

# Linear Algebra I - Homework problems

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*If problem **n** from **exercise\_x.pdf** is assigned as homework then it will be referred to as **x.n**.*

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**1.8** It was already shown in class that  $\mathbb{C}^{\mathbb{R}} = \{f : \mathbb{R} \rightarrow \mathbb{C}\}$  forms a vectorspace over  $\mathbb{R}$  with the same operations defined here. We show that  $V$  is a subspace and hence a vectorspace itself. For all  $\alpha \in \mathbb{R}$  and  $f, g \in V$  then for all  $t \in \mathbb{R}$ ,

$$\overline{(\alpha f + g)(t)} = \overline{\alpha f(t) + g(t)} = \overline{\alpha} \cdot \overline{f(t)} + \overline{g(t)} = \alpha \cdot \overline{f(-t)} + \overline{g(-t)} = (\alpha f + g)(-t)$$

and thus  $\alpha f + g \in V$  and we are done as according to what was discussed in class,  $V$  is a subspace if for all  $\alpha \in \mathbb{R}$  and  $f, g \in V$  its the case that  $\alpha f + g \in V$  too. An example of a function with some non real outputs in  $V$  is  $f(t) = it$ .

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**2.11** We have seen that if  $A \subseteq B$  are subsets of  $V$  then  $Sp(A) \subseteq Sp(B)$  and using this if  $A, B$  are any subsets of  $V$  (not the ones used to state the result earlier) that as  $A, B \subseteq A \cup B$  we have that  $Sp(A), Sp(B) \subseteq Sp(A \cup B)$  and thus  $Sp(A) \cup Sp(B) \subseteq Sp(A \cup B)$  and the second bulleted claim follows similarly from the fact that  $A \cap B \subseteq A, B$ . The last claim is false, we can take disjoint sets  $A, B$  such that  $Sp(A) = Sp(B) \neq \{0\}$ , for example in  $V = \mathbb{R}^2$  take  $A = \{(0, 1), (1, 0)\}$  and  $B = \{(0, -1), (-1, 0)\}$ .

**2.17** By definition, if  $W_1 + W_2 = V$  then for all  $v \in V$  there are some  $w_1 \in W_1$  and  $w_2 \in W_2$  such that  $w_1 + w_2 = v$ . We claim that these are unique. Say there are  $(w_1, w_2), (u_1, u_2) \in W_1 \times W_2$  such that  $w_1 + w_2 = v = u_1 + u_2$ . Then we see that  $w_1 + w_2 = u_1 + u_2$  and thus  $W_1 \ni w_1 - u_1 = u_2 - w_2 \in W_2$  as these are vectorspaces, thus  $w_1 - u_1, u_2 - w_2 \in W_1 \cap W_2$  as both of these are in both of these subsets. Now as  $\{0\} = W_1 \cap W_2$  we see that  $w_1 - u_1 = 0 = u_2 - w_2$  thus  $w_1 = u_1$  and  $w_2 = u_2$  thus the  $w_1, w_2$  we get are unique.

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**3.9** As per the condition on  $U$  we see that all  $(x_1, x_2, x_3, x_4, x_5) \in U$  can be written in the form  $(3x_2, x_2, 7x_4, x_4, x_5)$  where  $x_2, x_4, x_5 \in \mathbb{R}$ . It can be easily seen that  $U$  is a subspace via the definition and it is also spanned by the linearly independant set  $\{(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)\}$  and hence this is a basis for  $U$ .

**3.11** If  $\alpha \in \mathbb{R}$  then  $\{x_1, x_2 + \alpha x_1, \dots, x_n + \alpha x_1\}$  is linearly independant and hence a basis iff for  $\beta_1, \dots, \beta_n \in \mathbb{R}$ ,

$$\beta_1 x_1 + \beta_2 (x_2 + \alpha x_1) + \dots + \beta_n (x_n + \alpha x_1) = 0 \implies \forall k, \beta_k = 0$$

we can rewrite this as,

$$(\beta_1 + \alpha(\beta_2 + \dots + \beta_n))x_1 + \beta_2 x_2 + \dots + \beta_n x_n = 0$$

But as  $\{x_1, \dots, x_n\}$  was already a basis and hence linearly independant, we see that,  $\beta_1 + \alpha(\beta_2 + \dots + \beta_n) = \beta_2 = \dots = \beta_n = 0$  and thus  $\beta_1 = -\alpha(0 + \dots + 0) = 0$  as well and thus this new set is linearly independant and thus also a basis being

of size  $n$ . Now we can clearly take  $\alpha$  large enough and get a basis where all the vectors have all positive coordinates.

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**4.5** For the only if part we see that if  $S \oplus T_1 = V = S \oplus T_2$  then  $\dim(T_1) = \dim(T_2) = \dim(V) - \dim(S)$  by the modular law. For the sake of contradiction (we will contradict the fact that  $T_1 \cap T_2 = \{0\}$ ) assume that  $\dim S < \dim V/2$ , this implies that  $\dim T_1 = \dim T_2 > \dim V/2$ . Now take some basis  $\{a_1, \dots, a_m\}$  for  $T_1$  and another basis  $\{b_1, \dots, b_m\}$  for  $T_2$ , we now see that as the set  $\{a_1, \dots, a_m, b_1, \dots, b_m\}$  has size strictly greater than  $2 \times \dim V/2 = \dim V$  and thus it must be linearly dependant because the size of some linearly independant set is atmost that of some spanning set and in this case the basis with  $\dim V$  elements spans  $V$  so they have size atmost  $\dim V$ . We can thus find scalars  $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$  not all zero such that

$$\sum_{i=1}^m \alpha_i a_i + \sum_{i=1}^m \beta_i b_i = 0 \implies T_1 \ni \sum_{i=1}^m \alpha_i a_i = \sum_{i=1}^m (-\beta_i) b_i \in T_2$$

Here not all the scalars are zero so wlog say  $\alpha_1 \neq 0$ , now this implies that the left side isn't zero itself as otherwise we would contradict the linear independance of elements of the basis of  $T_1$  i.e.  $\{a_1, \dots, a_m\}$ . Thus we found a nonzero vector  $v = \sum_{i=1}^m \alpha_i v_i \in T_1 \cap T_2$  which contradicts the fact that  $T_1 \cap T_2 = \{0\}$ . From this we have that  $\dim T_1 = \dim T_2 \leq \dim V/2$  which implies that  $\dim S = \dim V - \dim T_1 \geq \dim V - \dim V/2 = \dim V/2$ . For the if part we will provide a construction of such  $T_1$  and  $T_2$ . First take some basis  $\{v_1, \dots, v_n\}$  for  $S$  and extend it to a basis  $\{v_1, \dots, v_n, \dots, v_{n+m}\}$  where  $m$  might be zero, in which case taking  $T_1 = T_2 = \{0\}$  suffices as  $S$  is the whole space anyways and all the conditions hold. Now assume  $m \geq 1$  and let  $T_1$  be the subspace spanned by  $\{v_{n+1}, \dots, v_{n+m}\}$ . We can see that  $T_1 + S$  is clearly  $V$ . Now we prove that  $T_1 \cap S = \{0\}$ . If we had some vector  $v = \sum_{i=1}^n \alpha_i v_i = \sum_{i=n+1}^{n+m} \beta_i v_i \in S \cap T_1$  then

$$\sum_{i=1}^n \alpha_i v_i + \sum_{j=n+1}^{n+m} (-\beta_j) v_j = 0 \implies \alpha_1 = \dots = \alpha_n = -\beta_{n+1} = \dots = -\beta_{n+m} = 0$$

by the linear independance of  $\{v_1, \dots, v_{n+m}\}$  as its a basis. This implies that  $v = \sum_{i=1}^n \alpha_i v_i = \sum_{i=1}^n 0 \cdot v_i = \sum_{i=1}^n 0 = 0$  and thus  $S \cap T_1 = \{0\}$  so by the definition of a direct sum,  $S \oplus T_1 = \{0\}$ . Now we claim that, the subspace  $T_2$  spanned by  $\{v_{n+1} + v_1, \dots, v_{n+m} + v_m\}$  is also a complement of  $S$  and we have that  $T_1 \cap T_2 = 0$