Lebesgue Integration: Limits

Limits and the Integral

Question:

- Are the following two quantities equal?

$$\int \lim_{n\to\infty} f_n \, d\mu \quad \text{vs} \quad \lim_{n\to\infty} \int f_n \, d\mu$$

In generally? No!

Example to keep in mind: $f_n(x) = n\,\mathbf{1}_{(0,\,1/n)}(x)$

Fatou's Lemma

$$f_n \geq 0 \;\Rightarrow\; \int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

Intuition:

- Limits can lose mass
- Limits cannot create mass

$$\liminf f_n = \lim_{n\to\infty} \inf_{k\geq n} f_k$$

- Only mass that persists eventually can be captures and transient spikes are discarded.

Canonical Example: Escaping mass

$$f_n(x) = n\,\mathbf{1}_{(0,\,1/n)}(x)$$

- $f_n(X) \geq 0$
- For every $x \geq 0,\ f_n(x) = 0$ eventually
- Hence $\liminf f_n(x) = 0 \implies \int \liminf f_n(x) = 0$

$$\int_0^1 f_n(x)\,dx = \int_0^{1/n} n\,dx = 1, \text{ so } \liminf_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \geq \int_0^1 \liminf_{n\to\infty} f_n(x)\,dx = 0$$

Monotone Convergence Theorem (MCT)

MCT: $0 \le f_1 \le f_2 \le \cdots, f_n \uparrow f \;\Rightarrow\; \int f_n \, d\mu \uparrow \int f \, d\mu$

$$\int f \, d\mu = \int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu$$

Since $f_n \le f$:

$$\limsup_{n\to\infty} \int f_n \, d\mu \le \int f \, d\mu.$$

$$\int f \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu \le \limsup_{n\to\infty} \int f_n \, d\mu \le \int f \, d\mu$$

Hence equality.

Dominated Convergence Theorem (DCT)

$$f_n \to f \text{ a.e.}, \ |f_n| \le g, \ \int g \, d\mu < \infty \ \Rightarrow \ \int f_n \, d\mu \to \int f \, d\mu$$

$$g + f_n \ge 0 \Rightarrow \int (g+f) \, d\mu \le \liminf_{n\to\infty} \int (g+f_n) \, d\mu \Rightarrow \int f \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu$$

$$g - f_n \ge 0 \Rightarrow \int (g-f) \, d\mu \le \liminf_{n\to\infty} \int (g-f_n) \, d\mu \Rightarrow \int f \, d\mu \le \limsup_{n\to\infty} \int f_n \, d\mu$$

$$\text{so} \quad \int f \, d\mu \le \liminf \int f_n \, d\mu \le \limsup \int f_n \, d\mu \le \int f \, d\mu$$

Bounded Convergence Theorem

$$\mu(E) < \infty,\ f_n = 0 \text{ on } E^c,\ |f_n| \le M,\ f_n \to f \text{ a.e.} \ \Rightarrow\ \int f_n\, d\mu \to \int f\, d\mu$$

$$|f_n| \le M\mathbf{1}_E, \quad \int M\mathbf{1}_E\, d\mu = M\mu(E) < \infty \quad \Rightarrow \quad \text{apply DCT}$$