Matematinė analizė

* Integralai — Apskaičiuokite integralus

I var.

1. $$\int \frac{3^x + 5^{2x}}{7^x}\,dx = \int \left(\frac{3}{7}\right)^x dx + \int \left(\frac{25}{7}\right)^x dx = \frac{\left(\frac{3}{7}\right)^x}{\ln\frac{3}{7}} + \frac{\left(\frac{25}{7}\right)^x}{\ln\frac{25}{7}} + C$$

2. $$\int \frac{dx}{1+\cos x} = \int \frac{\frac{dx}{2}}{2\cos^2\frac{x}{2}} = \frac{1}{2}\cdot 2\int \frac{dx}{\cos^2\frac{x}{2}} = \operatorname{tg}\frac{x}{2} + c \ ?$$

3. $$\int \operatorname{ctg}^2 x\,dx = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx - \int dx =$$
$$= -\operatorname{ctg} x - x + c$$

4. $$\int \frac{\cos x}{3+\sin x}\,dx = \int \frac{d(\sin x)}{3+\sin x} = \int \frac{d(\sin x + 3)}{3+\sin x} = \int \frac{dt}{t} =$$
$$= \ln t + c = \ln(\sin x + 3)$$

5. $$\int \frac{dx}{\operatorname{tg}^4 x \cos^2 x} = \int \frac{d(\operatorname{tg} x)}{\operatorname{tg}^4 x} = \int \operatorname{tg}^{-4} d(\operatorname{tg} x) = -\frac{1}{3}\operatorname{tg}^{-3} x + c$$

6. $$\int \frac{\sin(\ln x)}{x}\,dx = \int \sin(\ln x)\,d(\ln x) = -\cos(\ln x) + c$$

7. $$\int \frac{1}{1-x^2}\ln\left|\frac{1+x}{1-x}\right| dx = -\int \frac{1}{x^2-1}\ln\left|\frac{1+x}{1-x}\right| dx =$$
$$= -\int \ln\left|\frac{1+x}{1-x}\right| d\left(\frac{1}{2}\ln\left|\frac{1-x}{1+x}\right|\right) = \frac{1}{2}\int \ln\left|\frac{1+x}{1-x}\right| d\left(\ln\left|\frac{1+x}{1-x}\right|\right) = \ ?$$
$$= \frac{1}{2}\int t\,dt = \frac{1}{4}\ln^2\left|\frac{1+x}{1-x}\right| + c$$

8. $$\int x^2 e^{-x^3} dx = \int e^{-x^3} d\left(\frac{x^3}{3}\right) = -\frac{1}{3}\int e^{-x^3} d(-x^3) = -\frac{1}{3}\int e^t dt =$$
$$= -\frac{1}{3}e^t + c = -\frac{1}{3}e^{-x^3} + c$$

9. $$\int \frac{e^{-\alpha x}}{1+e^{-2\alpha x}}\,dx = -\frac{1}{\alpha}\,?\cdot\int \frac{d(e^{-\alpha x})}{1+e^{-2\alpha x}} = -\frac{1}{\alpha}\cdot\int \frac{dt}{1+t^2} =$$
$$= -\frac{1}{\alpha}\cdot 1\cdot \operatorname{arctg} t + c, \text{ kai } t = e^{-\alpha x}$$

10. $$\int \frac{x^3+2}{x^4+8x+2}\,dx = \frac{1}{4}\int \frac{d\left(\frac{x^4}{4}+2x\right)}{\frac{x^4}{4}+2x+\frac{1}{2}} = \frac{1}{4}\cdot\int \frac{d\left(\frac{x^4}{4}+2x+\frac{1}{2}\right)}{\frac{x^4}{4}+2x+\frac{1}{2}} =$$
$$= \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}\ln t + c = \frac{1}{4}\ln\left(\frac{x^4}{4}+2x+\frac{1}{2}\right) + c$$

11. $$\int \frac{dx}{1+\sqrt{1+x}} = \int \frac{2t\,dt}{1+t} = 2\int \frac{(t+1)-1}{(1+t)}\,dt = 2\int dt - 2\int \frac{d(t+1)}{t+1} =$$
$$= 2t - 2\ln(t+1) + c = 2\sqrt{1+x} - 2\ln(\sqrt{1+x}+1) + c$$

$t = \sqrt{1+x}$
$x = t^2 - 1$
$dx = d(t^2-1) = 2t\,dt$

(12) $\int \frac{e^{2x}}{1+e^x}dx = \int \frac{t^{\cancel{2}}}{1+t}\cdot\frac{dt}{\cancel{t}} = \int\frac{t}{1+t}dt = \int \frac{(t+1)-1}{(t+1)}dt =$

$t = e^x$ $\qquad = \int dt - \int \frac{d(t+1)}{t+1} =$

$x = \log_e t = \ln t$

$dx = d(\log_e t) = d(\ln t) = \frac{dt}{t}$ $\qquad = t - \ln(t+1) + c = e^x - \ln(e^x+1) + c$

(13) $\int\frac{dx}{e^{\frac{x}{2}}+e^x} = \int\frac{dx}{e^{\frac{x}{2}}(1+e^{\frac{x}{2}})} = \int\frac{e^{-\frac{x}{2}}}{1+e^{\frac{x}{2}}}dx = -2\int\frac{e^{-\frac{x}{2}}d(-\frac{x}{2})}{1+\frac{1}{e^{-\frac{x}{2}}}} = -2\int\frac{de^{-\frac{x}{2}}}{1+\frac{1}{e^{-\frac{x}{2}}}} =$?

$= -2\int\frac{dt}{1+\frac{1}{t}} = -2\int\frac{(t+1)-1}{t+1}dt = -2\int dt + 2\int\frac{dt}{t+1} = -2t + 2\ln(t+1) + c =$?

$= -2e^{-\frac{x}{2}} + 2\ln(e^{-\frac{x}{2}}+1) + c$

(14) $\int \cos^5 x\cdot\sqrt{\sin x}\,dx = \int (1-t^2)^2\sqrt{t}\frac{dt}{\sqrt{1-t^2}} = \int(1-t^2)^2\sqrt{t}\,dt = \int(t^{\frac{1}{2}} - 2t^{\frac{5}{2}} + t^{\frac{9}{2}})dt =$

$x = \arcsin t$ $\qquad = \frac{2}{3}t^{\frac{3}{2}} - \frac{4}{7}t^{\frac{7}{2}} + \frac{2}{11}t^{\frac{11}{2}} + c$?

$dx = d(\arcsin t) = \frac{dt}{\sqrt{1-t^2}}$

(15) $\int\frac{dx}{x\sqrt{x^2+a^2}} = \int\frac{\cancel{t}dt}{(\sqrt{t^2-a^2})\cancel{t}} = \int\frac{dt}{t^2-a^2} = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + c = \frac{1}{2a}\ln\left|\frac{\sqrt{x^2-a^2}-a}{\sqrt{x^2-a^2}+a}\right| + c$

$t = \sqrt{x^2+a^2}$

$x = \sqrt{t^2-a^2}$

$dx = d(\sqrt{t^2-a^2}) = \frac{\cancel{2}t}{\cancel{2}(\sqrt{t^2-a^2})}dt = \frac{tdt}{\sqrt{t^2-a^2}}$

(16) $\int\frac{dx}{(1+x^2)^2} = \int\frac{dt}{\cos^2 x}\cdot\frac{1}{(1+tg^2t)^2} = \int\frac{1}{(1+\frac{\sin^2 t}{\cos^2 t})^2}\cdot\frac{dt}{\cos^2 t} = \int\frac{1}{\frac{1}{\cos^4 t}\cdot\cos^2 t}dt =$

$t = \operatorname{arctg} x$ $\qquad = \int\cos^2 t\,dt = \frac{1}{2}\int(1+\cos(2t))\,dt =$

$x = tg\,t$

$dx = d\,tg\,t = \frac{dt}{\cos^2 x}$ $\qquad = \frac{1}{2}\int dt + \frac{1}{4}\int\cos(2t)\,d2t$? $= \frac{1}{2}t + \frac{1}{4}\sin(2t) + c$

(17) $\int\sqrt{5-4\sin x}\,\cos x\,dx = \int\sqrt{5-4\sin x}\,d\sin x = \int\sqrt{5-4t}\,dt = -\frac{1}{4}\int\sqrt{5-4t}\,d(5-4t) =$

$= -\frac{1}{4}\cdot\frac{2}{3}(5-4t)^{\frac{3}{2}} + c = -\frac{1}{6}(5-4\sin)^{\frac{3}{2}} + c$?

Integravimo dalimis formulė: (2.5 teig. 17 psl.)

$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx = \int f(x)dg(x) = f(x)g(x) - \int g(x)df(x)$

Apytaisė skaičiuoti integralus

(1.) $\int x\sin x\,dx = -x\cos x + \int\cos x\cdot 1\,dx = -x\cos x + \sin x + c = -\int x\,d\cos x =$ II sąv.

$= -x\cos x + \int\cos x\,dx = -x\cos x + \sin x + c$

$G(x) = \int g'(x)dx$

(2) $\int x\operatorname{arctg} x\,dx = \frac{1}{2}\int\operatorname{arctg} x\,dx^2 = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int x^2\,d\operatorname{arctg} x =$

$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int\frac{x^2+1-1}{x^2+1}dx = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{dx}{x^2+1} =$

$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}x + \frac{1}{2}\operatorname{arctg} x + c$

$x^2 = t$

$$③ \int x^3 e^{-x^2} dx = \frac{1}{2}\int x^2 e^{-x^2} dx^2 = \frac{1}{2}\int t e^{-t} dt = -\frac{1}{2}\int t \, d e^{-t} =$$

$$= -\frac{1}{2}\left(\int t e^{-t} - \int e^{-t} dt\right) = -\frac{1}{2}\left(t e^{-t} + e^{-t}\right) + c = -\frac{1}{2}\left(x^2 e^{-x^2} + e^{-x^2}\right) + c$$

$n \geq 0$

$$④ \int x^n \ln x \, dx = \frac{1}{n+1}\int \ln x \, d(x^{n+1}) = \ln x \cdot x^{n+1} \cdot \frac{1}{n+1} - \frac{1}{n+1}\int$$?

$$\int x^{n+1} d(\ln x) = \ln x \cdot x^{n+1} \cdot \frac{1}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c$$

$$⑤ \int x^2 3^x dx = \frac{1}{\ln 3} \cdot \int x^2 d 3^x = \frac{1}{\ln 3}\left(x^2 3^x - \int 3^x dx^2\right) =$$

$$= \frac{1}{\ln 3}\left(x^2 3^x - 2\int 3^x x \, dx\right) = \frac{1}{\ln 3}\left(x^2 3^x - \frac{2}{\ln 3}\int x \, d 3^x\right) =$$

$$= \frac{1}{\ln 3}\left(x^2 3^x - \frac{2}{\ln 3} x 3^x + \frac{2}{\ln 3} \cdot \frac{3^x}{\ln 3}\right) + c = \frac{x^2 3^x}{\ln 3} - \frac{2x 3^x}{(\ln 3)^2} + \frac{2 \cdot 3^x}{(\ln 3)^3} + c$$

$$⑥ \int \frac{\ln^2(x)}{x^2} dx = \int \ln^2(x) \, d\frac{x^{-1}}{-1} = -\int \ln^2(x) \, d\frac{1}{x} = -\left(\ln^2(x) \cdot \frac{1}{x} - \right.$$

$$\left. - \int \frac{1}{x} d \ln^2 x\right) = \int \frac{1}{x} d \ln^2(x) - \frac{\ln^2(x)}{x} = \int \frac{2 \ln x}{x^2} dx - \frac{\ln^2 x}{x} =$$?

$$= 2\int \frac{\ln x}{x^2} dx - \frac{\ln^2(x)}{x} = 2\int \ln x \, d\frac{x^{-1}}{-1} - \frac{\ln^2(x)}{x} =$$

$$= -2\int \ln x \, d\frac{1}{x} - \frac{\ln^2(x)}{x} = -2\left(\ln x \frac{1}{x} - \int \frac{1}{x} d \ln x\right) - \frac{\ln^2(x)}{x} =$$

$$= -2\left(\frac{\ln x}{x} - \int x^{-2} dx\right) - \frac{\ln^2(x)}{x} = -2\left(\frac{\ln x}{x} + \frac{1}{x}\right) - \frac{\ln^2 x}{x}$$

$$⑦ \int \arcsin x \, dx = x \arcsin x - \int x \, d(\arcsin x) = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx =$$

$$= x \arcsin x + \frac{1}{2}\int \frac{d(-x^2+1)}{\sqrt{1-x^2}} = x \arcsin x + \frac{1}{2} \cdot \frac{2}{1}\sqrt{1-x^2} + c =$$

$$= x \arcsin x + \sqrt{1-x^2} + c$$

$$⑧ \int \ln(x) dx = \ln x \cdot x - \int x \cdot \ln(x)' dx = x \ln x - \int x \cdot \frac{1}{x} dx =$$

$$= x \ln x - \int dx = x \ln x - x + c$$

$$⑨ \int x \cos^2(x) dx = \frac{1}{2}\int x(1 + \cos(2x)) dx = \frac{1}{2}\left(x\left(x + \frac{\sin(2x)}{2}\right) - \int 1 \cdot \frac{\sin(2x)}{2}\right) =$$

?

$$= \left(x^2 + \frac{x \sin(2x)}{2} - \int \frac{\sin(2x)}{2} = x^2 + \frac{x \sin(2x)}{2} + \cos x =\right.$$

$$= \frac{x^2}{4} + \frac{1}{4}\int x \, d(\sin(2x)) = \frac{x^2}{4} + \frac{x \sin(2x)}{4} - \frac{1}{4}\int \sin(2x) dx \Leftrightarrow$$

$$\frac{x^2}{4} + \frac{x \sin(2x)}{4} + \frac{\cos(2x)}{8} + c$$

$$⑩ \int \sin(\ln(x)) dx = x \sin(\ln(x)) - \int x \cos \ln x \cdot \frac{1}{x} =$$

?

$$= x \sin(\ln(x)) - \int \cos(\ln(x)) = x \sin(\ln(x)) - x \cos(\ln(x)) - \int x \sin$$

$$= x \sin(\ln(x)) - \cos(\ln(x))$$

$$\int \frac{a}{bx+c}\,dx = a\int\frac{dx}{bx+c} = \frac{a}{b}\int\frac{d(bx+c)}{bx+c} = \frac{a}{b}\ln(bx+c)+C$$ III sav.

$$\int \frac{a}{(bx+c)^\lambda}\,dx = \frac{a}{b}\int\frac{d(bx+c)}{(bx+c)^\lambda} = \frac{a}{b}\int t^{-\lambda}\,dt = \frac{a}{b}\cdot\frac{t^{1-\lambda}}{1-\lambda}+C$$

① $\int \frac{3x+8}{(x-2)(x+5)}\,dx$

$$\frac{A}{x-2}+\frac{B}{x+5} = \frac{Ax+5A+Bx-2B}{(x-2)(x+5)}$$

$$\begin{matrix} A+B=3 \\ 5A-2B=8 \end{matrix} \Rightarrow \begin{cases} A=3-B \\ 5(3-B)-2B=8 \end{cases} \Rightarrow 15-5B-2B=8B=8 \Rightarrow B=1$$

$B=1$
$A=2$

$$\int\frac{2}{x-2}\,dx + \int\frac{1}{x+5}\,dx = \ln|x+5| + 2\ln|x-2| + C$$

② $\int\frac{5x^2+1}{x(x^2-1)}\,dx = \int\frac{3}{x-1}\,dx + \int\frac{3}{x+1}\,dx - \int\frac{1}{x}\,dx$

$A=-1$
$B=3$
$C=3$

* $\int \mathrm{tg}^2 x\,dx = \int\frac{\sin^2 x}{\cos^2 x}\,dx = \int\frac{1-\cos^2 x}{\cos^2 x}\,dx = \int\frac{dx}{\cos^2 x} - \int dx =$
$= \mathrm{tg}\,x - x + C$

$\int\frac{P_n(x)}{Q_m(x)}\,dx$ – neapibrėžtųjų koef. metodas III sav.

Apskaičiuokite integralus

① $\int\frac{3x+8}{(x-2)(x+5)}\,dx = \int\frac{A}{x-2}\,dx + \int\frac{B}{x+5}\,dx = 2\int\frac{dx}{x-2} + \int\frac{dx}{x+5}$ (=)

$$\frac{A}{x-2}+\frac{B}{x+5} = \frac{Ax+5A+Bx-B2}{(x-2)(x+5)} = \frac{3x+8}{(x-2)(x+5)}$$

$$\begin{cases} A+B=3 \\ 5A-2B=8 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=1 \end{cases}$$ (=) $2\ln|x-2| + \ln|x+5| + C$

② $\int\frac{5x^2+1}{x(x^2-1)}\,dx = \int\frac{A}{x}\,dx + \int\frac{Bx+C}{x^2-1} = -\int\frac{dx}{x} + 6\int\frac{x\,dx}{x^2-1}$ (=)

$$\frac{A}{x}+\frac{Bx+C}{x^2-1} = \frac{Ax^2-A+Bx^2+Cx}{x(x^2-1)} = \frac{5x^2+1}{x(x^2-1)}$$

$$\begin{cases} A+B=5 \\ C=0 \\ -A=1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=6 \end{cases}$$ (=) $-\ln x + 3\int\frac{d(x^2-1)}{x^2-1} = -\ln x + 3\ln|x^2-1| + C$

③ $\int \frac{5x^2+1}{(x-1)^2(x-2)}\,dx = \int \frac{A}{(x-1)^2}dx + \int \frac{B}{x-1}\,dx + \int \frac{C}{x-2}\,dx \ominus$

$\frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x-2} = \frac{Ax-2A+Bx^2-3Bx+2B+Cx^2-2Cx+C}{(x-1)^2(x-2)} \Rightarrow$

$\Rightarrow \begin{cases} B+C=5 \\ A-3B-2C=0 \\ -2A+2B+C=1 \end{cases} \Rightarrow \begin{cases} B=5-C \\ A=15-C \\ C=21 \end{cases} \Rightarrow \begin{cases} B=-16 \\ A=-6 \\ C=21 \end{cases}$

$\ominus -6\int \frac{d(x-1)}{(x-1)^2} - 16\int \frac{dx}{x-1} + 21\int \frac{dx}{x-2} =$

?

$= \frac{6}{x-1} - 16\ln|x-1| + 21\ln|x-2| + C$

④ $\int \frac{6+8x-x^2}{x^3+3x^2+2x}\,dx = \int \frac{6+8x-x^2}{x(x+2)(x+1)}\,dx = \int \frac{A}{x}dx + \int \frac{B}{x+2}dx + \int \frac{C}{x+1}dx \ominus$

$\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+1} = \frac{A(x+2)(x+1)+Bx(x+1)+Cx(x+2)}{x(x+2)(x+1)} =$

$= \frac{Ax^2+3Ax+2A+Bx^2+Bx+Cx^2+2Cx}{x(x+2)(x+1)} \Rightarrow$

$\begin{cases} A+B+C=-1 \\ 3A+B+2C=8 \\ 2A=6 \end{cases} \Rightarrow \begin{cases} B+C=-4 \\ B+2C=-1 \\ A=3 \end{cases} \Rightarrow \begin{cases} B=-4-C \\ -4-C+2C=-1 \\ A=3 \end{cases} \Rightarrow$

$\begin{cases} B=-7 \\ C=3 \\ A=3 \end{cases}$

$\ominus 3\int \frac{dx}{x} - 7\int \frac{d(x+2)}{x+2} + 3\int \frac{d(x+1)}{x+1} = 3\ln|x| - 7\ln|x+2| + 3\ln|x+1| + C$

⑤ $\int \frac{x^2-12}{x(x+4)(x-3)}\,dx = \int \frac{A}{x}dx + \int \frac{B}{(x+4)}dx + \int \frac{C}{(x-3)}\,dx \ominus$

$\frac{A}{x} + \frac{B}{x+4} + \frac{C}{x-3} = \frac{A(x+4)(x-3)+Bx(x-3)+Cx(x+4)}{x(x+4)(x-3)} =$

$= \frac{Ax^2+Ax-12A+Bx^2-3Bx+Cx^2+4Cx}{x(x+4)(x-3)} \Rightarrow$

$\begin{cases} A+B+C=1 \\ A-3B+4C=0 \\ -12A=-72 \end{cases} \Rightarrow \begin{cases} B+C=-5 \\ -3B+4C=-6 \\ A=6 \end{cases} \Rightarrow \begin{cases} B=-5-C \\ -3(-5-C)+4C=-6 \\ A=6 \end{cases} \Rightarrow \begin{cases} B=-2 \\ C=-3 \\ A=6 \end{cases}$

$(=)\ 6\int \frac{dx}{x} - 2\int \frac{d(x+4)}{x+4} - 3\int \frac{d(x-3)}{x-3} = 6\ln|x| - 2\ln|x+4| - 3\ln|x-3| + C$

6) $\int \frac{x^3}{x^8+3}\,dx = \frac{1}{4}\int \frac{d(x^4)}{x^8+3} = \frac{1}{4}\int \frac{dt}{t^2+3} = \frac{1}{4}\cdot\frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{t}{\sqrt{3}} + c\ (=)$

$t = x^4$

$(=)\ \frac{\sqrt{3}}{12}\,\mathrm{arctg}\,\frac{\sqrt{3}\,x^4}{3} + c$

7) $\int \frac{x}{x^8-1}\,dx = \frac{1}{2}\int \frac{d(x^2)}{x^8-1} = \frac{1}{2}\int \frac{dt}{t^4-1} = \frac{1}{2}\int \frac{dt}{(t^2-1)(t^2+1)}\ (=)$

$t = x^2$

$(=)\ \frac{1}{2}\left(\int \frac{A}{t^2-1} + \int \frac{B}{t^2+1}\right) = \frac{1}{2}\cdot\frac{1}{2}\int \frac{dt}{t^2-1} - \frac{1}{2}\cdot\frac{1}{2}\int \frac{dt}{t^2+1}\ (=)$

$\frac{A}{t^2-1} + \frac{B}{t^2+1} = \frac{At^2+A+Bt^2-B}{(t^2-1)(t^2+1)}$

$\begin{cases} A + B = 0 \\ A - B = 1 \end{cases} = \begin{cases} 1 + B + B = 0 \\ A = 1 + B \end{cases} \Rightarrow \begin{cases} B = -\frac{1}{2} \\ A = \frac{1}{2} \end{cases}$

$(=)\ \frac{1}{4}\int \frac{dt}{t^2-1} - \frac{1}{4}\int \frac{dt}{t^2+1} = \frac{1}{4}\cdot\frac{1}{2}\cdot\ln\left|\frac{t-1}{t+1}\right| - \frac{1}{4}\,\mathrm{arctg}\,t + c =$

$= \frac{1}{8}\ln\left|\frac{x^2-1}{x^2+1}\right| - \frac{1}{4}\,\mathrm{arctg}\,x^2 + C$

8) $\int \frac{4x-10}{x^2-6x+10}\,dx = \int \frac{4x-10}{(x-3)^2+1}\,d(x-3) = \int \frac{4t+2}{t^2+1}\,dt = 4\int \frac{t\,dt}{t^2+1} + 2\int \frac{dt}{t^2+1}\ (=)$

$t = (x-3)$

$(=)\ 2\int \frac{d(t^2+1)}{t^2+1} + 2\,\mathrm{arctg}\,t = 2\ln|t^2+1| + 2\,\mathrm{arctg}\,t + c =$

$= 2\ln|(x-3)^2+1| + 2\,\mathrm{arctg}\,(x-3) + c$

9) $\int \frac{x^3}{(x-1)^{100}}\,dx = \int \frac{(x-1)^3+3x^2-3x+1}{(x-1)^{100}}\,dx = \int \frac{d(x-1)}{(x-1)^{97}} + \int \frac{3x^2-3x+1}{(x-1)^{100}}\,dx\ (=)$

? $(x-1)^3 = x^3 \underline{- 3x^2 + 3x - 1}$

$(=)\ -\frac{1}{96}\cdot\frac{1}{(x-1)^{96}} + \int \frac{3(x-1)^2 + 3x - 2}{(x-1)^{100}}\,dx = -\frac{1}{96}\cdot\frac{1}{(x-1)^{96}} + 3\int \frac{dx}{(x-1)^{98}} +$

$+ 3\int \frac{x-1}{(x-1)^{100}}\,dx + \int \frac{dx}{(x-1)^{100}} = -\frac{1}{96}\cdot\frac{1}{(x-2)^{96}} - \frac{3}{97}\cdot\frac{1}{(x-1)^{97}} - \frac{3}{98}\cdot\frac{1}{(x-1)^{98}} - \frac{1}{99}\cdot\frac{1}{(x-1)^{99}} + C$

(10) $\displaystyle\int \frac{dx}{(x^2-2x+2)^2} = \int \frac{d(x-1)}{((x-1)^2+1)^2} = \int \frac{dt}{(t^2+1)^2} = \int \frac{du}{(tg\,u^2+1)^2\cos^2 u} \ominus$

$t = (x-1)$
$t = tg\,u$
$u = arctg\,t$
$dt = d\,tg\,u = \frac{du}{\cos^2 u}$

$\ominus \displaystyle\int \frac{du}{\left(\frac{\sin^2 u+\cos^2 u}{\cos^2 u}\right)^2 \cos^2 u} = \int \frac{du}{\left(\frac{1}{\cos^2 u}\right)^2 \cos^2 u} \ominus$

$\ominus \displaystyle\int \cos^2 u\,du = \frac{1}{2}\int (1+\cos(2u))\,du \ominus$

$\ominus \displaystyle\frac{u}{2} + \frac{1}{4}\sin(2u) + C = \frac{arctg(x-1)}{2} + \frac{1}{4}\sin(2\,arctg(x-1)) + C$

(11) $\displaystyle\int \frac{x^3-4x^2+2x+10}{x^2-6x+10}\,dx = \int \frac{(x^3-6x^2+10x)+2x^2-8x+10}{(x-3)^2+1}\,dx =$

$\displaystyle= \int x\,dx + \int \frac{2x^2-12x+20+12x-20-8x+10}{(x-3)^2+1}\,dx =$

$\displaystyle= \frac{x^2}{2} + 2\int dx + \int \frac{4x-10}{x^2-6x+10}\,dx = \frac{x^2}{2} + 2x + \int \frac{4(x-3)+2}{(x-3)^2+1}\,dx =$

$\displaystyle= \frac{x^2}{2} + 2x + 2\int \frac{d((x-3)^2+1)}{(x-3)^2+1} + 2\int \frac{d(x-3)}{(x-3)^2+1} =$

$\displaystyle= \frac{x^2}{2} + 2x + 2\ln|(x-3)^2+1| + 2\,arctg(x-3) + C$

(12) $\displaystyle\int \frac{x^{2n-1}}{x^n+1}\,dx = \int \frac{x^n \cdot x^{n-1}}{x^n+1}\,dx = \frac{1}{n}\int \frac{x^n\,dx^n}{x^n+1} = \frac{1}{n}\int \frac{(t+1-1)\,dt}{t+1} \ominus$

$t = x^n$

$\ominus \displaystyle\frac{1}{n}\int \frac{d(t+1)}{t+1} = \frac{t}{n} - \frac{1}{n}\ln|t+1| + C = \frac{x^n}{n} - \frac{1}{n}\ln|x^n+1| + C$

(13) $\displaystyle\int \frac{1-x^7}{x(1+x^7)}\,dx = \int \frac{A}{x}\,dx + \int \frac{Bx^6+C}{1+x^7}\,dx = \int \frac{dx}{x} + \int \frac{-2x^6}{1+x^7}\,dx \ominus$

$\displaystyle\frac{A+Ax^7+Bx^7+Cx}{x(1+x^7)} = \frac{1-x^7}{x(1+x^7)} \Rightarrow \begin{cases} A=1 \\ C=0 \\ A+B=-1 \\ B=-2 \end{cases}$

$\ominus \displaystyle\ln|x| - 2\int \frac{x^6\,dx}{1+x^7} = \ln|x| - 2\cdot\frac{1}{7}\int \frac{d(x^7+1)}{x^7+1} = \ln|x| - \frac{2}{7}\ln|x^7+1| + C$

*

$\displaystyle\int arctg(x^2)\,dx = x\,arctg\,x^2 - \int \frac{2x^2}{1+x^4}\,dx =$

$\displaystyle A = \int \frac{2x^2}{(1+2x^2+x^4)-(\sqrt{2}x)^2}\,dx = \int \frac{2x^2}{(x^2+1)^2-(\sqrt{2}x)^2}$ nepabaigtas

IV sav

1) $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx \quad (ad-bc \neq 0)$

Tinka keitinys: $\sqrt[n]{\frac{ax+b}{cx+d}} = t$ arba $\sqrt[n]{\frac{ax+b}{cx+d}} = \frac{1}{t}$

2) $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{r_1}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{r_s}\right) dx$

$\sqrt[m]{\frac{ax+b}{cx+d}} = t$, m yra $r_1, \ldots r_s$ kartotinis.

3) $\int R(x, \sqrt{ax^2+bx+c})\,dx$

$\sqrt{ax^2+bx+c} = t \pm \sqrt{a}\,x$, $a>0$

$\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$, $c>0$

$\sqrt{(x-\alpha)(x-\beta)} = t(x-\alpha)$, $\alpha \neq \beta$

4) $\int R(x, \sqrt{a^2-x^2})\,dx$

$x = a \sin t$

5) $\int R(x, \sqrt{a^2+x^2})\,dx$

$x = a\,\mathrm{tg}\,t$

6) $\int R(x, \sqrt{x^2-a^2})\,dx$

$x = \frac{a}{\cos t}$

apskaičiuokite integralus

① $\int \frac{dx}{\sqrt{3x-x^2-2}} = \int \frac{d(x-\frac{3}{2})}{\sqrt{0{,}25-(x-\frac{3}{2})^2}} = \arcsin\left(\frac{(x-\frac{3}{2})}{\frac{1}{2}}\right) + C$

$\cancel{3x-x^2-2>0}$

$\cancel{x^2-3x+2<0}$

② $\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}\,dx = \int \frac{1+t^2}{1-t^2} \cdot t \cdot \frac{-4t}{(1+t^2)^{\cancel{2}}}\,dt = -4\int \frac{t^2}{(1-t^2)(1+t^2)^{\cancel{2}}}\,dt =$

$\sqrt{\frac{1-x}{1+x}} = t$

$(1+x)t^2 = 1-x$

$xt^2 + x = 1-t^2$

$x = \frac{1-t^2}{1+t^2}$

$dx = \frac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2}\,dt = \frac{-4t}{(1+t^2)^2}\,dt$

$\frac{t^2}{(1-t^2)(1+t^2)} = \frac{At+B}{1-t^2} + \frac{Ct+D}{1+t^2} = \frac{At+At^3+B+Bt^2+Ct+D-Ct^3-Dt^2}{(1-t^2)(1+t^2)} \Rightarrow$

$\Rightarrow \begin{cases} A-C=0 \\ B-D=1 \\ A+C=0 \\ B+D=0 \end{cases} \Rightarrow A=0;\ B=\frac{1}{2};\ C=0;\ D=-\frac{1}{2}$

$\circleddash\; -2\int \frac{dt}{1-t^2} + 2\int \frac{dt}{1+t^2} = \ln\left|\frac{t-1}{t+1}\right| + 2\,\mathrm{arctg}\, t + C$, kiu $t=\sqrt{\frac{1-x}{1+x}}$

(3) $\int \frac{\sqrt[6]{x}-1}{x(1+\sqrt[3]{x})}\,dx = \int \frac{t-1}{t^6(1+t^2)}\cdot 6t^5\,dt = 6\int \frac{t^5(t-1)}{t^6(1+t^2)}\,dt \;\circleddash$

$\sqrt[6]{x} = t$

$\sqrt[3]{x} = t^2$

$x = t^6$

$dx = 6t^5\,dt$

da wstawia w liczniku $\frac{1}{2}$

$\circleddash\; 6\int \frac{t-1}{t(1+t^2)}\,dt = -6\int\frac{dt}{t} + 6\int \frac{t+1}{t^2+1}\,dt \;\circleddash$

$\frac{t-1}{t(1+t^2)} = \frac{A}{t} + \frac{Bt+C}{(1+t^2)} = \frac{A + At^2 + Bt^2 + Ct}{t(1+t^2)} \Rightarrow$

$\begin{cases} A + B = 0 \\ A = -1 \\ C = 1 \end{cases} \Rightarrow \begin{cases} B = 1 \\ A = -1 \\ C = 1 \end{cases}$

$\circleddash\; -6\ln(t) + 3\int\frac{d(t^2+1)}{t^2+1} + 6\int\frac{dt}{t^2+1} = -6\ln(t) + 3\ln(t^2+1) + 6\,\mathrm{arctg}(t) + C$,

kiu $t = \sqrt[6]{x}$

(4) $\int \frac{dx}{x+\sqrt{x^2+x+1}} = 2\int \frac{1}{t}\cdot\frac{t^2+t+1}{(1+2t)^2}\,dt = 2\int\frac{dt}{t} - 6\int\frac{t+1}{(1+2t)^2}\,dt \;\circleddash$

$\sqrt{x^2+x+1} = t - x$

$\cancel{x^2} + x + 1 = t^2 - 2tx + \cancel{x^2}$

$x + 2tx = t^2 - 1$

$x(1+2t) = t^2 - 1$

$x = \frac{t^2-1}{1+2t}$

$dx = \frac{2t\cdot(1+2t) - (t^2-1)\cdot 2}{(1+2t)^2} = \frac{2t + 4t^2 - 2t^2 + 2}{(1+2t)^2} = \cancel{\quad} =$

$= \frac{2t^2 + 2t + 2}{(1+2t)^2}\,dt$

$\frac{t^2+t+1}{t(1+2t)^2} = \frac{A}{t} + \frac{Bt+C}{(1+2t)^2} = \frac{A + 4At + 4At^2 + Bt^2 + Ct}{t(1+2t)^2} \Rightarrow$

$\begin{cases} 4A + B = 1 \\ 4A + C = 1 \\ A = 1 \end{cases} \Rightarrow \begin{cases} B = -3 \\ C = -3 \\ A = 1 \end{cases}$

$$\stackrel{\circ}{=} 2\ln(t) - 3\int\frac{(2t+1)+1}{(1+2t)^2}\,dt = 2\ln(t) - 3\int\frac{dt}{1+2t} - 3\int\frac{dt}{1+2t} =$$

$$= 2\ln(t) - \frac{3}{2}\ln(1+2t) + \frac{3}{2}\int\frac{d(2t+1)}{(1+2t)^2} = 2\ln(t) - \frac{3}{2}\ln(2t+1) + \frac{3}{2}\cdot\frac{1}{2t+1} + C_1$$

kur $t = \ldots$

⑤ $$\int\frac{dx}{x^2(x+\sqrt{1+x^2})} = \int\frac{1}{\cos^2 t}\cdot\frac{dt}{\frac{\sin^2 t}{\cos^2 t}\left(\frac{\sin t}{\cos t}+\sqrt{1+\frac{\sin^2 t}{\cos^2 t}}\right)} = \int\frac{1}{\cos^2 t}\cdot\frac{dt}{\frac{\sin^2 t}{\cos^2 t}\left(\frac{\sin t+1}{\cos^2 t}\right)} \stackrel{\circ}{=}$$

$x = \operatorname{tg} t$

$dx = \frac{dt}{\cos^2 t}$

$u = \sin t$

$$\stackrel{\circ}{=} \int\frac{\cos t\,dt}{\sin^2 t(1+\sin t)} = \int\frac{d(\sin t)}{\sin^2 t(1+\sin t)} = \int\frac{du}{u^2(1+u)} = -\int\frac{du}{u} + \int\frac{du}{u^2} + \int\frac{du}{1+u} \stackrel{\circ}{=}$$

$$\frac{1}{u^2(1+u)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{1+u} = Au + Au^2 + B + Bu + Cu^2$$

$$\begin{cases} A+C=0 \\ A+B=0 \\ B=1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=1 \\ C=1 \end{cases}$$

$$\stackrel{\circ}{=} -\ln|u| - \frac{1}{u} + \ln|1+u| + C, \text{ kai } u = \sin(\operatorname{arctg} x)$$

1) $\int R(\sin x, \cos x)\,dx$

Visada tinka keitinys

$\operatorname{tg}\frac{x}{2} = t$. Tada $x = 2\operatorname{arctg} t$; $dx = \frac{2dt}{1+t^2}$; $\sin x = \frac{2t}{1+t^2}$; $\cos x = \frac{1-t^2}{1+t^2}$

2) $R(\sin x, \cos x) = R(-\sin x, -\cos x)$

$\operatorname{tg} x = t$; $\sin x = \frac{t}{\sqrt{1+t^2}}$; $\cos x = \frac{1}{\sqrt{1+t^2}}$

3) $R(-\sin x, \cos x) = -R(\sin x, \cos x)$

$\cos x = t$

4) $R(\sin x, -\cos x) = -R(\sin x, \cos x)$

$\sin x = t$

atskiri atvejai