Université de Genève

Département d'informatique

Imagerie Numérique — FT of typical signals

TP Class N° 9 — March 19, 2024

Warning : The implementation of the sinc(x) function in Numpy is different from the one you've seen in class :

- In class : $\text{sinc}(x) = \frac{\sin(x)}{x}$
- In Numpy : $\text{np.sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

Be aware of this difference when you code. In the TPs, we will use the definition you saw in class.

We try to be compliant with the course and use small letters $f(t)$ for functions in the time domain and capital letters $F(w)$ for their Fourier Transform.

Exercise 1. Fourier Transform properties *(1.5 points)*

In this exercise, you will prove formulas from page 111 Theme 7 of the course.

(a) The FT of a Gaussian filter is given by : (you don't need to prove this result)

$$g(t,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}, \qquad G(w,\sigma) = \frac{\sqrt{2\pi}}{\sigma} \cdot g(w,\sigma^{-1})$$

- Start by computing the Laplacian of Gaussian :

$$\frac{d^2}{dt^2} g(t,\sigma) = (\frac{t^2}{\sigma^4} - \frac{1}{\sigma^2}) \cdot g(t,\sigma)$$

- Use the time derivative property of the FT and the formula for the FT of the Gaussian to prove that :

$$\mathcal{F}\left((\frac{t^2}{\sigma^4} - \frac{1}{\sigma^2}) \cdot g(t,\sigma) \right) = -\frac{\sqrt{2\pi}}{\sigma} w^2 \cdot g(w,\sigma^{-1})$$

(b) A Gabor filter is the product of a Gaussian signal with a cosine signal :

$$k(t) = \cos(w_0 t) \cdot g(t,\sigma)$$

Use the product theorem and your knowledge of FT for cosine signals to prove that :

$$K(w) = \frac{\sqrt{2\pi}}{2\sigma} \left(g(w - w_0, \sigma^{-1}) + g(w + w_0, \sigma^{-1}) \right)$$

(c) An unsharp mask is given by the following formula :

$$u(t) = (1+\gamma)\delta(t) - \gamma g(t,\sigma)$$

Use the linearity of the FT to prove that :

$$U(w) = (1+\gamma) - \gamma \frac{\sqrt{2\pi}}{\sigma} \cdot g(w,\sigma^{-1})$$

Exercise 2. Continuous signal filtering *(1.5 points)*

Remember from semester 1, that filtering a signal $f(t)$ with a filter $h(t)$ consists in computing the convolution :

$$(f * h)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t - \tau) d\tau$$

Using the convolution theorem for FT, we can see that filtering corresponds to multiplication in the frequency domain : $\mathcal{F}(f * h)(w) = F(w) \cdot H(w)$. The goal is now for you to explore the effect of filtering a continuous signal $f(t)$, with the filters from exercise 1 viewed in frequency domain.

For the illustration, you can take the signal $F(w) = \text{sinc}^2(w)$.

(a) On four different graphs, represent :

- The Gaussian filter $g(t, \sigma)$ with $\sigma = 1$,
- its Fourier transform $G(w, \sigma)$,
- The signal $F(w)$,
- The product $G(w) \cdot F(w)$.

By refering to the convolution theorem, explain the following sentence :
"Filtering a signal with a Gaussian blur kills high frequencies."

(b) Do the same for the Gabor filter with $w_0 = 10$ and explain the following sentence :
"Filtering with a Gabor filter at frequency w_0 only keeps the parts of the signal which have a frequency close to w_0."

(c) Repeat again the same steps for the unsharp mask with $\gamma = 1.5$ and explain :
"The Unsharp mask filtering sharpens the signal by increasing high frequency components."

Hint : Use the range $t \in [-10, 10]$ and $\omega \in [-20, 20]$ for all three parts (a) - (c)

Exercise 3. The Dirac comb *(1.5 points)*
The Dirac comb or train of impulses is the following "function":

$$\text{Ш}_T(t) = \sum_{k \in \mathbb{Z}} \delta(t - k \cdot T)$$

(a) Explain why sampling a function at rate $1/T$ is equivalent to multiplying it with $\text{Ш}_T(t)$.

(b) Given any function $f(t)$ defined on $\mathbb{R}$, show that

$$(f * \text{Ш}_T)(t) = \sum_{k \in \mathbb{Z}} f(t - kT).$$

Explain why this function is periodic of period T.
Reminder : The delta function has the property that $f(t) * \delta(t - y) = f(t - y)$.

(c) Illustrate point (b) with the triangular function:

$$h(t) = \begin{cases} 1 - |t| & \text{if } |t| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For various $T = 1,\ 1.5,\ 2,\ 4$, plot on the same graph :

- The shifted functions $h(t - kT)$ on an interval $t \in [-5, 5]$ for each $k \in \mathbb{Z}$.
- The periodic signal $(h * \text{Ш}_T)(t)$

(d) You've seen in the course that

- $\mathcal{F}(\text{Ш}_T) = \frac{2\pi}{T} \cdot \text{Ш}_{\frac{2\pi}{T}}$
- $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$
- $\mathcal{F}(f \cdot g) = \frac{1}{2\pi} \mathcal{F}(f) * \mathcal{F}(g)$

Use these facts and points (a), (b) to explain the following :

i. The Fourier transform of a periodic function is discrete (non-zero only at isolated points). Give an example.

ii. The Fourier transform of a sampled signal (non-zero only at points $\{0, \pm T, \pm 2T, ...\}$ for some $T \in \mathbb{R}$) is a periodic function. Give an example.

iii. If $f(t)$ is both sampled and periodic, then its transform is also sampled and periodic and vice-versa. Give an example.
Remark : This last result plays a central role in the definition of the DFT.

Hint : Start with any signal $f(t)$ and consider the Fourier Transform of $f * \text{Ш}_T$, $f \cdot \text{Ш}_T$ and $(f \cdot \text{Ш}_T) * \text{Ш}_S$.

Exercise 4. Shannon-Nyquist bound *(1.5 points)*

Consider a function $f(t)$ and its Fourier transform $\mathcal{F}(f)(w) = F(w)$.

(a) Sampling f by using the Dirac comb III_T, we showed the following in the class :

$$\mathcal{F}(f \cdot \text{III}_T)(\omega) \quad = \quad \frac{1}{T} \sum_{k \in \mathbb{Z}} F(\omega - k \cdot \omega_0), \quad \text{where } \omega_0 = \frac{2\pi}{T}.$$

Using this formula, explain the phenomenon of folding.

Hint : If this formula looks intimidating, just put $\omega = 0$ and write out the sum explicitely.

(b) Suppose now that $F(w) = 0$, if $|w| > W$. What is the biggest sampling T_{max} such that the folding does not appear ?

Remark : We call $2\pi/T_{max}$ the Shannon-Nyquist rate.

(c) Consider the function $f(t) = \text{sinc}^2(\frac{t}{2})$, which has FT $F(w) = h(w)$ the triangular function (see exercise 3).

- Find W and T_{max}.
- Plot on the same graph, the function $f(t)$ and its sampled version $f_s(t)$ for $T = \frac{\pi}{2}, \pi, 2\pi, 4\pi$.
- On a separate graph, plot the Fourier transform $F_s(w)$ of the sampled signal $f_s(t)$ for each T.
  Hint : Use a range $t \in [-20, 20]$ and a range $w \in [-10, 10]$
- Comment on the signal preservation when sampling if $T < T_{max}$ or $T > T_{max}$.

(d) **(Bonus)** Given a sampled signal $f_s(t)$ at time interval T, one can reconstruct the original signal $f(t)$ by interpolation using the Shannon-Whittaker interpolation formula :

$$f_{approx}(t) = \sum_{k \in \mathbb{Z}} f(k \cdot T) \cdot \text{sinc}(\pi \cdot \frac{t - kT}{T})$$

- Using the same functions as in part (c), for each T, visualise on a same plot, the original function $f(t)$ and its approximation $f_{approx}(t)$.
- Comment your result in view of the Shannon-Nyquist rate.

Submission

Please archive your report and codes in "Name_Surname.zip" (replace "Name" and "Surname" with your real name), and upload to "`Assignments/TP8 FT of typical signals`" on `https://moodle.unige.ch` before Thursday, April 18 2024, 23:59 PM. Note, **the assessment is based not only on your code, but also on your report, which should include your answers to all questions and the experimental results.**