

# **Analysis I**

Home Assignment V

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# 1 Home Assignment V (Due: Dec 4, 2025)

**Problem 1.1.** (Discrete L'Hospital): Let  $\{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1}$  be sequences of non-zero real numbers converging to 0. Suppose  $b_n > b_{n+1}$  for all  $n$  and  $v := \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$  exists as a real number. Show that  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists and equals  $v$ . Give one such example.

*Solution.* Given any  $\varepsilon > 0$  there exists  $N$  such that  $\forall n > N$ ,

$$v - \varepsilon < \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < v + \varepsilon \implies (v - \varepsilon)(b_{n+1} - b_n) > a_{n+1} - a_n > (v + \varepsilon)(b_{n+1} - b_n)$$

Where the signs flipped because  $b_{n+1} - b_n < 0$  as stated in the problem. We sum both sides of this inequality for  $n, n+1, \dots, m$  to get,

$$\begin{aligned} (v - \varepsilon) \sum_{k=n}^m (b_{k+1} - b_k) &> \sum_{k=n}^m (a_{k+1} - a_k) > (v + \varepsilon) \sum_{k=n}^m (b_{k+1} - b_k) \\ \implies (v - \varepsilon)(b_{m+1} - b_n) &> a_{m+1} - a_n > (v + \varepsilon)(b_{m+1} - b_n) \end{aligned}$$

Thus this holds for all  $m \geq n$  and taking the limits as  $m \rightarrow \infty$  on all of the terms in this inequality we get,

$$(v - \varepsilon)(-b_n) \geq -a_n \geq (v + \varepsilon)(-b_n) \implies v - \varepsilon \leq \frac{a_n}{b_n} \leq v + \varepsilon$$

which is enough to conclude that  $a_n/b_n \rightarrow v$  as well. For an example consider  $a_n = \sin(n)/n - 1$  and  $b_n = 1/n^2$ , both of these tend to 0 and  $b_n$  decreases monotonically. Now using Taylor series we can easily show that,  $a_{n+1} - a_n = \frac{-1}{6n^3} + O(n^{-5})$  hence,

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \xrightarrow{n \rightarrow \infty} \frac{-1}{6}$$

Thus using the result we see that,

$$\frac{a_n}{b_n} \xrightarrow{n \rightarrow \infty} \frac{-1}{6}$$

□

**Problem 1.2.** Suppose  $k \in \mathbb{N}$  and  $b_1, b_2, \dots, b_k$  are strictly positive real numbers. Show that (i)  $\lim_{n \rightarrow \infty} b_j^{\frac{1}{n}} = 1$ . (ii)  $\lim_{n \rightarrow \infty} (b_1^n + b_2^n + \dots + b_k^n)^{\frac{1}{n}} = b$  where  $b = \max\{b_j : 1 \leq j \leq k\}$ .

*Solution.* For (i), let  $x > 0$  and consider the two cases :  $x \geq 1$  and  $x < 1$ . For the first case, by Bernoulli's inequality we see that,

$$x = (x^{1/n})^n = (1 + (x^{1/n} - 1))^n \geq 1 + n(x^{1/n} - 1) \implies x^{1/n} - 1 \leq \frac{x - 1}{n}$$

Here we have used the fact that  $x^p \geq 1$  for any positive  $p$ , making  $x^{1/n} - 1$  nonnegative. Thus we can use the squeeze theorem on the following inequality to conclude,

$$0 \leq x^{1/n} - 1 \leq \frac{x - 1}{n}$$

And when  $0 < x < 1$  we can just consider  $1/x > 1$  and get that  $1/x^{1/n} \xrightarrow{n \rightarrow \infty} 1$  and using algebra of limits we get that  $1/\lim x^{1/n}$  is also 1 which makes  $\lim x^{1/n} = 1$ . To prove (ii) we will also use squeeze theorem. As  $b$  is the maximum of these  $b_i$  we have,

$$\begin{aligned} b^n &< b_1^n + \dots + b_k^n \leq b^n + \dots + b^n = kb^n \\ \implies b &< (b_1^n + \dots + b_k^n)^{1/n} \leq k^{1/n}b \end{aligned}$$

Using the fact  $k^{1/n} \rightarrow 1$ , which is proven in the immediate next exercise, we can conclude using squeeze theorem.  $\square$

**Problem 1.3.** Show that  $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$ .

*Solution.* For large enough ( $n \geq 3$ ) we can say that  $n^{1/n} \geq 1$  and then using the binomial theorem

$$\begin{aligned} n &= (1 + (n^{1/n} - 1))^n = \sum_{k=0}^n \binom{n}{k} (n^{1/n} - 1)^k \geq \frac{n(n-1)}{2} (n^{1/n} - 1)^2 \\ \implies 1 &\leq n^{1/n} \leq 1 + \sqrt{\frac{2}{n-1}} \end{aligned}$$

And yet again we conclude using the squeeze theorem.  $\square$

**Problem 1.4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ , be a continuous function satisfying  $f(3x) = f(x), \forall x \in \mathbb{R}$ . Show that  $f$  is a constant function.

*Solution.* For all real  $x$ , by the condition we have  $f(x) = f(3 \cdot (x/3)) = f(x/3)$  and continuing for  $n$  steps we have  $f(x) = f(x/3^n)$ . Now from the continuity of  $f$ ,

$$f(x) = \lim_{n \rightarrow \infty} f(x/3^n) = f\left(\lim_{n \rightarrow \infty} x/3^n\right) = f(0)$$

which makes  $f$  a constant function as  $\forall x \in \mathbb{R}, f(x) = f(0)$ .  $\square$

**Problem 1.5.** Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying  $g(x+y) = g(x) + g(y), \forall x, y \in \mathbb{R}$ . Show that  $g = cx$  for some  $c \in \mathbb{R}$ .

*Solution.* Just as in Problem 2.10. (problem 10 in the 2nd assignment) we can show that for any rational  $r$  we have  $f(r) = f(1) \cdot r$  and as given any real number  $x$  there exists a sequence of rationals  $\{x_n\}$  converging to it we see that,

$$f(x) = f\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(1) \cdot x_n = f(1) \cdot \lim_{n \rightarrow \infty} x_n = f(1) \cdot x$$

where we used the continuity of  $f$  in the second inequality and a basic fact from the algebra of limits in the fourth. Thus we see that  $c = f(1)$  and  $f(x) = f(1) \cdot x$  for all  $x \in \mathbb{R}$   $\square$

**Problem 1.6.** Let  $I$  be an interval in  $\mathbb{R}$ . A function  $f : I \rightarrow \mathbb{R}$ , is said to be convex if  $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \forall x < y$  in  $I, 0 < \lambda < 1$ . (i) Show that if  $f : (0, 1) \rightarrow \mathbb{R}$  is convex then for  $0 < s < t < u < 1$ ,

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

(ii) Show that if  $f : (0, 1) \rightarrow \mathbb{R}$ , is convex then it is continuous. (iii) Show that a convex function  $g : [0, 1] \rightarrow \mathbb{R}$ , need not be continuous.

*Solution.* Rearranging the first inequality we get the following equivalent inequality,

$$\begin{aligned} \frac{f(t)}{t-s} - \frac{f(u)}{u-s} &\leq \frac{f(s)}{t-s} - \frac{f(s)}{u-s} = \left( \frac{u-t}{(t-s)(u-s)} \right) f(s) \\ \iff f(t) &\leq \frac{u-t}{u-s} \cdot f(s) + \frac{t-s}{u-s} \cdot f(u) \end{aligned}$$

Which is true from the definition of convexity via  $\lambda = (u-t)/(u-s) \in (0, 1)$ . The second inequality is proven similarly. For (ii), we will show that for any  $x \in (0, 1)$ , the right and left limits of  $f(y)$  as  $y$  tends to  $x$  is zero which will imply that the limit exists and is zero as well, proving continuity. Without loss of generality assume that  $x < y$ , then choose  $u, v, w \in (0, 1)$  such that  $u < x < y < v$ . Now by the previous result we see that,

$$\begin{aligned} \frac{f(x) - f(u)}{x-u} &\leq \frac{f(y) - f(x)}{y-x} \leq \frac{f(v) - f(y)}{v-y} \leq \frac{f(w) - f(v)}{w-v} \\ \implies \frac{f(x) - f(u)}{x-u} \cdot (y-x) &\leq f(y) - f(x) \leq (y-x) \cdot \frac{f(v) - f(y)}{v-y} \leq (y-x) \cdot \frac{f(w) - f(v)}{w-v} \end{aligned}$$

Now, using the squeeze theorem, as  $y$  tends to  $x$  from the right we get,

$$\lim_{y \rightarrow x; y > x} (f(y) - f(x)) = 0$$

and the same can be said for the left limit using a very similar argument and we are done. For (iii) we can consider the following function,

$$f(x) := \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}$$

To show that this is convex we will consider the inequality as in the definition on all points  $x, y \in [0, 1]$ . The inequality is clearly always true for  $0 < x, y$  as the function is constant on  $(0, 1]$  and when some  $x, y$  is 0, wlog say  $x = 0$ , then,

$$f(\lambda x + (1-\lambda)y) = f((1-\lambda)y) \leq \underbrace{\lambda f(x)}_{=1} + (1-\lambda)f(y)$$

for both  $y = 0$  and  $y > 0$  as in the first case its  $1 \leq 1$  and in the second, as  $1-\lambda > 0$  its  $0 \leq 1$ . □

**Problem 1.7.** Show that there is no continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , such that for every  $y \in \mathbb{R}$ , there are exactly two real numbers  $x_1, x_2$ , such that  $f(x_1) = f(x_2) = y$ .

*Solution.* For the sake of contradiction assume that there is some  $f$  that is continuous and also satisfies said criterion. Let  $x < y$  be the reals such that  $f(x) = f(y) = 0$ . Without loss of generality we can assume  $x < y$ . Now consider the interval  $[x, y]$ , as  $f$  is continuous it must have some it must achieve its extrema in this interval at some points in the interval. Unless there is an extrema which is achieved only in  $(x, y)$  we can easily see that  $f$  is constant in this interval which gives us much more than two, infact infinitely

many reals where  $f$  is zero which is not allowed as per the assumption. Again without loss of generality assume that we reach a maxima at  $u \in (x, y)$ . We claim that  $u$  is the unique such point in  $[x, y]$ . For a proof of this claim, for the sake of contradiction assume without loss of generality that we also reach a maxima at  $v \in (u, y]$ . Now  $f$  can not be continuous on the interval  $[u, v]$  so there is some  $w \in (u, v)$  such that  $0 < f(w) < f(u)$  where the first inequality can be achieved due to continuity because by construction  $f(u) = f(v) > 0$  and if we did not have some  $w$  it would produce jump discontinuities in  $(u, v)$ . Now for any  $\lambda \in (f(w), f(u))$  we can find a real  $z$  with  $f(z) = \lambda$  in all three of the intervals :  $(x, u), (u, v), (v, y)$  by IVT, which contradicts the assumption as there should be exactly two such  $z$  and the claim is proved. By the condition on  $f$ , there must be another  $g \notin [x, y]$  such that  $f(u) = f(g)$ . Without loss of generality assume that  $g > y$ , then for any  $\lambda \in (0, f(u))$  we can find some  $z$  with  $f(z) = \lambda$  from all of these three intervals :  $(x, u), (u, y), (y, g)$  by IVT which is a contradiction and we are done.  $\square$

**Problem 1.8.** Fix  $n \in \mathbb{N}$ . Let  $x_1, x_2, \dots, x_n$  be  $n$  distinct real numbers and let  $y_1, y_2, \dots, y_n$  be another  $n$ -tuple of not-necessarily distinct real numbers. Show that there is a unique polynomial  $p$  of degree  $(n - 1)$  such that  $p(x_k) = y_k$  for  $1 \leq k \leq n$ . Hint: Consider

$$p(x) = \sum_{j=1}^n y_j \frac{\prod_{i \neq j} (x - x_i)}{\prod_{i \neq j} (x_j - x_i)}.$$

*Solution.* The polynomial  $p$  provided in the hint is clearly of degree at most  $n - 1$  and when evaluated at  $x_i$ , all but the  $i$ 'th term in the sum vanish and in this term the numerator and denominator of the fraction cancel out leaving only  $y_i$  and thus  $p(x_i) = y_i$ . Assume  $p, q$  both have degree at most  $n - 1$  and they both are polynomials which satisfy the condition in the problem, then we see that  $p - q$  has  $n$  zeroes.  $p - q$  clearly has degree at most  $n - 1$  and thus for it to have  $n > n - 1$  zeroes, by the fundamental theorem of algebra we see that  $p - q$  must be identically zero i.e.  $p = q$  and hence such a polynomial is unique.  $\square$

**Problem 1.9.** Let  $a < b$  be real numbers. Let  $f : (a, b) \rightarrow \mathbb{R}$  be a twice differentiable function. Show that for any  $x_0 \in (a, b)$

$$f''(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h))}{h^2}.$$

*Solution.*  $f$  being twice differentiable makes it continuous and thus the numerator tends to 0 as  $h \rightarrow 0$  and the denominator also tends to zero as  $h \rightarrow 0$  and also the derivative of the denominator is nonzero near 0, except at 0. Let the numerator be  $X(h)$  and denominator  $Y(h)$ , then,

$$\begin{aligned} \frac{X'(h)}{Y'(h)} &= \frac{f'(x_0 + h) - f'(x_0 - h)}{2h} \\ &= \frac{f'(x_0 + h) - f'(x_0)}{2h} - \frac{f'(x_0) - f'(x_0 - h)}{2h} \xrightarrow{h \rightarrow 0} \frac{f''(x_0)}{2} + \frac{f''(x_0)}{2} = f''(x_0) \end{aligned}$$

and we can thus conclude via L'Hopital's rule.  $\square$

**Problem 1.10.** (i) For  $x \in \mathbb{R}$ , show that  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges absolutely. (ii) Define  $e : \mathbb{R} \rightarrow \mathbb{R}$  by  $e(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ . Show that  $e(x + y) = e(x)e(y), \forall x, y \in \mathbb{R}$ .

*Solution.* We can say, using Problem 2.9. that for a sequence of positive numbers  $x_n$  that converge to  $x$ , that  $\sqrt[n]{x_1 x_2 \dots x_n} \xrightarrow{n \rightarrow \infty} x$  using that result on  $\log(x_n)$  and using the continuity of  $\log$  on its domain. We know from Problem 2.5. that the limit  $(1 + 1/n)^n$  as  $n$  tends to infinity exists and using the binomial theorem we clearly see that  $(1 + 1/n)^n \geq 1 + n(1/n) = 2$  so the limit is positive, call this  $\text{not}_e$ . Now consider the product,

$$\prod_{k=1}^n \left(1 + \frac{1}{k}\right)^k = \frac{2^1}{1^1} \cdot \frac{3^2}{2^2} \dots \frac{(n+1)^n}{n^n} = \frac{(n+1)^n}{n!}$$

Using the results mentioned above we have,

$$\frac{n+1}{(n!)^{1/n}} = \sqrt[n]{\prod_{k=1}^n \left(1 + \frac{1}{k}\right)^k} \xrightarrow{n \rightarrow \infty} \text{not}_e$$

For (i), we will show that for any  $x \in \mathbb{R}$  this series is absolutely convergent using the root test,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|x^n/n!|} &= \lim_{n \rightarrow \infty} \frac{|x|}{(n!)^{1/n}} = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} \cdot \frac{n+1}{(n!)^{1/n}} \\ &= \left( \lim_{n \rightarrow \infty} \frac{|x|}{n+1} \right) \cdot \lim_{n \rightarrow \infty} \frac{n+1}{(n!)^{1/n}} = 0 \cdot \text{not}_e = 0 < 1 \end{aligned}$$

so we are done. For (ii), using the cauchy product theorem for product of sums, atleast one of which is absolutely convergent we have,

$$e(x)e(y) = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = e(x+y)$$

□

**Problem 1.11.** Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be the function  $h(x) = \sin(x)$  (Here we are assuming your familiarity with trigonometric functions). Show that the remainder term in Taylor's theorem converges to 0 as  $n \rightarrow \infty$  for every  $x_0$  and  $x$ .

*Solution.* If we keep taking the derivative of  $h$  we can easily see that  $|h^{(n)}(x)|$  is either  $|\sin x|$  or  $|\cos x|$ , eitherways its bounded by 1. Now given any  $x, x_0$  the  $n$ -th remainder term, say  $R_n$  tends to zero as follow,

$$|R_n| \leq \frac{|x_0 - x|^n}{n!} \cdot \sup_{x \in \mathbb{R}} |h^{(n)}(x)| = \frac{|x - x_0|^n}{n!} \xrightarrow{n \rightarrow \infty} 0$$

just as we did in Problem 5.10..

□

**Problem 1.12.** Consider the function  $h : [0, \infty) \rightarrow \mathbb{R}$  defined by  $h(x) = \frac{x}{1+x^2}$ . Determine the set  $\{h(x) : x \in [0, \infty)\}$ .

*Solution.* Clearly  $h(x) \geq 0$  with equality at 0 and using AM-GM,  $h(x) \leq 1/2$  with equality at  $x = 1$ . By IVT we can say that it also achieves every value in between and hence the set is  $[0, 1/2]$

□

**Problem 1.13.** Let  $v : \mathbb{R} \rightarrow \mathbb{R}$  be the function  $v(x) = x^3 - 6x^2 + 9x$ , for all  $x$ . Determine the set:  $\{x : v(x) > 0\}$ .

*Solution.* We can factor this into  $x(x-3)^2$ , so it is zero at 0 and 2. For  $x \neq 0, 3$  it has the same sign as  $x$  as  $(x-3)^2$  is positive. Thus the set is  $(0, \infty) \setminus \{3\}$ .  $\square$

**Problem 1.14.** Let  $f : [0, 1] \rightarrow [0, 1]$  be a continuous strictly increasing bijection. Assume that  $x < f(x)$  for all  $x \in (0, 1)$ . Fix  $x_0 \in (0, 1)$ . Define  $x_n$  for  $n \in \mathbb{Z}$  by  $x_n = f^n(x_0)$ . Show that: (i)  $0 < x_m < x_n < 1$  for  $m < n$  in  $\mathbb{Z}$ ; (ii)  $\lim_{n \rightarrow \infty} x_n = 1$  and  $\lim_{n \rightarrow -\infty} x_n = 0$ ; (iii) For every  $n \in \mathbb{Z}$ ,  $f$  maps  $[x_n, x_{n+1}]$  bijectively to  $[x_{n+1}, x_{n+2}]$ .

*Solution.* As  $f$  is strictly increasing we see that we must have  $f(0) = 0$  and  $f(1) = 1$  and as its also a bijection, 0 or 1 never appear in  $f$  orbits i.e.  $\{f^n(x) : n \in \mathbb{Z}\}$  of any points in  $(0, 1)$ . Thus we can see that for any  $m < n$ ,

$$f^n(x_0) = \underbrace{f(f^{n-1}(x_0) > f^{n-1}(x_0) > \dots > f^m(x_0))}_{n-m \text{ inequalities}}$$

and as 1 and 0 never appear in the orbit of  $x_0$  we see that  $0 < f^n(x_0) < 1$  for all  $n \in \mathbb{Z}$  and together it reads  $0 < x_m < x_n < 1$ , we have thus proved (i). For (ii), as  $n \rightarrow \infty$ ,  $\{x_n\}$  must converge to a limit as its increasing and bounded above, say it converges to  $x$ , then  $0 < x \leq 1$ . Using the continuity of  $f$  we have can show that this is a fixed point as follows

$$f(x) = f\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} x_n = x$$

so it has to be 1. We can similarly show that  $x_n \xrightarrow{n \rightarrow -\infty} x_n = 0$  as well. For every  $n \in \mathbb{Z}$  we see that  $x_n < x_{n+1}$  and thus  $x_{n+1} = f(x_n) < f(x_{n+1}) = x_{n+2}$  so we have  $f([x_n, x_{n+1}]) \subseteq [x_{n+1}, x_{n+2}]$  using the fact that  $f$  is increasing. Using IVT we can say that we have equality here i.e.  $f([x_n, x_{n+1}]) = [x_{n+1}, x_{n+2}]$  and the  $f$  is clearly still a bijection here, this proves (iii).  $\square$

**Problem 1.15.** Consider the set up of the previous question. Fix  $y_0 \in (x_0, x_1)$ . Set  $y_n = f^n(y_0)$  for  $n \in \mathbb{Z}$ . Let  $h : [x_0, y_0] \rightarrow [y_0, x_1]$  be a continuous strictly increasing bijection. Define  $g : [0, 1] \rightarrow [0, 1]$  by  $g(0) = 0, g(1) = 1$ ,

$$g(t) = h(t) \quad \forall t \in [x_0, y_0];$$

$$g(t) = f \circ h^{-1}(t) \quad \forall t \in [y_0, x_1];$$

and more generally, for  $n \in \mathbb{Z}$ , define

$$g(t) = f^n \circ h \circ f^{-n}(t) \quad \forall t \in [x_n, y_n]$$

and

$$g(t) = f^{n+1} \circ h^{-1} \circ f^{-n}(t) \quad \forall t \in [y_n, x_{n+1}].$$

Then show that: (i) For every  $n$ ,  $g$  maps  $[x_n, y_n]$  bijectively to  $[y_n, x_{n+1}]$  and it maps  $[y_n, x_{n+1}]$  bijectively to  $[x_{n+1}, y_{n+1}]$ . (ii)  $g$  is a strictly increasing continuous bijection. (iii)  $f = g \circ g$ . This shows that  $f$  has infinitely many square roots.

*Solution.* We know that inverses of continuous bijections in  $\mathbb{R}$  are also continuous bijections and both of these are strictly increasing. From the previous problem we can see that  $f^{-n}$  bijectively maps  $[x_n, y_n]$  to  $[x_0, y_0]$ , after this  $h$  maps  $[x_0, y_0]$  to  $[y_0, x_1]$  bijectively and at the end  $f^n$  maps this bijectively to  $[y_n, x_{n+1}]$ , thus  $g$  maps  $[x_n, y_n]$



bijectively to  $[y_n, x_{n+1}]$ . Using similar arguments we can show that  $g$  also bijectively maps  $[y_n, x_{n+1}]$  to  $[x_{n+1}, y_{n+1}]$  and (i) is proven. We can see that

$$\mathfrak{J} := \{[x_n, y_n), [y_n, x_{n+1}) : n \in \mathbb{Z}\} \cup \{\{0\}, \{1\}\}$$

is a partition of  $[0, 1]$ , this is because  $\{[x_n, x_{n+1}) : n \in \mathbb{Z}\} \cup \{\{0\}, \{1\}\}$  is a partition (this can be deduced from parts (i) and (ii) of [Problem 5.14.](#)) and we just partition each of the intervals into two intervals. As  $g(0) = 0, g(1) = 1$  and,

$$g([x_n, y_n)) = [x_{n+1}, y_n] \setminus \{g^{-1}(y_n)\} = [y_n, x_{n+1})$$

As,  $g(y_n) = f^n(h(f^{-n}(y_n))) = f^n(h(y_0)) = f^n(x_1) = x_{n+1}$ . Similarly, for the other type of intervals, as,  $g(x_{n+1}) = f^{n+1}(h^{-1}(f^{-n}(x_{n+1}))) = f^{n+1}(h^{-1}(x_1)) = f^{n+1}(y_0) = y_{n+1}$  we get,

$$g([y_n, x_{n+1})) = [x_{n+1}, y_{n+1})$$

Now, looking at  $g$  as a set function defined by  $g(S) := \{g(x) : x \in S\}$  we can clearly see that  $g$  is a bijection on  $\mathfrak{J}$  and as its a bijection on each set in  $\mathfrak{J}$  which is a partition of  $[0, 1]$ , its clearly a bijection on  $[0, 1]$  as a whole. Thus we have shown that  $g$  is continuous and a bijection, thus it must be strictly increasing as well hence proving (ii). For (iii), firstly  $(g \circ g)(0) = 0 = f(0)$  and  $(g \circ g)(1) = 1 = f(1)$ , now if some  $x \in (0, 1)$  is in an interval of form  $[x_n, y_n]$  then when  $g$  is first applied to it, it acts as  $f^n \circ h \circ f^{-n}$  and sends it to the interval  $[y_n, x_{n+1}]$  as seen in (ii) and thus on the second application  $g$  acts as  $f^{n+1} \circ h^{-1} \circ f^{-n}$  as in the definition of  $g$ , thus,

$$\begin{aligned} (g \circ g)(x) &= (f^{n+1} \circ h^{-1} \circ f^{-n} \circ f^n \circ h \circ f^{-n})(x) = (f^{n+1} \circ h^{-1} \circ (f^{-n} \circ f^n) \circ h \circ f^{-n})(x) \\ &= (f^{n+1} \circ h^{-1} \circ \text{Id} \circ h \circ f^{-n})(x) = (f^{n+1} \circ h^{-1} \circ (\text{Id} \circ h) \circ f^{-n})(x) = (f^{n+1} \circ h^{-1} \circ h \circ f^{-n})(x) \\ &= (f^{n+1} \circ (h^{-1} \circ h) \circ f^{-n})(x) = (f^{n+1} \circ \text{Id} \circ f^{-n})(x) = (f^{n+1} \circ (\text{Id} \circ f^{-n}))(x) = (f^{n+1} \circ f^{-n})(x) \\ &= f(x) \end{aligned}$$

and when  $x$  is in an interval of form  $[y_n, x_{n+1}]$  using similar arguments we have that,

$$(g \circ g)(x) = \underbrace{(f^{n+1} \circ h \circ f^{-(n+1)})}_{\text{as } g(x) \in [x_{n+1}, y_{n+1}]} \circ f^{n+1} \circ h^{-1} \circ f^{-n}(x) = (f^{n+1} \circ f^{-n})(x) = f(x)$$

Thus we have shown that  $g \circ g = f$  on  $[0, 1]$  and we are done.  $\square$