Convolution VS correlation

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Convolution: formula

$$(f * w)(x, y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s, t) f(x - s, y - t)$$

Correlation: formula

$$(f \circ w)(x, y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s, t) f(x + s, y + t)$$

Differences between convolution and correlation

	Convolution	Correlation
Associativity	$f * (h_1 * h_2) = (f * h_1) * h_2$	No
Commutativity	$f * h = h * f$	No
Distributivity	$f * (h_1 + h_2) = f * h_1 + f * h_2$	$f \circ (h_1 + h_2) = f \circ h_1 + f \circ h_2$

Differences between convolution and correlation

To make convolution, we have to flip (180) the filter and then to apply correlation

Convolution	Correlation
filtering operations	pattern matching

Distributivity of convolution

$$
\begin{aligned}
&(f * (h_1 + h_2))(x, y) \\
&= \sum_s \sum_t (h_1 + h_2)(s, t) f(x - s, y - t) \\
&= \sum_s \sum_t (h_1(s, t) + h_2(s, t)) f(x - s, y - t) \\
&= \sum_s \sum_t \left(h_1(s, t) f(x - s, y - t) + h_2(s, t) f(x - s, y - t) \right) \\
&= \sum_s \sum_t h_1(s, t) f(x - s, y - t) + \sum_s \sum_t h_2(s, t) f(x - s, y - t) \\
&= (f * h_1)(x, y) + (f * h_2)(x, y)
\end{aligned}
$$

Distributivity of correlation

$$
\begin{aligned}
&(f \circ (h_1 + h_2))(x, y) \\
&= \sum_s \sum_t (h_1 + h_2)(s, t) f(x + s, y + t) \\
&= \sum_s \sum_t (h_1(s, t) + h_2(s, t)) f(x + s, y + t) \\
&= \sum_s \sum_t \left(h_1(s, t) f(x + s, y + t) + h_2(s, t) f(x + s, y + t)\right) \\
&= \sum_s \sum_t h_1(s, t) f(x + s, y + t) + \sum_s \sum_t h_2(s, t) f(x + s, y + t) \\
&= (f \circ h_1)(x, y) + (f \circ h_2)(x, y)
\end{aligned}
$$

Associativity

$$f * (h_1 * h_2) = (f * h_1) * h_2?$$

$$f * (h_1 * h_2)(x, y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} (h_1 * h_2)(s, t) f(x - s, y - t)$$

$$= \sum_{s=-a}^{a} \sum_{t=-b}^{b} \left(\sum_{u=-a}^{s} \sum_{v=-b}^{t} h_2(u, v) h_1(s - u, t - v) \right) f(x - s, y - t)$$

$$= \sum_{-a \le u \le s \le a} \sum_{-b \le v \le t \le b} h_2(u, v) h_1(s - u, t - v) f(x - s, y - t)$$

$$= \sum_{-a \le u \le s \le a} \sum_{-b \le v \le t \le b} h_2(u, v) h_1(s - u, t - v) f(x - s, y - t)$$

$$= \sum_{u=-a}^{a} \sum_{v=-b}^{b} h_2(u, v) \left(\sum_{s=u}^{a} \sum_{t=v}^{b} h_1(s - u, t - v) f(x - s, y - t) \right)$$

Commutativity

$$(f * w)(x, y) = \sum_s \sum_t w(s, t) f(x - s, y - t)$$

If we define s = x - s and t = y - t, we have:

$$\sum_s \sum_t w(s, t) f(x{-}s, y{-}t) = \sum_{x-s} \sum_{y-t} w(x{-}s, y{-}t) f(s, t) = (w{*}f)(x, y)$$