Expected Value and Inequalities

Expected Value

Expectation is Lebesgue integration with respect to a probability measure.
Let $(\Omega, \mathcal{F}, P)$ be a probability space. If $X \geq 0$,

$$\mathbb{E}[X] := \int_\Omega X \, dP \in [0, \infty].$$

Expectation may be infinite.

Positive and Negative Parts

For any real-valued X,

$$X^+ = \max(X, 0), \qquad X^- = \max(-X, 0).$$

Define

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$$

whenever at least one term is finite.

Basic Properties

Expectation inherits properties of the integral.
If expectations exist:

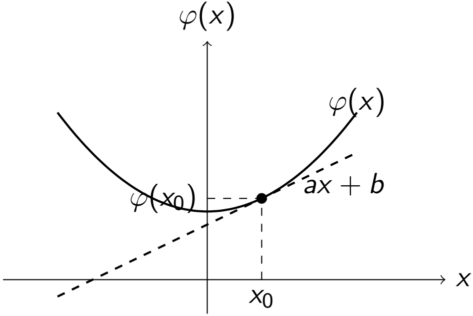
- Linearity: $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- Affine invariance: $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
- Monotonicity: $X \leq Y \Rightarrow \mathbb{E}[X] \leq \mathbb{E}[Y]$

Theorem. If φ is convex and expectations exist, then

$$\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)].$$

Expectation interacts predictably with convex functions.

Key geometric fact (supporting line)



A convex function admits a supporting (tangent) line at every point.

Proof of Jensen's Inequality

Let

$$x_0 := \mathbb{E}[X].$$

From the supporting-line inequality,

$$\varphi(X) \geq aX + b \quad \text{a.s.}$$

Taking expectations and using linearity,

$$\mathbb{E}[\varphi(X)] \;\geq\; a\,\mathbb{E}[X] + b \;=\; ax_0 + b \;=\; \varphi(x_0) \;=\; \varphi(\mathbb{E}[X]).$$

This proves the result.

Jensen example: $g(x) = x^2 - 3x + 3$

Let X satisfy $P(X = 1) = P(X = 3) = \frac{1}{2}$.

Compute the mean:

$$\mathbb{E}[X] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2.$$

Compute both sides of Jensen:

$$g(1) = 1 - 3 + 3 = 1, \qquad g(3) = 9 - 9 + 3 = 3,$$

so

$$\mathbb{E}[g(X)] = \frac{1}{2}g(1) + \frac{1}{2}g(3) = \frac{1}{2}(1 + 3) = 2.$$

Also

$$g(\mathbb{E}[X]) = g(2) = 4 - 6 + 3 = 1.$$

Thus

$$g(\mathbb{E}[X]) = 1 \; < \; 2 = \mathbb{E}[g(X)].$$

Hölder's Inequality

Theorem. Let $p, q \geq 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|Y|^q < \infty$, then

$$\mathbb{E}|XY| \leq \|X\|_p \, \|Y\|_q, \qquad \|X\|_p := (\mathbb{E}|X|^p)^{1/p}.$$

Proof Idea for Hölder

Young's inequality for $a, b \geq 0$:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Apply pointwise to

$$a = \frac{|X|}{\left(\mathbb{E}|X|^p\right)^{1/p}}, \qquad b = \frac{|Y|}{\left(\mathbb{E}|Y|^q\right)^{1/q}},$$

then integrate.

Proof of Hölder (via Young + normalization)

If $\left(\mathbb{E}|X|^p\right)^{1/p} = 0$ or $\left(\mathbb{E}|Y|^q\right)^{1/q} = 0$, then $XY = 0$ a.s. and the claim is trivial. Assume both are positive and define

$$U := \frac{|X|}{\left(\mathbb{E}|X|^p\right)^{1/p}}, \qquad V := \frac{|Y|}{\left(\mathbb{E}|Y|^q\right)^{1/q}}.$$

Then $\mathbb{E}[U^p] = 1$ and $\mathbb{E}[V^q] = 1$.

Apply Young pointwise:

$$UV \le \frac{U^p}{p} + \frac{V^q}{q} \quad \text{a.s.}$$

Taking expectations,

$$\mathbb{E}[UV] \le \frac{1}{p}\,\mathbb{E}[U^p] + \frac{1}{q}\,\mathbb{E}[V^q] = \frac{1}{p} + \frac{1}{q} = 1.$$

Finally,

$$\mathbb{E}|XY| = \left(\mathbb{E}|X|^p\right)^{1/p}\left(\mathbb{E}|Y|^q\right)^{1/q}\,\mathbb{E}[UV] \le \left(\mathbb{E}|X|^p\right)^{1/p}\left(\mathbb{E}|Y|^q\right)^{1/q}. \qquad \square$$

Cauchy–Schwarz Inequality

Corollary (Hölder, $p = q = 2$). If $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$, then

$$\mathbb{E}|XY| \leq (\mathbb{E}[X^2])^{1/2}(\mathbb{E}[Y^2])^{1/2}.$$

Cauchy–Schwarz is the L^2 case of Hölder.

Remark on Cauchy–Schwarz

There is also a direct proof using

$$\mathbb{E}[(X + tY)^2] \geq 0 \quad \text{for all } t \in \mathbb{R},$$

which shows that expectation behaves like an inner product.
We will rely on the Hölder formulation.

Cauchy–Schwarz: direct proof

Proof. For any $t \in \mathbb{R}$,

$$0 \leq \mathbb{E}[(X + tY)^2] = \mathbb{E}[X^2] + 2t\,\mathbb{E}[XY] + t^2\,\mathbb{E}[Y^2].$$

Thus the quadratic

$$Q(t) := \mathbb{E}[X^2] + 2t\,\mathbb{E}[XY] + t^2\,\mathbb{E}[Y^2]$$

satisfies $Q(t) \geq 0$ for all $t \in \mathbb{R}$. Hence its discriminant must be nonpositive:

$$(2\mathbb{E}[XY])^2 - 4\,\mathbb{E}[X^2]\mathbb{E}[Y^2] \leq 0.$$

Therefore,

$$|\mathbb{E}[XY]| \leq \big(\mathbb{E}[X^2]\big)^{1/2}\big(\mathbb{E}[Y^2]\big)^{1/2}. \qquad \square$$

Markov Inequality

Theorem (Markov). If $X \geq 0$ and $a > 0$, then

$$P(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

Proof. On the event $\{X \geq a\}$ we have $X \geq a$, so

$$X \;\geq\; a\,\mathbf{1}_{\{X \geq a\}} \quad \text{a.s.}$$

Taking expectations and using linearity and monotonicity,

$$\mathbb{E}[X] \;\geq\; a\,\mathbb{E}\left[\mathbf{1}_{\{X \geq a\}}\right] = a\,P(X \geq a).$$

Divide by a to obtain the result. $\square$

Apply Markov to X^2.
If $\mathbb{E}[X^2] < \infty$ and $a > 0$,

$$P(|X| \geq a) = P(X^2 \geq a^2) \leq \frac{\mathbb{E}[X^2]}{a^2}.$$

Chebyshev is a corollary of Markov.