline{G})$ *of* $E(G)$ *in* $\mathbf{V_{(2)}}$, *i.e.* uv *is an edge of* $\overline{G}$ *iff* uv *is not an edge of* G. *A graph* G *is said to be self-complementary if* $G = \overline{G}$.

The complent of $\overline{K_n}$ of the complete graph of order n is clearly the empty graph of order n.

8.2 Removal of Edge

Definition 8.2 *Let* $\mathrm{G(V,E)}$ *be a graph and* $F \subset E(G)$. *Then the graph* $H = (V, E - F)$ *with the same vertex set as* G *and edge set* $E - F$ *is said to be obtained from* G *by removing the edges in the set* F. *It is denoted by* $G - F$. *If* F *consists of a single edge* $\mathbf{e} \in G$, *the graph obtained by removing* $\mathbf{e}$ *is denoted by* $G - e$.

Definition 8.3 (Isolated Vertex) *A vertex* v *of a graph* G *which is not adjacent with any other vertex is called an* **Isolated vertex** *of* G.

For $F \subseteq E$ or $e \in E$, $G - F$ or $G - e$ may contain isolated vertices which are not isolated vertices of G. The graphs obtained by removing these newly created isolated vertices from $G - F$ or $G - e$ are denoted by $G \backslash F$ or $G \backslash e$.

8.3 Removal of Vertex

Definition 8.4 *Let* $\mathrm{G(V,E)}$ *be a graph and* $v \in V(G)$. *Let* $E_v(G)$ *be the set of all edges incident with* v. *Then the graph* $H = (V - \{v\}, E - E_v)$ *is said to be obtained from* G *by the removal of the vertex* v *and is denoted by* $G - v$, *If* $U \subset V$, *the graph obtained by removing the vertices of* G *in* U, *in any order is denoted by* $G - U$.

If H is a *subgraph* of G, we denote $G - V(H)$ by $G - H$ and $G - E(H)$ by $\overline{H(G)}$. The latter is called the *relative complement* of H in G.

8.4 Addition of Edge

Definition 8.5 *Let* $\mathrm{G(V,E)}$ *be a graph and* f *an edge of* $\overline{G}$. *Then the graph* $H = (V, E \cup \{f\})$ *is said to be obtained from* G *by the addition of the edge* f *and is denoted by* $G + f$. *If* F *is a subset of edges of* G, *the graph obtained from* G *by adding the edges of* F *in any order is denoted by* $G + F$.

When the particular edge $f \in G$ is not specified, one may still use the notation $G + f$ if all graphs obtained from G by the addition of any edge f of $\overline{G}$ are isomorphic.

8.5 Addition of a Vertex

Definition 8.6 *Let* $\mathrm{G(V,E)}$ *be a graph and* $v \notin V$. *Then the graph* H *with vertex set* $V \cup \{v\}$ *and edge set* $E \cup \{uv | u \in V\}$ *is said to be obtained from* G *by adding a vertex* v, *and is denoted by* $G + v$

Thus $G + v$ is obtained from G by adding a new vertex and joining it all vertices of G.

8.6 Contraction of Graph

Definition 8.7 (Contraction of Graph) *Let* $\mathbf{G(V,E)}$*be a graph and* $\boldsymbol{G}$ **contract** $\boldsymbol{e}$ *is the graph* $\boldsymbol{G/e}$ *obtained by deleting* $\mathbf{e}$ *from* $\boldsymbol{E(G)}$ *, deleting* $\boldsymbol{v, w}$ *from* $\boldsymbol{V(G)}$ *and then adding a new vertex* $\boldsymbol{z}$ *which is incident to all edges in* $\boldsymbol{E(G)e}$ *which were incident to* $\boldsymbol{v}$ *or* $\boldsymbol{w}$ *.*

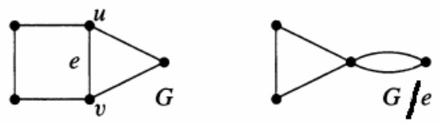


Figure 11: Contraction of a Graph

Note :

1. If $\boldsymbol{e} \in \boldsymbol{E(G)}$ is a **loop** then $\boldsymbol{E(G)} - \boldsymbol{e} = \boldsymbol{E(G)/e}$.
2. A **contraction** of $\mathbf{G(V,E)}$is any graph which can be obtained by *recursively contracting edges* in $\mathbf{G(V,E)}$.
3. A minor of a graph, $\mathbf{G(V,E)}$is any graph which can be obtained by **recursively deleting or contracting edges and deleting isolated vertices** from $\mathbf{G(V,E)}$.

8.7 Joining/Union of Graphs

Definition 8.8 (Union of Graphs) *Let* $\mathbf{G(V,E)}$*and* $\boldsymbol{H} = (\boldsymbol{U}, \boldsymbol{F})$ *be two graphs with disjoint vertex sets i.e* $\boldsymbol{V} \cap \boldsymbol{U} = \boldsymbol{\phi}$*. Then the join of* $\boldsymbol{G}$ *and* $\boldsymbol{H}$ *denoted by* $\boldsymbol{G} \vee \boldsymbol{H}$ *has vertex set* $\boldsymbol{V} \cup \boldsymbol{U}$ *and edge set* $\boldsymbol{E} \cup \boldsymbol{F} \cup [\boldsymbol{V}, \boldsymbol{U}]$*, where* $[\boldsymbol{V}, \boldsymbol{U}] = \{\boldsymbol{uv} | \boldsymbol{u} \in \boldsymbol{U}, \boldsymbol{v} \in \boldsymbol{V}\}$ *are all the set of edges joining every vertex of* $\boldsymbol{V}$ *to every vertex of* $\boldsymbol{U}$*.*

From the above definition, $\boldsymbol{G} + \boldsymbol{v} = \boldsymbol{G} \vee \boldsymbol{K_1} = \boldsymbol{G} \vee \overline{\boldsymbol{K_1}}$.

8.8 Intersection of Graphs

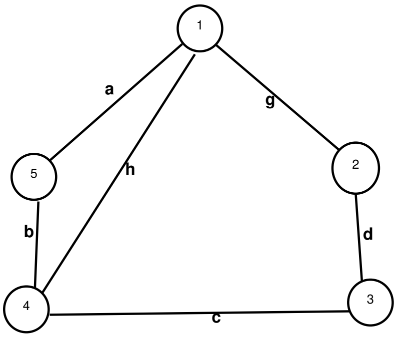
Definition 8.9 *The intersection of* $\boldsymbol{H_1(V_1, E_1)}$ *and* $\boldsymbol{H_2(V_2, E_2)}$ *is the subgraph of* $\mathbf{G(V,E)}$*given by* $\boldsymbol{H_1 \bigcap H_2} = (\boldsymbol{V_1 \bigcap V_2}, \boldsymbol{E_1 \bigcap E_2})$.

9 Representation of Graphs

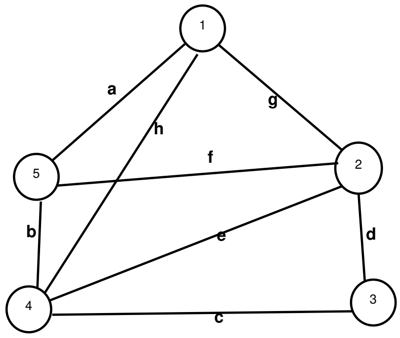
Graphs are usually labelled. Vertices and/or Edges can be labelled.

Labelling of graphs can be done based on

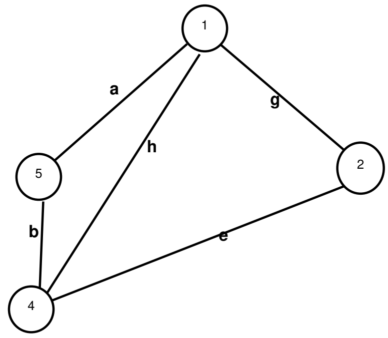
- Vertices
  - To give semantic meaning e.g. Places to visit in TSP or Autoroute
  - Labels can be arbitrary or change to prove some relationship between graphs
  - When we describe edges we usually refer to sets of vertices
- Edges



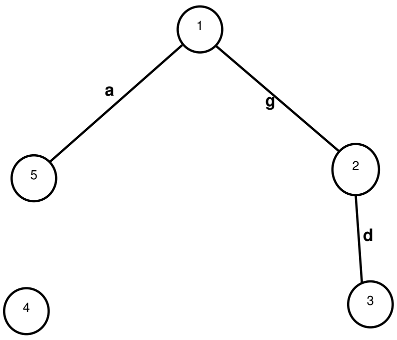
(a) Edge Deletion : $\boldsymbol{G - e}$



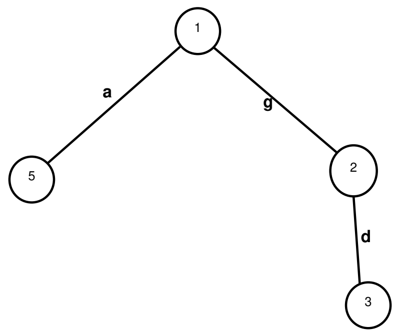
(b) Edge Addition : $\boldsymbol{G + f}$



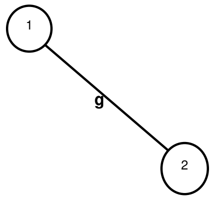
(c) Vertex Deletion : $\boldsymbol{G - 3}$



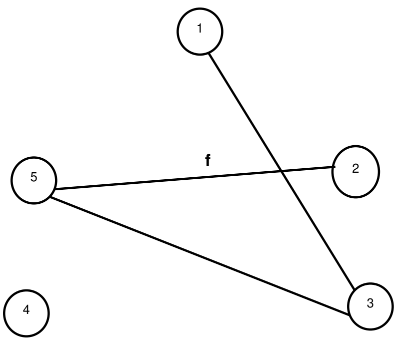
(d) Multiple Edge Deletions : $\boldsymbol{G - \{43, 42, 41, 45\}}$



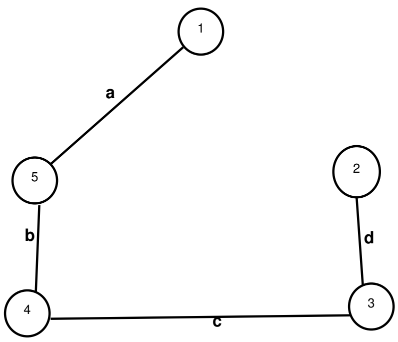
(e) Vertex Deletion : $\boldsymbol{G - 4 = G \backslash \{43, 42, 41, 45\}}$



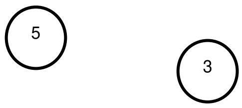
(f) Multiple Vertex Deletions : $\boldsymbol{G - \{3, 4, 5\}}$



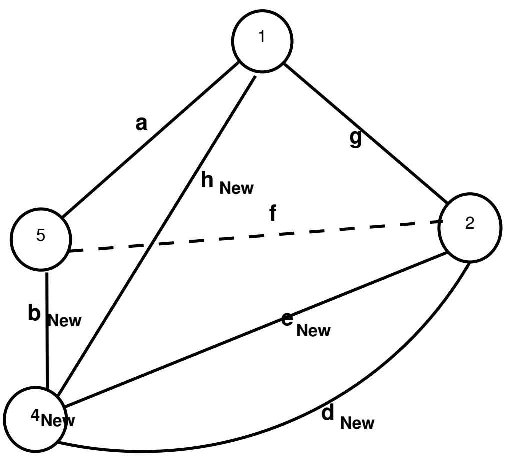
(g) Complement of Graph :$\boldsymbol{G - E + \overline{E} = \overline{G}}$



(h) Complement of Graph :$\boldsymbol{G - E(H) = \overline{H(G)}}$



(i) Graph Deletion : $\boldsymbol{G - V(H) = G - H}$



(j) Edge Contract : $\boldsymbol{G/c}$

Figure 12: Operations on Graphs

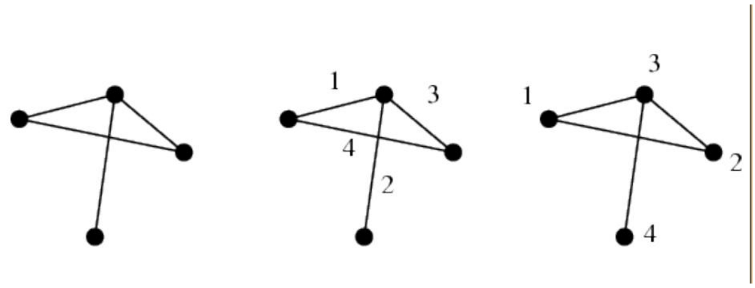
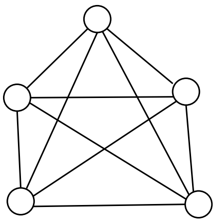
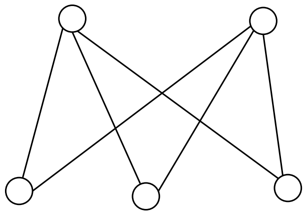


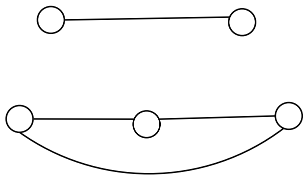
Figure 13: Labeled and Unlabeled Graph



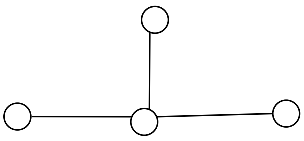
(a) Complete Graph - Kuratowski Graph $\boldsymbol{K_5}$



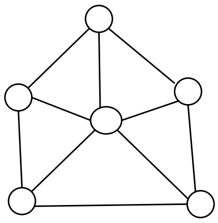
(b) Kuratowski Graph $\boldsymbol{K_{23}}$



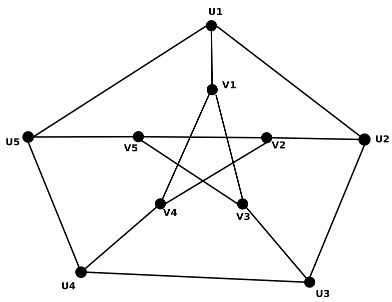
(c) $\overline{\boldsymbol{K_{23}}}$



(d) Star : $\boldsymbol{K_{13}}$



(e) Wheel :$\boldsymbol{W_6}$



(f) Peterson Graph

Figure 14: Special Graphs

- We use graphs to represent data, encode knowledge or enforce relationships between data
- Numbers usually represent weights, distances or cost of some relationship between the 2 vertices
- Graph Theory enumerates these weights in many ways to attempt to solve a problem:
  * Minimum cost - shortest path
  * Maximum cost
  * Max-Min costs in game playing