220A Discussion 2

1. Let $\Omega = \mathbb{R}$ and define
$$\mathcal{F}_1 = \sigma\big((-\infty, a] : a \in \mathbb{Q}\big), \qquad \mathcal{F}_2 = \sigma\big((-\infty, b] : b \in \mathbb{R} \setminus \mathbb{Q}\big).$$
   (a) Describe $\mathcal{F}_1$ and $\mathcal{F}_2$ explicitly.

   (b) Compute $\mathcal{F}_1 \cap \mathcal{F}_2$.

   (c) Is it true that
$$\sigma(\mathcal{F}_1 \cap \mathcal{F}_2) = \mathcal{F}_1 \cap \mathcal{F}_2?$$
   (d) Compare $\sigma(\mathcal{F}_1 \cup \mathcal{F}_2)$ with the Borel σ-field on $\mathbb{R}$.

2. Let $F : \mathbb{R} \to [0, 1]$ be a distribution function, and let μ_F be the probability measure induced by F, i.e.
$$F(x) = \mu_F((-\infty, x]) \quad \text{for all } x \in \mathbb{R}.$$
   (a) Show that for any $x \in \mathbb{R}$,
$$\mu_F(\{x\}) = F(x) - \lim_{y \uparrow x} F(y).$$
   (b) Prove that F is continuous at x if and only if $\mu_F(\{x\}) = 0$.

   (c) Give an example of a distribution function that is continuous everywhere except at one point, and identify the corresponding value of $\mu_F(\{x\})$ at that point.