Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester

Analysis I — Instructor: B V Rajarama Bhat
Home Assignment III — Due Date : Oct. 10, 2025

(1) A binary expansion $0.b_1b_2b_3\cdots$ is said to be *periodic* if there exist natural numbers M and p such that $b_{k+p} = b_k$ for all $k \geq M$. Show that a binary expansion of a real number in the interval $[0,1]$ is periodic if and only if it is a rational number. Use this to show that the set of all rational numbers in $[0,1]$ is countable.

(2) Determine limsup and liminf of the following sequences of real numbers:

$$(i)\ \{(-1)^n 5 + (-\frac{1}{2})^n 7 :\ n \in \mathbb{N}\}.$$

$$(ii)\ \{\frac{n}{2^n} - \frac{n}{3^n} :\ n \in \mathbb{N}\}.$$

$$(iii)\ \{\frac{n+6}{n^2-2n-8} :\ n \in \mathbb{N}\}.$$

(3) Suppose $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ are two bounded sequences of real numbers. Show that $\{a_n + b_n\}_{n\in\mathbb{N}}$ is a bounded sequence of real numbers and
(i) $\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(a_n + b_n)$;
(ii) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
Give examples to show that equality may not hold in (i) or (ii).

(4) Show that a continuous function $g : [0,1] \to (0,1)$ can not be surjective. Give an example of a surjective continuous function $h : (0,1) \to [0,1]$.

(5) Let $k : [0,1] \to [0,1]$ be a continuous function. Show that there exists $t \in [0,1]$ such that $k(t) = 1 - t^2$.

(6) Prove or disprove: Suppose A is a non-empty subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ has the property that $\{f(a_n)\}_{n\in\mathbb{N}}$ is Cauchy whenever $\{a_n\}_{n\in\mathbb{N}}$ is Cauchy in A. Then f is uniformly continuous.

(7) Let A be a non-empty subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ be uniformly continuous functions. Prove or disprove: (i) $af + bg$ is uniformly continuous for every $a, b \in \mathbb{R}$; (ii) fg is uniformly continuous on A. (iii) If $g(x) \neq 0$ for every $x \in \mathbb{A}$, then $\frac{f}{g}$ is uniformly continuous on A.

(8) Prove or disprove: Let A, B be non-empty subsets of $\mathbb{R}$. Suppose $f : A \to \mathbb{R}, g : A \to \mathbb{R}$ are uniformly continuous functions such that $f(A) \subseteq B$. Then $h = g \circ f$ is uniformly continuous.

(9) Show that $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x^3 - 5x,\ x \in \mathbb{R}$, is not uniformly continuous.

(10) Give an example of a bijection $f : \mathbb{R} \to \mathbb{R}$, where f is differentiable at every point in $\mathbb{R}$ but f^{-1} is not differentiable at some point. (Hint: Think of polynomials). Prove your claim.