Slice Sampling: An Introduction[1]

Arkajyoti Bhattacharjee* Max Cartwright* Rezoanoor Rahman*

*Department of Statistics
The Ohio State University

April 19, 2024

THE OHIO STATE UNIVERSITY
COLLEGE OF ARTS AND SCIENCES

[1] Main Reference: Neal (2003)

Contents

1. Introduction
2. The idea of slice sampling
3. Single-variable slice sampling
4. Stepping-out Procedure
5. Doubling Procedure
6. Other Variations
7. Merits and Demerits of Slice Sampling
8. Simulation Examples

Motivation

- Sampling process *known*: use **Gibbs** sampling (Gelfand (2000))
- Sampling process *unknown*, envelope distribution *known*: Use **Rejection** Sampling (Forsythe (1972)).
- Sampling process *unknown*, envelop distribution *unknown*: Use **Adaptive Rejection** Sampling (Gilks and Wild (1992)), provided the density is $\log$-concave.
- What if the density is **not** $\log$-concave?

The idea of slice sampling

- Sample from $x \in \mathbb{R}^n$, with $p(x) \propto f(x)$.
- Introduce **auxiliary** variable, y, and define

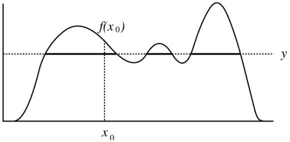
$$(x, y) \sim Uniform(U),$$

where $U = \{(x, y) : 0 < y < f(x)\}$.

- **Joint density:**

$$p(x, y) = \begin{cases} 1/Z, & \text{if } 0 < y < f(x) \\ 0, & \text{otherwise} \end{cases}$$

where $Z = \int f(x)dx$.



- **Marginal density:**

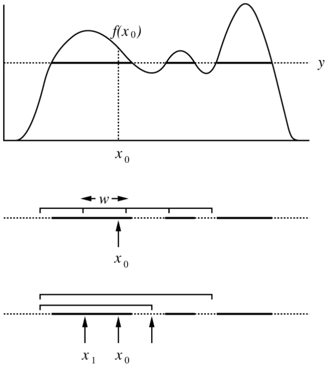
$$p(x) = \int_0^{f(x)} (1/Z)dy = f(x)/Z$$

- Sample jointly for (x, y), and then ignore y.

[1] Figure 4 adapted from Neal (2003).

Single-variable slice sampling

1. Draw
$$y \sim U(0, f(x_0)).$$
   Define **horizontal slice**:
$$S = \{x : y < f(x)\}.$$
   Note: $x_0 \in S$.
2. Find an interval, $I = (L, R)$, around x_0 containing all/or much, of the slice.
3. Draw the new point, x_1, from the part of the slice within this interval.



The Stepping-out Procedure

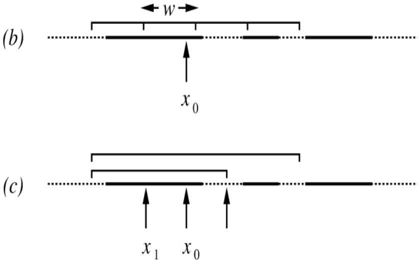


Figure: (b) An interval of width w is randomly positioned around x_0, and then expanded in steps of size w until both ends are outside the slice. (c) A new point, x_1, is found by picking uniformly from the interval until a point inside the slice is found. Points picked that are outside the slice are used to shrink the interval.[2]

[2] Figure 1 adapted from Neal (2003).

Stepping-out Algorithm

Algorithm 1 The "stepping out" procedure for finding an interval around the current point.

Require: $f =$ function proportional to the density
Require: $x_0 =$ the current point
Require: $y =$ the vertical level defining the slice
Require: $w =$ estimate of the typical size of a slice
Require: $m =$ integer limiting the size of a slice to mw
Ensure: $(L, R) =$ the interval found

$U \sim \text{Uniform}(0, 1)$
$L \leftarrow x_0 - w \times U$
$R \leftarrow L + w$
$V \sim \text{Uniform}(0, 1)$
$J \leftarrow \text{Floor}(m \times V)$
$K \leftarrow (m - 1) - J$
while $J > 0$ **and** $y < f(L)$ **do**
  $L \leftarrow L - w$
  $J \leftarrow J - 1$
end while
while $K > 0$ **and** $y < f(R)$ **do**
  $R \leftarrow R + w$
  $K \leftarrow K - 1$
end while

The Doubling Procedure

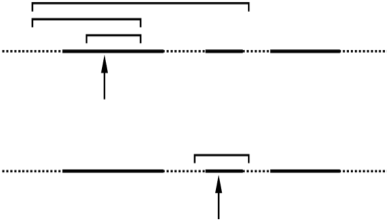


Figure: In *top* row, the initial interval is doubled twice, until both ends are outside the slice. In *bottom* row, where the start state is different, and the initial interval's ends are already outside the slice, no doubling is done.[3]

[3] Figure 2 adapted from Neal (2003).

Doubling Algorithm

Algorithm 2 The "doubling" procedure for finding an interval around the current point. *Note* that it is possible to save some computation in the second and later iterations of the loop, since only one of $f(L)$ and $f(R)$ will have changed from the previous iteration.

Require: $f =$ function proportional to the density
Require: $x_0 =$ the current point
Require: $y =$ the vertical level defining the slice
Require: $w =$ estimate of the typical size of a slice
Require: $p =$ integer limiting the size of a slice to $2pw$
Ensure: $(L, R) =$ the interval found

$U \sim \text{Uniform}(0, 1)$
$L \leftarrow x_0 - w \times U$
$R \leftarrow L + w$
$K \leftarrow p$
while $K > 0$ **and** $(y < f(L)$ or $y < f(R))$ **do**
  $V \sim \text{Uniform}(0, 1)$
  if $V < \frac{1}{2}$ **then**
    $L \leftarrow L - (R - L)$
  else
    $R \leftarrow R + (R - L)$
  end if
  $K \leftarrow K - 1$
end while

Other Variations of Slice Sampling

- **Multivariate Slice Sampling**
  - Utilizes hyperrectangles to sample from multivariate distributions.
  - Hyperrectangle dimensions vary based on the distribution's characteristics.
  - Efficient for exploring high-dimensional spaces.
  - Simplifies sampling from complex multivariate distributions.
- **Overrelaxed Slice Sampling**
  - Suppresses random walks by updating variables using "overrelaxed" updates.
  - New values are chosen on the opposite side of the mode from the current value.
  - Improves sampling efficiency by reducing random walk behavior.
  - Particularly useful when conditional distributions are Gaussian.
- **Reflective Slice Sampling**
  - Similar to bouncing off boundaries in physics.
  - Reflects off slice boundaries to guide sampling.
  - Specializes in uniform distributions.
  - Ensures the rotationally symmetric distribution's independence from x.

Merits and Demerits of Slice Sampling

Merits:

- **No Tuning Parameters:** Slice sampling does not require tuning parameters like step sizes or proposal distributions.
- **Robustness:** It can handle complex and multimodal distributions without getting stuck in local modes.
- **Ease of Implementation:** Simple to implement and easy to understand compared to some other sampling methods.

Demerits:

- **Efficiency:** Slice sampling can be inefficient in high-dimensional spaces due to the need for high-dimensional slices.
- **Sampling from High-Dimensional Distributions:** In high-dimensional spaces, determining good intervals for slice sampling can be challenging.
- **Slice Width:** The width of the slice can significantly affect the efficiency of sampling.

Merits and Demerits of Slice Sampling

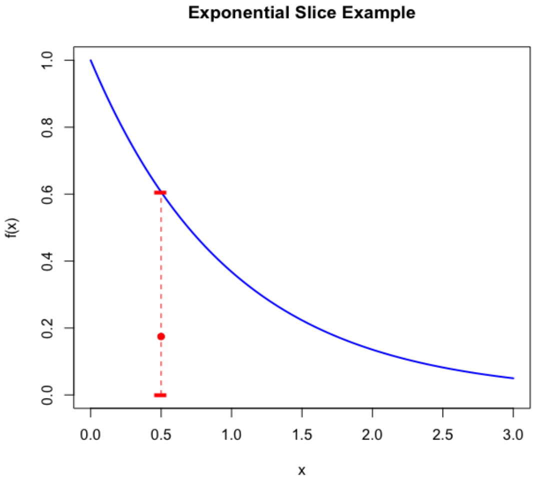
	Derivatives needed?	How critical is tuning?	Retrospective tuning allowed?	Can suppress random walks?
Single-variable methods				
ARS/ARMS	No (but helpful)	Low–Medium	If log concave	No
Single-variable Metropolis	No	Medium	No	No
Single-variable slice sampling	No	Low	If unimodal	No
Overrelaxed slice sampling	No (but helpful)	Low	If unimodal and endpoints exact	Yes
Multivariate methods				
Multivariate Metropolis	No	Medium–High	No	No
Dynamical methods	Yes	High	No	Yes
Slice sampling with hyperrectangles	No	Low–Medium	No	No
Slice sampling with Gaussian crumbs	Possibly helpful	Low–Medium	No	No
Reflective slice sampling	Yes	Medium–High	No	Yes

Figure: Characteristics of some general-purpose Markov chain sampling methods.[4]

[4] Figure 3 adapted from Neal (2003).

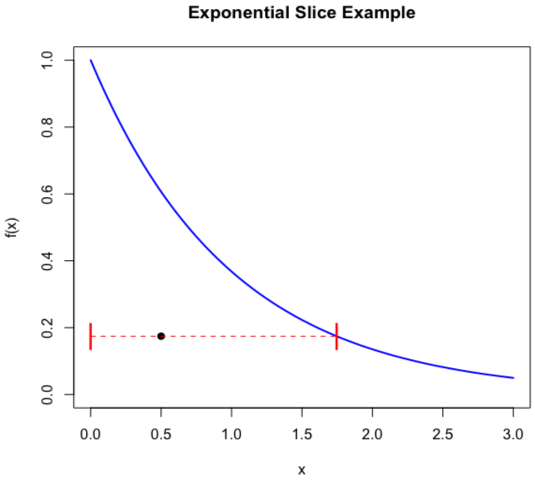
Exponential(1) Example

1. Initialize $x_0 = 0.5$
2. Draw $y_0 \sim U(0, f(0.5))$
3. $(x_0, y_0) = (0.5, 0.174)$



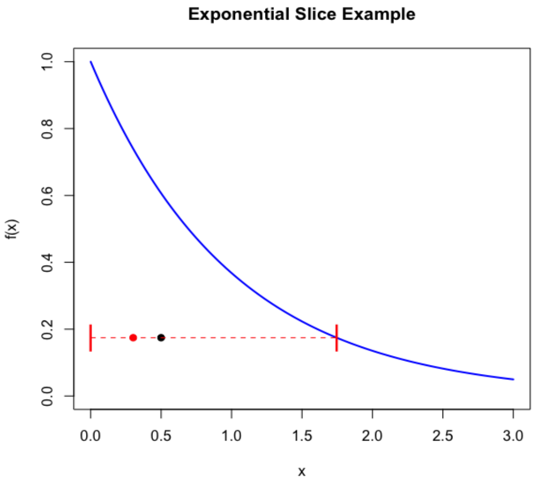
Exponential(1) Example

1. Draw $x_1 \sim U(0, -log(y_0))$



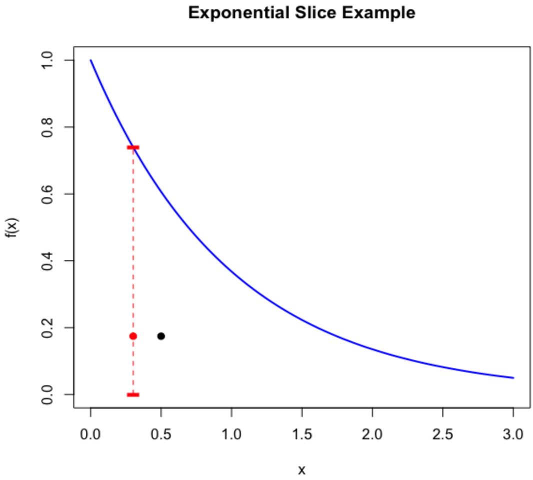
Exponential(1) Example

1. $x_1 = 0.302$



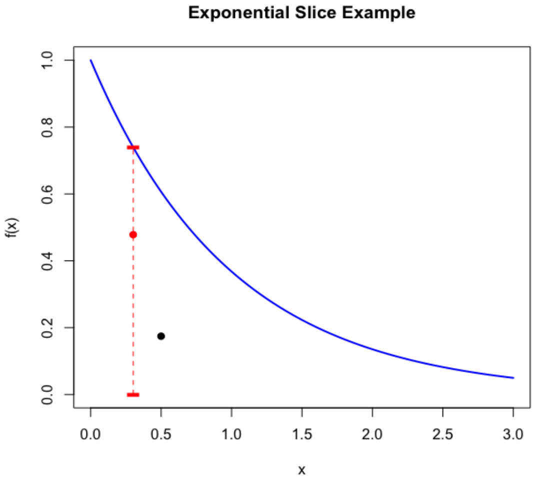
Exponential(1) Example

1. Draw $y_1 \sim U(0, f(x_1))$



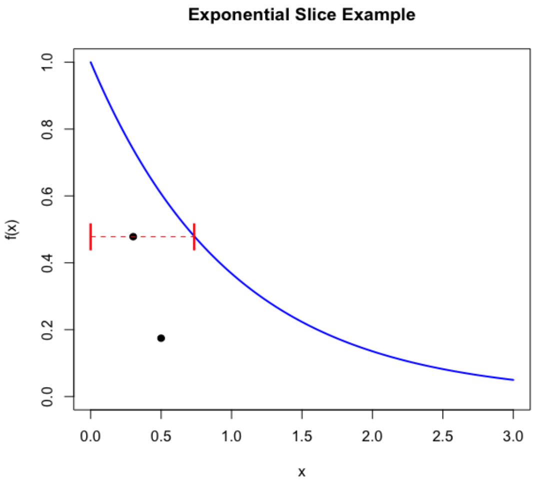
Exponential(1) Example

1. $(x_1, y_1) = (0.5, 0.174)$



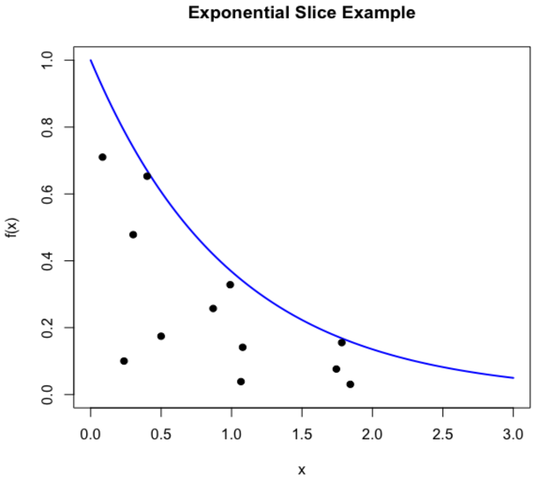
Exponential(1) Example

1. Keep repeating process n times...



Exponential(1) Example

1. 12 example (x_i, y_i) points



Exponential(1) Example

1. Histogram of $n = 10^5$ sample

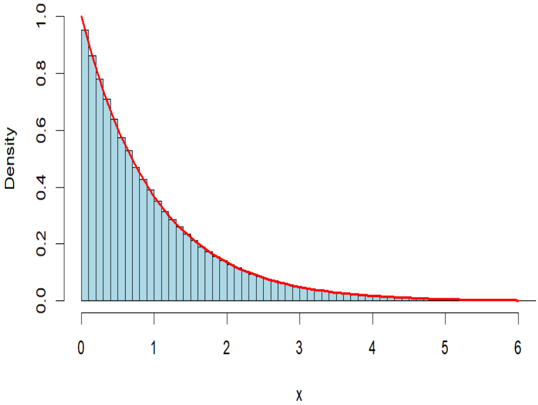


Figure: Histogram of Exponential(1) Slice Samples.

Bimodal Normal Example using Stepping-out

$$f(x) = \frac{1}{2}\mathcal{N}(-10, 36) + \frac{1}{2}\mathcal{N}(15, 4)$$

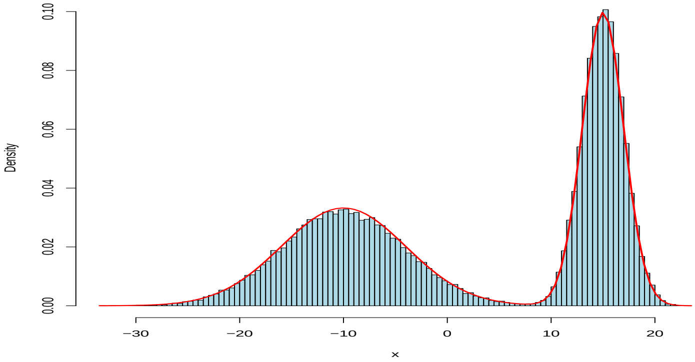


Figure: Histogram of slice sampling from bimodal normal distribution using stepping-out.

Bimodal Normal Example using Doubling

$$f(x) = \frac{1}{2}\mathcal{N}(-10, 36) + \frac{1}{2}\mathcal{N}(15, 4)$$

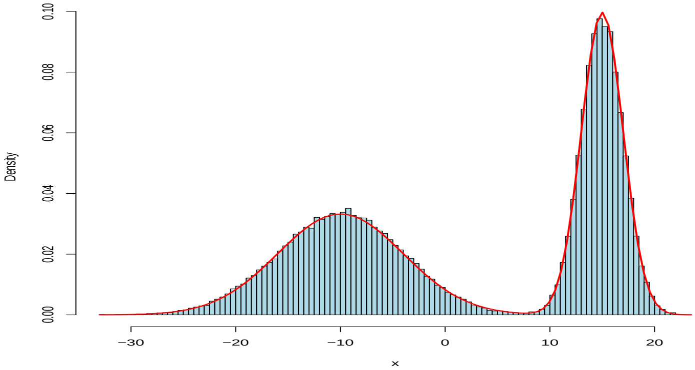


Figure: Histogram of slice sampling from bimodal normal distribution using doubling.

References

Forsythe, G. E. (1972). Von neumann's comparison method for random sampling from the normal and other distributions. *Mathematics of Computation*, 26(120):817–826.

Gelfand, A. E. (2000). Gibbs sampling. *Journal of the American statistical Association*, 95(452):1300–1304.

Gilks, W. R. and Wild, P. (1992). Adaptive rejection sampling for gibbs sampling. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 41(2):337–348.

Neal, R. M. (2003). Slice sampling. *The annals of statistics*, 31(3):705–767.