Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester

Analysis I — Instructor: B V Rajarama Bhat
Home Assignment V — Due Date : Nov. 2, 2025

(1) (Discrete L'Hospital): Let $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$ be sequences of non-zero real numbers converging to 0. Suppose $b_n > b_{n+1}$ for all n and $v := \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ exists as a real number. Show that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and equals v. Give one such example.

(2) Suppose $k \in \mathbb{N}$ and $b_1, b_2, \ldots, b_k$ are strictly positive real numbers. Show that (i) $\lim_{n\to\infty} b_1^{\frac{1}{n}} = 1$. (ii) $\lim_{n\to\infty}(b_1^n + b_2^n + \cdots + b_k^n)^{\frac{1}{n}} = b$ where $b = \max\{b_j : 1 \leq j \leq k\}$.

(3) Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$, be a continuous function satisfying
$$f(3x) = f(x),\ \forall x \in \mathbb{R}.$$
Show that f is a constant function.

(5) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying
$$g(x+y) = g(x) + g(y),\ \forall x, y \in \mathbb{R}.$$
Show that $g = cx$ for some $c \in \mathbb{R}$.

(6) Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$, is said to be *convex* if
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),\ \forall x < y \text{ in } I, 0 < \lambda < 1.$$
(i) Show that if $f : (0,1) \to \mathbb{R}$ is convex then for $0 < s < t < u < 1$,
$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}.$$
(ii) Show that if $f : (0,1) \to \mathbb{R}$, is convex then it is continuous.
(iii) Show that a convex function $g : [0,1] \to \mathbb{R}$, need not be continuous.

(7) Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$, such that for every $y \in \mathbb{R}$, there are exactly two real numbers x_1, x_2, such that $f(x_1) = f(x_2) = y$.

(8) Fix $n \in \mathbb{N}$. Let $x_1, x_2, \ldots, x_n$ be n distinct real numbers and let $y_1, y_2, \ldots, y_n$ be another n-tuple of not-necessarily distinct real numbers. Show that there is unique polynomial p of degree $(n-1)$ such that $p(x_k) = y_k$ for $1 \leq k \leq n$. Hint: Consider
$$p(x) = \sum_{j=1}^{n} y_j \frac{\Pi_{i\neq j}(x - x_i)}{\Pi_{i\neq j}(x_j - x_i)}.$$

(9) Let $a < b$ be real numbers. Let $f : (a,b) \to \mathbb{R}$ be a twice differentiable function. Show that for any $x_0 \in (a,b)$
$$f''(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}.$$

P.T.O.

(10) (i) For $x \in \mathbb{R}$, show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely. (ii) Define $e : \mathbb{R} \to \mathbb{R}$ by $e(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ x \in \mathbb{R}$. Show that

$$e(x+y) = e(x)e(y),\ \forall x, y \in \mathbb{R}.$$

(11) Let $h : \mathbb{R} \to \mathbb{R}$ be the function $h(x) = \sin(x)$ (Here we are assuming your familiarity with trigonometric functions). Show that the remainder term in Taylor's theorem converges to 0 as $n \to \infty$ for every x_0 and x.

(12) Consider the function $h : [0, \infty) \to \mathbb{R}$ defined by $h(x) = \frac{1}{1+x^2}$. Determine the set

$$\{h(x) : x \in [0, \infty)\}.$$

(13) Let $v : \mathbb{R} \to \mathbb{R}$ be the function $v(x) = x^3 - 6x^2 + 9x$, for all x. Determine the set:

$$\{x : v(x) > 0\}.$$

(14) Let $f : [0, 1] \to [0, 1]$ be a continuous strictly increasing bijection. Assume that $x < f(x)$ for all $x \in (0, 1)$. Fix $x_0 \in (0, 1)$. Define x_n for $n \in \mathbb{Z}$ by $x_n = f^n(x_0)$. Show that:
 (i) $0 < x_m < x_n < 1$ for $m < n$ in $\mathbb{Z}$;
 (ii) $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} x_{-n} = 0$;
 (iii) For every $n \in \mathbb{Z}$, f maps $[x_n, x_{n+1}]$ bijectively to $[x_{n+1}, x_{n+2}]$.

(15) Consider the set up of the previous question. Fix $y_0 \in (x_0, x_1)$. Set $y_n = f^n(y_0)$ for $n \in \mathbb{Z}$, Let $h : [x_0, y_0] \to [y_0, x_1]$ be a continuous strictly increasing bijection. Define $g : [0, 1] \to [0, 1]$ by $g(0) = 0, g(1) = 1$,

$$g(t) = h(t)\ \forall t \in [x_0, y_0];$$
$$g(t) = f \circ h^{-1}(t)\ \forall t \in [y_0, x_1];$$

and more generally, for $n \in \mathbb{Z}$, define

$$g(t) = f^n \circ h \circ f^{-n}(t)\ \forall t \in [x_n, y_n]$$

and

$$g(t) = f^{n+1} \circ h^{-1} \circ f^{-n}(t)\ \forall t \in [y_n, x_{n+1}].$$

Then show that:
 (i) For every n, g maps $[x_n, y_n]$ bijectively to $[y_n, x_{n+1}]$ and it maps $[y_n, x_{n+1}]$ bijectively to $[x_{n+1}, y_{n+1}]$.
 (ii) g is a strictly increasing continuous bijection.
 (iii) $f = g \circ g$.
 This shows that f has infinitely many square roots.