Lebesgue integral

What is integration?

Integration is weighted aggregation of function values.

$$\int f\, d\mu \;=\; \sum \text{(value of } f\text{)} \times \text{(weight)}$$

Examples:

- Geometry: weight = length / area
- Probability: weight = probability
- Counting: weight = number of occurrences

Depending on context, this means: area under a curve, total mass, expected value, average behavior

Riemann Integration: The Classical Answer

Riemann integration:

- Partitions the domain
- Approximates the area under the curve via limits of rectangles
- Works well for continuous / piecewise smooth functions

Failure 1: Discontinuous Everywhere

$$f(x) = \mathbf{1}_{\mathbb{Q}\cap[0,1]}(x)$$

- Upper Riemann sums = 1
- Lower Riemann sums = 0
- Riemann integral does not exist
- But $f(x) = 0$ almost everywhere; probabilistically, this function is trivial.

Failure 2: Poor Stability Under Limits

- $f_n(x) = \mathbf{1}_{A_n}(x), \qquad A_n = \bigcup_{k=1}^{\infty} \left(q_k - \frac{1}{n} 2^{-k}, q_k + \frac{1}{n} 2^{-k} \right)$. $\{q_k\}_{k \geq 1}$ is an enumeration of $\mathbb{Q} \cap [0,1]$.
- Pointwise, $\quad f_n(x) \longrightarrow f(x) := \mathbf{1}_{\mathbb{Q} \cap [0,1]}(x)$.
- Each f_n is Riemann integrable, but f is not.

The Core Issue

Riemann integration:

- Treats all points equally
- Cannot ignore negligible sets
- Has weak convergence behavior

What we want instead?

- Ignore sets of measure zero
- Be stable under limits
- Work naturally with probability measures

The Lebesgue Viewpoint

- We only care about where $f(x)$ value occurs.
- Partition the *range* instead of the domain

$$\int f\,d\mu = \sum_{\text{levels}} (\text{value}) \times (\text{mass of level set})$$

$$\int f\,d\mu = \sum_{a} a\,\mu\big(\{\omega : f(\omega) = a\}\big)$$

But, we cannot possible index all levels, right?

Construction of Lebesgue Integral

- Step 1: Simple Functions
- Step 2: Bounded Functions on Finite Measure Sets
- Step 3: Nonnegative Measurable Functions
- Step 4: General Integrable Functions

Step 1: Simple Functions

A simple function: $\phi(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{A_i}$, where A_i are disjoint, $\mu(A_i) < \infty$.

This is the atomic case: finite-valued, finite-mass.

Definition: $\int \phi \, d\mu := \sum_{i=1}^{n} a_i \, \mu(A_i)$.

Step 2: Bounded Functions on Finite Measure Sets

- **Assume:** $\mu(E) < \infty, \quad f$ is bounded, $\quad f = 0$ on E^c .
- **Definition:** $\displaystyle\int f\, d\mu := \sup_{\phi \le f} \int \phi\, d\mu = \inf_{\psi \ge f} \int \psi\, d\mu,$ where ϕ, ψ are simple functions supported on E.
- If ϕ, ψ are simple functions with $\phi \le f \le \psi,$ then $\displaystyle\int \phi\, d\mu \;\le\; \int f\, d\mu \;\le\; \int \psi\, d\mu$
- Because we can approximate f **arbitrarily closely from above and below** by simple functions, and the gap between the two approximations has **vanishing total mass.**

Nonnegative Measurable Functions

For $f \geq 0$ **:** $$\int f\,d\mu = \sup\left\{\int h\,d\mu :\ 0 \leq h \leq f,\ h \text{ bounded},\ \mu(\{h > 0\}) < \infty\right\}.$$

Equivalent operational view: $$\int f\,d\mu = \lim_{n\to\infty} \int (f \wedge n)\,\mathbf{1}_{E_n}\,d\mu\,.$$

Every admissible h in the definition is eventually dominated by some $(f \wedge n)\mathbf{1}_{E_n}$, and the mass lost outside E_n vanishes since h has finite support.

Hence no lower approximation is missed in the limit.

Step 4: General Integrable Functions

Define: $$f^+ = \max(f, 0), \qquad f^- = \max(-f, 0)\,.$$

We say f is integrable if: $$\int |f|\, d\mu < \infty,$$

Define: $$\int f\, d\mu := \int f^+\, d\mu - \int f^-\, d\mu\,.$$

This avoids undefined expressions like $\infty - \infty$.

Theorem: Properties of the Lebesgue Integral

If f, g are integrable, then:

1. **Linearity** $\displaystyle\int (af + bg)\, d\mu = a \int f\, d\mu + b \int g\, d\mu\,.$

2. **Positivity** $f \geq 0 \;\Rightarrow\; \displaystyle\int f\, d\mu \geq 0.$

3. **Monotonicity** $g \leq f$ a.e. $\;\Rightarrow\; \displaystyle\int g\, d\mu \leq \int f\, d\mu\,.$

4. **Equality a.e.** $f = g$ a.e. $\;\Rightarrow\; \displaystyle\int f\, d\mu = \int g\, d\mu\,.$

5. **Triangle inequality** $\displaystyle\left| \int f\, d\mu \right| \leq \int |f|\, d\mu\,.$

How to prove these?

- Start with simple functions.
- For bounded functions, replace by simple under- and over-approximations. The property is preserved because:
    - inequalities respect sup/inf,
    - linearity respects limits.
- For nonnegative functions, use truncation and localization: $\int f\, d\mu \approx \int (f \wedge n)\, \mathbf{1}_{E_n}\, d\mu$. Properties pass through monotone limits.
- Write $f := f^+ - f^-$, and use integrability.