Analysis I

Home Assignment V

Arkaraj Mukherjee
B.Math., First Year, ISI Bangalore

November 2, 2025

Contents

1 Home Assignment V (Due: Nov 2, 2025)

Problem 1.1. (Discrete L'Hospital): Let $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$ be sequences of non-zero real numbers converging to 0. Suppose $b_n > b_{n+1}$ for all n and $v := \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ exists as a real number. Show that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and equals v. Give one such example.

Solution. Given any $\varepsilon > 0$ there exists N such that $\forall n > N$,

$$v - \varepsilon < \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < v + \varepsilon \implies (v - \varepsilon)(b_{n+1} - b_n) > a_{n+1} - a_n > (v + \varepsilon)(b_{n+1} - b_n)$$

Where the signs flipped because $b_{n+1} - b_n < 0$ as stated in the problem. We sum both sides of this inequality for $n, n+1, \ldots, m$ to get,

$$(v - \varepsilon)\sum_{k=n}^{m}(b_{k+1} - b_k) > \sum_{k=n}^{m}(a_{k+1} - a_k) > (v + \varepsilon)\sum_{k=n}^{m}(b_{k+1} - b_k)$$

$$\implies (v - \varepsilon)(b_{m+1} - b_n) > a_{m+1} - a_n > (v + \varepsilon)(b_{m+1} - b_n)$$

Thus this holds for all $m \geq n$ and taking the limits as $m \to \infty$ on all of the terms in this inequality we get,

$$(v - \varepsilon)(-b_n) \geq -a_n \geq (v + \varepsilon)(-b_n) \implies v - \varepsilon \leq \frac{a_n}{b_n} \leq v + \varepsilon$$

which is enough to conclude that $a_n/b_n \longrightarrow v$ as well. For an example consider $a_n = 1/n^2$ and $b_n = 1/n$, both of these tend to 0 and b_n decreases strictly. Now, $a_{n+1} - a_n = -(2n+1)/(n^4 + 2n^3 + n^2)$ and $b_{n+1} - b_n = -1/(n^2 + n)$. Thus,

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{(2n+1)(n^2+n)}{n^4 + 2n^3 + n^2} = 0$$

and we can verify,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^2}{1/n} = 0$$

clearly as well. □

Problem 1.2. Suppose $k \in \mathbb{N}$ and $b_1, b_2, \ldots, b_k$ are strictly positive real numbers. Show that (i) $\lim_{n\to\infty} b_j^{\frac{1}{n}} = 1$. (ii) $\lim_{n\to\infty}(b_1^n + b_2^n + \cdots + b_k^n)^{\frac{1}{n}} = b$ where $b = \max\{b_j : 1 \leq j \leq k\}$.

Solution. For (i), let $x > 0$ and consider the two cases : $x \geq 1$ and $x < 1$. For the first case, by bernoulli's inequality we see that,

$$x = (x^{1/n})^n = (1 + (x^{1/n} - 1))^n \geq 1 + n(x^{1/n} - 1) \implies x^{1/n} - 1 \leq \frac{x-1}{n}$$

Here we have used the fact that $x^p \geq 1$ for any positive p, making $x^{1/n} - 1$ nonnegative. Thus we can use the squeeze theorem on the following inequality to conclude,

$$0 \leq x^{1/n} - 1 \leq \frac{x-1}{n}$$

And when $0 < x < 1$ we can just consider $1/x > 1$ and get that $1/x^{1/n} \xrightarrow{n\to\infty} 1$ and using algebra of limits we get that $1/\lim x^{1/n}$ is also 1 which makes $\lim x^{1/n} = 1$. To prove (ii) we will also use squeeze theorem. As b is the maximum of these b_i we have,

$$b^n < b_1^n + \ldots + b_k^n \leq b^n + \ldots + b^n = kb^n$$

$$\implies b < (b_1^n + \ldots + b_k^n)^{1/n} \leq k^{1/n} b$$

Using the fact $k^{1/n} \longrightarrow 1$, which is proven in the immediate next exercise, we can conclude using squeeze theorem. $\square$

Problem 1.3. Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Solution. For large enough($n \geq 3$) we can say that $n^{1/n} \geq 1$ and then using the binomial theorem

$$n = (1 + (n^{1/n} - 1))^n = \sum_{k=0}^{n} \binom{n}{k} (n^{1/n} - 1)^k \geq \frac{n(n-1)}{2} (n^{1/n} - 1)^2$$

$$\implies 1 \leq n^{1/n} \leq 1 + \sqrt{\frac{2}{n-1}}$$

And yet again we conclude using the squeeze theorem. $\square$

Problem 1.4. Let $f : \mathbb{R} \to \mathbb{R}$, be a continuous function satisfying $f(3x) = f(x), \forall x \in \mathbb{R}$. Show that f is a constant function.

Solution. For all real x, by the condition we have $f(x) = f(3 \cdot (x/3)) = f(x/3)$ and continuing for n steps we have $f(x) = f(x/3^n)$. Now from the continuity of f,

$$f(x) = \lim_{n\to\infty} f(x/3^n) = f\left(\lim_{n\to\infty} x/3^n\right) = f(0)$$

which makes f a constant function as $\forall x \in \mathbb{R}, f(x) = f(0)$. $\square$

Problem 1.5. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $g(x + y) = g(x) + g(y), \forall x, y \in \mathbb{R}$. Show that $g = cx$ for some $c \in \mathbb{R}$.

Solution. Just as in Problem 2.10. (problem 10 in the 2nd assignment) we can show that for any rational r we have $g(r) = g(1) \cdot r$ and as given any real number x there exists a sequence of rationals $\{x_n\}$ converging to it we see that,

$$g(x) = g\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} g(1) \cdot x_n = g(1) \cdot \lim_{n\to\infty} x_n = g(1) \cdot x$$

where we used the continuity of g in the second inequality and a basic fact from the algebra of limits in the fourth. Thus we see that $c = g(1)$ and $g(x) = g(1) \cdot x$ for all $x \in \mathbb{R}$ $\square$

Problem 1.6. Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$, is said to be convex if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \forall x < y$ in $I, 0 < \lambda < 1$. (i) Show that if $f : (0, 1) \to \mathbb{R}$ is convex then for $0 < s < t < u < 1$,

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

(ii) Show that if $f : (0,1) \to \mathbb{R}$, is convex then it is continuous. (iii) Show that a convex function $g : [0,1] \to \mathbb{R}$, need not be continuous.

Solution. Rearranging the first inequality we get the following equivalent inequality,

$$\frac{f(t)}{t-s} - \frac{f(u)}{u-s} \leq \frac{f(s)}{t-s} - \frac{f(s)}{u-s} = \left(\frac{u-t}{(t-s)(u-s)}\right) f(s)$$

$$\iff f(t) \leq \frac{u-t}{u-s} \cdot f(s) + \frac{t-s}{u-s} \cdot f(u)$$

Which is true from the definiton of convexity via $\lambda = (u-t)/(u-s) \in (0,1)$. The second inequality is proven similarly. For (ii), we will show that for any $x \in (0,1)$, the right and left limits of $f(y)$ as y tends to x is zero which will imply that the limit exists and is zero as well, proving continuity. Without loss of generality assume that $x < y$, then choose $u, v, w \in (0,1)$ such that $u < x < y < v$. Now by the previous result we see that,

$$\frac{f(x)-f(u)}{x-u} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(v)-f(y)}{v-y} \leq \frac{f(w)-f(v)}{w-v}$$

$$\implies \frac{f(x)-f(u)}{x-u} \cdot (y-x) \leq f(y)-f(x) \leq (y-x) \cdot \frac{f(v)-f(y)}{v-y} \leq (y-x) \cdot \frac{f(w)-f(v)}{w-v}$$

Now, using the squeeze theorem, as y tends to x from the right we get,

$$\lim_{y \to x; y > x} (f(y) - f(x)) = 0$$

and the same can be said for the left limit using a very similar arguement and we are done. For (iii) we can consider the following function,

$$f(x) := \begin{cases} 0 \text{ if } 0 < x \leq 1 \\ 1 \text{ if } x = 0 \end{cases}$$

To show that this is convex we will consider the inequality as in the definition on all points $x, y \in [0,1]$. The inequality is clearly always true for $0 < x, y$ as the function is constant on $(0,1]$ and when some x, y is 0, wlog say $x = 0$, then,

$$f(\lambda x + (1-\lambda)y) = f((1-\lambda)y) \leq \underbrace{\lambda f(x)}_{=1} + (1-\lambda) f(y)$$

for both $y = 0$ and $y > 0$ as in the first case its $1 \leq 1$ and in the second, as $1 - \lambda > 0$ its $0 \leq 1$. □

Problem 1.7. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$, such that for every $y \in \mathbb{R}$, there are exactly two real numbers x_1, x_2, such that $f(x_1) = f(x_2) = y$.

Solution. For the sake of contradiction assume that there is some f that is continuous and also satisfies said criterion. Let $x < y$ be the reals such that $f(x) = f(y) = 0$. Without loss of generality we can asssume $x < y$. Now consider the interval $[x, y]$, as f is continuous it must have some it must achieve its extrema in this interval at some points in the interval. Unless there is an extrema which is achieved only in (x, y) we can easily see that f is constant in this interval which gives us much more than two, infact infinitely many reals where f is zero which is not allowed as per the assumption. Again without loss of generality assume that we reach a maxima at $u \in (x, y)$. We claim that u is the unique such point in $[x, y]$. For a proof of this claim, for the sake of contradiction asssume without loss of generality that we also reach a maxima at $v \in (u, y]$. f can not be constant on the interval $[u, v]$ so there is some $w \in (u, v)$ such that $0 < f(w) < f(u)$ where the first inequality can be achieved due to continuity because by construction $f(u) = f(v) > 0$ and if we did not have some w it would produce jump discontinuities in (u, v). Now for any $\lambda \in (f(w), f(u))$ we can find a real z with $f(z) = \lambda$ in all three of the intervals : $(x, u), (u, v), (v, y)$ by IVT, which contradicts the assumption as there should be exactly two such z and the claim is proved. By the condition on f, there must be another $g \notin [x, y]$ such that $f(u) = f(g)$. Without loss of generality assume that $g > y$, then for any $\lambda \in (0, f(u))$ we can find some z with $f(z) = \lambda$ from all of these three intervals : $(x, u), (u, y), (y, g)$ by IVT which is a contradiction and we are done. □

Problem 1.8. Fix $n \in \mathbb{N}$. Let $x_1, x_2, \ldots, x_n$ be n distinct real numbers and let $y_1, y_2, \ldots, y_n$ be another n-tuple of not-necessarily distinct real numbers. Show that there is a unique polynomial p of degree $(n-1)$ such that $p(x_k) = y_k$ for $1 \leq k \leq n$. Hint: Consider

$$p(x) = \sum_{j=1}^{n} y_j \frac{\prod_{i \neq j}(x - x_i)}{\prod_{i \neq j}(x_j - x_i)}.$$

Solution. The polynomial p provided in the hint is clearly of degree atmost $n - 1$ and when evaluated at x_i, all but the i'th term in the sum vanish and in this term the numerator and denominator of the fraction cancel out leaving only y_i and thus $p(x_i) = y_i$. Assume p, q both have degree atmost $n - 1$ and they both are polynomials which satisfy the condition in the problem, then we see that $p - q$ has n zeroes. $p - q$ clearly has degree atmost $n - 1$ and thus for it to have $n > n - 1$ zeroes, by the fundamental theorem of algebra we see that $p - q$ must be identically zero i.e. $p = q$ and hence such a polynomial is unique. □

Problem 1.9. Let $a < b$ be real numbers. Let $f : (a, b) \to \mathbb{R}$ be a twice differentiable function. Show that for any $x_0 \in (a, b)$

$$f''(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}.$$

Solution. f being twice differentiable makes it continuous and thus the numerator tends to 0 as $h \to 0$ and the denominator also tends to zero as $h \to 0$ and also the derivative of the denominator is nonzero near 0, except at 0. Let the numerator be $X(h)$ and denominator $Y(h)$, then,

$$\frac{X'(h)}{Y'(h)} = \frac{f'(x_0 + h) - f'(x_0 - h)}{2h}$$

$$= \frac{f'(x_0 + h) - f'(x_0)}{2h} - \frac{f'(x_0) - f'(x_0 - h)}{2h} \xrightarrow{h \to 0} \frac{f''(x_0)}{2} + \frac{f''(x_0)}{2} = f''(x_0)$$

and we can thus conclude via LHopital's rule. □

Problem 1.10. (i) For $x \in \mathbb{R}$, show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely. (ii) Define $e : \mathbb{R} \to \mathbb{R}$ by $e(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Show that $e(x + y) = e(x)e(y), \forall x, y \in \mathbb{R}$.

Solution. For (i), we will show that for any $x \in \mathbb{R}$ this series is absolutely convergent using the ratio test,

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

so we are done. For (ii), using the cauchy product theorem for product of sums, atleast one of which is absolutely convegrent we have,

$$e(x)e(y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = e(x+y)$$

□

Problem 1.11. Let $h : \mathbb{R} \to \mathbb{R}$ be the function $h(x) = \sin(x)$ (Here we are assuming your familiarity with trigonometric functions). Show that the remainder term in Taylor's theorem converges to 0 as $n \to \infty$ for every x_0 and x.

Solution. If we keep taking the derivative if h we can easily see that $|h^{(n)}(x)|$ is either $|\sin x|$ or $|\cos x|$, eitherways its bounded by 1. Now given any x, x_0 the $n-th$ remainder term, say R_n tends to zero as follows,

$$|R_n| \leq \frac{|x_0 - x|^n}{n!} \cdot \sup_{x\in\mathbb{R}} |h^{(n)}(x)| = \frac{|x - x_0|^n}{n!} \xrightarrow{n\to\infty} 0$$

just as we did in Problem 5.10.. □

Problem 1.12. Consider the function $h : [0, \infty) \to \mathbb{R}$ defined by $h(x) = \frac{1}{1+x^2}$. Determine the set $\{h(x) : x \in [0, \infty)\}$.

Solution. Clearly h is strictly decreasing with $h(0) = 1$ and $\lim_{x\to\infty} h(x) = 0$ so the set is $(0, 1]$ which we get by using IVT as we can find arbitrary small positive values of h at very large inputs and $h(0) = 0$. □

Problem 1.13. Let $v : \mathbb{R} \to \mathbb{R}$ be the function $v(x) = x^3 - 6x^2 + 9x$, for all x. Determine the set: $\{x : v(x) > 0\}$.

Solution. We can factor this into $x(x-3)^2$, so it is zero at 0 and 3. For $x \neq 0, 3$ it has the same sign as x as $(x-3)^2$ is positive. Thus the set is $(0, \infty) \setminus \{3\}$. □

Problem 1.14. Let $f : [0, 1] \to [0, 1]$ be a continuous strictly increasing bijection. Assume that $x < f(x)$ for all $x \in (0, 1)$. Fix $x_0 \in (0, 1)$. Define x_n for $n \in \mathbb{Z}$ by $x_n = f^n(x_0)$. Show that: (i) $0 < x_m < x_n < 1$ for $m < n$ in $\mathbb{Z}$; (ii) $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to-\infty} x_n = 0$; (iii) For every $n \in \mathbb{Z}$, f maps $[x_n, x_{n+1}]$ bijectively to $[x_{n+1}, x_{n+2}]$.

Solution. As f is strictly increasing we see that we must have $f(0) = 0$ and $f(1) = 1$ and as its also a bijection, 0 or 1 never appear in f orbits i.e. $\{f^n(x) : n \in \mathbb{Z}\}$ of any points in $(0, 1)$. Thus we can see that for any $m < n$,

$$f^n(x_0) = \underbrace{f(f^{n-1}(x_0)) > f^{n-1}(x_0) > \ldots > f^m(x_0)}_{n-m \text{ inequalities}}$$

and as 1 and 0 never appear in the orbit of x_0 we see that $0 < f^n(x_0) < 1$ for all $n \in \mathbb{Z}$ and together it reads $0 < x_m < x_n < 1$, we have thus proved (i). For (ii), as $n \to \infty$, $\{x_n\}$ must converge to a limit as its increasing and bounded above, say it converges to x, then $0 < x \leq 1$. Using the continuity of f we have can show that this is a fixed point as follows

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$$

so it has to be 1. We can similarly show that $x_n \xrightarrow{n\to-\infty} 0$ as well. For every $n \in \mathbb{Z}$ we see that $x_n < x_{n+1}$ and thus $x_{n+1} = f(x_n) < f(x_{n+1}) = x_{n+2}$ so we have $f([x_n, x_{n+1}]) \subseteq [x_{n+1}, x_{n+2}]$ using the fact that f is increasing. Using IVT and the fact that $x_{n+1} = f(x_n), x_{n+2} = f(x_{n+1})$ we can say that we have equality here i.e. $f([x_n, x_{n+1}]) = [x_{n+1}, x_{n+2}]$ and the f is clearly still a bijection here, this proves (iii). $\square$

Problem 1.15. Consider the set up of the previous question. Fix $y_0 \in (x_0, x_1)$. Set $y_n = f^n(y_0)$ for $n \in \mathbb{Z}$. Let $h : [x_0, y_0] \to [y_0, x_1]$ be a continuous strictly increasing bijection. Define $g : [0,1] \to [0,1]$ by $g(0) = 0, g(1) = 1$,

$$g(t) = h(t) \quad \forall t \in [x_0, y_0];$$

$$g(t) = f \circ h^{-1}(t) \quad \forall t \in [y_0, x_1];$$

and more generally, for $n \in \mathbb{Z}$, define

$$g(t) = f^n \circ h \circ f^{-n}(t) \quad \forall t \in [x_n, y_n]$$

and

$$g(t) = f^{n+1} \circ h^{-1} \circ f^{-n}(t) \quad \forall t \in [y_n, x_{n+1}].$$

Then show that: (i) For every n, g maps $[x_n, y_n]$ bijectively to $[y_n, x_{n+1}]$ and it maps $[y_n, x_{n+1}]$ bijectively to $[x_{n+1}, y_{n+1}]$. (ii) g is a strictly increasing continuous bijection. (iii) $f = g \circ g$. This shows that f has infinitely many square roots.

Solution. h being a continuous bijection forces $h(x_0) = y_0$ and $h(y_0) = x_1$, this will be used here on. We know that inverses of continuous bijections in $\mathbb{R}$ are also continuous bijections and both of these are strictly increasing. From the previous problem we can see that f^{-n} bijectively maps $[x_n, y_n]$ to $[x_0, y_0]$, after this h maps $[x_0, y_0]$ to $[y_0, x_1]$ bijectively and at the end f^n maps this bijectively to $[y_n, x_{n+1}]$, thus g maps $[x_n, y_n]$ bijectively to $[y_n, x_{n+1}]$. Using similar arguements we can show that g also bijectively maps $[y_n, x_{n+1}]$ to $[x_{n+1}, y_{n+1}]$ and (i) is proven. We can see that

$$\mathfrak{I} := \{[x_n, y_n), [y_n, x_{n+1}) : n \in \mathbb{Z}\} \cup \{\{0\}, \{1\}\}$$

is a partition of $[0,1]$, this is because $\{[x_n, x_{n+1}) : n \in \mathbb{Z}\} \cup \{\{0\}, \{1\}\}$ is a partition (this can be deduced from parts (i) and (ii) of Problem 5.14.) and we just partition each of the intervals into two intervals. If we were to write these out in order(visually speaking) then $[0,1]$ would look like this,

$$\ldots [y_{-2}, x_{-1}) \cup [x_{-1}, y_{-1}) \cup [y_{-1}, x_0) \cup [x_0, y_0) \cup [y_0, x_1) \cup [x_1, y_1) \cup [y_1, x_2) \ldots$$

As $g(0) = 0, g(1) = 1$ and,

$$g([x_n, y_n)) = [y_n, x_{n+1}] \setminus \{g^{-1}(y_n)\} = [y_n, x_{n+1})$$

As, $g(y_n)= f^n(h(f^{-n}(y_n))) = f^n(h(y_0)) = f^n(x_1) = x_{n+1}$. Similarly, for the other type of intervals, as, $g(x_{n+1}) = f^{n+1}(h^{-1}(f^{-n}(x_{n+1})) = f^{n+1}(h^{-1}(x_1)) = f^{n+1}(y_0) = y_{n+1}$ we get,

$$g([y_n, x_{n+1})) = [x_{n+1}, y_{n+1})$$

Now, looking at g as a set function defined by $g(S) := \{g(x) : x \in S\}$ we can clearly see that g is a bijection on $\mathfrak{I}$ and as its a bijection on each set in $\mathfrak{I}$ which is a partition of $[0, 1]$, its clearly a bijection on $[0, 1]$ as a whole. Thus we have shown that g is a bijection. But we still have to show that it is continous, which it is on the smaller intervals we partitioned the larger interval $[0, 1]$ into, thus for it to be continuous on the larger interval we need to check continuity at the points where one of these smaller intervals end and the other begins i.e. x_n and y_n for $n \in \mathbb{Z}$. At x_n, we arrive at the left limit for g i.e. $g(x_n-)$ by using its definition on $[y_{n-1}, x_n)$ so due to continuity on this interval this is,

$$g(x_n-) = f^{(n-1)+1}(h^{-1}(f^{-(n-1)}(x_n))) = f^n(h^{-1}(x_1) = f^n(y_0) = y_n$$

And now the right limit will use the definition of g on $[x_n, y_n)$ and thus similarly,

$$g(x_n+) = f^n(h(f^{-n}(x_n))) = f^n(h(x_0)) = f^n(y_0) = y_n$$

So its continuous at the points x_n. The continuity at the points y_n can be proved very similarly. Thus it must be strictly increasing as well hence proving (ii). For (iii), firstly $(g \circ g)(0) = 0 = f(0)$ and $(g \circ g)(1) = 1 = f(1)$, now if some $x \in (0, 1)$ is in an interval of form $[x_n, y_n]$ then when g is first applied to it, it acts as $f^n \circ h \circ f^{-n}$ and sends it to the interval $[y_n, x_{n+1}]$ as seen in (ii) and thus on the second application g acts as $f^{n+1} \circ h^{-1} \circ f^{-n}$ as in the definition of g, thus,

$$(g \circ g)(x) = (f^{n+1} \circ h^{-1} \circ f^{-n} \circ f^n \circ h \circ f^{-n})(x) = (f^{n+1} \circ h^{-1} \circ (f^{-n} \circ f^n) \circ h \circ f^{-n})(x)$$

$$= (f^{n+1} \circ h^{-1} \circ \mathrm{Id} \circ h \circ f^{-n})(x) = (f^{n+1} \circ h^{-1} \circ (\mathrm{Id} \circ h) \circ f^{-n})(x) = (f^{n+1} \circ h^{-1} \circ h \circ f^{-n})(x)$$

$$= (f^{n+1} \circ (h^{-1} \circ h) \circ f^{-n})(x) = (f^{n+1} \circ \mathrm{Id} \circ f^{-n})(x) = (f^{n+1} \circ (\mathrm{Id} \circ f^{-n}))(x) = (f^{n+1} \circ f^{-n})(x)$$
$$= f(x)$$

and when x is in an interval of form $[y_n, x_{n+1}]$ using similar arguements we have that,

$$(g \circ g)(x) = (\underbrace{f^{n+1} \circ h \circ f^{-(n+1)}}_{\text{as } g(x) \in [x_{n+1}, y_{n+1}]} \circ f^{n+1} \circ h^{-1} \circ f^{-n})(x) = (f^{n+1} \circ f^{-n})(x) = f(x)$$

Thus we have shown that $g \circ g = f$ on $[0, 1]$ and we are done. □