1. Are the following functions cumulative distribution functions?

   - $F_1(x) = \begin{cases} 0 & \text{if } x < -5 \\ x & \text{if } -5 \le x \le 0.5 \\ 1 & \text{if } x > 0.5 \end{cases}$
   - $F_2(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x), -\infty < x < +\infty$
   - $F_3(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \ge 0 \end{cases}$
   - $F_4(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{\Gamma(\alpha)} \int_0^x y^{\alpha-1} e^{-y} dy & \text{if } x \ge 0 \end{cases}$

2. Let X be a random variable having the distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{2}{3} & \text{if } 0 \le x < 1 \\ \frac{7-6c}{6} & \text{if } 1 \le x < 2 \\ \frac{4c^2-9c+6}{4} & \text{if } 2 \le x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$$

   where c is a real constant.
   (a) Find the value of constant c.(Ans : 1/4)
   (b) Using the distribution function, find $P(\{1 < X < 2\}), P(\{2 \le X < 3\}), P(\{0 < X \le 1\}), P(\{1 \le X \le 2\})$ $P(\{X \ge 3\})$, and $P(\{X = 2.5\})$.(Ans: $0, 1/12, 1/4, 1/3, 0, 0$.)
   (c) Find the conditional probabilities $P(\{X = 1\}|\{1 \le X \le 2\}), P(\{1 \le X < 2\}|\{X > 1\})$, and $P(\{1 \le X \le 2\}|\{X = 1\})$. (Ans: $3/4, 0, 1$.)
   (d) Find the PMF of X.

3. Let X be a random variable having CDF

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \left(1 - (1-p)^{[x]}\right) & \text{if } x \ge 0 \end{cases}$$

   Determine whether X is DRV or CRV. Find the PMF /PDF , whatever applicable, of X.

4. Let X be a random variable with CDF

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{2} & \text{if } 0 \le x < 1 \\ \frac{x+2}{6} & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$$

   (a) Using the distribution function, evaluate $P(\{X = 1\}), P(\{1 < X < 2\}), P(\{1 \le X < 2\}), P(\{1 < X \le 2\}),$, $P(\{1 \le X \le 2\})$ and $P(\{X \ge 1\})$. (Ans: $0, 1/6, 1/2, 1/2, 1/2, 1/2$)
   (b) Is the RV X a DRV?
   (c) Is the RV X a CRV?

5. Let X be a CRV with PDF

$$f_X(x) = \begin{cases} k - |x| & \text{if } |x| < 0.5 \\ 0 & \text{otherwise} \end{cases}$$

where $k \in \mathbb{R}$.
(a) Find the value of constant k.(Ans : 5/4.)
(b) Using the PDF, evaluate $P(\{X < 0\}), P(\{X \leq 0\}), P\left(\left\{0 < X \leq \frac{1}{4}\right\}\right), P\left(\left\{0 \leq X < \frac{1}{4}\right\}\right)$, and $P\left(\left\{-\frac{1}{8} \leq X \leq \ \frac{1}{4}\right\}\right)$.(Ans : $1/2, 1/2, 9/32, 9/32, 25/32$.)
(c) Find the conditional probabilities $P\left(\left\{X > \frac{1}{4}\right\} | \left\{|X| > \frac{2}{5}\right\}\right)$ and $P\left(\frac{1}{10} < X < 1| \left\{\frac{1}{10} < X < \frac{1}{5}\right\}\right)$ (Ans: $1/2, 1)$
(d) Find the CDF of X.

6. For the bivariate negative binomial distribution, the PMF is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{(x+y+k-1)!}{x!y!(k-1)!}\theta_1^x \theta_2^y (1 - \theta_1 - \theta_2)^k & \text{if } x \in \{0, 1, 2, \ldots\}, y \in \{0, 1, 2, \ldots\} \\ 0 & \text{otherwise} \end{cases}$$

k is a positive integer, $0 < \theta_1 < 1, 0 < \theta_2 < 1$, and $0 < \theta_1 + \theta_2 < 1$. Find both the marginal distributions.

7. For the bivariate beta random vector (X, Y) having PDF

$$f_{X,Y}(x, y) = \begin{cases} \frac{\Gamma(\theta_1+\theta_2+\theta_3)}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3)} x^{\theta_1 - 1} y^{\theta_2 - 1} (1 - x - y)^{\theta_3 - 1} & \text{if } x > 0, y > 0, x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\theta_i > 0, i = 1, 2, 3$. Find both the marginal PDFs.

8. The joint PDF of (X, Y) is given by

$$f_{X,Y}(x, y) = \begin{cases} 4xy & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal PDFs of X and Y.
(b) Verify whether X and Y are independent.
(c) Find $P(\{0 < X < 0.5, 0.25 < Y < 1\})$ and $P(\{X + Y < 1\})$

9. Let $X = (X_1, X_2, X_3)$ be a random vector with joint PDF

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right)} \left(1 + x_1 x_2 x_3 e^{-\frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right)}\right) \quad \text{If } (x_1, x_2, x_3) \in \mathbb{R}^3$$

(a) Are X_1, X_2, and X_3 independent?
(b) Are X_1, X_2, and X_3 pairwise independent?