The Combination Lemma for curves

Γ_1, Γ_2 – sets of Jordan arcs.

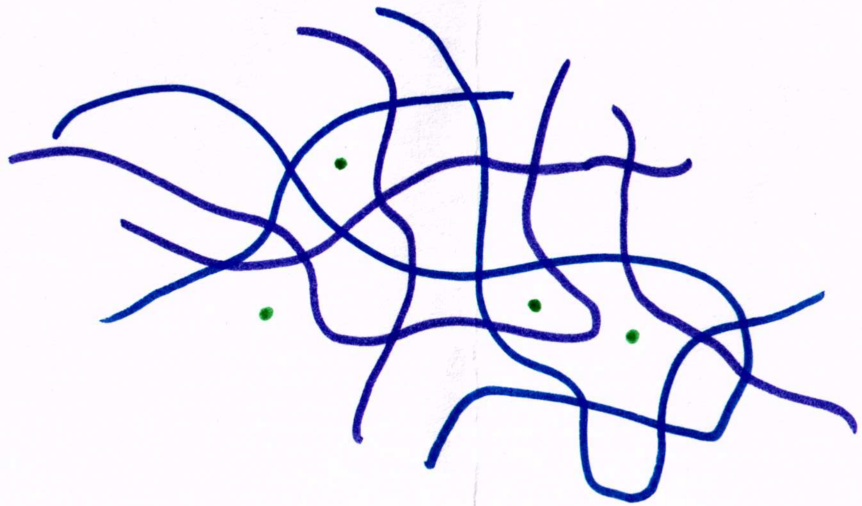
P, $|P| = k$ – marking points.

Let $A(\Gamma_1)$, $A(\Gamma_2)$ have complexities r & b for marked faces in $A(\Gamma_1)$ and $A(\Gamma_2)$, respectively.

$A(\Gamma_1 \cup \Gamma_2)$ has complexity $O(r+b+k)$ for marked faces in $A(\Gamma_1 \cup \Gamma_2)$.

GSS, DCG, 4: 491–521, 1989

Guibas et. al.



Applications.

Use recursion for estimating "many faces" complexity in an arrangement of Jordan arcs.

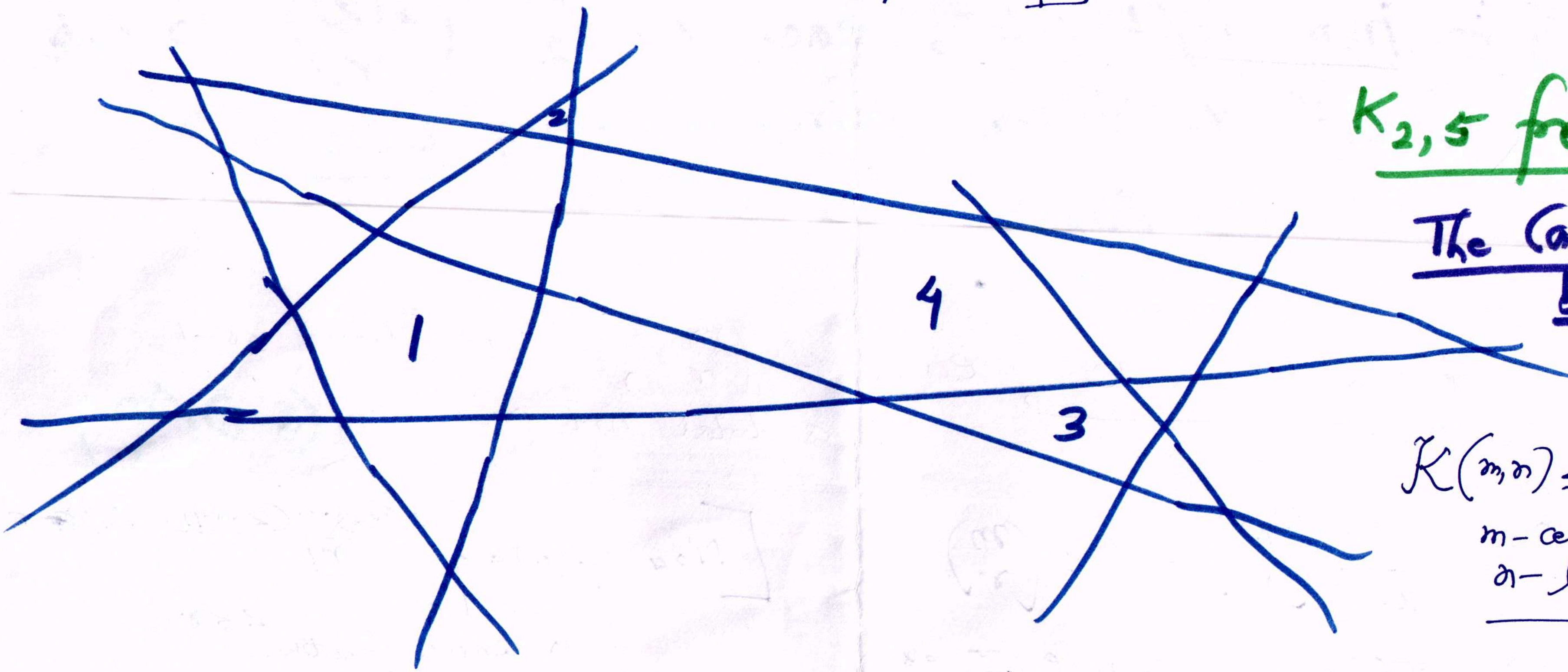
So, $(s-1)\binom{m}{r} \geq n \binom{|E(G_{m,n})|/n}{r}$ $\left[f(|G_{m,n}|/n) \triangleq \binom{|G_{m,n}|/n}{r} \right]$

$$\Rightarrow (s-1)\, m^r \geq n \left(\frac{E(G_{m,n})}{n} - r + 1 \right)^r$$

[denominators $r!$ get balanced)

$$\Rightarrow (s-1)^{1/r}\, m n^{1-\frac{1}{r}} + (r-1)n \geq |E(G_{m,n})|$$

□



$K_{2,5}$ free

The Canham bound

(1969)

$K(m,n) \leq c(m\sqrt{n} + n)$

m- cells

n- lines

Kővári, Sós, Turán (1954) & Erdős [x]

$G_{m,n}$ bipartite $m \times n$ vertices, no $K_{r,s}$ as a subgraph

$$|E(G_{m,n})| \le c_{r,s}\,(m n^{1-\frac{1}{r}} + n)$$

$c_{r,s}$ – Constant, function of only r & s, not m & n.

Proof: From right side, each $x \in V_2$ $\binom{d(x)}{r}$ r-tuple W with x (W,x) pairs, totalling exactly

$$\sum_{x \in V_2} \binom{d(x)}{r}.$$

From left side, each of $\binom{m}{r}$, (W,x) pairs $\le s-1$, so a total not exceeding $(s-1)\binom{m}{r}$

So,
$$\sum_{x \in V_2} \binom{d(x)}{r} \le (s-1)\binom{m}{r}$$

[Now $f(z) = \begin{cases} \dfrac{z(z-1)\cdots(z-r+1)}{r!} & z \ge r \\ 0 & z \le r-1 \end{cases}$ is a convex function.

$$\Rightarrow \sum_{x \in V_2} f(d(x)) \ge n f\left(\sum \frac{d(x)}{n}\right) \rightarrow \boxed{\text{Jensen}}$$
$$= n f\left(\frac{|E(G_{m,n})|}{n}\right)$$

Crossing number theorem

$G(V,E)$, no multiple edges.

$$cr(G) \geq \frac{1}{64} \cdot \frac{|E|^3}{|V|^2} - |V| .$$

$cr(K_5) = 1$. $cr(K_4) = 0$.

For non-planar graphs G, $cr(G) > 0$.

$\Rightarrow$ $cr(G) \geq |E| - 3|V|$. If $|E| > 3|V|$ and we had $< |E| - 3|V|$ crossings then for each crossing drop an edge (at most $|E| - 3|V|$ edges dropped), still having a planar graph with $|E| > 3|V|$ edges*, a contradiction $\square$

* A graph with more that $3|V| - 6$ edges has $cr(G) > 0$, non-planar