Convolution VS correlation

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Convolution: formula

$$(f * w)(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x-s, y-t)$$

Correlation: formula

$$(f \circ w)(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x+s, y+t)$$

Associativity

$$f * (h_1 * h_2) = (f * h_1) * h_2?$$

$$f * (h_1 * h_2)(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} (h_1 * h_2)(s,t) f(x-s, y-t)$$

$$= \sum_{s=-a}^{a} \sum_{t=-b}^{b} \left(\sum_{u=-a}^{s} \sum_{v=-b}^{t} h_2(u,v) h_1(s-u, t-v) \right) f(x-s, y-t)$$

$$= \sum_{-a \leq u \leq s \leq a} \sum_{-b \leq v \leq t \leq b} h_2(u,v) h_1(s-u, t-v) f(x-s, y-t)$$

$$= \sum_{-a \leq u \leq s \leq a} \sum_{-b \leq v \leq t \leq b} h_2(u,v) h_1(s-u, t-v) f(x-s, y-t)$$

$$= \sum_{u=-a}^{a} \sum_{v=-b}^{b} h_2(u,v) \left(\sum_{s=u}^{a} \sum_{t=v}^{b} h_1(s-u, t-v) f(x-s, y-t) \right)$$

$$= \sum_{u=-c}^{c} \sum_{v=-d}^{d} h_2(u,v) \left(\sum_{s=-a}^{a} \sum_{t=-b}^{b} h_1(s-u, t-v) f(x-s, y-t) \right)$$

if we remplace s by s + u and t by t + v, we have:

$$= \sum_{u=-c}^{c} \sum_{v=-d}^{d} h_2(u,v) \left(\sum_{(s+u)=-(a+c)}^{a+c} \sum_{(t+v)=-(b+d)}^{b+d} h_1(s,t) f(x-(s+u), y-(t+v)) \right)$$

$$= \sum_{u=-c}^{c} \sum_{v=-d}^{d} h_2(u,v) \left((f * h_1)(x-u, y-v) \right)$$

$$= ((f * h_1) * h_2)(x,y)$$

$$f * (h_1 * h_2)(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x-s, y-t)$$

and

$$(h_1 * h_2)(x,y) = \sum_{u=-a}^{a} \sum_{v=-b}^{b} h_2(u,v) h_1(x-u, y-v)$$

Commutativity

$$(f * w)(x,y) = \sum_{s} \sum_{t} w(s,t) f(x-s, y-t)$$

If we define s = x - s and t = y - t, we have:

$$\sum_{s} \sum_{t} w(s,t) f(x-s, y-t) = \sum_{x-s} \sum_{y-t} w(x-s, y-t) f(s,t) = (w * f)(x,y)$$

Distributivity of convolution

$$(f * (h_1 + h_2))(x,y)$$

$$= \sum_{s} \sum_{t} (h_1 + h_2)(s,t) f(x-s, y-t)$$

$$= \sum_{s} \sum_{t} (h_1(s,t) + h_2(s,t)) f(x-s, y-t)$$

$$= \sum_{s} \sum_{t} h_1(s,t) f(x-s, y-t) + h_2(s,t) f(x-s, y-t)$$

$$= \sum_s \sum_t h_1(s,t) f(x-s, y-t) + \sum_s \sum_t h_2(s,t) f(x-s, y-t)$$

$$= (f * h_1)(x,y) + (f * h_2)(x,y)$$

Distributivity of correlation

$$(f \circ (h_1 + h_2))(x,y)$$

$$= \sum_s \sum_t (h_1 + h_2)(s,t) f(x+s, y+t)$$

$$= \sum_s \sum_t (h_1(s,t) + h_2(s,t)) f(x+s, y+t)$$

$$= \sum_s \sum_t h_1(s,t) f(x+s, y+t) + h_2(s,t) f(x+s, y+t)$$

$$= \sum_s \sum_t h_1(s,t) f(x+s, y+t) + \sum_s \sum_t h_2(s,t) f(x+s, y+t)$$

$$= (f \circ h_1)(x,y) + (f \circ h_2)(x,y)$$