STAT 220A — Homework 2
Solution Discussion

Problem 1: Setup and goal

Let $F(x) = P(X \le x)$ be a continuous distribution function and define

$$Y := F(X).$$

Goal: Show that $Y \sim \mathrm{Unif}(0,1)$, i.e.

$$P(Y \le y) = y \quad \text{for all } y \in [0,1].$$

Since F may not have a classical inverse, define the generalized inverse

$$F^{-1}(y) := \inf\{x \in \mathbb{R} : \ F(x) \ge y\}.$$

Key step: relating events

Fix $y \in (0,1)$. We claim that

$$\{F(X) \leq y\} = \{X \leq F^{-1}(y)\}.$$

($\subset$) If $X \leq F^{-1}(y)$, then by monotonicity of F,

$$F(X) \leq F(F^{-1}(y)).$$

($\supset$) If $F(X) \leq y$ but $X > F^{-1}(y)$, then $X \geq F^{-1}(y) + \varepsilon$ for some $\varepsilon > 0$. By definition of F^{-1},

$$F(F^{-1}(y) + \varepsilon) \geq y,$$

implying $F(X) \geq y$, a contradiction.

Thus the two events are equal.

Conclusion of Problem 1

By continuity of F,

$$F(F^{-1}(y)) = y, \qquad y \in (0,1).$$

Hence

$$P(Y \leq y) = P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y.$$

For $y = 0$, $P(Y \leq 0) = 0$ since $Y \geq 0$ a.s. For $y = 1$, $P(Y \leq 1) = 1$ since $Y \leq 1$ a.s.

Conclusion: $Y \sim \mathrm{Unif}(0,1)$.

Problem 2: Setup and strategy

Let f_1, f_2 be integrable and suppose

$$\int_A f_1 \, d\mu = \int_A f_2 \, d\mu \quad \text{for all } A \in \mathcal{A}.$$

Define $g := f_1 - f_2$. Then

$$\int_A g \, d\mu = 0 \quad \text{for all } A \in \mathcal{A}.$$

Goal: Show $g = 0$ almost everywhere.

Positive deviations cannot have mass

For $n \in \mathbb{N}$, define

$$A_n := \{g > 1/n\}.$$

If $\mu(A_n) > 0$, then

$$\int_{A_n} g \, d\mu \geq \frac{1}{n}\mu(A_n) > 0,$$

contradicting $\int_{A_n} g \, d\mu = 0$.

Thus $\mu(A_n) = 0$ for all n.

Similarly, defining

$$B_n := \{g < -1/n\},$$

we obtain $\mu(B_n) = 0$ for all n.

Conclusion of Problem 2

Observe that

$$\{g \neq 0\} = \bigcup_{n=1}^{\infty} (A_n \cup B_n).$$

Since countable unions of null sets have measure zero,

$$\mu(\{g \neq 0\}) = 0.$$

Conclusion: $f_1 = f_2$ almost everywhere.

Problem 3: Statement and plan

Let $f \geq 0$ be bounded and measurable. Show that for $p > 0$,

$$\int_\Omega f^p \, d\mu = \int_0^\infty p\lambda^{p-1} \, \mu(\{f > \lambda\}) \, d\lambda.$$

Plan:

1. Verify the identity for indicator functions.
2. Extend to simple functions.
3. Pass to general bounded f by monotone convergence.

Step 1: Indicator functions

Let $f = \mathbf{1}_E$. Then

$$\int_\Omega f^p \, d\mu = \mu(E).$$

On the right-hand side,

$$\mu(\{f > \lambda\}) = \begin{cases} \mu(E), & 0 \le \lambda < 1, \\ 0, & \lambda \ge 1. \end{cases}$$

Hence

$$\int_0^\infty p\lambda^{p-1} \mu(\{f > \lambda\}) \, d\lambda = \int_0^1 p\lambda^{p-1} \mu(E) \, d\lambda = \mu(E).$$

Step 2: Simple functions

Let

$$f = \sum_{i=1}^{n} a_i \mathbf{1}_{E_i}, \qquad a_i \geq 0.$$

- The identity holds for each indicator.
- Both sides are linear in f.

Thus the identity holds for all nonnegative simple functions.

Step 3: General bounded functions

Let (φ_n) be nonnegative simple functions such that

$$\varphi_n \uparrow f.$$

By the Monotone Convergence Theorem,

$$\int_\Omega f^p \, d\mu = \lim_{n\to\infty} \int_\Omega \varphi_n^p \, d\mu.$$

Similarly,

$$\mu(\{\varphi_n > \lambda\}) \uparrow \mu(\{f > \lambda\}),$$

so MCT applies again to the λ-integral.

Conclusion: The identity holds for all bounded $f \geq 0$.

Problem 4: Setup

Consider

$$\lim_{n\to\infty} \int_0^1 (1 - e^{-x^2/n}) x^{-1/2}\, dx.$$

For fixed $x \in (0, 1]$, we have $x^2/n \to 0$, hence

$$1 - e^{-x^2/n} \to 0.$$

Thus the integrand converges pointwise to 0.

Dominating function

For $t \geq 0$, $1 - e^{-t} \leq t$. Applying this with $t = x^2/n$ gives

$$0 \leq 1 - e^{-x^2/n} \leq \frac{x^2}{n} \leq x^2.$$

Therefore

$$0 \leq (1 - e^{-x^2/n})x^{-1/2} \leq x^{3/2}.$$

The function $x^{3/2}$ is integrable on $(0, 1)$:

$$\int_0^1 x^{3/2}\,dx = \frac{2}{5} < \infty.$$

Conclusion of Problem 4

By the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^1 (1 - e^{-x^2/n})x^{-1/2}\, dx = \int_0^1 0\, dx = 0.$$

Final answer: The limit is $\boxed{0}$.