220A Discussion 6

1. Let $A, B \geq 0$ be random variables and define
$$W := (A - B)^+.$$
   (a) Prove that $\mathbb{E}[W] = 0$ if and only if
$$\mathbb{P}(A \leq B) = 1.$$
   (b) Does independence of A and B play any role in part (a)? Explain.
   (c) Give an example (with explicit distributions) such that
$$\mathbb{E}[A - B] = 0 \quad \text{but} \quad \mathbb{E}[(A - B)^+] > 0.$$
   Interpret what this means probabilistically.
   (d) Is it possible that
$$\mathbb{E}[(A - B)^+] = 0 \quad \text{and} \quad \mathbb{E}[(B - A)^+] = 0,$$
   while A and B are not equal almost surely? Justify your answer carefully.

   Discussion. Comment on the difference between $\mathbb{E}[A - B] = 0$ and $\mathbb{E}[(A - B)^+] = 0$. Why is the latter a strictly stronger statement?

2. Let X, Y be independent, square-integrable random variables. Define
$$U = X + Y, \qquad V = X - Y.$$
   (a) Compute $\mathrm{Cov}(U, V)$ in terms of $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.
   (b) Under what condition on $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$ are U and V uncorrelated?
   (c) Are U and V necessarily independent? Give a concrete non-Gaussian example showing that they need not be independent even when they are uncorrelated.
   (d) Suppose now that (X, Y) are jointly Gaussian and independent. Are U and V independent in this case? Explain your reasoning.

   Discussion. Explain clearly which parts of your reasoning use independence and which parts rely only on second-moment calculations.