TP8_FS_and_FT

March 7, 2024

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Imagerie Numérique 2024 Printemps

March 8, 2024

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TP Class N°8 - Fourier Series and Fourier Transforms

Instructions :

- This TP should be completed and uploaded on Moodle before **Thursday 21 March 2024, 23h59**.
- The name of the file you upload should be **TP8_name_surname.ipynb**.
- If you need to include attached files to you TP, please archive them together in a folder named **TP8_name_surname.zip**.

0.0.1 Exercise 1 : Fourier Series expansion

(2 points)

We consider the following function (square function):

$$\tilde{f}(t) = \begin{cases} 1 & \text{if } t \in [-1, 1] \\ -1 & \text{if } t \in [1, 3] \end{cases}$$

Repeated periodically on all $\mathbb{R}$.

(a) Compute the periode T and the frequency ω_0 and show, by computing the integral, that

$$a_k = \begin{cases} (-1)^{(k-1)/2} \cdot \frac{4}{k\pi} & \text{if k is odd} \\ 0 & \text{if k is even} \end{cases}$$

Hint : Use the formula on page 42 but choose the interval of integration wisely to cut the integral into two parts. You may also need trigonometric formulas from Theme 7 page 30.

> *Write your answer here using latex*
> *Please use the > symbol at the beginning of each answer line*
> *To differentiate them from the instructions*

(b) Explain why $b_k = 0$ and compute F_k. Write the function $\tilde{f}(t)$ both in sinusoidal form and exponential form.

Write your answer using latex

(c) We write

$$f_N(t) = \frac{1}{2}a_0 + \sum_{k=1}^{N} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t).$$

Implement the functions $\tilde{f}(t)$ and $f_N(t)$ in Python and plot them on the same graph for $t \in [-5, 5]$. Do several plots for $N = 0, 1, 3, 5, 10, 100$.

[1]:
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def f_tilda(t):
    pass

def f(t, N):
    pass
```

(d) Do a small research and explain what is the Gibbs phenomenon.

Write your answer here

0.1 Exercise 2 : Fourier series and Fourier transform

(2 points)

We consider the following triangular function :

$$h(t) = \begin{cases} 1 - |t| & \text{if } |t| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute its Fourier transform $H(w)$, using integration by part. Why is it a real-valued function ? What is the value $H(0)$?

Hint : Start by decomposing $e^{-itw} = cos(tw) - j \cdot sin(wt)$ in the integral and use symmetry of the functions to show that,

$$H(w) = 2 \cdot \int_0^1 (1 - t)\cos(wt)dt$$

Write your answer here

(b) Fix a real number $T \geq 2$, and consider $h|_{[-T/2,T/2]}$. Define h_T as the periodic extension of this function on all $\mathbb{R}$.

On separate graphs, visualize $h(t)$, $h_2(t)$, $h_4(t)$ and $h_8(t)$ for $t \in [-10, 10]$.

[ ]:
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(c) Using a result seen in the course, compute the Fourier coefficients F_k for $k \in \mathbb{Z}$ from the Fourier transform.
Note that your result will depend on the period T.

Write your answer here

(d) Plot on the same graph the function $H(w)$ and the points $(\frac{2\pi k}{T}, T \cdot F_k)$ for

$$|k| \leq 3T \text{ and } w \in [-20, 20].$$

Do a different graph for each choice of $T = 2, 4, 8, 50, 100$.
Interprete the result based on the theory seen in class.

[ ]:

0.2 Exercise 3 : Fourier Transform of Gaussian signal

(2 points)

Using complex analysis, one can show that that the Fourier transform of a Gaussian function is again a Gaussian function. Namely :

$$g(t) = e^{-\alpha t^2} \implies G(w) = \sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{w^2}{4\alpha}}$$

for any real number $\alpha > 0$.

(a) Represent both functions side-by-side on two different plots for $\alpha = 0.1, 1, 2, 10$. Explain the scaling effect that you observe, based on the property 7 of the FT (Theme 7 : p.80).

[ ]:

(b) The energy of a function $f(x)$ is given by

$$E(f) = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Using your knowledge of Gaussian integrals, compute the energy of both $g(t)$ and $G(w)$ for a real $\alpha > 0$.
Show that $E(g) = \frac{1}{2\pi} E(G)$.
Hint : Starting from $E(g)$ and $E(G)$, do a change of variable and use the equality :

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Write your answer here

(c) Write a python function *represent_complex_signal*(x, y), where

- x is a real numpy array representing an interval range in $\mathbb{R}$.
- y is a complex numpy array defined by $y = f(x)$.

The function should plot four different representations of the pairs (x, y). Namely :

- Real part : using the pair \$(x, \$ $np.real(y))$
- Imaginary part : using the pair \$(x, \$ $np.imag(y))$
- Magnitude : using the pair \$(x, \$ $np.abs(y))$
- Phase : using the pair \$(x, \$ $np.angle(y))$

[3]:
```
def represent_complex_signal(x,y):
    pass
```

(d) For the rest of the exercise, we fix $\alpha = 1$. We define $h(t) = g(t-1)$.

- Compute explicitely $H(w)$ using the time shift property of the FT.
- Represent both $G(w)$ and $H(w)$ using the function you implemented in part (c).

Compare similarities and differences and explain them based on the course.

Write your answer here

(e) We define $i(t) = h(-t)$.

- Compute explicitely $I(w)$ using the correct time-reversal property of the FT.
- Represent both $I(w)$ and $H(w)$ using part (c).

Compare them and explain the differences based on the course.

Write your answer here