220A Discussion 4

Throughout, all random variables are defined on a common probability space $(\Omega, \mathcal{F}, P)$.

1. Let $X \geq 0$ be a random variable with tail distribution

$$P(X > t) = \frac{1}{1+t}, \qquad t \geq 0.$$

Define the truncated random variables

$$X_n := X \wedge n = \min(X, n).$$

(a) Compute $\mathbb{E}[X_n]$.

(b) Determine whether $\mathbb{E}[X]$ is finite or infinite.

(c) Does $\mathbb{E}[X_n] \to \mathbb{E}[X]$? In what sense?

(d) Which convergence theorem explains your conclusion?

Hint. If $Z \geq 0$ is a nonnegative random variable, then

$$\mathbb{E}[Z] = \int_0^\infty P(Z > t)\, dt.$$

(This identity will be justified rigorously later using Tonelli's theorem.)

2. Let $U \sim \text{Uniform}(0, 1)$ and define

$$X_n := n\, \mathbf{1}_{(0,1/n)}(U).$$

(a) Compute $\mathbb{E}[X_n]$.

(b) Find the almost sure limit of X_n.

(c) Compare

$$\lim_{n\to\infty} \mathbb{E}[X_n] \quad \text{and} \quad \mathbb{E}\Big[\lim_{n\to\infty} X_n\Big].$$

(d) For each of the following results — Fatou's lemma, the Dominated Convergence Theorem, the Monotone Convergence Theorem, and the Bounded Convergence Theorem — determine whether it applies to this sequence. If it applies, state the conclusion it yields. If it does not apply, identify precisely which assumption fails.