Imagery numérique

Theme 7
The 2D Discrete Fourier Transform

Content of course

Semester 1

Theme 1: Introduction to image processing

Theme 2: The HVS perception and color

Theme 3: Image acquisition and sensing

Theme 4: Histograms and point operations

Theme 5: Geometric operations

Theme 6: Spatial filters

Content of course

Semester 2

Theme 7: The 2D Discrete Fourier Transform

Theme 8: Frequency domain filtering

Theme 9: Multiresolution Transforms

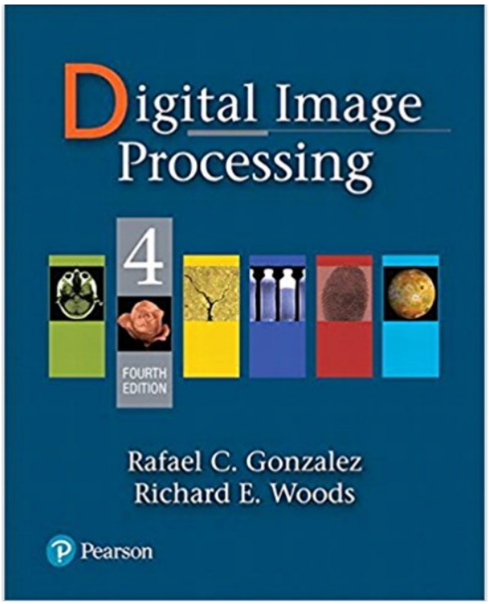
Theme 10: Lossless image coding

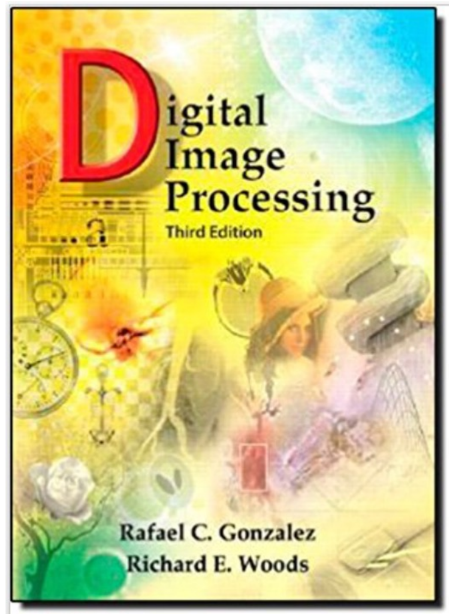
Theme 11: Lossy image compression

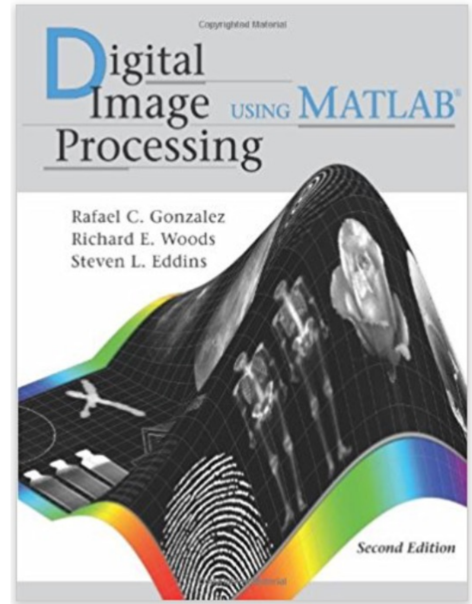
Theme 12: Image denoising

Theme 13: Image restoration

Recommended books







YouTube lectures

Intro to Digital Image Processing (ECSE-4540) Lectures, Spring 2015

by Prof. Rich Radke from Rensselaer Polytechnic Institute

Roadmap for the next lectures

Theme 7: The 2D Discrete Fourier Transform

Chapter 1: Basics of DSP, complex numbers

Chapter 2: Continuous periodic signals – FS

Chapter 3: Continuous aperiodic signals -FT (cont. time – cont. frequency)

Chapter 4: Sampling:

Chapter 4.1: Sampling in time domain: Continuous aperiodic band-limited signals – (sampling in time domain) – cont. frequency – discrete time (DTCF FT)

Chapter 4.2: Sampling in frequency domain: Continuous aperiodic band-limited signals – (sampling in both time and frequency domain) – discrete frequency – discrete time – discrete FT (DFT)

Chapter 4.3: Extension to 2D DFT

Theme 8: Frequency domain filtering, sampling and aliasing

Theme 9: Multiresolution transforms – beyond DFT

Theme 11: Part II. Machine learnable (or data dependent) transforms (in Theme 11: Lossy image compression)

Why do we need image transforms?

Transforms – what for?

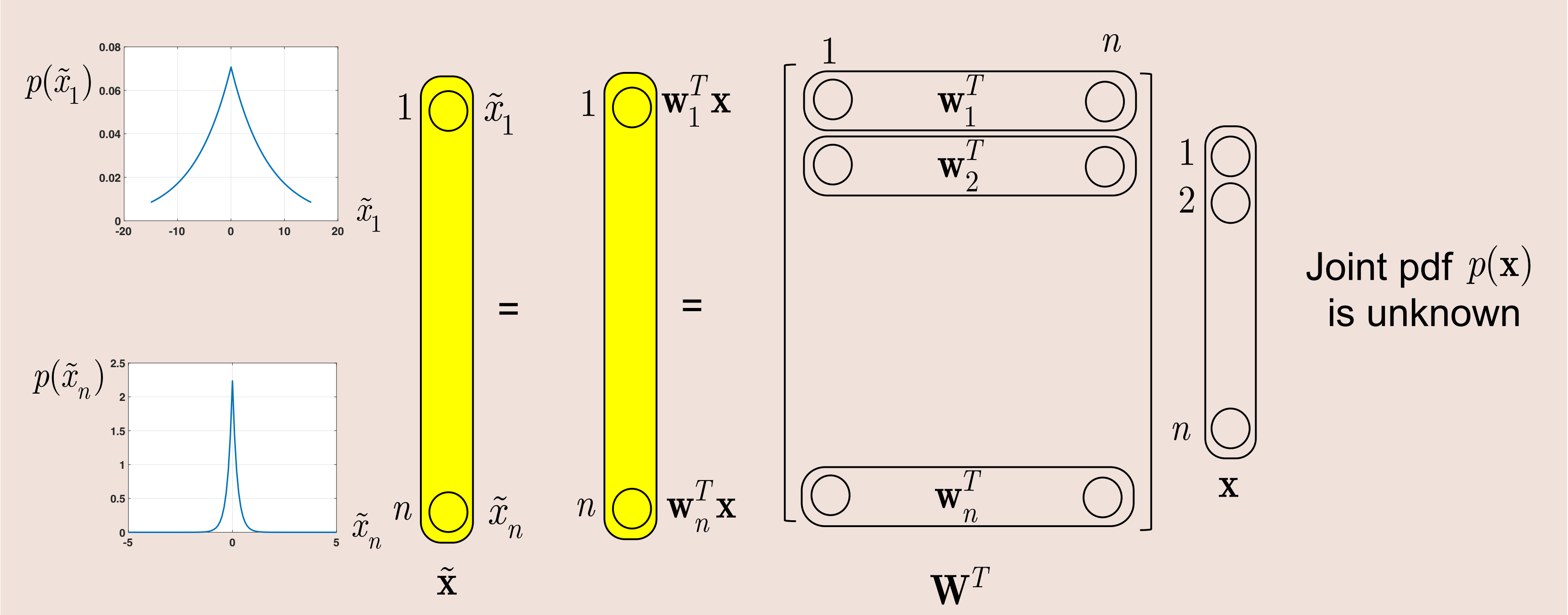
- Simple statistical modeling
- Analysis/Synthesis decomposition
- Interpretability
- Separation of image and distortion/noise statistics and features
- Link to the HVS
- Complexity and efficient computations including parallelization
- Usefulness for many applications: Compression, Recognition, Restoration, Denoising, etc.

Motivation for transform domain processing

- **Main idea**: assume $\mathbf{x}$ can be transformed to some domain where all pixels will be independent, i.e.:

$$p(\mathbf{x}) = p(x_1, x_2, \cdots, x_n) = p(x_1)p(x_2 | x_1)\cdots p(x_n | x_{n-1}, \cdots, x_1) \Rightarrow p(\tilde{\mathbf{x}}) = p(\tilde{x}_1)p(\tilde{x}_2)\cdots p(\tilde{x}_n)$$

- such that we can uniquely transform back to the original vector



Roadmap for the next lectures

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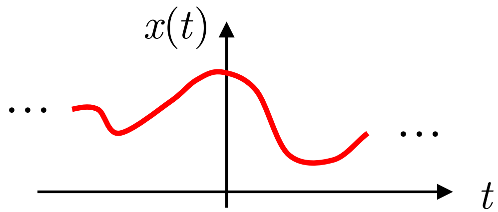
Theme 8: Frequency domain filtering, sampling and aliasing

Theme 9: Unitary (data independent) transforms – beyond DFT

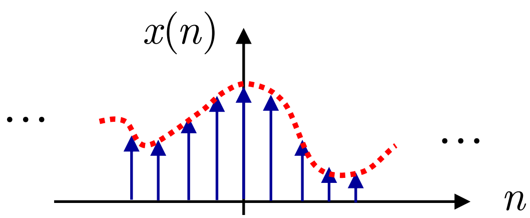
Theme 11: Part II. Machine learnable (or data dependent) transforms (in Theme 11: Lossy image compression)

Recall: introduction into DSP

Continuous signals



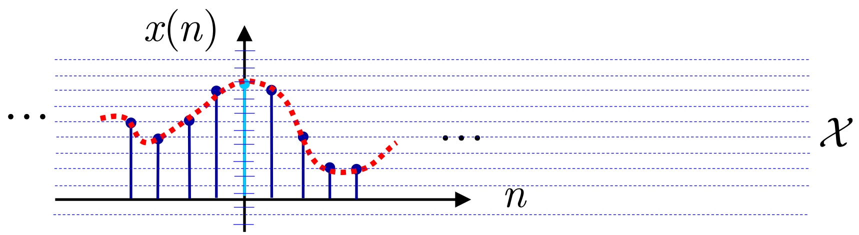
Discrete signals



The discrete signals are defined only a discrete grid $n = 0, \pm 1, \pm 2, \cdots$

Information between the samples is lost!

Discrete quantized signals=digital signals



Additionally, the signal can be quantized, i.e., each sample might take a value from some alphabet $x(n) \in \mathcal{X}$

Information between the quantization levels is lost!

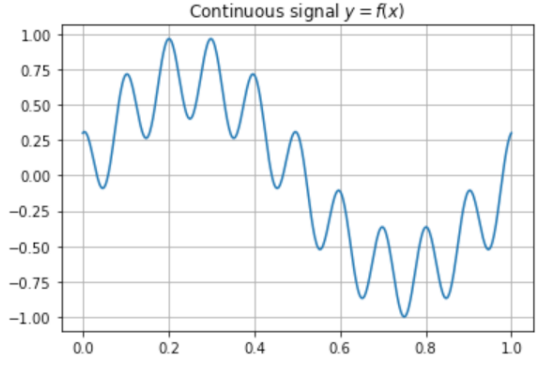
Basic DSP operations

- **Examples in Python**

Continuous signal

```
[ ] import numpy as np
    import matplotlib.pyplot as plt
    import matplotlib

    def f(x):
      return .7*np.sin(2*np.pi*x) + 0.3 * np.cos(20*np.pi*x)
```

```
[ ] x = np.linspace(0, 1, 500)
    y = f(x)

    plt.plot(x,y)
    plt.grid()
    plt.title(r'Continuous signal $y = f(x)$')
    plt.show()
```

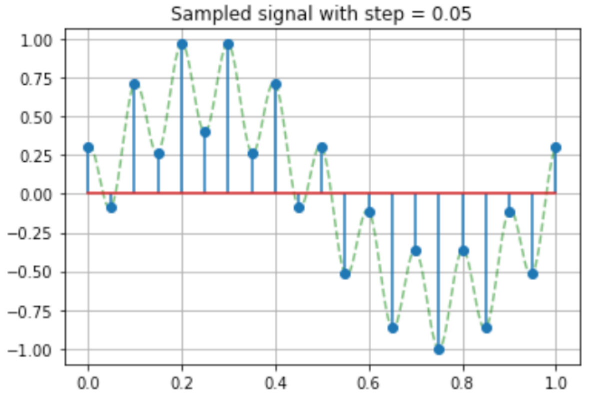


Basic DSP operations

- **Examples in Python**

Sampled signal

```
sample_step = 0.05

x_discrete = np.linspace(0, 1, 1+int(1/sample_step))
y_discrete = f(x_discrete)

plt.plot(x,y, 'g--', alpha=.5)
plt.stem(x_discrete, y_discrete, use_line_collection=True)
plt.grid()
plt.title(f'Sampled signal with step = {sample_step}')
plt.show()
```

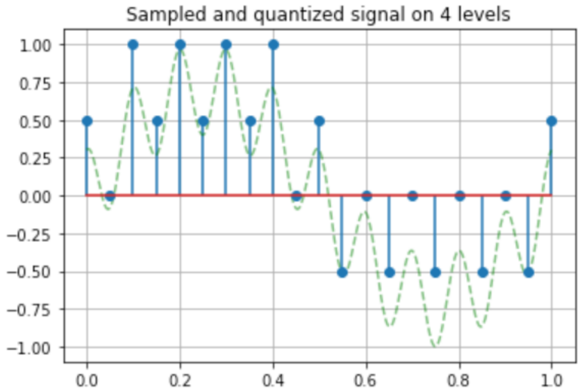


Basic DSP operations

- **Examples in Python**

▾ Quantized and sampled signal

```
[ ] quantize_levels = 4
    bins = np.linspace(-1,1,quantize_levels+1)[1:]

    y_quantized = np.digitize(y_discrete, bins)

    plt.plot(x,y, 'g--', alpha=.5)
    plt.stem(x_discrete, bins[y_quantized], use_line_collection=True)
    plt.grid()
    plt.title(f'Sampled and quantized signal on {quantize_levels} levels')
    plt.show()

    print('bins = ', bins)
```

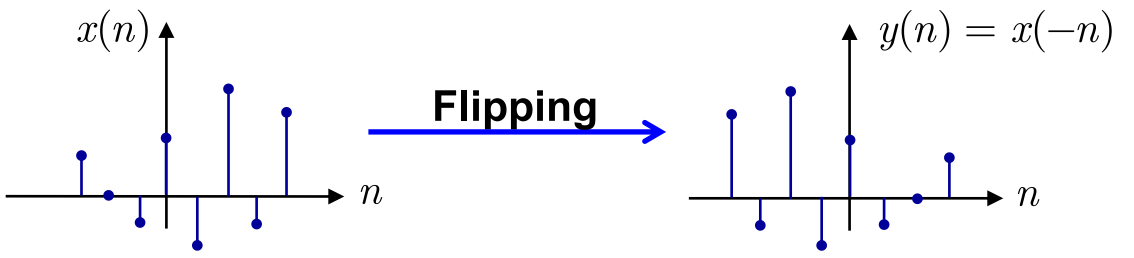



```
bins =  [-0.5  0.   0.5  1. ]
```

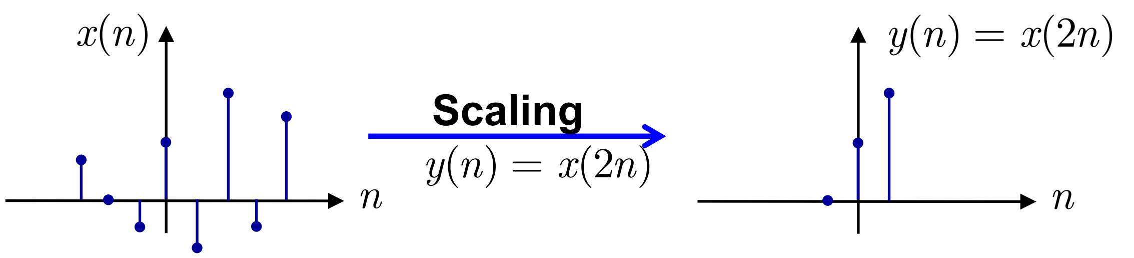
Basic DSP operations

Flipping

The signal is mirrored around the vertical-axe.



Scaling

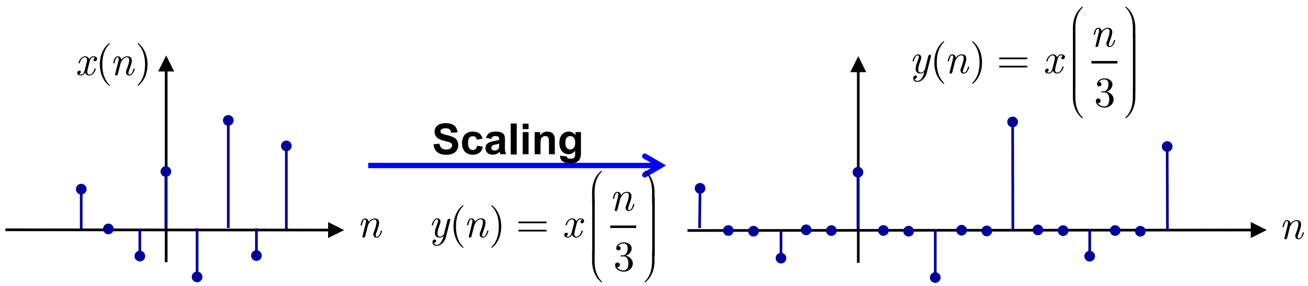


n	$x(2n)$	$y(n)$
0	$x(0)$	$y(0)$
1	$x(2)$	$y(1)$
2	$x(4)$	$y(2)$
3	$x(6)$	$y(3)$
4	$x(8)$	$y(4)$

Hint: think about scaling as a matrix by vector multiplication

Basic DSP operations

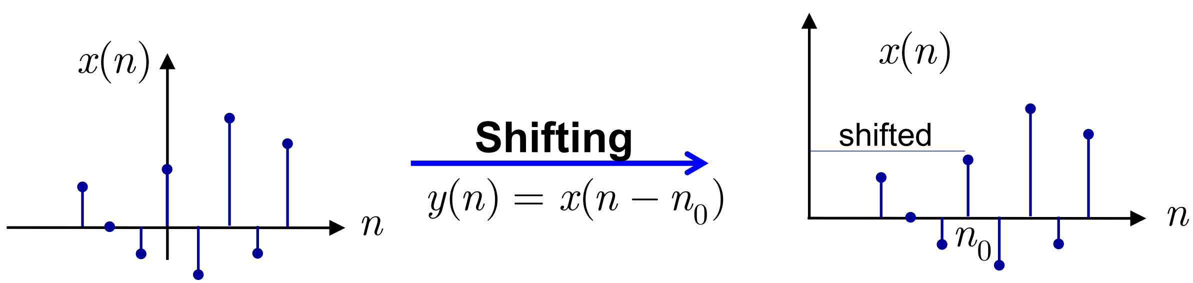
Scaling (*cont*)



n	$x(n/3)$	$y(n)$
0	$x(0)$	$y(0)$
1	0	$y(1)$
2	0	$y(2)$
3	$x(1)$	$y(3)$
4	0	$y(4)$

Shifting

The signal is "delayed"



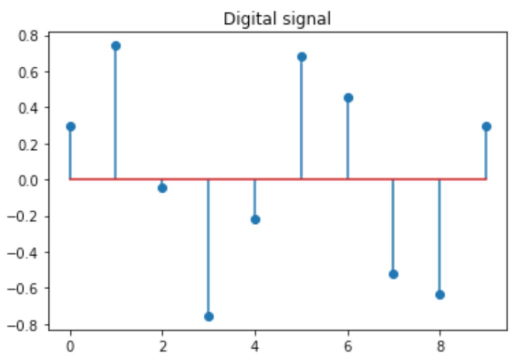
n	$x(n-n_0)$	$y(n)$
0	$x(0)$	$y(n_0)$
1	$x(1)$	$y(n_0+1)$

Basic DSP operations

- **Examples in Python**

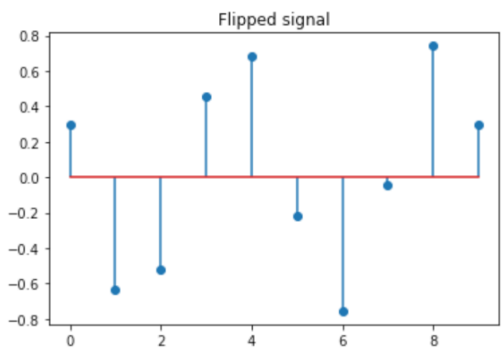
Digital signal

```
[ ] t = np.linspace(-1,1,10)
    s = f(t)
    plt.stem(s, use_line_collection=True)
    plt.title('Digital signal')
    plt.show()
```



Flipping

```
[ ] s_flipped = s[::-1]

    plt.stem(s_flipped, use_line_collection=True)
    plt.title('Flipped signal')
    plt.show()
```

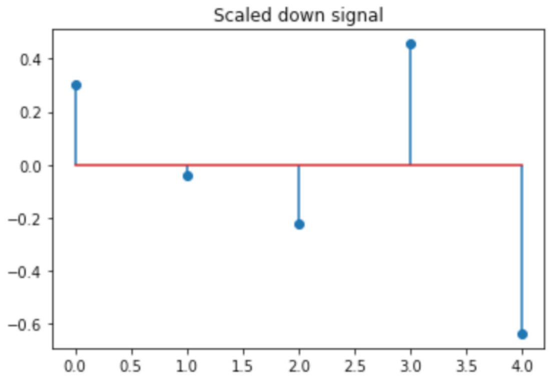


Basic DSP operations

- **Examples in Python**

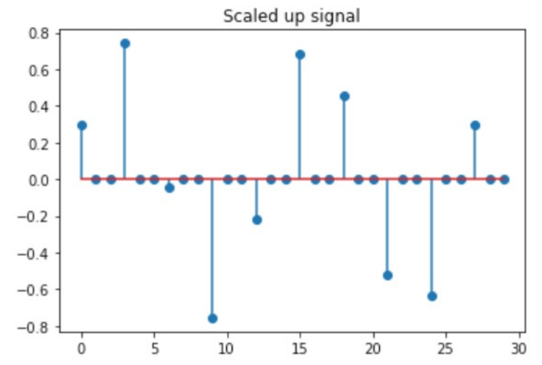
Downscaling

```
[ ] s_scaled_down = s[::2]
    plt.stem(s_scaled_down, use_line_collection=True)
    plt.title('Scaled down signal')
    plt.show()
```



Upscaling

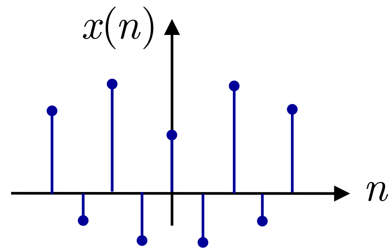
```
[ ] s_scaled_up = np.zeros(3*len(s))
    s_scaled_up[::3] = s
    plt.stem(s_scaled_up, use_line_collection=True)
    plt.title('Scaled up signal')
    plt.show()
```



Basic DSP operations

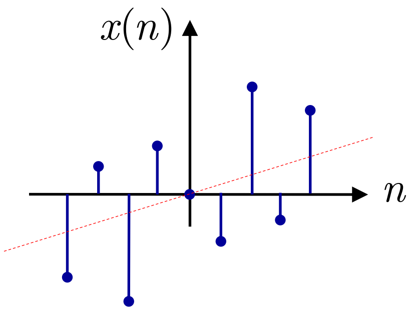
Even signal

$$x(n) = x(-n)$$



Odd signal

$$x(n) = -x(-n)$$



Computing even and odd parts of a given signal

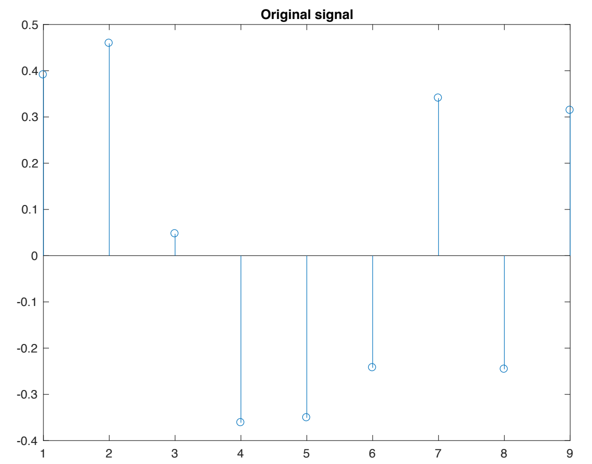
$$Even\left[x(n)\right] = \frac{1}{2}\left(x(n) + (x(-n)\right)$$

$$Odd\left[x(n)\right] = \frac{1}{2}\left(x(n) - (x(-n)\right)$$

Note: even and odd signals are important for real and imaginary parts of the Fourier transform

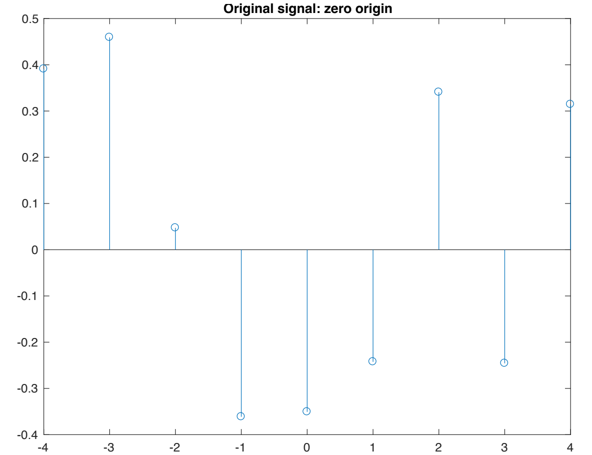
Basic DSP operations

```
% Even and odd signal decomposition

x=rand(1,9)-0.5;

% Show plot
figure(1); stem(x); title('Original signal')
```



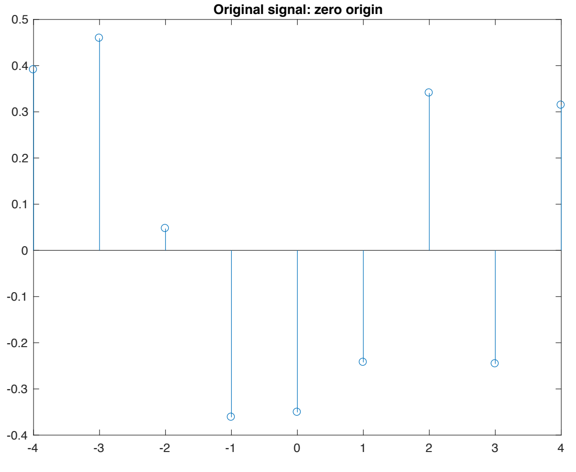
Visualization wrt 0

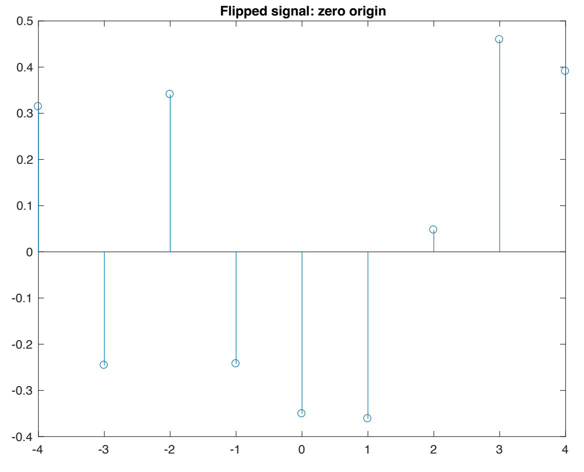
```
% Show plot
figure(2); stem([-4:4],x); title('Original signal: zero origin')
```



Basic DSP operations

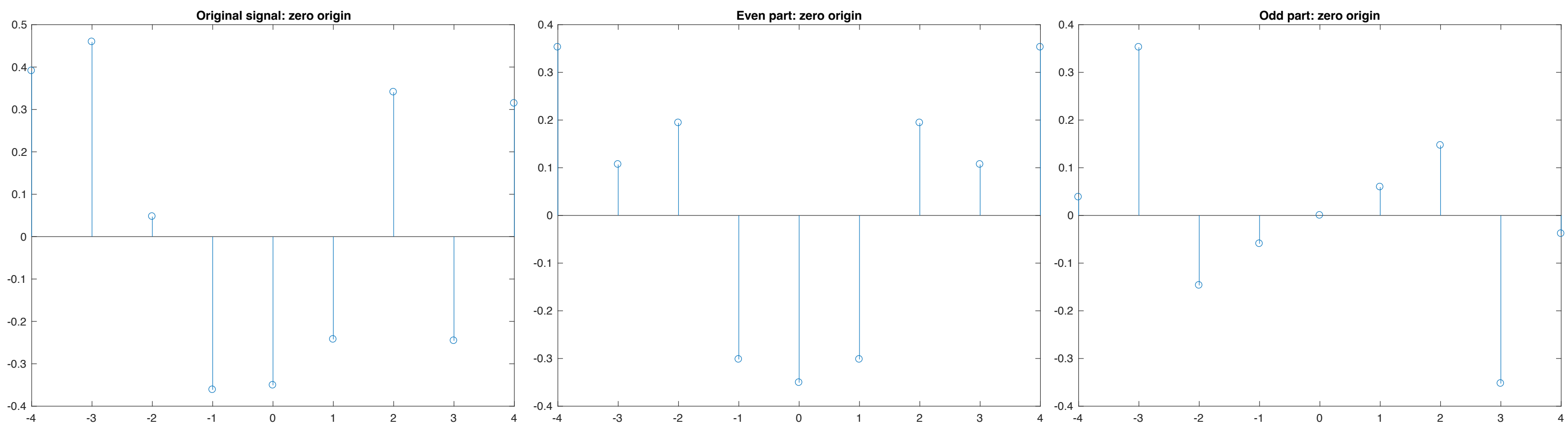
```
% Flipped signal
flipped_x=fliplr(x);
% Show plot
figure(2);stem([-4:4],x); title('Original signal: zero origin');
figure(3);stem([-4:4],flipped_x); title('Flipped signal: zero origin')
```





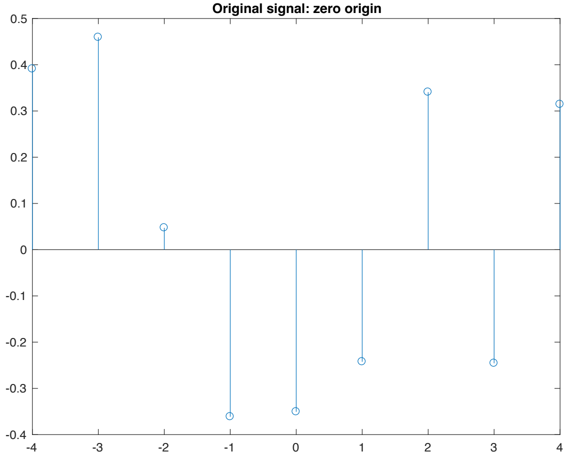
Basic DSP operations

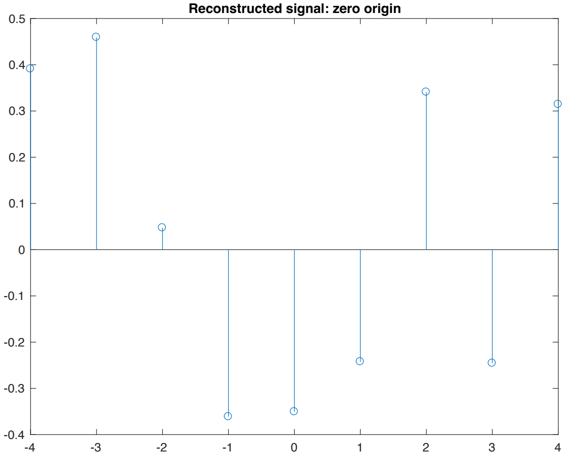
```
evx=1/2*(x+flipped_x);
odx=1/2*(x-flipped_x);
% Show plot
figure(4);stem([-4:4],x); title('Original signal: zero origin');
figure(5);stem([-4:4],evx); title('Even part: zero origin')
figure(6);stem([-4:4],odx); title('Odd part: zero origin')
```



Basic DSP operations

```
% Reconstructed signal
xr=(evx+odx);
figure(7);stem([-4:4],x); title('Original signal: zero origin');
figure(8);stem([-4:4],xr); title('Reconstructed signal: zero origin')
```



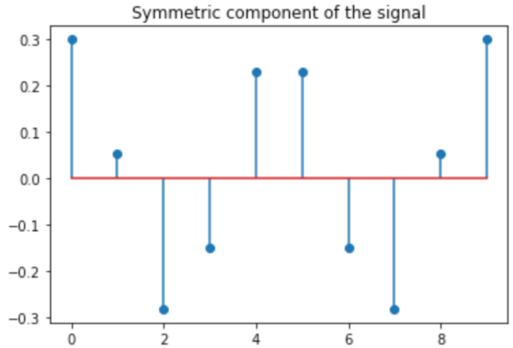


Basic DSP operations

- **Examples in Python**

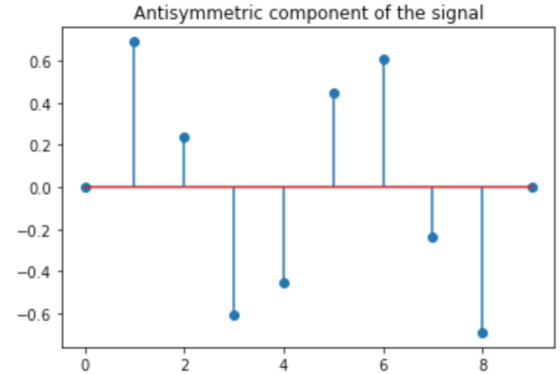
Symmetric component

```
[ ] even_s = (s + s[::-1]) / 2
    plt.stem(even_s, use_line_collection=True)
    plt.title('Symmetric component of the signal')
    plt.show()
```



Antisymmetric component

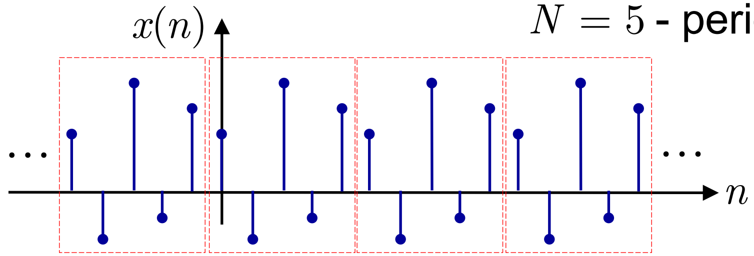
```
[ ] odd_s = (s - s[::-1]) / 2
    plt.stem(odd_s, use_line_collection=True)
    plt.title('Antisymmetric component of the signal')
    plt.show()
```



Basic DSP operations

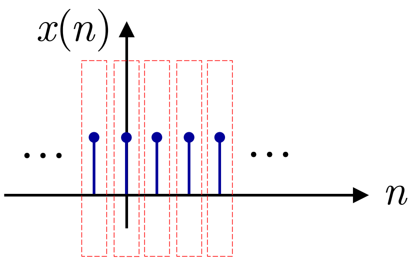
Periodic signals

$$x(n) = x(n+N)$$



$N = 5$ - period

Constant signal is periodic with the period $N = 1$

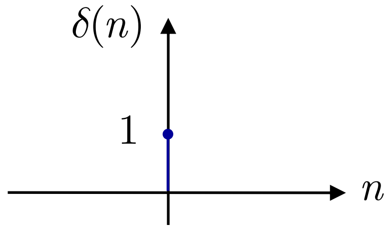


Note: periodic signals play a fundamental role for the discrete Fourier transform

Special signals

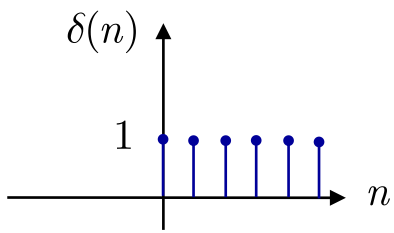
Delta function

$$\delta(n) = \begin{cases} 1, \; n = 0, \\ 0, \;\; \text{otherwise} \end{cases}$$



Unit step function

$$u(n) = \begin{cases} 1, \; n \geq 0, \\ 0, \;\; n<0 \end{cases}$$



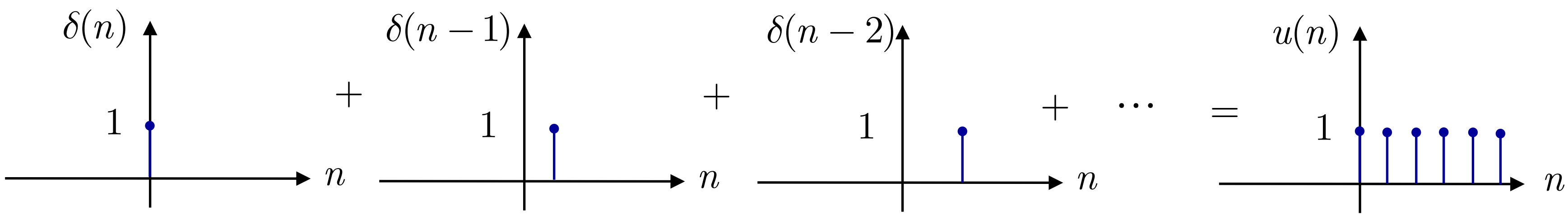
Continuous delta-function

$$\delta(t) = \begin{cases} \infty, \; t = 0, \\ 0, \;\; \text{otherwise} \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Special signals

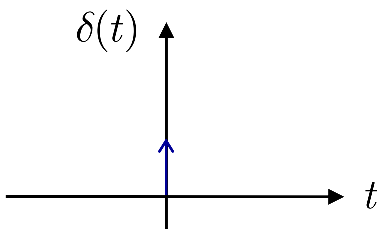
Delta and unit step functions are related

$$u(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \cdots = \sum_{k=0}^{\infty} \delta(n-k)$$

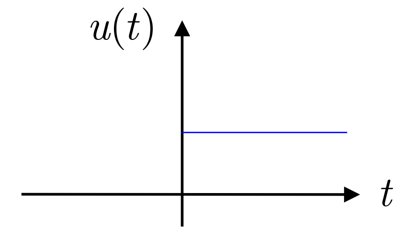


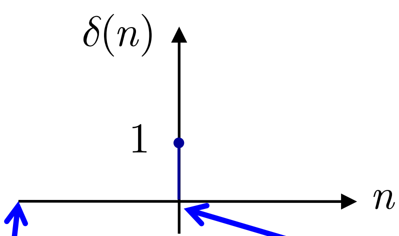
Special signals

Link to continuous delta-function

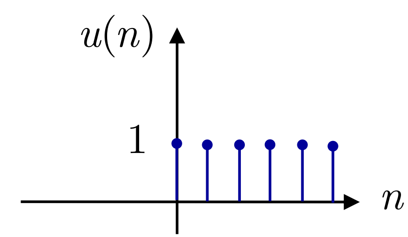


$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$





$$\sum_{k=-\infty}^{n} \delta(k) = u(n)$$



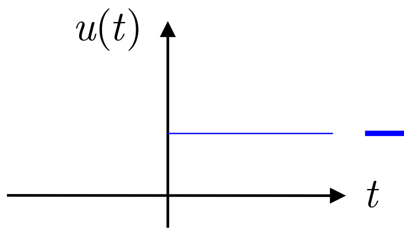
$$n = -3: \sum_{k=-\infty}^{-3} \delta(k) = 0 = u(-3) \qquad n = 0: \sum_{k=-\infty}^{0} \delta(k) = 1 = u(0)$$

$$n = -1: \sum_{k=-\infty}^{-1} \delta(k) = 0 = u(-1) \qquad n = 1: \sum_{k=-\infty}^{1} \delta(k) = 1 = u(1)$$

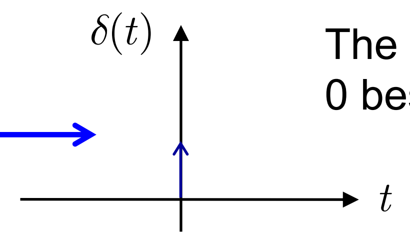
Once we hit 1, we start counting it. Like CDF.

Special signals

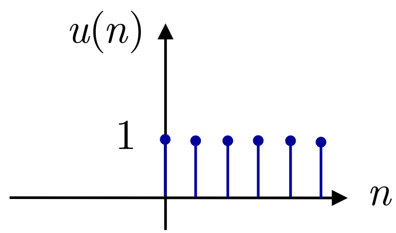
Reverse order

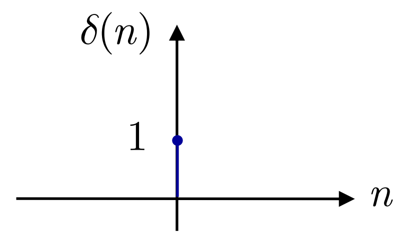


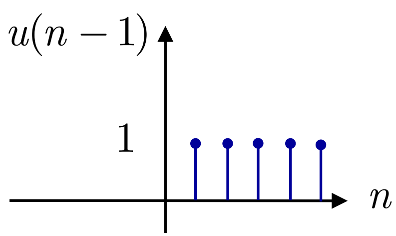
$$\delta(t) = \frac{du(t)}{dt}$$



The derivative is everywhere 0 besides the transition point







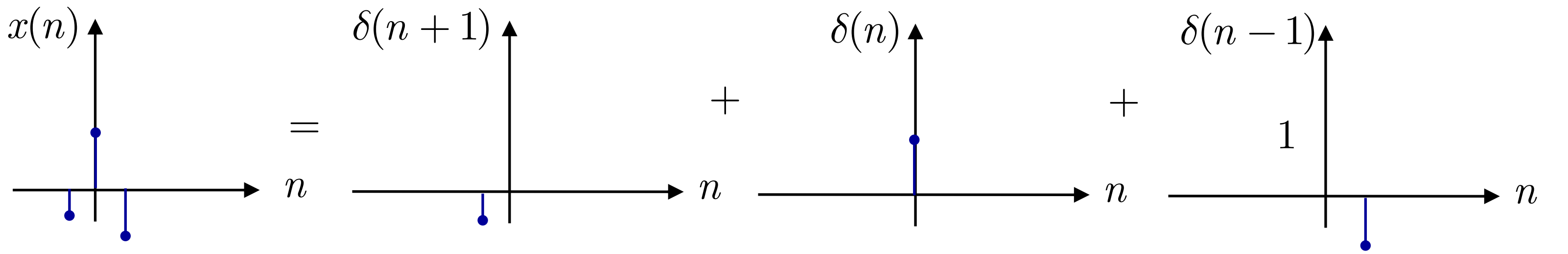
Technically, we can compute the delta-function as a difference between the shifted unit step functions

$$\delta(n) = u(n) - u(n-1) \longrightarrow \delta(t) = \frac{du(t)}{dt}$$

an approximation of the derivative

Construction of signal from delta-function

Any signal can be approximated as a sum of scaled and shifted delta-functions



$$x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1)$$

Superposition basis of delta-functions

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)}$$

Note: this is related to convolution

Sampling property

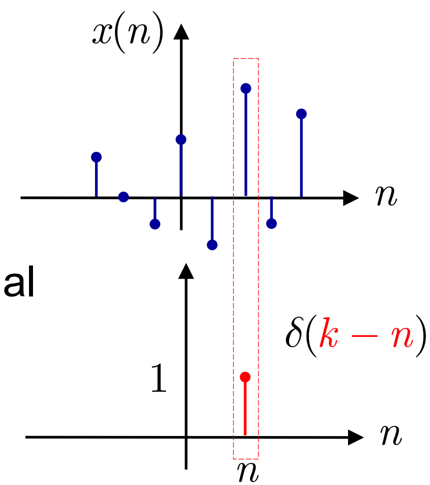
Alternative interpretation: sampling or filtering properly of delta-function

$$x(n) = \sum_{k=-\infty}^{\infty} x\left(k\right)\delta(k-n)$$

Suppose given

We want to get a signal

in the position n



The delta-function is zero besides the position n that is a shifted version $\delta(k-n)$

$$x(n) = x\left(n\right)\delta(n-n) = x\left(n\right)\delta(0) = x\left(n\right)$$

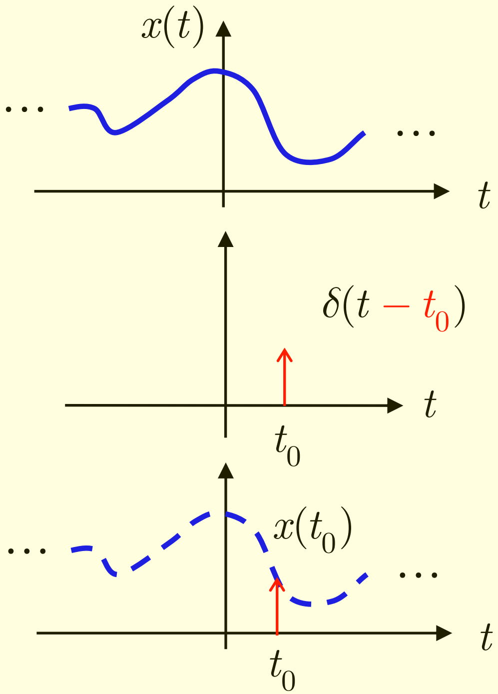
with zeros for all $k \neq n$

Note: pay your attention to the arguments of delta-function

Sampling property

This is also similar to the continuous analog

$$x(n) = \sum_{k=-\infty}^{\infty} x\left(k\right)\delta(k-n)$$



$$x(t_0) = \int_{-\infty}^{-\infty} x\left(t\right)\delta(t-t_0)\,dt$$

Complex numbers: a recall

Cartesian/rectangular coordinates

$$\boxed{C = R + jI}$$

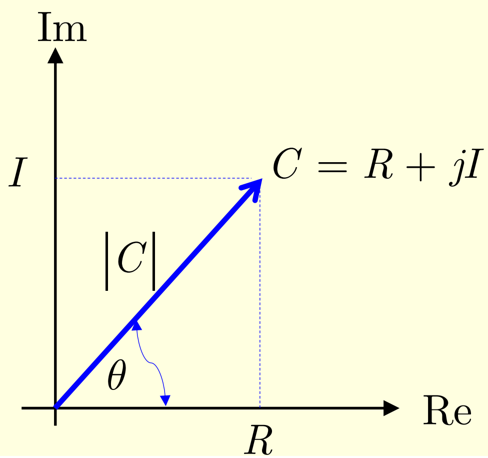
$$R = \mathrm{Re}(C) \qquad j = \sqrt{-1}$$

$$I = \mathrm{Im}(C)$$

Conjugate of C

$$C^* = R - jI$$

Polar coordinates



$|C| = \sqrt{R^2 + I^2}$ - length of vector or **magnitude**

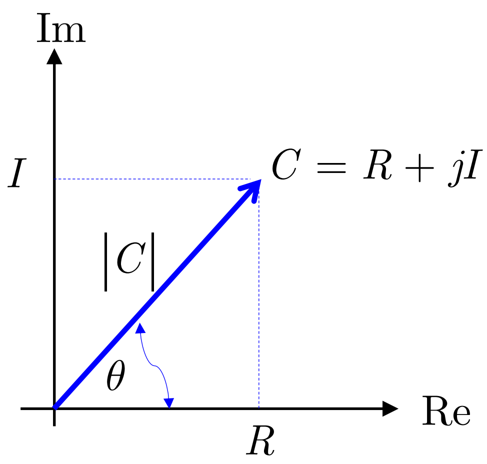
$\tan(\theta) = I/R \Rightarrow \theta = \arctan(I/R) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ - **phase**

For both positive and negative real and imaginary parts, $\theta = \in [-\pi, \pi]$. Multiply by sign of these components.

Matlab function: **atan2(I, R)**

Complex numbers: a recall

Polar coordinates



$$R = |C|\cos(\theta)$$
$$I = |C|\sin(\theta)$$

The **Euler's formula**

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \qquad e = 2.718$$

$$C = |C|e^{j\theta} = |C|\big(\cos(\theta) + j\sin(\theta)\big)$$

$$C = \underbrace{|C|\cos(\theta)}_{\mathrm{Re}} + j\underbrace{|C|\sin(\theta)}_{\mathrm{Im}}$$

Complex numbers: a recall

Complex numbers

```
[ ] import numpy as np
    import matplotlib.pyplot as plt
    import matplotlib
    from matplotlib.patches import Arc
```

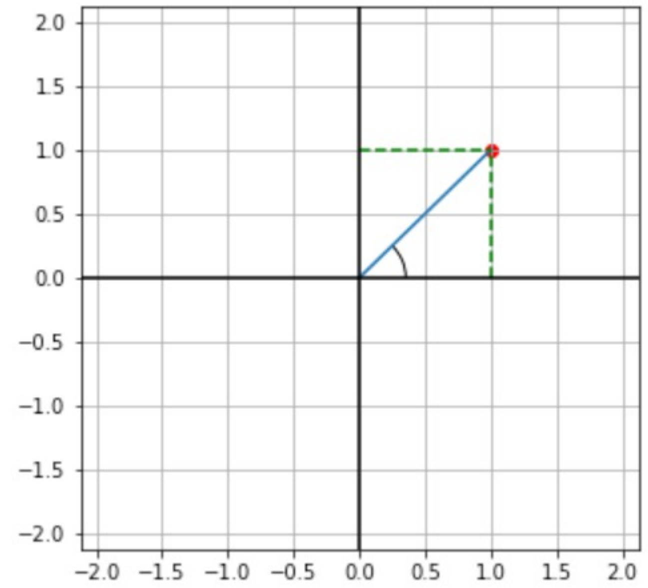
Rectangular form

```
[ ] x = 1
    y = 1

    z = x + 1j*y

    m = np.abs(z)

    print(z, type(z))
    print("")
    print(f'real : {z.real:.02f}')
    print(f'imaginary : {z.imag:.02f}')
    print(f'module : {np.abs(z):.02f} ')
    print(f'phase (radians) : {np.angle(z):.02f}')
```

```
(1+1j) <class 'complex'>

real : 1.00
imaginary : 1.00
module : 1.41
phase (radians) : 0.79
```

Complex numbers: a recall

```
def plot_complex_number(z):

  fig = plt.figure(figsize=(5,5))
  ax = fig.add_subplot(1,1,1)


  m = np.abs(z)

  plt.plot([0, z.real], [0, z.imag], color='tab:blue')
  plt.plot([z.real, z.real], [0, z.imag], 'g--')
  plt.plot([0, z.real], [z.imag, z.imag], 'g--')
  plt.scatter(z.real, z.imag, color='red')

  arc = Arc((0, 0), m / 2, m / 2, 0, 0, np.angle(z, deg=True), color='black', lw=1)
  ax.add_patch(arc)

  plt.axhline(y=0, color='k')
  plt.axvline(x=0, color='k')
  plt.xlim(-1.5*m, 1.5*m)
  plt.ylim(-1.5*m, 1.5*m)
  plt.grid()

  fig.show()


plot_complex_number(z)
```

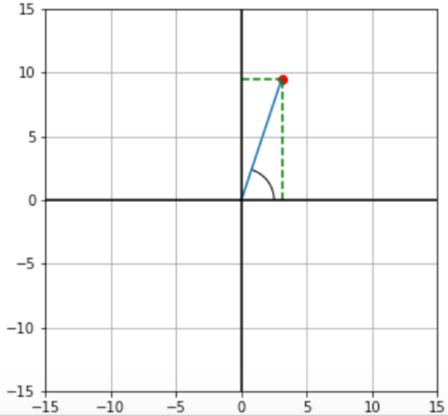


Complex numbers: a recall

Polar form

```
r = 10
theta = 2*np.pi / 5

z = r*np.exp(1j*theta)

print(z.real, z.imag)

print(r*np.cos(theta), r*np.sin(theta))

plot_complex_number(z)
```

```
3.0901699437494745 9.510565162951535
3.0901699437494745 9.510565162951535
```

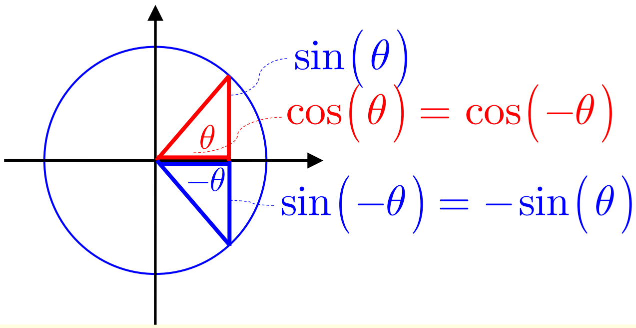


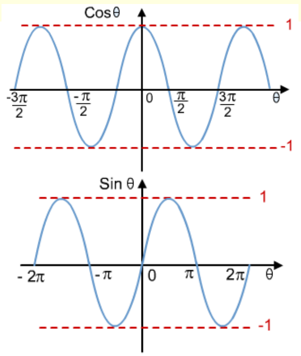
Complex numbers

Conversion formulas

$$\cos(\theta) = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) = \frac{1}{2}\left(\cos(\theta) + j\sin(\theta) + \cos(-\theta) + j\sin(-\theta)\right)\Bigg| \begin{matrix}\cos(-\theta) = \cos(\theta) \\ \sin(-\theta) = -\sin(\theta)\end{matrix}$$

$$= \frac{1}{2}\left(\cos(\theta) + j\sin(\theta) + \cos(\theta) - j\sin(\theta)\right) = \frac{1}{2}2\cos(\theta) = \cos(\theta)$$





$$\sin(\theta) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) = \frac{1}{2j}\left(\cos(\theta) + j\sin(\theta) - \cos(-\theta) - j\sin(-\theta)\right)\Bigg| \begin{matrix}\cos(-\theta) = \cos(\theta) \\ \sin(-\theta) = -\sin(\theta)\end{matrix}$$

$$= \frac{1}{2j}\left(\cos(\theta) + j\sin(\theta) - \cos(\theta) + j\sin(\theta)\right) = \frac{1}{2j}2j\sin(\theta) = \sin(\theta)$$

Sinusoidal function – time domain representation

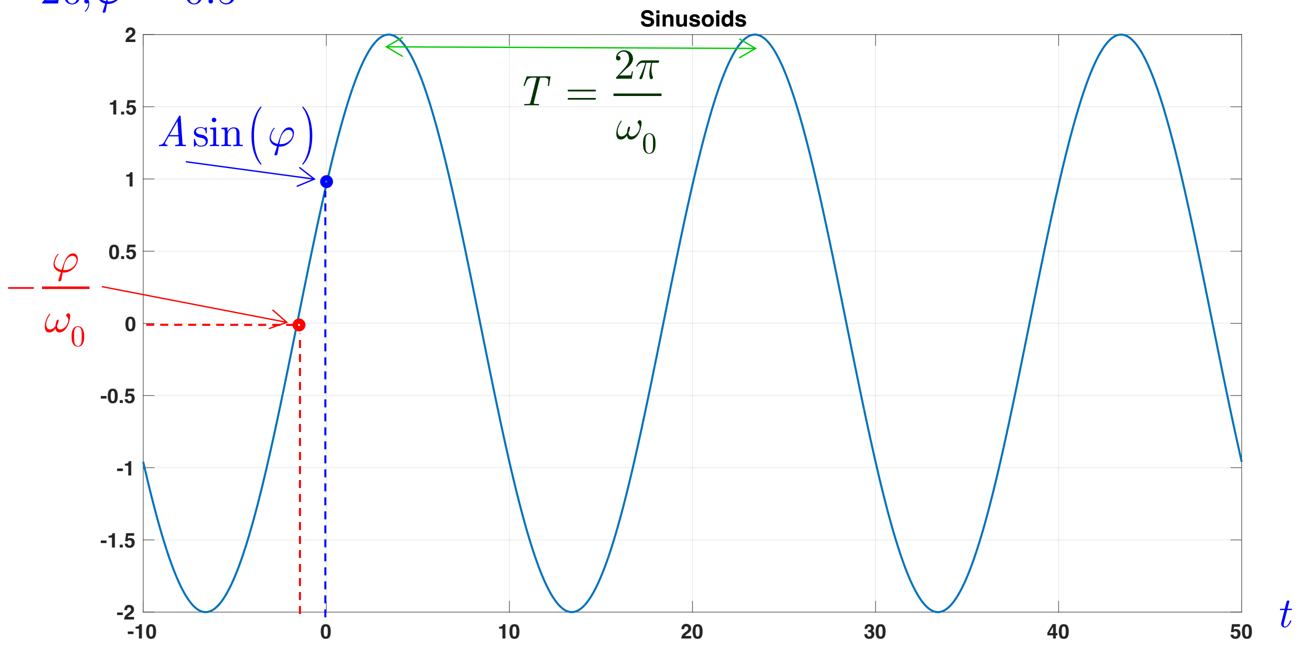
Sinusoids – periodic functions

$$f(t) = A\sin\left(\omega_0 t + \varphi\right)$$

Amplitude Frequency Phase

$$T = \frac{2\pi}{\omega_0}$$ - period

$A = 2, T = 20, \varphi = 0.5$



Sinusoidal function – time domain representation

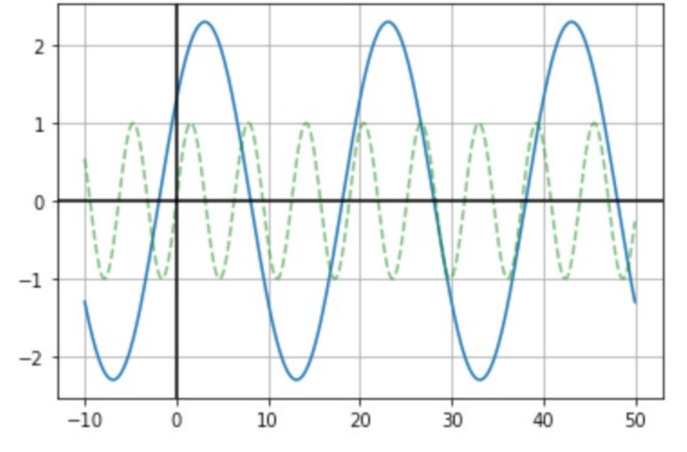
Sinusoidal functions

```python
def f(t, amplitude, frequency, phase):
  return amplitude * np.sin(frequency * t + phase)
```

```python
t = np.linspace(-10,50, 500)

amplitude = 2.3
frequency = np.pi / 10
phase = 0.6

period = 2 * np.pi / frequency
y = f(t, amplitude, frequency, phase)
```

```python
print(f'Amplitude A = {amplitude:.02f}')
print(f'Frequency w = {frequency:.02f}')
print(f'Phase phi = {phase:.02f}')
print(f'Period T = {period:.02f}')

plt.plot(t,y)
plt.plot(t, np.sin(t), 'g--', alpha=.5)
plt.axhline(y=0, color='k')
plt.axvline(x=0, color='k')
plt.grid()
plt.show()
```

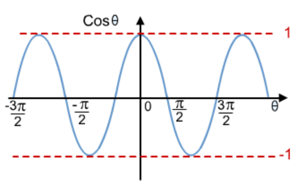
```
Amplitude A = 2.30
Frequency w = 0.31
Phase phi = 0.60
Period T = 20.00
```

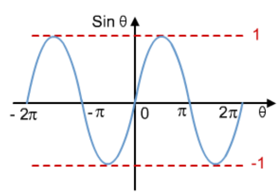


Complex numbers

Recall: sin can turn into cos with a proper phase shift

Shift by one quarter period	Shift by one half period[9]	Shift by full periods[10]	Period
$\sin(\theta \pm \frac{\pi}{2}) = \pm\cos\theta$	$\sin(\theta + \pi) = -\sin\theta$	$\sin(\theta + k\cdot 2\pi) = +\sin\theta$	2π
$\cos(\theta \pm \frac{\pi}{2}) = \mp\sin\theta$	$\cos(\theta + \pi) = -\cos\theta$	$\cos(\theta + k\cdot 2\pi) = +\cos\theta$	2π
$\tan(\theta \pm \frac{\pi}{4}) = \frac{\tan\theta \pm 1}{1 \mp \tan\theta}$	$\tan(\theta + \frac{\pi}{2}) = -\cot\theta$	$\tan(\theta + k\cdot\pi) = +\tan\theta$	π
$\csc(\theta \pm \frac{\pi}{2}) = \pm\sec\theta$	$\csc(\theta + \pi) = -\csc\theta$	$\csc(\theta + k\cdot 2\pi) = +\csc\theta$	2π
$\sec(\theta \pm \frac{\pi}{2}) = \mp\csc\theta$	$\sec(\theta + \pi) = -\sec\theta$	$\sec(\theta + k\cdot 2\pi) = +\sec\theta$	2π
$\cot(\theta \pm \frac{\pi}{4}) = \frac{\cot\theta \pm 1}{1 \mp \cot\theta}$	$\cot(\theta + \frac{\pi}{2}) = -\tan\theta$	$\cot(\theta + k\cdot\pi) = +\cot\theta$	π

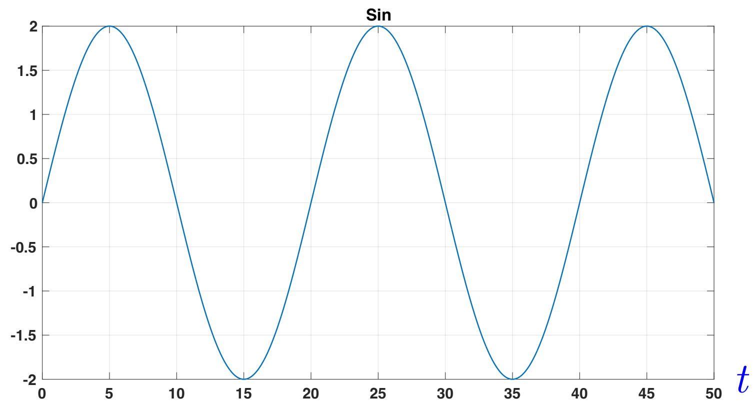




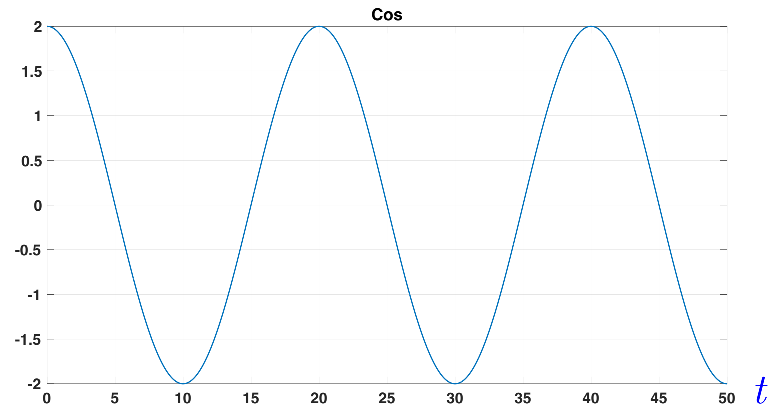
https://en.wikipedia.org/wiki/List_of_trigonometric_identities

Complex numbers

$$f(t) = 2\sin\left(\omega_0 t + 0\right)$$



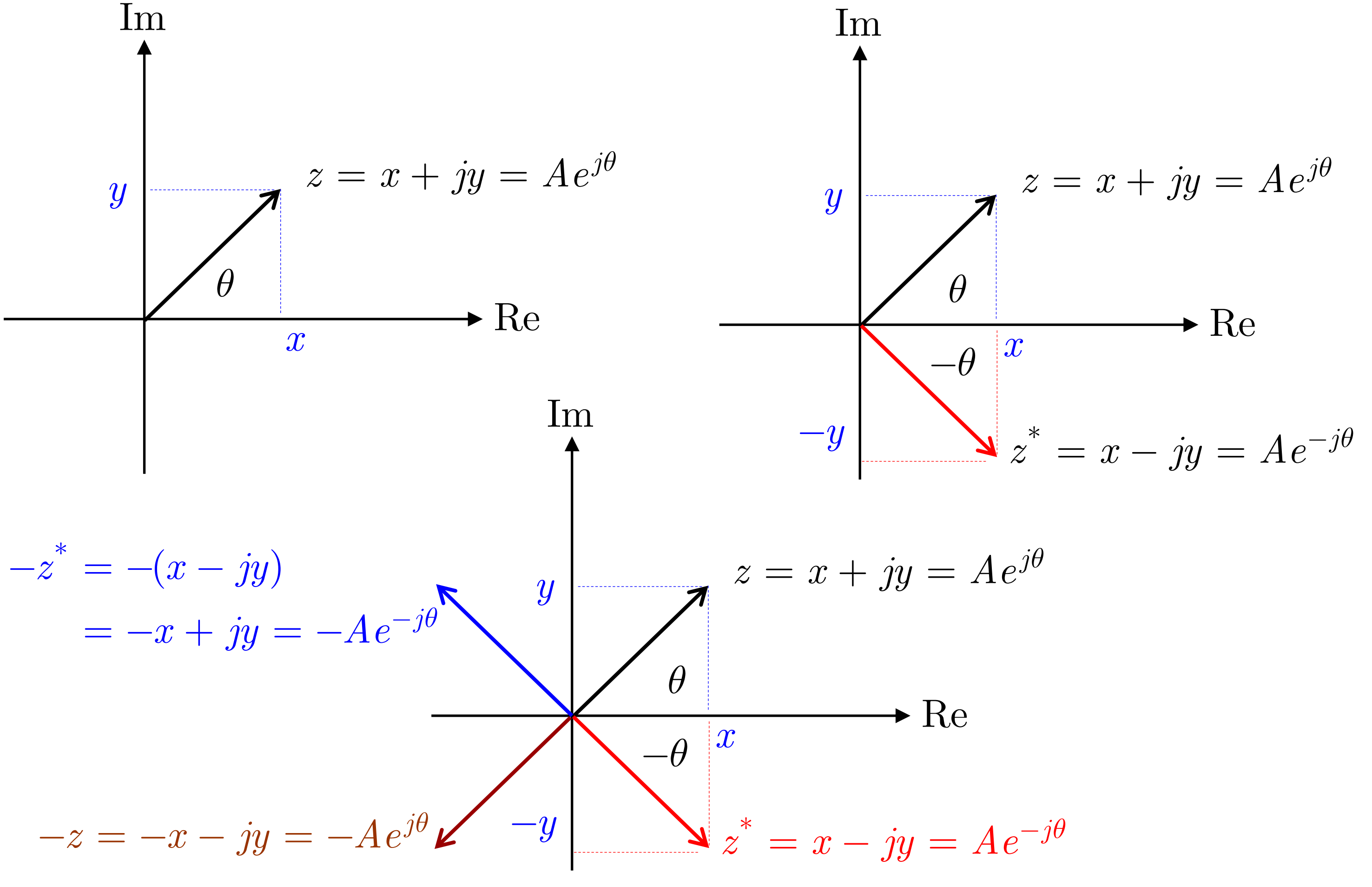
$$f(t) = 2\sin\left(\omega_0 t + \pi/2\right) = 2\cos(\omega_0 t)$$



$$T = \frac{2\pi}{\omega_0}$$

Polar domain representation

Consider a complex number $z = x + jy = Ae^{j\theta}, \theta = w_0 t, w_0 t = const$

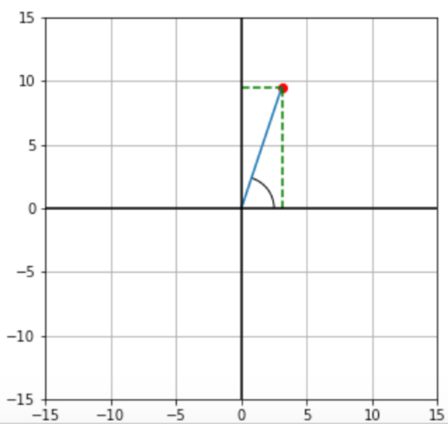


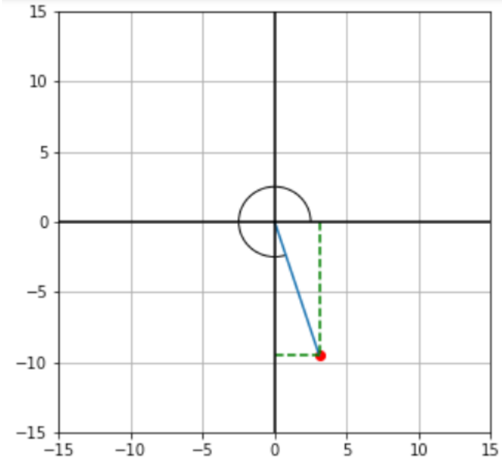
Complex numbers: a recall

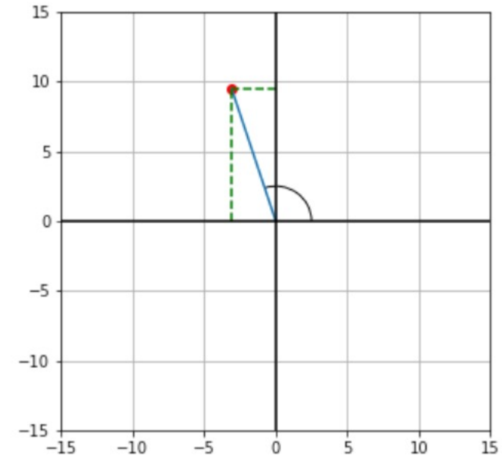
Polar form

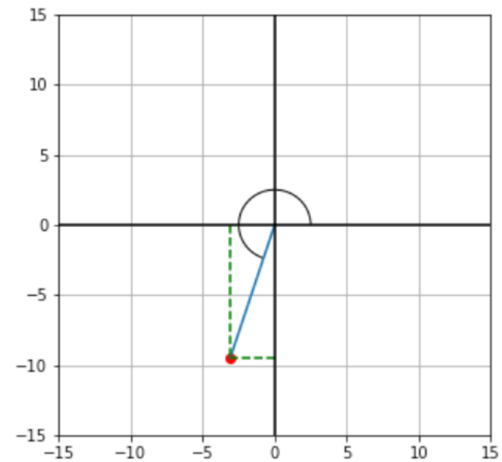
```python
r = 10
theta = 2*np.pi / 5

z = r*np.exp(1j*theta)

print(z.real, z.imag)

print(r*np.cos(theta), r*np.sin(theta))

plot_complex_number(z)
plot_complex_number(z.conjugate())
plot_complex_number(-z)
plot_complex_number(-z.conjugate())
```

```
3.0901699437494745 9.510565162951535
3.0901699437494745 9.510565162951535
```



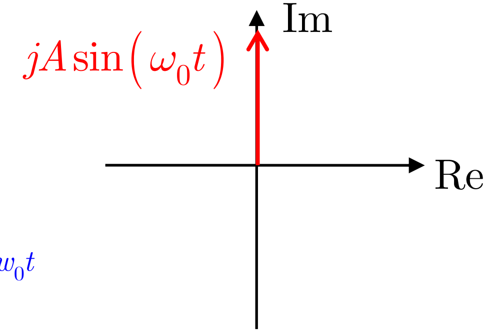






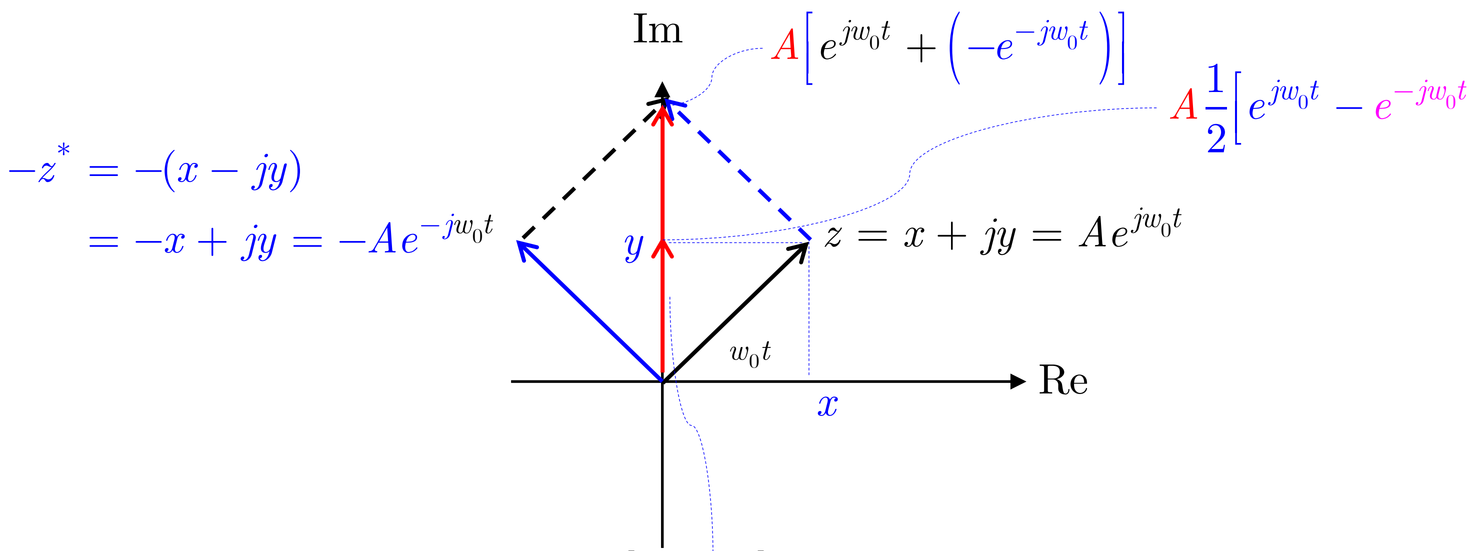
Polar domain representation

Consider a complex number $jA\sin(w_0 t)$



From conversion formulas:

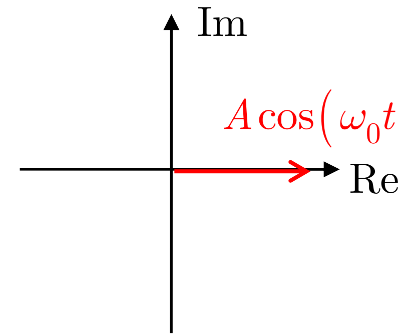
$$jA\sin\left(\omega_0 t\right) = A\frac{1}{2}\left[e^{jw_0 t} - e^{-jw_0 t}\right] = \frac{A}{2}e^{jw_0 t} - \frac{A}{2}e^{-jw_0 t}$$



Equivalently: $jA\sin(w_0 t) = \mathrm{Im}\left[Ae^{jw_0 t}\right] = \mathrm{Im}\left[A(\cos(w_0 t) + j\sin(w_0 t))\right]$

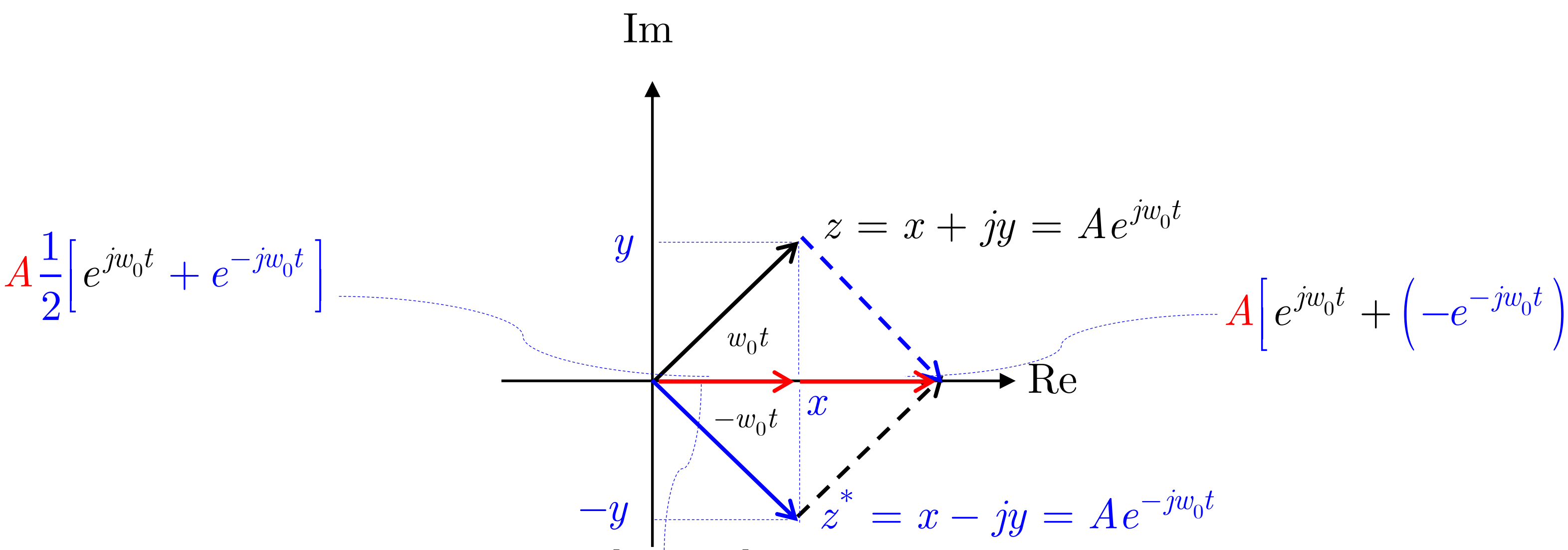
Polar domain representation

Consider a real number $A\cos(w_0 t)$



From conversion formulas:

$$A\cos\left(\omega_0 t\right) = A\frac{1}{2}\left[e^{jw_0 t} + e^{-jw_0 t}\right] = \frac{A}{2}e^{jw_0 t} + \frac{A}{2}e^{-jw_0 t}$$



Equivalently: $A\cos(w_0 t) = \mathrm{Re}\left[Ae^{jw_0 t}\right] = \mathrm{Re}\left[A(\cos(w_0 t) + j\sin(w_0 t))\right]$

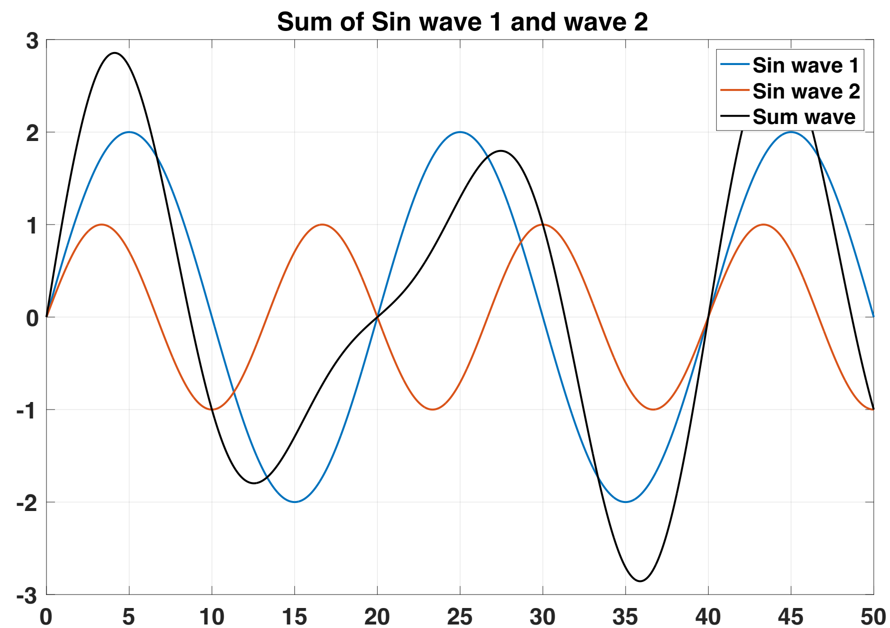
Roadmap

Theme 7: The 2D Discrete Fourier Transform

- Chapter 1: Basics of DSP, complex numbers
- Chapter 2: Continuous periodic signals – FS
- Chapter 3: Continuous aperiodic signals -FT (cont. time – cont. frequency)
- Chapter 4: Sampling:
  - Chapter 4.1: Sampling in time domain: Continuous aperiodic band-limited signals – (sampling in time domain) – cont. frequency – discrete time (DTCF FT)
  - Chapter 4.2: Sampling in frequency domain: Continuous aperiodic band-limited signals – (sampling in both time and frequency domain) – discrete frequency – discrete time – discrete FT (DFT)
  - Chapter 4.3: Extension to 2D DFT

Fourier – main idea behind function approximation

Fourier's main idea of periodic signal approximation: using a weighted sum of sine and cosine functions, one can obtain any signal shape

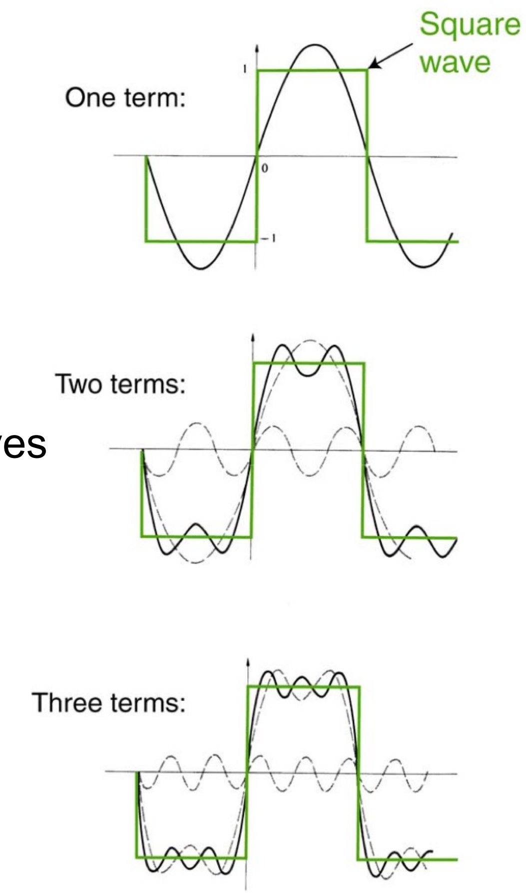


Joseph Fourier

https://www.britannica.com/biography/Joseph-Baron-Fourier

Fourier – main idea behind function approximation

Square wave as a sum of sine waves

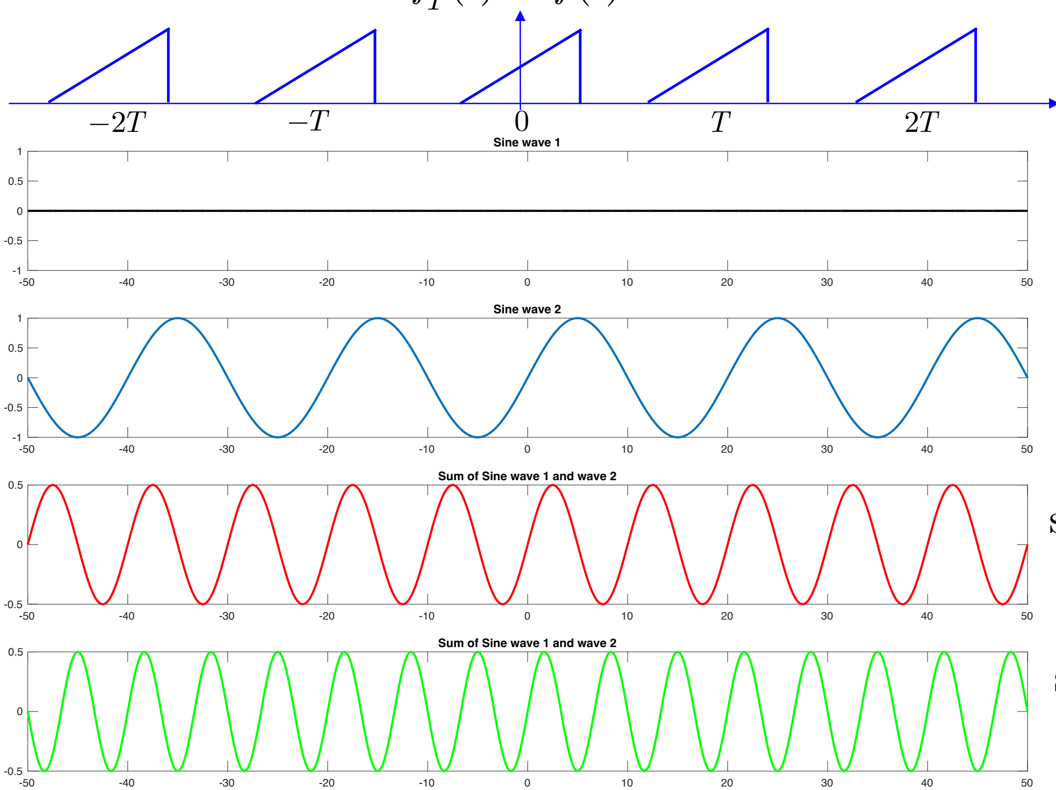


Note: we consider only periodic functions!

General formulation

Given a periodic signal $f_T(t) = \tilde{f}(t)$

$$w_0 = \frac{2\pi}{T}$$



a_0 — $\sin(0)$

$\pm a_1$ — $\sin(w_0 t)$

$\pm a_2$ — $\sin(2w_0 t)$

$\pm a_3$ — $\sin(3w_0 t)$

$$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

*sine functions are only shown

The idea is to use the periodic functions of the same period T as the signal to be approximated by the infinitive series.

General formulation of FS

The **Fourier series (FS)** representation of a real **periodic function** $\tilde{f}(t)$ is based on the summation of harmonically related sinusoidal components.

If the period is T, then the harmonics are sinusoids with frequencies that are inter multiples of ω_0, that is the k th harmonic component has a frequency $k\omega_0 = k2\pi/T$ and can be written as

$$\tilde{f}_k(t) = A_k \sin\left(k\omega_0 t + \phi_k\right)$$
$$= a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

a_k, b_k are real coefficients

Link between these two forms

$$A_k \sin\left(k\omega_0 t + \phi_k\right) = \underbrace{A_k \sin(\phi_k)}_{a_k} \cos(k\omega_0 t) + \underbrace{A_k \cos(\phi_k)}_{b_k} \sin(k\omega_0 t)$$

$$\text{Recall: } \sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta; \quad \alpha = k\omega_0 t,\ \beta = \phi_k$$

General formulation of FS: sinusoidal form

Here

$$\left.\begin{aligned} a_k &= A_k \sin(\phi_k) \\ b_k &= A_k \cos(\phi_k) \end{aligned}\right\} \Rightarrow \begin{cases} A_k = \sqrt{{a_k}^2 + {b_k}^2} \\ \phi_k = \tan^{-1} \dfrac{a_k}{b_k} \end{cases}$$

The FS can approximate any periodic function $\tilde{f}(t)$ by an infinite sum of harmonics

$$\boxed{\begin{aligned} \tilde{f}(t) &= \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \\ &= \frac{1}{2} a_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega_0 t + \phi_k) \end{aligned}}$$

(1)

(2)

General formulation of FS: exponential form

The third form (completely equivalent) is based on the conversion formulas leading to the complex exponents:

$$j = \sqrt{-1}$$

$$j^2 = \sqrt{-1}\sqrt{-1} = (-1)^1 = -1$$

$$\frac{1}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$$

$$\begin{aligned}
\tilde{f}_k(t) &= a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \\
&= \frac{a_k}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) + \frac{b_k}{2j}\left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right) \\
&= \underbrace{\frac{1}{2}\left(a_k - jb_k\right)}_{F_k} e^{jk\omega_0 t} + \underbrace{\frac{1}{2}\left(a_k + jb_k\right)}_{F_{-k}} e^{-jk\omega_0 t}
\end{aligned}$$

For all harmonics

$$\boxed{\Rightarrow \tilde{f}(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t}} \qquad (3)$$

F_k are complex coefficients

Summary on the FS

	Sinusoidal formulation	Exponential formulation
Synthesis (inverse FS)	$\tilde{f}(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$	$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} F_k e^{jkw_0 t}$
Analysis (direct FS)	$a_k = \frac{2}{T}\int_{-T/2}^{T/2} \tilde{f}(t)\cos(k\omega_0 t)dt$ $b_k = \frac{2}{T}\int_{-T/2}^{T/2} \tilde{f}(t)\sin(k\omega_0 t)dt$	$F_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{f}(t)e^{-jk\omega_0 t}dt$

We will see next that

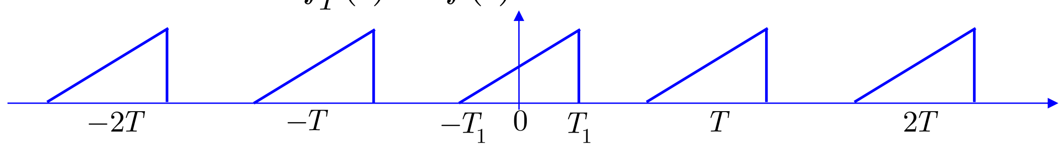
$F_k = \frac{1}{T}F\left(k\omega_0\right)$ - samples of Fourier transform at points $k\omega_0, k = 0, \pm 1, \cdots$

Note: a continuous periodic function is represented by discrete periodic samples F_k

General formulation: link to aperiodic functions

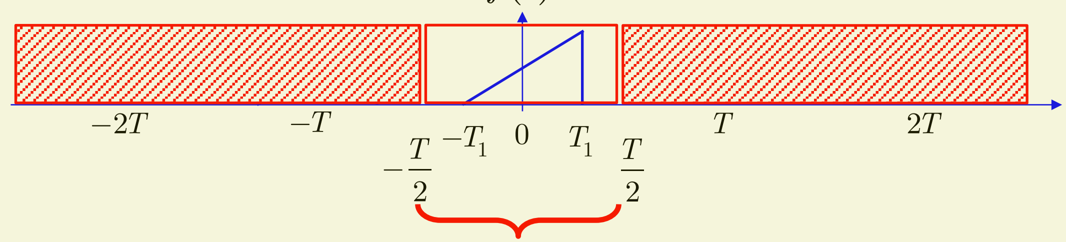
- Given a **periodic** signal

$$f_T(t) = \tilde{f}(t)$$



If the approximation of a periodic signal works, it can be extended to the approximation of **aperiodic** signals

$$f(t)$$



(a) Considering only this interval

(b) Letting $T \to \infty$

FS: a link to even and odd functions

$$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t}$$

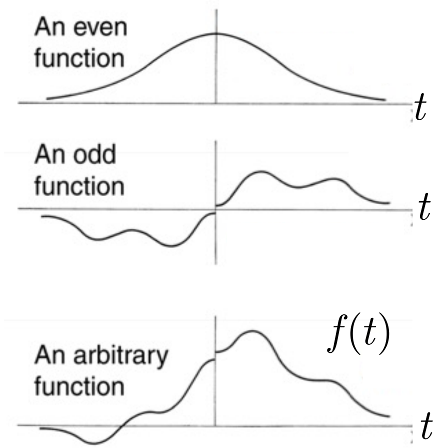
$$F_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{f}(t) e^{-jk\omega_0 t} dt$$ - complex coefficients

Reminder: any function can be written as the sum of an even and an odd function

$$Even\left[f(t)\right] = \frac{1}{2}\left(f(t) + (f(-t)\right)$$

$$Odd\left[f(t)\right] = \frac{1}{2}\left(f(t) - (f(-t)\right)$$

$$\Rightarrow f(t) = Even\left[f(t)\right] + Odd\left[f(t)\right]$$

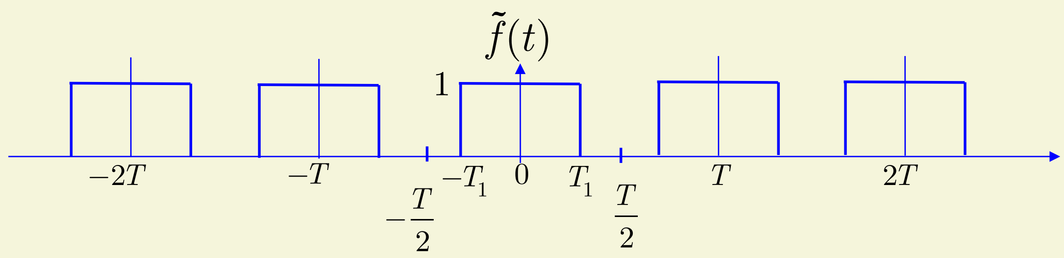


$$Even\left[f(t)\right] = Even\left[f(-t)\right]$$

$$Odd\left[f(-t)\right] = -Odd\left[f(t)\right]$$

Link to cos and sin basis functions

Example: a periodic square wave/signal



$$w_0 = \frac{2\pi}{T}$$

$$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} F_k e^{jkw_0 t}$$

$$F_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{f}(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T_1}^{T_1} 1 \cdot e^{-jk\omega_0 t} dt = \frac{2}{T} \frac{\sin(k\omega_0 T_1)}{k\omega_0}$$

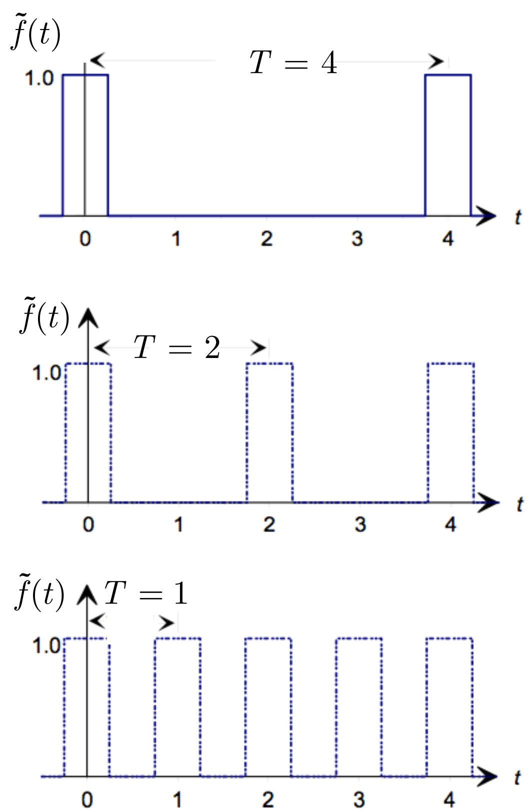
$$F_k = \frac{2T_1}{T} \frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}$$

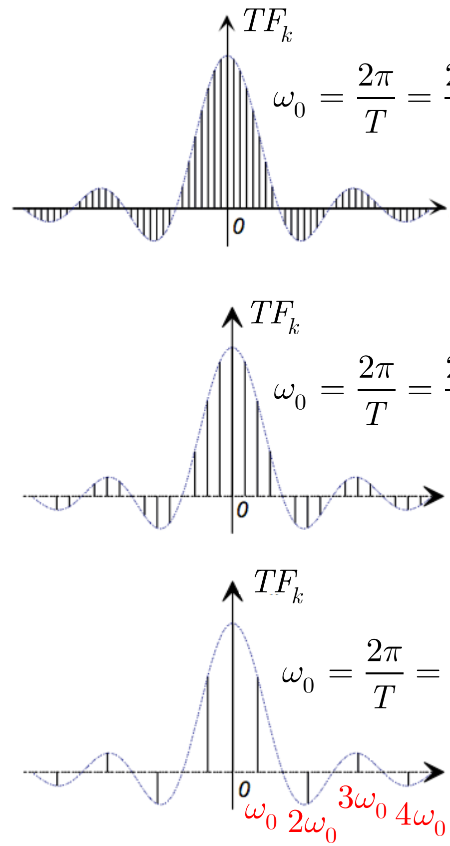
Remark: $\frac{\sin(x)}{x} \triangleq \mathrm{sinc}(x)$ - we will see its properties later

Example: a periodic square wave/signal

Line spectra: visualization of F_k

$$F_k = \frac{2T_1}{T}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}$$





$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}\mathrm{rad/sec}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi\mathrm{rad/sec}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi\mathrm{rad/sec}$$

http://web.mit.edu/2.151/www/Handouts/FreqDomain.pdf

Summary: FS of periodic functions

- The FS applies only to the periodic functions with a period T
- The FS coefficients represent samples with a step $k\omega_0 = k\frac{2\pi}{T}, k = 0, \pm 1, \cdots$
- Any periodic function can be represented by the FS coefficients F_k
- The link to aperiodic functions is to assume that $T \to \infty$
  - In turns, it is equivalent to $\omega_0 \to 0$
  - Intuition for the next step:
    - Aperiodic functions have “continuous” spectra instead of discrete FS samples
    - This forms a basis of Fourier Transform

Roadmap

Theme 7: The 2D Discrete Fourier Transform

Chapter 1: Basics of DSP, complex numbers

Chapter 2: Continuous periodic signals – FS

$\boxed{\text{Chapter 3: Continuous aperiodic signals -FT (cont. time – cont. frequency)}}$

Chapter 4: Sampling:

Chapter 4.1: Sampling in time domain: Continuous aperiodic band-limited signals – (sampling in time domain) – cont. frequency – discrete time (DTCF FT)

Chapter 4.2: Sampling in frequency domain: Continuous aperiodic band-limited signals – (sampling in both time and frequency domain) – discrete frequency – discrete time – discrete FT (DFT)

Chapter 4.3: Extension to 2D DFT

Theme 8: Frequency domain filtering, sampling and aliasing

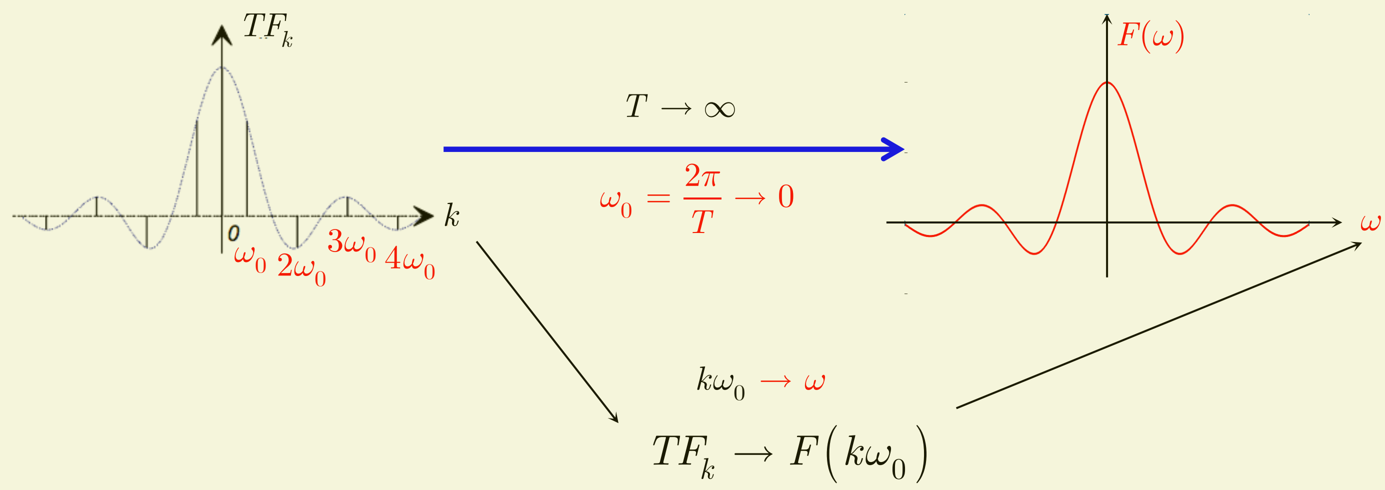
Theme 9: Unitary (data independent) transforms – beyond DFT

Theme 11: Machine learnable (or data dependent) transforms (in Theme 11: Lossy image compression)

Intuitive introduction of Fourier transfrom (FT)

FS representation of **periodic** function

FT of **aperiodic** function



FS and FT: as infinite period

$$F_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{f}(t)e^{-jk\omega_0 t}dt$$

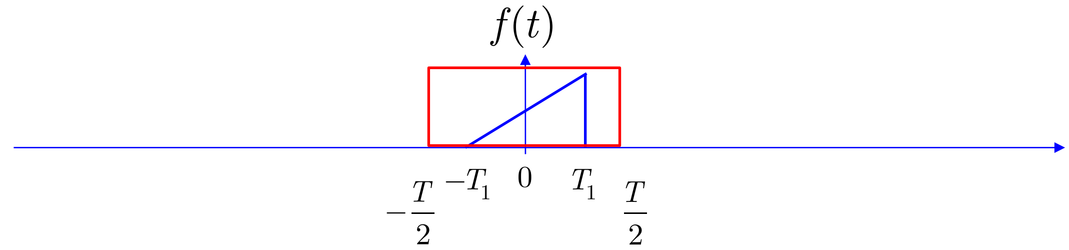
$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} f(t)e^{-jk\omega_0 t}dt = \frac{1}{T}F(k\omega_0)$$

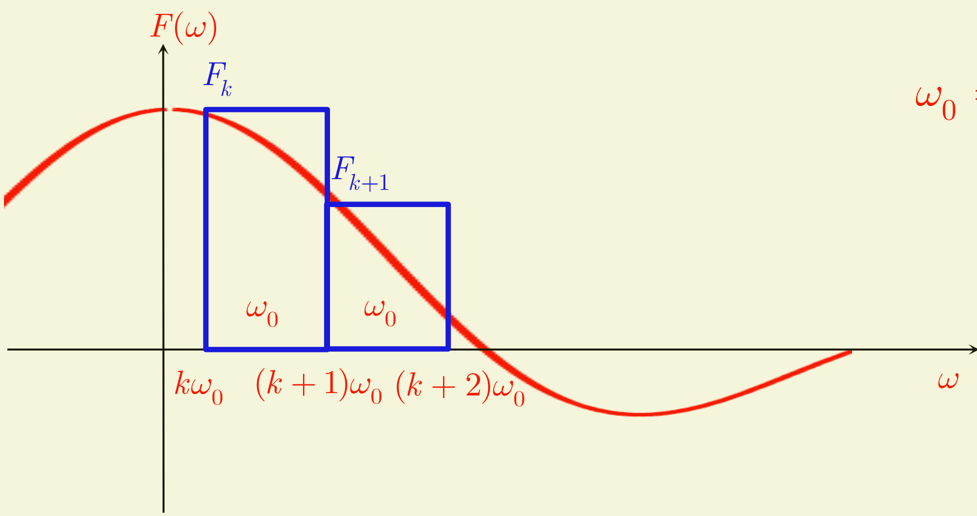
$$k\omega_0 = \omega$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

Link between Fourier series and Fourier transform



FS and FT: as discrete approximation



$$\omega_0 = \frac{2\pi}{T} \rightarrow T = \frac{2\pi}{\omega_0}$$

$$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T} F(k\omega_0)}_{F_k} e^{jk\omega_0 t}$$

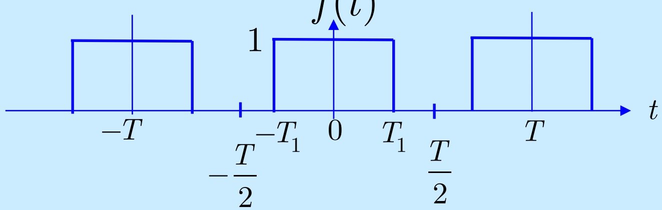
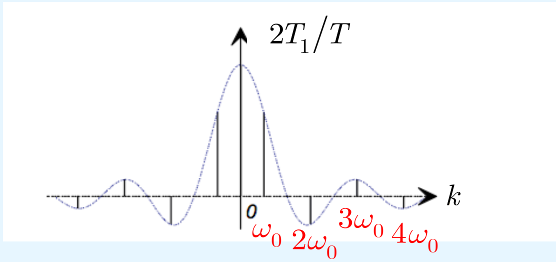
$$= \frac{1}{2\pi}\left[\sum_{k=-\infty}^{\infty} F(k\omega_0) e^{jk\omega_0 t}\right]\omega_0$$

If $\omega_0 \rightarrow 0$, then $\sum \rightarrow \int$ & $k\omega_0 \rightarrow \omega$

$$\underbrace{\tilde{f}(t)}_{periodic} \rightarrow \boxed{\underbrace{f(t)}_{aperiodic} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{jwt} dw} \qquad \boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt}$$

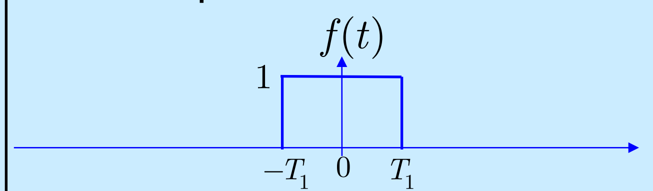
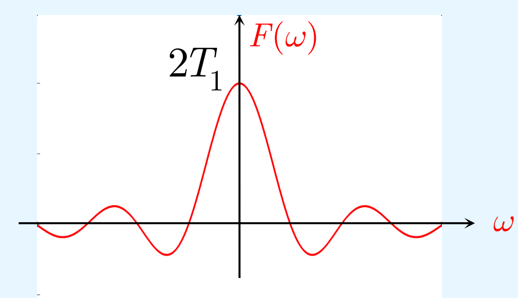
FS and FT: summary

For a periodic function $\tilde{f}(t)$, which is a periodic extension of $f(t)$, with the period $T = 2\pi/\omega_0$

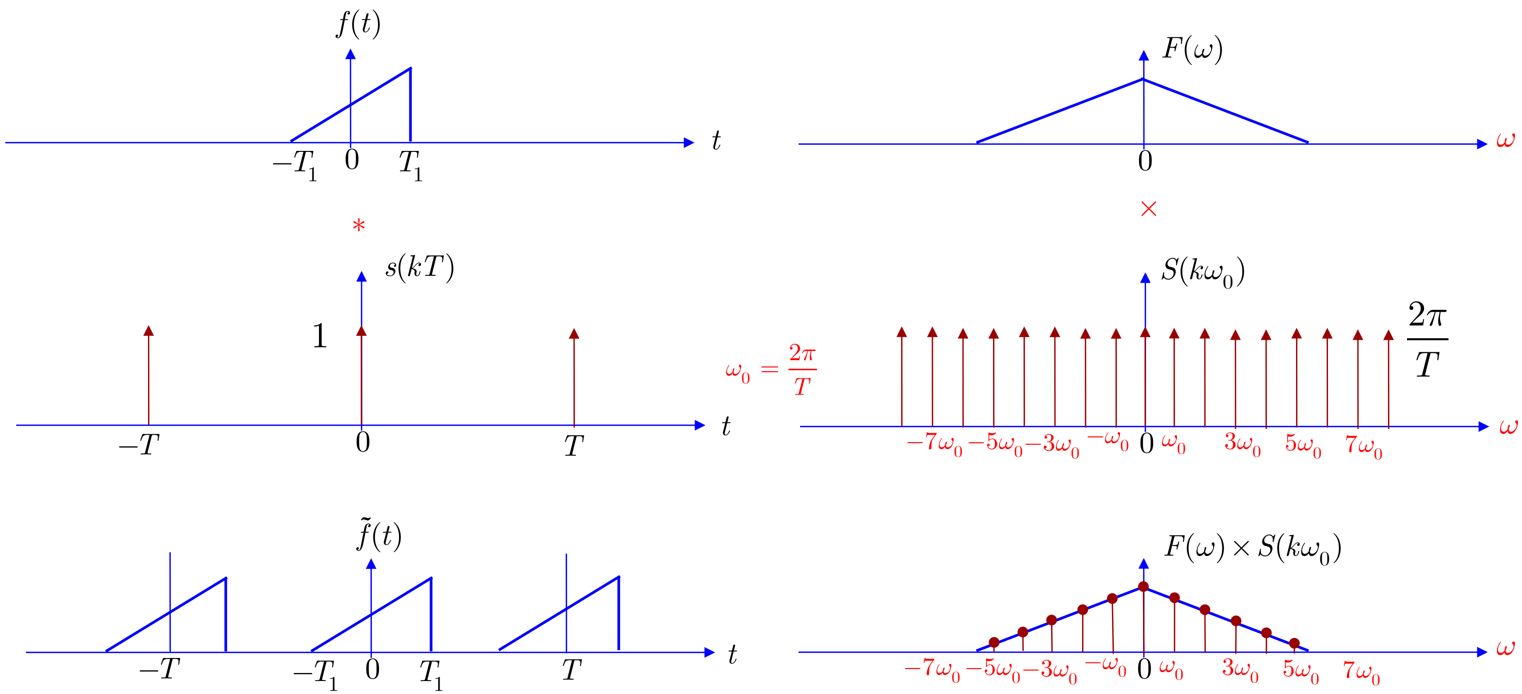
Series (periodic function)	Example function	Plot
Synthesis $\tilde{f}(t) = \sum_{k=-\infty}^{\infty} F_k e^{jkw_0 t}$	$\tilde{f}(t) = \tilde{\Pi}\left(\frac{t}{2T_1}\right)$ periodic extension	Periodic function 
Analysis $F_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{f}(t) e^{-jk\omega_0 t} dt$ $= \frac{1}{T} F(k\omega_0)$	$F_k = \frac{2T_1}{T}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}$	

FS and FT: summary

For an aperiodic function $f(t)$, such that $\int_{-\infty}^{\infty} |f(t)| dt < \infty$

FT (aperiodic function)	Example function	Plot
Synthesis $$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$	$$f(t) = \begin{cases} 1, \|t\| < T_1, \\ 0, otherwise \end{cases}$$ $$f(t) = \Pi\left(\frac{t}{2T_1}\right)$$	Aperiodic function 
Analysis $$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$	$$F(\omega) = 2T_1 \mathrm{sinc}\left(\omega T_1\right)$$	

FS and FT: alternative interpretation (see DFT)



Periodic $\Leftrightarrow$ Discrete

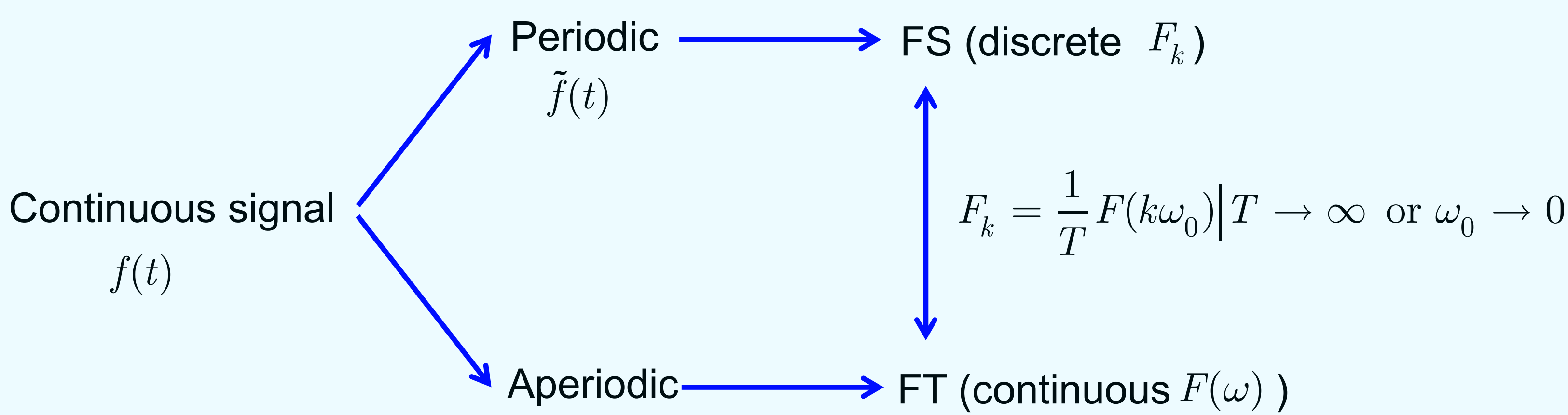
Discrete $\Leftrightarrow$ Periodic

$$F_k = \frac{1}{T} F(k\omega_0) \Big| T \to \infty$$

$$\frac{1}{T} = 2\pi\omega_0$$

compensates the preservation of the total area due to sampling

FS and FT: summary



The Fourier Transform – formal definition

We consider functions of one continuous variable (continues time domain).

The considered functions are:

- continuous in t
- aperiodic
- bounded $\int_{-\infty}^{\infty} \left| f(t) \right| dt < \infty$
- *we do allow the FT of the periodic signals, assuming that it will be a discrete FS

Direct Fourier transform (analysis)

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt}$$

$w = 2\pi\mu$

Units of w are cycles/unit of t

$F(\omega)$ is a complex function

Inverse Fourier transform (synthesis)

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega}$$

The Fourier Transform of Functions

Remark about an alternative form:

Gonzalez

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega} \rightarrow \begin{aligned}\omega &= 2\pi\mu \\ d\omega &= 2\pi d\mu\end{aligned} \rightarrow \boxed{f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}\,d\mu}$$

Direct Fourier transform (trigonometric form using Euler's formula)

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)(\cos(\omega t) - j\sin(\omega t))\,dt} \qquad \boxed{F(\omega) = \Im\left\{f(t)\right\}}$$

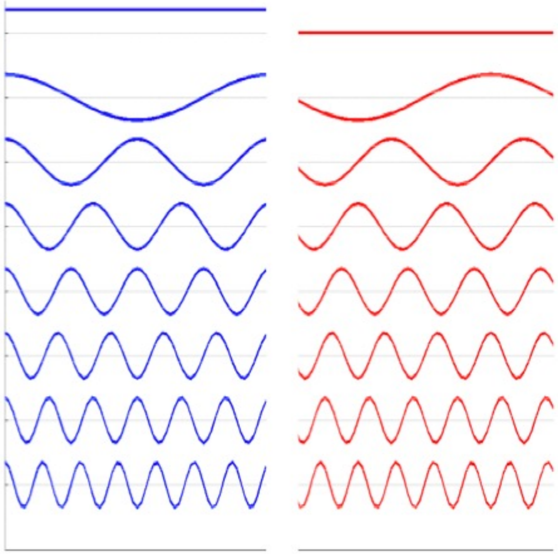
$$F(\omega) = \underbrace{\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt}_{\text{Real part}} - j\underbrace{\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt}_{\text{Imaginary part}}$$

One can consider $F(\omega)$ as a projection of $f(t)$ onto cosine and sine basis functions.

The Fourier Transform of Functions

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \longrightarrow F(\omega) = \int_{-\infty}^{\infty} f(t)(\cos(\omega t) - j\sin(\omega t))\, dt$$

$$F(\omega) = \underbrace{\int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt}_{\text{Real part}} - j \underbrace{\int_{-\infty}^{\infty} f(t)\sin(\omega t)\, dt}_{\text{Imaginary part}}$$



Fourier transform: properties

1. **Linearity**

$$\Im\left\{af_1(t)+bf_2(t)\right\}=a\Im\left\{f_1(t)\right\}+b\Im\left\{f_2(t)\right\}=aF_1(\omega)+bF_2(\omega)$$

2. **Time shift**

$$\Im\left\{f(t\pm t_0)\right\}=F(\omega)e^{\pm j\omega t_0}$$

$$f(t)\rightarrow f(t\pm t_0)$$

$$F(\omega)\rightarrow F(\omega)e^{\pm j\omega t_0}$$

$$\Im\left\{f(t\pm t_0)\right\}=\int_{-\infty}^{\infty} f(t\pm t_0)e^{-j\omega t}\,dt\left|\begin{array}{l} t'=t\pm t_0 \\ t=t'\mp t_0 \\ dt=dt' \end{array}\right.=\int_{-\infty}^{\infty} f(t')e^{-j\left(\omega t'\mp t_0\right)}\,dt'$$

$$=e^{\pm j\omega t_0}\int_{-\infty}^{\infty} f(t')e^{-j\omega t'}\,dt'=e^{\pm j\omega t_0}F(\omega)$$

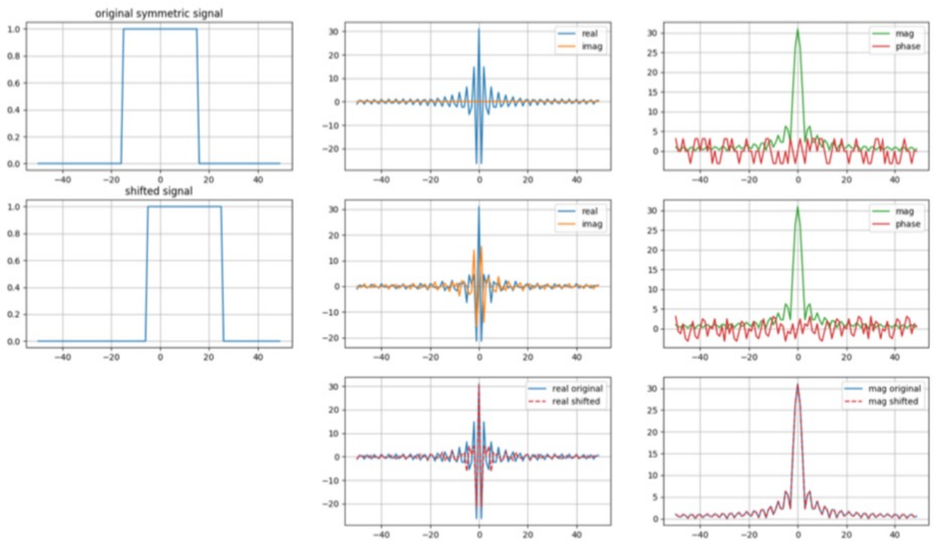
The magnitude stays the same but the phase is shifted

Fourier transform: properties

2. **Time shift (cont.)**

$$\Im\left\{f(t \pm t_0)\right\} = F(\omega)e^{\pm j\omega t_0}$$

$f(t)$-real and even



Magnitude is the same; only phase is shifted by $e^{\pm j\omega t_0}$

Real part is the same; the zero imaginary part of real signal becomes an odd one.

Fourier transform: properties

3. **Frequency shift**

$$\Im^{-1}\left\{F(\omega \pm \omega_0)\right\} = f(t)e^{\mp j\omega_0 t}$$

$$F(\omega) \to F(\omega \pm \omega_0)$$

$$f(t) \to f(t)e^{\mp j\omega_0 t}$$

$$\Im^{-1}\left\{F(\omega \pm \omega_0)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega \pm \omega_0)e^{j\omega t}\, d\omega \left| \begin{array}{l} \omega' = \omega \pm \omega_0 \\ \omega = \omega' \mp \omega_0 \\ d\omega = d\omega' \end{array} \right. = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega')e^{jt(\omega' \mp \omega_0)}\, d\omega'$$

$$= e^{\mp j\omega_0 t}\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega')e^{jt\omega'}\, d\omega' = f(t)e^{\mp j\omega_0 t}$$

Fourier transform: properties

4. **Time reversal: real signals**

$$\Im\left\{f(-t)\right\} = F^*(\omega)$$

$$\Im\left\{f(-t)\right\} = \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}\,dt \left| \begin{array}{l} t' = -t \\ t = -t' \\ dt = -dt' \end{array} \right. = -\int_{\infty}^{-\infty} f(t')e^{j\omega t'}\,dt' = \int_{-\infty}^{\infty} f(t')e^{j\omega t'}\,dt'$$

Reminder:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \longrightarrow \Im\left\{f(-t)\right\} = \left[\int_{-\infty}^{\infty} f^*(t')e^{-j\omega t'}\,dt'\right]^*$$

Since $f(t)$ is real, then $f^*(t) = f(t)$

$$\Im\left\{f(-t)\right\} = \left[\underbrace{\int_{-\infty}^{\infty} f(t')e^{-j\omega t'}\,dt'}_{F(\omega)}\right]^* = F^*(\omega)$$

Fourier transform: properties

5. **Time reversal and conjugation: complex signals**

$$\Im\left\{f^*(-t)\right\} = F^*\left(\omega\right)$$

$$\Im\left\{f^*(-t)\right\} = \int_{-\infty}^{\infty} f^*(-t)e^{-j\omega t}\,dt \left| \begin{array}{l} t' = -t \\ t = -t' \\ dt = -dt' \end{array} \right. = -\int_{\infty}^{-\infty} f^*(t')e^{j\omega t'}\,dt' = \int_{-\infty}^{\infty} f^*(t')e^{j\omega t'}\,dt'$$

$$\Im\left\{f^*(-t)\right\} = \left[\underbrace{\int_{-\infty}^{\infty} f(t')e^{-j\omega t'}\,dt'}_{F(\omega)}\right]^* = F^*(\omega)$$

Fourier transform: properties

5. **Time reversal and NO conjugation: complex signals**

$$\Im\left\{f(-t)\right\} = F\left(-\omega\right)$$

$$\Im\left\{f(-t)\right\} = \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}\,dt \left|\begin{array}{l} t' = -t \\ t = -t' \\ dt = -dt' \end{array}\right. = -\int_{\infty}^{-\infty} f(t')e^{j\omega t'}\,dt' = \int_{-\infty}^{\infty} f(t')e^{j\omega t'}\,dt'$$

Change of variables: $\left|\omega = -\omega'\right.$

$$\Im\left\{f(-t)\right\} = \int_{-\infty}^{\infty} f(t')e^{-j\omega' t'}\,dt' = F(\omega') = F(-\omega)$$

Fourier transform: properties

5. **Cont: Conjugation of complex signals (no time reversal)**

$$\Im\left\{f^*(t)\right\} = F^*\left(-\omega\right)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \Im\left\{f(t)\right\}$$

$$F(-\omega) = \int_{-\infty}^{\infty} f(t)\underbrace{e^{-j(-\omega)t}}_{e^{j\omega t}}\,dt$$

$$F^*(-\omega) = \left[\int_{-\infty}^{\infty} f(t)\underbrace{e^{-j(-\omega)t}}_{e^{j\omega t}}\,dt\right]^* = \int_{-\infty}^{\infty} f^*(t)\underbrace{\left[e^{j\omega t}\right]^*}_{e^{-j\omega t}}\,dt = \Im\left\{f^*(t)\right\}$$

$$F(\omega) = F_R(\omega) + jF_I(\omega)$$

$$F^*(\omega) = F_R(\omega) - jF_I(\omega)$$

$$F^*(-\omega) = F_R(-\omega) - jF_I(-\omega)$$

Fourier transform: properties

5. **Cont: Conjugation of complex signals**

$$\Im\left\{f^*(t)\right\} = F^*\left(-\omega\right)$$

- If $f(t)$ is real, then $f(t) = f^*(t)$

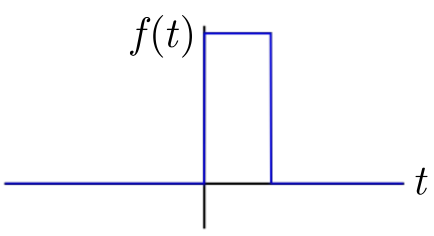
$$\Im\left\{f(t)\right\} = \Im\left\{f^*(t)\right\}$$

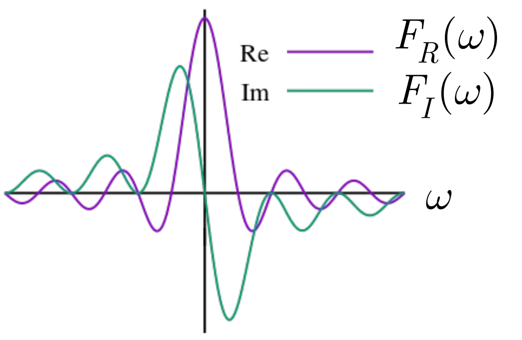
$$F\left(\omega\right) = F^*\left(-\omega\right)$$ - conjugated symmetry of spectrum

$$F(\omega) = F_R(\omega) + jF_I(\omega) \qquad F^*(-\omega) = F_R(-\omega) - jF_I(-\omega)$$

$F_R(\omega) = F_R(-\omega)$ - even

$F_I(\omega) = -F_I(-\omega)$ - odd





Fourier transform: properties

Summary on time reversal

Real valued signal	Fourier Transform
$f(t)$	$F(\omega)$
$f(-t)$	$F^*(\omega)$
-	-
-	-

Complex valued signal	Fourier Transform
$f(t)$	$F(\omega)$
$f^*(-t)$	$F^*(\omega)$
$f^*(t)$	$F^*(-\omega)$
$f(-t)$	$F(-\omega)$

Fourier transform: properties

6. **Even and odd complex signals and spectra**

Even signal $\Im\left\{"f(t)=f(-t)"\right\} \Leftrightarrow F(\omega)=F(-\omega)$

Odd signal $\Im\left\{"f(t)=-f(-t)"\right\} \Leftrightarrow F(\omega)=-F(-\omega)$

If the signal is even $f(t)=f(-t)$, then according to the time reversal property:

$$F(\omega)=\Im\left\{f(t)\right\}=\underbrace{\Im\left\{f(-t)\right\}=F(-\omega)}_{\text{time reversal for complex signals}}$$

If the signal is odd $f(t)=-f(-t)$, then according to the time reversal property:

$$F(\omega)=\Im\left\{f(t)\right\}=\underbrace{\Im\left\{-f(-t)\right\}=-F(-\omega)}_{\text{time reversal for complex signals}}$$

Fourier transform: properties

7. **Time and frequency scaling**

$$\Im\left\{f(at)\right\} = \frac{1}{a}F\left(\frac{\omega}{a}\right) \Leftrightarrow \Im\left\{af(at)\right\} = F\left(\frac{\omega}{a}\right)$$

$$\Im\left\{f(at)\right\} = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}\,dt \left| \begin{array}{l} u = at, a > 0 \\ t = u/a \\ dt = 1/a\,du \end{array} \right. = \int_{-\infty}^{\infty} f(u)e^{-j\omega u/a}d(u/a) = \frac{1}{a}F\left(\frac{\omega}{a}\right)$$

Any a:
$$\Im\left\{f(at)\right\} = \frac{1}{|a|}F\left(\frac{\omega}{|a|}\right)$$

Fourier transform: properties

8. **Convolution theorem**

$$\Im\left\{f(t)*h(t)\right\} = F\left(\omega\right)H\left(\omega\right) \quad (a)$$

Product theorem

$$\Im\left\{f(t)h(t)\right\} = \frac{1}{2\pi}F\left(\omega\right)*H\left(\omega\right) \quad (b)$$

$$y(t) = f(t)*h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

$$(a) \quad \Im\left\{y(t)\right\} = \Im\left\{f(t)*h(t)\right\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau\right]e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}\,dt\right]d\tau = \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}\underbrace{\left[\int_{-\infty}^{\infty} h(t-\tau)\underbrace{e^{-j\omega t}e^{j\omega\tau}}_{e^{-j\omega(t-\tau)}}\underbrace{d(t-\tau)}_{dt}\right]}_{H(\omega)}d\tau$$

$$= F(\omega)H(\omega)$$

Fourier transform: properties

9. **Convolution theorem – product theorem**

$$\Im\left\{f(t) * h(t)\right\} = F\left(\omega\right)H\left(\omega\right) \quad (a)$$

Product theorem

$$\boxed{\Im\left\{f(t)h(t)\right\} = \frac{1}{2\pi}F\left(\omega\right) * H\left(\omega\right) \quad (b)}$$

$$(b) \quad \Im\left\{f(t)h(t)\right\} = \int_{-\infty}^{\infty}\left[f(t)h(t)\right]e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega')e^{j\omega' t}\,d\omega'\right]}_{f(t)}h(t)e^{-j\omega t}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega')\left[\int_{-\infty}^{\infty}h(t)e^{j\omega' t}e^{-j\omega t}\,dt\right]d\omega' = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega')\left[\int_{-\infty}^{\infty}h(t)e^{-j(\omega-\omega')t}\,dt\right]d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega')H(\omega-\omega')\,d\omega' = \frac{1}{2\pi}F(\omega) * H(\omega)$$

Fourier transform: properties

10. **Correlation theorem**

$$\Im\left\{r_{xh}(t)\right\} = \Im\left\{f(t)\circ h(t)\right\} = F\left(\omega\right)^{*} H\left(\omega\right)$$

$$\Im\left\{r_{ff}(t)\right\} = \Im\left\{f(t)\circ f(t)\right\} = F\left(\omega\right)^{*} F\left(\omega\right) = \left|F\left(\omega\right)\right|^{2}$$

$$y(t) = f(t) * h(t) = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau \qquad \Im\left\{f(t) * h(t)\right\} = F(\omega)H(\omega)$$

$$r_{fh}(t) = f(t)\circ h(t) = \int_{-\infty}^{+\infty} f^{*}(\tau)h(t+\tau)d\tau \qquad \Im\left\{f(t)\circ h(t)\right\} = F\left(\omega\right)^{*} H\left(\omega\right)$$

$$\Im\left\{f(t)\circ h(t)\right\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f^{*}(\tau)h(t+\tau)\,d\tau\right]e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f^{*}(\tau)\left[\int_{-\infty}^{\infty} h(t+\tau)e^{-j\omega t}\,dt\right]d\tau \Big| \times e^{-j\omega\tau}e^{j\omega\tau}$$

$$= \int_{-\infty}^{\infty} \underbrace{f^{*}(\tau)e^{j\omega\tau}}_{\left[f(\tau)e^{-j\omega\tau}\right]^{*}} \underbrace{\left[\int_{-\infty}^{\infty} h(t+\tau)\underbrace{e^{-j\omega t}e^{-j\omega\tau}}_{e^{-j\omega(t+\tau)}}\underbrace{dt}_{d(t+\tau)}\right]}_{H(\omega)} d\tau$$

$$= F(\omega)^{*} H(\omega)$$

Fourier transform: properties

Auto-correlation theorem

$$\Im\left\{r_{ff}(t)\right\} = \Im\left\{f(t) \circ f(t)\right\} = F^*\left(\omega\right)F\left(\omega\right) = \left|F\left(\omega\right)\right|^2 = S_x(\omega)$$

$$r_{ff}(t) = f(t) \circ f(t) = \int_{-\infty}^{+\infty} f^*(\tau)f(t+\tau)d\tau$$

Fourier transform: properties

11. **Time derivative**

$$\Im\left\{\frac{d^n}{dt^n} f(t)\right\} = \left(j\omega\right)^n F\left(\omega\right)$$

$$\frac{d}{dt} f(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) \frac{d}{dt} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[j\omega F(\omega)\right] e^{j\omega t}\, d\omega = \Im^{-1}\left\{j\omega F(\omega)\right\}$$

$$\Im\left\{\frac{d}{dt} f(t)\right\} = j\omega F(\omega)$$

Fourier transform: summary of properties

$f(t) - \text{complex}$	$F(\omega)$
$f(t) - \text{real}$	$F^*(\omega) = F(-\omega)$
$f(t) - \text{imagery}$	$F^*(-\omega) = -F(\omega)$
$f(t) - \text{real}$	$F_R(\omega)$-even;$F_I(\omega)$-odd

Fourier transform: summary of properties

Linearity

$$\Im\left\{af_1(t)+bf_2(t)\right\}=a\Im\left\{f_1(t)\right\}+b\Im\left\{f_2(t)\right\}=aF_1(\omega)+bF_2(\omega)$$

Time shift

$$\Im\left\{f(t\pm t_0)\right\}=F(\omega)e^{\pm j\omega t_0}$$

Frequency shift

$$\Im^{-1}\left\{F(\omega\pm\omega_0)\right\}=f(t)e^{\mp j\omega_0 t}$$

Time reversal

$$\Im\left\{f^*(-t)\right\}=F*(\omega)-\text{for complex signals;}$$

$$\Im\left\{f(-t)\right\}=F*(\omega)-\text{for complex signals}$$

Time and frequency scaling

$$\Im\left\{f(at)\right\}=\frac{1}{a}F\left(\frac{\omega}{a}\right)\Leftrightarrow\Im\left\{af(at)\right\}=F\left(\frac{\omega}{a}\right)$$

Complex conjugation

$$\Im\left\{f^*(t)\right\}=F^*\left(-\omega\right)$$

Fourier transform: summary of properties

Convolution theorem

$$\Im\left\{f(t) * h(t)\right\} = F\left(\omega\right)H\left(\omega\right) \quad (a)$$

Product theorem

$$\Im\left\{f(t)h(t)\right\} = \frac{1}{2\pi}F\left(\omega\right) * H\left(\omega\right) \quad (b)$$

Correlation and auto-correlation theorem

$$\Im\left\{r_{xh}(t)\right\} = \Im\left\{f(t) \circ h(t)\right\} = F^*\left(\omega\right)H\left(\omega\right)$$

$$\Im\left\{r_{ff}(t)\right\} = \Im\left\{f(t) \circ f(t)\right\} = F^*\left(\omega\right)F\left(\omega\right) = \left|F\left(\omega\right)\right|^2$$

Time derivative

$$\Im\left\{\frac{d^n}{dt^n}f(t)\right\} = \left(j\omega\right)^n F\left(\omega\right)$$

Summary of FT properties

Superposition: The Fourier transform of a sum of signals is the sum of their Fourier transforms. Thus, the Fourier transform is a linear operator.

Shift: The Fourier transform of a shifted signal is the transform of the original signal multiplied by a *linear phase shift* (complex sinusoid).

Reversal: The Fourier transform of a reversed signal is the complex conjugate of the signal's transform.

Convolution: The Fourier transform of a pair of convolved signals is the product of their transforms.

Correlation: The Fourier transform of a correlation is the product of the conjugate of the first transform times the transform of the second one.

Multiplication: The Fourier transform of the product of two signals is the convolution of their transforms.

Differentiation: The Fourier transform of the derivative of a signal is that signal's transform multiplied by the frequency. In other words, differentiation linearly emphasizes (magnifies) higher frequencies.

Summary of FT properties

Domain scaling: The Fourier transform of a stretched signal is the equivalently compressed (and scaled) version of the original transform and *vice versa*.

Real images: The Fourier transform of a real-valued signal is symmetric around the origin. This fact can be used to save space and to double the speed of image FFTs by packing alternating scanlines into the real and imaginary parts of the signal being transformed.

Parseval's Theorem: The energy (sum of squared values) of a signal is the same as the energy of its Fourier transform.

Fourier transform: typical signals

1. **FT of delta-function**

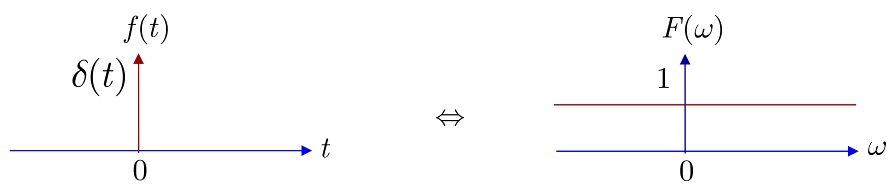
$$f(t) = \delta(t)$$

$$F(\omega) = \Im\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt$$

$$= e^{j\omega 0} = 1 \text{ (for all } \omega)$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)\,dt = f(t_0)$$

$$t_0 = 0; \int_{-\infty}^{\infty} \delta(t - 0)f(t)\,dt = f(0)$$



Fourier transform: typical signals

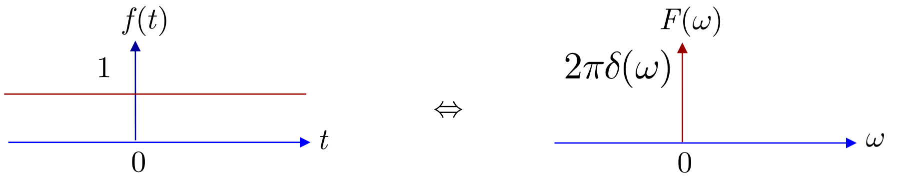
2. **FT of constant signal**

$$\Im\left\{1\right\} = 2\pi\delta(\omega)$$

$$\Im\left\{f(t)\right\} = \int_{-\infty}^{\infty} 1\cdot e^{-j\omega t}\,dt = \frac{1}{-j\omega}e^{-j\omega t}\Big|_{\omega=-\infty}^{\omega=\infty} \rightarrow$$ difficult to evaluate directly

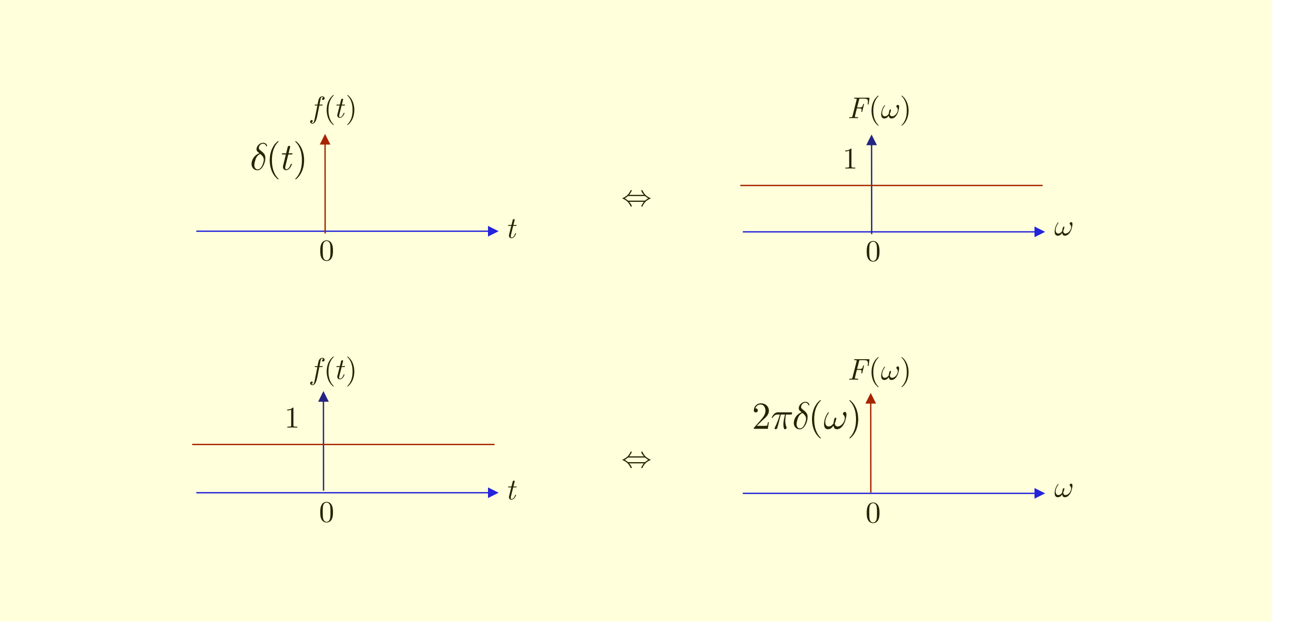
We use the duality to show that

$$\Im^{-1}\left\{\delta(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)e^{j\omega t}\,dt = \frac{1}{2\pi}e^{j\omega 0} = \frac{1}{2\pi}\cdot 1 \Rightarrow 1 = \Im^{-1}\left\{2\pi\delta(\omega)\right\}$$



Fourier transform: typical signals

Duality



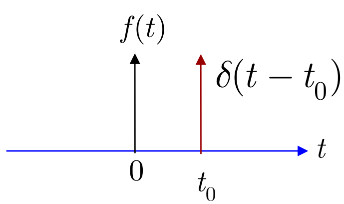
Fourier transform: typical signals

2. **FT of time shifted delta-function**

$$f(t) = \delta(t - t_0)$$

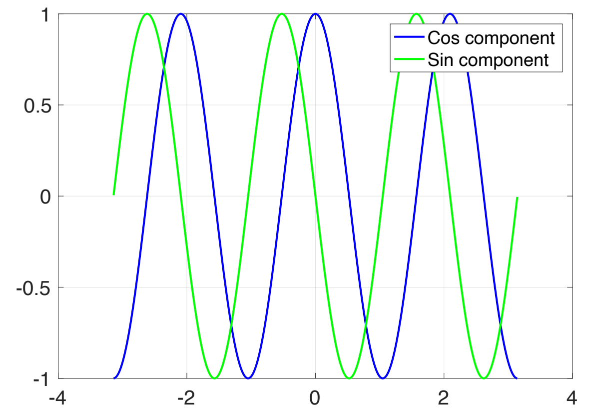
$$F(\omega) = \Im\left\{\delta(t - t_0)\right\} = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t}\, dt$$

$$= e^{-j\omega t_0} = \cos(\omega t_0) - j\sin(\omega t_0)$$ shifted phase



$\Leftrightarrow$

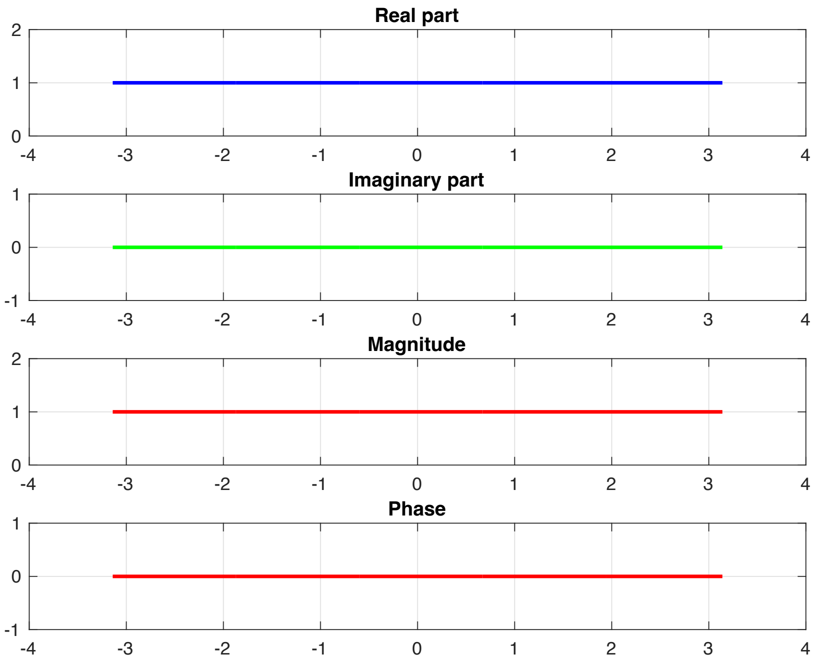
$$e^{-j\omega t_0} = \cos(\omega t_0) - j\sin(\omega t_0)$$



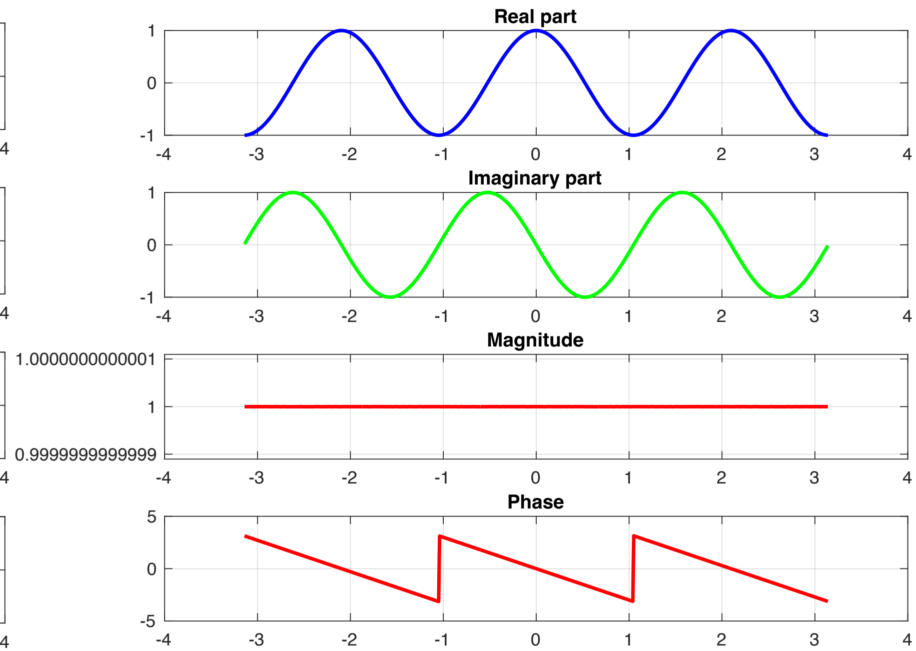
Attention: to the sign of sine component

Fourier transform: typical signals

$t_0 = 0$



$t_0 = 3$



Remarks:

- Magnitude is always equal to 1
- Phase is delayed by time shift

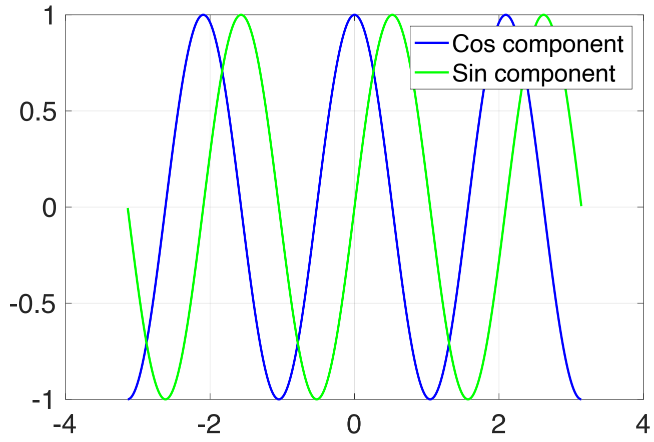
Fourier transform: typical signals

3. **FT of frequency shifted delta-function**

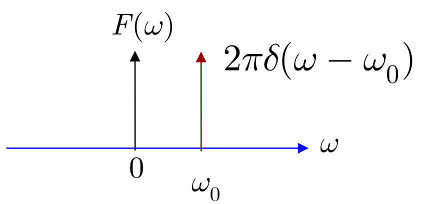
$$F(\omega) = \delta(\omega - \omega_0)$$

$$\Im^{-1}\left\{\delta(\omega-\omega_0)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}\,d\omega = \frac{1}{2\pi}e^{j\omega_0 t} \Rightarrow \Im\left\{e^{j\omega_0 t}\right\} = 2\pi\delta(\omega-\omega_0)$$

$$e^{j\omega t_0} = \cos(\omega t_0) + j\sin(\omega t_0)$$



$\Leftrightarrow$



Compare with the sign of time shifted delta function

Fourier transform: typical signals

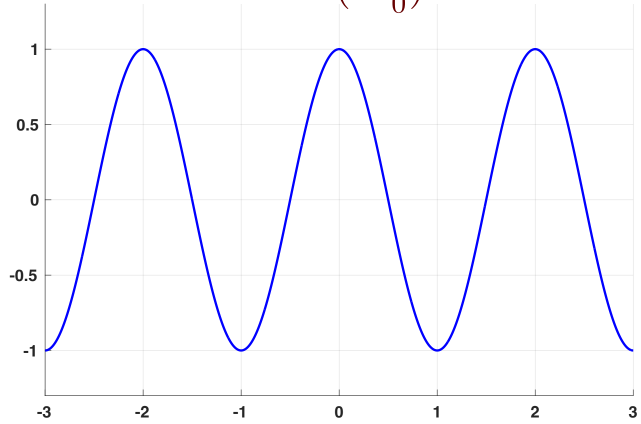
4. **FT of cosine function**

$$f(t) = \cos(\omega_0 t)$$

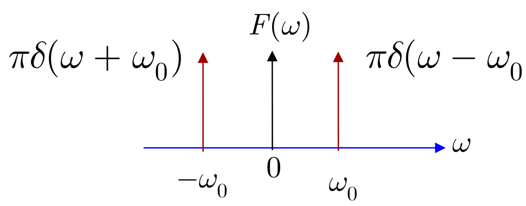
$$\Im\left\{\cos(\omega_0 t)\right\} = \frac{1}{2}\Im\left\{e^{j\omega_0 t} + e^{-j\omega_0 t}\right\} = \frac{1}{2}\Im\left\{e^{j\omega_0 t}\right\} + \frac{1}{2}\Im\left\{e^{-j\omega_0 t}\right\}$$
$$= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$\Im\left\{e^{j\omega_0 t}\right\} = 2\pi\delta(\omega - \omega_0)$$

$\cos(\omega t_0)$



$\Leftrightarrow$



Fourier transform: typical signals

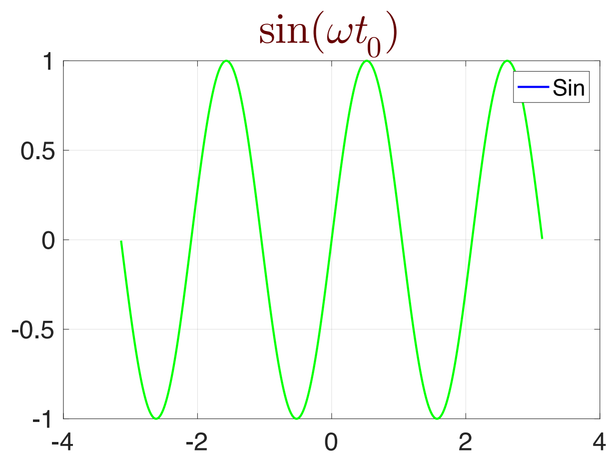
5. **FT of sine function**

$$\Im\left\{e^{j\omega_0 t}\right\} = 2\pi\delta(\omega-\omega_0)$$

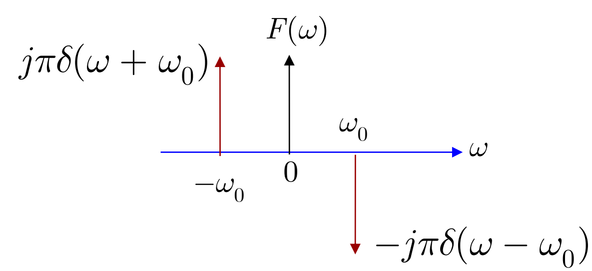
$$f(t) = \sin(\omega_0 t)$$

$$\Im\left\{\cos(\omega_0 t)\right\} = \frac{1}{2j}\Im\left\{e^{j\omega_0 t} - e^{-j\omega_0 t}\right\} = \frac{1}{2j}\Im\left\{e^{j\omega_0 t}\right\} - \frac{1}{2j}\Im\left\{e^{-j\omega_0 t}\right\}\Bigg|\times\frac{j}{j}$$

$$= -\frac{j}{2}\Im\left\{e^{j\omega_0 t}\right\} + \frac{j}{2}\Im\left\{e^{-j\omega_0 t}\right\} = -j\pi\delta(\omega-\omega_0) + j\pi\delta(\omega+\omega_0)$$

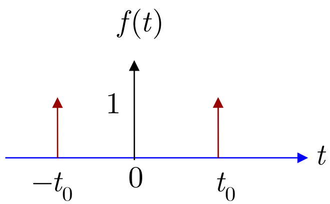


$\Leftrightarrow$



Fourier transform: typical signals

6. **FT of two symmetrically located delta-functions**

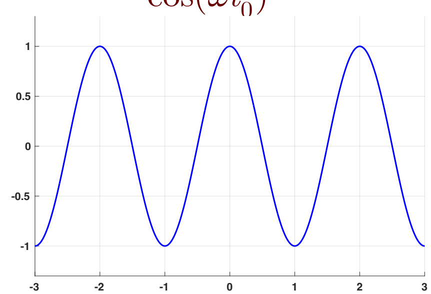


$$\Im\left\{\delta(t-t_0)+\delta(t+t_0)\right\}=\cos(\omega t_0)$$

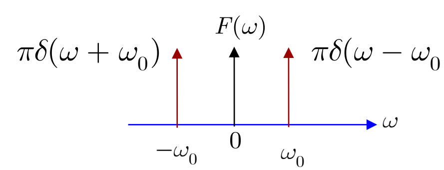
$$\begin{aligned}\Im\left\{\delta(t-t_0)+\delta(t+t_0)\right\}&=\Im\left\{\delta(t-t_0)\right\}+\Im\left\{\delta(t+t_0)\right\}=e^{-j\omega t_0}+e^{j\omega t_0}\\&=(\cos(\omega t_0)-j\sin(\omega t_0))+(\cos(\omega t_0)+j\sin(\omega t_0))\\&=2\cos(\omega t_0)\end{aligned}$$

See the duality with the spectrum of cosine function

$\cos(\omega t_0)$

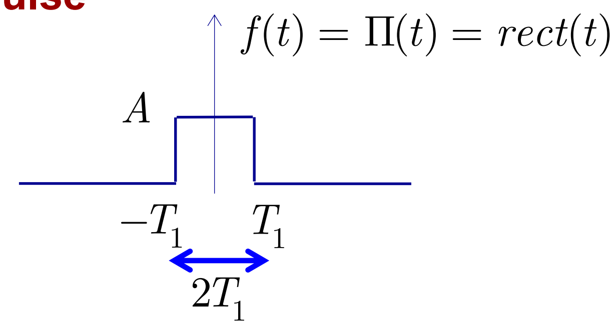


$\Leftrightarrow$



Fourier transform: typical signals

7. **FT of a square pulse**



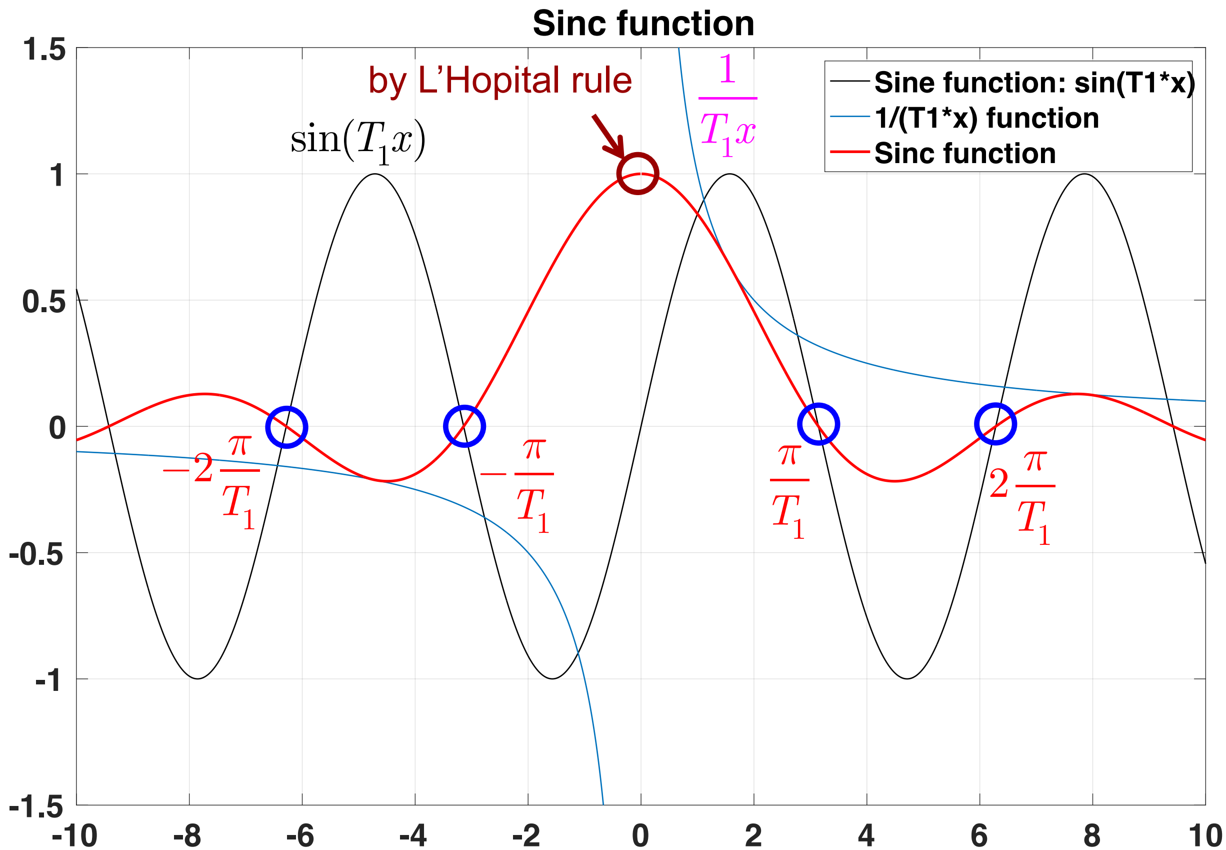
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = A \int_{-T_1}^{T_1} 1 e^{-j\omega t}\, dt$$

$$= A \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-T_1}^{T_1} = A \frac{1}{-j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1} \right] = A \frac{1}{j\omega}\left[e^{j\omega T_1} - e^{-j\omega T_1} \right]$$

$$= 2A \frac{1}{\omega} \sin(\omega T_1) \Rightarrow 2A\, T_1 \frac{\sin(\omega T_1)}{\omega T_1} = 2A\, T_1 \mathrm{sinc}(\omega T_1)$$

Sinc function

$\mathrm{sinc}(T_1 x) = \sin(T_1 x)/(T_1 x)$



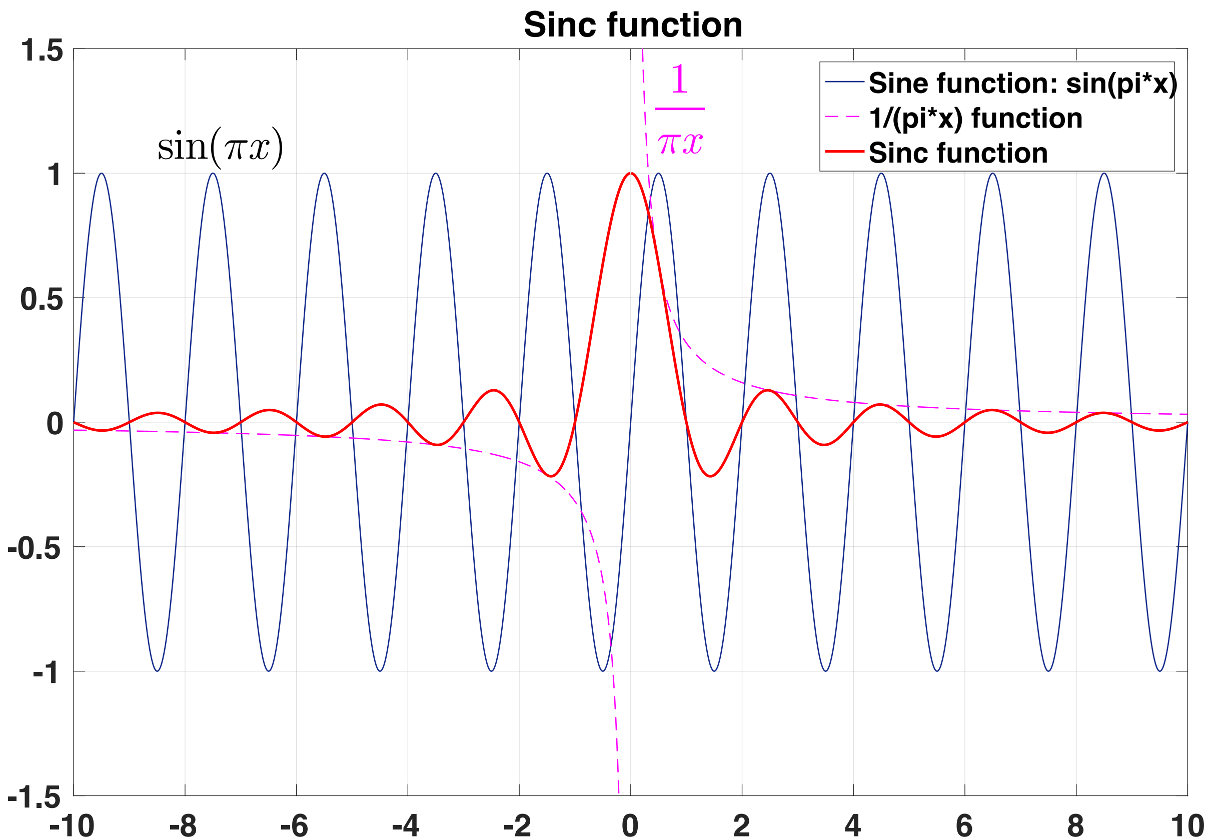
$T_1 = 1$

$\pi = 3.14$

$\mathrm{sinc}(T_1 x) = 0$ for all $\pm\frac{\pi}{T_1}, \pm 2\frac{\pi}{T_1}, \pm 3\frac{\pi}{T_1}, \cdots$

Sinc function

$$\mathrm{sinc}(\pi x) = \sin(\pi x)/\pi x$$

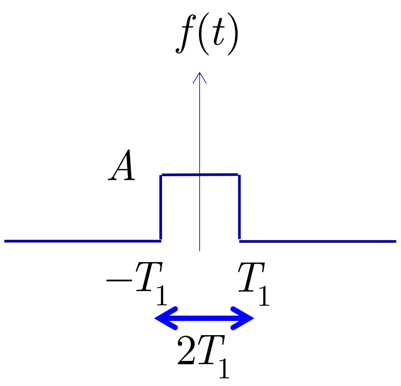


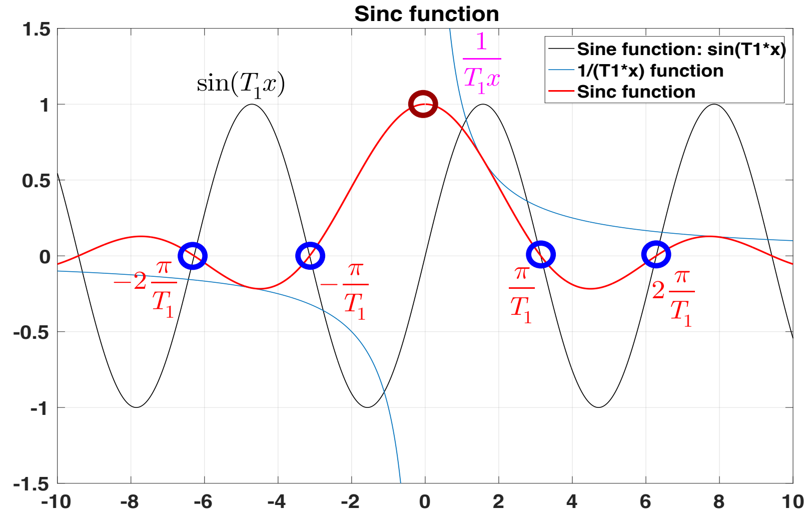
$T_1 = \pi$

$\mathrm{sinc}(\pi x) = 0$ for all integers except $x = 0$ $\rightarrow \pm\frac{\pi}{T_1}, \pm 2\frac{\pi}{T_1}, \pm 3\frac{\pi}{T_1}, \cdots$

Sinc function

$$\text{sinc}(T_1 x) = \sin(T_1 x)/(T_1 x)$$



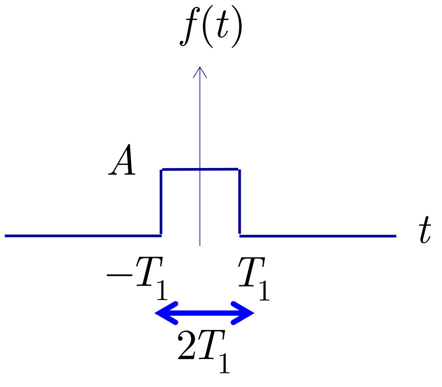


Interpretation: $\frac{\text{Full period of } 2\pi}{\text{Duration of signal } 2T_1} = \frac{2\pi}{2T_1} \rightarrow \frac{\pi}{T_1}$

Remark (Gonzalez): $\omega = 2\pi\mu$ $\quad \frac{2\pi}{2T_1} \Rightarrow \frac{1}{2T_1}$

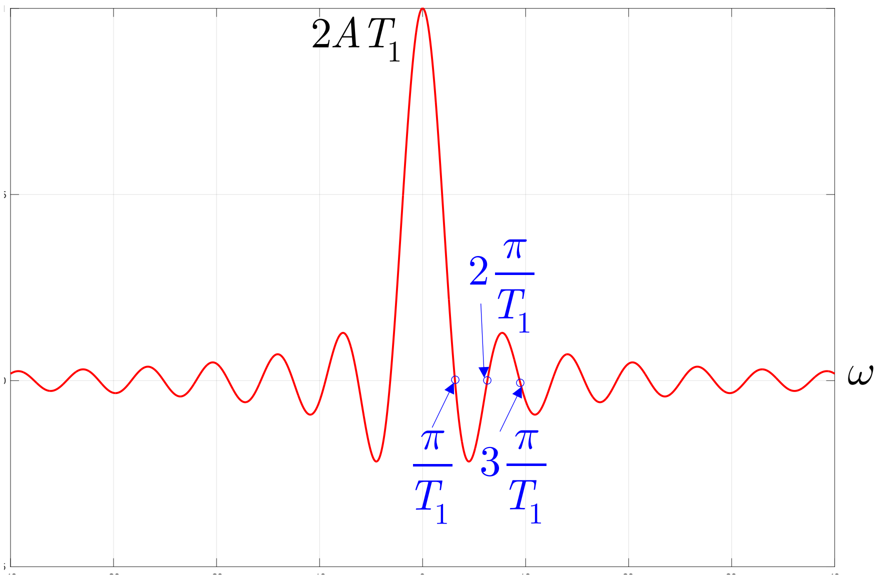
$\mu = \omega/2\pi$

Fourier transform: typical signals



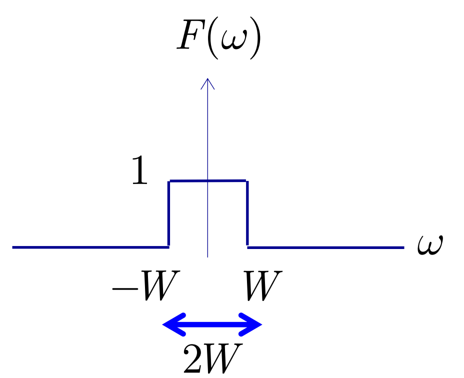
$\Leftrightarrow$

$F(\omega) = 2AT_1 \text{sinc}(\omega T_1)$



Fourier transform: examples

8. **FT of an "ideal lowpass filter"**

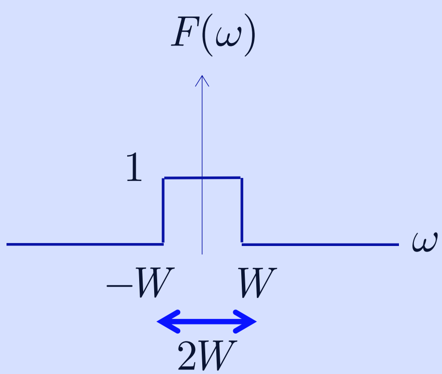


$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-W}^{W} 1e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\frac{1}{jt}e^{j\omega t}\Big|_{-W}^{W} = \frac{1}{2\pi}\frac{1}{jt}\left[e^{jtW} - e^{-jtW}\right] = \frac{1}{\pi t}\underbrace{\frac{1}{2j}\left[e^{jtW} - e^{-jtW}\right]}_{\sin(tW)}$$

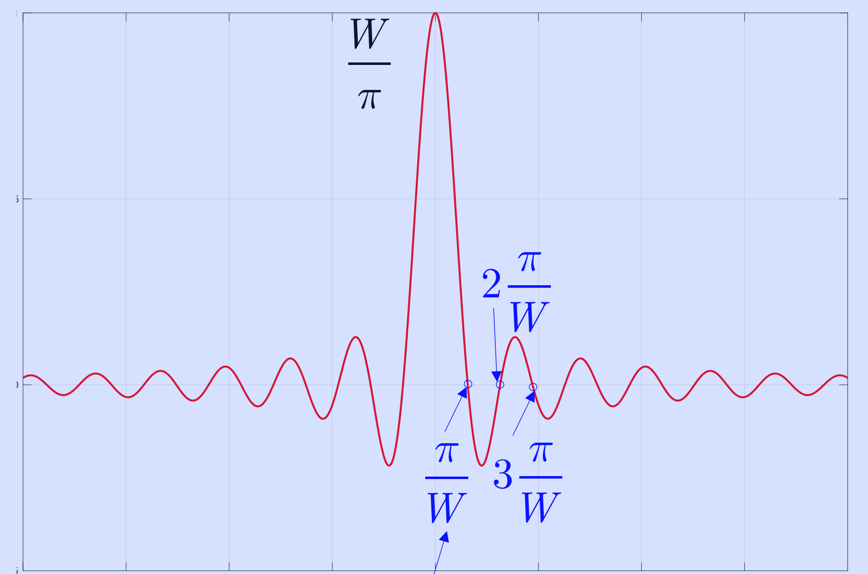
$$= \frac{1}{\pi t}\sin(tW) \Rightarrow \frac{W}{\pi}\frac{\sin(tW)}{tW} = \frac{W}{\pi}\mathrm{sinc}(tW)$$

Fourier transform: typical signals



$\Leftrightarrow$

$$f(t) = \frac{W}{\pi}\,\mathrm{sinc}(tW)$$



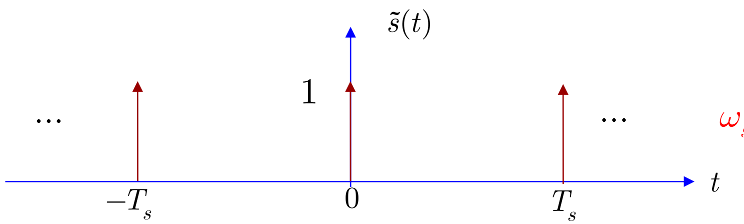
Remark (Gonzalez):

$$\omega = 2\pi\mu \qquad W = 2\pi V \qquad \frac{\pi}{W} \to \frac{\pi}{2\pi V} = \frac{1}{2V}$$

$$\mu = \omega/2\pi$$

Fourier transform: examples

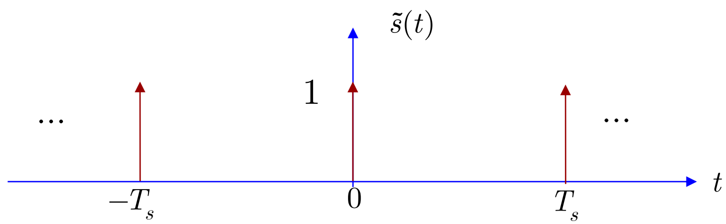
9. **FT of "train of delta-pulses"**

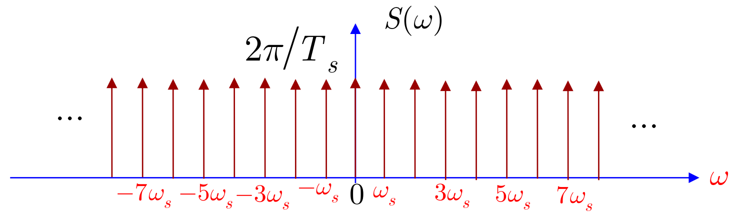


$$\omega_s = \frac{2\pi}{T_s}$$

$$\tilde{s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

$$\Im\left\{ \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \right\} = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$





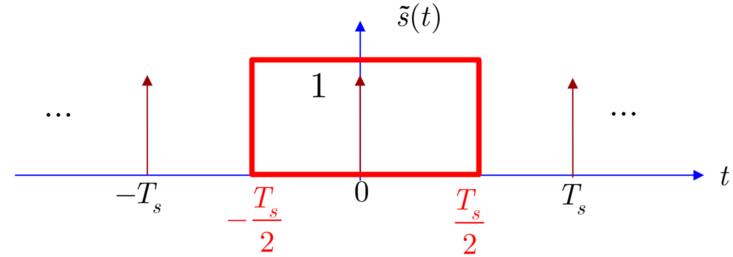
Fourier transform: typical signals

Proof

1. Represent a periodic function $\tilde{s}(t)$ as the FS:

$$\tilde{s}(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\omega_s t} \qquad \omega_s = \frac{2\pi}{T_s}$$ is a fundamental frequency

where $$S_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \tilde{s}(t) e^{-j\omega t} dt$$



$$S_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t) e^{-j\omega t} dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t-0) e^{-j\omega t} dt = \frac{1}{T_s} e^{-j\omega t}\Big|_{t=0} = \frac{1}{T_s} e^{-j\omega 0} = \frac{1}{T_s}$$

$$\Rightarrow \tilde{s}(t) = \sum_{k=-\infty}^{\infty} S_k e^{jk\omega_s t} = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

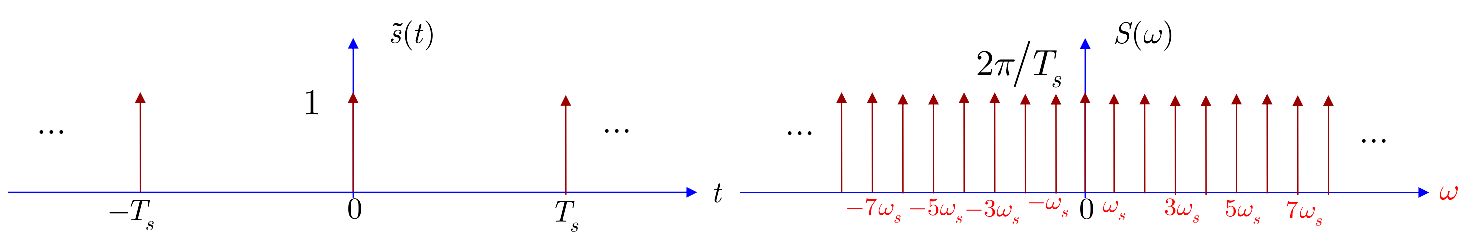
Fourier transform: typical signals

2. Apply the FT to this representation of $\tilde{s}(t)$:

$$S(\omega) = \int_{-\infty}^{\infty} \tilde{s}(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left[\frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t} \right] e^{-j\omega t} dt = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} \left[e^{jk\omega_s t} \right] e^{-j\omega t} dt}_{\Im\left\{e^{jk\omega_s t}\right\} = 2\pi\delta(\omega - k\omega_s)}$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - k\omega_s) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \Rightarrow [\text{Gonzalez}] : \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_s} \right)$$

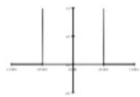
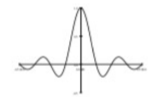
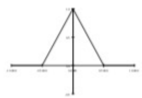
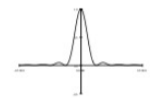
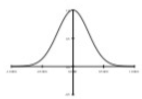
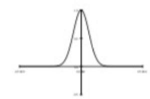
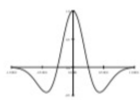
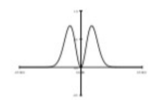
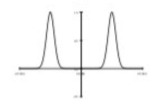
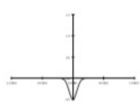
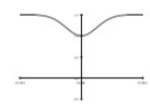
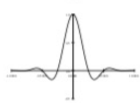
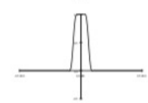
$$\Im\left\{ \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \right\} = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$



Fourier transform: examples for HW

1. Gaussian signal
2. Triangular signal
3. Time bounded by square weighted cosinusoidal/sinusoidal signal
4. Time bounded by Gaussian weighed cosinusoidal/sinusoidal signal – a basis of Gabor filters

Summary of useful signals

Name		Signal		Transform	
impulse		$\delta(x)$	$\Leftrightarrow$	1	
shifted impulse		$\delta(x-u)$	$\Leftrightarrow$	$e^{-j\omega u}$	
box filter		$\mathrm{box}(x/a)$	$\Leftrightarrow$	$a\mathrm{sinc}(a\omega)$	
tent		$\mathrm{tent}(x/a)$	$\Leftrightarrow$	$a\mathrm{sinc}^2(a\omega)$	
Gaussian		$G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega;\sigma^{-1})$	
Laplacian of Gaussian		$(\frac{x^2}{\sigma^4}-\frac{1}{\sigma^2})G(x;\sigma)$	$\Leftrightarrow$	$-\frac{\sqrt{2\pi}}{\sigma}\omega^2 G(\omega;\sigma^{-1})$	
Gabor		$\cos(\omega_0 x)G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega\pm\omega_0;\sigma^{-1})$	
unsharp mask		$(1+\gamma)\delta(x)$ $-\gamma G(x;\sigma)$	$\Leftrightarrow$	$(1+\gamma)-$ $\frac{\sqrt{2\pi}\gamma}{\sigma}G(\omega;\sigma^{-1})$	
windowed sinc		$\mathrm{rcos}(x/(aW))$ $\mathrm{sinc}(x/a)$	$\Leftrightarrow$	(see Figure 3.29)	

Summary on FS, FT and Discrete Time FT

Periodic signals

Fourier Series (FS)

$\tilde{f}(t)$ periodic with T

$$\omega_0 = 2\pi/T$$

$$F_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{f}(t) e^{-jk\omega_0 t} dt$$

$$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} F_k e^{jkw_0 t}$$

Aperiodic cont. signals

Fourier Transform (FT)

$$f(t) : -\infty < t < \infty$$

$$F(\omega) : -\infty < \omega < \infty$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Our next target

Aperiodic discrete-time signals in time domain

Roadmap

Theme 7: The 2D Discrete Fourier Transform

- Chapter 1: Basics of DSP, complex numbers
- Chapter 2: Continuous periodic signals – FS
- Chapter 3: Continuous aperiodic signals -FT (cont. time – cont. frequency)
- Chapter 4: Sampling:
  - Chapter 4.1: Sampling in time domain: Continuous aperiodic band-limited signals – (sampling in time domain) – cont. frequency – discrete time (DTCF FT)
  - Chapter 4.2: Sampling in frequency domain: Continuous aperiodic band-limited signals – (sampling in both time and frequency domain) – discrete frequency – discrete time – discrete FT (DFT)
  - Chapter 4.3: Extension to 2D DFT

Theme 8: Frequency domain filtering, sampling and aliasing

Theme 9: Unitary (data independent) transforms – beyond DFT

Theme 11: Machine learnable (or data dependent) transforms (in Theme 11: Lossy image compression)

Roadmap

We have considered:

1. Periodic functions in the continuous-time domain - FS - (discrete Fourier series coefficients)
2. Aperiodic functions in the continuous-time domain - FT - (continuous Fourier spectra)

Next, we will consider:

1. Aperiodic functions in the **discrete**-time domain - FT - (continuous Fourier spectra – periodicity): discrete-time FT
2. Aperiodic functions in the **discrete**-frequency domain - FT - (**discrete** Fourier spectra – periodicity): discrete-time and -frequency FT or simply discrete FT

Roadmap

Theme 7: The 2D Discrete Fourier Transform

Chapter 1: Basics of DSP, complex numbers

Chapter 2: Continuous periodic signals – FS

Chapter 3: Continuous aperiodic signals -FT (cont. time – cont. frequency)

Chapter 4: Sampling:

Chapter 4.1: Sampling in time domain: Continuous aperiodic band-limited signals – (sampling in time domain) – cont. frequency – discrete time (DTCF FT)

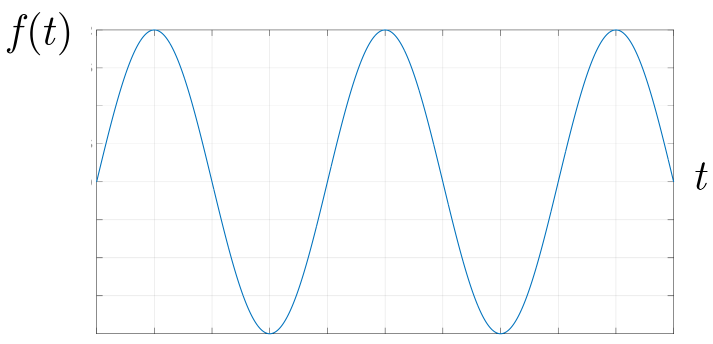
Chapter 4.2: Sampling in frequency domain: Continuous aperiodic band-limited signals – (sampling in both time and frequency domain) – discrete frequency – discrete time – discrete FT (DFT)

Chapter 4.3: Extension to 2D DFT

Discrete-time domain FT: sampling

Intuition on sampling

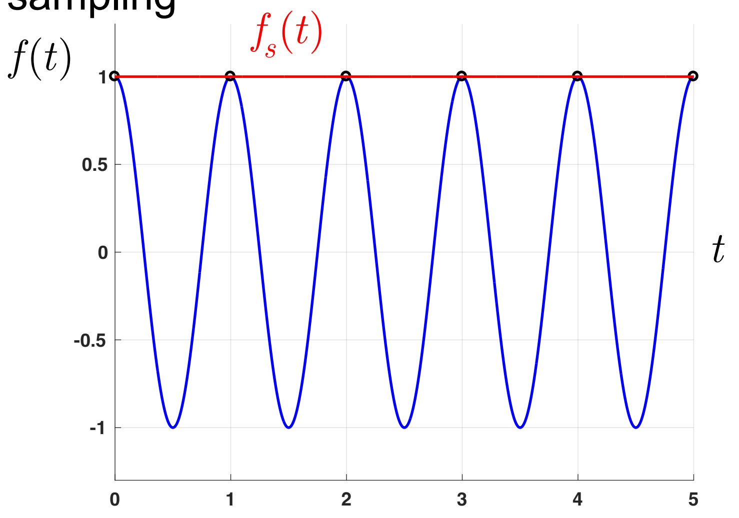
Given a **continuous-time** signal of some frequency $\omega_0 = 2\pi / T_0$



Goal (sampling): to represent this signal by the equally spaced samples (discrete-time representation)

Discrete-time domain FT: sampling

1. “Very slow” sampling

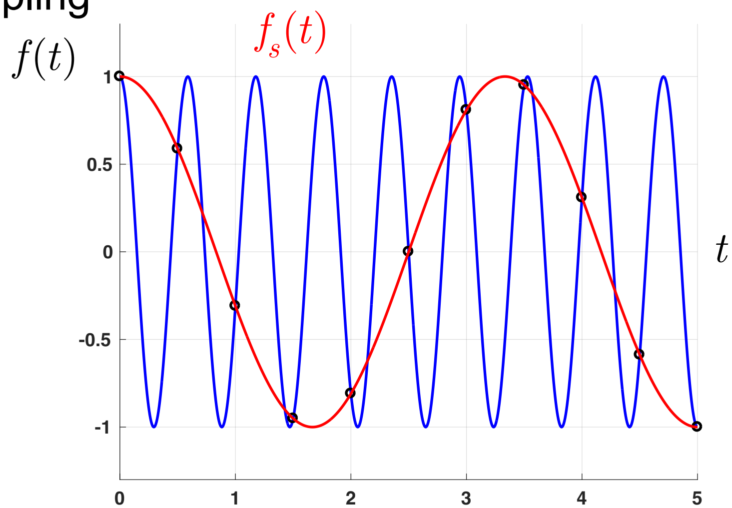


Problem: the obtained samples do not look like an original signal!

Our goal is to retain as much as possible information about the original signal

Discrete-time domain FT: sampling

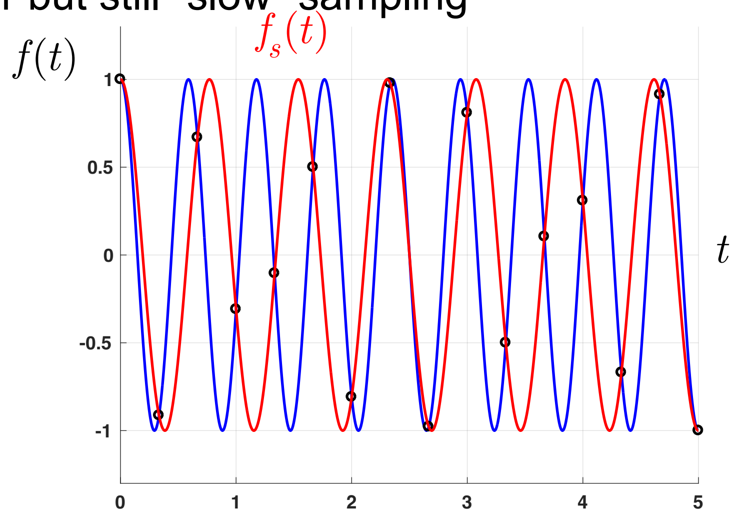
2. "Slow" sampling



Problem: the obtained samples look as a signal with lower frequency

Discrete-time domain FT: sampling

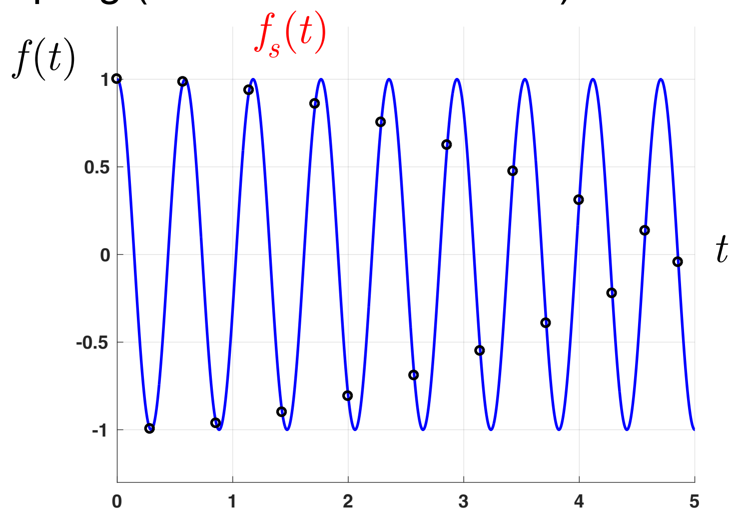
3. Even higher but still "slow" sampling



Problem: still not a correct signal

Discrete-time domain FT: sampling

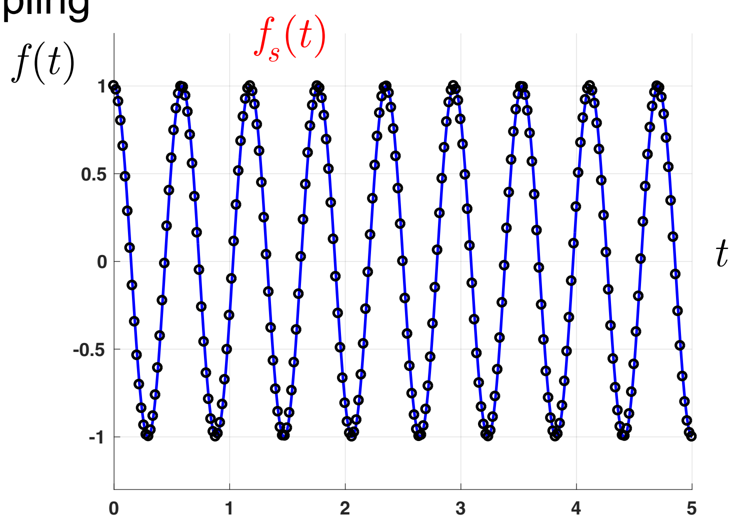
4. Just OK sampling (both curves coincides)



Problem: fine shape representation

Discrete-time domain FT: sampling

5. “Over” sampling



Problem: shape is preserved but too many samples are taken without a real need

Question: how to sample correctly?   How many samples are needed?

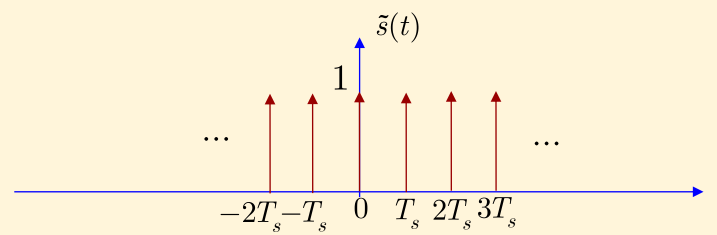
Impulse sampling

Let $f(t)$ be a **continuous-time** signal that has $f(t) \Leftrightarrow F(\omega)$

We will sample this signal using an impulse train (train of delta-functions)

$$f_s(t) = f(t)\tilde{s}(t)$$

where $\tilde{s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$



$$\omega_s = \frac{2\pi}{T_s}$$

In the frequency domain we have (based on property 8 of product of two signals)

$$\Im\left\{f(t)h(t)\right\} = \frac{1}{2\pi} F\left(\omega\right) * H\left(\omega\right) \quad (b)$$

$$F_s(\omega) = \frac{1}{2\pi} F\left(\omega\right) * S\left(\omega\right)$$

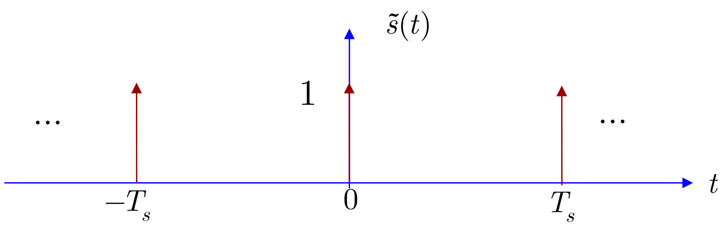
$$F\left(\omega\right) \Leftrightarrow f(t)$$

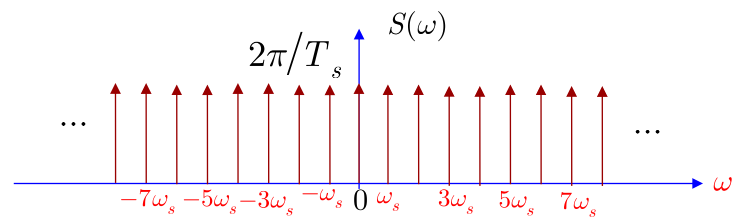
$$S\left(\omega\right) \Leftrightarrow \tilde{s}(t)$$

Impulse sampling

Recall from the property 9 of typical signals (the train of delta functions)

$$\Im\left\{\sum_{k=-\infty}^{\infty}\delta(t-kT_s)\right\}=\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)$$



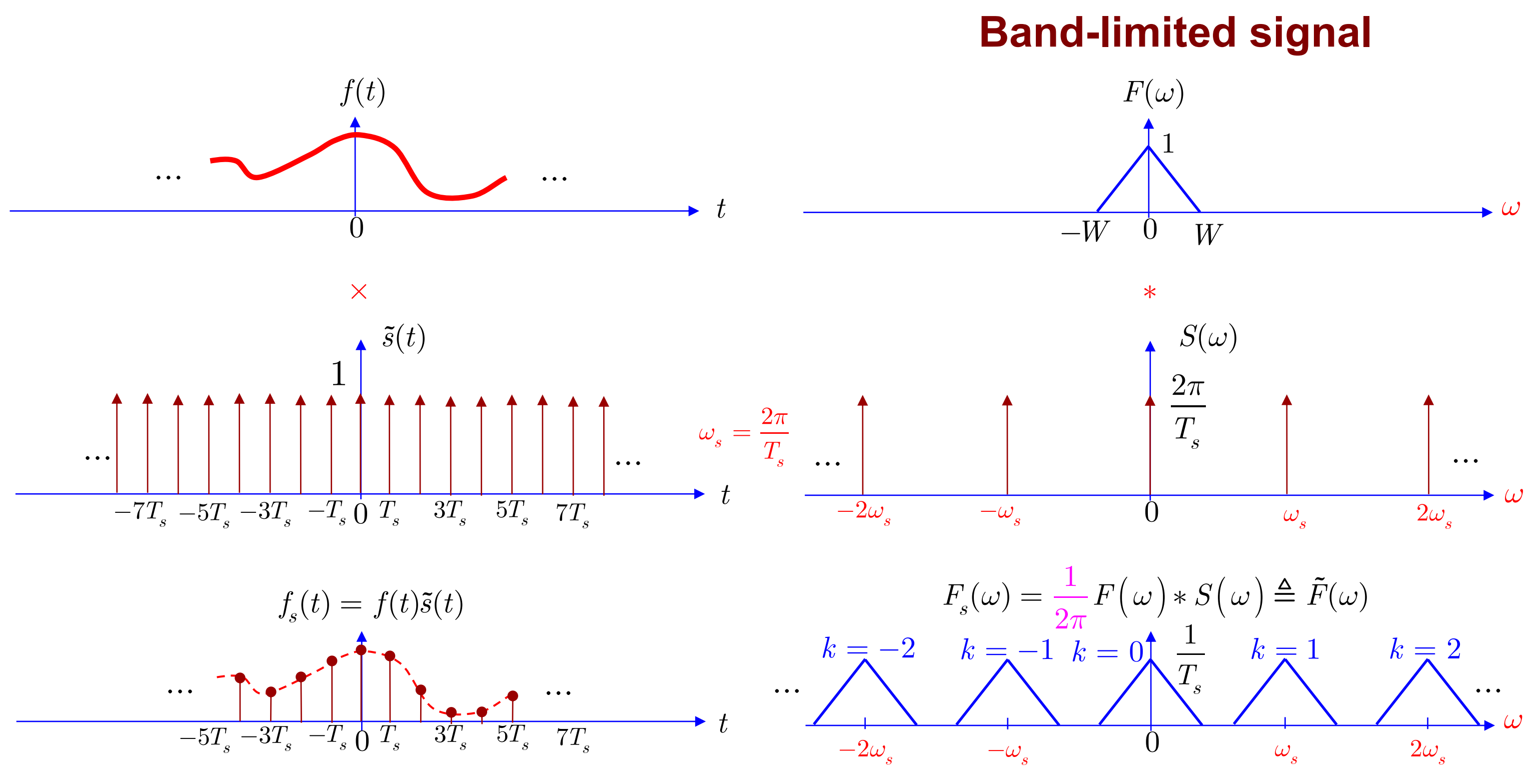


Thus:

$$F_s(\omega)=\frac{1}{2\pi}F(\omega)*S(\omega)=\frac{2\pi}{T_s}\frac{1}{2\pi}F(\omega)*\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)=\frac{1}{T_s}F(\omega)*\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)$$

$$\boxed{F_s(\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}F(\omega-k\omega_s)}$$

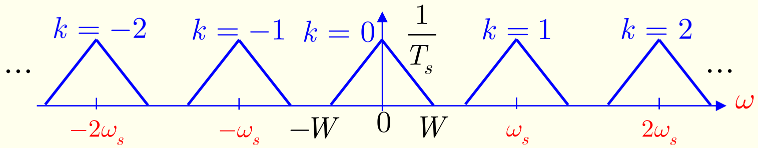
Impulse sampling – graphical interpretation



Impulse sampling – avoiding aliasing

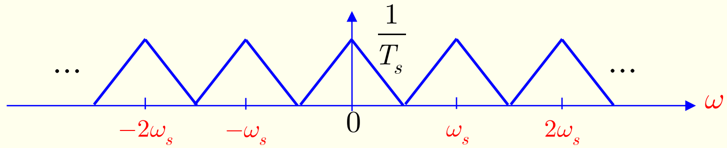
Effects of sampling

$$F_s(\omega) = \frac{1}{2\pi} F(\omega) * S(\omega) \triangleq \tilde{F}(\omega)$$



Oversampling

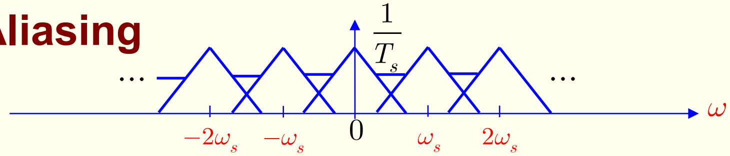
(sampling over the Nyquist rate)



Just properly sampled

(sampling at the Nyquist rate)

Aliasing



Undersampling

(sampling below the Nyquist rate)

Nyquist rate

Unique conditions

$$\omega_s/2 > W \Rightarrow \omega_s > 2W$$

$$\omega_s = \frac{2\pi}{T_s} > 2W \Rightarrow T_s < \frac{\pi}{W}$$

The sampling theorem

The sampling theorem: Let $f(t) \overset{\Im}{\Leftrightarrow} F(\omega)$ represents *a band-limited signal*, such that $F(\omega) = 0$ for $|\omega| > W$. The unique reconstruction of $f(t)$ is possible from its sampled version $f_s(t)$, if the sampling frequency $\omega_s / 2 \geq W \Rightarrow \omega_s \geq 2W$.

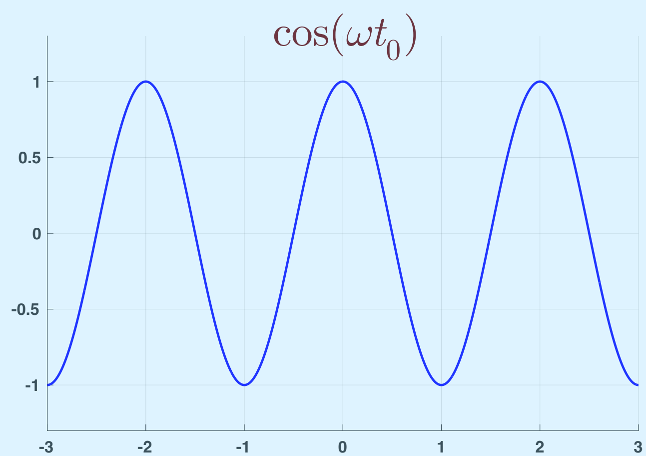
The Nyquist rate: $2W$ is the minimum sampling rate required by the Sampling Theorem.

Sampling cos/sin signals

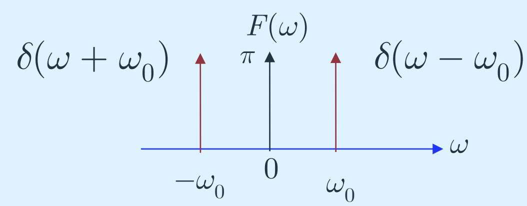
The FT:

Let $f(t)=\cos(\omega_0 t)$ $\quad \omega_0=\frac{2\pi}{T_0}$ $\quad \Im\left\{\cos(\omega_0 t)\right\}=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$

Assume $T_0=2, \omega_0=\frac{2\pi}{2}=\pi$



$\Leftrightarrow$



Compute FT of impulse sampled signal for $\omega_s=\frac{2\pi}{T_s}$ sampling frequency

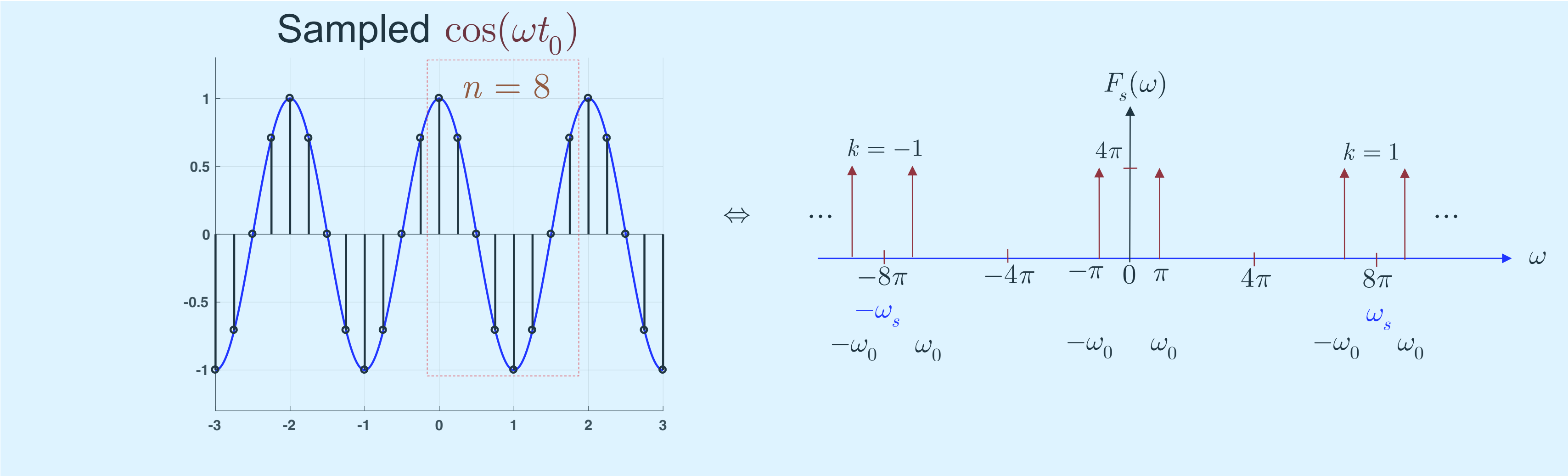
$$F_s(\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}F(\omega-k\omega_s)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty}\left[\pi\delta(\omega-\omega_0-k\omega_s)+\pi\delta(\omega+\omega_0-k\omega_s)\right]$$

Sampling cos/sin signals

Cosine wave with the period $T_0 = 2, \omega_0 = \frac{2\pi}{2} = \pi$

Sampling interval $T_s = \frac{1}{4} \to \omega_s = \frac{2\pi}{T_s} = 8\pi$

Number of samples per period $n = \frac{T_0}{T_s} = \frac{2}{1/4} = 8$



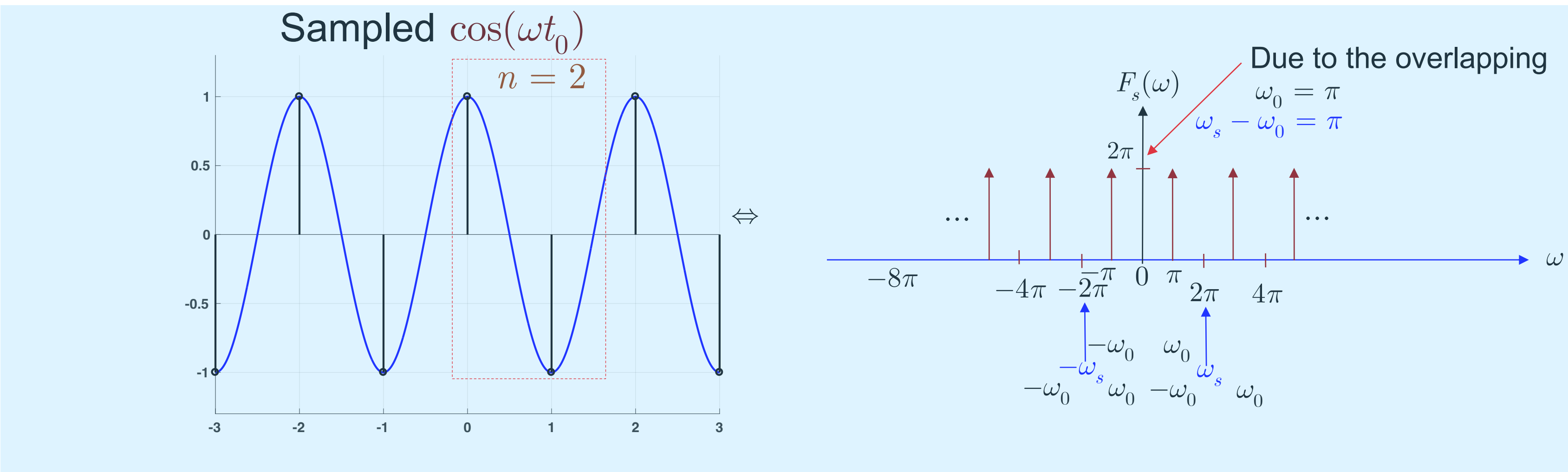
$$F_s(\omega) = \frac{\pi}{T_s} \sum_{k=-\infty}^{\infty} \left[\delta(\omega - \omega_0 - k\omega_s) + \delta(\omega + \omega_0 - k\omega_s) \right]$$

Sampling cos/sin signals

Cosine wave with the period $T_0 = 2, \omega_0 = \frac{2\pi}{2} = \pi$

Sampling interval $T_s = 1 \rightarrow \omega_s = \frac{2\pi}{T_s} = 2\pi$

Number of samples per period $n = \frac{T_0}{T_s} = \frac{2}{1} = 2$

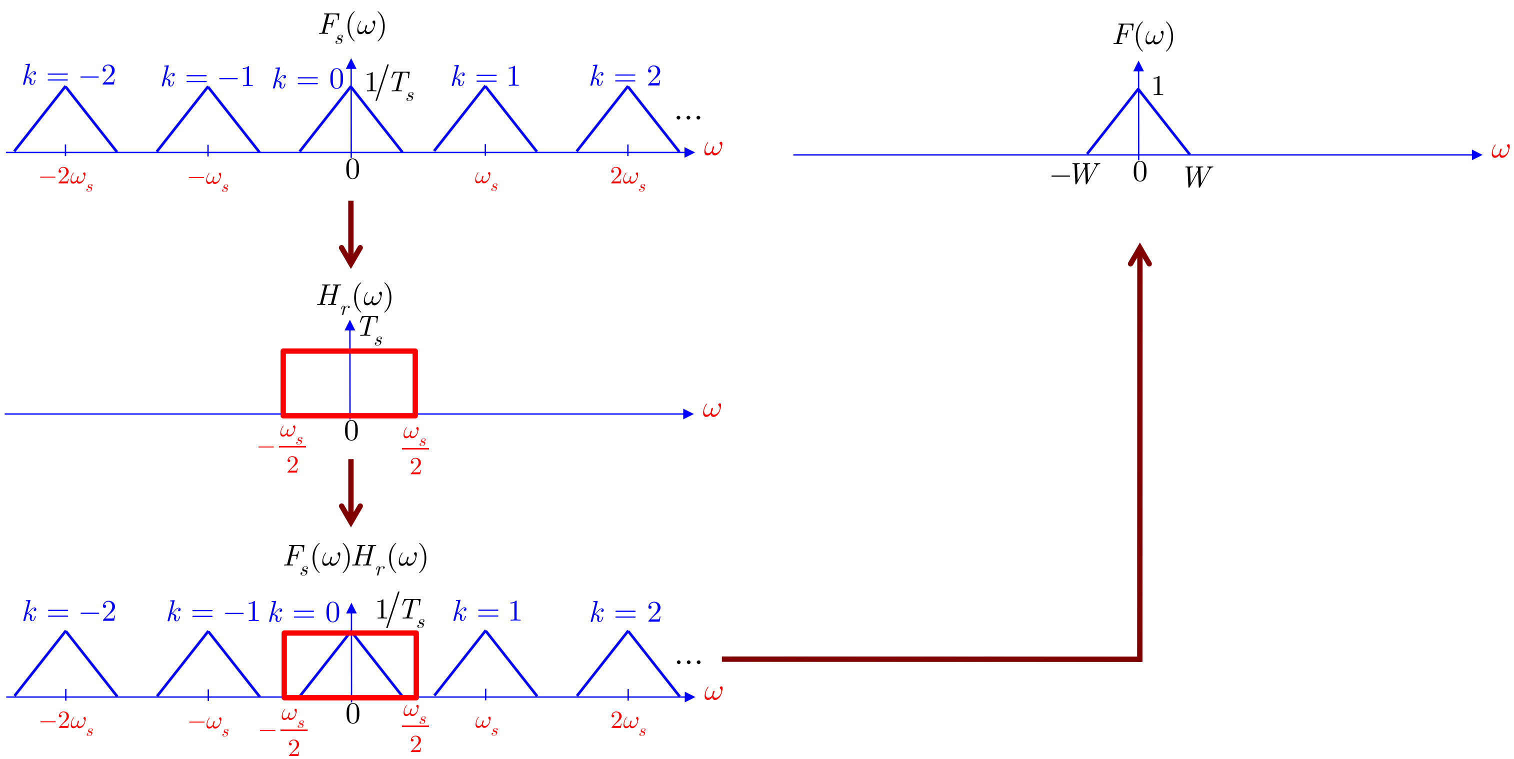


$$F_s(\omega) = \frac{\pi}{T_s} \sum_{k=-\infty}^{\infty} \left[\delta(\omega - \omega_0 - k\omega_s) + \delta(\omega + \omega_0 - k\omega_s) \right]$$

 https://www.youtube.com/watch?v=OFIQwsodJrg

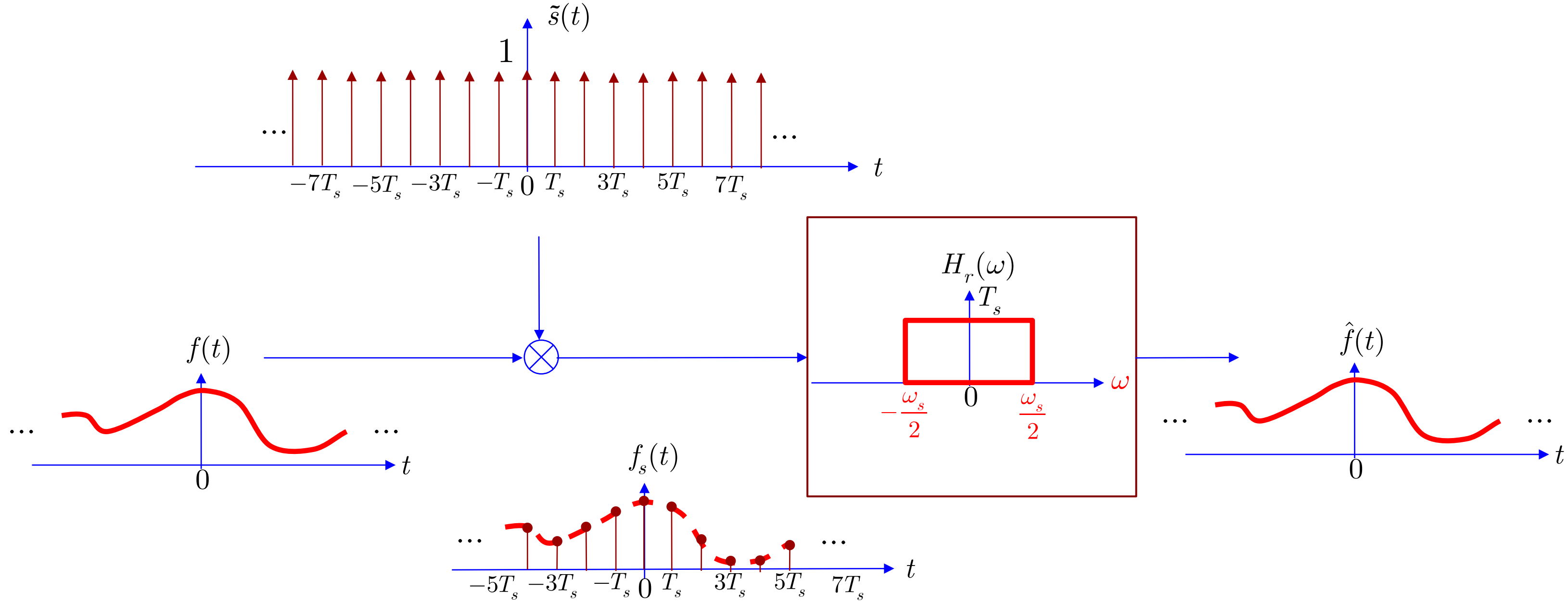
Ideal reconstruction

Goal: to return from the the sampled (periodic) spectrum to the original one



Ideal band-limited interpolation

Sampling and ideal reconstruction

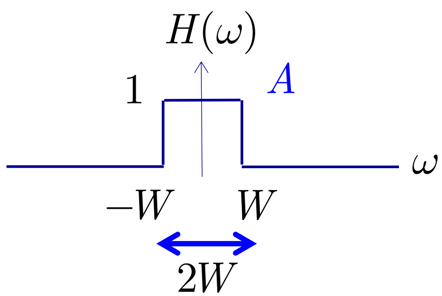


Ideal band-limited interpolation

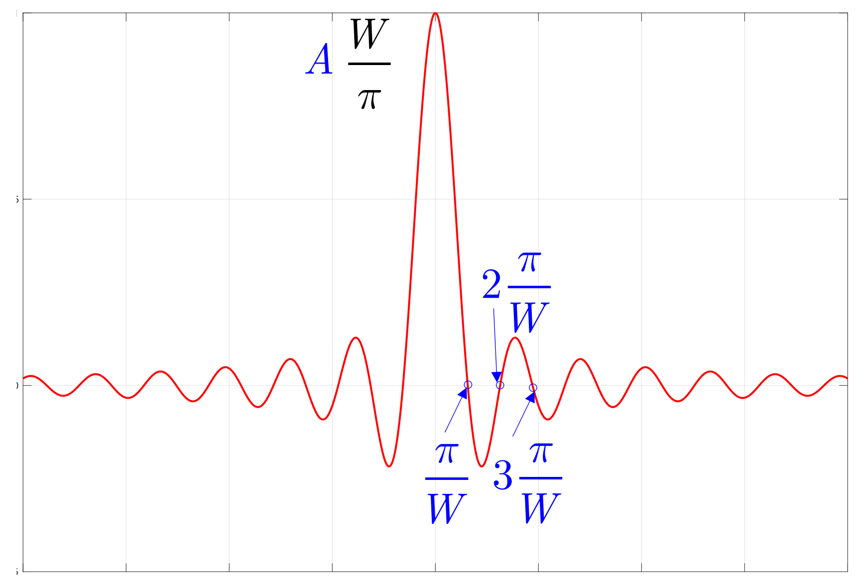
Ideal band-limited interpolation

Note: we know that

$$h(t) = A\frac{W}{\pi}\mathrm{sinc}(tW)$$



$\Leftrightarrow$



In our case: $A = T_s$

$$W = \frac{\omega_s}{2} \qquad W = \frac{2\pi}{T_s}\frac{1}{2} = \frac{\pi}{T_s}$$

$$\Rightarrow h_r(t) = T_s \frac{\omega_s}{2}\frac{1}{\pi}\mathrm{sinc}\left(\frac{\omega_s t}{2}\right) = \mathrm{sinc}\left(\frac{\omega_s t}{2}\right)$$

$$T_s = \frac{2\pi}{\omega_s}$$

Ideal band-limited interpolation

Recovered original spectrum is: $\hat{F}(\omega) = F(\omega) = F_s(\omega)H_r(\omega)$

$$\Rightarrow \hat{f}(t) = f_s(t) * h_r(t)$$

$$h_r(t) = \mathrm{sinc}\left(\frac{\omega_s t}{2}\right)$$

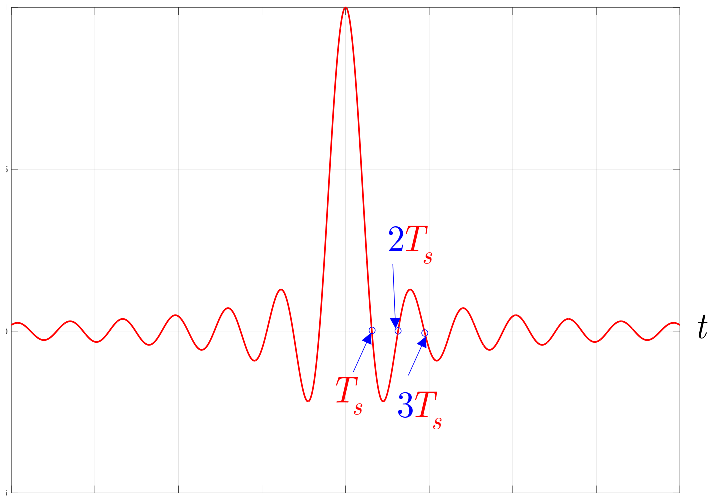
$$f_s(t) = f(t)\tilde{s}(t) = f(t)\sum_{k=-\infty}^{\infty}\delta(t - kT_s) = \sum_{k=-\infty}^{\infty} f(kT_s)\delta(t - kT_s)$$

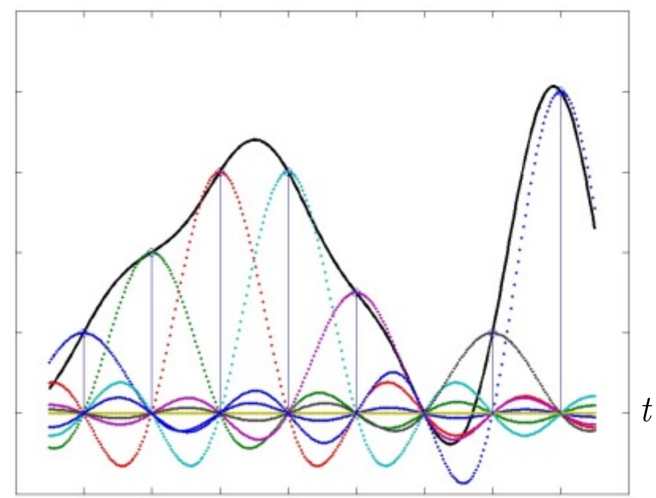
$$\Rightarrow \hat{f}(t) = f_s(t) * h_r(t) = \sum_{k=-\infty}^{\infty} f(kT_s)\delta(t - kT_s) * \mathrm{sinc}\left(\frac{\omega_s t}{2}\right)$$

$$= \sum_{k=-\infty}^{\infty} f(kT_s)\mathrm{sinc}\left(\frac{\omega_s (t - kT_s)}{2}\right)$$

Infinite weighted sum of time-shifted sinc functions

Ideal band-limited interpolation





We consider this system to be “**ideal**” since we have an *infinite time extent.*

It is unpractical. The extent is limited that leads to the distortions – thus non-ideal.

Discrete-Time FT (DT-FT)

Definition of DT-FT

Discrete-Time FT (DT-FT)

$$f_n \quad : n = ..., -2, -1, 0, 1, 2, ...$$

$$\tilde{F}(\omega) : -\pi < \omega < \pi$$

$$\tilde{F}(\omega) = \sum_{n=-\infty}^{\infty} f_n e^{-j\omega n}$$

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega$$

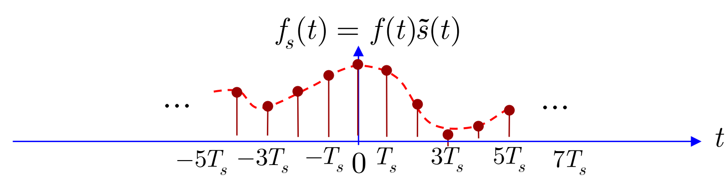
Discrete-Time FT via sampling

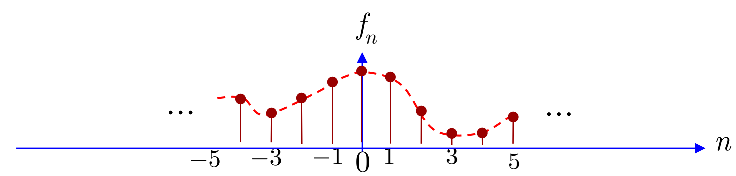
Sampling

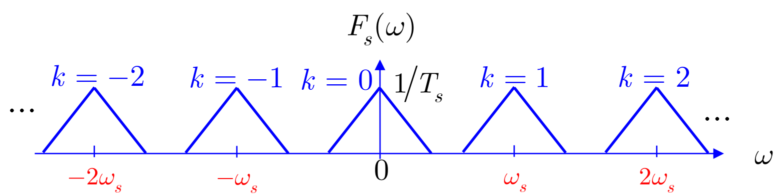
(we start with a cont. time signal and sample it; Here we keep the real time scale)

DT-FT

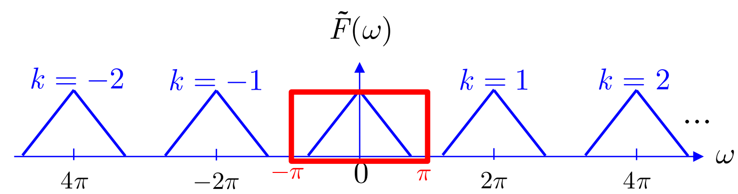
(we start with a discrete time signal with *no reference* to the real time scale)







$$-\infty < \omega < \infty$$



$$-\pi < \omega < \pi$$

In spectrum, we have a periodicity with

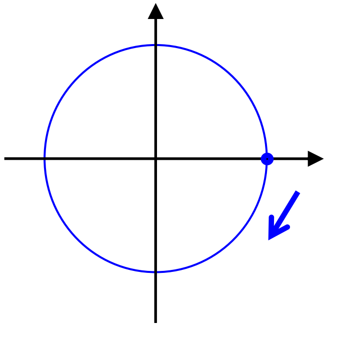
$\omega_s = 2\pi/T_s$ $\longrightarrow$ No absolute time scale just relative one

$$T_s = 1 \rightarrow \omega_s = 2\pi/T_s = 2\pi$$

Discrete-Time FT

- CT-FT has a frequency range $-\infty < \omega < \infty$
- DT-FT has a frequency range $-\pi < \omega < \pi$

Periodicity: $$\begin{aligned}\tilde{F}(\omega + 2\pi) &= \sum_{n=-\infty}^{\infty} f_n e^{-j(\omega+2\pi)n} \\ &= \boxed{\sum_{n=-\infty}^{\infty} f_n e^{-j\omega n}}\, \underbrace{e^{-j2\pi n}}_{1} \quad \Big| n \text{ is integer} \\ &= \tilde{F}(\omega)\end{aligned}$$

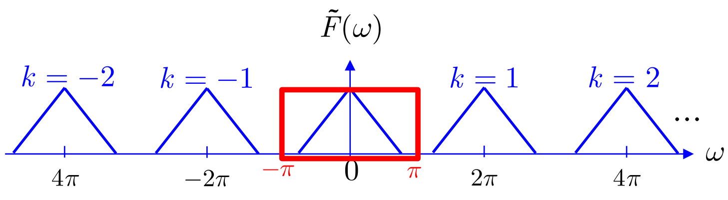


For every $n = 0, \pm 1, \pm 2, \cdots$ we make the entire circle and return to the initial point. That is why in the discrete case the fundamental frequency is periodic with the period 2π .

Summary on FS, FT and DT-FT

Periodic signals	Aperiodic cont. signals	Aperiodic discrete signals
Fourier Series (FS)	Fourier Transform (FT)	Discrete-Time FT (DT-FT)
$\tilde{f}(t)$ periodic with T $\omega_0 = 2\pi/T$	$f(t) : -\infty < t < \infty$ $F(\omega) : -\infty < \omega < \infty$	$f_n \; : n = \ldots, -2, -1, 0, 1, 2, \ldots$ $\tilde{F}(\omega) : -\pi < \omega < \pi$
$F_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{f}(t) e^{-jk\omega_0 t} dt$ $\tilde{f}(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t}$	$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$ $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$	$\tilde{F}(\omega) = \sum_{n=-\infty}^{\infty} f_n e^{-j\omega n}$ $f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{F}(\omega) e^{j\omega n} d\omega$
	Continuous time (CT-FT)	Discrete samples (no reference to the real time scale)

Summary on FS, FT and Discrete Time FT

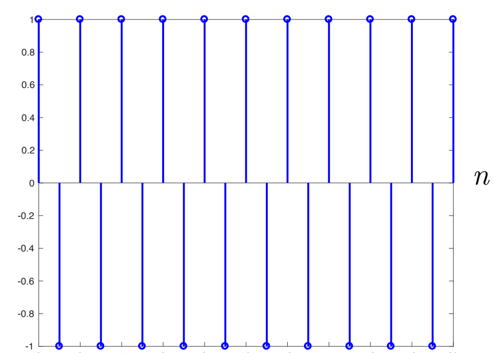


$$-\pi < \omega < \pi$$

- Note on maximum achievable «digitized» frequency in the DT-FT and weights:

$$\tilde{F}(\omega) = \sum_{n=-\infty}^{\infty} f_n \boxed{e^{-j\omega n}} \quad \rightarrow e^{-j\omega n}\Big|_{\omega=\pi} = e^{-j\pi n}$$

$$= \cos(\pi n) - j\sin(\pi n) = (-1)^n$$



Summary on FS, FT and Discrete Time FT

Periodic signals	Aperiodic cont. signals	Aperiodic discrete signals
Fourier Series (FS)	Fourier Transform (FT)	Discrete-Time FT (DT-FT)
$\tilde{f}(t)$ periodic with T $\omega_0 = 2\pi/T$	$f(t): -\infty < t < \infty$ $F(\omega): -\infty < \omega < \infty$	$f_n \; : n = \ldots, -2, -1, 0, 1, 2, \ldots$ $\tilde{F}(\omega): -\pi < \omega < \pi$
$F_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{f}(t) e^{-jk\omega_0 t} dt$	$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$	$\tilde{F}(\omega) = \sum_{n=-\infty}^{\infty} f_n e^{-j\omega n}$
$\tilde{f}(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t}$	$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$	$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{F}(\omega) e^{j\omega n} d\omega$

discrete relative samples (FS coefficients F_k)

periodic (FS signal $\tilde{f}(t)$)

periodic (DT-FT $\tilde{F}(\omega)$)

discrete relative samples (DT-FT samples f_n)

Time sampling

See an “analogy” (or somehow a duality)

Roadmap

We have considered:

1. Periodic functions in the continuous-time domain - FS - (discrete Fourier series coefficients)
2. Aperiodic functions in the continuous-time domain - FT - (continuous Fourier spectra)
3. Aperiodic functions in the **discrete**-time domain - FT - (continuous Fourier spectra – periodicity): discrete-time FT

Next, we will consider:

4. Aperiodic functions in the **discrete**-time domain - FT - (**discrete** Fourier spectra – periodicity): discrete FT (**DFT**)

Roadmap

Theme 7: The 2D Discrete Fourier Transform

Chapter 1: Basics of DSP, complex numbers

Chapter 2: Continuous periodic signals – FS

Chapter 3: Continuous aperiodic signals -FT (cont. time – cont. frequency)

Chapter 4: Sampling:

Chapter 4.1: Sampling in time domain: Continuous aperiodic band-limited signals – (sampling in time domain) – cont. frequency – discrete time (DTCF FT)

Chapter 4.2: Sampling in frequency domain: Continuous aperiodic band-limited signals – (sampling in both time and frequency domains) – discrete frequency – discrete time – discrete FT (DFT)

Chapter 4.3: Extension to 2D DFT

Summary on FS, FT and Discrete Time FT

Discrete-Time FT (DT-FT)

$$f_n \; : n = \dots, -2, -1, 0, 1, 2, \dots$$

$$\tilde{F}(\omega) : -\pi < \omega < \pi$$

$$\tilde{F}(\omega) = \sum_{n=-\infty}^{\infty} f_n e^{-j\omega n}$$

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) e^{j\omega n} d\omega$$

1. Finite number of samples in time domain
2. Discretization in frequency domain

Discrete Fourier Transform (DFT)

For f_n of length M, set $\omega_d = 2\pi / M$

$$f_n \; : n = 0, 1, \dots, M-1$$

$$F_k : k = 0, 1, \dots, M-1$$

$$F_k = \sum_{n=0}^{M-1} f_k e^{-jk\omega_d n}$$

$$f_n = \frac{1}{M} \sum_{k=0}^{M-1} F_k e^{jk\omega_d n}$$

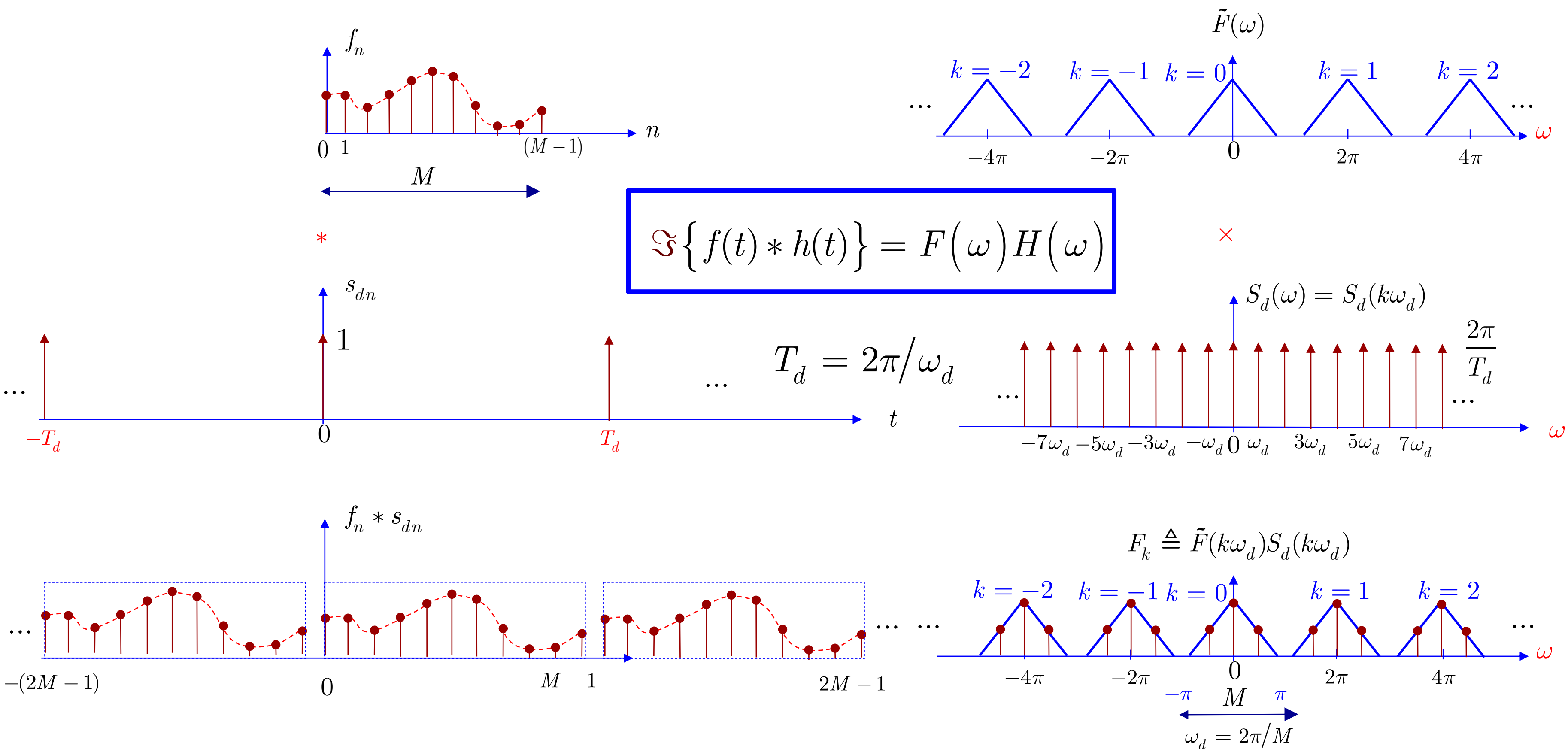
Discrete time but

continuous and periodic frequency spectrum

Discrete both time signal (finite number of samples) and frequency spectrum

DFT – graphical interpretation

Consider discrete signal $f_n \; : n = 0,1,\ldots,M-1$



$$\boxed{\Im\left\{ f(t) * h(t) \right\} = F\left(\omega\right) H\left(\omega\right)}$$

DFT – sampling in frequency domain

- Sample in the frequency domain with step ω_d
- Therefore, the signal will be periodic in the time domain with the period $T_d = 2\pi/\omega_d$
- Thus:

Limited in time by M samples

$$\tilde{F}(\omega) = \sum_{n=-\infty}^{\infty} f_n e^{-j\omega n} \longrightarrow F_k = \tilde{F}(\omega)\Big|_{\omega = k2\pi/M} = \sum_{n=0}^{M-1} f_n e^{-j[k2\pi/M]n}$$

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{F}(\omega) e^{j\omega n} d\omega \longrightarrow f_n = \frac{1}{M}\sum_{n=0}^{M-1} F_k e^{j[k2\pi/M]n}$$

Interval of 2π is sampled by M samples with $\omega_d = 2\pi/M$

New notations

New notations

Discrete Fourier Transform (DFT)
For f_n of length M, set $\omega_d = 2\pi/M$

$$f_n \; : n = 0,1,\ldots,M-1$$
$$F_k \; : k = 0,1,\ldots,M-1$$
$$F_k = \sum_{n=0}^{M-1} f_k e^{-jk\omega_d n}$$
$$f_n = \frac{1}{M}\sum_{k=0}^{M-1} F_k e^{jk\omega_d n}$$

Notations for discrete variables

$\rightarrow$

Discrete Fourier Transform (DFT)
For $f(x)$ of length M, set $\omega_d = 2\pi/M$

$$f(x) : x = 0,1,\ldots,M-1$$
$$F(u) : u = 0,1,\ldots,M-1$$
$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}$$
$$f(x) = \frac{1}{M}\sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}$$

Notations for image processing

x, y - notations of discrete coordinates in images

Interpretation of DFT

- Let's consider the direct DFT

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \quad \text{with} \quad \begin{aligned} f(x) &: x = 0,1,...,M-1 \\ F(u) &: u = 0,1,...,M-1 \end{aligned}$$

- The exponential terms

$$e^{-j2\pi xu/M} \longrightarrow e^{-j2\pi u(x+mM)/M} = e^{-j2\pi ux/M} \underbrace{e^{-j2\pi um}}_{1 \text{for any } m} = e^{-j2\pi ux/M}$$

Therefore, it might take only M complex exponential values of period M

DFT

- We have a discrete time – discrete frequency signal

$$f(x) : x = 0,1,...,M-1$$
$$F(u) : u = 0,1,...,M-1$$

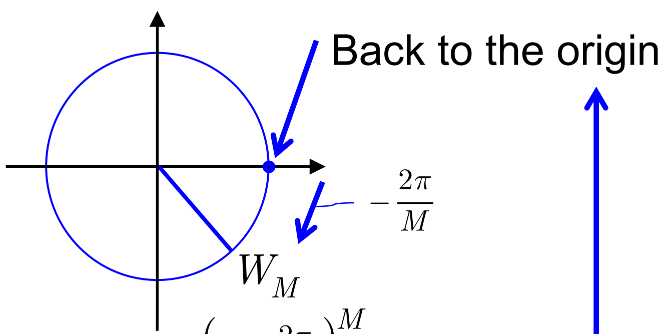
- Implicitly, both representations are periodical

$$f(x) = f(x + mM) \qquad m = 0, \pm 1, \pm 2, \cdots$$
$$F(u) = F(u + mM)$$

- Often a new notation is used

$$\boxed{F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}}$$

$$W_M = e^{-j\frac{2\pi}{M}}$$



For the complete length $W_M = \left(e^{-j\frac{2\pi}{M}} \right)^M \rightarrow e^{-j2\pi} = 1$

DFT in matrix form

- Direct DFT

$$F(u) = \sum_{x=0}^{M-1} f(x) W_M^{ux}, u = 0,1,\ldots, M-1$$

$$\underbrace{\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ F(M-1) \end{bmatrix}}_{\substack{M\times 1 \\ complex}} = \underbrace{\begin{bmatrix} W_M^{0\cdot 0} & W_M^{0\cdot 1} & W_M^{0\cdot 2} & \cdots & W_M^{0\cdot(M-1)} \\ W_M^{1\cdot 0} & W_M^{1\cdot 1} & W_M^{1\cdot 2} & \cdots & W_M^{1\cdot(M-1)} \\ W_M^{2\cdot 0} & W_M^{2\cdot 1} & W_M^{2\cdot 2} & \cdots & W_M^{2\cdot(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_M^{(M-1)\cdot 0} & W_M^{(M-1)\cdot 1} & W_M^{(M-1)\cdot 2} & \cdots & W_M^{(M-1)\cdot(M-1)} \end{bmatrix}}_{\substack{M\times M \\ complex}} \underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(M-1) \end{bmatrix}}_{\substack{M\times 1 \\ real(images)}}$$

$\mathbf{F}$ $\mathbf{W}$ $\mathbf{f}$

$$\boxed{\mathbf{F} = \mathbf{Wf}}$$

DFT in matrix form

- Direct DFT

$$F(u) = \sum_{x=0}^{M-1} f(x) W_M^{ux}, u = 0,1,...,M-1$$

$$\underbrace{\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ F(M-1) \end{bmatrix}}_{\substack{M\times 1 \\ complex}} = \underbrace{\begin{bmatrix} W_M^0 & W_M^0 & W_M^0 & \cdots & W_M^0 \\ W_M^0 & W_M^1 & W_M^2 & \cdots & W_M^{(M-1)} \\ W_M^0 & W_M^2 & W_M^4 & \cdots & W_M^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_M^0 & W_M^{1(M-1)} & W_M^{2(M-1)} & \cdots & W_M^{(M-1)^2} \end{bmatrix}}_{\substack{M\times M \\ complex}} \underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(M-1) \end{bmatrix}}_{\substack{M\times 1 \\ real(images)}}$$

$$\boxed{\mathbf{F} = \mathbf{W}\mathbf{f}}$$

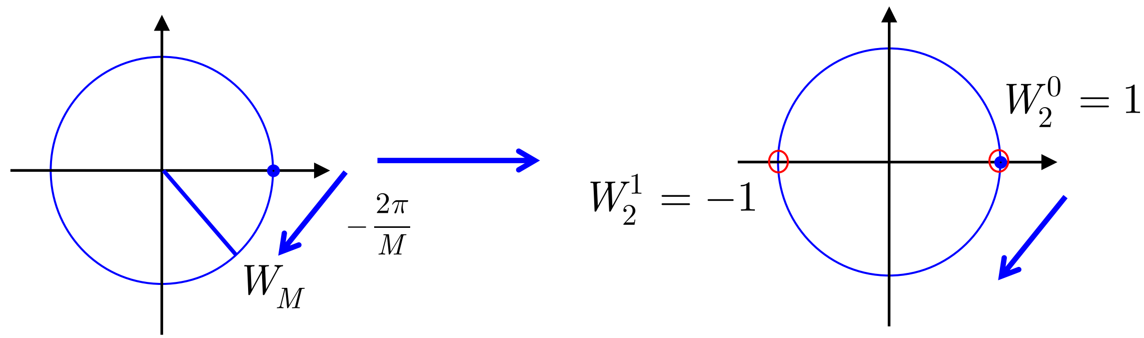
DFT in matrix form - examples

- Direct DFT

$$M = 2$$

$$W_2^0 = e^{-j\frac{2\cdot 0\pi}{2}} = e^0 = 1$$

$$W_2^1 = e^{-j\frac{2\cdot 1\pi}{2}} = e^{-j\pi} = -1$$



$$\begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

DFT in matrix form - examples

- Direct DFT

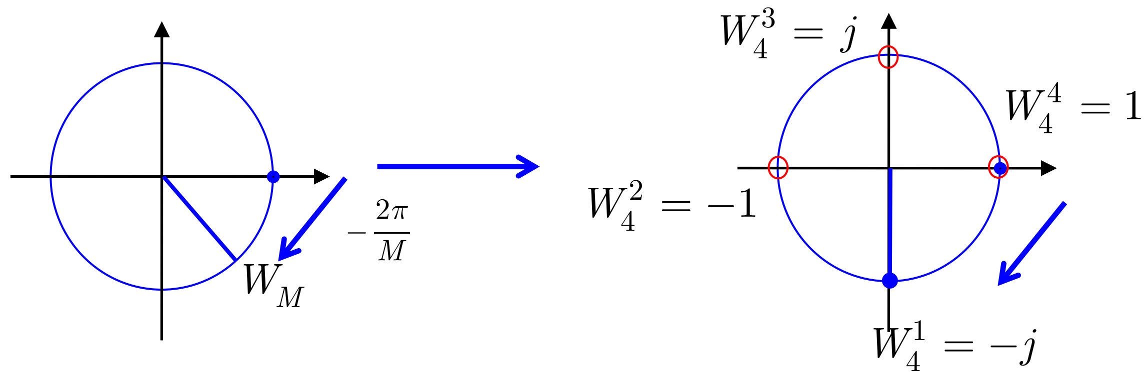
$$M = 4$$

$$W_4^1 = e^{-j\frac{2\cdot 1\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$$

$$W_4^2 = e^{-j\frac{2\cdot 2\pi}{4}} = e^{-j\pi} = -1$$

$$W_4^3 = e^{-j\frac{2\cdot 3\pi}{4}} = e^{-j\frac{3\pi}{2}} = j$$

$$W_4^4 = e^{-j\frac{2\cdot 4\pi}{4}} = e^{-j2\pi} = 1$$



Note: $\mathbf{w}_r(j) = \mathbf{w}_c^T(j)$

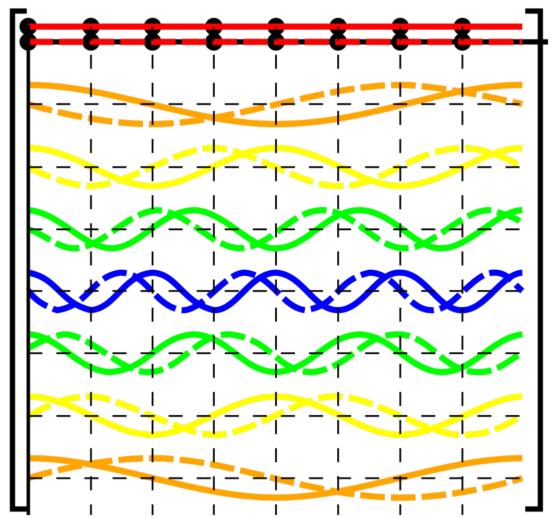
$$\begin{array}{cc} & \mathbf{w}_c(j) \\ \mathbf{w}_r(j) & \underbrace{\begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}}_{\substack{4\times 4 \\ complex}} \end{array} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

DFT in matrix form basis vectors

- Direct DFT

$$M = 8$$

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ F(4) \\ F(5) \\ F(6) \\ F(7) \end{bmatrix} =$$



$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \end{bmatrix}$$

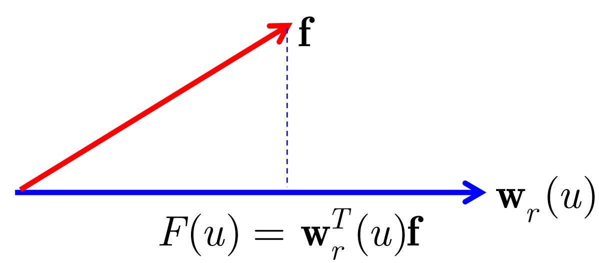
Complex exponents are presented via cosine and sine functions

DFT in matrix form – interpretation of basis vectors

- Direct DFT – interpretation as projections

$$
\underbrace{\begin{bmatrix} F(0) \\ F(1) \\ \boxed{F(2)} \\ \vdots \\ F(M-1) \end{bmatrix}}_{\substack{M\times 1 \\ complex}}
=
\underbrace{\begin{bmatrix} \mathbf{w}_c(2) = \mathbf{w}_r^T(2) & & & & \\ W_M^0 & W_M^2 & W_M^4 & \cdots & W_M^{2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_M^0 & W_M^{1(M-1)} & W_M^{2(M-1)} & \cdots & W_M^{(M-1)^2} \end{bmatrix}}_{\substack{M\times M \\ complex}}
\underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(M-1) \end{bmatrix}}_{\substack{M\times 1 \\ real(images)}}
$$

$$F(u) = \mathbf{w}_r^T(u)\mathbf{f}$$



Matlab: **fft**

DFT in matrix form

- Inverse DFT

$$f(x)=\frac{1}{M}\sum_{u=0}^{M-1}F(u)W_M^{-ux},\ x=0,1,\dots,M-1$$

$$\underbrace{\begin{bmatrix} f(0)\\ f(1)\\ f(2)\\ \vdots\\ f(M-1)\end{bmatrix}}_{\substack{M\times 1\\ real(images)}}=\frac{1}{M}\underbrace{\begin{bmatrix} W_M^0 & W_M^0 & W_M^0 & \cdots & W_M^0\\ W_M^0 & W_M^{-1} & W_M^{-2} & \cdots & W_M^{-(M-1)}\\ W_M^0 & W_M^{-2} & W_M^{-4} & \cdots & W_M^{-2(M-1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ W_M^0 & W_M^{-1(M-1)} & W_M^{-2(M-1)} & \cdots & W_M^{-(M-1)^2}\end{bmatrix}}_{\substack{M\times M\\ complex}}\underbrace{\begin{bmatrix} F(0)\\ F(1)\\ F(2)\\ \vdots\\ F(M-1)\end{bmatrix}}_{\substack{M\times 1\\ complex}}$$

$$\boxed{\mathbf{f}=\frac{1}{M}\mathbf{W}^{-1}\mathbf{F}}$$

$$\mathbf{W}^{-1}=\mathbf{W}^{H}$$

$$\frac{1}{M}\mathbf{W}^{H}\mathbf{W}=\frac{1}{M}\mathbf{W}\mathbf{W}^{H}=\mathbf{I}$$

$\mathbf{W}$ is a unitary (orthogonal) matrix

Matlab: **ifft**

DFT - orthogonally

- The orthogonality of DFT
- The columns of matrix $\mathbf{W}$

$$\mathbf{w}_c(u) = \left[e^{j\frac{2\pi}{M}xu} \middle| x = 0, 1, \ldots, M-1 \right]^T$$

form an orthogonal basis over the set of M-dimensional complex vectors:

$$\mathbf{w}_c^T(u)\mathbf{w}_c^*(u') = \sum_{x=0}^{M-1} \left(e^{j\frac{2\pi}{M}xu} \right) \left(e^{j\frac{2\pi}{M}x(-u')} \right) = \sum_{x=0}^{M-1} e^{j\frac{2\pi}{M}x(u-u')} = M\delta_{uu'}$$

where $\delta_{uu'}$ is the Kronecker delta

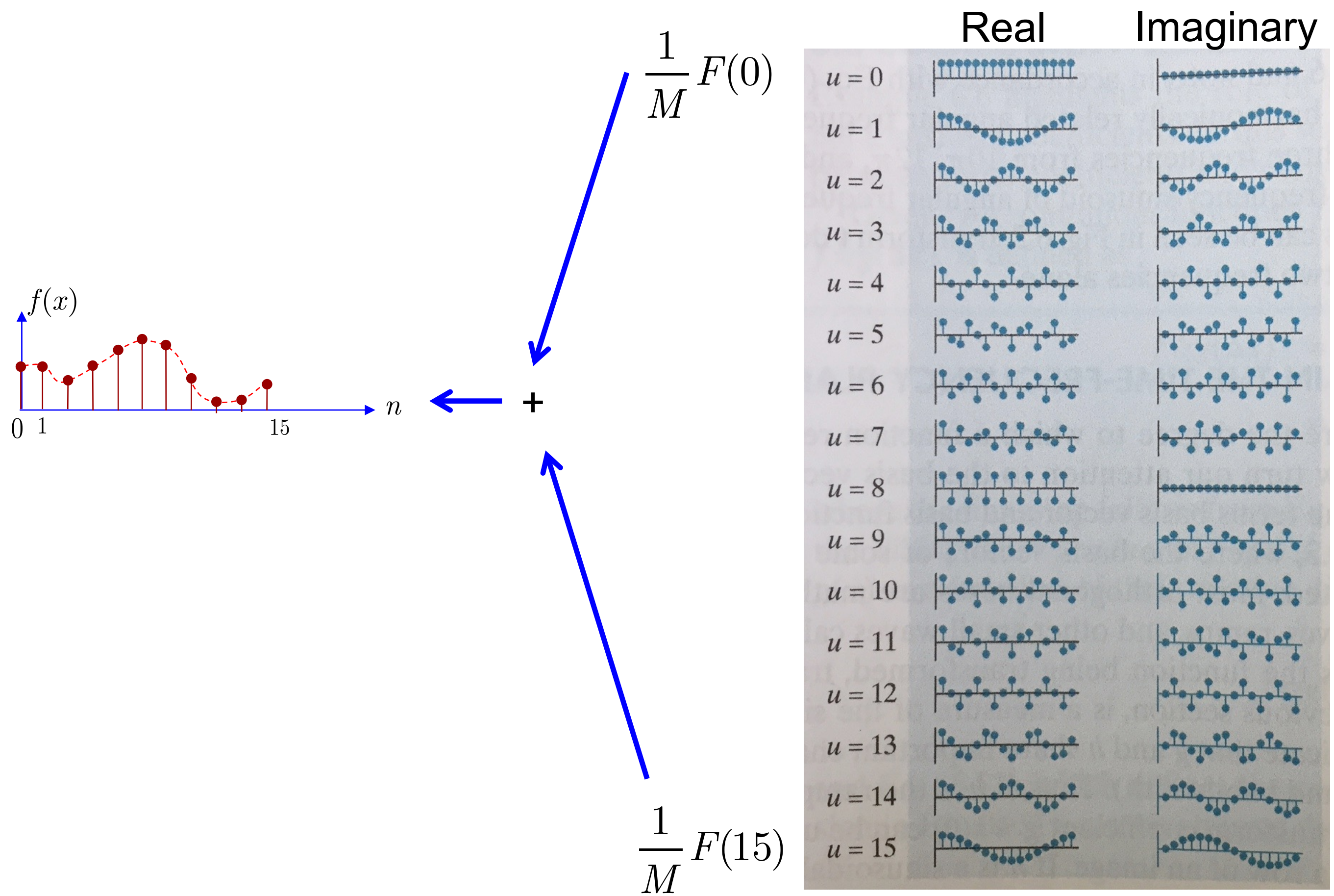
DFT in matrix form – interpretation

- Inverse DFT – interpretation as synthesis

$$\underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(M-1) \end{bmatrix}}_{\substack{M\times 1 \\ real(images)}} = \frac{1}{M}F(0)\underbrace{\begin{bmatrix} W_M^0 \\ W_M^0 \\ W_M^0 \\ \vdots \\ W_M^0 \end{bmatrix}}_{\substack{M\times 1 \\ \text{DC component} \\ \text{0 - frequency}}} + \frac{1}{M}F(1)\underbrace{\begin{bmatrix} W_M^0 \\ W_M^{-1} \\ W_M^{-2} \\ \vdots \\ W_M^{-1(M-1)} \end{bmatrix}}_{\substack{M\times 1 \\ 1st\text{ component} \\ \text{1 - frequency}}} + \cdots + \frac{1}{M}F(M-1)\underbrace{\begin{bmatrix} W_M^0 \\ W_M^{-(M-1)} \\ W_M^{-2(M-1)} \\ \vdots \\ W_M^{-(M-1)^2} \end{bmatrix}}_{\substack{M\times 1 \\ Mth\text{ component} \\ (M-1)\text{ - frequency}}}$$

DFT in matrix form – interpretation

- Inverse DFT – interpretation as synthesis



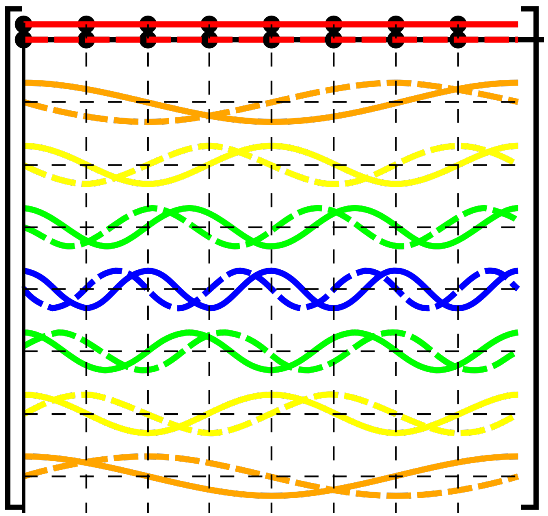
DFT in matrix form – interpretation

- Matrix interpretation $\mathbf{F} = \mathbf{W}\mathbf{f}$

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}$$

$M = 8$

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ F(4) \\ F(5) \\ F(6) \\ F(7) \end{bmatrix} =$$

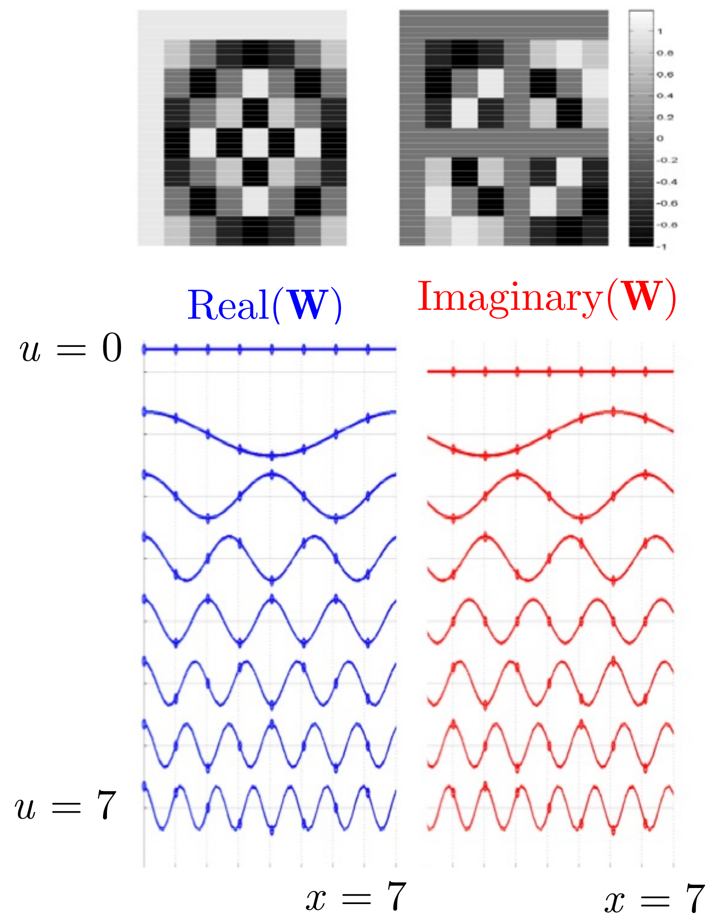


$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \\ f(7) \end{bmatrix}$$

complex

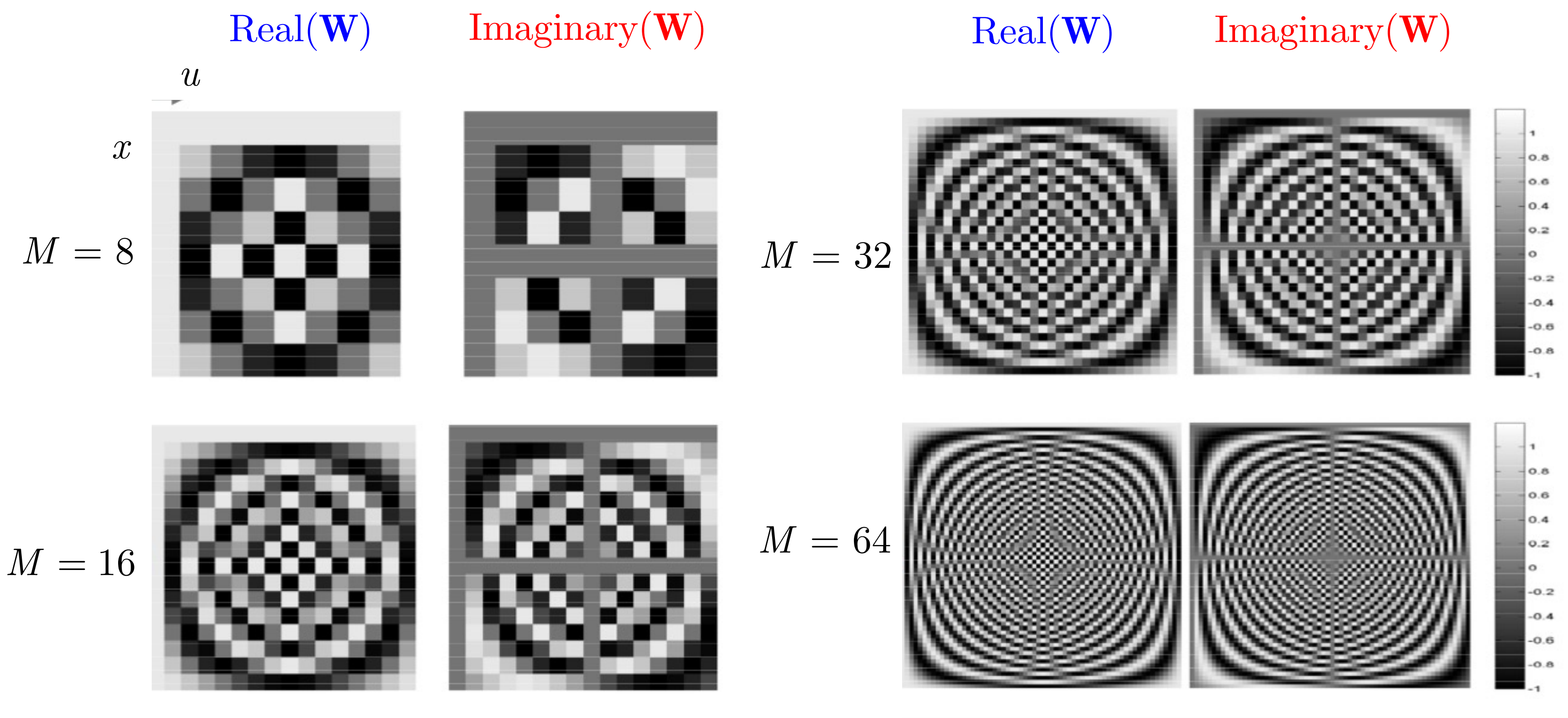
real

Complex exponents are presented via cosine and sine functions



DFT in matrix form – interpretation

- Matrix interpretation $\mathbf{F} = \mathbf{W}\mathbf{f}$



DFT based convolution

- The convolution theorem for the continium domain

$$\Im\left\{ f(t) * h(t) \right\} = F\left(\omega\right) H\left(\omega\right)$$

also applies to the discrete case.

- Remark: The only difference for the discrete domain is that the convolution is **cyclic**.

$$\Im\left\{ f(x) \circledast h(x) \right\} = F\left(u\right) H\left(u\right)$$

The signals are of length M

- Next: we will see the diference between the linear convolution and cyclic convolution and how to use the DFT for that.

Circulant convolution

$$\underbrace{\begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \\ g(M-1) \end{bmatrix}}_{M\times 1} = \underbrace{\begin{bmatrix} h(0) & h(M-1) & h(M-2) & \cdots & h(1) \\ h(1) & h(0) & h(M-1) & \cdots & h(2) \\ h(2) & h(1) & h(0) & \cdots & h(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(M-1) & h(M-2) & h(M-3) & \cdots & h(0) \end{bmatrix}}_{M\times M} \underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(M-1) \end{bmatrix}}_{M\times 1}$$

- The diagonal elements of cyclic matrix are the same
- This matrix is know as **a circulant matrix**

$$\underbrace{\begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \\ g(M-1) \end{bmatrix}}_{M\times 1} = \underbrace{\begin{bmatrix} h(0) & h(M-1) & h(M-2) & \cdots & h(1) \\ h(1) & h(0) & h(M-1) & \cdots & h(2) \\ h(2) & h(1) & h(0) & \cdots & h(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(M-1) & h(M-2) & h(M-3) & \cdots & h(0) \end{bmatrix}}_{M\times M} \underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(M-1) \end{bmatrix}}_{M\times 1}$$

DFT: example linear vs circulant convolution

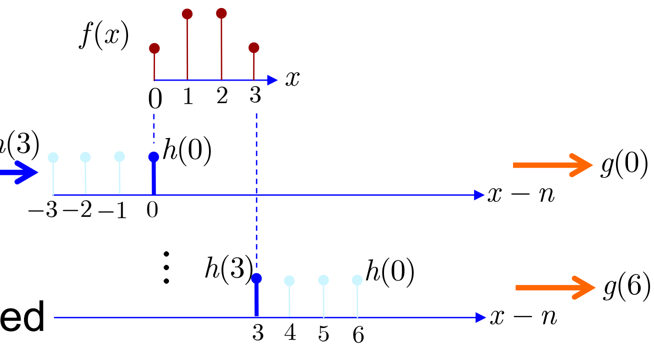
- Consider **linear convolution**

$$\mathbf{f}=\left[f(0),f(1),f(2),f(3)\right]^T$$

$$\mathbf{h}=\left[h(0),h(1),h(2),h(3)\right]^T \longrightarrow \begin{array}{c} h(x-n) \\ \text{flipped} \end{array} \longrightarrow$$

\+

shifted



$$g(0) = f(0)h(0)$$

$$g(1) = f(0)h(1) + f(1)h(0)$$

$$g(2) = f(0)h(2) + f(1)h(1) + f(2)h(0)$$

$$\longrightarrow \underset{0\le x\le 6}{g(x)} = f(x) * h(x) = \sum_{n=0}^{(M_f-1)=3} h(x-n)f(n)$$

Length of output $M_f + M_h - 1 = 7$

$$\underbrace{\begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \\ g(6) \end{bmatrix}}_{7\times 1} = \underbrace{\begin{bmatrix} h(0) & 0 & 0 & 0 \\ h(1) & h(0) & 0 & 0 \\ h(2) & h(1) & h(0) & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & h(3) \end{bmatrix}}_{7\times 4} \underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}}_{4\times 1}$$

DFT: example linear vs circulant convolution

- Consider **circulant convolution**

$$\mathbf{f}=\left[f(0),f(1),f(2),f(3)\right]^T$$

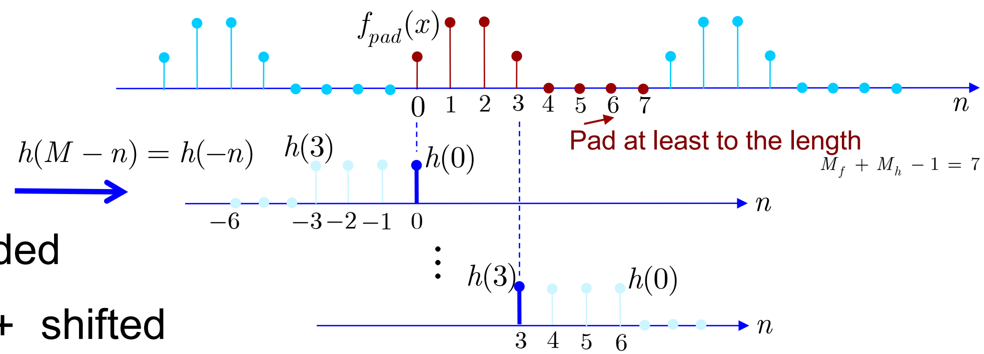
$$\mathbf{h}=\left[h(0),h(1),h(2),h(3)\right]^T$$

Flipped

\+ padded

\+ shifted

Periodically extended signals



$$g(x) = f(x) \circledast h(x) = \sum_{n=0}^{(M_f+M_h-1)-1=6} h_{pad}(x-n) f_{pad}(n)$$

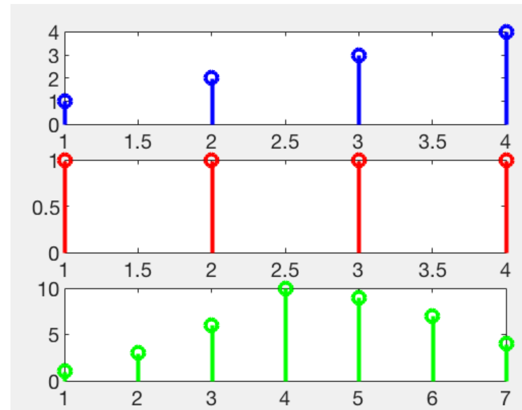
$$\underbrace{\begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ \vdots \\ g(6) \end{bmatrix}}_{7\times 1} = \underbrace{\left[\begin{array}{cccc|ccc} h(0) & 0 & 0 & 0 & h(3) & h(2) & h(1) \\ h(1) & h(0) & 0 & 0 & & h(3) & h(2) \\ h(2) & h(1) & h(0) & 0 & & & h(3) \\ \vdots & \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & h(3) & h(2) & h(1) & h(0) \end{array}\right]}_{7\times 7} \underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{7\times 1}$$

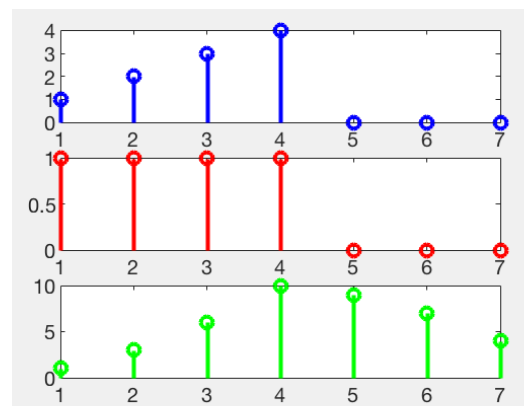
padded (columns $h(3)\;h(2)\;h(1)$ …) — padded (entries $0, 0, 0$ of $\mathbf{f}$)

$$\mathbf{g} = \mathbf{H}\mathbf{f}$$

DFT: example linear vs circular convolution

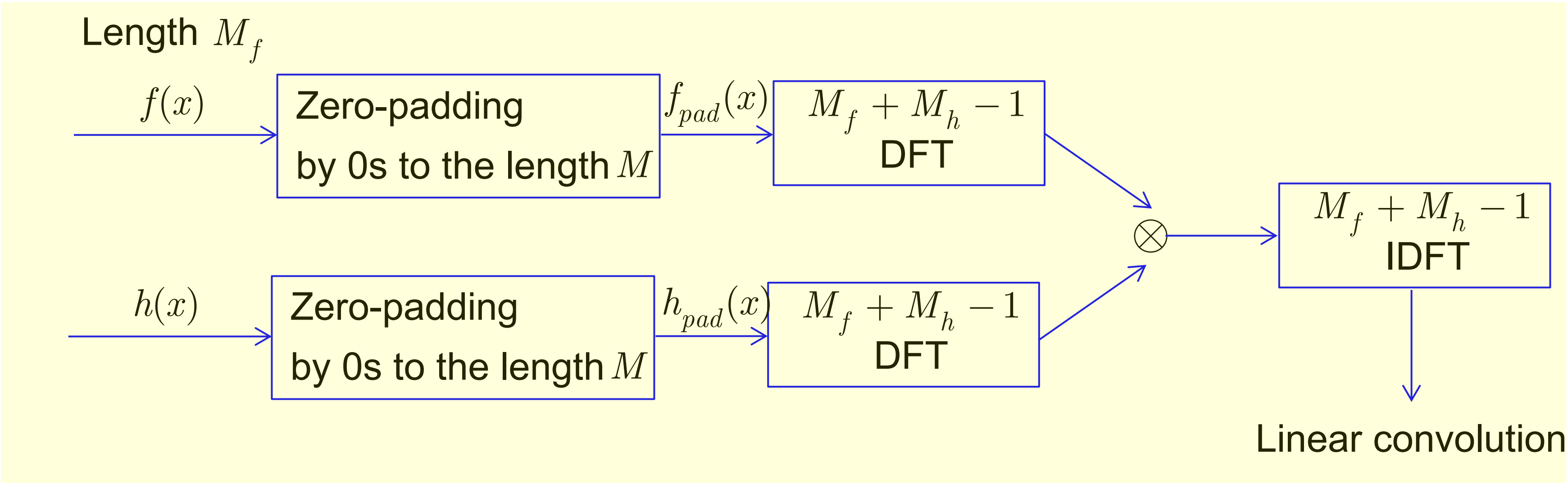
```
%Linear vs DFT
close all
clear all

x=1:3;
f = [1 2 3 4];
h = [1 1 1 1];
g_lin = conv(f,h);

x_f=1:length(f);
x_h=1:length(h);
x_g=1:length(f)+length(h)-1;

figure;
set(gca, 'fontsize', 14);
subplot(3,1,1);stem(x_f,f, 'b', 'linewidth', 2);
subplot(3,1,2);stem(x_h,h, 'r', 'linewidth', 2);
subplot(3,1,3);stem(x_g, g_lin, 'g', 'linewidth', 2);

% Padding

fpad = [f zeros(1,length(f)+length(h)-1  - length(f))];
hpad = [h zeros(1,length(f)+length(h)-1  - length(h))];
g_circ = ifft(fft(fpad).*fft(hpad));

figure;
set(gca, 'fontsize', 14);
subplot(3,1,1);stem(fpad, 'b', 'linewidth', 2);
subplot(3,1,2);stem(hpad, 'r', 'linewidth', 2);
subplot(3,1,3);stem(g_circ, 'g', 'linewidth', 2);
```





Summary: linear convolution via DFT

- **Linear convolution via DFT**



$$g(x) = f(x) * h(x) = \sum_{n=0}^{(M_f-1)} h(x-n)f(n)$$

- Matlab: **cconv** internally uses the same DFT-based procedure

  ccirc = **cconv**(f,h,M);

DFT – elementary typical signals

- Consider **1-pulse discrete M-length signal**

$$f(x) = \begin{cases} 1, x = 0, \\ 0, 1 \le x \le M-1 \end{cases}$$

- Its M-length DFT is given by

$$F(u) = \sum_{x=0}^{M-1} f(x) W_M^{ux} = f(0) W_M^0 = 1$$

$$u = 0,1,\ldots,M-1$$

Analogy to FT: the discrete spectrum corresponds to the constant spectrum of delta-function

$$\delta(x) \Leftrightarrow 1$$
$$1 \Leftrightarrow \delta(x)$$

DFT – elementary typical signals

- Consider **time shifted 1-pulse discrete M-length signal**

$$f(x) = \begin{cases} 1, x = m, \\ 0, 1 \le x \le m-1, m+1 \le x \le M-1 \end{cases}$$

- Its M-length DFT is given by

$$F(u) = \sum_{x=0}^{M-1} f(x) W_M^{ux} = f(m) W_M^{um} = W_M^{um}$$
$$u = 0,1,\ldots,M-1$$

Analogy to FT: the discrete spectrum corresponds to the complex exponent (phase)

$$\delta(x-m) \Leftrightarrow W_M^{um}$$

DFT – elementary typical signals

- Consider **discrete cosine/sine M-length signals**

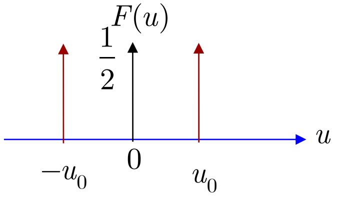
$$f(x) = \cos(2\pi x\, u_0 / M)$$
$$f(x) = \sin(2\pi x\, u_0 / M)$$

- We use the conversion formulas

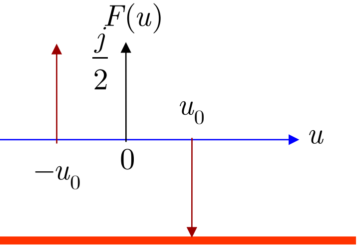
$$\cos(2\pi x\, u_0 / M) = \frac{1}{2}\left[e^{j2\pi x u_0/M} + e^{-j2\pi x u_0/M}\right] = \frac{1}{2}\left[W_M^{-xu_0} + W_M^{xu_0}\right]$$
$$\sin(2\pi x\, u_0 / M) = \frac{1}{2j}\left[e^{j2\pi x u_0/M} - e^{-j2\pi x u_0/M}\right] = \frac{1}{2j}\left[W_M^{-xu_0} - W_M^{xu_0}\right]$$

- Their M-length DFTs are given by

$$\cos(2\pi x\, u_0 / M) \Leftrightarrow \frac{1}{2}\Big(\delta(u+u_0) + \delta(u-u_0)\Big)$$



$$\sin(2\pi x\, u_0 / M) \Leftrightarrow \frac{j}{2}\Big(\delta(u+u_0) - \delta(u-u_0)\Big)$$



Next

- Extension to 2D DFT
- Image filtering in DFT domain