Indian Statistical Institute, Bangalore

B. Math.

First Year, First Semester

Analysis I — Instructor: B V Rajarama Bhat

Home Assignment II — Due Date : Sept. 4, 2025

(1) Take
$$C = \{2 - \frac{1}{n} : n \in \mathbb{N}\} \bigcup \{5 - \frac{1}{n} : n \in \mathbb{N}\}.$$
Show that every non-empty subset of C has a minimal element. Determine as to whether the same property holds for D, where
$$D = \{5 - \frac{1}{n} - \frac{1}{m^2} : m, n \in \mathbb{N}\}.$$

(2) Find infimum and supremum of following subsets of the real line:
$$A_1 = \{3 + (-1)^n \frac{1}{n} : n \in I\!N\}.$$
$$A_2 = \{x^2 + 1 : 0 \leq x \leq 1\}.$$

(3) Let A, B non-empty, bounded subsets of $\mathbb{R}$. Take $A+B = \{a+b : a \in A, b \in B\}$ and $A - B = \{a - b : a \in A, b \in B\}$ and $A.B = \{a.b : a \in A, b \in B\}$. Show that these sets are bounded. Which of the following statements are true and which are not true in general (prove your claim):
   (a) $\sup(A \bigcup B) = \max\{\sup A, \sup B\}$;
   (b) $\sup(A \bigcap B) = \min\{\sup A, \sup B\}$;
   (c) $\sup(A + B) = \sup A + \sup B$.
   (d) $\sup(A - B) = \sup A - \sup B$;
   (e) $\sup(A.B) = \sup(A). \sup(B)$.

(4) Let $\{t_n\}_{n \geq 1}$ be a sequence defined by
$$t_1 = 2, t_{n+1} = \frac{1}{2}(t_n + \frac{2}{t_n}) \text{ for } n \geq 1.$$
Show that $\{t_n\}_{n \geq 1}$ is a convergent sequence, converging to $\sqrt{2}$.

(5) Show that $\lim_{n \to \infty} (1 + \frac{1}{n})^n$ exists. (Hint: Prove that it is a montonic bounded sequence.)

(6) Show that there exists a unique positive real number x such that $x^3 = 2$.

(7) Prove that the following sequences are convergent:
$$a_n = (1 - \frac{1}{2})(1 - \frac{2}{3}) \cdots (1 - \frac{1}{n+1}), \ n \in \mathbb{N};$$
$$b_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}, \ n \in \mathbb{N}.$$

P.T.O.

(8) Prove that the sequence

$$c_n = 5 + (-1)^n(2 + \frac{1}{n})$$

is not convergent.

(9) Suppose $\{x_n\}$ is a real sequence. For $n \geq 1$ define averages

$$y_n = \frac{1}{n}\sum_{k=1}^{n} x_k.$$

Show that if $\{x_n\}$ converges then $\{y_n\}$ converges, however the converse is not true.

(10) Find all functions $h; \mathbb{R} \to \mathbb{R}$, satisfying $h(x + y) = h(x) + h(y)$ and $h(x.y) = h(x).h(y)$ for all x, y in $\mathbb{R}$. (Hint: You may need order properties and completeness of $\mathbb{R}$.)