Exercise sheet 3

Only the exercises marked [HW] need to be submitted.
Vectors will be denoted in bold unlike scalars.

1. Show that a subset A of a vector space is linearly dependent if and only if there exists $\mathbf{x} \in A$ such that $\mathbf{x} \in \mathrm{Sp}(A - \{\mathbf{x}\})$

2. Let A be a linearly independent subset and $\mathbf{y} \notin A$. Then show that $A \cup \{\mathbf{y}\}$ is linearly dependent if and only if $\mathbf{y} \in \mathrm{Sp}(A)$.

3. Prove that the real vector space consisting of all continuous real valued functions on the interval $[0, 1]$ is infinite dimensional.

4. Find all the maximal linearly independent subsets of $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_5\}$, where $\mathbf{x}_1 = (1, 1, 0, 1), \mathbf{x}_2 = (1, 2, -1, 0), \mathbf{x}_3 = (1, 0, 1, 2), \mathbf{x}_4 = (0, 1, 1, 1)$ and $\mathbf{x}_5 = (2, 0, 2, 4)$.

5. Let $\mathbf{x}_1, \cdots, \mathbf{x}_k$ be vectors in F^n and let $\mathbf{y}_i$ the vector in F^{n-1} formed by the first $n-1$ components of $\mathbf{x}_i$, $i = 1, 2, \cdots, k$. Show that if $\mathbf{y}_1, \cdots, \mathbf{y}_k$ are linearly indepedent in F^{n-1} then $\mathbf{x}_1, \mathbf{x}_2 \cdots, \mathbf{x}_k$ are linearly independent in F^n. Is the converse true? Why?

6. Show that $\{1, \sqrt{2}\}$ and $\{\sqrt{2}, \sqrt{3}, \sqrt{6}\}$ are linearly independent over $\mathbb{Q}$ and that $\{\sqrt{2}, \sqrt{3}, \sqrt{12}\}$ is linearly dependent over $\mathbb{Q}$.

7. Let $Span(A) = S$. Then show that no proper subset of A generates S if and only if A is linearly independent.

8. Let S and T be subspaces of a finite dimensional vector space V such that $S \subseteq T$. Then show that $\dim(S) \leq \dim(T)$ with equality if and only if $S = T$.

9. [HW 3, due August 22] Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, \cdots, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

   Find a basis of U.

10. Extend $\{(1, 1, 2, 1, 0), (1, 0, 1, 3, 1)\}$ to a basis of $\mathbb{R}^5$

11. [HW 3, due August 22] If a subspace S of $\mathbb{R}^n$ has a basis $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$ such that all components of $\mathbf{x}_1$ are strictly positive, show that S has a basis such that all components of each vector in B are strictly positive.

12. Let $A \subseteq V$. If one vector in $Span(A)$ can be expressed uniquely as a linear combination form A then show A is linearly independent, and so is a basis of $Span(A)$.

13. Prove or disprove: If B is a basis of V and S is a subspace of V then B contains a basis of S.

14. Find three vectors in $\mathbb{R}^3$ which are linearly dependent , and are such that any two of them are linearly dependent.

15. Let $x_1, \cdots, x_n$ be fixed real numbers.

    (a) Show that $l_1(t), \cdots, l_n(t)$ form a basis of $\mathcal{P}_n$, where $l_i(t) = \prod_{j \neq i}(t - x_j)$. Show that any $f(t) \in \mathcal{P}_n$ can be written as $\sum_{i=1}^{n} \alpha_i l_i(t)$, where $\alpha_i = f(x_1)/l_i(x_i)$ (this is *Lagrange interpolation formula*).

    (b) Show that $\psi_1(t), \cdots, \psi_n(t)$ form a basis of $\mathcal{P}_n$, where $\psi_1(t) = 1$ and $\psi_i(t) = \prod_{j=1}^{i-1}(t - x_j)$ for $i = 2, \cdots, n$ (this leads to *Newton's divided difference formula*)