# Lecture 10: Minimum Spanning Trees and Prim's Algorithm

CLRS Chapter 23

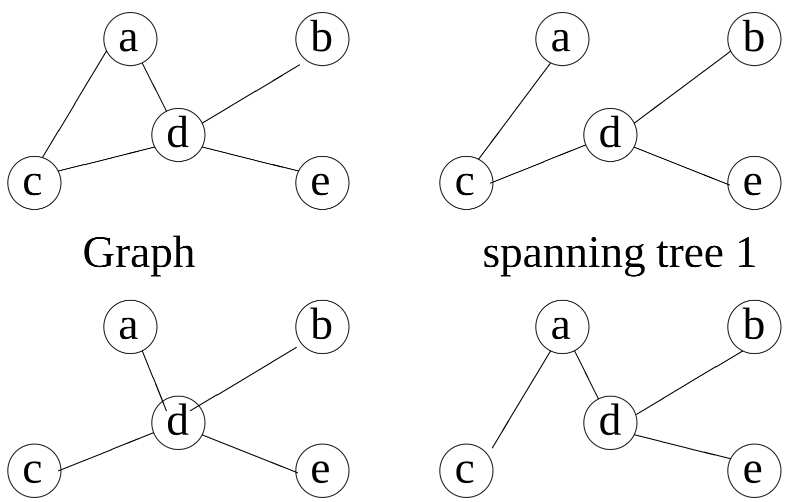
## Outline of this Lecture

- Spanning trees and minimum spanning trees.
- The minimum spanning tree (MST) problem.
- Prim's algorithm for the MST problem.
  - The algorithm
  - Correctness
  - Implementation + Running Time

# Spanning Trees

**Spanning Trees:** A subgraph $T$ of a undirected graph $G = (V, E)$ is a spanning tree of $G$ if it is a tree and contains every vertex of $G$.

**Example:**




Graph    spanning tree 1

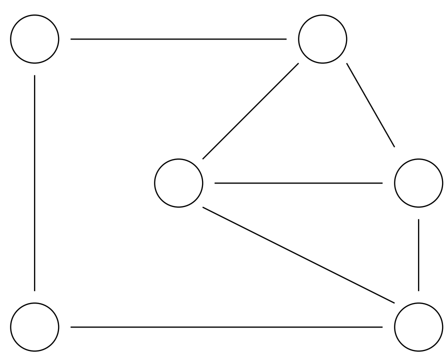
spanning tree 2    spanning tree 3

## Spanning Trees

**Theorem:** Every connected graph has a spanning tree.

**Question:** Why is this true?

**Question:** Given a connected graph $G$, how can you find a spanning tree of $G$?

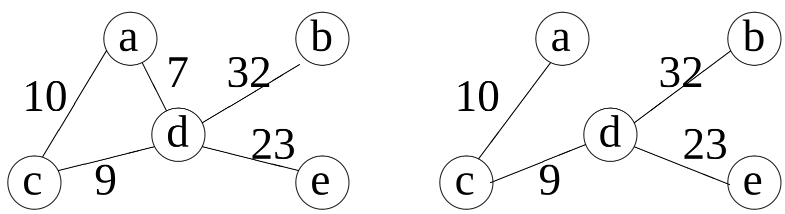


# Weighted Graphs

**Weighted Graphs:** A weighted graph is a graph, in which each edge has a weight (some real number).

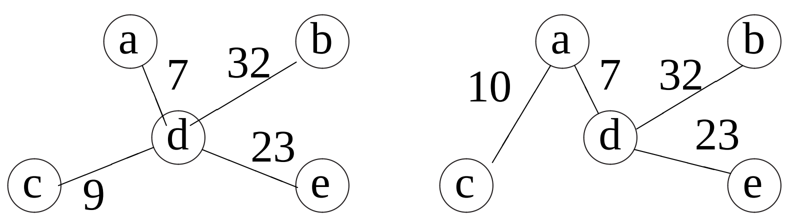
**Weight of a Graph:** The sum of the weights of all edges.

**Example:**




weighted graph

Tree 1. w=74



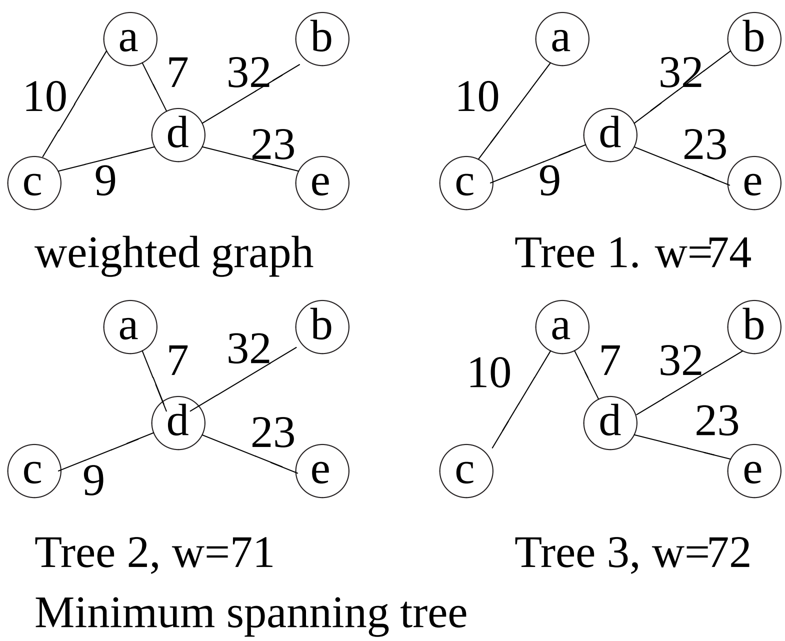

Tree 2, w=71
Minimum spanning tree

Tree 3, w=72

## Minimum Spanning Trees

A **Minimum Spanning Tree** in an undirected connected weighted graph is a spanning tree of minimum weight (among all spanning trees).

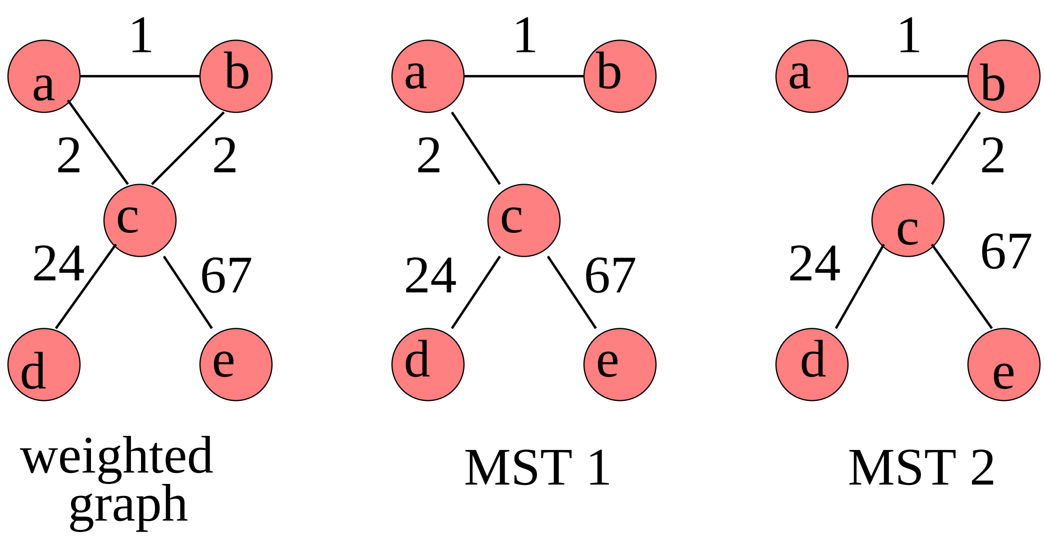
**Example:**



# Minimum Spanning Trees

**Remark:** The minimum spanning tree may not be unique. However, if the weights of all the edges are pairwise distinct, it is indeed unique (we won't prove this now).

**Example:**



# Minimum Spanning Tree Problem

**MST Problem:** Given a connected weighted undirected graph $G$, design an algorithm that outputs a minimum spanning tree (MST) of $G$.

**Question:** What is most intuitive way to solve?

**Generic approach:** A tree is an acyclic graph.
Idea is to start with an empty graph and try to add edges one at a time, always making sure that what is built remains acyclic.

We introduce two greedy algorithms (Prim's and Kruskal's algorithms) for computing a MST.
They differ in how to choose edges to add.

*Greedy: make the cheapest possible choice in each step.*

## What is Prim's Algorithm?

- A greedy algorithm for the MST problem.

- Looks very much like Dijkstra's algorithm:
  Grow a Tree

  – Start by picking any vertex $r$ to be the root of the tree.

  – While the tree does not contain
    all vertices in the graph
    find shortest edge leaving the tree
    and it to the tree .

- Running time is $O((|V| + |E|) \log |V|)$.

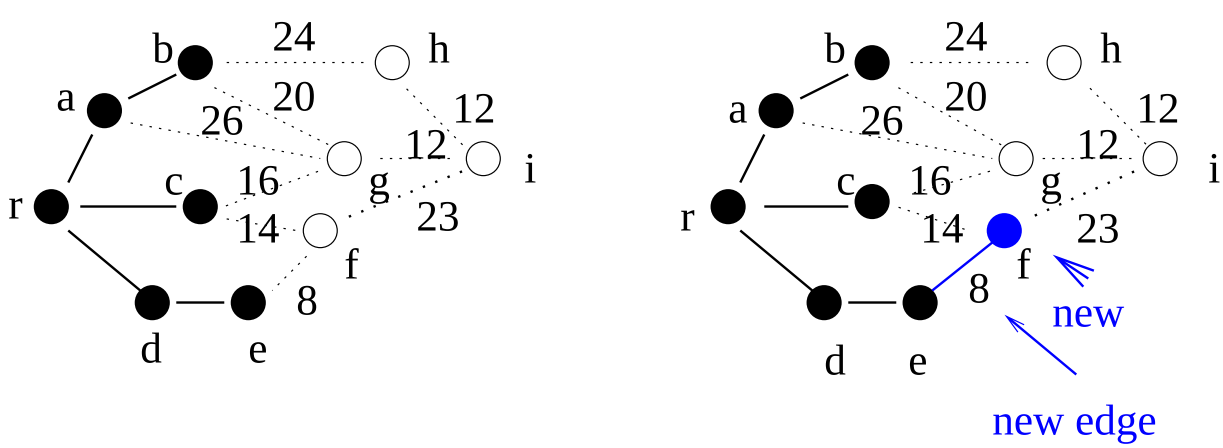
## More Details

**Step 0:** Choose any element $r$; set $S = \{r\}$ and $A = \emptyset$. (Take $r$ as the root of our spanning tree.)

**Step 1:** Find a lightest edge such that one endpoint is in $S$ and the other is in $V \setminus S$. Add this edge to $A$ and its (other) endpoint to $S$.

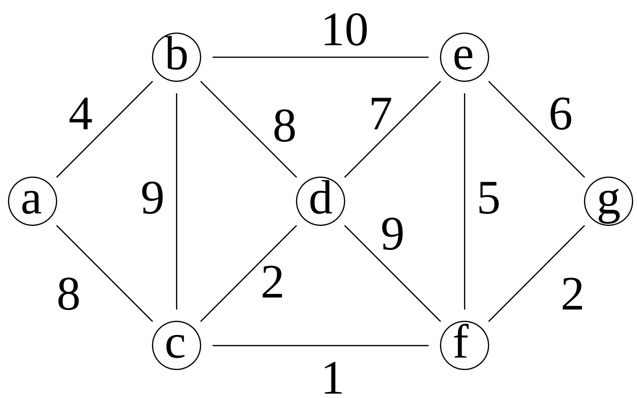
**Step 2:** If $V \setminus S = \emptyset$, then stop & output (minimum) spanning tree $(S, A)$. Otherwise go to Step 1.

The idea: expand the current tree by adding the lightest (shortest) edge leaving it and its endpoint.

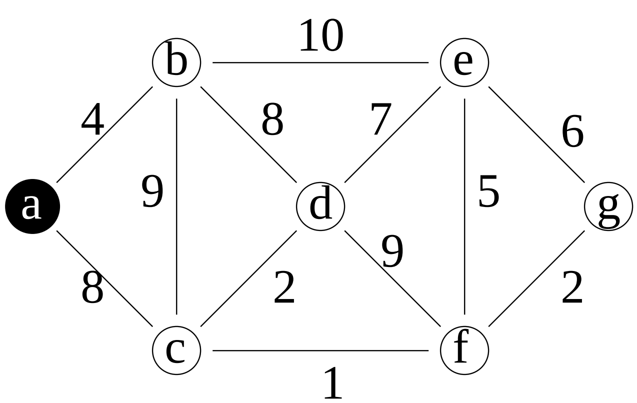


## Prim's Algorithm

# Worked Example




Connected graph



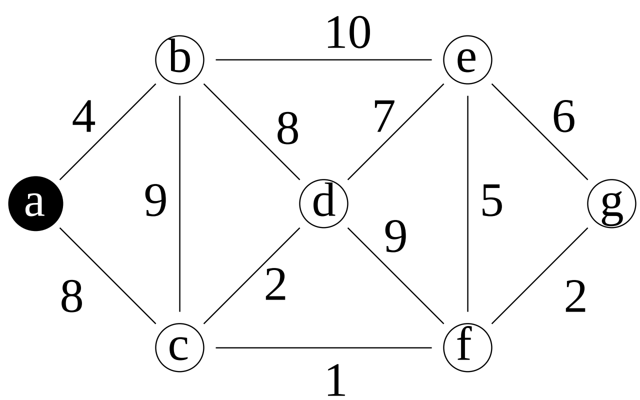

Step 0

S={a}

V \ S = {b,c,d,e,f,g}

lightest edge = {a,b}

## Prim’s Example – Continued



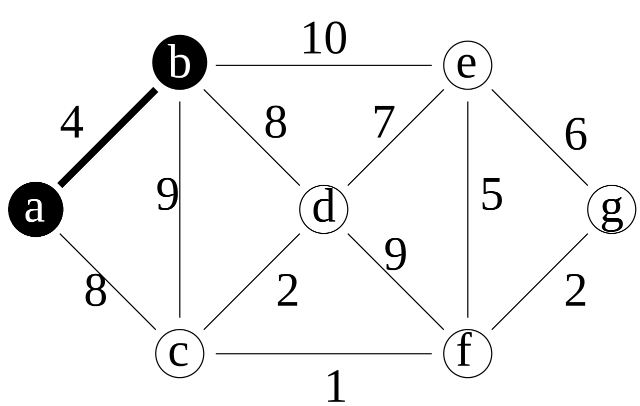

Step 1.1 before

S={a}

V \ S = {b,c,d,e,f,g}

A={}

lightest edge = {a,b}




Step 1.1 after

S={a,b}

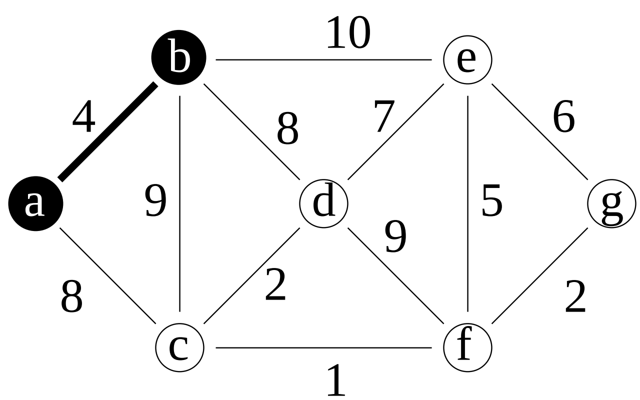
V \ S = {c,d,e,f,g}

A={{a,b}}

lightest edge = {b,d}, {a,c}

# Prim's Algorithm

## Prim's Example – Continued



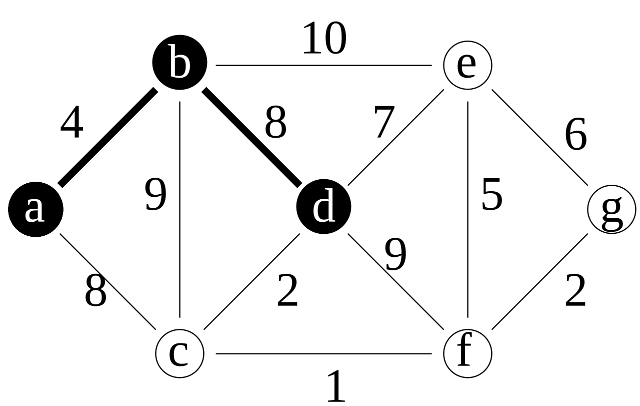

Step 1.2 before

S={a,b}

V \ S = {c,d,e,f,g}

A={{a,b}}

lightest edge = {b,d}, {a,c}




Step 1.2 after

S={a,b,d}

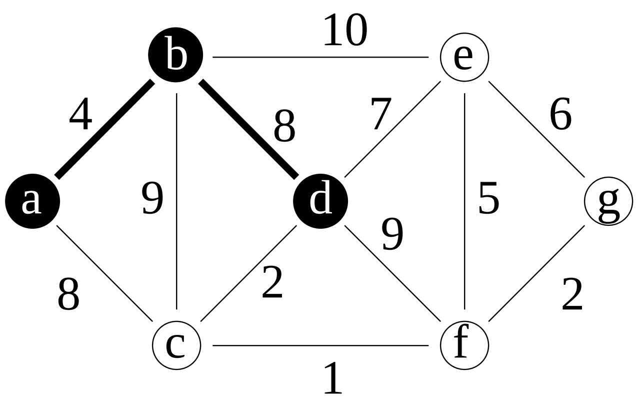
V \ S = {c,e,f,g}

A={{a,b},{b,d}}

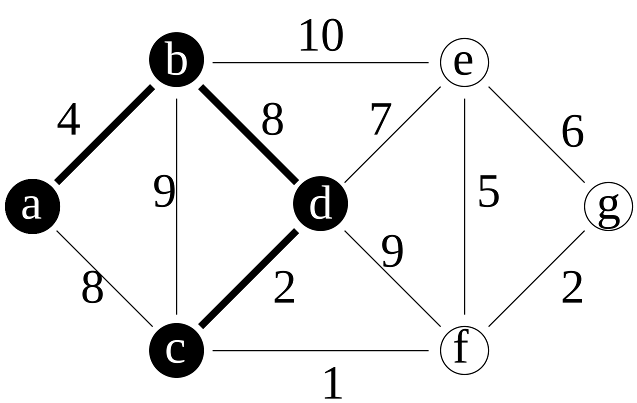
lightest edge = {d,c}

# Prim's Algorithm

## Prim's Example – Continued

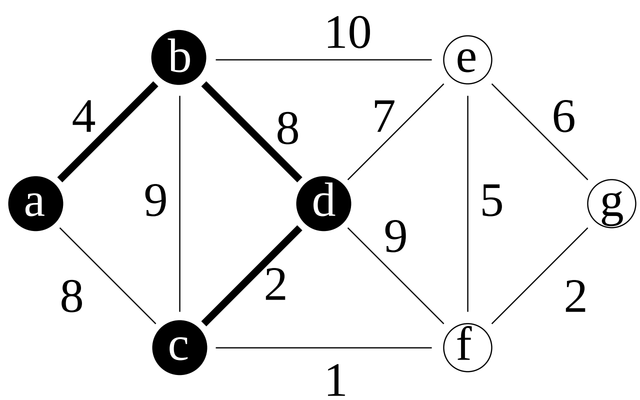



Step 1.3 before
S={a,b,d}
V \ S = {c,e,f,g}
A={{a,b},{b,d}}
lightest edge = {d,c}




Step 1.3 after
S={a,b,c,d}
V \ S = {e,f,g}
A={{a,b},{b,d},{c,d}}
lightest edge = {c,f}

## Prim’s Example – Continued

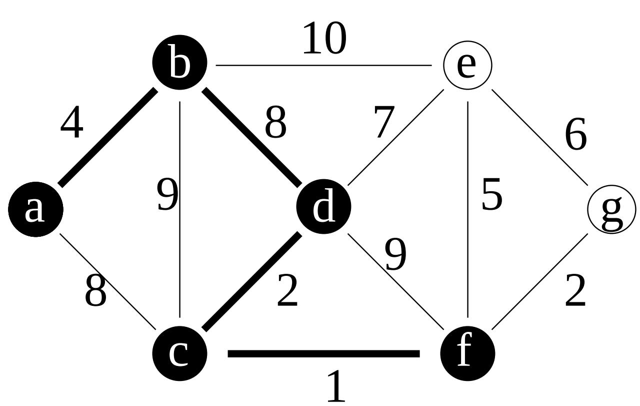



Step 1.4 before

S={a,b,c,d}

V \ S = {e,f,g}

A={{a,b},{b,d},{c,d}}
lightest edge = {c,f}



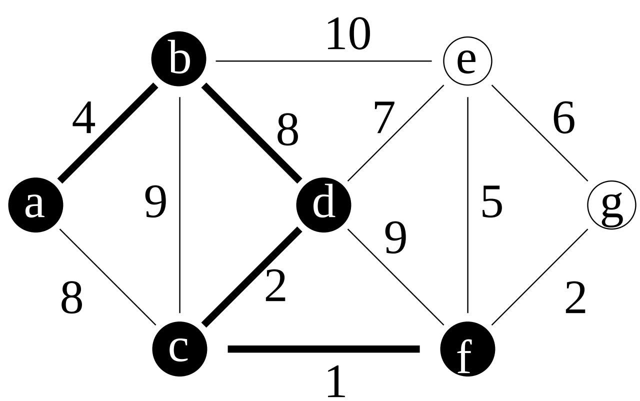

Step 1.4 after

S={a,b,c,d,f}
V \ S = {e,g}

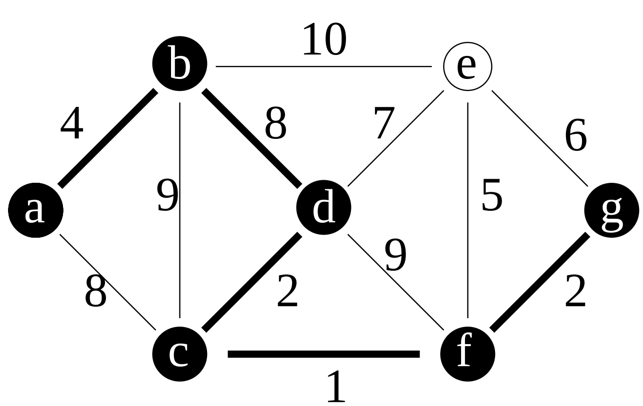
A={{a,b},{b,d},{c,d},{c,f}}

lightest edge = {f,g}

## Prim's Example – Continued

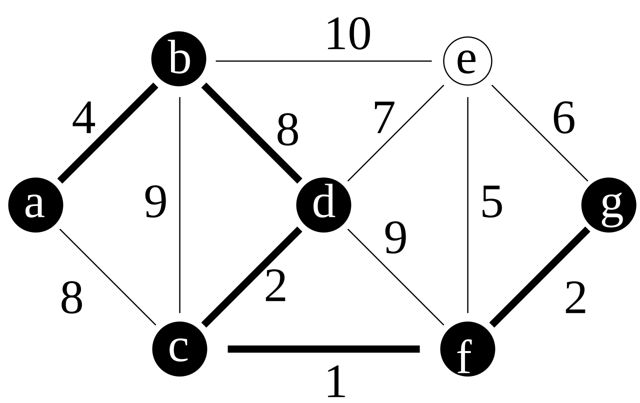



Step 1.5 before
S={a,b,c,d,f}
V \ S = {e,g}
A={{a,b},{b,d},{c,d},{c,f}}
lightest edge = {f,g}

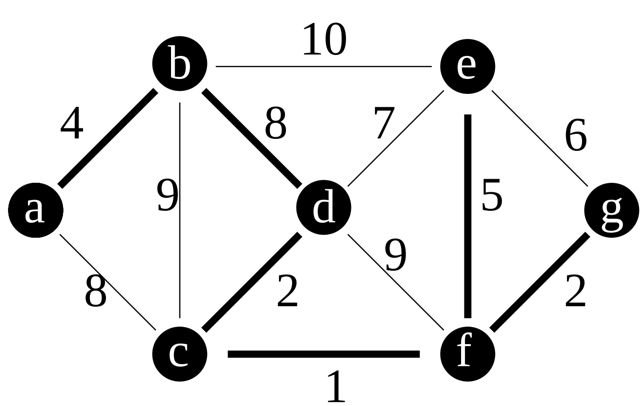



Step 1.5 after
S={a,b,c,d,f,g}
V \ S = {e}
A={{a,b},{b,d},{c,d},{c,f},
{f,g}}
lightest edge = {f,e}

## Prim’s Example – Continued




Step 1.6 before
S={a,b,c,d,f,g}
V \ S = {e}
A={{a,b},{b,d},{c,d},{c,f},
{f,g}}
lightest edge = {f,e}




Step 1.6 after
S={a,b,c,d,e,f,g}
V \ S = {}
A={{a,b},{b,d},{c,d},{c,f},
{f,g},{f,e}}

MST completed

# Recall Idea of Prim's Algorithm

**Step 0:** Choose any element $r$ and set $S = \{r\}$ and $A = \emptyset$. (Take $r$ as the root of our spanning tree.)

**Step 1:** Find a lightest edge such that one endpoint is in $S$ and the other is in $V \setminus S$. Add this edge to $A$ and its (other) endpoint to $S$.

**Step 2:** If $V \setminus S = \emptyset$, then stop and output the minimum spanning tree $(S, A)$.
Otherwise go to Step 1.

## Questions:

- Why does this produce a **Minimum** Spanning Tree?

- How does the algorithm find the lightest edge and update $A$ efficiently?

- How does the algorithm update $S$ efficiently?

## Correctness of Prim's Algorithm

**Lemma:** Let $(S, A)$ be a subtree of a MST of an undirected graph $G = (V, E)$, where $S \subset V$ and $A \subset E$. Let $e = \{u, v\}$ be an edge such that

(1) $u \in S$ and $v \in V \setminus S$;
(2) $e$ has lowest weight among all the edges between a vertex in $S$ and a vertex in $V \setminus S$.

Then $(S \cup \{v\}, A \cup \{e\})$ is a subtree of a MST.

**Proof:** Let $T$ be a MST of $G$ that contains $(S, A)$.
If $e$ is an edge of $T$, we are done.

Suppose that $e$ is not an edge of $T$.
There is a unique path from $u$ to $v$ in $T$. There must be at least one edge $e' = \{u', v'\}$ in the path such that $u' \in S$ and $v' \in V \setminus S$. By (2) above,

$$W(e) \le W(e'). \qquad (*)$$

Consider the new tree $T' := (T \cup \{e\}) \setminus \{e'\}$. Since $T$ is MST,

$$W(T) \le W(T') = W(T) - W(e') + W(e)$$

and so $W(e') \le W(e)$. Combined with $(*)$, this proves that $W(e') = W(e)$, and so $W(T') = W(T)$. Therefore $T'$ is also a MST, and $T'$ contains $(S \cup \{v\}, A \cup \{e\})$.

# Correctness of Prim's Algorithm

**Lemma:** Let $(S, A)$ be a subtree of a MST of an undirected graph $G = (V, E)$, where $S \subset V$ and $A \subset E$. Let $e = \{u, v\}$ be an edge such that

(1) $u \in S$ and $v \in V \setminus S$;
(2) $e$ has the lowest weight among all the edges between a vertex in $S$ and a vertex in $V \setminus S$.

Then $(S \cup \{v\}, A \cup \{e\})$ is a subtree of a MST.

We can now prove the correctness of Prim's algorithm by induction.

When the algorithm starts, $(\{r\}, \emptyset)$ is definitely a subtree of a MST of $G$ (why ).

At each step the algorithm chooses an edge $e = \{u, v\}$ that satisfies (1) and (2) so, from the lemma, $(S \cup \{v\}, A \cup \{e\})$ remains a subtree of some MST of $G$.

In particular, when the algorithm ends, $S = V$ and $A$ is a tree on $V$. We know from above that $(S, A)$ is a subtree of some MST of $G$ but, since $A$ itself is a tree on $G$, this means that $A$ itself is a MST.

## Prim's Algorithm

**Question:** How does the algorithm update $S$ efficiently?

**Answer:** Color the vertices. Initially all are white. Change the color to black when the vertex is moved to $S$. Use color[$v$] to store color.

**Question:** How does the algorithm find the lightest edge and update $A$ efficiently?

**Answer:**
(a) Use a priority queue to find the lightest edge.
(b) Use pred[$v$] to update $A$.

## Reviewing Priority Queues

**Priority Queue** is a data structure (can be implemented as a heap) which supports the following operations:

**insert($u, key$):**

Insert $u$ with the key value $key$ in $Q$.

**u = extractMin():**

Extract the item with the minimum key value in $Q$.

**decreaseKey($u, new$-$key$):**

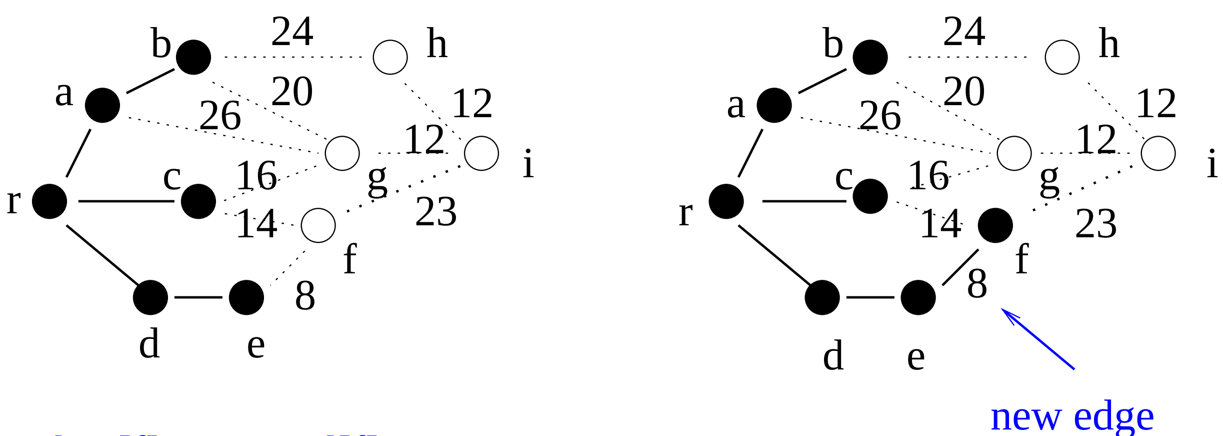
Decrease $u$'s key value to $new$-$key$.

**Remark:** Priority Queues can be implemented so that each operation takes time $O(\log |Q|)$. See CLRS!

## Using a Priority Queue to Find the Lightest Edge

Each item of the queue is a triple $(u, pred[u], key[u])$, where

- $u$ is a vertex in $V \setminus S$,
- $key[u]$ is the weight of the lightest edge from $u$ to any vertex in $S$, and
- $pred[u]$ is the endpoint of this edge in $S$. The array is used to build the MST tree.




key[f] = 8, pred[f] = e

key[i] = infinity, pred[i] = nil

key[g] = 16, pred[g] = c

key[h] = 24, pred[h] = b

$\longrightarrow$ f has the minimum key

key[i] = 23, pred[i] = f

After adding the new edge and vertex f, update the key[v] and pred[v] for each vertex v adjacent to f

## Description of Prim's Algorithm

**Remark:** $G$ is given by adjacency lists. The vertices in $V \setminus S$ are stored in a priority queue with key=value of lightest edge to vertex in $S$.

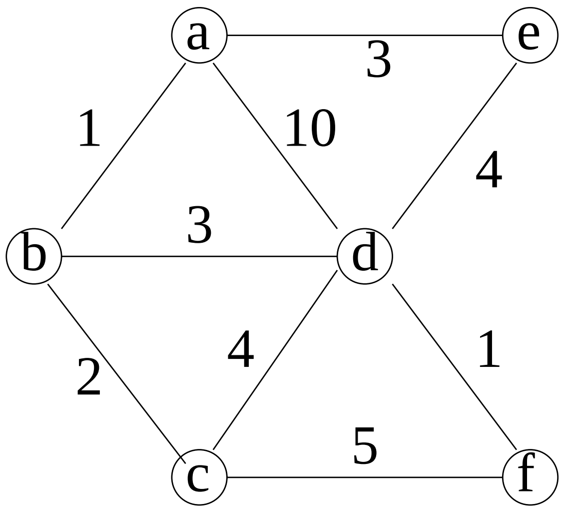
```
Prim(G, w, r)
{   for each u ∈ V                                  initialize
    {   key[u] = +∞;
        color[u] = W;
    }
    key[r] = 0;                                     start at root
    pred[r] = NIL;
    Q = new PriQueue(V);                            put vertices in Q
    while(Q is nonempty)                            until all vertices in MST
    {   u=Q.extraxtMin();                           lightest edge
        for each (v ∈ adj[u])
        {   if ((color[v] == W)&(w[u, v] < key[v]))
                key[v] = w[u, v];                   new lightest edge
                Q.decreaseKey(v, key[v]);
                pred[v] = u;
        }
        color[u] = B;
    }
}
```

When the algorithm terminates, $Q = \emptyset$ and the MST is

$$T = \{\{v, pred[v]\} : v \in V \setminus \{r\}\}.$$

The pred pointers define the MST as an inverted tree rooted at $r$.

# Example for Running Prim's Algorithm




| u | a | b | c | d | e | f |
|---|---|---|---|---|---|---|
| key[u] | | | | | | |
| pred[u] | | | | | | |

## Analysis of Prim's Algorithm

Let $n = |V|$ and $e = |E|$. The data structure PriQueue supports the following two operations: (See CLRS)

- $O(\log n)$ to extract each vertex from the queue.
  Done once for each vertex $= O(n \log n)$.

- $O(\log n)$ time to decrease the key value of neighboring vertex.
  Done at most once for each edge $= O(e \log n)$.

Total cost is then

$$O((n + e) \log n)$$

## Analysis of Prim's Algorithm – Continued

```
Prim(G, w, r)  {
    for each (u in V)                                   
    {
        key[u] = +infinity;                             2n
        color[u] = white;
    }

    key[r] = 0;                                         1
    pred[r] = nil;                                      1
    Q = new PriQueue(V);                                n

    while (Q. nonempty())                   1
    {
        u = Q.extractMin();                 O(log n)
        for each (v in adj[u])
        {
            if ((color[v] == white) &       1
                 (w(u,v) < key[v])          1   O(deg(u) log n)
            {
                key[v] = w(u, v);           1
                Q.decreaseKey(v, key[v]);   O(log n)
                pred[v] = u;                1
            }
        }
        color[u] = black;                   1
    }
}
```

$$\sum_{u \in V} [O(\log n) + O(\deg(u) \log n)]$$

## Analysis of Prim's Algorithm – Continued

So the overall running time is

$$
\begin{aligned}
& T(n,e) \\
= \; & 3n + 2 + \sum_{u \in V} [O(\log n) + O(\deg(u) \log n)] \\
= \; & 3n + 2 + O\left[(\log n) \sum_{u \in V} (1 + \deg(u))\right] \\
= \; & 3n + 2 + O[(\log n)(n + 2e)] \\
= \; & O[(\log n)(n + 2e)] \\
= \; & O[(\log n)(n + e)] \\
= \; & O[(|V| + |E|) \log |V|].
\end{aligned}
$$