GRADIENT, DIVERGENCE and CURL

The vector differential operator del, written $\vec{\nabla}$, is defined by

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \equiv \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

Gradient of a scalar point function :

Let $\phi(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then the gradient of ϕ, written $\vec{\nabla}\phi$ or gradϕ, is defined by

$$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

Note that $\vec{\nabla}\phi$ defines a vector field.

Divergence of a vector point function :

Let $\vec{V} = V_1\vec{i} + V_2\vec{j} + V_3\vec{k}$ be defined and differentiable at each point (x, y, z) in a certain region of space. Then the divergence of $\vec{\nabla}$, written $\vec{\nabla}.\vec{V}$ or div$\vec{V}$, is defined by

$$\vec{\nabla}.\vec{V} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right).\left(V_1\vec{i} + V_2\vec{j} + V_3\vec{k}\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Note that $\vec{\nabla}.\vec{V} \neq \vec{V}.\vec{\nabla}$

Curl of a vector point function:

If $\vec{V} = V_1\vec{i} + V_2\vec{j} + V_3\vec{k}$ is a differentiable vector field then the curl or rotation of $\vec{V}$, written $\vec{\nabla} \times \vec{V}$, curl$\vec{V}$ or rot$\vec{V}$, is defined by

$$\vec{\nabla} \times \vec{V} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(V_1\vec{i} + V_2\vec{j} + V_3\vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

Directional derivative: Let f be a scalar point function of the cartesian co-ordinates (x, y, z) of a point. Then the partial derivatives $\frac{\partial f}{\partial x}$ which gives the rate of increase of f along the x-axis, is called the directional derivative of f along the x-axis. Similar meaning are given to the partial derivatives $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

We consider further two points P and P', where a uniform point function f is continuous and has values f and $f + \delta f$. Let δs be the distance PP' from P to P'. If $\lim_{\delta s \to 0} \frac{\delta f}{\delta s}$ be finite, then this limit is called the directional derivative of f at P in the direction from P to P' and is denoted by $\frac{df}{ds}$, which is a scalar.

Expressions for directional derivative :

Consider a scalar point function $f(\vec{r})$ or $f(x,y,z)$ in the neighbourhood of the point $P_0(x_0,y_0,z_0)$ of position vector $\vec{r}_0$, where it is continuous and differentiable. A straight line through P_0 in the direction of the unit vector $\vec{e} = \cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}$ has the vector equation $\vec{r} = \vec{r}_0 + s\vec{e}$, where the scalar s=$\left|\overrightarrow{PP_0}\right|$>0

The parametric equations of the straight line are

$x = x_0 + s\cos\alpha,\ y = y_0 + s\cos\beta,\ z = z_0 + s\cos\gamma,$ …………..(1)

Along the line, $f(x,y,z)$ is a function of s alone and we have,

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

or, by (1), $\frac{df}{ds} = \cos\alpha\frac{\partial f}{\partial x} + \cos\beta\frac{\partial f}{\partial y} + \cos\gamma\frac{\partial f}{\partial z}$ …………..(2)

If the partial derivatives be computed at P_0, then (2) gives the directional derivative of $f(x,y,z)$ at P_0 in the direction of $\vec{e}$.

Now, $\frac{df}{ds} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right).\left(\cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}\right)$

$= \vec{\nabla} f\,.\,\vec{e}$

= the directional derivative of f in the direction of $\vec{e}$

Solenoidal vector: A vector is said to be solenoidal if its divergence is zero.

Irrotational vector: A vector $\vec{A}$ is said to be irrotational if $\vec{\nabla} \times \vec{A} = \vec{0}$ i.e., if curl $\vec{A} = \vec{0}$.

Formulae involving $\vec{\nabla}$: If $\vec{A}$ and $\vec{B}$ are differentiable vector functions, and ϕ and ψ are differentiable scalar function of position (x,y,z), then

1. $\vec{\nabla}(\phi+\psi) = \vec{\nabla}\phi + \vec{\nabla}\psi$ or grad$(\phi+\psi)$=gradϕ+gradψ.

2. $\vec{\nabla}.\left(\vec{A}+\vec{B}\right) = \vec{\nabla}.\vec{A} + \vec{\nabla}.\vec{B}$ or div$\left(\vec{A}+\vec{B}\right)$=div$\vec{A}$+div$\vec{B}$

3. $\vec{\nabla}\times\left(\vec{A}+\vec{B}\right) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{B}$ or curl$\left(\vec{A}+\vec{B}\right)$= curl$\vec{A}$+curl$\vec{B}$

4. $\vec{\nabla}.\left(\phi\vec{A}\right) = \left(\vec{\nabla}\phi\right).\vec{A} + \phi\left(\vec{\nabla}.\vec{A}\right)$

5. $\vec{\nabla}\times\left(\phi\vec{A}\right) = \left(\vec{\nabla}\phi\right)\times\vec{A} + \phi\left(\vec{\nabla}\times\vec{A}\right)$

6. $\vec{\nabla}.\left(\vec{A}\times\vec{B}\right) = \vec{B}.\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}.\left(\vec{\nabla}\times\vec{B}\right)$

7. $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\left(\vec{B}.\vec{\nabla}\right)\vec{A}-\vec{B}\left(\vec{\nabla}.\vec{A}\right)-\left(\vec{A}.\vec{\nabla}\right)\vec{B}+\vec{A}\left(\vec{\nabla}.\vec{B}\right)$

8. $\vec{\nabla}\left(\vec{A}.\vec{B}\right)=\left(\vec{B}.\vec{\nabla}\right)\vec{A}+\left(\vec{A}.\vec{\nabla}\right)\vec{B}+\vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)+\vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)$

9. $\vec{\nabla}.\left(\vec{\nabla}\phi\right)=\vec{\nabla}^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$ where $\vec{\nabla}^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

10. $\vec{\nabla}\times\left(\vec{\nabla}\phi\right)=\vec{0}$**.** The curl of the gradient of ϕ is zero.

11. $\vec{\nabla}.\left(\vec{\nabla}\times\vec{A}\right)=0$**.** The divergence of the curl of $\vec{A}$ is zero.

12. $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right)-\vec{\nabla}^2\vec{A}$

Worked out exercises:

Exercise1: Find $\vec{\nabla}\phi$ if (a) $\phi=\ln|\vec{r}|$, (b) $\phi=\frac{1}{r}$, where $r=|\vec{r}|$.

☺**.** (a) $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$. Then $|\vec{r}|=\sqrt{x^2+y^2+z^2}$, $\phi=\ln|\vec{r}|=\frac{1}{2}\left(x^2+y^2+z^2\right)$

$$\vec{\nabla}\phi=\frac{1}{2}\vec{\nabla}\left\{\ln\left(x^2+y^2+z^2\right)\right\}$$

$$=\frac{1}{2}\left[\vec{i}\frac{\partial}{\partial x}\left\{\ln\left(x^2+y^2+z^2\right)\right\}+\vec{j}\frac{\partial}{\partial y}\left\{\ln\left(x^2+y^2+z^2\right)\right\}+\vec{k}\frac{\partial}{\partial z}\left\{\ln\left(x^2+y^2+z^2\right)\right\}\right]$$

$$=\frac{1}{2}\left\{\vec{i}\frac{2x}{x^2+y^2+z^2}+\vec{j}\frac{2y}{x^2+y^2+z^2}+\vec{k}\frac{2z}{x^2+y^2+z^2}\right\}$$

$$=\frac{x\vec{i}+y\vec{j}+z\vec{k}}{x^2+y^2+z^2}=\frac{\vec{r}}{r^2}$$ where $r=|\vec{r}|$.

(b) $$\vec{\nabla}\phi=\vec{\nabla}\left(\frac{1}{r}\right)=\vec{\nabla}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$=\vec{i}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)+\vec{j}\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)+\vec{k}\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$=\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}2x\right\}\vec{i}+\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}2y\right\}\vec{j}$$

$$+\left\{-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}2z\right\}\vec{k}=\frac{-x\vec{i}-y\vec{j}-z\vec{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}=-\frac{\vec{r}}{r^3}$$

Exercise2: Show that $\vec{\nabla}r^n=nr^{n-2}\vec{r}$.

☺**.** $\vec{\nabla}r^n=\vec{\nabla}\left(\sqrt{x^2+y^2+z^2}\right)^n=\vec{\nabla}\left(x^2+y^2+z^2\right)^{\frac{n}{2}}$

$$=\vec{i}\,\frac{\partial}{\partial x}\left\{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}\right\}+\vec{j}\,\frac{\partial}{\partial y}\left\{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}\right\}+\vec{k}\,\frac{\partial}{\partial z}\left\{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}\right\}$$

$$=\vec{i}\left\{\frac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}2x\right\}+\vec{j}\left\{\frac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}2y\right\}$$

$$+\vec{k}\left\{\frac{n}{2}\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}2z\right\}=n\left(x^2+y^2+z^2\right)^{\frac{n}{2}-1}\left(x\vec{i}+y\vec{j}+z\vec{k}\right)$$

$$=n\left(r^2\right)^{\frac{n}{2}-1}\vec{r}=nr^{n-2}\vec{r}$$

Exercise3: Show that $\vec{\nabla}\phi$ is a vector perpendicular to the surface $\phi(x,y,z)=c$, where c is a constant.

☺. Let $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ be the position vector to any point $P(x,y,z)$ on the surface. Then $d\vec{r}=dx\vec{i}+dy\vec{j}+dz\vec{k}$ lies in the tangent plane to the surface at P.

But $d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=0$

or, $\left(\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}\right).\left(dx\vec{i}+dy\vec{j}+dz\vec{k}\right)=0$

or, $\vec{\nabla}\phi\,.\,d\vec{r}=0$, so that $\vec{\nabla}\phi$ is perpendicular to $d\vec{r}$ and therefore to the surface.

Exercise4: Find a unit normal to the surface $x^2y+2xz=4$ at the point $(2,-2,3)$.

☺. $\vec{\nabla}\left(x^2y+2xz\right)=(2xy+2z)\,\vec{i}+x^2\,\vec{j}+2x\,\vec{k}$.

At $(2,-2,3)$, $\vec{\nabla}\left(x^2y+2xz\right)=-2\vec{i}+4\vec{j}+4\vec{k}$

Then a unit normal to the surface $x^2y+2xz=4$ at $(2,-2,3)$ is

$$\frac{-2\vec{i}+4\vec{j}+4\vec{k}}{\sqrt{(-2)^2+4^2+4^2}}=-\frac{1}{3}\vec{i}+\frac{2}{3}\vec{j}+\frac{2}{3}\vec{k}.$$

Another unit normal is $\frac{1}{3}\vec{i}-\frac{2}{3}\vec{j}-\frac{2}{3}\vec{k}$ having direction opposite to above.

Exercise5: Show that greatest rate of charge of ϕ, i.e. the maximum directional derivative, takes place in the direction of, and has the magnitude of the vector $\vec{\nabla}\phi$.

☺. Now, $\frac{d\phi}{ds}=\frac{\partial\phi}{\partial x}\frac{dx}{ds}+\frac{\partial\phi}{\partial y}\frac{dy}{ds}+\frac{\partial\phi}{\partial z}\frac{dz}{ds}$

$$=\left(\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}\right).\left(\frac{dx}{ds}\vec{i}+\frac{dy}{ds}\vec{j}+\frac{dz}{ds}\vec{k}\right)$$

$$=\vec{\nabla}\phi\,.\,\frac{d\vec{r}}{ds}$$

$\therefore \dfrac{d\phi}{ds}$ is the projection of $\vec{\nabla}\phi$ in the direction $\dfrac{d\vec{r}}{ds}$.

This projection will be a maximum when $\vec{\nabla}\phi$ and $\dfrac{d\vec{r}}{ds}$ have the same direction. Then the maximum value of $\dfrac{d\phi}{ds}$ takes place in the direction of $\vec{\nabla}\phi$ and its magnitude is $\left|\vec{\nabla}\phi\right|$.

Exercise6: Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\vec{i} - \vec{j} - 2\vec{k}$.

☺. $\vec{\nabla}\phi = \vec{\nabla}\left(x^2yz + 4xz^2\right) = \left(2xyz + 4z^2\right)\vec{i} + x^2z\vec{j} + \left(x^2y + 8xz\right)\vec{k}$

At $(1,-2,-1)$, $\vec{\nabla}\phi = 8\vec{i} - \vec{j} - 10\vec{k}$

The unit vector in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$ is

$$\vec{a} = \frac{2\vec{i} - \vec{j} - 2\vec{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}$$

Then the required directional derivative is

$$\vec{\nabla}\phi\,.\,\vec{a} = \left(8\vec{i} - \vec{j} - 10\vec{k}\right).\left(\frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}\right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

Exercise7:(a) In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum ?

(b) What is the magnitude of this maximum ?

☺. $\vec{\nabla}\phi = \vec{\nabla}\left(x^2yz^3\right) = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$

At $(2,1,-1)$, $\vec{\nabla}\phi = -4\vec{i} - 4\vec{j} + 12\vec{k}$

(a) the directional derivative is a maximum in the direction $\vec{\nabla}\phi = -4\vec{i} - 4\vec{j} + 12\vec{k}$.

(b)the magnitude of this maximum is

$$\left|\vec{\nabla}\phi\right| = \sqrt{(-4)^2 + (-4)^2 + (12)^2} = \sqrt{176} = 4\sqrt{11}.$$

Exercise8: Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2,-1,2)$

☺. The angle between the surface at the point is the angle between the normal to the surface at the point.

A normal to $x^2 + y^2 + z^2 = 9$ at $(2,-1,2)$ is

$\vec{\nabla}\phi_1 = \vec{\nabla}\left(x^2 + y^2 + z^2\right) = 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = 4\vec{i} - 2\vec{j} + 4\vec{k}$.

A normal to $z = x^2 + y^2 - 3$ or $x^2 + y^2 - z = 3$ at $(2,-1,2)$ is

$\vec{\nabla}\phi_2 = \vec{\nabla}(x^2 + y^2 - z) = 2x\vec{i} + 2y\vec{j} - \vec{k} = 4\vec{i} - 2\vec{j} - \vec{k}$

$\left(\vec{\nabla}\phi_1\right).\left(\vec{\nabla}\phi_2\right) = \left|\vec{\nabla}\phi_1\right|\left|\vec{\nabla}\phi_2\right|\cos\theta$, where θ is the required angle.

Then $\left(4\vec{i} - 2\vec{j} + 4\vec{k}\right).\left(4\vec{i} - 2\vec{j} - \vec{k}\right) = \left|4\vec{i} - 2\vec{j} + 4\vec{k}\right|\left|4\vec{i} - 2\vec{j} - \vec{k}\right|\cos\theta$

or, $16+4-4=\sqrt{(4)^2+(-2)^2+(4)^2}\sqrt{(4)^2+(-2)^2+(-1)^2}\cos\theta$

or, $\cos\theta=\dfrac{16}{6\sqrt{21}}=\dfrac{8\sqrt{21}}{63}=0.5819$; thus the acute angle is $\theta=\cos^{-1}(0.5819)=54^0 25'$.

Exercise9: Given $\phi=2x^3y^2z^4$. (a) Find $\vec{\nabla}.\vec{\nabla}\phi$ (or divgrad ϕ). (b) Show that $\vec{\nabla}.\vec{\nabla}\phi=\vec{\nabla}^2\phi$, where $\vec{\nabla}^2\phi\equiv\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.

☺. (a) $\vec{\nabla}\phi=\vec{i}\,\dfrac{\partial}{\partial x}\left(2x^3y^2z^4\right)+\vec{j}\,\dfrac{\partial}{\partial y}\left(2x^3y^2z^4\right)+\vec{k}\,\dfrac{\partial}{\partial z}\left(2x^3y^2z^4\right)$

$$=6x^2y^2z^4\vec{i}+4x^3yz^4\vec{j}+8x^3y^2z^3\vec{k}.$$

Then $\vec{\nabla}.\vec{\nabla}\phi=\left(\dfrac{\partial}{\partial x}\vec{i}+\dfrac{\partial}{\partial y}\vec{j}+\dfrac{\partial}{\partial z}\vec{k}\right).\left(6x^2y^2z^4\vec{i}+4x^3yz^4\vec{j}+8x^3y^2z^3\vec{k}\right)$

$$=\frac{\partial}{\partial x}\left(6x^2y^2z^4\right)+\frac{\partial}{\partial y}\left(4x^3yz^4\right)+\frac{\partial}{\partial z}\left(8x^3y^2z^3\right)$$

$$=12xy^2z^4+4x^3z^4+24x^3y^2z^2$$

(b) $\vec{\nabla}.\vec{\nabla}\phi=\left(\dfrac{\partial}{\partial x}\vec{i}+\dfrac{\partial}{\partial y}\vec{j}+\dfrac{\partial}{\partial z}\vec{k}\right).\left(\dfrac{\partial\phi}{\partial x}\vec{i}+\dfrac{\partial\phi}{\partial y}\vec{j}+\dfrac{\partial\phi}{\partial z}\vec{k}\right)$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

$$=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=\vec{\nabla}^2\phi.$$

Exercise10: Prove that $\vec{\nabla}^2\left(\dfrac{1}{r}\right)=0$

☺. $\vec{\nabla}^2\left(\dfrac{1}{r}\right)=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}\right)\left(\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=-\frac{\partial}{\partial x}\left[x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right]$$

$$=\frac{3}{2}x\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}(2x)-\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$=\frac{3x^2-x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=\frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

and $\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)=\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

Then by addition $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$

$$=\frac{2x^2-y^2-z^2+2y^2-z^2-x^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}=0.$$

The equation $\vec{\nabla}^2\phi=0$ is called Laplace's equation. It follows that $\phi=\frac{1}{r}$ is a solution of this equation.

Exercise11: Prove (a) $\vec{\nabla}\times\left(\vec{A}+\vec{B}\right)=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$

(b) $\vec{\nabla}\times\left(\phi\vec{A}\right)=\vec{\nabla}\phi\times\vec{A}+\phi\left(\vec{\nabla}\times\vec{A}\right)$

☺. Let $\vec{A}=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$, $\vec{B}=B_1\vec{i}+B_2\vec{j}+B_3\vec{k}$.

Now, $\vec{\nabla}\times\left(\left(A_1+B_1\right)\vec{i}+\left(A_2+B_2\right)\vec{j}+\left(A_3+B_3\right)\vec{k}\right)$

$$=\left[\frac{\partial}{\partial y}\left(A_3+B_3\right)-\frac{\partial}{\partial z}\left(A_2+B_2\right)\right]\vec{i}+\left[\frac{\partial}{\partial z}\left(A_1+B_1\right)-\frac{\partial}{\partial x}\left(A_3+B_3\right)\right]\vec{j}$$

$$+\left[\frac{\partial}{\partial x}\left(A_2+B_2\right)-\frac{\partial}{\partial y}\left(A_1+B_1\right)\right]\vec{k}$$

$$=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\vec{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\vec{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\vec{k}+\left(\frac{\partial B_3}{\partial y}-\frac{\partial B_2}{\partial z}\right)\vec{i}+\left(\frac{\partial B_1}{\partial z}-\frac{\partial B_3}{\partial x}\right)\vec{j}$$

$$+\left(\frac{\partial B_2}{\partial x}-\frac{\partial B_1}{\partial y}\right)\vec{k}=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}.$$

(b) Let $\vec{A}=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$

$$\vec{\nabla}\times\left(\phi\vec{A}\right)=\vec{\nabla}\times\left(\phi A_1\vec{i}+\phi A_2\vec{j}+\phi A_3\vec{k}\right)$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \phi A_1 & \phi A_2 & \phi A_3\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(\phi A_3)-\frac{\partial}{\partial z}(\phi A_2)\right]\vec{i}+\left[\frac{\partial}{\partial z}(\phi A_1)-\frac{\partial}{\partial x}(\phi A_3)\right]\vec{j}+\left[\frac{\partial}{\partial x}(\phi A_2)-\frac{\partial}{\partial y}(\phi A_1)\right]\vec{k}$$

$$=\left[\frac{\partial\phi}{\partial y}A_3+\phi\frac{\partial A_3}{\partial y}-\frac{\partial\phi}{\partial z}A_2-\phi\frac{\partial A_2}{\partial z}\right]\vec{i}+\left[\frac{\partial\phi}{\partial z}A_1+\phi\frac{\partial A_1}{\partial z}-\frac{\partial\phi}{\partial x}A_3-\phi\frac{\partial A_3}{\partial x}\right]\vec{j}$$

$$+\left[\frac{\partial\phi}{\partial x}A_2+\phi\frac{\partial A_2}{\partial x}-\frac{\partial\phi}{\partial y}A_1-\phi\frac{\partial A_1}{\partial y}\right]\vec{k}$$

$$=\phi\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\vec{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\vec{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\vec{k}\right]$$

$$+\left[\left(\frac{\partial \phi}{\partial y}A_3-\frac{\partial \phi}{\partial z}A_2\right)\vec{i}+\left(\frac{\partial \phi}{\partial z}A_1-\frac{\partial \phi}{\partial x}A_3\right)\vec{j}+\left(\frac{\partial \phi}{\partial x}A_2-\frac{\partial \phi}{\partial y}A_1\right)\vec{k}\right]$$

$$=\phi\left(\vec{\nabla}\times\vec{A}\right)+\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$$

$$=\phi\left(\vec{\nabla}\times\vec{A}\right)+\left(\vec{\nabla}\phi\times\vec{A}\right)$$

Exercise12: Prove (a) $\vec{\nabla}\times\left(\vec{\nabla}\phi\right)=\vec{0}$ (curlgradϕ = $\vec{0}$),

(b) $\vec{\nabla}.\left(\vec{\nabla}\times\vec{A}\right)=0$ (div curl $\vec{A}=0$)

☺**.** (a) $\vec{\nabla}\times\left(\vec{\nabla}\phi\right)=\vec{\nabla}\times\left(\frac{\partial \phi}{\partial x}\vec{i}+\frac{\partial \phi}{\partial y}\vec{j}+\frac{\partial \phi}{\partial z}\vec{k}\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)\right]\vec{i}+\left[\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right)\right]\vec{j}+\left[\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)\right]\vec{k}$$

$=\left(\frac{\partial^2 \phi}{\partial y\partial z}-\frac{\partial^2 \phi}{\partial z\partial y}\right)\vec{i}+\left(\frac{\partial^2 \phi}{\partial z\partial x}-\frac{\partial^2 \phi}{\partial x\partial z}\right)\vec{j}+\left(\frac{\partial^2 \phi}{\partial x\partial y}-\frac{\partial^2 \phi}{\partial y\partial x}\right)\vec{k}=\vec{0}$, provided we assume that ϕ has continuous second order partial derivatives so that the order of differentiation is immaterial.

(b) Let $\vec{A}=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$

Then, $\vec{\nabla}.\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}.\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$

$$=\vec{\nabla}.\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\vec{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\vec{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\vec{k}\right]$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

$$=\frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}+\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial y\partial x}+\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial z\partial y}$$

$=0$, assuming that $\vec{A}$ has continuous second order partial derivatives.

Exercise13: Find $\operatorname{curl}\left(\vec{r}f(r)\right)$, where $f(r)$ is differentiable.

☺**.** $\operatorname{curl}\left(\vec{r}f(r)\right)=\vec{\nabla}\times\left(\vec{r}f(r)\right)$

$$=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$=\left(z\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial z}\right)\vec{i}+\left(x\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial x}\right)\vec{j}+\left(y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}\right)\vec{k}$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}=\frac{\partial f}{\partial r}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)=\frac{\partial f}{\partial r}\frac{2x}{2\sqrt{x^2+y^2+z^2}}=\frac{xf'}{\sqrt{x^2+y^2+z^2}}=\frac{xf'}{r}$$

Similarly, $\dfrac{\partial f}{\partial y}=\dfrac{yf'}{\sqrt{x^2+y^2+z^2}}=\dfrac{yf'}{r}$ and $\dfrac{\partial f}{\partial z}=\dfrac{zf'}{r}$

Therefore, $\operatorname{curl}\left(\vec{r}f(r)\right)=\left(\dfrac{zyf'}{r}-\dfrac{yzf'}{r}\right)\vec{i}+\left(\dfrac{xzf'}{r}-\dfrac{zxf'}{r}\right)\vec{j}+\left(\dfrac{yxf'}{r}-\dfrac{xyf'}{r}\right)\vec{k}=\vec{0}$

Exercise14: Prove $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=-\vec{\nabla}^2\vec{A}+\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right)$

☺**.** Let $\vec{A}=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\times\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$=\vec{\nabla}\times\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\vec{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\vec{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\vec{k}\right]$$

$$=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial A_3}{\partial y}-\dfrac{\partial A_2}{\partial z} & \dfrac{\partial A_1}{\partial z}-\dfrac{\partial A_3}{\partial x} & \dfrac{\partial A_2}{\partial x}-\dfrac{\partial A_1}{\partial y} \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\right]\vec{i}+\left[\frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)-\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\right]\vec{j}$$

$$+\left[\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\right]\vec{k}$$

$$=\left(-\frac{\partial^2 A_1}{\partial y^2}-\frac{\partial^2 A_1}{\partial z^2}\right)\vec{i}+\left(-\frac{\partial^2 A_2}{\partial z^2}-\frac{\partial^2 A_2}{\partial x^2}\right)\vec{j}+\left(-\frac{\partial^2 A_3}{\partial x^2}-\frac{\partial^2 A_3}{\partial y^2}\right)\vec{k}$$

$$+\left(\frac{\partial^2 A_2}{\partial y\partial x}+\frac{\partial^2 A_3}{\partial z\partial x}\right)\vec{i}+\left(\frac{\partial^2 A_3}{\partial z\partial y}+\frac{\partial^2 A_1}{\partial x\partial y}\right)\vec{j}+\left(\frac{\partial^2 A_1}{\partial x\partial z}+\frac{\partial^2 A_2}{\partial y\partial z}\right)\vec{k}$$

$$=\left(-\frac{\partial^2 A_1}{\partial x^2}-\frac{\partial^2 A_1}{\partial y^2}-\frac{\partial^2 A_1}{\partial z^2}\right)\vec{i}+\left(-\frac{\partial^2 A_2}{\partial x^2}-\frac{\partial^2 A_2}{\partial y^2}-\frac{\partial^2 A_2}{\partial z^2}\right)\vec{j}+$$

$$\left(-\frac{\partial^2 A_3}{\partial x^2}-\frac{\partial^2 A_3}{\partial y^2}-\frac{\partial^2 A_3}{\partial z^2}\right)\vec{k}+\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_2}{\partial y\partial x}+\frac{\partial^2 A_3}{\partial z\partial x}\right)\vec{i}$$

$$+\left(\frac{\partial^2 A_1}{\partial x\partial y}+\frac{\partial^2 A_2}{\partial y^2}+\frac{\partial^2 A_3}{\partial z\partial y}\right)\vec{j}+\left(\frac{\partial^2 A_1}{\partial x\partial z}+\frac{\partial^2 A_2}{\partial y\partial z}+\frac{\partial^2 A_3}{\partial z^2}\right)\vec{k}$$

$$=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(A_1\vec{i}+A_2\vec{j}+A_3\vec{k}\right)+\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)\right\}\vec{i}$$

$$+\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)\right\}\vec{j}+\left\{\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)\right\}\vec{k}$$

$$=-\vec{\nabla}^2\vec{A}+\vec{\nabla}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)$$

$$=--\vec{\nabla}^2\vec{A}+\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right)$$

Exercise15: Prove $\vec{\nabla}.\left(\vec{A}\times\vec{B}\right)=\vec{B}.\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}.\left(\vec{\nabla}\times\vec{B}\right)$

☺**.** Let $\vec{A}=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$, $\vec{B}=B_1\vec{i}+B_2\vec{j}+B_3\vec{k}$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}$$

$$=\left(A_2B_3-B_2A_3\right)\vec{i}+\left(A_3B_1-B_3A_1\right)\vec{j}+\left(A_1B_2-A_2B_1\right)\vec{k}$$

$$\therefore\ \vec{\nabla}.\left(\vec{A}\times\vec{B}\right)=\frac{\partial}{\partial x}\left(A_2B_3-B_2A_3\right)+\frac{\partial}{\partial y}\left(A_3B_1-B_3A_1\right)+\frac{\partial}{\partial z}\left(A_1B_2-B_1A_2\right)$$

$$=B_1\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+B_2\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+B_3\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)-A_1\left(\frac{\partial B_3}{\partial y}-\frac{\partial B_2}{\partial z}\right)$$

$$-A_2\left(\frac{\partial B_1}{\partial z}-\frac{\partial B_3}{\partial x}\right)-A_3\left(\frac{\partial B_2}{\partial x}-\frac{\partial B_1}{\partial y}\right)$$

$$=\left(B_1\vec{i}+B_2\vec{j}+B_3\vec{k}\right).\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\vec{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\vec{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\vec{k}\right]$$

$$-\left(A_1\vec{i}+A_2\vec{j}+A_3\vec{k}\right).\left[\left(\frac{\partial B_3}{\partial y}-\frac{\partial B_2}{\partial z}\right)\vec{i}+\left(\frac{\partial B_1}{\partial z}-\frac{\partial B_3}{\partial x}\right)\vec{j}+\left(\frac{\partial B_2}{\partial x}-\frac{\partial B_1}{\partial y}\right)\vec{k}\right]$$

$$=\vec{B}.\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}-\vec{A}.\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ B_1 & B_2 & B_3\end{vmatrix}$$

$$=\vec{B}.\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}.\left(\vec{\nabla}\times\vec{B}\right)$$

Exercise16: Prove $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\left(\vec{B}.\vec{\nabla}\right)\vec{A}-\vec{B}\left(\vec{\nabla}.\vec{A}\right)-\left(\vec{A}.\vec{\nabla}\right)\vec{B}+\vec{A}\left(\vec{\nabla}.\vec{B}\right)$

☺**.** Let $\vec{A}=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$, $\vec{B}=B_1\vec{i}+B_2\vec{j}+B_3\vec{k}$

and $\vec{C}=\vec{A}\times\vec{B}=C_1\vec{i}+C_2\vec{j}+C_3\vec{k}$.

Then $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\vec{\nabla}\times\vec{C}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ C_1 & C_2 & C_3\end{vmatrix}$

$$=\left(\frac{\partial C_3}{\partial y}-\frac{\partial C_2}{\partial z}\right)\vec{i}+\left(\frac{\partial C_1}{\partial z}-\frac{\partial C_3}{\partial x}\right)\vec{j}+\left(\frac{\partial C_2}{\partial x}-\frac{\partial C_1}{\partial y}\right)\vec{k}$$

$$=\left(\frac{\partial C_2}{\partial x}\vec{k}-\frac{\partial C_3}{\partial x}\vec{j}\right)+\left(\frac{\partial C_3}{\partial y}\vec{i}-\frac{\partial C_1}{\partial y}\vec{k}\right)+\left(\frac{\partial C_1}{\partial z}\vec{j}-\frac{\partial C_2}{\partial z}\vec{i}\right)$$

$$=\vec{i}\times\frac{\partial\vec{C}}{\partial x}+\vec{j}\times\frac{\partial\vec{C}}{\partial y}+\vec{k}\times\frac{\partial\vec{C}}{\partial z}$$

[Since, $\vec{i}\times\frac{\partial\vec{C}}{\partial x}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 0\\ \frac{\partial C_1}{\partial x} & \frac{\partial C_2}{\partial x} & \frac{\partial C_3}{\partial x}\end{vmatrix}=\frac{\partial C_2}{\partial x}\vec{k}-\frac{\partial C_3}{\partial x}\vec{j}$ etc.]

$$=\sum\left(\vec{i}\times\frac{\partial\vec{C}}{\partial x}\right)$$

$$=\sum\left\{\vec{i}\times\frac{\partial}{\partial x}\left(\vec{A}\times\vec{B}\right)\right\}$$

$$=\sum\left[\vec{i}\left\{\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)+\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)\right\}\right]$$

$$=\sum\left\{\vec{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right\}+\sum\left[\vec{i}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)\right]$$

$$=\sum\left\{\left(\vec{i}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{A}\right\}-\sum\left\{\left(\vec{i}\cdot\vec{A}\right)\frac{\partial\vec{B}}{\partial x}\right\}+\sum\left\{\left(\vec{i}\cdot\vec{B}\right)\frac{\partial\vec{A}}{\partial x}\right\}-\sum\left\{\left(\vec{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B}\right\}\ldots\ldots(1)$$

$$\sum\left\{\left(\vec{i}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{A}\right\}=\left(\frac{\partial B_1}{\partial x}+\frac{\partial B_2}{\partial y}+\frac{\partial B_3}{\partial z}\right)\vec{A}=\left(\vec{\nabla}\cdot\vec{B}\right)\vec{A}\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Similarly, $\sum\left\{\left(\vec{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B}\right\}=\left(\vec{\nabla}\cdot\vec{A}\right)\vec{B}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$

$$\sum\left\{\left(\vec{i}\cdot\vec{A}\right)\frac{\partial\vec{B}}{\partial x}\right\}=A_1\frac{\partial\vec{B}}{\partial x}+A_2\frac{\partial\vec{B}}{\partial y}+A_3\frac{\partial\vec{B}}{\partial z}$$

$$=\left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)\vec{B}=\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}\quad\ldots\ldots\ldots..(4)$$

Similarly, $\sum\left\{\left(\vec{i}\cdot\vec{B}\right)\frac{\partial\vec{A}}{\partial x}\right\}=\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}\ldots\ldots\ldots\ldots.(5)$

Using (2),(3),(4) and (5) in (1) we have,

$$\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\left(\vec{B}.\vec{\nabla}\right)\vec{A}-\vec{B}\left(\vec{\nabla}.\vec{A}\right)-\left(\vec{A}.\vec{\nabla}\right)\vec{B}+\vec{A}\left(\vec{\nabla}.\vec{B}\right)$$

Exercise17: Prove that

$$\vec{\nabla}\left(\vec{A}.\vec{B}\right)=\left(\vec{B}.\vec{\nabla}\right)\vec{A}+\left(\vec{A}.\vec{\nabla}\right)\vec{B}+\vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)+\vec{A}\times\vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)$$

☺**.** Let $\vec{A}=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$, $\vec{B}=B_1\vec{i}+B_2\vec{j}+B_3\vec{k}$

$$\vec{\nabla}\left(\vec{A}.\vec{B}\right)=\sum\left\{\vec{i}\,\frac{\partial}{\partial x}\left(\vec{A}.\vec{B}\right)\right\}=\sum\left\{\vec{i}\left(\frac{\partial\vec{A}}{\partial x}.\vec{B}\right)\right\}+\sum\left\{\vec{i}\left(\vec{A}.\frac{\partial\vec{B}}{\partial x}\right)\right\}\text{...........}(1)$$

Now, $\vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{B}\times\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$

$$=\vec{B}\times\left\{\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\vec{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\vec{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\vec{k}\right\}$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ B_1 & B_2 & B_3\\ \frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\end{vmatrix}$$

$$=\left\{B_2\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)-B_3\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\right\}\vec{i}+\left\{B_3\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)-B_1\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\right\}\vec{j}$$

$$+\left\{B_1\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)-B_2\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\right\}\vec{k}$$

$$=\left(\frac{\partial A_1}{\partial x}B_1+\frac{\partial A_2}{\partial x}B_2+\frac{\partial A_3}{\partial x}B_3\right)\vec{i}+\left(\frac{\partial A_1}{\partial y}B_1+\frac{\partial A_2}{\partial y}B_2+\frac{\partial A_3}{\partial y}B_3\right)\vec{j}+$$

$$\left(\frac{\partial A_1}{\partial z}B_1+\frac{\partial A_2}{\partial z}B_2+\frac{\partial A_3}{\partial z}B_3\right)\vec{k}-\left(B_1\frac{\partial A_1}{\partial x}+B_2\frac{\partial A_1}{\partial y}+B_3\frac{\partial A_1}{\partial z}\right)\vec{i}$$

$$-\left(B_1\frac{\partial A_2}{\partial x}+B_2\frac{\partial A_2}{\partial y}+B_3\frac{\partial A_2}{\partial z}\right)\vec{j}-\left(B_1\frac{\partial A_3}{\partial x}+B_2\frac{\partial A_3}{\partial y}+B_3\frac{\partial A_3}{\partial z}\right)\vec{k}$$

$$=\left(\frac{\partial\vec{A}}{\partial x}.\vec{B}\right)\vec{i}+\left(\frac{\partial\vec{A}}{\partial y}.\vec{B}\right)\vec{j}+\left(\frac{\partial\vec{A}}{\partial z}.\vec{B}\right)\vec{k}-\left(B_1\frac{\partial}{\partial x}+B_2\frac{\partial}{\partial y}+B_3\frac{\partial}{\partial z}\right)\left(A_1\vec{i}+A_2\vec{j}+A_3\vec{k}\right)$$

$$=\sum\left(\frac{\partial\vec{A}}{\partial x}.\vec{B}\right)\vec{i}-\left(\vec{B}.\vec{\nabla}\right)\vec{A}$$

$$\therefore\sum\left(\frac{\partial\vec{A}}{\partial x}.\vec{B}\right)\vec{i}=\vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)+\left(\vec{B}.\vec{\nabla}\right)\vec{A}\text{...........}(2)$$

Similarly, $\sum\left(\vec{A}.\frac{\partial\vec{B}}{\partial x}\right)\vec{i}=\vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)+\left(\vec{A}.\vec{\nabla}\right)\vec{B}$ ....(3)

Using (2),(3) in (1), we have

$$\vec{\nabla}\left(\vec{A}.\vec{B}\right)=\vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)+\left(\vec{B}.\vec{\nabla}\right)\vec{A}+\vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)+\left(\vec{A}.\vec{\nabla}\right)\vec{B}$$

$$=\left(\vec{B}.\vec{\nabla}\right)\vec{A}+\left(\vec{A}.\vec{\nabla}\right)\vec{B}+\vec{B}\times\left(\vec{\nabla}\times\vec{A}\right)+\vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)$$

Exercise18: Prove $\vec{\nabla}\left(\frac{F}{G}\right)=\frac{G\vec{\nabla}F-F\vec{\nabla}G}{G^2}$ if $G\neq 0$.

☺. $\vec{\nabla}\left(\frac{F}{G}\right)=\left\{\frac{\partial}{\partial x}\left(\frac{F}{G}\right)\right\}\vec{i}+\left\{\frac{\partial}{\partial y}\left(\frac{F}{G}\right)\right\}\vec{j}+\left\{\frac{\partial}{\partial z}\left(\frac{F}{G}\right)\right\}\vec{k}$

$$=\frac{G\frac{\partial F}{\partial x}-F\frac{\partial G}{\partial x}}{G^2}\vec{i}+\frac{G\frac{\partial F}{\partial y}-F\frac{\partial G}{\partial y}}{G^2}\vec{j}+\frac{G\frac{\partial F}{\partial z}-F\frac{\partial G}{\partial z}}{G^2}\vec{k}$$

$$=\frac{G\left\{\left(\frac{\partial F}{\partial x}\right)\vec{i}+\left(\frac{\partial F}{\partial y}\right)\vec{j}+\left(\frac{\partial F}{\partial z}\right)\vec{k}\right\}-F\left\{\left(\frac{\partial G}{\partial x}\right)\vec{i}+\left(\frac{\partial G}{\partial y}\right)\vec{j}+\left(\frac{\partial G}{\partial z}\right)\vec{k}\right\}}{G^2}$$

$$=\frac{G\vec{\nabla}F-F\vec{\nabla}G}{G^2}$$

Exercise19: Find the constants a and b so that the surface $ax^2-byz=(a+2)x$ will be orthogonal to the surface $4x^2y+z^3=4$ at the point $(1,-1,2)$.

☺. The given surfaces are $ax^2-byz=(a+2)x$ i.e., $ax^2-(a+2)x-byz=0$ and $4x^2y+z^3=4$.

Let $\phi_1=ax^2-(a+2)x-byz$ and $\phi_2=4x^2y+z^3$.

Then $\vec{\nabla}\phi_1=(2ax-a-2)\vec{i}-bz\vec{j}-by\vec{k}$ and $\vec{\nabla}\phi_2=8xy\vec{i}+4x^2\vec{j}+3z^2\vec{k}$

At $(1,-1,2)$, $\vec{\nabla}\phi_1=(a-2)\vec{i}-2b\vec{j}+b\vec{k}$ and $\vec{\nabla}\phi_2=-8\vec{i}+4\vec{j}+12\vec{k}$

The given surfaces will be orthogonal if $\vec{\nabla}\phi_1$ and $\vec{\nabla}\phi_2$ are orthogonal, that is if $\vec{\nabla}\phi_1.\vec{\nabla}\phi_2=0$ or if $-8(a-2)-8b+12b=0$

or if $-8a+4b+16=0$ or, if $2a-b=4$ ....(1)

Since $(1,-1,2)$ lies on the surface $ax^2-(a+2)x-byz=0$, therefore we have $a-(a+2)+2b=0$ or, $b=1$ and from (1) we have $a=\frac{5}{2}$

Thus the required values of a,b are $\frac{5}{2},1$

Exercise20: Prove $\vec{\nabla}^2r^n=n(n+1)r^{n-2}$ where n is a constant.

☺. Since $r=\sqrt{x^2+y^2+z^2}$ i.e., $r^2=x^2+y^2+z^2$

Therefore, $\frac{\partial r}{\partial x}=\frac{x}{r},\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$

$$\frac{\partial}{\partial x}\left(r^n\right)=nr^{n-1}\frac{\partial r}{\partial x}=nr^{n-1}\frac{x}{r}=nr^{n-2}x$$

$$\frac{\partial^2}{\partial x^2}\left(r^n\right)=\frac{\partial}{\partial x}\left(nr^{n-2}x\right)=n(n-2)r^{n-3}\frac{x}{r}x+nr^{n-2}=n(n-2)r^{n-4}x^2+nr^{n-2}$$

Similarly, $\frac{\partial^2}{\partial y^2}\left(r^n\right)=n(n-2)r^{n-4}y^2+nr^{n-2}$

and $\frac{\partial^2}{\partial z^2}\left(r^n\right)=n(n-2)r^{n-4}z^2+nr^{n-2}$

Therefore $\vec{\nabla}^2 r^n=\frac{\partial^2}{\partial x^2}\left(r^n\right)+\frac{\partial^2}{\partial y^2}\left(r^n\right)+\frac{\partial^2}{\partial z^2}\left(r^n\right)$

$$=n(n-2)r^{n-4}\left(x^2+y^2+z^2\right)+3nr^{n-2}$$
$$=n(n-2)r^{n-2}+3nr^{n-2}=n(n+1)r^{n-2}$$

Exercise21: (a) Prove $\vec{\nabla}^2 f(r)=\frac{d^2 f}{dr^2}+\frac{2}{r}\frac{df}{dr}$. (b) Find $f(r)$ such that $\vec{\nabla}^2 f(r)$=0.

☺**.** (a) Since $r=\sqrt{x^2+y^2+z^2}$ i.e., $r^2=x^2+y^2+z^2$

Therefore, $\frac{\partial r}{\partial x}=\frac{x}{r},\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$

$$\frac{\partial}{\partial x}\left(f(r)\right)=\frac{df}{dr}\frac{\partial r}{\partial x}=\frac{x}{r}\frac{df}{dr}$$

$$\frac{\partial^2}{\partial x^2}\left(f(r)\right)=\frac{\partial}{\partial x}\left(\frac{x}{r}\frac{df}{dr}\right)=\frac{1}{r}\frac{df}{dr}-\frac{x}{r^2}\frac{\partial r}{\partial x}\frac{df}{dr}+\frac{x}{r}\frac{d^2 f}{dr^2}\frac{\partial r}{\partial x}=\frac{1}{r}\frac{df}{dr}-\frac{x^2}{r^3}\frac{df}{dr}+\frac{x^2}{r^2}\frac{d^2 f}{dr^2}$$

Similarly, $\frac{\partial^2}{\partial y^2}\left(f(r)\right)=\frac{1}{r}\frac{df}{dr}-\frac{y^2}{r^3}\frac{df}{dr}+\frac{y^2}{r^2}\frac{d^2 f}{dr^2}$

and $\frac{\partial^2}{\partial z^2}\left(f(r)\right)=\frac{1}{r}\frac{df}{dr}-\frac{z^2}{r^3}\frac{df}{dr}+\frac{z^2}{r^2}\frac{d^2 f}{dr^2}$

$$\vec{\nabla}^2\left(f(r)\right)=\frac{\partial^2}{\partial x^2}\left(f(r)\right)+\frac{\partial^2}{\partial y^2}\left(f(r)\right)+\frac{\partial^2}{\partial z^2}\left(f(r)\right)$$
$$=\frac{3}{r}\frac{df}{dr}-\frac{x^2+y^2+z^2}{r^3}\frac{df}{dr}+\frac{x^2+y^2+z^2}{r^2}\frac{d^2 f}{dr^2}=\frac{2}{r}\frac{df}{dr}+\frac{d^2 f}{dr^2}$$

(b) $\vec{\nabla}^2 f(r)$=0 implies $\frac{2}{r}\frac{df}{dr}+\frac{d^2 f}{dr^2}=0$

or, $\frac{2}{r}\frac{df}{dr}+\frac{d}{dr}\left(\frac{df}{dr}\right)=0$ or, $\frac{\frac{d}{dr}\left(\frac{df}{dr}\right)}{\frac{df}{dr}}=-\frac{2}{r}$

Integrating, $\log\frac{df}{dr}=-2\log r+\log C=\log\frac{C}{r^2}$ i.e., $\frac{df}{dr}=\frac{C}{r^2}$

Integrating again we have, $f(r)=-\frac{C}{r}+A=A+\frac{B}{r}$, where $A,B(-C)$ are constants.

Exercise22: Find the vorticity of the vector field $\vec{\alpha}=3y\vec{i}+4zx\vec{j}$ at the point $(0,2,1)$ in the positive direction of the z-axis.

☺. $\vec{\nabla}\times\vec{\alpha}=\vec{\nabla}\times\left(3y\vec{i}+4zx\vec{j}\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3y & 4zx & 0\end{vmatrix}=-4x\vec{i}+\left(4z-3\right)\vec{k}$

At (0,2,1), $\vec{\nabla}\times\vec{\alpha}=\vec{k}$. The unit vector along the positive direction of z-axis is $\vec{k}$. Thus the vorticity along the positive direction of z-axis is $\vec{k}.\vec{k}=1$

Exercise23: Prove that $\vec{a}.\vec{\nabla}\left(\vec{b}.\vec{\nabla}\left(\frac{1}{r}\right)\right)=\frac{3\left(\vec{a}.\vec{r}\right)\left(\vec{b}.\vec{r}\right)}{r^5}-\frac{\left(\vec{a}.\vec{b}\right)}{r^3}$ and where $\vec{a}$ and $\vec{b}$ are constant vectors and $r=\left|\vec{r}\right|$

☺. Since $r=\left|\vec{r}\right|=\sqrt{x^2+y^2+z^2}$, therefore $\frac{\partial r}{\partial x}=\frac{x}{r},\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$.

$$\vec{\nabla}\left(\frac{1}{r}\right)=\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\vec{i}+\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\vec{j}+\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\vec{k}$$

$$=\left(-\frac{1}{r^2}\right)\frac{x}{r}\vec{i}+\left(-\frac{1}{r^2}\right)\frac{y}{r}\vec{j}+\left(-\frac{1}{r^2}\right)\frac{z}{r}\vec{k}$$

$$=-\frac{\vec{r}}{r^3}$$

$$\vec{b}.\vec{\nabla}\left(\frac{1}{r}\right)=-\frac{\vec{b}.\vec{r}}{r^3}$$

$$\vec{\nabla}\left(\vec{b}.\vec{\nabla}\left(\frac{1}{r}\right)\right)=-\frac{\partial}{\partial x}\left(\frac{\vec{b}.\vec{r}}{r^3}\right)\vec{i}-\frac{\partial}{\partial y}\left(\frac{\vec{b}.\vec{r}}{r^3}\right)\vec{j}-\frac{\partial}{\partial z}\left(\frac{\vec{b}.\vec{r}}{r^3}\right)\vec{k}$$

$$=-\vec{b}.\left\{\frac{1}{r^3}\frac{\partial}{\partial x}\left(\vec{r}\right)-\frac{3}{r^4}\frac{x}{r}\vec{r}\right\}\vec{i}-\vec{b}.\left\{\frac{1}{r^3}\frac{\partial}{\partial y}\left(\vec{r}\right)-\frac{3}{r^4}\frac{y}{r}\vec{r}\right\}\vec{j}-\vec{b}.\left\{\frac{1}{r^3}\frac{\partial}{\partial z}\left(\vec{r}\right)-\frac{3}{r^4}\frac{z}{r}\vec{r}\right\}\vec{k}$$

$$=-\frac{1}{r^3}\left\{\left(\vec{b}.\vec{i}\right)\vec{i}+\left(\vec{b}.\vec{j}\right)\vec{j}+\left(\vec{b}.\vec{k}\right)\vec{k}\right\}+\frac{3\left(\vec{b}.\vec{r}\right)}{r^5}\left(x\vec{i}+y\vec{j}+z\vec{k}\right)=-\frac{\vec{b}}{r^3}+\frac{3\left(\vec{b}.\vec{r}\right)}{r^5}\vec{r}$$

Therefore, $\vec{a}.\vec{\nabla}\left(\vec{b}.\vec{\nabla}\left(\frac{1}{r}\right)\right)=-\frac{\left(\vec{a}.\vec{b}\right)}{r^3}+\frac{3\left(\vec{a}.\vec{r}\right)\left(\vec{b}.\vec{r}\right)}{r^5}$

Exercise24: Prove that $r^n\vec{r}$ is irrotational for all values of n.

☺. Since $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ and $\left|\vec{r}\right|=\sqrt{x^2+y^2+z^2}$,

therefore $\frac{\partial r}{\partial x}=\frac{x}{r},\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$

Now $\vec{\nabla}\times\left(r^n\vec{r}\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ r^n x & r^n y & r^n z\end{vmatrix}$

$$=\left\{\frac{\partial}{\partial y}\left(r^n z\right)-\frac{\partial}{\partial z}\left(r^n y\right)\right\}\vec{i}+\left\{\frac{\partial}{\partial z}\left(r^n x\right)-\frac{\partial}{\partial x}\left(r^n z\right)\right\}\vec{j}+\left\{\frac{\partial}{\partial x}\left(r^n y\right)-\frac{\partial}{\partial y}\left(r^n x\right)\right\}\vec{k}$$

$$=\left(nr^{n-1}z\frac{y}{r}-nr^{n-1}y\frac{z}{r}\right)\vec{i}+\left(nr^{n-1}x\frac{z}{r}-nr^{n-1}z\frac{x}{r}\right)\vec{j}+\left(nr^{n-1}y\frac{x}{r}-nr^{n-1}x\frac{y}{r}\right)\vec{k}=\vec{0}$$

This shows that $r^n\vec{r}$ is irrotational for all values of n

Exercise25: (a) Find the constants a,b,c so that

$\vec{v}=(x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+cy+2z)\vec{k}$ is irrotational.

(b) Show that $\vec{v}$ can be expressed as the gradient of a scalar function.

☺**.** (a) $\text{curl}\,\vec{v}=\vec{\nabla}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+2y+az & bx-3y-z & 4x+cy+2z\end{vmatrix}$

$$=(c+1)\vec{i}+(a-4)\vec{j}+(b-2)\vec{k}$$

This equal to zero vector when $a=4$, $b=2$, $c=-1$

and $\vec{v}=(x+2y+4z)\vec{i}+(2x-3y-z)\vec{j}+(4x-y+2z)\vec{k}$

(b) Assume $\vec{v}=\vec{\nabla}\phi=\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}$

Then we have, $\frac{\partial\phi}{\partial x}=x+2y+4z$

$$\frac{\partial\phi}{\partial y}=2x-3y-z$$

$$\frac{\partial\phi}{\partial z}=4x-y+2z$$

$$d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$$

$$=(x+2y+4z)dx+(2x-3y-z)dy+(4x-y+2z)dz$$

$$=xdx-3ydy+2zdz+2(ydx+xdy)+4(zdx+xdz)-(zdy+ydz)$$

$$=d\left(\frac{x^2}{2}-\frac{3y^2}{2}+z^2+2xy+4zx-yz\right)$$

$\therefore\ \phi=\frac{x^2}{2}-\frac{3y^2}{2}+z^2+2xy+4zx-yz+c$, where c is a constant.

Exercises:

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1. Define curl and divergence of a vector quantity.
2. Find the curl and divergence of the vector $\vec{v}=\frac{\hat{r}}{r}$, where $\hat{r}$ is the unit vector in the direction of $\vec{r}$ and $r=|\vec{r}|$
3. Find the maximum value of the directional derivative of $\phi=x^2+z^2-y^2$ at the point $(1,3,2)$. Find also the direction in which it occurs.
4. If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A}\times\vec{B}$ is solenoidal.
5. If $f(r)$ be differentiable, prove that $f(r)\,\vec{r}$ is irrotational.

6. If $\vec{\nabla}.\vec{E}=0$, $\vec{\nabla}.\vec{H}=0$, $\vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{H}}{\partial t}$, $\vec{\nabla}\times\vec{H}=\dfrac{\partial\vec{E}}{\partial t}$, show that $\vec{E}$ and $\vec{H}$ satisfy $\vec{\nabla}^2 u=\dfrac{\partial^2 u}{\partial t^2}$

7. Show that $\vec{A}=\left(6xy+z^3\right)\vec{i}+\left(3x^2-z\right)\vec{j}+\left(3xz^2-y\right)\vec{k}$ is irrotational. Find ϕ such that $\vec{A}=\vec{\nabla}\phi$

8. Show that $\vec{E}=\dfrac{\vec{r}}{r^2}$ is irrotational. Find ϕ such that $\vec{E}=-\vec{\nabla}\phi$ and such that $\phi(a)=0$ where $a>0$.