Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester

Linear Algebra II — Instructor: B V Rajarama Bhat
Home Assignment I — Due Date : February 1, 2026

In the following for any permutation σ, $\epsilon(\sigma)$ denotes the signature of σ.

(1) Consider a permutation $\sigma := (2,5)(1,4,3)$ of $\{1,2,3,4,5,6\}$. Write down the matrices P^{σ}, $P^{\sigma^{-1}}$. Verify that $\det(P^{\sigma}) = \epsilon(\sigma)$ for this permutation σ.

(2) Let A be a real matrix of order 3×3 and let r_1, r_2 and r_3 be the rows of A. Let P be the parallelogram in $\mathbb{R}^3$ whose vertices are $0, r_1, r_2, r_3, r_1+r_2, r_2+r_3, r_1+r_3$ and $r_1+r_2+r_3$. Prove that the volume of P is the absolute value of the $\det(A)$. (Hint: Compare the effect of an elementary row operation on the volume and on $\det(A)$.)

(3) Let σ be a permutation in S_n. Let $s(\sigma)$ denote the number of inversions, that is, the number of pairs $\{i,j\}$ such that $\sigma(i) > \sigma(j)$ when $1 \leq i < j \leq n$. Show that $\epsilon(\sigma) = (-1)^{s(\sigma)}$. Verify $\prod_{1\leq i<j\leq n}(x_{\sigma(j)} - x_{\sigma(i)}) = \epsilon(\sigma)\prod_{1\leq i<j\leq n}(x_j - x_i)$, for any complex numbers $x_1, x_2, \ldots, x_n$.

(4) Show that for $n \geq 3$, every even permutation in S_n is a product of 3-cycles of the form $(i, i+1, i+2)$.

(5) Let A be a complex square matrix. A vector v is called an eigenvector of A if $Av = dv$ for some $d \in \mathbb{C}$. Show that for an invertible matrix S, $S^{-1}AS$ is a diagonal matrix if and only if columns of S are eigenvectors of A.

(6) If A is an $n \times n$ matrix over $\mathbb{C}$. Show that $|\det(A)| \leq \prod_{i=1}^{n}(\sum_{j=1}^{n}|a_{ij}|)$.

(7) Let $f(x) = (\lambda_1 - x)(\lambda_2 - x)\cdots(\lambda_n - x)$ and let $A := \begin{bmatrix} \lambda_1 & a & a & a & \cdots & a & a \\ b & \lambda_2 & a & a & \cdots & a & a \\ b & b & \lambda_3 & a & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b & b & b & b & \cdots & \lambda_{n-1} & a \\ b & b & b & b & \cdots & b & \lambda_n \end{bmatrix}$.

Show that $\det(A) = \frac{bf(a) - af(b)}{b-a}$ when $a \neq b$. Also find $\det(A)$ when $a = b$.

(8) **Prove or disprove:** Let A be a square matrix of size $n \times n$ such that all entries are either 1 or -1. Then $\det(A)$ is divisible by 2^{n-1}.

(9) The **resultant** is a determinant used to determine whether two polynomials have a common root or not. Consider polynomials $p(x) = a_2x^2 + a_1x + a_0$ and $q(x) = b_1x + b_0$. The Sylvester matrix of these two polynomials is defined as:

$$A := \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & 0 \\ 0 & b_1 & b_0 \end{bmatrix}.$$

Show that $\det(A) = 0$ if and only if p, q have a common root.

(10) Let A, B, C and D be square matrices of same size and let $P := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Prove or disprove: if $BD = DB$ then $\det(P) = \det(DA - BC)$.