STAT 220A — Midterm Exam

Time: 2 hours Open notes

This exam is graded out of 115 points; final scores will be capped at 100. You may use the statement of any earlier part of a question in later parts, even if you did not complete the earlier part.

1. **(20 points)** Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables.

   (a) Define
   $$\mathcal{F}_0 := \bigcup_{k=1}^{\infty} \sigma(X_1, \ldots, X_k).$$
   Show that $\mathcal{F}_0$ is a field (algebra of sets). (10 points)

   (b) Conclude that
   $$\sigma(\mathcal{F}_0) = \sigma(X_1, X_2, \ldots).$$
   (10 points)

2. **(50 points) Finite vs. countable additivity on the cofinite algebra**

   Let Ω be an infinite set, and define
   $$\mathcal{F} = \{A \subset \Omega : A \text{ is finite or } A^c \text{ is finite}\}.$$

Define $\lambda : \mathcal{F} \to \{0, 1\}$ by

$$\lambda(A) = \begin{cases} 0, & \text{if } A \text{ is finite,} \\ 1, & \text{if } A^c \text{ is finite.} \end{cases}$$

(a) Show that λ is finitely additive on $\mathcal{F}$, i.e. if $A, B \in \mathcal{F}$ are disjoint, then

$$\lambda(A \cup B) = \lambda(A) + \lambda(B).$$

(12 points)

(b) Assume that Ω is countably infinite. Show that λ is not countably additive. *(Hint: consider singleton sets.)* (13 points)

(c) Assume that Ω is uncountable. Let $(A_n)_{n\geq 1} \subset \mathcal{F}$ be pairwise disjoint such that

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

Show that

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n).$$

Hint: First show that at most one of the complements $A_1^c, A_2^c, \ldots$ can be finite. (25 points)

3. **(20 points)** Evaluate the limit

$$\lim_{n\to\infty} \int_0^1 \Big(1 - \cos(x/\sqrt{n})\Big)\, x^{-1/2}\, dx.$$

4. **(25 points)** Let F be a distribution function on $\mathbb{R}$. For $x \in \mathbb{R}$, define the jump size

$$\Delta F(x) := F(x) - F(x^-), \qquad F(x^-) := \lim_{t\uparrow x} F(t).$$

You may assume that $F(x^-)$ exists for all x and that F is discontinuous at x if and only if $\Delta F(x) > 0$.

Let

$$D := \{x \in \mathbb{R} : \Delta F(x) > 0\}$$

be the set of discontinuities of F.

(a) For $k \in \mathbb{N}$, define

$$D_k := \{x \in \mathbb{R} : \Delta F(x) \geq 1/k\}.$$

Show that for any finite subset $\{x_1, \ldots, x_m\} \subset D_k$ with $x_1 < \cdots < x_m$,

$$\sum_{j=1}^{m} \Delta F(x_j) \leq F(x_m) - F(x_1^-) \leq 1.$$

(10 points)

(b) Deduce that D_k is finite. (8 points)

(c) Conclude that D is at most countable. (7 points)

Best of luck.