Exercise sheet 4

Only the exercises marked [HW] need to be submitted.
Vectors will be denoted in bold unlike scalars.

1. Show that the distributive law

$$S \cap (T + W) = (S \cap T) + (S \cap W)$$

is false for subspaces. However prove that it holds whenever $T \subseteq S$ or $W \subseteq S$.

2. Let S and T be subspaces of $\mathbb{R}^4$ given by

$$S = \{(x_1, x_2, x_3, x_4) : 3x_1 + x_2 + x_3 + x_4 = 0,\ x_1 - x_3 + 2x_4 = 0\}$$
$$T = \{(x_1, x_2, x_3, x_4) : 5x_1 + 2x_2 + 3x_3 = 0,\ x_2 + x_3 + x_4 = 0\}.$$

(a) Obtain a basis for each of $S \cap T, S, T$ and $S + T$.

(b) Verify the modular law for S and T

(c) Extend the basis of $S + T$ you obtained in the first part to a basis of $\mathbb{R}^4$

(d) Express $S + T$ and $S \cap T$ in the same form as S and T.

3. Let S and T be complementary subspaces of V. Any $\mathbf{v} \in V$ can be written uniquely as $\mathbf{x}+\mathbf{y}$ where $\mathbf{x} \in S$ and $\mathbf{y} \in T$. Then $\mathbf{x}$ is called the projection of $\mathbf{v}$ into S along T. Let us denote $\mathbf{x} = P_{S,T}(\mathbf{v})$. Show

(a) The projection of $\mathbf{v}$ into T along S is $\mathbf{v} - P_{S,T}(\mathbf{v})$.

(b) $\mathbf{v} \in S$ if and only if $P_{S,T}(\mathbf{v}) = \mathbf{v}$.

(c) $\mathbf{v} \in T$ if and only if $P_{S,T}(\mathbf{v}) = \mathbf{0}$.

(d) $P_{S,T}(P_{S,T}(\mathbf{v})) = P_{S,T}(\mathbf{v})$

(e) $P_{S,T}$ is a linear map on V.

4. We say that the sum $S_1 + \cdots + S_k$ of the subspaces $S_1, S_2 \cdots, S_k$ is *direct* if any vector $\mathbf{x}$ in $S_1 + \cdots + S_k$ can be expressed uniquely as $\mathbf{x}_1 + \cdots + \mathbf{x}_k$ with $\mathbf{x}_i \in S_i$ for all i. We then write $S_1 + \cdots + S_k = S_1 \oplus S_2 \oplus + \cdots \oplus S_k$. Show the following are equivalent

- $S_1 + \cdots + S_k$ is direct.
- $(S_1 + \cdots + S_i) \cap S_{i+1} = \{\mathbf{0}\}$ for $i = 1, 2, \cdots, k - 1$.
- $\mathbf{0} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$, $\mathbf{x}_i \in S_i, i = 1, 2, \cdots, k$ implies $\mathbf{x}_i = \mathbf{0}$ for $i = 1, 2, \cdots, k$.
- $dim(S_1 + \cdots + S_k) = \sum_{i=1}^{k} dim(S_i)$.

5. [HW 4, due August 29] Show that a non-trivial subspace S of V has two complements T_1, T_2 with $T_1 \cap T_2 = \{\mathbf{0}\}$ if and only if $dim(S) \geq dim(V)/2$.

6. Let V, W be vector spaces over F, and let $T : V \to W$ be an isomorphism. Show that $T^{-1} : W \to V$ is an isomorphism.

7. Show that two vector spaces over F are isomorphic if and only if they have the same dimension.

8. Consider exercise 7 from exercise sheet 1 with Ω a finite set of size n. Then show that V is isomorphic to F^n where $F = \mathbf{Z}_2$.
HINT: Consider whether each element of Ω is in the subset or not.