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Asymmetric Cryptography (Public Key Cryptography)

Mathematical Foundations

Fundamental Theorem of Arithmetic and Euler's Totient Function

Asymmetric cryptography relies on solid mathematical foundations from number theory. Two concepts are essential:

Fundamental Theorem of Arithmetic: Every positive integer greater than 1 can be written uniquely (up to order) as a product of prime powers:

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_m^{e_m}$$

Euler's Totient Function $\phi(n)$: Number of positive integers smaller than n that are coprime with n.

To compute $\phi(n)$:

$$\phi(n) = \prod_{i=1}^{m} p_i^{e_i} \cdot \left(1 - \frac{1}{p_i}\right)$$

Important special case: If $n = p \cdot q$ with p and q prime, then:

$$\phi(n) = (p-1)(q-1)$$

Original Text

Mathematical Foundations

Fundamental Theorem of Arithmetic: Every positive integer n can be written uniquely (up to order) as a product of powers of distinct prime numbers p_i:

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_m^{e_m}$$

Euler's Totient Function: Let $n \in \mathbb{Z}^+$, the **Euler's totient function** $\phi(n)$ is equal to the number of positive integers smaller than n that are **relatively prime** to n.
Calculation of Euler's totient function: According to the fundamental theorem of arithmetic, every integer $n > 1$ can be written as:

$$n = \prod_{i=1}^{m} p_i^{e_i}$$

then $\phi(n)$ is calculated as:

$$\phi(n) = \prod_{i=1}^{m} \left(p_i^{e_i} - p_i^{e_i-1}\right)$$

In particular, if $n = p \cdot q$ with p and q prime, then:

$$\phi(n) = (p-1)(q-1)$$

Quick Revision

- **Unique decomposition**: every integer = product of prime numbers
- $\phi(n)$: counts integers $< n$ coprime with n
- **Key for RSA**: if $n = pq$ (primes) then $\phi(n) = (p-1)(q-1)$

Euler's Theorem and Fermat's Little Theorem

These theorems are at the heart of RSA and other asymmetric algorithms.

Euler's Theorem: If $n \in \mathbb{Z}^+$ and $a \in \mathbb{Z}$ with $\gcd(a, n) = 1$, then:

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

Fermat's Little Theorem (special case if $n = p$ prime): If $a \in \mathbb{Z}$ and p prime does not divide a:

$$a^{p-1} \equiv 1 \pmod{p}$$

Important applications:

1. **Exponent reduction**: If n is a product of distinct primes and $r \equiv s \pmod{\phi(n)}$, then:

$$a^r \equiv a^s \pmod{n}$$

2. **Calculation of inverses**: $a^{\phi(n)-1}$ is the inverse of a modulo n. In particular, if p is prime, a^{p-2} is the inverse of a modulo p.

Original Text

Mathematical Foundations (II)

Euler's Theorem: Let $n \in \mathbb{Z}^+$ and $a \in \mathbb{Z}$ with $\gcd(a, n) = 1$, then we have:

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

Fermat's Little Theorem (special case of Euler's theorem if n is prime): Let $a \in \mathbb{Z}$ and p a prime number such that p does not divide a, then we have:

$$a^{p-1} \equiv 1 \pmod{p}$$

Note that since p is prime, we have $\phi(p) = p - 1$.
Exponent reduction $\bmod \phi(n)$: If n is the product of distinct primes and $r, s \in \mathbb{Z}$ such that $r \equiv s \pmod{\phi(n)}$ then $\forall a \in \mathbb{Z}$:

$$a^r \equiv a^s \pmod{n}$$

Application of Euler's Theorem to inverse calculation: From Euler's theorem, we have that:

$$a \cdot a^{\phi(n)-1} \equiv 1 \pmod{n}$$

which means that $a^{\phi(n)-1}$ is the **inverse of** a **modulo** n. In particular, a^{p-2} is the inverse of a modulo n if p is prime.

Quick Revision

- **Euler's Theorem**: $a^{\phi(n)} \equiv 1 \pmod{n}$
- **Fermat**: special case if p prime: $a^{p-1} \equiv 1 \pmod{p}$
- **Modular inverse**: $a^{-1} \equiv a^{\phi(n)-1} \pmod{n}$
- **Base of RSA**: enables encryption/decryption with exponents

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Multiplicative Groups and Generators

Multiplicative group $\mathbb{Z}_n^*$: Set of elements of $\mathbb{Z}_n$ coprime with n:

$$\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$$

If n is prime: $\mathbb{Z}_n^* = \{1, 2, \dots, n - 1\}$

Order of an element: Smallest positive integer t such that $a^t \equiv 1 \pmod{n}$

Generator: An element α is a generator of $\mathbb{Z}_n^*$ if its order is $\phi(n)$. Then $\mathbb{Z}_n^*$ is said to be **cyclic**.

Properties of generators:

1. $\mathbb{Z}_n^*$ has a generator iff $n = 2, 4, p^k$ or $2p^k$ (with p prime, $p \neq 2$ and $k \geq 1$)
2. If p is prime, $\mathbb{Z}_p^*$ always has a generator
3. If α is a generator, all elements can be written as: $\mathbb{Z}_n^* = \{\alpha^i \bmod n \mid 0 \leq i < \phi(n)\}$
4. The number of generators is $\phi(\phi(n))$

Generator test

- α is a generator of $\mathbb{Z}_n^*$ iff for every prime p dividing $\phi(n)$, $\alpha^{\phi(n)/p} \not\equiv 1 \pmod{n}$
- if $n = 2p + 1$ is a "safe prime" with p prime: α is a generator iff $\alpha^2 \not\equiv 1 \pmod{n}$ and $\alpha^p \not\equiv 1 \pmod{n}$

Original Text

Mathematical Foundations (III)

Definition: The **multiplicative group of** $\mathbb{Z}_n$, denoted $\mathbb{Z}_n^*$ is:

$$\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$$

In particular, if n is prime: $\mathbb{Z}_n^* = \{a \mid 1 \leq a \leq n - 1\}$
The **number of elements or order** of the multiplicative group $\mathbb{Z}_n^*$ is $\phi(n)$ (by definition of ϕ).
Definition: Let $a \in \mathbb{Z}_n$, the **order of** a is the smallest positive integer t for which:

$$a^t \equiv 1 \pmod{n}$$

Definition: Let $\alpha \in \mathbb{Z}_n^*$, if the order of α is $\phi(n)$, then α is a **generator of** $\mathbb{Z}_n^*$. When a group $\mathbb{Z}_n^*$ has a generator, it is said to be **cyclic**.
Properties of generators:

- $\mathbb{Z}_n^*$ has a generator iff $n = 2, 4, p^k$ or $2p^k$, with p prime, $p \neq 2$ and $k \geq 1$. In particular, if p is prime, $\mathbb{Z}_p^*$ has a generator.
- If α is a generator of $\mathbb{Z}_n^*$, then all elements of $\mathbb{Z}_n^*$ can be written as:

$$\mathbb{Z}_n^* = \{\alpha^i \bmod n \mid 0 \leq i \leq \phi(n) - 1\}$$

- The number of generators of $\mathbb{Z}_n^*$ is $\phi(\phi(n))$.

- α is a generator of $\mathbb{Z}_n^*$ iff for every prime p dividing $\phi(n)$, we have:

$$\alpha^{\phi(n)/p} \not\equiv 1 \pmod{n}$$

In particular if n is a prime of the form $n = 2p + 1$ with p prime (such n is called a **safe prime**), α is a generator of $\mathbb{Z}_n^*$ iff $\alpha^2 \not\equiv 1 \pmod{n}$ and $\alpha^p \not\equiv 1 \pmod{n}$.

Quick Revision

- $\mathbb{Z}_n^*$: elements coprime with n, cardinality $= \phi(n)$
- **Generator**: element of order $\phi(n)$ (generates the entire group)
- **Crucial for DH and ElGamal**: security based on discrete logarithm in cyclic group
- **Safe prime**: $n = 2p + 1$ with p and n prime

Fast Exponentiation

Efficient computation of $a^k \bmod n$ in polynomial time, essential for all asymmetric algorithms.

Principle: Use the binary representation of the exponent k.

Example: Computation of $2^{644} \bmod 645$

1. Binary representation: $(644)_{10} = (1010000100)_2$
2. Compute successive powers of 2 modulo 645:
   - $2^1 \bmod 645$
   - $2^2 \bmod 645$
   - $2^4 \bmod 645$
   - $2^8 \bmod 645$
   - ...
   - $2^{512} \bmod 645$
3. Combine according to bits set to 1: $2^{644} = 2^{512} \cdot 2^{128} \cdot 2^4$

Complexity: $O(\log^3 n)$ - very efficient!

Application: Computation of the inverse using Euler's theorem in polynomial time.

Alternative: **Extended Euclidean algorithm** to find x such that $ax \equiv 1 \pmod{n}$ by solving $ax - kn = 1 = \gcd(a, n)$. Complexity also $O(\log^3 n)$.

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Fast Exponentiation

Fast exponentiation: Using the binary representation of a number, we can compute powers very efficiently.
Example: computation of $2^{644} \bmod 645$
$(644)_{10} = (1010000100)_2$
Now, we compute the exponents corresponding to the powers of 2, namely:

$$2^1 \bmod 645, \quad 2^2 \bmod 645, \quad 2^4 \bmod 645, \quad \dots, \quad 2^{512} \bmod 645$$

From the binary representation, we compute:

$$2^{644} = 2^{512+128+4} = 2^{512} \cdot 2^{128} \cdot 2^4 = 160 \cdot 153 \cdot 6 \bmod 645$$

The complexity of this algorithm fast exponentiation is $O(\log^3 n)$.
By relying on **Euler's theorem**, the computation of the **inverse of a number** in such a group is therefore performed in polynomial time.
The extended Euclidean algorithm can also be used to find an x such that:

$$ax \equiv 1 \pmod n$$

since this congruence can be written as: $ax - 1 = kn$ and therefore:

$$ax - kn = 1 = \gcd(a, n)$$

The complexity of this algorithm is also $O(\log^3 n)$.

Quick Revision

- **Idea**: binary representation of the exponent
- **Complexity**: $O(\log^3 n)$ - polynomial!
- **Essential**: makes RSA, ElGamal, DH practical
- **Alternative**: extended Euclidean algorithm for inverses

Chinese Remainder Theorem (CRT)

The CRT allows solving systems of simultaneous congruences, with important applications in cryptography.

Theorem: Let $n_1, n_2, \dots, n_t \in \mathbb{Z}^+$ pairwise coprime ($\gcd(n_i, n_j) = 1$ if $i \neq j$) and $a_1, a_2, \dots, a_t \in \mathbb{Z}$. Then the system:

$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \\ \vdots \\ x \equiv a_t \pmod{n_t} \end{cases}$$

has a unique solution $x \bmod N$ with $N \coloneqq n_1 \cdot n_2 \cdots n_t$.

Gauss's algorithm (1801) to compute x:

$$x = \sum_{i=1}^{t} a_i N_i M_i \bmod N$$

with:

- $N_i = N/n_i$
- $M_i = N_i^{-1} \bmod n_i$ (modular inverse)

Complexity: $O(\log^3 n)$ - polynomial!

Cryptographic applications:

1. Acceleration of RSA computations (use p and q separately)
2. Secret sharing (secret sharing schemes)
3. Certain attacks on RSA (if small exponent and multiple messages)

> **i** Original Text
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> **Chinese Remainder Theorem**
>
> The **Chinese Remainder Theorem** (3rd century!) allows solving linear systems of simultaneous congruences. It solves problems raised in ancient Chinese puzzles. It was, for example, about finding a number that produces a remainder of 1 when divided by 3, of 2 when divided by 5 and of 3 when divided by 7… It was also used to calculate the exact moment of alignment of several celestial bodies having different orbits (and therefore periods).
> **Chinese Remainder Theorem**: Let $n_1, n_2, \dots, n_t \in \mathbb{Z}^+$ be pairwise coprime (i.e., $\gcd(n_i, n_j) = 1, \forall i \neq j$) and $a_1, a_2, \dots, a_t \in \mathbb{Z}$. Then, the system of congruences:

$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \\ \vdots \\ x \equiv a_t \pmod{n_t} \end{cases}$$

has a unique solution $x \bmod N := n_1 n_2 \cdots n_t$
Gauss's algorithm (1801) for the computation of x:

$$x = \sum_{i=1}^{t} a_i N_i M_i \bmod N$$

with $N_i = N/n_i$ and $M_i = N_i^{-1} \bmod n_i$.
The **complexity** of this algorithm is $O(\log^3 n)$.
It is therefore possible in **polynomial time** to go from congruences mod n_i to congruences mod N!

Quick Revision

- **Solves**: systems of congruences with pairwise coprime moduli
- **Unique solution**: modulo product of moduli
- **Complexity**: $O(\log^3 n)$ (polynomial)
- **Crypto usage**: RSA optimization, attacks if small exponent

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Basic Problems and Complexity

Classification of Hard Problems

The security of asymmetric cryptography relies on mathematical problems reputed to be hard:

Generic problems:

1. **Factorization (FACTP)**: Given n, find its factorization into prime numbers
   - Base of **RSA** and **Rabin**
2. **Discrete Logarithms (DLP)**: Given prime p, a generator $\alpha \in \mathbb{Z}_p^*$ and $\beta \in \mathbb{Z}_p^*$, find x such that:
$$\alpha^x \equiv \beta \pmod{p}$$
   - Base of **ElGamal** and **Diffie-Hellman**

3. **Square Root modulo composite (SQROOTP)**: Given composite n and a quadratic residue a, find $\sqrt{a} \bmod n$
    - Base of **Rabin**

Specific problems:

1. **RSA Problem (RSAP)**: Given $n = pq$, e with $\gcd(e, \phi(n)) = 1$ and c, find m such that $m^e \equiv c \pmod{n}$
2. **Diffie-Hellman Problem (DHP)**: Given prime p, generator α, $\alpha^a \bmod p$ and $\alpha^b \bmod p$, find $\alpha^{ab} \bmod p$

Proven equivalences:

- **DHP** **DLP** (equivalent under certain conditions)
- **RSAP** **FACTP** (proven equivalent for the generic case)
- **SQROOTP** **FACTP**

> Original Text
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> **Basic Problems**
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> **Main generic problems**:
>
> - **Factorization (FACTP)**: Given a positive integer n, find its factorization into prime numbers.
> - **Discrete Logarithms (DLP)**: Given a prime number p, a generator $\alpha \in \mathbb{Z}_p^*$ and an element $\beta \in \mathbb{Z}_p^*$, find the integer x, $0 \leq x \leq p-2$, such that: $\alpha^x \equiv \beta \pmod{p}$.
> - **Square Root in $\mathbb{Z}_n$ if n is composite (SQROOTP)**: Given a composite integer n and a quadratic residue a, find the square root of a mod n.
>
> **Specific problems** (proper to an encryption system):
>
> - **RSA (RSAP)**: Given a positive integer $n = pq$, a positive integer e with $\gcd(e, (p-1)(q-1)) = 1$ and an integer c, find an integer m with $m^e \equiv c \pmod{n}$.
> - **Diffie-Hellman (DHP)**: Given a prime number p, a generator $\alpha \in \mathbb{Z}_p^*$ and the elements $\alpha^a \bmod p$ and $\alpha^b \bmod p$, find $\alpha^{ab} \bmod p$.
>
> **Proven results**:
>
> - **DHP** **DLP** (Equivalent under certain conditions)
> - **RSAP** **FACTP** (Proven equivalent for the generic problem)
> - **SQROOTP** **FACTP**

Quick Revision

- **FACTP**: factor $n \to$ base of RSA/Rabin
- **DLP**: find discrete logarithm $\to$ base ElGamal/DH
- **SQROOTP**: square root mod composite $\to$ Rabin
- **Equivalences**: breaking = solving the base problem

Factorization Techniques

The security of RSA depends on the difficulty of factoring large numbers.

Exponential time methods: $O(\exp(c \cdot \ln(n)))$

- Trial Division (successive division)
- Sieve of Eratosthenes (2nd century BC)
- Fermat's Method (~1650)
- Pollard's ρ Method (1975)
- Pollard's $p-1$ Method (1974)

Sub-exponential time methods: $O(\exp(c \cdot (\ln(n))^{1/3}))$

- Continued Fractions (1975)
- **Quadratic Sieve (1981)** - very effective in practice
- **Number Field Sieve - NFS (1990)** - currently the fastest
- General Number Field Sieve - GNFS (2006)

Polynomial time methods:

- **Shor's Algorithm** (1994): $O(\log^c n)$ on **quantum computer**

Current records (2020):

- Largest number factored: **RSA-829** (250 digits, 829 bits)
- Computation time: 2700 core-years (Intel Xeon Gold 6130 CPUs)
- Method: General Number Field Sieve

Implications:

- RSA keys < 1024 bits: **vulnerable**
- RSA keys 1024 bits: **limits** (states with significant resources)
- Recommendation: **2048 bits minimum** (3072-4096 for long term)

Original Text

Classical Factoring Techniques and New Developments

Exponential time: $O(\exp(c \cdot \ln(n)))$

- Trial Division
- Eratosthenes' Sieve (II B.C.)
- Fermat's Difference of Squares Method (~1650)
- Square Form Factorization (1971)
- Pollard's p-1 method (1974)
- Pollard's Rho Method (1975)

Sub-exponential time: $O(\exp(c \cdot (\ln(n))^{1/3}))$

- Continued Fractions (1975)
- **Quadratic Sieve (1981)**
- **Number Field Sieve - NFS (1990)**
- **General Number Field Sieve - GNFS (2006)**

Polynomial time:

- **Shor's Algorithm in a Quantum Computer (1994)**: $O(\log^c n)$

Recent developments:

- Bernstein's specific NFS computer to factor a 1536-bit number would take the same time as a 512-bit computation on a conventional machine
- **Largest factorization to date (2020)**: RSA-829 (250-digit number) using NFS
- Total computation time: **2700 core-years** (Intel Xeon Gold 6130 CPUs at 2.1GHz)

Factorization on quantum computer:

- Significant problems (errors, dispersion, etc.)
- 2001: 7-qubit computer (IBM Almaden)
- Feasibility of a computer with millions of qubits... ?

Quick Revision

- **Sub-exponential**: NFS currently the fastest
- **Record 2020**: RSA-829 (829 bits) in 2700 core-years
- **Recommendation**: keys 2048 bits for RSA
- **Future threat**: quantum computers (Shor)

The RSA Algorithm

RSA Operation (Encryption/Decryption)

RSA (Rivest-Shamir-Adleman, 1978) is the most used asymmetric algorithm.

Key generation:

1. Choose two **large** prime numbers p and q (1024 bits each)
2. Compute $n := p \cdot q$ and $\phi(n) = (p-1)(q-1)$
3. Choose encryption exponent e with:
   - $1 < e < \phi(n)$
   - $\gcd(e, \phi(n)) = 1$
4. Compute decryption exponent d such that:

$$e \cdot d \equiv 1 \pmod{\phi(n)}$$

   (using extended Euclidean algorithm or fast exponentiation)

Resulting keys:

- **Public** key: (n, e)
- **Private** key: d (keep p and q secret too!)

Encryption (by Bob, to Alice):

1. Obtain authentic public key (n, e) of Alice
2. Transform plaintext into integers $m_i \in [0, n-1]$
3. Compute ciphertexts: $c_i := m_i^e \bmod n$
4. Send the c_i to Alice

Decryption (by Alice):

- Use private key d to compute:
$$m_i = c_i^d \bmod n$$

Proof of operation:

$$c^d \equiv (m^e)^d \equiv m^{ed} \pmod{n}$$

Since $ed \equiv 1 \pmod{\phi(n)}$, there exists k such that $ed = 1 + k\phi(n)$, therefore:

$$c^d \equiv m^{1+k\phi(n)} \equiv m \cdot (m^{\phi(n)})^k \equiv m \cdot 1^k \equiv m \pmod{n}$$

(by Euler's theorem)

Original Text

RSA Encryption/Decryption Procedure and Proof

Key generation:

- Each entity (A) creates a key pair (public and private) as follows:
  - A chooses the size of the modulus n (e.g., size$(n) = 1024$ or size$(n) = 2048$).
  - A generates two prime numbers p and q of large size $(n/2)$.
  - A computes $n := pq$ and $\phi(n) = (p-1)(q-1)$.
  - A generates the encryption exponent e, with $1 < e < \phi(n)$ such that $\gcd(e, \phi(n)) = 1$.
  - A computes the decryption exponent d, such that: $ed \equiv 1 \pmod{\phi(n)}$ using the extended Euclidean algorithm or fast exponentiation.
- The pair (n, e) is A's **public** key; d is A's **private** key.

Encryption:

- Entity B obtains (n, e), the **authentic** public key of A.
- B transforms its plaintext into a series of integers m_i, such that $m_i \in [0, n-1]$ $\forall i$.
- B computes the ciphertext $c_i := m_i^e \bmod n$, $\forall i$ using fast exponentiation.
- B sends to A all the ciphertexts c_i.

Decryption:

- A uses its private key to compute the plaintexts $m_i = c_i^d \bmod n$.

Proof: Let m be the plaintext and c the ciphertext with $c := m^e \bmod n$, we need to prove: $m \overset{!}{=} c^d \bmod n$
Substituting c by its value we obtain:

$$c^d \bmod n = m^{ed} \bmod n \quad (*)$$

but, we know that:

$$ed \equiv 1 \pmod{\phi(n)}$$

and therefore by definition of congruences, there exists an integer k with:

$$ed - 1 = k\phi(n)$$

substituting in (*):

$$c^d \equiv m^{k\phi(n)+1} \equiv m^{k\phi(n)} \cdot m \pmod{n}$$

If $\gcd(m, n) = 1$, we have by **Euler's theorem**:

$$m^{\phi(n)} \equiv 1 \pmod{n}$$

therefore:

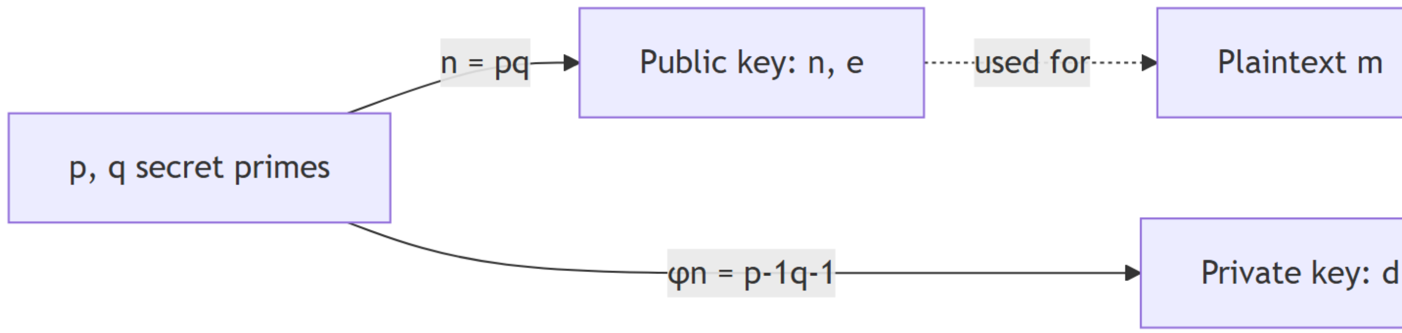
$$c^d \equiv (m^{\phi(n)})^k \cdot m \equiv m \pmod{n}$$

Q.E.D. !

If $\gcd(m, n) \neq 1$, m is necessarily a multiple of p or q (very unlikely case...), we can show by doing the calculations mod p and mod q that the congruence remains true.

Quick Revision

- **Public key**: (n, e) with $n = pq$
- **Private key**: d such that $ed \equiv 1 \pmod{\phi(n)}$
- **Encryption**: $c = m^e \bmod n$
- **Decryption**: $m = c^d \bmod n$
- **Security**: based on difficulty of factoring n



RSA Security

Equivalence RSA problem   Factorization:

- Finding d factoring n (proven equivalent)
- Decrypting without d is **not proven** as hard as factoring, but...
- No method faster than factoring is known

Factorization complexity:

- Fastest methods: $O(\exp(c \cdot (\ln(n))^{1/3}))$ (sub-exponential)
- Computationally impossible for $n \geq 1024$ bits
- **Current recommendation**: 2048 bits minimum (3072-4096 for long-term security)

Choice of exponents:

- **Encryption exponent** e:
  - Often **small** for speed: $e = 3, 17, 65537$ (common)
  - Caution: if e too small AND $m < n^{1/e}$, attack possible (e-th root in $\mathbb{Z}$)
  - Solution: **randomization** (padding) of the message
- **Decryption exponent** d:
  - Must be **large**: at least half the size of n
  - If d small: vulnerable to Wiener's attack

Performance consequence:

- **Fast encryption** (e small)
- **Slow decryption** (d large)

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> **RSA: Security**
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> The **RSAP** problem of finding m from c is not proven to be as hard as factorization but…:
>
> - We can prove that if we find d we can easily compute p and q. This is equivalent to saying that **factoring** n **and finding** d **require equivalent computational effort**.
> - We know that the fastest methods for factoring have a **sub-exponential complexity** $O(\exp(c \cdot (\ln(n))^{1/3}))$. The problem therefore remains **computationally impossible** for modulus ≥ 1048 bits (2048 bits is a frequent choice for long-term security…).
> - To improve encryption speed, we tend to choose **relatively small exponents** e (typically: $e := 3$, $e := 17$ and $e := 19$). However, it has been proven that computing an i-th root (with small i) modulo a composite n can be significantly easier than factoring n. On the other hand, in 2008 it was proven that the generic RSA problem is equivalent to factorization.
> - The **decryption exponent** d **must imperatively be large** (at least half the size

of n) to guarantee the system's security.

- Consequently, **encryption is normally significantly faster than decryption** since the exponents used are much smaller!

Quick Revision

- **Security**: based on difficulty of FACTP (factorization)
- **Recommended size**: $n \geq 2048$ bits
- **Small** e: fast encryption (3, 17, 65537)
- **Large** d: at least size(n)/2
- **Separate keys**: encryption   signature

Attacks on RSA

Attack on small exponent with same message

If the same message m is sent to 3 recipients with $e = 3$:

- $c_1 \equiv m^3 \pmod{n_1}$
- $c_2 \equiv m^3 \pmod{n_2}$
- $c_3 \equiv m^3 \pmod{n_3}$

The **Chinese Remainder Theorem** gives a unique solution $x \bmod n_1 n_2 n_3$ such that:

$$x \equiv c_1 \pmod{n_1}, \quad x \equiv c_2 \pmod{n_2}, \quad x \equiv c_3 \pmod{n_3}$$

If $m^3 < n_1 n_2 n_3$ (often true), then $x = m^3$ in $\mathbb{Z}$ and we can compute m by simply taking the integer cube root!

Protection: always randomize the message before encryption (OAEP padding)

Attack if message small

If $m < n^{1/e}$, then $m^e < n$, so $c = m^e$ (in $\mathbb{Z}$, not modulo). We can directly compute the e-th root!

Protection: padding mandatory

Multiplicative property

$$E(m_1) \cdot E(m_2) \equiv (m_1 \cdot m_2)^e \equiv E(m_1 \cdot m_2) \pmod n$$

Allows chosen-ciphertext attacks and blind signatures.

General attack

The most effective method remains **factoring** n (if parameters well chosen and implementation correct).

> **Original Text**
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> **RSA: Attacks**
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> When we want to encrypt the **same message for a group of correspondents**, it is advisable to introduce variations (**randomization**) before encryption to avoid the following attack:
> Assume we compute ciphertexts c_1, c_2, c_3 from the same plaintext m and the same exponent $e := 3$ addressed to three entities with modulus: n_1, n_2, n_3.
> The **Chinese Remainder Theorem** tells us that there exists a solution $x \bmod n_1 n_2 n_3$, such that:
>
> $$x \equiv c_1 \pmod{n_1}, \quad x \equiv c_2 \pmod{n_2}, \quad x \equiv c_3 \pmod{n_3}$$
>
> But if m does not change for the three encryptions, we have that $x = m^3 \bmod n_1 n_2 n_3$ and, moreover: $m^3 < n_1 n_2 n_3$. We can, therefore, find m by computing the **integer cube root** of m^3, knowing that for this calculation there exist efficient algorithms!
> More generally, if $m < n^{1/e}$, we can apply fast algorithms (in $\mathbb{Z}$) to compute the e-th roots of m^e. It is therefore advisable to perform **"randomization" of m before encrypting**!
> **The multiplicative property of RSA**: $(m_1 m_2)^e \equiv m_1^e \cdot m_2^e \equiv c_1 \cdot c_2 \pmod n$
> gives rise to **dangerous vulnerabilities** (see blind signatures).
> Assuming parameters are correctly chosen and the implementation has no flaws, **the most effective method to "break" the generic RSA algorithm remains factoring n**.

> **Quick Revision**
>
> - **Same message, small e**: CRT allows extracting m!
> - **Message too small**: $m < n^{1/e} \to$ direct root
> - **Multiplicative property**: $E(m_1) \cdot E(m_2) = E(m_1 m_2)$
> - **Protection**: always padding/randomization (OAEP)

The ElGamal Algorithm

Asymmetric system (1985) based on the **discrete logarithm problem (DLP)**.

Keys:

- Choose prime p, generator $\alpha \in \mathbb{Z}_p^*$, secret a
- Compute $y = \alpha^a \bmod p$
- **Public**: (p, α, y) | **Private**: a

Encryption: For message m, choose unique random k

- $\gamma = \alpha^k \bmod p$
- $\delta = m \cdot y^k \bmod p$
- Send (γ, δ)

Decryption: $m = \delta \cdot \gamma^{-a} \bmod p$

> Original Text
>
> ### ElGamal Encryption/Decryption Procedure
>
> **Key generation**
> Each entity (A) creates a key pair (public and private) as follows:
>
> - A generates a prime number p (len(p) 1024 bits) and a **generator** α of the multiplicative group $\mathbb{Z}_p^*$
> - A generates a random number a, such that $1 \leq a \leq p-2$ and computes $y := \alpha^a \bmod p$
> - The **public key** of A is (p, α, y), the **private key** of A is a
>
> **Encryption**
>
> - Entity B obtains $(p, \alpha, \alpha^a \bmod p)$, the authentic public key of A
> - B transforms its plaintext into a series of integers m_i, such that $m_i \in [0, p-1]\ \forall i$
> - For each message m_i:
>   - B generates a **unique** random number k, such that $1 \leq k \leq p-2$
>   - B computes $\gamma := \alpha^k \bmod p$ and $\delta := m_i \cdot (\alpha^a)^k \bmod p$ and sends the ciphertext $c := (\gamma, \delta)$
>
> **Decryption**
>
> - A uses its private key a to compute $\gamma^{p-1-a} \bmod p$ (note that: $\gamma^{p-1-a} \equiv \gamma^{-a} \equiv \alpha^{-ak} \bmod p$)
> - A retrieves the plaintext by computing: $\delta \cdot \gamma^{-ak} \bmod p$

Quick Revision

Base: DLP in $\mathbb{Z}_p^*$
Ciphertext: $(\alpha^k, m \cdot y^k)$
Security: k must be unique and large
Disadvantage: doubles message size

Essential Remarks

- **Proof**: $\delta \cdot \gamma^{-a} = m \cdot (\alpha^a)^k \cdot (\alpha^k)^{-a} = m \bmod p$
- **Security**: based on DLP (complexity sub-exponential close to factorization)
- **Exponents**: k and a must be large (otherwise vulnerable to baby-step giant-step)
- **Reuse prohibited**: if k repeated, $\delta_1/\delta_2 = m_1/m_2$ reveals the messages
- **Major disadvantage**: $\times 2$ expansion of ciphertext size
- **Generalization**: works on $GF(2^n)$ or elliptic curves

Original Text - Remarks

Proof that the scheme works: If $s \equiv k^{-1}(m_h - ar) \bmod (p-1)$, we have that: $m_h \equiv (ar + ks) \bmod (p-1)$ and $v_2 = \alpha^{H(m)} \bmod p$. If, as we wish to show $m_h = H(m)$, by reducing exponents mod $(p-1)$, we can rewrite v_2: $v_2 \equiv \alpha^{ar+ks} \bmod p$. On the other hand: $v_1 = y^r \alpha^{rs} \equiv \alpha^{ar}\alpha^{ks} \equiv \alpha^{ar+ks} \bmod p$.
The ElGamal procedure is based on the difficulty of computing **discrete logarithms modulo a prime number** (DLP problem) even though it has not been proven to be strictly equivalent to this problem.
The **most efficient algorithms** known have a sub-exponential complexity very close to that of factorization (we often use the same algorithms).
The **chosen exponents** (k, a) must be large because there exist efficient algorithms to compute discrete logarithms modulo a prime number when the exponent is small (baby-step giant-step algorithm).
A **disadvantage of ElGamal** is that it multiplies the ciphertext length by 2.
It is **essential** for the security of the procedure that the random number k is not repeated, otherwise: let (γ_1, δ_1) and (γ_2, δ_2) be the two generated ciphertexts, we have that $\delta_1/\delta_2 = m_1/m_2$ and consequently, it is trivial to recover one plaintext from the other.
The ElGamal procedure can be **generalized** to other groups like $GF(2^n)$ or elliptic curves.

Quick Revision - Remarks

Equivalence: based on DLP (not proven equivalent)
k **unique**: CRITICAL - otherwise m_1/m_2 revealed
Key size: large exponents necessary
Extensions: $GF(2^n)$, elliptic curves

Rabin Algorithm

Asymmetric system **equivalent to factorization** (provably secure).

Keys:

- Generate two primes p, q (1024 bits total), compute $n = pq$
  - **Public**: n
  - **Private**: (p, q)

Encryption: $c = m^2 \bmod n$

Decryption:

- Compute the 4 square roots of $c \bmod n$ (via roots mod p and mod q)
- Identify the correct message by redundancy

Original Text

Rabin Encryption/Decryption Procedure

Key generation
Each entity (A) creates a key pair (public and private) as follows:

- A generates two random prime numbers p and q of large size (len(pq) 1024)
- A computes $n := pq$
- The **public key** of A is n, the **private key** of A is (p, q)

Encryption

- Entity B obtains n, the authentic public key of A
- B transforms its plaintext into a series of integers m_i, such that $m_i \in [0, n-1]\ \forall i$
- B computes $c_i = m_i^2 \bmod n$ for each message m_i
- B sends all the ciphertexts c_i to A

Decryption

- A uses its private key (p, q) to retrieve the **4 solutions** of the equation: $c_i = x^2 \bmod n$ using **efficient algorithms** to compute square roots mod$p$ and mod$q$
- A determines either by an **additional indication** from B, or by **redundancy analysis** which of the 4 messages m_1, m_2, m_3, m_4 is the original plaintext

Quick Revision

Base: SQROOTP (square root mod composite)
Advantage: proven equivalent to factorization
Problem: 4 possible solutions, requires redundancy
Vulnerability: chosen-ciphertext attack reveals factors

Essential Remarks

- **Proven security**: SQROOTP FACTP (only algorithm with proven equivalence)
- **Chosen-ciphertext attack**: if A decrypts $c = m^2 \bmod n$ chosen by adversary M
  - M receives a root m_x among 4 possible
  - If $m \neq m_x \bmod n$ (prob. 0.5), then $\gcd(m - m_x, n)$ gives a factor of n
- **Solution**: require sufficient redundancy to identify unique solution without ambiguity

Original Text - Remarks

The Rabin procedure is based on the **impossibility of finding square roots modulo a composite of unknown factorization** (SQROOTP problem).
The **main interest** of this algorithm lies in the fact that it has been **proven to be equivalent to factorization** (SQROOTP FACTP). This algorithm therefore belongs to the **provably secure** category for any passive attack.
Active attacks can, in some cases, compromise the algorithm's security. More precisely, if we mount the following **chosen ciphertext** attack:

- The attacker M generates an m and sends to A the ciphertext $c = m^2 \bmod n$.
- A responds with a root m_x among the 4 possible m_1, m_2, m_3, m_4.
- If $m \neq m_x \bmod n$ (probability 0.5), M repeats with a new m.
- Otherwise, A computes $\gcd(m - m_x, n)$ and thus obtains one of the two factors of n.

This attack could be **avoided** if the procedure required **sufficient redundancy** in the plaintexts allowing A to identify without ambiguity which of the possible solutions is the original plaintext. In this case, A would always respond with m and discard the other solutions that do not have the predefined level of redundancy.

Quick Revision - Remarks

Unique: only algorithm proven equivalent to FACTP
Attack: chosen-ciphertext gives factors (prob. 0.5)
Countermeasure: mandatory redundancy in messages

Comparison RSA - ElGamal - Rabin

Criterion	RSA	ElGamal	Rabin
Problem	RSAP	DLP	SQROOTP
Security	Equiv. factorization (generic case)	Based on DLP	**Proven** factorization
Expansion	1:1	**1:2**	1:1
Decryption	Deterministic	Deterministic	**4 solutions**
Signature	Yes	Yes	Yes (with precautions)

Elliptic Curves (Basic Idea)

Fundamental Concept

An **elliptic curve** E is defined by: $y^2 = x^3 + ax + b$ (with discriminant $4a^3 + 27b^2 \neq 0$).

Key operation: Point addition

- Geometrically: draw a line between two points P and Q, find the 3rd intersection point, then take its symmetric
- Forms a **commutative group** with point at infinity $\mathcal{O}$ as identity
- **Scalar multiplication**: $kP = P + P + ... + P$ (k times)

Cryptographic advantage:

- The **ECDLP problem**: finding k such that $Q = kP$ is very difficult (exponential effort)
- **Shorter keys** for same security as in $\mathbb{Z}_p^*$

> **Original Text - Definition**
>
> An **elliptic curve** is a set of points E defined by the equation: $y^2 = x^3 + ax + b$, with x, y, a and b rational numbers, integers or integers modulo m ($m > 1$). The set E also contains a "point at infinity" denoted $\mathcal{O}$. The point $\mathcal{O}$ is not on the curve but it is the identity element of E.
> We will choose for our calculations elliptic curves that do not have multiple roots or, in other words, curves where the **discriminant** $4a^3 + 27b^2 \neq 0$.

> **Quick Revision - Concept**
>
> **Equation**: $y^2 = x^3 + ax + b$
> **Structure**: group with $\mathcal{O}$
> **Operation**: geometric addition
> **Hard problem**: ECDLP

Addition on Elliptic Curves

Let $P := (x, y) \in E$, we define $-P := (x, -y)$ (symmetric with respect to the x-axis). We have $P + (-P) = \mathcal{O}$.

For two points $P, Q \in E$ with $Q \neq -P$, we define $P + Q := R$ where $-R$ is the 3rd intersection point between the curve and the line passing through P and Q.

For **doubling**: $2P = R$ where $-R$ is the intersection point of the curve with the tangent to the curve at point P.

> **Original Text - Addition**
>
> Let $P := (x, y) \in E$, we define $-P$ as $-P := (x, -y)$. Graphically, $-P$ is the symmetric point of P with respect to the x-axis. Note that $P + (-P) = \mathcal{O}$.
> Let two points $P, Q \in E$, such that $Q \neq -P$, we define the addition $P + Q := R$ where $R \in E$ such that $-R$ is the 3rd intersection point between the curve and the line passing through P and Q.
> The set E with $\oplus$ defines a **commutative group** for addition.
> Let $P \in E$, the point $2P = R$, such that $-R$ is the intersection point of the curve with the line tangent to the curve at point P.

Quick Revision - Addition

Inverse: $-P = (x, -y)$
Addition: 3rd intersection point + symmetry
Doubling: tangent + symmetry
Property: commutative group

ECDLP and Cryptographic Advantages

When the elliptic curve is defined over the field $\mathbb{Z}_p$ with p a large prime ($y^2 \equiv x^3+ax+b \bmod p$), the computation of $k \in \mathbb{Z}_p$ such that $Q = kP$ with (P, Q) known is **very difficult** (exponential effort). This problem is the **Elliptic Curve Discrete Logarithm Problem (ECDLP)**.

Main advantage: key sizes much smaller for equivalent security.

Original Text - ECDLP and Advantages

When the elliptic curve is defined over the field $\mathbb{Z}_p$ with p a large prime number ($y^2 \equiv x^3 + ax + b \bmod p$), the computation of $k \in \mathbb{Z}_p$ such that $Q = kP$ with (P, Q) known, is very difficult (requires exponential effort). This problem is known as: **Elliptic Curve Discrete Logarithm Problem (ECDLP)**.
The **main advantage** of public cryptography based on elliptic curves is that the size of the numbers used (and therefore, keys) is smaller.
This is due to the **increased complexity** of computations on E_p (elliptic curve defined over field $\mathbb{Z}_p$) compared to usual fields such as $\mathbb{Z}_p$ or $GF(2^m)$.
The **representation of a plaintext as points** of the curve remains a complex operation.
In October 2003, the **US National Security Agency (NSA)** purchased a patent from Certicom for the use of elliptic curve cryptography.
In September 2013 Claus Diem showed that under certain conditions the ECDLP problem could be solved in **sub-exponential time**.

Quick Revision - ECDLP

Problem: finding k in $Q = kP$ (exponential)
Gain: keys ~6-10× shorter
Limit: representing messages as points difficult
NSA: adopted in 2003

Key Size Comparison Table

AES (symmetric)	RSA/DH	Elliptic Curves	Ratio
56 bits	512 bits	112 bits	1:4.6
80 bits	1024 bits	160 bits	1:6.4
112 bits	2048 bits	224 bits	1:9.1
128 bits	3072 bits	256 bits	1:12
256 bits	15360 bits	512 bits	1:30

> **Original Text - Table**
>
> This table shows the key size ratios compared to RSA for equivalent security.
> *(Table extracted from original document)*

ElGamal on Elliptic Curves

Direct Adaptation

Replace operations in $\mathbb{Z}_p^*$ with operations on E_p

Keys:

- Choose curve E_p and point $P_0 \in E_p$ of large order
- Secret x, compute $P_a = xP_0$
- **Public**: (E_p, P_0, P_a) | **Private**: x

Encryption: For message $m_i \in E_p$

- Choose random k
- $\gamma = kP_0$, $\delta = kP_a + m_i$
- Send (γ, δ)

Decryption: $m_i = \delta - x\gamma$

> **Original Text - ElGamal EC**
>
> **Key generation**
> Each entity (A) creates a key pair (public and private) as follows:
>
> - A chooses an elliptic curve E_p with p, a large prime number (len(p) bits) and a point $P_0 \in E_p$.
> - A generates a random number x, such that $1 \leq x \leq p$ and computes $P_a = xP_0$

(multiplication by a scalar on E_p, for which efficient algorithms exist).
- The public key of A is (E_p, P_0, P_a), the private key of A is x.

Encryption
Entity B obtains (E_p, P_0, P_a), the authentic public key of A.

- B transforms its plaintext into a series of integers m_i, such that $m_i \in E_p$ for all i.
- For each message m_i:
  - B generates a **unique** random number k, such that $1 \leq k \leq p$.
  - B computes $\gamma := kP_0$ and $\delta := kP_a + m_i$ and sends the ciphertext $c := (\gamma, \delta)$.

Decryption

- A uses its private key x to compute: $x\gamma = xkP_0 = kP_a$.
- A retrieves the plaintext by computing: $\delta - kP_a = kP_a + m_i - kP_a = m_i$.

The security of the scheme relies on **ECDLP**!
It is also necessary to **authenticate** the exchanged public parts to avoid the previously described man-in-the-middle attacks.
The properties of the protocol are identical to the $\mathbb{Z}_p^*$ case.

Quick Revision - ElGamal EC

Principle: same as ElGamal on E_p
Operations: + and scalar multiplication on points
Security: ECDLP
Authentication: necessary against MitM
Advantage: short keys