STAT 220A — Homework 1
Solution Discussion

Problem 1: Statement and strategy

Claim. A field $\mathcal{F}$ is a σ-field if and only if it is a monotone class.

Strategy.

- One direction is immediate from definitions.
- The other direction uses the field property to reduce countable unions to monotone unions.

σ-field implies monotone class

Assume $\mathcal{F}$ is a σ-field.

- $\mathcal{F}$ is closed under countable unions.
- Hence for any increasing sequence $A_1 \subset A_2 \subset \cdots$,

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

- For a decreasing sequence $B_1 \supset B_2 \supset \cdots$,

$$\bigcap_{n=1}^{\infty} B_n = \left(\bigcup_{n=1}^{\infty} B_n^c \right)^c \in \mathcal{F}.$$

Conclusion. Every σ-field is a monotone class.

Monotone class + field implies σ-field

Now assume $\mathcal{F}$ is a field and a monotone class.

Let $A_1, A_2, \ldots \in \mathcal{F}$ be arbitrary. Define

$$B_k := \bigcup_{n=1}^{k} A_n.$$

- Because $\mathcal{F}$ is a field, each $B_k \in \mathcal{F}$.
- The sequence (B_k) is increasing.
- By closure under monotone unions,

$$\bigcup_{k=1}^{\infty} B_k = \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

Conclusion. $\mathcal{F}$ is closed under countable unions, hence a σ-field.

Problem 2: Setup and key idea

Let $\mathcal{H}$ be a semi-field on Ω and $\overline{\mathcal{H}}$ the collection of all finite disjoint unions of sets in $\mathcal{H}$.

Key idea.

- Finite disjoint unions do not add new "σ-information".
- The σ-field generated by $\mathcal{H}$ already contains them.

Problem 2: First inclusion

Since $\mathcal{H} \subset \overline{\mathcal{H}}$, any σ-field containing $\overline{\mathcal{H}}$ also contains $\mathcal{H}$.

By minimality of generated σ-fields,

$$\sigma(\mathcal{H}) \subset \sigma(\overline{\mathcal{H}}).$$

Problem 2: Reverse inclusion

Let $A \in \overline{\mathcal{H}}$. Then

$$A = \bigcup_{i=1}^{n} H_i, \qquad H_i \in \mathcal{H}, \quad H_i \text{ disjoint.}$$

- Each H_i belongs to $\sigma(\mathcal{H})$.
- $\sigma(\mathcal{H})$ is closed under finite unions.

Therefore $A \in \sigma(\mathcal{H})$, and

$$\sigma(\overline{\mathcal{H}}) \subset \sigma(\mathcal{H}).$$

Conclusion. $\sigma(\mathcal{H}) = \sigma(\overline{\mathcal{H}})$.

Problem 3: Definitions and goal

Define

$$\mathcal{F} = \{A : \ A \text{ finite or } A^c \text{ finite}\}, \quad \mathcal{C} = \{A : \ A \text{ countable or } A^c \text{ countable}\}.$$

Goal. Show that $\sigma(\mathcal{F}) = \mathcal{C}$.

Problem 3: Inclusion $\sigma(\mathcal{F}) \subset \mathcal{C}$

- Finite sets are countable.
- Cofinite sets are cocountable.
- $\mathcal{C}$ is a σ-field.

Thus $\mathcal{F} \subset \mathcal{C}$, and by minimality,

$$\sigma(\mathcal{F}) \subset \mathcal{C}.$$

Problem 3: Inclusion $\mathcal{C} \subset \sigma(\mathcal{F})$

If A is countable, write

$$A = \bigcup_{n=1}^{\infty} \{a_n\},$$

where each singleton $\{a_n\}$ is finite and hence in $\mathcal{F}$.

By closure under countable unions,

$$A \in \sigma(\mathcal{F}).$$

If A^c is countable, then $A = (A^c)^c \in \sigma(\mathcal{F})$.

Conclusion. $\sigma(\mathcal{F}) = \mathcal{C}$.

Problem 4: Setup and plan

Let $\mu : \mathcal{F} \to [0, \infty]$ be finitely additive and countably subadditive.

Plan.

1. Show finite additivity implies monotonicity.
2. Use monotonicity to obtain superadditivity on disjoint unions.
3. Combine with subadditivity to get equality.

Problem 4: Finite additivity implies monotonicity

Let $A \subset B$. Then

$$B = A \cup (B \setminus A), \qquad A \cap (B \setminus A) = \varnothing.$$

By finite additivity,

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

Thus μ is monotone.

Problem 4: Superadditivity on disjoint unions

Let (E_n) be disjoint sets in $\mathcal{F}$.

For each N,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \geq \mu\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} \mu(E_n),$$

where the equality follows from finite additivity.

Letting $N \to \infty$,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \geq \sum_{n=1}^{\infty} \mu(E_n).$$

Problem 4: Conclusion

By countable subadditivity,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Combining both inequalities yields

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Conclusion. μ is countably additive.