STAT 220A — Midterm Exam: Official Solutions

1. **(20 points)** Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables. Define

$$\mathcal{F}_0 := \bigcup_{k=1}^{\infty} \sigma(X_1, \ldots, X_k).$$

Solution. *For each $k \geq 1$, set $\mathcal{F}_k := \sigma(X_1, \ldots, X_k)$. Then*

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots, \qquad \text{and} \qquad \mathcal{F}_0 = \bigcup_{k\geq 1} \mathcal{F}_k.$$

(a) $\mathcal{F}_0$ ***is a field (algebra).***

We check the axioms.

(i) $\Omega \in \mathcal{F}_0$. *Since each $\mathcal{F}_k$ is a σ-field, $\Omega \in \mathcal{F}_k$ for all k, hence $\Omega \in \mathcal{F}_0$.*

(ii) Closed under complements. *If $A \in \mathcal{F}_0$, then $A \in \mathcal{F}_k$ for some k. Since $\mathcal{F}_k$ is a σ-field, $A^c \in \mathcal{F}_k \subset \mathcal{F}_0$.*

(iii) Closed under finite unions. *If $A, B \in \mathcal{F}_0$, choose k, ℓ such that $A \in \mathcal{F}_k$ and $B \in \mathcal{F}_\ell$. Let $m = \max\{k, \ell\}$. Because the sequence is increasing, $A, B \in \mathcal{F}_m$. Since $\mathcal{F}_m$ is a σ-field, $A \cup B \in \mathcal{F}_m \subset \mathcal{F}_0$.*

Thus $\mathcal{F}_0$ is a field.

(b) $\sigma(\mathcal{F}_0) = \sigma(X_1, X_2, \ldots)$.

First, for each k, $\mathcal{F}_k \subset \sigma(X_1, X_2, \ldots)$, hence

$$\mathcal{F}_0 = \bigcup_{k\geq 1} \mathcal{F}_k \subset \sigma(X_1, X_2, \ldots),$$

and therefore

$$\sigma(\mathcal{F}_0) \subset \sigma(X_1, X_2, \ldots).$$

Conversely, each X_k is measurable w.r.t. $\mathcal{F}_k \subset \mathcal{F}_0$, hence each X_k is measurable w.r.t. $\sigma(\mathcal{F}_0)$. Therefore the σ-field generated by all X_k must be contained in $\sigma(\mathcal{F}_0)$:

$$\sigma(X_1, X_2, \ldots) \subset \sigma(\mathcal{F}_0).$$

Combining inclusions gives $\sigma(\mathcal{F}_0) = \sigma(X_1, X_2, \ldots)$.

2. **(50 points) Finite vs. countable additivity on the cofinite algebra.**

Let Ω be an infinite set and

$$\mathcal{F} = \{A \subset \Omega : \ A \text{ is finite or } A^c \text{ is finite}\}.$$

Define $\lambda : \mathcal{F} \to \{0, 1\}$ by

$$\lambda(A) = \begin{cases} 0, & A \text{ finite,} \\ 1, & A^c \text{ finite.} \end{cases}$$

Solution. *(a)* ***Finite additivity.*** *Let $A, B \in \mathcal{F}$ be disjoint. We show*

$$\lambda(A \cup B) = \lambda(A) + \lambda(B).$$

First note: it is impossible that both A^c and B^c are finite. Indeed, if A^c and B^c are finite then $A \cap B$ is cofinite because

$$(A \cap B)^c = A^c \cup B^c$$

would be finite, so $A \cap B \neq \varnothing$ (since Ω is infinite), contradicting $A \cap B = \varnothing$. Hence at most one of A, B can be cofinite.

Now consider cases.

Case 1: A and B finite. *Then $A \cup B$ is finite, so $\lambda(A \cup B) = 0 = \lambda(A) + \lambda(B)$.*

Case 2: A^c is finite (so $\lambda(A) = 1$). *Since $A \cap B = \varnothing$, we have $B \subset A^c$, hence B is finite and $\lambda(B) = 0$. Also*

$$(A \cup B)^c = A^c \cap B^c \subset A^c$$

so $(A \cup B)^c$ is finite and $\lambda(A \cup B) = 1 = 1 + 0 = \lambda(A) + \lambda(B)$. The case B^c finite is symmetric.

Thus λ is finitely additive.

(b) ***Not countably additive when Ω is countably infinite.*** *Assume Ω is countably infinite. Enumerate $\Omega = \{\omega_1, \omega_2, \ldots\}$ and set $A_n = \{\omega_n\}$. Then $A_n \in \mathcal{F}$ and the A_n are pairwise disjoint with $\bigcup_{n \geq 1} A_n = \Omega$. But $\lambda(A_n) = 0$ for all n, so $\sum_{n \geq 1} \lambda(A_n) = 0$, while $\lambda(\Omega) = 1$ because $\Omega^c = \varnothing$ is finite. Hence*

$$\lambda\Big(\bigcup_{n \geq 1} A_n\Big) = \lambda(\Omega) = 1 \neq 0 = \sum_{n \geq 1} \lambda(A_n),$$

so λ is not countably additive.

(c) ***Countably additive when Ω is uncountable (for disjoint unions whose union lies in $\mathcal{F}$).*** *Assume Ω is uncountable. Let $(A_n)_{n \geq 1} \subset \mathcal{F}$ be pairwise disjoint and let*

$$A := \bigcup_{n=1}^{\infty} A_n, \qquad \textit{with } A \in \mathcal{F}.$$

We show

$$\lambda(A) = \sum_{n=1}^{\infty} \lambda(A_n).$$

Claim: At most one of the complements A_n^c can be finite. *If A_i^c and A_j^c are finite with $i \neq j$, then*

$$(A_i \cap A_j)^c = A_i^c \cup A_j^c$$

is finite, so $A_i \cap A_j$ is cofinite and therefore nonempty, contradicting disjointness. $\square$

Now consider two cases.

Case 1: A is finite. *Since $A = \bigcup_n A_n$ is a union of pairwise disjoint sets and A is finite, only finitely many A_n can be nonempty, and every A_n must be finite. Thus $\lambda(A_n) = 0$ for all n, and $\lambda(A) = 0$, so the desired equality holds.*

Case 2: A^c is finite (so A is cofinite). *We show there exists n_0 with $A_{n_0}^c$ finite. If not, then each $A_n \in \mathcal{F}$ must be finite (since it is not cofinite). Therefore A is a countable union of finite sets, hence A is countable. But then A^c is uncountable (because Ω is uncountable), contradicting that A^c is finite. So there exists n_0 with $\lambda(A_{n_0}) = 1$.*

By the claim, for all $n \neq n_0$, A_n^c cannot be finite, hence A_n is finite and $\lambda(A_n) = 0$. Therefore

$$\sum_{n=1}^{\infty} \lambda(A_n) = 1.$$

Also A^c is finite, so $\lambda(A) = 1$. Hence $\lambda(A) = \sum_{n \geq 1} \lambda(A_n)$.

This proves countable additivity in the stated sense.

3. **(20 points)** Evaluate

$$\lim_{n\to\infty} \int_0^1 \big(1 - \cos(x/\sqrt{n})\big) x^{-1/2}\, dx.$$

Solution. *For each fixed $x \in (0,1]$, we have $x/\sqrt{n} \to 0$ as $n \to \infty$, hence*

$$1 - \cos(x/\sqrt{n}) \to 0.$$

Thus the integrand converges pointwise to 0 on $(0,1]$.

To apply dominated convergence, use the standard inequality (valid for all $u \in \mathbb{R}$)

$$0 \leq 1 - \cos u \leq \frac{u^2}{2}.$$

With $u = x/\sqrt{n}$ this gives

$$0 \leq (1 - \cos(x/\sqrt{n}))x^{-1/2} \leq \frac{1}{2} \cdot \frac{x^2}{n} \cdot x^{-1/2} = \frac{1}{2n}\, x^{3/2}.$$

For all $n \geq 1$,

$$(1 - \cos(x/\sqrt{n}))x^{-1/2} \leq \frac{1}{2} x^{3/2},$$

and the function $x \mapsto \frac{1}{2}x^{3/2}$ is integrable on $(0,1)$:

$$\int_0^1 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_0^1 = \frac{2}{5} < \infty.$$

Therefore, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_0^1 (1-\cos(x/\sqrt{n}))x^{-1/2}\,dx = \int_0^1 \lim_{n\to\infty}(1-\cos(x/\sqrt{n}))x^{-1/2}\,dx = \int_0^1 0\,dx = 0.$$

4. **(25 points)** Let F be a distribution function on $\mathbb{R}$ and define

$$\Delta F(x) := F(x) - F(x^-), \qquad F(x^-) := \lim_{t\uparrow x} F(t).$$

Assume $F(x^-)$ exists for all x and F is discontinuous at x iff $\Delta F(x) > 0$. Let

$$D := \{x \in \mathbb{R} : \Delta F(x) > 0\}.$$

For $k \in \mathbb{N}$, define

$$D_k := \{x \in \mathbb{R} : \Delta F(x) \geq 1/k\}.$$

Solution. *(a)* ***Bound on the sum of jumps over a finite subset of D_k.***

Let $\{x_1,\dots,x_m\} \subset D_k$ with $x_1 < \cdots < x_m$. Since F is nondecreasing, for each $j \geq 2$ we have $x_{j-1} < x_j$ and thus

$$F(x_j^-) \geq F(x_{j-1}),$$

because the left limit at x_j is the supremum of values $F(t)$ for $t < x_j$, in particular it is at least $F(x_{j-1})$.

Hence for each $j \geq 2$,

$$\Delta F(x_j) = F(x_j) - F(x_j^-) \leq F(x_j) - F(x_{j-1}),$$

and also $\Delta F(x_1) = F(x_1) - F(x_1^-)$.

Summing these inequalities gives

$$\begin{aligned}\sum_{j=1}^m \Delta F(x_j) &= \Delta F(x_1) + \sum_{j=2}^m \Delta F(x_j)\\ &\leq \big(F(x_1) - F(x_1^-)\big) + \sum_{j=2}^m \big(F(x_j) - F(x_{j-1})\big)\\ &= F(x_m) - F(x_1^-).\end{aligned}$$

Since F is a distribution function, $0 \leq F \leq 1$, so $F(x_m) - F(x_1^-) \leq 1$. Thus

$$\sum_{j=1}^m \Delta F(x_j) \leq F(x_m) - F(x_1^-) \leq 1.$$

(b) D_k ***is finite.*** *Assume for contradiction that* D_k *is infinite. Then for each* $m \in \mathbb{N}$ *we can choose distinct points* $x_1 < \cdots < x_m$ *in* D_k*. By definition of* D_k*, each* $\Delta F(x_j) \geq 1/k$*, so*

$$\sum_{j=1}^{m} \Delta F(x_j) \geq \frac{m}{k}.$$

But part (a) gives $\sum_{j=1}^{m} \Delta F(x_j) \leq 1$*, hence* $\frac{m}{k} \leq 1$ *for all* m*, impossible for* $m > k$*. Therefore* D_k *must be finite.*

(c) D ***is at most countable.*** *If* $x \in D$*, then* $\Delta F(x) > 0$*, so choose* $k \in \mathbb{N}$ *such that* $\Delta F(x) \geq 1/k$ *(e.g. take* $k > \frac{1}{\Delta F(x)}$*). Hence* $x \in D_k$*. Therefore*

$$D \subset \bigcup_{k=1}^{\infty} D_k.$$

Conversely, if $x \in D_k$ *for some* k*, then* $\Delta F(x) \geq 1/k > 0$*, so* $x \in D$*. Thus*

$$D = \bigcup_{k=1}^{\infty} D_k.$$

Each D_k *is finite by part (b), and a countable union of finite sets is countable. Hence* D *is at most countable.*