(1) $\int \cos^7 x \, dx = \int \cos^6 x \, d\sin x = \int (1-\sin^2 x)^3 \, d\sin x = \int (1-t^2)^3 dt \;\circleddash$

$\sin x = t$

$(1-t^2)^3 = -t^6 + 3t^4 - 3t^2 + 1$

$\circleddash -\int t^6 dt + 3\int t^4 dt - 3\int t^2 dt + \int dt = -\frac{t^7}{7} + 3\frac{t^5}{5} - t^3 + t + C$

(2) $\int \frac{dx}{4-5\sin x} = \int \frac{2}{1+t^2} \cdot \frac{1}{4-5\left(\frac{2t}{1+t^2}\right)} dt = \int \frac{\cancel{2}}{\cancel{1+t^2}} \cdot \frac{\cancel{1+t^2}}{\cancel{2}(2t^2-5t+2)} dt \;\circleddash$

$tg\frac{x}{2} = t, \quad x = 2\,arctg\, t$

$dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}$

$\circleddash \int \frac{1}{(t-2)(2t-1)} dt = \int \frac{A}{t-2} dt + \int \frac{B}{2t-1} dt \;\circleddash$

$\frac{A}{t-2} + \frac{B}{2t-1} = \frac{2At - A + Bt - 2B}{(t-2)(2t-1)}$

$\begin{cases} 2A + B = 0 \\ -A - 2B = 1 \end{cases} \Rightarrow \begin{cases} B = -\frac{2}{3} \\ A = \frac{1}{3} \end{cases}$

$\circleddash \frac{1}{3}\int \frac{d(t-2)}{t-2} - \frac{2}{3}\cdot\frac{1}{2}\int \frac{d(2t-1)}{2t-1} = \frac{1}{3}\ln|t-2| - \frac{1}{3}\ln|2t-1| + C$

(3) $\int \frac{dx}{4\sin x + 3\cos x + 5} = \int \frac{2dt}{(1+t^2)\cdot\left(4\cdot\frac{2t}{1+t^2} + 3\cdot\frac{1-t^2}{1+t^2} + 5\frac{(1+t^2)}{1+t^2}\right)} \;\circleddash$

$tg\frac{x}{2} = t, \quad x = 2\,arctg\, t$

$dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$

$\circleddash \int \frac{\cancel{2}dt}{\cancel{2}(t^2+4t+4)} = \int \frac{d(t+2)}{(t+2)^2} = -\frac{1}{t+2} + C, \quad$ unde $t = tg\frac{x}{2}$

(4) $I_n = \int \sin^n(x)\,dx = -\int \sin^{n-1}(x)\,d\cos x = -\sin^{n-1}x\cos x + \int \cos x \, d\sin^{n-1}x =$

$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x \underbrace{\cos^2 x}_{1-\sin^2 x} dx = -\sin^{n-1}x\cos x +$

$+ (n-1)\underbrace{\int \sin^{n-2}x\,dx}_{I_{n-2}} - (n-1)\underbrace{\int \sin^n x\,dx}_{I_n} = -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$

$I_n = \frac{1}{n}\left(-\sin^{n-1}x\cos x + (n-1)I_{n-2}\right)$

$I_1 = \int \sin x\,dx = -\cos x + C$

$I_2 = \int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos(2x))\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

$I_4 = \frac{1}{4}\left(-\sin^3 x\cos x + 3\left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right)\right) + C$

V sav.

(1) $\int_0^{\pi} x\sin(x)\,dx = x\cdot(-\cos x) - \int_0^{\pi} -\cos(x)\,dx = -x\cos x + \int_0^{\pi}\cos x\,dx = -x\cos x + \sin x \Big|_0^{\pi} = -\pi\cos\pi + \sin\pi - (0\cdot\cos 0 + \sin 0) =$
$= -\pi\cdot(-1) + 0 = \pi$

(2) $\int_0^1 \arccos(x)\,dx = \int_0^1 \arccos(x)\cdot 1\,dx = \arccos x\cdot x - \int_0^1 x\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right)dx =$
$= \arccos x\cdot x + \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \arccos x\cdot x + \int -1\,dt \ominus$
$t = \sqrt{1-x^2}$
$\ominus \arccos x\cdot x - 1t = \arccos x\cdot x - 1\cdot\sqrt{1-x^2}\Big|_0^1 =$
$= \arccos 1\cdot 1 - \sqrt{1-1^2} - (\arccos 0\cdot 0 - \sqrt{1-0^2}) =$
$= 0\cdot 1 - 0 - \left(\frac{\pi}{2}\cdot 0 - 1\right) = 0 - 0 + 1 = 1$

(3) $\int_0^1 x(2-x^2)^{12}\,dx = \int_0^1 -\frac{t^{12}}{2}\,dt = -\frac{1}{2}\cdot\int t^{12}\,dt = -\frac{1}{2}\cdot\frac{t^{13}}{13} \ominus$
$t = 2-x^2$
$\ominus -\frac{1}{2}\cdot\frac{(2-x)^{13}}{13} = -\frac{(2-x)^{13}}{26} = -\frac{(2-1^2)^{13}}{26} + \frac{(2-0^2)^{13}}{26} =$
$= -\frac{1^{13}}{26} + \frac{2^{13}}{26} = \frac{2^{13}-1}{26}$

(4) $\int_{-1}^{2} x|x|\,dx =$

*** Apibrėžtinis integralas:**

V sav.

(1) $\int_0^{\pi} x\cdot\sin(x)\,dx = -\int_0^{\pi} x\,d(\cos x) = -\left(x\cdot\cos x\Big|_0^{\pi} - \int_0^{\pi}\cos x\,dx\right) =$
$= -x\cdot\cos x\Big|_0^{\pi} + \sin x\Big|_0^{\pi} = -\pi\cdot\cos\pi + 0 + \sin\pi - \sin 0 = \pi$

(2) $\int_0^1 \arccos x\,dx = x\cdot\arccos x\Big|_0^1 + \int_0^1 \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int_0^1 \frac{d(1-x^2)}{\sqrt{1-x^2}} =$
$= -\frac{1}{2}\int_1^0 \frac{dt}{\sqrt{t}} \quad (t = 1-x^2) = \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{t}} = \sqrt{t}\Big|_0^1 = 1$

dauginame iš − kad apsivertų rėžiai

(3) $\int_0^1 x(2-x^2)^{12}\,dx = -\frac{1}{2}\int_0^1 (2-x^2)^{12}\,d(2-x^2) \quad (2-x^2 = t;\ t_0 = 2-1 = 1,\ t_1 = 2-0 = 2)$
$= -\frac{1}{2}\int_2^1 t^{12}\,dt = -\frac{1}{2}\cdot\frac{t^{13}}{13}\Big|_2^1 = -\frac{1}{2}\cdot\frac{1}{13} + \frac{1}{2}\cdot\frac{2^{13}}{13} = \frac{2^{13}-1}{26}$

(4) $\int_{-1}^{2} x\cdot|x|\,dx = -\int_{-1}^{0} x^2\,dx + \int_0^2 x^2\,dx = -\frac{x^3}{3}\Big|_{-1}^{0} + \frac{x^3}{3}\Big|_0^2 =$
$= -\frac{1}{3} + \frac{8}{3} = \frac{7}{3}$

(5) $\int_{\frac{1}{e}}^{e} |\ln(x)|\,dx = -\int_{\frac{1}{e}}^{1}\ln x\,dx + \int_1^e \ln x\,dx = -(x\cdot\ln x - x)\Big|_{\frac{1}{e}}^{1} +$ $\qquad \ln\frac{1}{e} = -\ln e = -1$
$+ (x\cdot\ln x - x)\Big|_1^e = 1 - \frac{1}{e} - \frac{1}{e} + e - e + 1 = 2 - \frac{2}{e}$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

⑥ $\int_0^{100\pi} \sqrt{1-\cos(2x)}\,dx = \sqrt{2}\int_0^{100\pi} |\sin x|\,dx = 50\sqrt{2}\int_0^{2\pi} |\sin x|\,dx =$

$= 50\sqrt{2}\int_0^{\pi} \sin x\,dx - 50\sqrt{2}\int_{\pi}^{2\pi} \sin x\,dx = -50\sqrt{2}\cos x\Big|_0^{\pi} +$

$+ 50\sqrt{2}\cdot\cos x\Big|_{\pi}^{2\pi} = 50\sqrt{2} + 50\sqrt{2} + 50\sqrt{2} + 50\sqrt{2} = 200\sqrt{2}$

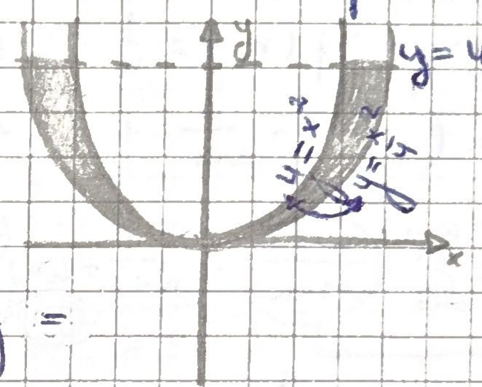
Apskaičiuokite figūros, apribotos kreivėmis, plotą:

⑦ $y = x^2;\ 4y = x^2;\ y = 4$

$y = \frac{x^2}{4}$

$x = \pm\sqrt{y}$

$x = \pm 2\sqrt{y}$



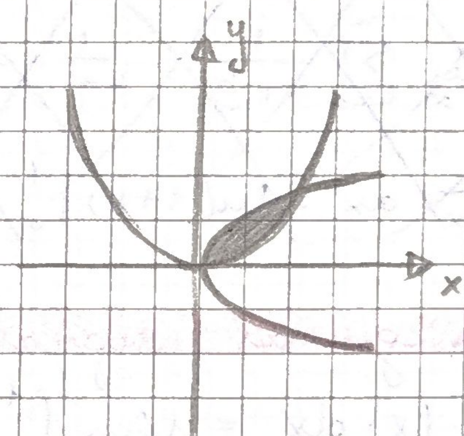
$I = \int_0^4 (2\sqrt{y} - \sqrt{y})\,dy = \int_0^4 \sqrt{y}\,dy =$

$= \frac{2}{3}\cdot y^{\frac{3}{2}}\Big|_0^4 = \frac{2}{3}\cdot 4^{\frac{2}{3}} = \frac{16}{3}$

~~arba $I = \int_0^4 \left(4 - \frac{x^2}{4}\right)dx - \int_0^2 (4 - x^2)\,dx$~~

$S = 2I = \frac{32}{3}$

⑧ $ax = y^2;\ ay = x^2;\ (a>0)$



$y = \pm\sqrt{ax}$

$y = \frac{x^2}{a}$

$\sqrt{ax} = \frac{x^2}{a}$

$ax = \frac{x^4}{a^2}$

$a^3 x = x^4$

$x = 0$ arba $x^3 = a^3$

$x = a$

$\int_0^a \left(\sqrt{ax} - \frac{x^2}{a}\right)dx = \frac{1}{a}\int_0^a (\sqrt{ax})\,d(ax) - \frac{1}{a}\int x^2\,dx =$

$= \frac{2}{3}(ax)^{\frac{3}{2}}\Big|_0^a - \frac{1}{3a}x^3\Big|_0^a = \frac{2}{3a}\cdot a^3 - \frac{1}{3a}\cdot a^3 = \frac{a^2}{3}$

*Rymano integralas

*Rymano integralas

$$\int_a^b f(x)\,dx = \lim_{\Delta x_k \to 0} \sum_{k=0}^{n-1} f(\xi_k)\,\Delta x_k \; ; \quad \xi_k \in [x_k, x_{k+1}]$$

① $$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\dots+\frac{n-1}{n^2}\right) = \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{k}{n^2} =$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n-1}\underbrace{\frac{1}{n}}_{\Delta x_k}\cdot\underbrace{\frac{k}{n}}_{\xi_k = x_k} = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$f(x)$ $\quad f\left(\frac{k}{n}\right) = \frac{k}{n} \Rightarrow f(x) = x$

② $$\lim_{n\to\infty}\left(\frac{1}{1+n}+\frac{1}{2+n}+\dots+\frac{1}{n+n}\right) = \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{n+1+k} \;\stackrel{\text{=}}{}$$

$\xi_k = x_k = \frac{k}{n+1}$

$\Delta x_k = \frac{1}{1+n}$

$$= \lim_{n\to\infty}\left(\sum_{k=0}^{n-1}\frac{1}{1+\frac{k}{n+1}}\cdot\frac{1}{1+n} - \frac{1}{1+\frac{n}{n+1}}\cdot\frac{1}{n+1}\right) =$$

$$= \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{1}{1+\frac{k}{n+1}}\cdot\frac{1}{1+n} - \frac{1}{2n+1}\right) = 0$$

$$\int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big|_0^1 = \ln 2$$

naudodamiesi apibrėžtiniu integralu

$$\lim_{a\downarrow 0} a\ln a = \lim_{a\downarrow 0}\frac{\ln a}{\frac{1}{a}} = \lim_{a\downarrow 0}\left(-\frac{\frac{1}{a}}{\frac{1}{a^2}}\right) = \lim_{a\downarrow 0}(-a) = 0$$

② $$\int_{-\infty}^{+\infty}\frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^0\frac{dx}{1+x^2} + \lim\int_a^b\frac{dx}{1+x^2} = \operatorname{arctg} x\Big|_{-\infty}^{+\infty} =$$

$$= \lim_{a\to\infty}\operatorname{arctg} a - \lim_{b\to-\infty}\operatorname{arctg} b = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

③ $$\int_0^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_0^\infty = \underbrace{\lim_{a\to\infty}\left(-\frac{1}{a}\right)}_{0} - \underbrace{\lim_{b\to 0}\left(-\frac{1}{b}\right)}_{\infty} = \infty \quad (\text{div.})$$

④ $$\int_0^1\frac{dx}{(2-x)\sqrt{1-x}} = -2\int_0^1\frac{d\sqrt{1-x}}{1+(\sqrt{1-x})^2} = -2\operatorname{arctg}\sqrt{1-x}\Big|_0^1 =$$

?

$$= \lim_{a\to 1} -2\operatorname{arctg}\sqrt{1-x} + 2\operatorname{arctg} 1 = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$$

arba

$t = \sqrt{1-x}$

(5) $$\int_{-1}^{2} \frac{dx}{\sqrt[3]{(x-1)^2}} = \int_{-1}^{1} \frac{dx}{\sqrt[3]{(x-1)^2}} + \int_{1}^{2} \frac{dx}{\sqrt[3]{(x-1)^2}} = 3(x-1)^{\frac{1}{3}}\Big|_{-1}^{1} + 3(x-1)^{\frac{1}{3}}\Big|_{1}^{2} =$$

$$\int \frac{dx}{\sqrt[3]{(x-1)^2}} = \int (x-1)^{-\frac{2}{3}}\, d(x-1) = 3(x-1)^{\frac{1}{3}} + C$$

$$= \lim_{a \uparrow 1} 3(a-1)^{\frac{1}{3}} - 3\cdot(-2)^{\frac{1}{3}} + 3\cdot 1 - \lim_{b \downarrow 1} 3(b-1)^{\frac{1}{3}} = 3\sqrt[3]{2} + 3$$

~~(6) $y = e^{a^3 x}$; $y = 0$, $a^2 + x^2 = 0$~~

Ištirkite integralų konvergavimą:

(11) $$\int_{2}^{\infty} \frac{x^2}{x^4 - x^2 + 1}\, dx \qquad \int_a^b f \le \int_a^b g$$

Imkime $f(x) = \frac{x^2}{x^4 - x^2 + 1}$ ir $g(x) = \frac{1}{x^2}$. Tada

$$\mu = \lim_{x \to \infty} \frac{x^2}{x^4 - x^2 + 1} \cdot x^2 = \lim_{x \to \infty} \frac{1}{1 - \frac{1}{x^2} + \frac{1}{x^4}} = 1$$

ir pagal 6.4 teiginį integralas konverguoja, nes $\int_2^{\infty} \frac{dx}{x^2}$ konverguoja.

(12) $$\int_0^{\infty} x^{p-1} e^{-x}\, dx$$

Nagrinėkime $A = \int_0^1 x^{p-1} e^{-x}\, dx$ ir $B = \int_1^{\infty} x^{p-1} e^{-x}\, dx$.

Integralas konverguos tada ir tik tada, jei konverguos A ir B.

A: $f(x) = x^{p-1} e^{-x}$, $g(x) = x^{p-1}$

$$\mu = \lim_{x \downarrow 0} \frac{x^{p-1} e^{-x}}{x^{p-1}} = \lim_{x \downarrow 0} e^{-x} = 1 \Rightarrow \text{konv. kai } -p+1 < 1,\ p > 0$$

B: $h(x) = \frac{1}{x^2}$

$$\mu = \lim_{x \to \infty} \frac{f(x)}{h(x)} = \lim_{x \to \infty} x^{p+1} e^{-x} = 0.\ \text{Todėl}$$

$$\int_1^{\infty} h(x) < +\infty \Rightarrow \int_1^{\infty} f(x) < +\infty$$

konverguoja, kai $p > 0$.

(13) $\int_0^{\infty} \frac{x^m}{1+x^n}\,dx \quad , n \geq 0$

Nagrinėkime du integralus $A = \int_0^1 \frac{x^m}{1+x^n}$ ir $B = \int_1^{\infty} \frac{x^m}{1+x^n}$.

A : $f(x) = \frac{x^m}{1+x^n}$; $g(x) = x^m$

$$\mu = \lim_{x \downarrow 0} \frac{f(x)}{g(x)} = \lim_{x \downarrow 0} \frac{1}{1+x^n} = 1$$

konverguoja, kai $m > -1$
diverguoja, kai $m \leq -1$

B : $h(x) = x^{m-n}$

$$\mu = \lim_{x \to \infty} \frac{f(x)}{h(x)} = \lim_{x \to \infty} \frac{\cancel{x^m}}{1+x^n} \cdot \frac{x^n}{\cancel{x^m}} = \lim_{x \to \infty} \frac{1}{\frac{1}{x^n}+1} = 1$$

konverguoja, kai $m - n < -1$
diverguoja kitus

konverguoja, kai $m > -1$ ir $n - m > 1$

(14) $\int_0^{\infty} \frac{\operatorname{arctg}(x)}{x^n}\,dx = \int_0^1 \frac{\operatorname{arctg}(x)}{x^n}\,dx + \int_1^{\infty} \frac{\operatorname{arctg}(x)}{x^n}\,dx =: A + B$

A : $f(x) = \frac{\operatorname{arctg}(x)}{x^n}$; $g(x) = \frac{1}{x^{n-1}}$

L'Hospital

$$\mu = \lim_{x \downarrow 0} \frac{\operatorname{arctg}(x)}{x^n} \cdot x^{n-1} = \lim_{x \downarrow 0} \frac{\operatorname{arctg}(x)}{x} = \lim_{x \downarrow 0} \frac{1}{1+x^2} = 1$$

$\int_0^1 \frac{dx}{x^{n-1}} < +\infty \iff n - 1 < 1$
$n < 2$

B : $h(x) = \frac{1}{x^n}$

$$\lim_{x \to \infty} \frac{f(x)}{h(x)} = \lim_{x \to \infty} \operatorname{arctg}(x) = \frac{\pi}{2}$$

$\int_1^{\infty} h(x)\,dx < +\infty \iff n > 1$

konverguoja, kai $1 < n < 2$
diverguoja kitus.

(15) $\int_1^{10} \frac{x^5}{(x^6+4)(x-1)}$

$g(x) = \frac{1}{x-1}$

$$\mu = \lim_{x \downarrow 1} \frac{x^5}{(x^6+4)(x-1)} \cdot (x-1) = \frac{1}{5} \in (0; +\infty)$$

$\int_1^{10} g(x)\,dx = \int_1^{10} \frac{dx}{x-1} = \ln|x-1| \Big|_1^{10}$ diverguoja $\Rightarrow$

$\Rightarrow \int_1^{10} f(x)\,dx$ diverguoja

$\int_{10}^{\infty} \frac{x^5}{(x^6+4)(x-1)}\,dx$?

$h(x) = \frac{1}{x^2}$

$\mu = \lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \frac{x^5}{(x^6+4)(x-1)} \cdot x^2 = \lim_{x\to\infty} \frac{1}{(1+\frac{4}{x^6})(1-\frac{1}{x})} = 1$

$\int_{10}^{\infty} \frac{dx}{x^2} < +\infty \Rightarrow \int_{10}^{\infty} f(x)\,dx$ konverguoja

(16) $\int_0^{\infty} \frac{\ln(1+x)}{x^n}\,dx = \int_0^1 \frac{\ln(1+x)}{x^n}\,dx + \int_1^{\infty} \frac{\ln(1+x)}{x^n}\,dx =: A + B$

$A:\ g(x) = \frac{1}{x^{n-1}}$

$\mu = \lim_{x\downarrow 0} \frac{f(x)}{g(x)} = \lim_{x\downarrow 0} \frac{\ln(1+x)}{x} \overset{L'H.t.}{=} \lim_{x\downarrow 0} \frac{1}{1+x} = 1$

$\int_0^1 \frac{dx}{x^{n-1}} < +\infty \iff n-1 < 1$, $n < 2$

Todėl ir $\int_0^1 f(x)\,dx$ konverguoja, kai $n < 2$.

$B:\ h(x) = \frac{\ln x}{x^n}$

$\mu = \lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \frac{\ln(1+x)}{\ln x} \overset{L'H.t.}{=} \lim_{x\to\infty} \frac{x}{1+x} = 1$

$\int_1^{\infty} \frac{\ln x}{x}\,dx = -\frac{1}{n-1}\int_1^{\infty} \ln x\, d\frac{1}{x^{n-1}} = -\frac{1}{n-1}\Big[\ln x \cdot \frac{1}{x^{n-1}}\Big|_1^{\infty} -$

$- \int_1^{\infty} \frac{dx}{x^n}\Big] < +\infty \iff n > 1$

VII sav. Ar konverguoja funkcijų sekos nurodytuose taškuose:

(1) $f_n(x) = nx^n$; $x = \frac{1}{2}$

$\lim_{n\to\infty} \frac{n}{2^n} = L'H.t. = \lim_{n\to\infty} \frac{1}{2^n \ln 2} = 0 < \infty$

konverguoja į 0.

(2) $f_n(x) = \frac{x^n}{n}$; $x = 2$

$\lim_{n\to\infty} \frac{2^n}{n} = L'H.t. = \infty$

diverguoja

Rasti funkcijų sekų ribas:

(3) $\lim_{n\to\infty} n \sin\left(\frac{x}{n}\right)$

$\lim_{n\to\infty} \frac{\sin(xn)}{n} \le \lim_{n\to\infty} \frac{1}{n} = 0$. Kita vertus:

$\lim_{n\to\infty} \frac{\sin(xn)}{n} \ge \lim_{n\to\infty} -\frac{1}{n} = 0$. Todėl $\lim_{n\to\infty} \frac{\sin(xn)}{n} = 0$

④ $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{1}{\left(\frac{n}{x}\right)}\right)^{\frac{n}{x}}\right)^x = e^x$

⑤ $\lim_{n\to\infty} n\left(x^{\frac{1}{n}}-1\right) = \ldots ? = \ln(x)$?

⑥ $\lim_{n\to\infty}\sqrt{x^2+\frac{1}{n}} = \sqrt{x^2} = |x|$

· $\lim_{n\to\infty}\cos\left(\frac{x}{n}\right) = \cos(0) = 1$

Raskite funkcijų eilučių konvergavimo sritis:

⑦ $\sum_{n=1}^{\infty} x^n$ – žinome kad tai begalinė geometrinė progresija, todėl konverguoja kai $x \in (-1;1)$.

Jei $x \geq 1$ arba $x < -1$ $\sum_{n=1}^{\infty} x^n = \infty$; $x = -1$ $\sum_{n=1}^{\infty} x^n$ – – diverguoja.

⑧ $\sum_{n=1}^{\infty}\frac{1}{x^2+n^2}$

fiksuojame $x_0 \in \mathbb{R}$ ir skaičiuojame kaip paprastą eilutę:

$\sum_{n=1}^{\infty}\frac{1}{x^2+n^2} \sim \frac{1}{n^2} \Rightarrow$ konverguoja $\forall x \in \mathbb{R}$

⑨ $\sum_{n=1}^{\infty}\sin^n\left(\frac{x}{2}\right)$

$|a_n| = \sin^n\left(\frac{x}{2}\right)$; $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\sin\left(\frac{x}{2}\right)\right| < 1$, kai

$\frac{x}{2} \neq \frac{\pi}{2}(2k+1)$, $k \in \mathbb{Z}$. Pagal Koši požymį konverguoja,

kai $x \neq \pi(2k+1)$, $k \in \mathbb{Z}$.

Jei $x = \pi$, tai $\sum_{n=1}^{\infty}\sin^n\left(\frac{x}{2}\right) = \sum_{n=1}^{\infty} 1^n = \infty$ arba

$= \sum_{n=1}^{\infty}(-1)^n$ – diverguoja

⑩ $\sum_{n=1}^{\infty}\frac{1}{n(x+2)^n}$

Pagal Dalambero požymį:

$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{n(x+2)^n}{(n+1)(x+2)^n(x+2)} = \lim_{n\to\infty}\left|\frac{1}{x+2+\frac{x}{n}+\frac{2}{n}}\right| =$

$= \left|\frac{1}{x+2}\right| < 1$

kai $|x+2| > 1 \Rightarrow x > -1$ arba $x < -3$

kai $x = -1$: $\sum_{n=1}^{\infty}\frac{1}{n(x+2)^n} = \sum_{n=1}^{\infty}\frac{1}{n} = \infty$

kai $x = -3$: $\sum_{n=1}^{\infty}\frac{1}{n(x+2)^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} < \infty$ (Abelio – Dirichlė / Leibnico)

$E = (-\infty; -3] \cup (-1; +\infty)$

(11) $\sum_{n=1}^{\infty} 10^{2n}(2x-3)^{2n-1}$

Pagal Dalambero arba Koši požymį:

$100(2x-3)^2 < 1 \iff \begin{cases} 2x-3 < \frac{1}{10} \\ 2x-3 > -\frac{1}{10} \end{cases} \quad \begin{cases} x < \frac{31}{20} \\ x > \frac{29}{20} \end{cases}$

Ribiniai netinka.

$E = \left(\frac{29}{20} ; \frac{31}{20}\right)$

(12) $\sum_{n=1}^{\infty} (x^n - x^{n+1})$

Teleskopinė pagal apibrėžimą konverguoja, kai $\lim_{n\to\infty}(x-x^n)$ konverguoja $E = (-1; 1]$

(13) $\sum_{n=1}^{\infty} ne^{-nx}$

Pagal Koši požymį:

$e^{-x} < 1 \iff x > 0$

$E = (0 ; \infty)$

(14) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{x-1}{x+1}\right)^n$

Pastebime, kad $\left|\frac{x-1}{x+1}\right| < 1$, kai $x > 0$,

$\left|\frac{x-1}{x+1}\right| > 1$, kai $x < 0$,

$\left|\frac{x-1}{x+1}\right| = 0$, kai $x = 1$,

$\left|\frac{x-1}{x+1}\right| = 1$, kai $x = 0$

Jei $x < 0$ netenkinama būtinoji konvergavimo sąlyga:
pažymim $p := \frac{x-1}{x+1}$, $|p| > 1$
$\lim_{n\to\infty} \frac{(-1)^n}{n} p^n \neq 0 \implies$ diverguoja, kai $x < 0$

Jei $x = 0$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{x-1}{x+1}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot (-1)^n =$
$= \sum_{n=1}^{\infty} \frac{1}{n}((-1)^n)^2 = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \implies$ div., kai $x = 0$

Jei $0 < x < 1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{x-1}{x+1}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (-1)^n}{n} \cdot \left(\frac{1-x}{1+x}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1-x}{1+x}\right)^n$ –
– begalinė geometrinė progresija, kuri konv., nes $0 < \left(\frac{1-x}{1+x}\right) < 1$, o
$\frac{1}{n} \cdot \left(\frac{1-x}{1+x}\right)^n < \left(\frac{1-x}{1+x}\right)^n$, todėl pagal palyginimo požymį $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1-x}{1+x}\right)^n$
konverguoja.

Jei $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{x-1}{x+1}\right)^n = \sum_{n=1}^{\infty} 0 = 0 \Rightarrow$ konverguoja

Jei $x > 1$. Pagal Abelio – Dirichlė (A–D) požymį:
$\left|\sum_{n=1}^{N} \frac{(-1)^n}{n}\right| < 1 \ \forall N \in \mathbb{N}$ ir $\left(\frac{x-1}{x+1}\right)^n$ monotoniškai artėja į 0.
(Parodyti!). Todėl $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{x-1}{x+1}\right)^n$ konverguoja.

konverguoja, kai $x \in E = (0; \infty)$

(15) $\sum_{n=1}^{\infty} \frac{1}{2n+1}\left(\frac{2x-1}{2x+1}\right)^n$

labai panašus į (14), $(-1)^n$ galima iškelti iš $\left(\frac{2x-1}{2x+1}\right)^n =$
$= (-1)^n\left(\frac{1-2x}{1+2x}\right)^n$, o $\sum_{n=1}^{\infty} \frac{1}{n}$ ir $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ abi elgiasi vienodai (remiantis palyginimo požymiu).

Ištirkite funkcijų sekų tolygų konvergavimą nurodytuose intervaluose:

(16) $f_n(x) = \frac{\sin(nx)}{n}$; $E = \mathbb{R}$

F-ja $f_n(x)$ pataškiui konverguoja į f-ją $f_0(x) = 0$.

t.y. $f_n(x) \to f_0(x) = 0$, $x \in \mathbb{R}$

$\sup_{x\in\mathbb{R}} |f_n(x) - f_0(x)| = \sup_{x\in\mathbb{R}} \left|\frac{\sin(nx)}{n} - 0\right| = \sup_{x\in\mathbb{R}} \left|\frac{\sin(nx)}{n}\right| \ominus$

f-ja $\frac{\sin(nx)}{n}$ pasiekia max ar min reikšmę perlinkio taške, todėl reikia rasti išvestinę ir ją prilyginti 0: $\left(\frac{\sin(nx)}{n}\right)' =$
$= \frac{1}{n} \cdot n \cdot \cos(nx) = \cos(nx) = 0 \Leftrightarrow nx = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{2n}$

$\ominus \sup_{x\in\mathbb{R}} \left|\frac{\sin(nx)}{n}\right| = \left|\frac{\sin(\frac{n\pi}{2n})}{n}\right| = \frac{\sin\frac{\pi}{2}}{n} = \frac{1}{n} = 0$, $n \to \infty$

T.y. $\sup_{x\in\mathbb{R}} |f_n(x) - f_0(x)| = 0$, $n \to \infty \Rightarrow f_n \overset{\mathbb{R}}{\rightrightarrows} f_0$

konverguoja tolygiai į $f_0(x) = 0$.

(17) $f_n(x) = x^n$, $E = [0; 1]$

$f_n(x) \to f(x) = \begin{cases} 0; & 0 \le x < 1 \\ 1; & x = 1 \end{cases}$

$\sup_{x\in[0;1]} |f_n(x) - f(x)| = \sup_{x\in[0;1]} \begin{cases} x^n; & 0 \le x < 1 \\ 0; & x = 1 \end{cases} = \sup_{x\in[0;1]} \begin{cases} 1^n \\ 0 \end{cases} = 1 \ne 0$

$f_n(x) = x^n$ yra didėjanti f-ja, $E = [0; 1]$, todėl sup bus didėjant x. Kadangi sup, ne max, statomės $x = 1$.

Todėl $f_n(x) \not\rightrightarrows f(x)$.