220A Homework 1

1. A *monotone class* is a collection $\mathcal{M}$ that is closed under countable monotone unions and countable monotone intersections, i.e.,

   - If $A_1, A_2, \ldots \in \mathcal{M}$ and $A_1 \subseteq A_2 \subseteq \cdots$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$.
   - If $B_1, B_2, \ldots \in \mathcal{M}$ and $B_1 \supseteq B_2 \supseteq \cdots$, then $\bigcap_{i=1}^{\infty} B_i \in \mathcal{M}$.

   Show that a field $\mathcal{F}$ is a $\sigma-$field if and only if it is a monotone class.

2. Let $\mathcal{H}$ be a semi-field on Ω, and let $\overline{\mathcal{H}}$ denote the collection of all finite disjoint unions of sets in $\mathcal{H}$. Show that

$$\sigma(\mathcal{H}) = \sigma(\overline{\mathcal{H}}).$$

3. Show that the "countable-cocountable" σ-field is the σ-field generated by the "finite-cofinite" field.

4. Suppose $\mathcal{F}$ is a σ-field on Ω, and $\mu : \mathcal{F} \mapsto [0, \infty]$ is a finitely additive and countably subadditive set function. Show that μ is also countably additive.
   Hint: Start with countably many disjoint sets and show that finite additivity implies countable superadditivity for disjoint sets.

Due: January 26 2026