APPENDIX THREE

Boolean Algebras, Boolean Rings, and Stone's Theorem

We saw in Sec. 2 that a *Boolean algebra of sets* can be defined as a class of subsets of a non-empty set which is closed under the formation of finite unions, finite intersections, and complements. Our purpose in this appendix is threefold: to define abstract Boolean algebras by means of lattices; to show that the theory of these systems can be regarded as part of the general theory of rings; and to prove the famous theorem of Stone, which asserts that every Boolean algebra is isomorphic to a Boolean algebra of sets.

The reader will recall that a *lattice* is a partially ordered set in which each pair of elements x and y has a greatest lower bound $x \wedge y$ and a least upper bound $x \vee y$, and that these elements are uniquely determined by x and y. It is easy to show (see Problem 8-5) that the operations $\wedge$ and $\vee$ have the following properties:

$$x \wedge x = x \quad \text{and} \quad x \vee x = x; \tag{1}$$
$$x \wedge y = y \wedge x \quad \text{and} \quad x \vee y = y \vee x; \tag{2}$$
$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad \text{and} \quad x \vee (y \vee z) = (x \vee y) \vee z; \tag{3}$$
$$(x \wedge y) \vee x = x \quad \text{and} \quad (x \vee y) \wedge x = x. \tag{4}$$

We shall see in the next paragraph that these properties are actually characteristic of lattices. Before proceeding further, however, we remark that

$$x \leq y \Leftrightarrow x \wedge y = x.$$

This fact serves to motivate the following discussion.

Let L be a non-empty set in which two operations $\wedge$ and $\vee$ are defined, and assume that these operations satisfy the above conditions. We shall prove that a partial order relation $\leq$ can be defined in L in such a way that L becomes a lattice in which $x \wedge y$ and $x \vee y$ are the greatest lower bound and least upper bound of x and y. Our first step is to notice that $x \wedge y = x$ and $x \vee y = y$ are equivalent; for if $x \wedge y = x$, then $x \vee y = (x \wedge y) \vee y = (y \wedge x) \vee y = y$, and similarly $x \vee y = y$ implies $x \wedge y = x$. We now define $x \leq y$ to mean that either $x \wedge y = x$ or $x \vee y = y$. Since $x \wedge x = x$, we have $x \leq x$ for every x. If $x \leq y$ and $y \leq x$, so that $x \wedge y = x$ and $y \wedge x = y$, then $x = x \wedge y = y \wedge x = y$. If $x \leq y$ and $y \leq z$, so that $x \wedge y = x$ and $y \wedge z = y$, then

$$x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x,$$

so $x \leq z$. This completes the proof that $\leq$ is a partial order relation. We now show that $x \wedge y$ is the greatest lower bound of x and y. Since $(x \wedge y) \vee x = x$ and $(x \wedge y) \vee y = (y \wedge x) \vee y = y$, we see that $x \wedge y \leq x$ and $x \wedge y \leq y$. If $z \leq x$ and $z \leq y$, so that $z \wedge x = z$ and $z \wedge y = z$, then $z \wedge (x \wedge y) = (z \wedge x) \wedge y = z \wedge y = z$, so $z \leq x \wedge y$. It is easy to prove, by similar arguments, that $x \vee y$ is the least upper bound of x and y.

This characterization of lattices brings the theory of these systems somewhat closer to ordinary abstract algebra, in which operations (instead of relations) are usually placed in the foreground.

A lattice is said to be *distributive* if it has the following properties:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{5}$$

and

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{6}$$

It is useful to know that (5) and (6) are equivalent to one another. For if (5) holds, then

$$\begin{aligned} (x \vee y) \wedge (x \vee z) &= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z] \\ &= x \vee [(x \vee y) \wedge z] \\ &= x \vee [(x \wedge z) \vee (y \wedge z)] \\ &= [x \vee (x \wedge z)] \vee (y \wedge z) \\ &= x \vee (y \wedge z), \end{aligned}$$

and a similar computation shows that (6) implies (5). We shall say that a lattice is *complemented* if it contains distinct elements 0 and 1 such that

$$0 \leq x \leq 1 \tag{7}$$

for every x (these elements are clearly unique when they exist), and if each element x has a *complement* x' with the property that

$$x \wedge x' = 0 \quad \text{and} \quad x \vee x' = 1. \tag{8}$$

We now define a *Boolean algebra* to be a complemented distributive lattice.

It is quite possible for an element of a complemented lattice to have many different complements. In a Boolean algebra, however, each