

# Indian Statistical Institute, Bangalore

B. Math.

First Year, First Semester

Analysis I

Home Assignment II

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Due Date : Sept. 4, 2025

(1) Take

$$C = \left\{ 2 - \frac{1}{n} : n \in \mathbb{N} \right\} \bigcup \left\{ 5 - \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Show that every non-empty subset of  $C$  has a minimal element. Determine as to whether the same property holds for  $D$ , where

$$D = \left\{ 5 - \frac{1}{n} - \frac{1}{m^2} : m, n \in \mathbb{N} \right\}.$$

(2) Find infimum and supremum of following subsets of the real line:

$$A_1 = \left\{ 3 + (-1)^n \frac{1}{n} : n \in \mathbb{N} \right\}.$$

$$A_2 = \{x^2 + 1 : 0 \leq x \leq 1\}.$$

(3) Let  $A, B$  non-empty, bounded subsets of  $\mathbb{R}$ . Take  $A+B = \{a+b : a \in A, b \in B\}$  and  $A-B = \{a-b : a \in A, b \in B\}$  and  $A.B = \{a.b : a \in A, b \in B\}$ . Show that these sets are bounded. Which of the following statements are true and which are not true in general (prove your claim):

- (a)  $\sup(A \cup B) = \max\{\sup A, \sup B\}$ ;
- (b)  $\sup(A \cap B) = \min\{\sup A, \sup B\}$ ;
- (c)  $\sup(A + B) = \sup A + \sup B$ .
- (d)  $\sup(A - B) = \sup A - \sup B$ ;
- (e)  $\sup(A.B) = \sup(A) \cdot \sup(B)$ .

(4) Let  $\{t_n\}_{n \geq 1}$  be a sequence defined by

$$t_1 = 2, t_{n+1} = \frac{1}{2}(t_n + \frac{2}{t_n}) \text{ for } n \geq 1.$$

Show that  $\{t_n\}_{n \geq 1}$  is a convergent sequence, converging to  $\sqrt{2}$ .

- (5) Show that  $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$  exists. (Hint: Prove that it is a monotonic bounded sequence.)
- (6) Show that there exists a unique positive real number  $x$  such that  $x^3 = 2$ .
- (7) Prove that the following sequences are convergent:

$$a_n = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{n+1}\right), \quad n \in \mathbb{N};$$

$$b_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}, \quad n \in \mathbb{N}.$$

P.T.O.

- (8) Prove that the sequence

$$c_n = 5 + (-1)^n \left(2 + \frac{1}{n}\right)$$

is not convergent.

- (9) Suppose  $\{x_n\}$  is a real sequence. For  $n \geq 1$  define averages

$$y_n = \frac{1}{n} \sum_{k=1}^n x_k.$$

Show that if  $\{x_n\}$  converges then  $\{y_n\}$  converges, however the converse is not true.

- (10) Find all functions  $h: \mathbb{R} \rightarrow \mathbb{R}$ , satisfying  $h(x+y) = h(x) + h(y)$  and  $h(x.y) = h(x).h(y)$  for all  $x, y$  in  $\mathbb{R}$ . (Hint: You may need order properties and completeness of  $\mathbb{R}$ .)