# Random variables

# Where We Left Off

**Last time:**

- We constructed probability measures on $\left(\mathbb{R}, \mathscr{B}(\mathbb{R})\right)$
- Starting from a distribution function $F$
- Using the Lebesgue–Stieltjes construction

**Todays question:**

- How do such probability measures arise in practice?

# Measurable Functions

- Let $(\Omega, \mathscr{A})$ and $(\Delta, \mathscr{D})$ be measurable spaces.

A function

$$f : \Omega \to \Delta$$

is measurable if

$$f(D)^{-1} \in \mathscr{A} \text{ for all } D \in \mathscr{D}, \text{ where } f(D)^{-1} := \{\omega \in \Omega : f(\omega) \in D\}$$

Measurability ensures that events defined via the function have well-defined probabilities.

# Random Variables

Let $(\Omega, \mathscr{A}, P)$ be a probability space.

A random variable is a measurable function

$$X : (\Omega, \mathscr{A}) \to \left(\mathbb{R}, \mathscr{B}\right).$$

So, a random variable is just a measurable function with values in $\mathbb{R}$.

$$X^{-1}(B) \in \mathscr{A} \text{ for all } B \in \mathscr{B}, \text{ where } X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\}$$

# Why Measurability Matters

- For any $x \in \mathbb{R}$,

$$\{X \leq x\} = X^{-1}\left((-\infty, x]\right)$$

- Since $(-\infty, x] \in \mathscr{B}$ and $X$ is a random variable, $\{X \leq x\} \in \mathscr{A}$, i.e., $\{X \leq x\}$ is measurable.

- Therefore, $P\{X \leq x\}$ is well-defined.

- Measurability allows us to assign probabilities to events involving $X$.

# Pushforward Measure

Let

$$X : (\Omega, \mathscr{A}, P) \to \left(\mathbb{R}, \mathscr{B}(\mathbb{R})\right)$$

Be a random variable.

Define a measure on $\left(\mathbb{R}, \mathscr{B}(\mathbb{R})\right)$ by

$$\mu_X(B) := P(X \in B) = P\left(X^{-1}(B)\right).$$

This measure is called the pushforward of $P$ by $X$.

# Where Does the Probability Come from?

- Probability is defined on the measurable space $(\Omega, \mathscr{A})$
- A random variable $X$ maps outcomes in $\Omega$ to values in $\mathbb{R}$
- We do not assume a probability measure on $\mathbb{R}$ a priori, instead, the random variable $X$ pushes probability from $\Omega$ to $\mathbb{R}$.

# Distribution Function of a Random Variable

- Define

$$F_X(x) := \mu_X\left((-\infty, x]\right) = P\left(X \leq x\right)$$

One can show that this function is:

- Is non-decreasing
- Is right-continuous
- $\lim_{x\to\infty} F_X(x) = 1, \lim_{x\to-\infty} F_X(x) = 0$

So this is a distribution function!! This is the distribution function of $X$.

# Does Every Distribution Function $F$ has a RV?

- We now know every random variable produces such a function $F$.

- Given every valid distribution function $F$, does there exist a random variable with that distribution? Yes!!

- If $F$ is nondecreasing, right-continuous, & $\lim_{x\to\infty} F_X(x) = 1, \lim_{x\to-\infty} F_X(x) = 0,$ then it is the distribution function of some random variable.

# Proof Idea

- The idea is to create an inverse of $F$
- $\Omega = (0,1); \mathscr{A} = \mathscr{B}\left(0,1)\right); P =$ lebesgue measure (length)
- If $\omega \in (0,1)$, define:

$$X(\omega) = \sup\left\{y : F(y) < \omega\right\}$$

Next, we show $\{\omega : X(\omega) \leq x\} = \{\omega : \omega \leq F(x)\}$, then the result follows by considering the probability of both sides above.

Proof intuition: We are proving $X$ indeed behaves like $F^{-1}$.

# Why This Matters?

- We call this $X$ the inverse of $F$ and denote it by

$$F^{-1}(\omega) := X(\omega) = \sup\left\{y : F(y) < \omega\right\}$$

- Distribution functions are not abstract objects
- They always correspond to random variables
- Used for generating random variable with distribution function $F$.

If $U \sim$ Uniform $(0,1)$, then $X := F^{-1}(U)$ satisfies $P(X \leq x) = F(x)$.
Thus $F^{-1}(U)$ has distribution function $F$.

# Closure Properties of Random Variables

Let $X_1, X_2, \cdots, X_n$ be random variables, then

- $Y := \sup_n X_n$ (pointwise supremum) is a random variable

Using similar arguments

- $\inf_n X_n$ , $\liminf_n X_n$, $\limsup_n X_n$ are random variables

In generally,

If $X_1, X_2, \cdots, X_n$ are random variables and $f : \mathbb{R}^n \to \mathbb{R}$ is measurable, then $f\left(X_1, X_2, \ldots, X_n\right)$ is a random variable.

# How Large Is the Class of Measurable Functions?

Very large! In particular, all of the following are measurable:

- All continuous functions; $f(x, y) = x + y$
- All polynomials $f(x) = x^2 + 2x + 5$
- All piecewise continuous functions (finitely or countably many pieces)
- Indicator functions of Borel sets $f(x) = \mathbf{1}_{[0,\infty]}(x)$
- Functions built from limits, sums, products, maxima, minima (i.e. statistics and estimators)

# Why This Matters?

This closure result guarantees that:

- Sums, products, and ratios (when defined) of random variables are random variables
- Statistics computed from data (means, maxima, regression coefficients) are random variables
- Limits of estimators are random variables