σ-fields

Where we left off

Last time:

- Probability models experiments that involve chance
- In discrete experiment settings, we assign probability to each individual outcome
- In continuous settings, probability cannot be assigned to individual outcomes
- Probability must be assigned to **sets of outcomes** (“questions”)

Todays question:

- What collections of sets are suitable for probability?

Probability spaces (preview)

A probability model consists of three ingredients:

- Ω: sample space (the set of all possible outcomes) each outcome corresponds to one element of Ω
- $\mathscr{A}$: collection of events; an **event** is a **subset of** $\mathbf{\Omega}$
- P: probability assigned to events

We write this as a triple: $(\Omega, \mathscr{A}, P)$

Todays question:

- What should $\mathscr{A}$ look like?

What should events allow us to do?

If an event makes sense, then:

- Its **complement** should also make sense (“What if it doesn’t happen?”)
- Combining events should make sense (“What if one of these happens?”)
- Repeating this process **countably many times** should still make sense

σ-fields: Definition

A collection $\mathscr{A}$ of subsets of Ω is called a $\boldsymbol{\sigma}$**-field** if:

1. $\Omega \in \mathscr{A}$
2. If $A \in \mathscr{A}$ then $A^c \in \mathscr{A}$
3. If $A_1, A_2, \ldots \in \mathscr{A}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$

Sets in $\mathcal{A}$ are called events.

A pair $(\Omega, \mathscr{A})$ is called a measurable space.

Examples of σ-fields

- Largest possible σ-field: All subsets of Ω
- Smallest possible σ-field: $\{\phi, \Omega\}$
- σ-field induced by a finite partition: Let $\{A_1, \ldots, A_N\}$ be a partition of Ω, then $\mathscr{F} = \left\{ \bigcup_{i \in I} A_i \ : \ I \subset \{1, \ldots, N\} \right\}$ is a σ-field.
- Let $\Omega \neq \phi$ and $\mathscr{G} := \left\{ A \subseteq \Omega \,|\, \text{Either } A \text{ or } A^c \text{ is countable} \right\}$, then $\mathscr{G}$ is a σ-field, and is called the "countable - cocountable" σ-field.

Properties

- σ-field is closed under countable intersections.
- $A, B \in \mathscr{A} \;\Rightarrow\; A \backslash B \in \mathscr{A}$
- If $A_n \uparrow A$ or $A_n \downarrow A$, and $A_1, A_2, \ldots \in \mathscr{A}$, then $A \in \mathscr{A}$

Generated σ-field

- Given $\Omega \neq \phi$ and a collection $\mathscr{S}$ of subsets of Ω, the σ-field generated by $\mathscr{S}$ is given by,

$$\sigma\left(\mathscr{S}\right) := \bigcap \left\{\mathscr{G} : \mathscr{G} \text{ is a } \sigma\text{-field on } \Omega, \mathscr{S} \subset \mathscr{G}\right\}$$

Properties:

- $\sigma\left(\mathscr{S}\right)$ is a σ-field.
- If any σ-field $\mathscr{H}$ contains $\mathscr{S}$, then it also contains $\sigma\left(\mathscr{S}\right)$; i.e. $\sigma\left(\mathscr{S}\right)$ is the smallest σ-field which contains $\mathscr{S}$.

The Borel σ-field on $\mathbb{R}$

Generated σ-field on $\mathbb{R}$:

$$\mathscr{B}(\mathbb{R}) = \sigma\left(\text{open intervals } (a, b) \subset \mathbb{R}\right)$$

The smallest σ-field on $\mathbb{R}$ containing all open intervals, called Borel σ-field.

What about other types of intervals?

- $\mathscr{O}$:= open intervals (a, b)
- $\mathscr{C}$:= closed intervals $[a, b]$
- $\mathscr{H}$:= half-open intervals $(a, b]$
- $\mathscr{D}$:= rays $(-\infty, x]$

$$\mathscr{B}(\mathbb{R}) = \sigma\left(\mathscr{O}\right) = \sigma\left(\mathscr{C}\right) = \sigma\left(\mathscr{H}\right) = \sigma\left(\mathscr{D}\right)$$

Is every interval covered? Yes!

- Half-open intervals: $[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b \right)$
- Unbounded intervals: $(a, \infty) = (-\infty, a]^c, [a, \infty), (-\infty, a)$
- Singleton: $\{x\} := \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n} \right)$

The Borel σ-field on $\mathbb{R}$

$\mathscr{B}(\mathbb{R})$ is the smallest σ-field on $\mathbb{R}$ containing all intervals. This is our default notion of ‘measurable sets’ on the real line.

- All familiar sets from calculus are Borel
- Not all subsets of $\mathbb{R}$ are Borel