220A Discussion 5

1. **(Indicators, joint pmf, and variance additivity).** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $A, B \in \mathcal{F}$. Define $X = \mathbf{1}_A$ and $Y = \mathbf{1}_B$. Assume A and B are independent.

   (a) Compute the joint pmf of (X, Y), i.e.,

   $$\mathbb{P}(X = i,\ Y = j), \qquad i, j \in \{0, 1\},$$

   and write your answers explicitly in terms of $p = \mathbb{P}(A)$ and $q = \mathbb{P}(B)$.

   (b) Using part (a), compute the distribution of

   $$Z = X + Y,$$

   i.e., find $\mathbb{P}(Z = 0)$, $\mathbb{P}(Z = 1)$, and $\mathbb{P}(Z = 2)$ in terms of p and q.

   (c) Prove that under independence,

   $$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

   Then give an explicit counterexample (i.e., specific events A, B on some probability space) showing that the identity can fail when A and B are *not* independent.

2. **(Moment factorization for indicators).** Let $A, B \in \mathcal{F}$ with $0 < \mathbb{P}(A), \mathbb{P}(B) < 1$ and define

   $$X = \mathbf{1}_A, \qquad Y = \mathbf{1}_B.$$

   (a) Compute $\mathbb{E}[X^m Y^n]$ for integers $m, n \geq 1$.

   (b) Show that the following are equivalent:

   (i) A and B are independent;

   (ii) for all integers $m, n \geq 1$,

   $$\mathbb{E}[X^m Y^n] = \mathbb{E}[X^m]\mathbb{E}[Y^n].$$