Some Discrete and Continuous distribution

Discrete :~

1. ~~Binomial Distri~~ Bernoulli Distribution :~

A random variable X is defined to have a Bernoulli distribution if it can take only two values 0 & 1, with probabilities $P(X=0) = 1-p$, $P(X=1) = p$, where $0 < p < 1$.

We write this as $X \sim Ber(p) / b(p)$.

☑ Expectation : $E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1)$
$= 0 \cdot (1-p) + 1 \cdot p = p$

☑ Variance : $E(X^2) = 0^2 \cdot P(X=0) + 1^2 P(X=1) = p$.

$\therefore Var(X) = E(X^2) - (E(X))^2 = p - p^2 = p(1-p)$.

2. Binomial Distribution :~

A random variable X is said to have a binomial Distribution with parameters p $(0<p<1)$ & n (a +ve integer) if its discrete mass function is given by,

$$f_X(x) = P(X=x) = {}^nC_x \, p^x q^{n-x}, \quad x = 0, 1, \dots n$$
$$= 0, \quad \text{otherwise}.$$

where $q = 1-p \Rightarrow p+q = 1$.

We write this, as $X \sim Bin(n, p)$.

☑ Expectation : $E(X) = \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$

$$= \sum_{x=1}^{n} x \cdot \frac{n}{x} \cdot {}^{n-1}C_{x-1}\, p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{n-x}$$

$$= np \sum_{r=0}^{n-1} {}^{n-1}C_r\, p^r q^{n-1-r},$$ putting r in place of $x-1$.

$$= np \cdot (p+q)^{n-1} = np.$$

Variance

$$E(X(X-1)) = \sum_{x=0}^{n} x(x-1)\, {}^nC_x\, p^x q^{n-x}.$$

$$= \sum_{x=2}^{n} \cancel{x}\,(x-\cancel{1})\, \frac{n(n-1)}{\cancel{x}\,(x-\cancel{1})}\, {}^{n-2}C_{x-2}\, p^x q^{n-x}$$

$$= n(n-1)\, p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{n-x}$$

$$= n(n-1)\, p^2 \sum_{r=0}^{n-2} {}^{n-2}C_r\, p^r q^{n-2-r}$$ [replacing $x-2$ as r]

$$= n(n-1)\, p^2\, (q+p)^{n-2}$$

$$= n(n-1)\, p^2$$

$$\therefore Var(X) = E(X(X-1)) - E(X)(E(X)-1)$$

$$= n(n-1)p^2 - np(np-1)$$

$$= \cancel{n^2p^2} - np^2 - \cancel{n^2p^2} + np$$

$$= np - np^2 = np(1-p) = npq.$$

3. Poisson Distribution :- A random variable X is said to have a poisson distribution with parameter $\mu\ (>0)$ if its mass funⁿ is given by -

$$f_X(x) = P(X=x) = e^{-\mu} \frac{\mu^x}{x!}, \quad \text{for } x = 0, 1, 2, \ldots\ldots$$
$$= 0, \quad \text{otherwise.}$$

we write $X \sim Poi(\mu)$.

▨ <u>Expectation</u> :~ !

$$E(X) = \sum_{x=0}^{\infty} x \, e^{-\mu} \frac{\mu^x}{x!}$$
$$= e^{-\mu} \sum_{x=1}^{\infty} x \frac{\mu^x}{x!}$$
$$= \mu e^{-\mu} \sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!} \Rightarrow \mu e^{-\mu} \sum_{r=0}^{\infty} \frac{\mu^r}{r!}$$
$$= \mu e^{-\mu} \sum_{r=0}^{\infty} \frac{\mu^r}{r!}$$
$$= \mu . e^{-\mu} . e^{\mu} = \mu.$$

~~$E(X^2)$~~

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) e^{-\mu} \frac{\mu^x}{x!}$$
$$= e^{-\mu} \mu^2 \sum_{x=2}^{\infty} \frac{\mu^{x-2}}{(x-2)!}$$
$$= e^{-\mu} \mu^2 . \sum_{r=0}^{\infty} \frac{\mu^r}{r!}$$
$$= e^{-\mu} . \mu^2 . e^{\mu} = \mu^2$$

$$\therefore Var(X) = E(X(X-1)) - E(X)(E(X)-1)$$
$$= \mu^2 - \mu(\mu-1)$$
$$= \mu^2 - \mu^2 + \mu = \mu.$$

☑ Continuous :~

1. Uniform Distribution :

A r.v. X is said to have a Uniform distribution on the interval $[a, b]$, $-\infty < a < b < \infty$, if it's probability density funⁿ is given by,

$$f_X(x) = \frac{1}{b-a}, \quad a < x \le b$$

$$= 0, \quad \text{otherwise}.$$

☑ We write, $X \sim U[a, b]$.

☑ Expectation : $E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx.$$

$$= \frac{1}{b-a} \cdot \left.\frac{x^2}{2}\right|_a^b$$

$$= \frac{1}{b-a} \cdot \left(\frac{b^2-a^2}{2}\right) = \frac{a+b}{2}.$$

Variance : $E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx.$

$$= \frac{1}{b-a} \cdot \left.\frac{x^3}{3}\right|_a^b$$

$$= \frac{1}{b-a} \cdot \frac{b^3-a^3}{3}$$

$$= \frac{1}{3} \cdot (b^2+ab+a^2)$$

$$\therefore Var(X) = E(X^2) - (E(X))^2 = \frac{1}{3}(a^2+ab+b^2) - \frac{1}{4}(a^2+2ab+b^2)$$

$$= \frac{1}{12}(a^2+b^2-2ab) = \frac{(b-a)^2}{12}$$

2. <u>Exponential Distribution</u> :- A r.v. X is said to have an exponential distribution with parameters $\alpha\ (\alpha>0)$. if its probability density funn is given by.

$$f_X(x) = \alpha e^{-\alpha x}, \quad x \geq 0.$$
$$= 0, \quad \text{otherwise}.$$

we write, $X \sim Exp(\alpha)$.

<u>Expectation</u> :-

$$E(X) = \alpha \int_{-\infty}^{\infty} f_X(x)\, dx$$

$$= \alpha . \int_0^{\infty} x e^{-\alpha x} dx$$

$$= \alpha . \frac{\Gamma(2)}{\alpha^2} = \alpha . \frac{1}{\alpha^2} = \frac{1}{\alpha}$$

<u>Variance</u> : $E(X^2) = \alpha \int_0^{\infty} x^2 e^{-\alpha x} dx = \frac{\alpha \Gamma(3)}{\alpha^3} = \frac{2}{\alpha^2}$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}.$$

3. <u>Normal Distribution</u> :- A r.v. X with parameters $\mu\ (-\infty<\mu<\infty)$ and $\sigma\ (>0)$ is said to have a normal distribution if its probability density funn is given by,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu^2)/2\sigma^2}$$

we write $X \sim Normal(\mu, \sigma) \,/\, Gaussian(\mu, \sigma)$.

[IV] Expectation :

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \int_{-\infty}^{\infty} (x-\mu+\mu)\, e^{-(x-\mu)^2/2\sigma^2}\, dx .$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \int_{-\infty}^{\infty} \underbrace{(x-\mu)\, e^{-(x-\mu)^2/2\sigma^2}}_{\text{odd fn}^n}\, dx + \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \mu \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$= 0 + \mu \cdot \frac{1}{\sqrt{2\pi}\cdot\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\, dx .$$

Put $\frac{x-\mu}{\sqrt{2}\,\sigma} = z \Rightarrow dx = \sqrt{2}\,\sigma\, dz .$

x	$-\infty$	∞
z	$-\infty$	∞

$$= \mu \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-z^2} \cdot \sqrt{2}.\sigma\, dz .$$

$$= \frac{\mu \cdot \cancel{\sqrt{2}}\, \cancel{\sigma}}{\cancel{\sqrt{2}}\sqrt{\pi} \cdot \cancel{\sigma}} \left(\int_{-\infty}^{\infty} e^{-z^2} dz \right)$$

$$= \frac{\mu}{\sqrt{\pi}} \cdot \left[\int_{-\infty}^{\infty} e^{-z^2} dz \right] = \frac{\mu}{\sqrt{\pi}} \cdot \Gamma\left(\tfrac{1}{2}\right) = \frac{\mu}{\cancel{\sqrt{\pi}}} \cdot \cancel{\sqrt{\pi}} = \mu .$$

$$\therefore \boxed{E(x) = \mu}$$

[ii] Variance:

$$\text{Var}(X) = E\left[(X - E(X))^2\right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx.$$

$\frac{x-\mu}{\sqrt{2}\,\sigma} = z$

$dx = \sqrt{2}\,\sigma\, dz.$

x	$-\infty$	∞
z	$-\infty$	∞

$\therefore x-\mu = \sqrt{2}\,\sigma.\, z.$

$\therefore (x-\mu)^2 = 2\sigma^2 . z^2$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} 2\sigma^2 z^2 . e^{-z^2} \sqrt{2}\,\sigma\, dz$$

$$= \frac{2\sigma^2 \sqrt{2}\,\cancel{\sigma}}{\sqrt{2\pi}\,.\cancel{\sigma}} \int_{-\infty}^{\infty} z^2 e^{-z^2} dz$$

$$= \frac{\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2} 2\, dz$$

$$= \frac{\sigma^2}{\sqrt{\pi}} \left[\int_{-\infty}^{\infty} z . e^{-z^2} 2z\, dz\right]$$

integration by parts.

$$= \frac{\sigma^2}{\sqrt{\pi}} \left[\cancel{-\left[z e^{-z^2}\right]_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} e^{-z^2} dz \right]$$

$$= \frac{\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{\sigma^2}{\sqrt{\pi}} . \sqrt{\pi} = \sigma^2.$$

$$\therefore \boxed{\text{Var}(X) = \sigma^2}.$$