# Measure

# Where we left off

**Last time:**

- Experiments → outcomes → events
- Probability must be assigned to **sets of outcomes** (“questions”)
- Events are modeled using $\sigma$-fields
- $\sigma$-fields tell us which questions are allowed

**Todays question:**

- How do we assign probabilities to events?

# What is a measure?

Let $(\Omega, \mathscr{A})$ be a measurable space.

A measure is a nonnegative countably additive set function; that is, a function

$$\mu : \mathscr{A} \rightarrow \mathbb{R}, \text{ with}$$

1. $\mu(A) \geq \mu\left(\phi\right) = 0$, for all $A \in \mathscr{A}$, and
2. If $A_i \in \mathscr{A}$ is a countable sequence of disjoint sets, then,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

The triple $\left(\Omega, \mathscr{A}, \mu\right)$ is a measure space.

If $\mu(\Omega) = 1$, we call $\mu$ a probability measure, $P$.

# Properties

Let $\mu$ be a measure on $(\Omega, \mathscr{A})$,

1. **monotonicity:** If $A \subset B$ then $\mu(A) \leq \mu(B)$.
2. **subadditivity:** If $A \subset \bigcup_{n=1}^{\infty} A_n$, then $\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$
3. **continuity from below:** If $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$.
4. **continuity from above:** If $A_n \downarrow A$, with $\mu(A_1) < \infty$, then $\mu(A_n) \downarrow \mu(A)$.

# The Construction Problem

- $\sigma$-fields are typically very large.
- To define a probability measure, we would need to:
  - assign a number to every event
  - verify countable additivity

This is conceptually clear, but practically impossible.

**Idea:** Define probabilities on a smaller, manageable collection of sets and extend them systematically.

# Fields

A collection $\mathscr{F}$ of subsets of Ω is called a $\sigma$**-field** if:

1. $\Omega \in \mathscr{F}$
2. If $A \in \mathscr{F}$ then $A^c \in \mathscr{F}$
3. If $A_1, A_2, \ldots A_n \in \mathscr{F}$ then $\bigcup_{i=1}^{n} A_i \in \mathscr{F}$

Difference from $\sigma$-fields: Closed under **finite** unions, not necessarily countable unions

# Examples of fields

Every $\sigma$-field is a field, but not every field is a $\sigma$-field.

- Trivial fields: All subsets of $\Omega$, $\left\{\phi, \Omega\right\}$
- Field induced by a finite partition: Let $\{A_1, \ldots, A_N\}$ be a partition of $\Omega$, then $\mathscr{F} = \left\{\bigcup_{i \in I} A_i \ : \ I \subset \{1, \ldots, N\}\right\}$ is a field.
- Let $\Omega \neq \phi$ and $\mathscr{G} := \left\{A \subseteq \Omega \,|\, \text{Either } A \text{ or } A^c \text{ is finite}\right\}$, then $\mathscr{G}$ is a field, and is called the "finite - cofinite" field. This is not a $\sigma$-field, why?
- $\mathscr{F} = \{$finite disjoint unions of intervals $(a, b] \subset (0,1]\}$ is a field.

# Why Fields Are Useful?

- Fields are **much smaller** than σ-fields
- They are often explicitly describable
- Probabilities can be defined on them directly

**Idea:** We can define a probability on a field $\mathscr{F}$ and then extend it to a probability on $\sigma\left(\mathscr{F}\right)$.

# From Fields to $\sigma$-fields: Carathéodory extension

Let $\mathscr{F}$ be a field of subsets of $\Omega$, and $\mathscr{A} = \sigma\left(\mathscr{F}\right)$. Suppose that $P$ is a probability on $\mathscr{F}$. Then $P$ has a unique extension as a probability to $\mathscr{A}$.

What this means:

- We do **not** need to define probabilities on the entire σ-field
- Defining them on a field is enough

# Carathéodory extension

Two aspects:

- Existence of a measure on $\sigma\left(\mathscr{F}\right)$ **Proof intuition:** We define the probability of any set as the *smallest total probability* of countable collections of simple sets that cover it. This agrees with the original probabilities on the field, and the sets where additivity holds form a σ-field containing the field, and hence contains $\sigma(\mathscr{F})$.

- Uniqueness. **Proof intuition:** The sets where the two measures agree form a σ-field containing the field; therefore they agree on the σ-field generated by the field