8.9. Method of variation of parameters.

This method as explained earlier for first order equations is used to find the complete primitive of a linear differential equation when its complementary function is known. We shall explain the method for a linear differential equation of second order, but it can be extended to linear equations of any order. The complete solution is obtained by varying the parameters of the complementary function.

Let us consider the general linear equation of second order

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X, \qquad \dots (1)$$

where P, Q and X are functions of x.

Let $\quad y=Au+Bv \qquad \dots (2)$

be the complementary function of the equation (1), where A and B are constants and u and v are functions of x, such that

$$\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu=0 \text{ and } \frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv=0. \qquad \dots (3)$$

Let us assume that $y=Au+Bv \qquad \dots (4)$

is the complete primitive of (1), where A and B are no longer constants but functions of x to be so chosen that (4) will satisfy (1).

Differentiating (4) with respect to x, we get

$$\frac{dy}{dx}=A\frac{du}{dx}+B\frac{dv}{dx}+u\frac{dA}{dx}+v\frac{dB}{dx}.$$

Let us choose A and B such that

$$u\frac{dA}{dx}+v\frac{dB}{dx}=0, \qquad \dots (5)$$

so that $\quad \frac{dy}{dx}=A\frac{du}{dx}+B\frac{dv}{dx}.$

Differentiating this once again, we have

$$\frac{d^2y}{dx^2}=A\frac{d^2u}{dx^2}+B\frac{d^2v}{dx^2}+\frac{dA}{dx}\frac{du}{dx}+\frac{dB}{dx}\frac{dv}{dx}.$$

Substituting these values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$A\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)+B\left(\frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv\right)$$
$$+\frac{dA}{dx}\frac{du}{dx}+\frac{dB}{dx}\frac{dv}{dx}=X$$

or, $\quad \frac{dA}{dx}\frac{du}{dx}+\frac{dB}{dx}\frac{dv}{dx}=X, \qquad \dots (6)$

by virtue of (3).

Solving for $\frac{dA}{dx}$ and $\frac{dB}{dx}$ from (5) and (6), we get

$$\frac{dA}{dx} = \frac{vX}{v\frac{du}{dx} - u\frac{dv}{dx}} \text{ and } \frac{dB}{dx} = -\frac{uX}{v\frac{du}{dx} - u\frac{dv}{dx}}.$$

Integrating, we get

$$A = C_1 + \int \frac{vX\,dx}{v\frac{du}{dx} - u\frac{dv}{dx}} \text{ and } B = C_2 - \int \frac{uX\,dx}{v\frac{du}{dx} - u\frac{dv}{dx}}.$$

Substituting these values of A and B in (4), we get the complete solution of the equation (1).

8.10. Illustrative Examples.

Ex. 1. *Solve, by the method of variation of parameters, the equation*

$$\frac{d^2y}{dx^2} + a^2y = \sec ax. \qquad [\text{ C. H. 1995 }]$$

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The complementary function of the equation is

$$A \cos ax + B \sin ax,$$

in which A and B are constants.

Now assume A and B to be functions of x in such a way that the given equation is satisfied by

$$y = A \cos ax + B \sin ax.$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -Aa \sin ax + Ba \cos ax + \cos ax \frac{dA}{dx} + \sin ax \frac{dB}{dx}.$$

We choose A and B such that

$$\cos ax \frac{dA}{dx} + \sin ax \frac{dB}{dx} = 0. \qquad \ldots \quad (1)$$

Therefore $\frac{dy}{dx} = -Aa \sin ax + Ba \cos ax$

and $\frac{d^2y}{dx^2} = -Aa^2 \cos ax - Ba^2 \sin ax - a \sin ax \frac{dA}{dx} + a \cos ax \frac{dB}{dx}.$

Putting these values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$-a \sin ax \frac{dA}{dx} + a \cos ax \frac{dB}{dx} = \sec ax. \qquad \ldots \quad (2)$$

Solving (1) and (2), we get

$$a \frac{dA}{dx} = -\tan ax \quad \text{and} \quad a \frac{dB}{dx} = 1.$$

Integrating, we get

$$A = \frac{1}{a^2} \log \cos ax + C_1 \quad \text{and} \quad B = \frac{x}{a} + C_2.$$

Hence the complete solution of the equation is

$$y = A \cos ax + B \sin ax$$

$$= C_1 \cos ax + C_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \log \cos ax.$$

Ex. 2. *Solve, by the method of variation of parameters,*

$$\frac{d^2y}{dx^2} + 4y = 4 \tan 2x. \qquad [\text{ V. H. 1991 }]$$

The complementary function of the equation is

$A \cos 2x + B \sin 2x$, A and B being constants.

Assume the complete solution to be

$y = A \cos 2x + B \sin 2x$, where A and B are functions of x.

Now $$\frac{dy}{dx} = -2A \sin 2x + 2B \cos 2x + \cos 2x \frac{dA}{dx} + \sin 2x \frac{dB}{dx}.$$

Choose A and B such that $\cos 2x \frac{dA}{dx} + \sin 2x \frac{dB}{dx} = 0. \qquad \ldots \quad (1)$

Therefore $$\frac{dy}{dx} = -2A \sin 2x + 2B \cos 2x.$$

Also $$\frac{d^2y}{dx^2} = -4A \cos 2x - 4B \sin 2x - 2 \sin 2x \frac{dA}{dx} + 2 \cos 2x \frac{dB}{dx}.$$

Putting these values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$\cos 2x \frac{dB}{dx} - \sin 2x \frac{dA}{dx} = 2 \tan 2x. \qquad \ldots \quad (2)$$

Solving (1) and (2) for $\frac{dA}{dx}$ and $\frac{dB}{dx}$, we get

$$\frac{dA}{dx} = -\frac{2\sin^2 2x}{\cos 2x} \quad \text{and} \quad \frac{dB}{dx} = 2\sin 2x .$$

Integrating, we get

$$A = -2\int \frac{\sin^2 2x}{\cos 2x}\,dx = -2\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$

$$= \sin 2x - \log(\sec 2x + \tan 2x) + C_1$$

and $\qquad B = 2\int \sin 2x\,dx = C_2 - \cos 2x .$

Hence the complete solution of the equation is

$$y = A\cos 2x + B\sin 2x$$

$$= \{\sin 2x - \log(\sec 2x + \tan 2x) + C_1\}\cos 2x + (C_2 - \cos 2x)\sin 2x .$$

Ex. 3. *Solve, by the method of variation of parameters,*

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}. \qquad \text{[V. H. 1992, 1997]}$$

The complementary function of the equation is

$$Ae^x + Be^{-x},$$

where A and B are constants.

Assume the complete solution of the equation as

$$y = Ae^x + Be^{-x},$$

in which A and B are functions of x and not constants.

Now $\qquad \frac{dy}{dx} = Ae^x - Be^{-x} + e^x\frac{dA}{dx} + e^{-x}\frac{dB}{dx}.$

Choose A and B such that

$$e^x\frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0. \qquad \ldots \quad (1)$$

Therefore $\qquad \frac{dy}{dx} = Ae^x - Be^{-x}$

and $\qquad \frac{d^2y}{dx^2} = Ae^x + Be^{-x} + e^x\frac{dA}{dx} - e^{-x}\frac{dB}{dx}.$

Putting these values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$e^x \frac{dA}{dx} - e^{-x} \frac{dB}{dx} = \frac{2}{1+e^x}. \qquad \ldots (2)$$

Solving (1) and (2), we get

$$\frac{dA}{dx} = \frac{e^{-x}}{1+e^x} \quad \text{and} \quad \frac{dB}{dx} = \frac{-e^x}{1+e^x}.$$

Integrating, we get

$$A = \int \frac{e^{-x}}{1+e^x}\, dx = \int \frac{dz}{z^2(1+z)}, \text{ putting } e^x = z$$

$$= \int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z} \right) dz = -\frac{1}{z} + \log \frac{1+z}{z} + C_1$$

$$= C_1 - e^{-x} + \log \frac{1+e^x}{e^x}$$

and $\quad B = -\int \frac{e^x}{1+e^x}\, dx = C_2 - \log\left(1+e^x\right).$

Hence the complete solution of the equation is

$$y = Ae^x + Be^{-x}$$

$$= C_1 e^x + C_2 e^{-x} + e^x \log \frac{1+e^x}{e^x} - e^{-x} \log\left(1+e^x\right) - 1.$$

Choose A and B such that $x\dfrac{dA}{dx}+x^{-1}\dfrac{dB}{dx}=0$. ... (1)

Then $\dfrac{dy}{dx}=A-Bx^{-2}$

and $\dfrac{d^2y}{dx^2}=\dfrac{dA}{dx}+2Bx^{-3}-x^{-2}\dfrac{dB}{dx}$.

Putting these values of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in the given equation, we get

$$\frac{dA}{dx}-x^{-2}\frac{dB}{dx}=e^x. \qquad \dots \quad (2)$$

Solving (1) and (2) for $\dfrac{dA}{dx}$ and $\dfrac{dB}{dx}$, we get

$$\frac{dA}{dx}=\frac{e^x}{2} \text{ and } \frac{dB}{dx}=-\frac{1}{2}x^2e^x.$$

Integrating, we get

$$A=C_1+\frac{1}{2}e^x \text{ and } B=-\frac{1}{2}e^x x^2+xe^x-e^x+C_2.$$

Hence the complete solution of the equation is

$$\begin{aligned} y &= Ax+Bx^{-1} \\ &= C_1x+C_2x^{-1}+\frac{1}{2}xe^x-\frac{1}{2}xe^x+e^x-x^{-1}e^x \\ &= C_1x+C_2x^{-1}+e^x-x^{-1}e^x. \end{aligned}$$

Examples VIII(C)

Apply the method of variation of parameters to solve the following equations (1 – 16) :

1. $\dfrac{d^2y}{dx^2}+y=\operatorname{cosec} x$.

2. $\dfrac{d^2y}{dx^2}+y=x$.

3. $\dfrac{d^2y}{dx^2}+9y=\sec 3x$.

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4. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \dfrac{e^x}{1+e^x}$. [C. H. 1998]

5. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{2x} + x^2$.

6. $(D^2 - 3D + 2)y = (1 + e^{-x})^{-1}$, $D \equiv \dfrac{d}{dx}$.

7. $(D^2 - 3D + 2)y = \cos(e^{-x})$.

8. $(D^2 - 2D + 1)y = e^{2x}(e^x + 1)^{-2}$.

9. $\dfrac{d^2y}{dx^2} + y = \sec x \tan x$.

Answers

1. $y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log \sin x$.
2. $y = C_1 \cos x + C_2 \sin x + x$.
3. $y = C_1 \cos 3x + C_2 \sin 3x + \frac{x}{3} \sin 3x + \frac{1}{9} \cos 3x \log \cos 3x$.
4. $y = C_1 e^{x} + C_2 e^{2x} - e^{x} + (e^{2x} - e^{x}) \log (1 + e^{-x})$.
5. $y = C_1 e^{x} + C_2 e^{2x} + xe^{2x} + \frac{3}{2} x + \frac{7}{4} + \frac{1}{2} x^{2} - e^{2x}$.
6. $y = C_1 e^{x} + C_2 e^{2x} + e^{x} (1 + e^{x}) \log (1 + e^{-x})$.
7. $y = C_1 e^{2x} + C_2 e^{x} - e^{2x} \cos (e^{-x})$.
8. $y = (C_1 + C_2 x) e^{x} + e^{x} \log (1 + e^{x})$.
9. $y = C_1 \cos x + C_2 \sin x + x \cos x + \sin \log \sec x$.
10. $y = (C_1 + C_2 x) e^{x} + \frac{1}{[illegible]} (x \cos x + \cos x - \sin x)$.

CHAPTER IX

SIMULTANEOUS EQUATIONS

9.1. Introduction.

So far we have considered only those differential equations which contain two variables, one independent and the other dependent. Now in this chapter we consider the methods of solutions of differential equations involving more than two variables. The simplest form of such equations is that in which the number of independent variables is one. The number of equations which will connect these variables will be equal to the number of dependent variables. We shall consider here ordinary equations with one independent variable and two dependent variables.

9.2. Simultaneous linear equations with constant coefficients.

First Method :

Let x, y be the dependent variables and t be the independent variable. The equations will involve derivatives of x and y with respect to t. Let us denote the operator $\frac{d}{dt}$ by the symbol D. Then the simultaneous linear equations to be solved will be of the form

$$f_1(D)\,x + f_2(D)\,y = T_1 \qquad \ldots \quad (1)$$

and

$$\phi_1(D)\,x + \phi_2(D)\,y = T_2, \qquad \ldots \quad (2)$$

where $f_1(D)$, $f_2(D)$, $\phi_1(D)$, $\phi_2(D)$ are all rational functions of D with constant coefficients and T_1, T_2 are functions of t, the independent variable.

To eliminate y, we operate (1) with $\phi_2(D)$ and (2) with $f_2(D)$. Then these equations become

$$\phi_2(D) f_1(D)\,x + \phi_2(D) f_2(D)\,y = \phi_2(D)\,T_1 \qquad \ldots \quad (3)$$

and

$$f_2(D)\,\phi_1(D)\,x + f_2(D)\,\phi_2(D)\,y = f_2(D)\,T_2. \qquad \ldots \quad (4)$$

Now, since $f_2(D)$ and $\phi_2(D)$ are rational functions of D with constant coefficients, we have

$$\phi_2(D) f_2(D)\,y = f_2(D)\,\phi_2(D)\,y.$$

Hence, subtracting (4) from (3), we get

$$\{\phi_2(D) f_1(D) - f_2(D)\phi_1(D)\} x = \phi_2(D) T_1 - f_2(D) T_2,$$

which is of the form $F(D)x = T(t)$. ... (5)

This equation (5) being a linear differential equation, can be solved to find x as a function of t. Now the value of y can be obtained as a function of t by substituting the value of x in either of the two equations. If, however, y be determined by an independent elimination, as the case of x, the values of x and y so obtained will have to be substituted in equation (1) or (2) and the arbitrary constants are adjusted so that the equations may be satisfied.

Note that the number of arbitrary constants in the complete solution of (1) and (2) will be equal to the degree of D in the polynomial $F(D)$ of (5).

Second method :

The two given equations connect t with the four quantities $x, y, \frac{dx}{dt}, \frac{dy}{dt}$. We differentiate them with respect to t and get four equations connecting $x, y, \frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$. From these four equations, we eliminate three quantities $y, \frac{dy}{dt}, \frac{d^2y}{dt^2}$. In this way an equation of the second order, in which x is the dependent variable and t is the independent variable, is obtained. This is solved to get x as a function of t. Then y is obtained by substituting this value of x in the equations already obtained.

Note that this method is applied only when the given simultaneous equations are of order one.

9.3. Illustrative Examples.

Ex. 1. *Solve :*

$$\frac{dx}{dt} - 7x + y = 0,$$

$$\frac{dy}{dt} - 2x - 5y = 0. \qquad \text{[B. H. 1991]}$$

Using the symbol D for $\frac{d}{dt}$, the given equations can be written as

$$(D - 7)x + y = 0, \qquad \dots (1)$$

$$-2x + (D - 5)y = 0. \qquad \dots (2)$$

Eliminating y between (1) and (2), we get

$$\{(D-5)(D-7)+2\}x=0$$

or, $$(D^2-12D+37)x=0.$$

The auxiliary equation is $m^2-12m+37=0$, giving $m=6\pm i$.

Therefore $$x=e^{6t}(C_1\cos t+C_2\sin t). \qquad \ldots (3)$$

Then we have $$\frac{dx}{dt}=e^{6t}(-C_1\sin t+C_2\cos t)$$
$$+6e^{6t}(C_1\cos t+C_2\sin t). \qquad \ldots (4)$$

Substituting for x and $\frac{dx}{dt}$ in the first of the given equations, we get

$$y=C_1e^{6t}\cos t+C_2e^{6t}\sin t+C_1e^{6t}\sin t-C_2e^{6t}\cos t$$
$$=e^{6t}\{(C_1-C_2)\cos t+(C_1+C_2)\sin t\}. \qquad \ldots (5)$$

Hence the complete solution is given by (3) and (5).

Ex. 2. *Solve :* $(4D+44)x+(9D+49)y=t,$

$$(3D+34)x+(7D+38)y=e^t, \text{ where } D\equiv\frac{d}{dt}.$$

Eliminating x between the two given equations, we get

$$\{(4D+44)(7D+38)-(3D+34)(9D+49)\}y$$
$$=(4D+44)e^t-(3D+34)t$$

or, $$(D^2+7D+6)y=48e^t-34t-3.$$

This is a linear equation of second order.

The auxiliary equation is $m^2+7m+6=0$, whence $m=-6,-1$.

Therefore the complementary function is

$$C_1e^{-6t}+C_2e^{-t}.$$

The particular integral is

$$\frac{1}{D^2+7D+6}(48e^t-34t-3)$$
$$=\frac{48}{14}e^t-\frac{1}{6}\left\{1+\left(\frac{7}{6}D+\frac{1}{6}D^2\right)\right\}^{-1}(34t+3)$$
$$=\frac{24}{7}e^t-\frac{1}{6}\left(1-\frac{7}{6}D\right)(34t+3)$$
$$=\frac{24}{7}e^t-\frac{1}{6}\left(34t+3-\frac{7}{6}\cdot 34\right)$$
$$=\frac{24}{7}e^t-\frac{17}{3}t+\frac{55}{9}.$$

Hence the general solution for y is

$$y = C_1 e^{-6t} + C_2 e^{-t} + \frac{24}{7} e^{t} - \frac{17}{3} t + \frac{55}{9}. \qquad \ldots \quad (1)$$

Therefore $$\frac{dy}{dt} = -6C_1 e^{-6t} - C_2 e^{-t} + \frac{24}{7} e^{t} - \frac{17}{3}. \qquad \ldots \quad (2)$$

Now, if we multiply the first equation by 3 and subtract the result from the second equation being multiplied by 4, we get

$$\frac{dy}{dt} + 4x + 5y = 4e^{t} - 3t. \qquad \ldots \quad (3)$$

Putting the values of y and $\frac{dy}{dt}$ from (1) and (2) in (3), we get

$$-6C_1 e^{-6t} - C_2 e^{-t} + \frac{24}{7} e^{t} - \frac{17}{3} + 5C_1 e^{-6t} + 5C_2 e^{-t}$$
$$+ \frac{120}{7} e^{t} - \frac{85}{3} t + \frac{275}{9} + 4x = 4e^{t} - 3t$$

or, $$4x = C_1 e^{-6t} - 4C_2 e^{-t} - \frac{116}{7} e^{t} + \frac{76}{3} t - \frac{224}{9}$$

or, $$x = \frac{1}{4} C_1 e^{-6t} - C_2 e^{-t} - \frac{29}{7} e^{t} + \frac{19}{3} t - \frac{56}{9}. \qquad \ldots \quad (4)$$

(1) and (4) constitute the solution of the given equations.

Ex. 3. *Solve* : $\frac{dx}{dt} + 5x + y = e^{t}$, $\frac{dy}{dt} - x + 3y = e^{2t}$. [*C. H. 1993*]

From the first equation, we have

$$y = e^{t} - \frac{dx}{dt} - 5x. \qquad \ldots \quad (1)$$

Putting (1) in the second equation, we get

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 4e^{t} - e^{2t},$$

which is a linear equation of second order in x.

Its complementary function is $(C_1 + C_2 t) e^{-4t}$.

The particular integral is

$$\frac{1}{(D+4)^2} 4e^{t} - \frac{1}{(D+4)^2} e^{2t}$$
$$= \frac{4}{25} e^{t} - \frac{1}{36} e^{2t}.$$

Therefore $$x = (C_1 + C_2 t)e^{-4t} + \frac{4}{25}e^{t} - \frac{1}{36}e^{2t}. \quad \ldots \quad (2)$$

Hence, from (1), we get

$$y = -(C_1 + C_2 + C_2 t)e^{-4t} + \frac{1}{25}e^{t} + \frac{7}{36}e^{2t}. \quad \ldots \quad (3)$$

(2) and (3) constitute the solution of the given equations.

Ex. 4. *Solve :* $\dfrac{d^2x}{dt^2} - 3x - 4y = 0,\quad \dfrac{d^2y}{dt^2} + x + y = 0.$

From the second equation, we have

$$x = -\frac{d^2y}{dt^2} - y. \quad \ldots \quad (1)$$

Putting (1) in the first equation, we get

$$\frac{d^4y}{dt^4} - 2\frac{d^2y}{dt^2} + y = 0,$$

whose solution is $$y = (C_1 + C_2 t)e^{t} + (C_3 + C_4 t)e^{-t}. \quad \ldots \quad (2)$$

Hence, from (1), we get

$$x = -2(C_1 + C_2 + C_2 t)e^{t} - 2(C_3 - C_4 + C_4 t)e^{-t}. \quad \ldots \quad (3)$$

(2) and (3) constitute the solution of the given equations.

Examples IX(A)

Solve the following equations (1 – 15) :

1. $\dfrac{dx}{dt} = -3x + 4y,$
$\dfrac{dy}{dt} = -2x + 3y.$

2. $\dfrac{dx}{dt} = 4x - 2y,$
$\dfrac{dy}{dt} = 5x + 2y.$

3. $\dfrac{dx}{dt} = 5x + 4y,$
$\dfrac{dy}{dt} = -x + y.$

4. $\dfrac{dx}{dt} = -wy,$
$\dfrac{dy}{dt} = wx.$

5. $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = 0,$
$\dfrac{dy}{dt} + 5x + 3y = 0.$

[C. H. 1992, 1994]

6. $\dfrac{dx}{dt} + 5x - 2y = e^{t},$
$\dfrac{dy}{dt} - x + 6y = e^{2t}.$

7. $2\frac{dx}{dt} - \frac{dy}{dt} + 2x + y = 11t$,

$2\frac{dx}{dt} + 3\frac{dy}{dt} + 5x - 3y = 2$.

8. $\frac{dx}{dt} + 2\frac{dy}{dt} + x + 7y = e^t - 3$,

$\frac{dy}{dt} - 2x + 3y = 12 - 3e^t$.

9. $\frac{d^2x}{dt^2} - \frac{dy}{dt} = 2x + 2t$,

$\frac{dx}{dt} + 4\frac{dy}{dt} = 3y$.

0

10. $\frac{d^2x}{dt^2} - 2\frac{dy}{dt} - x = e^t \cos t$,

$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} - y = e^t \sin t$.

0 11. $\frac{d^2x}{dt^2} + 2y = 0$,

$\frac{d^2y}{dt^2} - 2x = 0$.

12. $2\frac{d^2y}{dx^2} - \frac{dz}{dx} - 4y = 2x$,

$2\frac{dy}{dx} + 4\frac{dz}{dx} - 3z = 0$.

13. $t\frac{dx}{dt} + y = 0$,

$t\frac{dy}{dt} + x = 0$.

14. $t\,dx = (t - 2x)\,dt$,

$t\,dy = (tx + ty + 2x - t)\,dt$.

[C. H. 1995]

0 15. $\frac{dx}{dt} + \frac{2}{t}(x - y) = 1$,

$\frac{dy}{dt} + \frac{1}{t}(x + 5y) = t$.

16. Show that the integral of the equations $\frac{dx}{dt} = -2y$ and $\frac{dy}{dt} = x$ is given by $x^2 + 2y + 2c = 0$. [C. H. 1989]

Answers

1. $x = 2C_1 e^{-t} + C_2 e^t$, $y = C_1 e^{-t} + C_2 e^t$.

2. $x = e^{3t}(2C_1 \cos 3t + 2C_2 \sin 3t)$,
$y = e^{3t}\{C_1(\cos 3t + 3\sin 3t) + C_2(\sin 3t - 3\cos 3t)\}$.

3. $x = -2C_1 e^{3t} + C_2(1 + 2t)e^{3t}$,
$y = C_1 e^{3t} - C_2 t e^{3t}$.

4. $x = C_1 \cos wt + C_2 \sin wt$,
$y = C_1 \sin wt - C_2 \cos wt$.

5. $x = C_1 \cos t + C_2 \sin t,$
$y = -\frac{1}{2}(C_1 + 3C_2)\sin t + \frac{1}{2}(C_2 - 3C_1)\cos t.$

6. $x = C_1 e^{-4t} + C_2 e^{-7t} + \frac{7}{40} e^{t} + \frac{1}{27} e^{2t},$
$y = \frac{1}{2} C_1 e^{-4t} - C_2 e^{-7t} + \frac{1}{40} e^{t} + \frac{7}{54} e^{2t}.$

7. $x = C_1 e^{-\frac{11}{8}t} + 3t - 2,$
$y = 3 + 5t + \frac{6}{19} C_1 e^{-\frac{11}{8}t}.$

8. $x = e^{-4t}(C_1 \cos t + C_2 \sin t) + \frac{31}{26} e^{t} - \frac{93}{17},$
$y = -(C_1 + C_2) e^{-4t} \cos t + (C_1 - C_2) e^{-4t} \sin t - \frac{2}{13} e^{t} + \frac{6}{17}.$

9. $x = (C_1 + C_2 t) e^{t} + C_3 e^{-\frac{3}{2}t} - t,$
$y = -(C_1 + C_2 t) e^{t} + 3C_2 e^{t} - \frac{1}{6} C_3 e^{-\frac{3}{2}t} - \frac{1}{3}.$

10. $x = (C_1 + C_2 t)\cos t + (C_3 + C_4 t)\sin t + \frac{1}{25} e^{t}(4 \sin t - 3 \cos t),$
$y = -(C_1 + C_2 t)\sin t + (C_3 + C_4 t)\cos t + \frac{1}{25} e^{t}(3 \sin t + 4 \cos t).$

11. $x = (A \cos t + B \sin t) e^{t} + (C \cos t + D \sin t) e^{-t},$
$y = (A \sin t - B \cos t) e^{t} + (D \cos t - C \sin t) e^{-t}.$

12. $y = (C_1 + C_2 x) e^{x} + 3C_3 e^{-\frac{3}{2}x} - \frac{1}{2} x,$
$z = 2(3C_2 - C_1 - C_2 x) e^{x} - C_3 e^{-\frac{3}{2}x} - \frac{1}{3}.$

13. $x = C_1 t + C_2 t^{-1},$
$y = -C_1 t + C_2 t^{-1}.$

14. $x = \frac{t}{3} + \frac{C_2}{t^2},$
$y = C_1 e^{t} - \frac{1}{3}t - C_2 t^{-2}.$

15. $x = C_1 t^{-3} + C_2 t^{-4} + \frac{3}{10} t + \frac{1}{15} t^2,$
$y = -\frac{1}{2} C_1 t^{-3} - C_2 t^{-4} + \frac{2}{15} t^2 - \frac{1}{20} t.$