220A Homework 2

1. Show that if $F(x) = P(X \le x)$ is continuous then $Y = F(X)$ has a uniform distribution on $(0,1)$, that is, if $y \in [0,1]$, $P(Y \le y) = y$.

   Hint. F may not have a (classical) inverse. Define the generalized inverse
   $$F^{-1}(y) := \inf\{x \in \mathbb{R} : y \le F(x)\}.$$
   Relate the event $\{F(X) < y\}$ to events of the form $\{X < F^{-1}(y)\}$ and $\{X \le F^{-1}(y)\}$. Use continuity of F to show that $F(F^{-1}(y)) = y$ for $y \in (0,1)$. Consider separately the cases: (a) $y \in (0,1)$; (b) $y = 0$; (c) $y = 1$.

2. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space. Let f_1 and f_2 be real-valued measurable functions on Ω. Suppose that f_1 and f_2 are integrable on Ω and that
   $$\int_A f_1 \, d\mu = \int_A f_2 \, d\mu \quad \text{for every } A \in \mathcal{A}.$$
   Show that $f_1 = f_2$ almost everywhere on Ω.

   Hint. Let $g = f_1 - f_2$. For $n \in \mathbb{N}$, consider the sets
   $$A_n = \{g > 1/n\}, \qquad B_n = \{g < -1/n\}.$$
   Show that neither A_n nor B_n can have positive measure, and conclude that $g = 0$ almost everywhere.

3. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space. Let f be a real-valued, non-negative, bounded measurable function on Ω. Show that for any $p \in (0,\infty)$,
   $$\int_\Omega |f|^p \, d\mu = \int_0^\infty p\lambda^{p-1} \, \mu\big(\{\omega \in \Omega : |f(\omega)| > \lambda\}\big) \, d\lambda.$$

Hint. First verify the identity when f is an indicator function. Extend it to simple functions using linearity of the integral. Finally, pass to general bounded nonnegative f using a monotone (or bounded) convergence argument. Explain briefly why each extension step is valid.

4. Evaluate the limit

$$\lim_{n\to\infty} \int_0^1 \left(1 - e^{-x^2/n}\right) x^{-1/2}\, dx.$$

Hint. You may use a convergence theorem (Dominated/Monotone). A closed-form antiderivative is not expected. You may find it useful to note that

$$0 \le 1 - e^{-x^2/n} \le \frac{x^2}{n}.$$

Due: Feb 8 2026