Exercise sheet 2

Only the exercises marked [HW] need to be submitted.

Vectors will be denoted in bold unlike scalars.

1. Let $V = \mathbb{C}^{n\times n}$ be the space of $n \times n$ matrices with complex entries. Say $A \in V$ is Hermitian if $A_{jk} = \overline{A_{kj}}$ for all j, k. Show that the set of all $n \times n$ Hermitian matrices is not a subspace over $\mathbb{C}$, while it is a subspace over $\mathbb{R}$.

2. Show that $\{(x_1, \cdots, x_n) : x_1 + \cdots + x_n = 0\}$ is a subspace of $\mathbb{R}^n$. Show that $\{(x_1, x_2, x_3) : 2x_1 - 3x_2 + \sqrt{2}x_3 = 0, x_1 - 5x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

3. Let $\mathbf{x}$ and $\mathbf{y}$ be two fixed vectors in a vector spac V over F. Show that
$$S = \{\alpha\mathbf{x} + \beta\mathbf{y} : \alpha, \beta \in F\}$$
is a subspace of V.

4. Consider the vector space $\mathbb{R}$ over $\mathbb{Q}$. Show that $\mathbb{Q}$ is a subspace and so is $\{\alpha + \beta\sqrt{2} + \gamma\sqrt{3} : \alpha, \beta, \gamma \in \mathbb{Q}\}$.

5. Fix sets $Y \subseteq X$. Show that $\{f : f \in F^X \text{ and } f(x) = 0 \text{ for all } x \in Y\}$ is a subspace of F^X (functions from X to F). Show also that the set of all continuous functions and the set of all differentiable functions form subspaces of $\mathbb{R}^{\mathbb{R}}$.

6. Consider the vector space in Exercise 7 of Exercise Sheet 1. Show that for a non-empty subset $\mathbf{A}$ of Ω the set $\{\emptyset, \mathbf{A}\}$ is a subspace. For distinct non-empty subsets $\mathbf{A}$ and $\mathbf{B}$ of Ω, show that $\{\emptyset, \mathbf{A}, \mathbf{B}, \mathbf{A}\Delta\mathbf{B}\}$ is a subspace.

7. For any two subsets A, B of a vector space V show the following
   - A is a subspace of V if and only if $\mathrm{Sp}(A) = A$.
   - If $B \subset A$ then $\mathrm{Sp}(B) \subset \mathrm{Sp}(A)$.
   - $\mathrm{Sp}(\mathrm{Sp}(A)) = \mathrm{Sp}(A)$
   - If $A \subset B$ and $B \subset \mathrm{Sp}(A)$ then $\mathrm{Sp}(A) = \mathrm{Sp}(B)$.

8. Show that the only subspaces of $\mathbb{R}^2$ are $\{\mathbf{0}\}$, the lines through the origin and $\mathbb{R}^2$ itself. Show that the only subspaces of $\mathbb{R}^3$ are $\{\mathbf{0}\}$, the lines through the origin, the planes through the origin and $\mathbb{R}^3$ itself.

9. Consider the vector space $V = \mathbb{R}^{\mathbb{R}}$ and subsets $S = \{f \in V : f \text{ either non-decreasing or non-increasing}\}$, $T = \{f \in V : f(2) = f(5)^2\}$ and $W = \{f \in V : f(2) = f(5)\}$. Which of these are subspaces of V?

10. Let $V = \mathbb{R}^{\mathbb{N}}$ (here $\mathbb{N} = \{1, 2, \cdots\}$), and S be the set of f such that the sequence $(f(1), f(2), \cdots)$ converges. Is S a subspace of V?

11. [HW 2, due Aug 8] For any two subsets A and B of a vector space V, show that
    - $\mathrm{Sp}(A) \cup \mathrm{Sp}(B) \subseteq \mathrm{Sp}(A \cup B)$,
    - $\mathrm{Sp}(A) \cap \mathrm{Sp}(B) \subseteq \mathrm{Sp}(A \cap B)$,

    and that proper inclusion is possible in each. Prove or disprove
$$\mathrm{Sp}(A) \cap \mathrm{Sp}(B) \neq \{\mathbf{0}\} \Rightarrow A \cap B \neq \emptyset.$$

12. Let S and T be subspaces of V. Then prove that $S \cup T$ is a subspace if and only if either $S \subseteq T$ or $T \subseteq S$.

13. Let W be the set of all $(x_1, x_2 \cdots, x_5) \in \mathbb{R}^5$ which satisfy

$$\begin{aligned} 2x_1 - x_2 + \frac{4}{3}x_3 - x_4 &= 0 \\ x_1 + \frac{2}{3}x_3 - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0. \end{aligned}$$

Show that W is a subspace and find a set of vectors which spans W.

14. Let V be the vector space of $n \times n$ matrices over $\mathbb{R}$. Which of the following sets of matrices are subspaces of V?

- All invertible A
- All non-invertible A
- All A such that $AB = BA$, where B is a fixed matrix in V
- All A such that $A^2 = A$

15. If $S_1, S_2, \cdots, S_k$ are subsets of a vector space V, the set of sums $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ of vectors $\alpha_i \in S_i$ is called the *sum* of the subsets $S_1, S_2, \cdots, S_k$ and is denoted by $S_1 + \cdots + S_k$ or by $\sum_{i=1}^k S_i$. Show that if $W_1, W_2, \cdots W_k$ are subspaces of V, then so is the sum $W = W_1 + \cdots + W_k$, and moreover W contains each of the W_i. Therefore W is the subspace spanned by the union of $W_1, \cdots, W_k$.

16. Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let V_e be the subset of even functions: $f(-x) = f(x)$ and V_o be the subset of odd functions: $f(-x) = f(x)$.

- Prove that V_e and V_o are subspaces of V
- Prove that $V_e + V_o = V$.
- Prove that $V_e \cap V_o = \{\mathbf{0}\}$

17. [HW 2, due Aug 8] Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{\mathbf{0}\}$. Prove that for each vector $\mathbf{x}$ in V there are *unique* $\mathbf{x}_1 \in W_1$ and $\mathbf{x}_2 \in W_2$ such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$.