Combinatorics - Lecture 3
Pigeonhole Principle

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1 Pigeonhole Principle - Simple Form

Theorem 1.1 *If $(n+1)$ objects are put in n boxes, then at least one box contains two or more of the objects.*

Proof 1.1 *If each of the boxes contains at most one of the objects, then the total number of objects is at most n. Since, we start with $(n+1)$ objects, some objects contains at least two of the objects.*

Note :

- The principle/proof does not help in finding a box that contains two or more of the objects.
- The principle merely guarantees the existence of a box.
- The principle cannot be guaranteed if there are only (or fewer) objects.

Analogical Principles

1. If n objects are put into n boxes and no box is empty, then each box contains exactly one object.
2. If n objects are put into n boxes and no box gets more than one object, then each box has an object in it.

Applications

Example 1.1 *There are n married couples.*
How many of the $2n$ people must be selected in order to guarantee that one has selected a married couple.

Answer 1.1

Example 1.2 *Given m integers, $a_1, a_2, \ldots, a_m$, there exists integers k and l with $0 \leq k < l \leq m$ such that $a_{k+1} + a_{k+2} + \ldots + a_l$ is divisible by m.*

Less formally, there exists consecutive a's in the sequence $a_1, a_2, \ldots, a_m$ whose sum is divisible by m

Answer 1.2 *Let us consider the m-sums*

$$a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \ldots,\ \sum_{i=1}^{m} a_i$$

If any of thes sums is divisible by m, then the conclusion holds

- *Thus, we may suppose that each of these sums has a non-zero remainder m, and so has a remainder on $1, 2, \ldots, (m-1)$*
- *Since, there are m sums and $(m-1)$ remainders, two of the sums have the same remainder when divided by m*
- *$\therefore$, there are integers k and l, with $k < l$ such that*

$$a_1 + a_2 + \ldots + a_k = bm + r \tag{1}$$
$$a_1 + a_2 + \ldots + a_l = cm + r \tag{2}$$

Subtracting Eq (2 - 1) :

$$a_{k+1} + a_{k+2} + \ldots + a_l = (c-b)m \tag{3}$$

$\therefore$ $a_{k+1} + a_{k+2} + \ldots + a_l$ *is divisible by m*

Let $\boldsymbol{m} = \mathbf{7}$ *and let the integers be*

$$\boxed{\boldsymbol{a_1 = 2}}\ \boxed{\boldsymbol{a_1 = 2}}\ \boxed{\boldsymbol{a_2 = 4}}$$
$$\boxed{\boldsymbol{a_3 = 6}}\ \boxed{\boldsymbol{a_4 = 3}}\ \boxed{\boldsymbol{a_5 = 5}}\ \boxed{\boldsymbol{a_6 = 5}}\ \boxed{\boldsymbol{a_7 = 6}}$$

Thus, $\boldsymbol{k = 2, l = 5}$.
Computing the sums as :

$$\boxed{\boldsymbol{s_1 = a_1 = 2}}\ \boxed{\boldsymbol{s_2 = a_1 + a_2 = 6}}\ \boxed{\boldsymbol{s_3 = a_1 + a_2 + a_3 = 12}}\ \boxed{\boldsymbol{s_4 = a_1 + a_2 + a_3 + a_4 = 15}}$$
$$\boxed{\boldsymbol{s_5 = a_1 + a_2 + a_3 + a_4 + a_5 = 20}}\ \boxed{\boldsymbol{s_6 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 25}}$$
$$\boxed{\boldsymbol{s_7 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 31}}$$

Remainders, when divided by $\mathbf{7}$ *are*

$$\boxed{\boldsymbol{r_1 = 2}}\ \boxed{\boldsymbol{r_2 = 6}}\ \boxed{\boldsymbol{r_3 = 5}}\ \boxed{\boldsymbol{r_4 = 1}}$$
$$\boxed{\boldsymbol{r_5 = 6}}\ \boxed{\boldsymbol{r_6 = 4}}\ \boxed{\boldsymbol{r_7 = 3}}$$

Therefore, there are remainders equal to 6 i.e.
$\mathbf{6 + 3 + 5 = 14}$ *is divisible by* $\mathbf{7}$

Example 1.3 Chinese Remainder Theorem : *Let* $\boldsymbol{m}$ *and* $\boldsymbol{n}$ *be relatively prime positive integers and let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be integers where* $0 \leq a \leq (m-1)$ *and* $0 \leq b \leq (n-1)$*. Then, there is a positive integer* $\boldsymbol{x}$*, such that the* <u>*remainder*</u> *when*

(1) $\boldsymbol{x}$ *is divided by* $\boldsymbol{m}$ *is a i.e.* $x = pm + a$

(2) $\boldsymbol{x}$ *is divided by* $\boldsymbol{n}$ *is b i.e.* $x = qn + b$

for some integers p and q

Answer 1.3 *Let us consider the integers*

$$a, m + a, 2m + a, \ldots, (n-1)m + a$$

Each of the integers has remainder $\boldsymbol{a}$ *when divided by* $\boldsymbol{m}$.
Suppose that <u>*two of them*</u> *had the same remainder* $\boldsymbol{r}$ *when divided by* $\boldsymbol{n}$

Let the two numbers be

- $n = im + a$
- $n = jm + a$

$0 \leq i \leq j \leq (n-1)$

Thus, there are integers q_i *and* q_j *such that*

$$im + a = q_i n + r \tag{4}$$
$$jm + a = q_j n + r \tag{5}$$

- *i.e both are divided by* $\boldsymbol{n}$ *and they have a remainder.*

Subtracting Equations (5 - 4) :

$$(j - i)m = (q_j - q_i)n \tag{6}$$

Equation 7 states $\boldsymbol{n}$ *is a factor of the number* $(j-i)m$

- *Since* $\boldsymbol{n}$ *has no common factor other than 1 with* $\boldsymbol{m}$*, it follows that* $\boldsymbol{n}$ *is a factor of* $(j-i)$
- *However,* $0 \leq i < j \leq (n-1)$ *implies that* $0 < (j-i) \leq (n-1)$ *and hence* $\boldsymbol{n}$ *cannot be a factor of* $(j-i)$

This contradiction arises from our suppositions that two of the number

$$a, m + a, \ldots, (n-1)m + a$$

has the same remainder when divided by $\boldsymbol{n}$

Conclusion

Each of these $\boldsymbol{n}$-numbers has a different remainder when divided by $\boldsymbol{n}$

By the Pigeonhole principle, each of the $\boldsymbol{n}$-numbers $0, 1, \ldots, (n-1)$ occurs as a remainder; in particular b does.

Let $\boldsymbol{p}$ be an integer with $0 \leq p \leq (n-1)$ such that the number $x = pm + a$ has remainder b when divided by $\boldsymbol{n}$

Then, for some integer $\boldsymbol{q}$,

$$x = qn + b$$

.

$\therefore$, $x = pm + a$ and $x = qn + b$ and x has the required properties.

Note

The fact that a rational number (a/b) has a decimal expansion that eventually repeats is a consequence of the Pigeonhole Principle.

2 Pigeonhole Principle - Strong Form

Theorem 2.1 *Let $q_1, q_2, \ldots, q_n$ be a positive integers. If*

$$(q_1 + q_2 + \ldots + q_n - n + 1)$$

objects are put into n-boxes, then either the first box contains at least q_1 objects, or the second box contains q_2 objects, ... or the n^{th} box contains at least q_n objects.

Proof 2.1 *Suppose that we distribute $(q_1 + q_2 + \ldots + q_n - n + 1)$ objects among $\boldsymbol{n}$*

If for each $i = 1, 2, \ldots, n$, the i^{th} box contains fewer than q_i objects, then the total number of objects in all boxes does not exceed

$$(q_1 - 1) + (q_2 - 1) + \ldots + (q_n - 1) = q_1 + q_2 + \ldots + q_n - n \tag{7}$$

Since, this number is less than the number of objects distributed, we conclude that for some $i = 1, 2, \ldots, n$, the i^{th} box contains at least q_i objects.

- It is possible to distribute $\sum_{i=1}^{n} q_i - n$ objects, among n-boxes in such a way that for no $i = 1, 2, \ldots, n$ is it true that the i^{th} box contains q_i or more objects.

  This can be done by putting $(q_i - 1)$ objects in i^{th} box

- The simple form of the Pigeonhole Principle is obtained from the strong form by taking $q_i = 2 \ \forall i = 1, 2, \ldots, n$, then

$$\sum_{i=1}^{n} q_i - n + 1 = 2n - n + 1 = n + 1 \tag{8}$$

Example 2.1 *A basket of fruit is being arranged out of apples, bananas and oranges. What is the smallest number of pieces fruit that should be put in the basket in order to guarantee that either there are at least 8 apples / 6 bananas / 9 oranges.*

Answer 2.1 *By the Pigeonhole Principle - Strong Form, $(8+6+9)-3+1 = 21$ pieces of fruits, no matter how selected, will guarantee a basket of fruit with the desired properties.*

Example 2.2 *How many ordered pairs of integers (a, b) are needed to guarantee that there are (a_1, b_1) and (a_2, b_2) such that $a_1 \mod 5 = a_2 \mod 5$ and $b_1 \mod 5 = b_2 \mod 5$*

Answer 2.2

Solution 1. *There are 5 options for the congruence of $a \mod 5$ and 5 options for the congruence of $b \mod 5$. Therefore, there are 25 options for the congruence of the pair (a, b).*

If one had 26 options, then at least two will be in the same "congruence pair class".

However, with 25 it could be that each class has one pair, so if had 26 options, then at least one class will have 2

Solution 2. *One need to consider 25 different classes that both a, b will form. Thus the problem is now reduced to the Pigeonhole problem of $\lceil \frac{n}{25} \rceil = 2$*

By solving the equality of the ceiling function -

$$\begin{aligned} \left\lceil \frac{n}{25} \right\rceil - 1 < \frac{n}{25} &\leq \frac{n}{25} \\ \Rightarrow 2 - 1 < \frac{n}{25} &\leq 2 \\ \Rightarrow 25 < n &\leq 50 \end{aligned} \tag{9}$$

Therefore, there has to be $25 + 1 = 26$ by Pigeonhole principle

Example 2.3 *Let x be an irrational number. Show that for some positive integer j not exceeding the positive integer n, the absolute value of the difference between jx and the nearest integer to jx is less than $\frac{1}{n}$*

Answer 2.3 *Let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of the irrational number x, obvously $0 \leq \{x\} < 1$.*

Consider the $(n + 1)$ numbers $\{ax\}$, where $1 \leq a \leq n + 1$.

If these irrational numbers are put into the n-Pigeonholes $\left[0, \frac{1}{n}\right), \left[\frac{1}{n}, \frac{2}{n}\right), \ldots, \left[1 - \frac{1}{n}, 1\right)$, then there will be 1 Pigeonhole which must contain at least 2 of them.

Suppose, $\{ax\}$ and $\{bx\}$ belong to the same Pigeonhole, where $a > b$, which means $|\{ax\} - \{bx\}| < \frac{1}{n}$ with $(a - b) < x$.

Setting $j = (a - b)$ and $k = \lfloor ax \rfloor - \lfloor bx \rfloor$ gives us $|jx - k| < \frac{1}{n}$.

3 Averaging Principle

- If $(n(r - 1) + 1)$ objects are put into n-boxes, then at least one of the boxes contains r or more of the objects.

- **Theorem 3.1** *If the average of the n non-negative integers $m_1, m_2, \ldots, m_n$ is greater than $(r - 1)$, i.e.*

$$\frac{\sum_{i=1}^{n} m_i}{n} > r - 1$$

, then at least one of the integers is greater than or equal to r

Proof 3.1 *For $i = 1, 2, \ldots, n$, let m_i be the number of objects in the i^{th} box.*
Therefore, $\frac{\sum_{i=1}^{n} m_i}{n} = \frac{n(r-1)+1}{n} = (r - 1) + \frac{1}{n}$
Since, this average is greater than $(r - 1)$, one of the integers m_i is at least $\boldsymbol{r}$, i.e. one of the boxes contains at least r objects.

- If the average of n non-negative integers $m_1, m_2, \ldots, m_n$ is less than $(r + 1)$ i.e.

$$\frac{\sum m_i}{n} < (r + 1)$$

, then at least one of the integers is less than $(r + 1)$

- If the average of n non-negative integers, $m_1, m_2, \ldots, m_n$ is at least equal to r, then at least one of the integers $m_1, m_2, \ldots, m_n$ satisfies $m_i \geq r$.