220A Homework 3

1. Let $A \geq 0$ and $B \geq 0$ be independent random variables with $\mathbb{E}[A] < \infty$ and $\mathbb{E}[B] < \infty$. Interpret A as work arriving in one time period and B as service capacity. Define the backlog

$$W := (A - B)^+ = \max\{A - B, 0\}.$$

   (a) **(10)** Prove the tail-integral identity

$$\mathbb{E}[W] = \int_0^\infty \mathbb{P}(A > B + t)\, dt.$$

   (b) **(15)** Using only independence and Tonelli/Fubini, show that for each $t \geq 0$,

$$\mathbb{P}(A > B + t) = \int_0^\infty \mathbb{P}(A > b + t)\, dF_B(b),$$

   where F_B is the distribution function of B. Deduce that

$$\mathbb{E}[W] = \int_0^\infty \left(\int_0^\infty \mathbb{P}(A > b + t)\, dF_B(b) \right) dt.$$

   (c) **(20)** Assume additionally that $\mathbb{E}[A^2] < \infty$. Use Markov's inequality to prove

$$\mathbb{E}[W] \leq \mathbb{E}[A^2]\, \mathbb{E}\left[\frac{1}{B} \mathbf{1}\{B > 0\} \right] \quad \text{(possibly } = \infty).$$

   (d) **(15)** Prove the deterministic inequality (for all $a \geq 0$, $b > 0$)

$$(a - b)^+ \leq \frac{a^2}{4b}.$$

   Use this to deduce the sharper bound

$$\mathbb{E}[W] \leq \frac{1}{4}\, \mathbb{E}[A^2]\, \mathbb{E}\left[\frac{1}{B} \mathbf{1}\{B > 0\} \right],$$

   whenever the right-hand side is finite.

Hint. Use the identity

$$\mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z > t)\,dt \quad \text{for } Z \geq 0,$$

justify exchanges of integrals using Tonelli/Fubini, and use independence to express joint probabilities in terms of product measures.

2. Let $T_1, T_2 \geq 0$ be independent lifetimes of two components. Define the system lifetimes

$$T_S = \min(T_1, T_2) \qquad \text{(series system)}$$

and

$$T_P = \max(T_1, T_2) \qquad \text{(parallel system)}.$$

Let the survival functions be

$$S_i(t) = \mathbb{P}(T_i > t), \quad i = 1, 2.$$

(a) **(15)** Express $\mathbb{E}[T_S]$ and $\mathbb{E}[T_P]$ in terms of S_1 and S_2.

(b) **(10)** If $T_1, T_2 \overset{iid}{\sim} \mathrm{Exp}(\lambda)$, compute $\mathbb{E}[T_S]$ and $\mathbb{E}[T_P]$ explicitly.

(c) **(15)** Assume T_1, T_2 are i.i.d., nonnegative, integrable, and not almost surely constant. Prove that

$$\mathbb{E}[T_P] > \mathbb{E}[T_1] > \mathbb{E}[T_S].$$

Hint. For any nonnegative random variable Z,

$$\mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z > t)\,dt.$$

Apply this identity with $Z = \min(T_1, T_2)$ and $Z = \max(T_1, T_2)$. Rewrite the events

$$\{\min(T_1, T_2) > t\} \quad \text{and} \quad \{\max(T_1, T_2) > t\}$$

in terms of

$$\{T_1 > t\} \quad \text{and} \quad \{T_2 > t\},$$

and use independence.

Due: March 1 2026