

Kasteleyn's formula for domino tilings of rectangles

- presentation on Eric Stucky's expository paper -

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This is a presentation on Kasteleyn's formula for the number of domino tilings of a $m \times n$ rectangle heavily based on [Stucky 2015]. Although there are many connections between this and physics, we will not cover these here and instead direct the interested readers to [Kasteleyn 1961]. The key result we prove is : If $T(m, n)$ is the set of tilings of a $m \times n$ rectangle (wlog. m is even) and $H(\pi), V(\pi)$ the number of horizontal and vertical dominos in $\pi \in T(m, n)$ respectively, then for any complex x, y we have

$$\sum_{\pi \in T(m,n)} x^{H(\pi)} y^{V(\pi)} = 2^{\frac{mn}{2}} \prod_{k=1}^{m/2} \prod_{l=1}^n \left(x^2 \cos^2 \frac{\pi k}{m+1} + y^2 \cos^2 \frac{\pi l}{m+1} \right)^{\frac{1}{2}}$$

If we set $x = y = 1$ this gives us the number of such tilings.

The proof is separated into two major parts. In the first we deal with the combinatorial aspect where we translate the problem of counting tilings into that of counting specific permutations allowing us to use matrix methods¹ especially Pfaffians which seem to be just what we need here. For a specific $mn \times mn$ matrix D , the expression in the main theorem will be $\text{Pf}(D)$. We rely on the identity $\text{Pf}(K)^2 = \det(K)$ where K is skew symmetric of even dimension. First we will come up with a *guess* D' , which gives $\text{Per}(D')$ to be the expression we need but sadly the permanent is not as algebraically robust as determinants and we cannot calculate them as such, a big subpart of this part will be devoted to *fixing* D to get the right matrix. In the second part of the proof we will use kronecker products and (...left out till I read this...)

¹We have already seen things like the State matrix in the same context, tilings of hexagons with lozenges using determinants etc.

As discussed above we will try to connect tilings with dominos to permutations on $[mn]$. We will identify the usual unit cells in the $m \times n$ rectangle with numbers from $[mn]$ using the lexicographic ordering, starting from the bottom left corner. For example, when $m = n = 2$ we identify,

$$(1, 1) \leftrightarrow 1, (1, 2) \leftrightarrow 2, (2, 1) \leftrightarrow 3, (2, 2) \leftrightarrow 4$$

Now we line all the dominos up and label both of their ends with adjacent numbers as follows: Label the ends of the first domino with 1 and 2, the second with 3 and 4, ... upto $mn/2$ dominos. Then, we can associate to each tiling a permutation σ on $[mn]$ which maps k (some end of a domino) to the lexicographic rank of its position on the $m \times n$ rectangle.

It is easy to notice that a tiling can have multiple permutations corresponding to it (this is fixed over all tilings, finding how many corresponding permutations there are is a good exercise) and also that not every permutation corresponds to a tiling (if this weren't the case then this problem would be trivial), for example take (24) on [4]. We seek some way to identify an unique permutation corresponding with some tilings, something *canonical*. One way to do this is to put these restrictions on the permutations :

$\sigma(2k - 1) < \sigma(2k)$ and $\sigma(1) < \sigma(3) < \sigma(5) < \dots$. These hand-wavingly make sense because flipping a domino should be immaterial to the number of tilings so its good to fix the higher label(equivalently both) and also that switching *pairs* around just replaces two dominos which should also be immaterial - for example, on [4] we see that 1234 and 3412 correspond to the exact same tiling. Convince yourself that such permutations exist given any tiling, leaving uniqueness for exercise.

Definition 3.1 (Pfaffian)

For a $2n \times 2n$ skew symmetric matrix A , the pfaffian is defined as

$$\text{Pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn} \prod_{i=1}^n a_{\sigma(2i-1), \sigma(2i)}$$

However, there is a way to write the pfaffian with lesser terms. Let Π be the set of partitions of $[2n]$ into pairs. For such a partition $\pi = \{\{i_1, j_1\}, \dots, \{i_n, j_n\}\}$ where $i_1 < i_2 < i_3 < \dots$ and $j_k < j_k$ for all k , let σ_π be the permutation $i_1 j_1 i_2 j_2 \dots$. With the notation above, we claim that

$$\text{Pf}(A) = \sum_{\pi \in \Pi} \text{sgn}(\sigma_\pi) \prod_{i=1}^n a_{\sigma_\pi(2i-1), \sigma_\pi(2i)}$$

For a proof, notice that when the pairs in the partition are permuted and the elements in each pair also permuted, these correspond to $2^n n!$ permutations and that the summand remains constant upon switching position of two elements in a pair as A is skew symmetric and it stays constant over as the sign, being a homomorphism on S_{2n} , leads is multiplied by

$$\begin{aligned}\operatorname{sgn}((2i - 1, 2j - 1)(2i, 2j)) &= \operatorname{sgn}((2i - 1, 2j - 1)) \cdot \operatorname{sgn}((2i, 2j)) \\ &= (-1) \cdot (-1) = 1\end{aligned}$$

in cycle notation, when we switch the i -th and j -th pairs - making the summand stay constant all over these permutations and we are done.

Theorem 3.2 (spectral theorem for real skew symmetric matrices)

If A is a $2n \times 2n$ real skew symmetric matrix then the spectrum is of form $\{\pm i\lambda_k\}_{1 \leq k \leq n}$ and A is orthogonally similar to a block diagonal matrix H with diagonal entries being H_1, \dots, H_n with

$$H_k := \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix}$$

Proof.

The structure of the spectrum is a result from basic linear algebra. As skew symmetric matrices are normal we can diagonalise them into $D := Q^\top A Q = \text{diag}(i\lambda_1, -i\lambda_1, \dots)$ with Q orthogonal. Let,

$$P := \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$



(continued).

This is an orthogonal matrix. Let $L = \text{diag}(P, P, \dots)$ be a $2n \times 2n$ block diagonal matrix, this is also orthogonal and we see that

$$\begin{aligned} L^T H L &= \text{diag}(P^T, P^T, \dots) \text{diag}(H_1, H_2, \dots) \text{diag}(P, P, \dots) \\ &= \text{diag}(P^T H_1 P, P^T H_2 P, \dots) = \text{diag}(\text{diag}(i\lambda_1, -i\lambda_1), \dots) = D \end{aligned}$$

Thus, $A = Q^T L^T H L Q$ rearranging which we can conclude as the product of two orthogonal matrices is orthogonal. □

Theorem 3.3

For any $2n \times 2n$ skew symmetric matrix A and arbitrary matrix B we have, $\text{Pf}(BAB^T) = \text{Pf}(A)$.

For our use case however B is invertible and this version of the theorem can be proved via considering B to be a product of elementary matrices and using some of the determinant-like properties which can be found on wikipedia. Now we state the most important theorem we will use regarding pfaffians.

Theorem 3.4 (T. Muir. 1960)

If A is $2n \times 2n$ and skew symmetric then $\text{Pf}(A)^2 = \det(A)$

Proof.

Show that $P \in \mathbb{R}[x_1, \dots, x_m]$ and $P(\mathbb{R}^m) = \{0\} \implies P \equiv 0$ as an exercise. Thus it suffices to prove this theorem for all real matrices. We have seen that we can write $A = QHQ^\top$ with H the same as above, for some Q . Then, $\text{Pf}(A)^2 = \text{Pf}(H)^2 \det(Q)^2 = \det(A)^2$ with the second last equality left as another exercise. □

We first try to get a glimpse of the possible structure of D , only then rectifying our guess to the final matrix D which will give us $\text{Pf}(D) = |T(m, n)|$.

References I

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