Midterm Solutions Discussion

STATS 220A

Q1: Statement

Let $X_1, X_2, \ldots$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and define

$$\mathcal{F}_k := \sigma(X_1, \ldots, X_k), \qquad \mathcal{F}_0 := \bigcup_{k \geq 1} \mathcal{F}_k.$$

- Show $\mathcal{F}_0$ is an algebra (field).
- Show $\sigma(\mathcal{F}_0) = \sigma(X_1, X_2, \ldots)$.

Q1: $\mathcal{F}_0$ is an algebra (field)

Recall: an *algebra* $\mathcal{A}$ satisfies

$$A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}, \qquad A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}.$$

Take $A, B \in \mathcal{F}_0$. Then

$$\exists\, m, n \text{ s.t. } A \in \mathcal{F}_m,\ B \in \mathcal{F}_n.$$

Let $k := \max(m, n)$. Since the sequence is increasing, $\mathcal{F}_m \subseteq \mathcal{F}_k$ and $\mathcal{F}_n \subseteq \mathcal{F}_k$. So $A, B \in \mathcal{F}_k$, and because $\mathcal{F}_k$ is a σ-field,

$$A^c \in \mathcal{F}_k \subseteq \mathcal{F}_0, \qquad A \cup B \in \mathcal{F}_k \subseteq \mathcal{F}_0.$$

Hence $\mathcal{F}_0$ is an algebra.

Q1: Show $\sigma(\mathcal{F}_0) = \sigma(X_1, X_2, \dots)$

Let
$$\mathcal{G} := \sigma(X_1, X_2, \dots).$$

Step 1: $\sigma(\mathcal{F}_0) \subseteq \mathcal{G}$.

Since $\mathcal{F}_k = \sigma(X_1, \dots, X_k) \subseteq \sigma(X_1, X_2, \dots) = \mathcal{G}$ for every k,

$$\mathcal{F}_0 = \bigcup_{k \geq 1} \mathcal{F}_k \subseteq \mathcal{G}.$$

Because $\mathcal{G}$ is a σ-field containing $\mathcal{F}_0$, the smallest σ-field containing $\mathcal{F}_0$ must be contained in $\mathcal{G}$:
$$\sigma(\mathcal{F}_0) \subseteq \mathcal{G}.$$

Q1: Show $\sigma(\mathcal{F}_0) = \sigma(X_1, X_2, \dots)$ (cont.)

Step 2: $\mathcal{G} \subseteq \sigma(\mathcal{F}_0)$.

For each n, we have X_n measurable w.r.t. $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, hence also measurable w.r.t. $\mathcal{F}_0$? Not necessarily, but:

$$\mathcal{F}_n \subseteq \mathcal{F}_0 \subseteq \sigma(\mathcal{F}_0).$$

Therefore every event of the form $\{X_n \in B\}$ (with $B \in \mathcal{B}(\mathbb{R})$) lies in $\mathcal{F}_n \subseteq \sigma(\mathcal{F}_0)$.

But $\mathcal{G}$ is the *smallest* σ-field containing all such events $\{X_n \in B\}$, so

$$\mathcal{G} \subseteq \sigma(\mathcal{F}_0).$$

Combining both inclusions gives $\sigma(\mathcal{F}_0) = \mathcal{G}$.

Q2: Setup and statement

Let Ω be infinite and define the *cofinite algebra*

$$\mathcal{A} := \{A \subseteq \Omega : \; A \text{ is finite or } A^c \text{ is finite}\}.$$

Define $\lambda : \mathcal{A} \to [0,1]$ by

$$\lambda(A) = \begin{cases} 0, & A \text{ finite}, \\ 1, & A^c \text{ finite (i.e. } A \text{ cofinite).} \end{cases}$$

- (a) Show λ is finitely additive.
- (b) If Ω is countably infinite, show λ is not countably additive.
- (c) If Ω is uncountable, show: whenever (A_n) are disjoint in $\mathcal{A}$ and $\bigcup_n A_n \in \mathcal{A}$, then

$$\lambda\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \lambda(A_n).$$

Q2: First observations

- $\mathcal{A}$ is an algebra: closed under complements and finite unions.
- Every $A \in \mathcal{A}$ is either "small" (finite) or "almost everything" (cofinite).
- λ is a 0–1 set function: it does not see *how big* a cofinite set is; all cofinite sets get mass 1.

Q2(a): Finite additivity — what to show

Finite additivity means: if $A, B \in \mathcal{A}$ are disjoint, then

$$\lambda(A \cup B) = \lambda(A) + \lambda(B).$$

Because λ only takes values in $\{0, 1\}$, we can do a clean case split:

- both A, B finite
- one finite, one cofinite
- both cofinite (can this happen if disjoint?)

Q2(a): Case 1 — both A and B finite

If A and B are finite and disjoint, then $A \cup B$ is finite, so

$$\lambda(A) = 0, \quad \lambda(B) = 0, \quad \lambda(A \cup B) = 0,$$

and hence $\lambda(A \cup B) = \lambda(A) + \lambda(B)$.

Q2(a): Case 2 — one finite, one cofinite

Assume A is cofinite and B is finite, with $A \cap B = \varnothing$.
Then A^c is finite and since $B \subseteq A^c$ (disjointness forces $B \subseteq A^c$), $A^c \cup B$ is finite. But

$$(A \cup B)^c = A^c \cap B^c \subseteq A^c,$$

so $(A \cup B)^c$ is finite, meaning $A \cup B$ is cofinite.
Thus

$$\lambda(A) = 1, \quad \lambda(B) = 0, \quad \lambda(A \cup B) = 1,$$

so finite additivity holds.

Q2(a): Case 3 — can A and B both be cofinite and disjoint?

Suppose (for contradiction) that A and B are disjoint and both cofinite. Then A^c and B^c are finite. But if A and B are disjoint, then

$$\Omega = A^c \cup B^c \cup (A \cap B),$$

and since $A \cap B = \varnothing$,

$$\Omega = A^c \cup B^c,$$

a union of two finite sets, hence finite. Contradiction since Ω is infinite.

Therefore, *at most one* of two disjoint sets in $\mathcal{A}$ can be cofinite. This completes (a).

Q2(b): Countable additivity fails when Ω is countably infinite

Assume Ω is countably infinite. Enumerate $\Omega = \{\omega_1, \omega_2, \dots\}$ and define

$$A_n := \{\omega_n\}.$$

Then $A_n \in \mathcal{A}$ (finite), the sets are disjoint, and

$$\bigcup_{n=1}^{\infty} A_n = \Omega.$$

Compute both sides:

$$\lambda(\Omega) = 1, \qquad \sum_{n=1}^{\infty} \lambda(A_n) = \sum_{n=1}^{\infty} 0 = 0.$$

Thus λ is *not* countably additive on $\mathcal{A}$ when Ω is countably infinite.

Q2(b): Why this counterexample works

The moral:

- In a countable set, you can write Ω as a *countable union of finite (even singleton)* sets.
- λ assigns 0 to each finite piece, but assigns 1 to the whole space.
- This is exactly the signature of finite-but-not-countable additivity.

Q2(c): The uncountable case (restricted countable additivity)

Now assume Ω is uncountable. Let (A_n) be disjoint sets in $\mathcal{A}$ and assume additionally that

$$A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

We want to show

$$\lambda(A) = \sum_{n=1}^{\infty} \lambda(A_n).$$

Again, since values are 0 or 1, the sum can only be 0 or 1:

- sum is 1 if at least one A_n is cofinite
- sum is 0 if all A_n are finite

Q2(c): If some A_{n_0} is cofinite

If A_{n_0} is cofinite then $\lambda(A_{n_0}) = 1$ and all other A_n must be finite (disjointness rules out a second cofinite set).
Then

$$A = A_{n_0} \cup \Big(\bigcup_{n \neq n_0} A_n \Big),$$

where the second union is countable union of finite sets, hence *countable*.
But a cofinite set union anything is cofinite:

$$A^c \subseteq A_{n_0}^c,$$

so A^c is finite and $\lambda(A) = 1$.
Also $\sum_n \lambda(A_n) = 1$. So equality holds.

Q2(c): If all A_n are finite

Assume every A_n is finite. Then $A = \bigcup_n A_n$ is a countable union of finite sets, hence *countable*.
Therefore A^c is *uncountable* (since Ω is uncountable). In particular, A^c is not finite.
So A cannot be cofinite. Since $A \in \mathcal{A}$ by assumption, the only remaining possibility is that A is finite.
Thus $\lambda(A) = 0$. Also each $\lambda(A_n) = 0$, so $\sum_n \lambda(A_n) = 0$.

Q3: Statement

Evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{1-\cos\left(x/\sqrt{n}\right)}{x^{1/2}}\,dx.$$

Q3: Strategy

We will use the Dominated Convergence Theorem (DCT).
Define

$$f_n(x) := \frac{1 - \cos(x/\sqrt{n})}{x^{1/2}}, \qquad x \in (0, 1].$$

We need:

- pointwise limit $f_n(x) \to f(x)$
- an integrable dominating function g with $|f_n(x)| \leq g(x)$ for all n and a.e. x

Q3: Pointwise convergence

Fix $x \in (0, 1]$.
As $n \to \infty$, we have $x/\sqrt{n} \to 0$, and by continuity of cosine,

$$\cos(x/\sqrt{n}) \to \cos(0) = 1.$$

So

$$1 - \cos(x/\sqrt{n}) \to 0 \quad \Rightarrow \quad f_n(x) \to 0.$$

Thus the pointwise limit is $f(x) = 0$.

Q3: Find a domination bound (uniform in n)

Use a basic inequality:

$$1 - \cos(u) \le \frac{u^2}{2} \quad \text{for all } u \in \mathbb{R}.$$

Let $u = x/\sqrt{n}$. Then

$$1 - \cos(x/\sqrt{n}) \le \frac{1}{2} \cdot \frac{x^2}{n}.$$

Therefore

$$0 \le f_n(x) \le \frac{\frac{1}{2} \cdot \frac{x^2}{n}}{x^{1/2}} = \frac{1}{2n} x^{3/2}.$$

Since $n \ge 1$,

$$f_n(x) \le \frac{1}{2} x^{3/2} =: g(x) \quad \text{for all } n \ge 1.$$

Q3: Check g is integrable on $(0, 1]$

We have $g(x) = \frac{1}{2}x^{3/2}$, and

$$\int_0^1 g(x)\,dx = \frac{1}{2}\int_0^1 x^{3/2}\,dx = \frac{1}{2}\cdot\left[\frac{x^{5/2}}{5/2}\right]_0^1 = \frac{1}{2}\cdot\frac{2}{5} = \frac{1}{5} < \infty.$$

So $g \in L^1(0, 1)$.

Q3: Apply DCT

We have:

- $f_n(x) \to 0$ pointwise on $(0, 1]$
- $0 \leq f_n(x) \leq g(x)$ for all n, and g is integrable on $(0, 1)$

By DCT,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 \lim_{n\to\infty} f_n(x)\, dx = \int_0^1 0\, dx = 0.$$

$$\boxed{\lim_{n\to\infty} \int_0^1 \frac{1 - \cos\left(x/\sqrt{n}\right)}{x^{1/2}}\, dx = 0}.$$

Q4: Statement

Let $F(x) = \mathbb{P}(X \leq x)$ be a distribution function. Let

$$D := \{x \in \mathbb{R} : \ F(x) \text{ is discontinuous at } x\}.$$

Show that D is at most countable.

Q4: Facts about distribution functions

A distribution function F satisfies:

- F is nondecreasing
- F is right-continuous
- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

For any x, define the *jump size*:

$$\Delta F(x) := F(x) - F(x^-), \quad F(x^-) := \lim_{t \uparrow x} F(t).$$

Then F is discontinuous at x iff $\Delta F(x) > 0$.

Q4: Reformulate the goal

We want to show:

$$\{x : \Delta F(x) > 0\} \text{ is countable.}$$

Key idea: jumps of size at least ε can happen only finitely many times, because the total increase of F from $-\infty$ to ∞ is at most 1.

Q4: Step 1 — define large-jump sets

For each $n \in \mathbb{N}$ define

$$D_n := \{x \in \mathbb{R} : \ \Delta F(x) \geq 1/n\}.$$

Then clearly

$$D = \bigcup_{n=1}^{\infty} D_n,$$

because if $\Delta F(x) > 0$ then choose n with $1/n \leq \Delta F(x)$.
So it suffices to show each D_n is finite (or at least countable). Then D is countable union of finite sets, hence countable.

Q4(a): Bound on sum of jumps over finite subset of D_k

Fix $k \in \mathbb{N}$ and let

$$\{x_1, \ldots, x_m\} \subset D_k, \qquad x_1 < \cdots < x_m.$$

Recall

$$\Delta F(x) = F(x) - F(x^-).$$

For $j \geq 2$, since F is nondecreasing and $x_{j-1} < x_j$,

$$F(x_j^-) \geq F(x_{j-1}),$$

so

$$\Delta F(x_j) = F(x_j) - F(x_j^-) \leq F(x_j) - F(x_{j-1}).$$

Also

$$\Delta F(x_1) = F(x_1) - F(x_1^-).$$

Q4(a): Telescoping argument

Summing the inequalities:

$$\sum_{j=1}^{m} \Delta F(x_j) = \Delta F(x_1) + \sum_{j=2}^{m} \Delta F(x_j)$$

$$\leq \left(F(x_1) - F(x_1^-)\right) + \sum_{j=2}^{m} \left(F(x_j) - F(x_{j-1})\right).$$

The right-hand side telescopes:

$$\sum_{j=1}^{m} \Delta F(x_j) \leq F(x_m) - F(x_1^-).$$

Since $0 \leq F \leq 1$,

$$\sum_{j=1}^{m} \Delta F(x_j) \leq 1.$$

Q4(b): D_k is finite

Assume D_k is infinite.
Then for any $m \in \mathbb{N}$ we can choose distinct

$$x_1 < \cdots < x_m \in D_k.$$

By definition of D_k,

$$\Delta F(x_j) \geq \frac{1}{k}.$$

Thus

$$\sum_{j=1}^{m} \Delta F(x_j) \geq \frac{m}{k}.$$

But part (a) gives

$$\sum_{j=1}^{m} \Delta F(x_j) \leq 1.$$

So

$$\frac{m}{k} \leq 1 \quad \text{for all } m,$$

which is impossible for $m > k$.
Therefore D_k is finite.

Q4(c): D is at most countable

Let

$$D = \{x : \Delta F(x) > 0\}.$$

If $x \in D$, then $\Delta F(x) > 0$. Choose $k \in \mathbb{N}$ such that

$$\Delta F(x) \geq \frac{1}{k}.$$

Then $x \in D_k$.
Hence

$$D \subset \bigcup_{k=1}^{\infty} D_k.$$

Conversely, if $x \in D_k$ for some k, then $\Delta F(x) \geq 1/k > 0$, so $x \in D$.
Thus

$$D = \bigcup_{k=1}^{\infty} D_k.$$

Each D_k is finite, so D is countable.

Q4: Intuition (why this is unavoidable)

Think of F as a "budget" that can increase by at most 1 total.
If you have jumps:

- You can have *infinitely many* jumps, but their sizes must go to 0 fast enough (e.g. $\sum 2^{-k}$).
- You cannot have uncountably many positive jumps because each positive jump spends some nonzero budget.

This is the same core counting idea as: a summable collection of positive numbers has at most countably many nonzero terms.

Final checklist

- Q1: increasing union $\Rightarrow$ algebra; generate the full σ-field by taking $\sigma(\cdot)$
- Q2: finite additivity via casework; countable additivity fails on countable Ω; restricted property holds on uncountable Ω
- Q3: DCT with $1 - \cos u \leq u^2/2$ gives a clean dominator
- Q4: jumps $\geq 1/n$ can occur at most n times; union over n is countable