Construction of measure on $\mathbb{R}$

Where we left off

Last time:

- Experiments → outcomes → events
- Events are modeled using σ-fields
- It's difficult to assign probability to σ-fields
- We can define probability on fields and then uniquely extend to σ-fields (Carathéodory extension)

Todays question:

- How do we assign probabilities to fields in $\mathbb{R}$?

How do we define probability on $\mathbb{R}$?

- We want to define probability of the Borel σ-field $\mathscr{B}(\mathbb{R})$ on $\mathbb{R}$
- Obviously, we cannot assign probability to each element in $\mathscr{B}(\mathbb{R})$
- Fine! That's what Carathéodory extension is for!
  - We get hold of a field $\mathscr{F}$ such that $\sigma(\mathscr{F}) = \mathscr{B}(\mathbb{R})$,
  - Assign probability on elements of $\mathscr{F}$, and voila!
- Great! What are fields on $\mathbb{R}$?

What are fields on $\mathbb{R}$?

A natural starting point are the generators:

- $\mathscr{O}$:= open intervals (a, b)
- $\mathscr{C}$:= closed intervals $[a, b]$
- $\mathscr{H}$:= half-open intervals $(a, b]$
- $\mathscr{D}$:= rays $(-\infty, x]$

We know: $\mathscr{B}(\mathbb{R}) = \sigma\left(\mathscr{O}\right) = \sigma\left(\mathscr{C}\right) = \sigma\left(\mathscr{H}\right) = \sigma\left(\mathscr{D}\right)$

Are any of $\mathscr{O}, \mathscr{C}, \mathscr{H}$, or $\mathscr{D}$ a field? NO!!

So what are these? Semi-fields

A collection $\mathscr{S}$ of subsets of Ω is called a semi-field if:

1. $\Omega \in \mathscr{A}$
2. If $A, B \in \mathscr{S}$ then $A \cap B \in \mathscr{S}$
3. If $A \in \mathscr{S}$ then $A^c = \bigcup_{i=1}^{n} A_i$, where $A_1, A_2, \ldots, A_n \in \mathscr{S}$ are disjoint.

$\mathscr{H}$, the set of half open intervals is a semi-field.

How does this help us?

Semi-field
$\mathscr{H}$**, half-open**
intervals

Borel σ-field
$\mathscr{B}(\mathbb{R})$

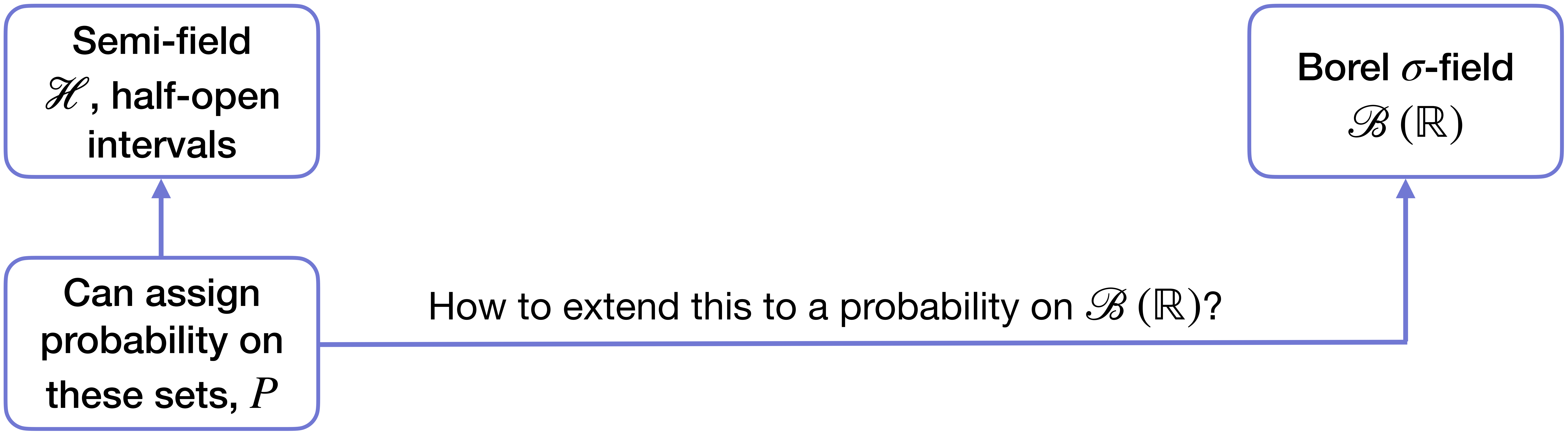
How does this help us?

Semi-field $\mathscr{H}$, half-open intervals

Can assign probability on these sets, P

Borel σ-field $\mathscr{B}(\mathbb{R})$

How does this help us?



How does this help us?

Semi-field $\mathscr{H}$, half-open intervals

$\bar{\mathscr{H}}$ finite union of disjoint sets in $\mathscr{H}$

Borel σ-field $\mathscr{B}(\mathbb{R})$

Can assign probability on these sets, P

How does this help us?

Semi-field $\mathscr{H}$, half-open intervals

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Borel σ-field $\mathscr{B}(\mathbb{R})$

Can assign probability on these sets, P

We can extend this to a probability on $\bar{\mathscr{H}}$

How does this help us?

$$
\begin{tikzcd}
\text{Semi-field } \mathscr{H}\text{, half-open intervals} & \bar{\mathscr{H}} \text{ finite union of disjoint sets in } \mathscr{H} & \text{Borel } \sigma\text{-field } \mathscr{B}(\mathbb{R}) \\
\text{Can assign probability on these sets, } P \arrow[u] \arrow[ur, "\text{We can extend this to a probability on } \bar{\mathscr{H}}"'] &
\end{tikzcd}
$$

$A \in \bar{\mathscr{H}} \implies A = \cup_{i=1}^{n} A_i$; disjoint $A_i \in \mathscr{H}$

Define $P^*(A) = \sum_{i=1}^{n} P\left(A_i\right)$; $\quad$ P^* is unique

How does this help us?

Semi-field $\mathscr{H}$, half-open intervals

Can assign probability on these sets, P

$\bar{\mathscr{H}}$ finite union of disjoint sets in $\mathscr{H}$

Can extend P to probability on these sets, $P*$

Borel σ-field $\mathscr{B}(\mathbb{R})$

How does this help us?

Semi-field $\mathscr{H}$, half-open intervals

Can assign probability on these sets, P

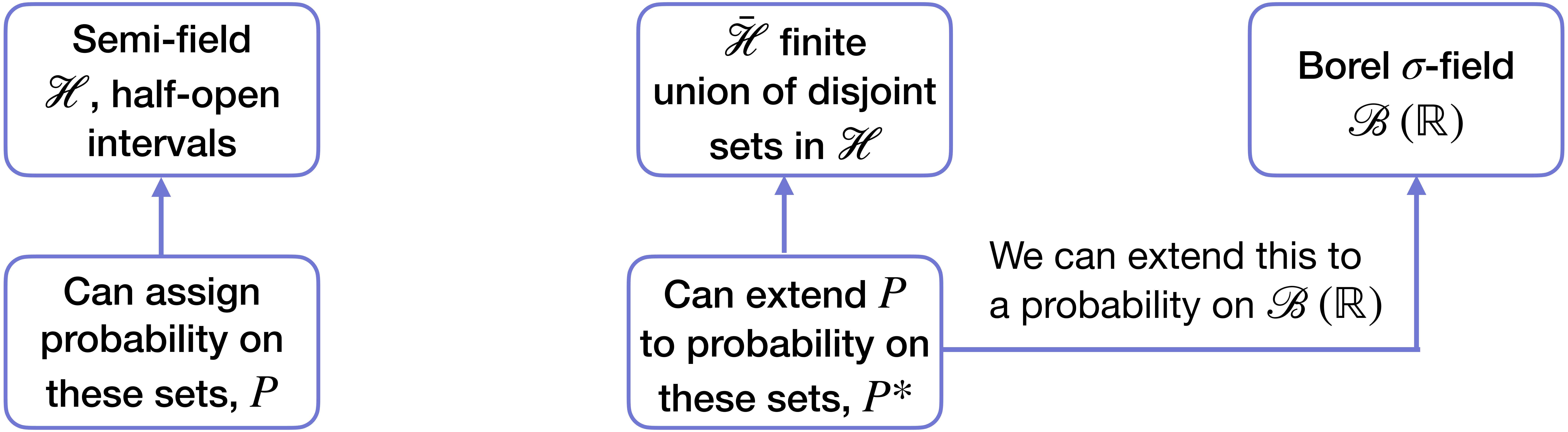
$\bar{\mathscr{H}}$ finite union of disjoint sets in $\mathscr{H}$

Can extend P to probability on these sets, $P*$

We can extend this to a probability on $\mathscr{B}(\mathbb{R})$

Borel σ-field $\mathscr{B}(\mathbb{R})$

How does this help us?



$\bar{\mathscr{H}}$ is a field that generates $\mathscr{B}(\mathbb{R})$,
we use Carathéodory extension

How does this help us?

Semi-field $\mathscr{H}$, half-open intervals

Can assign probability on these sets, P

$\bar{\mathscr{H}}$ finite union of disjoint sets in $\mathscr{H}$

Can extend P to probability on these sets, $P*$

Borel σ-field $\mathscr{B}(\mathbb{R})$

Can extend $P*$ to probability on $\mathscr{B}(\mathbb{R})$

So what's the recipe to define probability on $\mathscr{B}(\mathbb{R})$?

- Start with a semi-field, say $\mathscr{H}$, i.e., half-open intervals
- Define probability on all elements of $\mathscr{H}$
- First extend it to the field $\bar{\mathscr{H}}$ and then to $\mathscr{B}(\mathbb{R})$ using Carathéodory extension.

So, how do we define probability on half-open intervals?

Probability on half-open intervals

- We start with *any* $F : \mathbb{R} \to \mathbb{R}$, which is non-decreasing and right continuous, i.e., $\lim_{y \downarrow x} F(y) = F(x)$. F is called a Stieltjes measure function.
- For any $(a, b] \in \mathscr{H}$, assign the measure $\mu\big((a, b]\big) = F(b) - F(a)$
- Next, extend this measure on $\bar{\mathscr{H}}$ and then extend it to $\big(\mathbb{R}, \mathscr{B}(\mathbb{R})\big)$ using Carathéodory extension.

Probability on $\mathscr{B}(\mathbb{R})$

- Associated with each Stieltjes measure function F there is a *unique* measure μ on $\left(\mathbb{R}, \mathscr{B}(\mathbb{R})\right)$ with $\mu((a,b]) = F(b) - F(a)$
- This μ is the Lebesgue-Stieltjes measure
- If $F(x) = x$, then $\mu((a,b]) = b - a$ is the Lebesgue measure
- If a Stieltjes measure function additionally has $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$, then we call it a distribution function. In that case, the corresponding μ becomes a probability measure.