Exercise sheet 6

Only the exercises marked [HW] need to be submitted.
Vectors will be denoted in bold unlike scalars.

1. Let V be a vector space over F. A linear transformation from V to F is called a *linear functional*. Examples are

   - the map $\mathbf{x} \mapsto \mathbf{u}^T\mathbf{x}$ on F^n, where $\mathbf{u} \in F^n$ is a fixed vector
   - the map $g \mapsto \int_a^b g(t)dt$ from the space $C[a,b]$ of real continuous functions on $[a,b]$ to $\mathbb{R}$.
   - the map $p \mapsto p(t)$ from the space of polynomials, where t is a fixed element of F.

   Let $V^* = L(V, F)$ be the *dual space*, the space of all linear functionals from V to F.

   (a) Show that $\dim(V^*) = \dim(V)$.

   (b) Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ be a basis of V. Show that $\{f_1, \cdots, f_n\}$ is a basis of V^*, where $f_i(x_j) = \mathbf{1}\{i = j\}$. (here $\mathbf{1}\{i = j\}$ is 1 when $i = j$ and 0 otherwise). Show that for any $f \in V^*$ we have $f = \sum_{i=1}^n f(\mathbf{x}_i)f_i$, and for each vector $\mathbf{x} \in V$ we have $\mathbf{x} = \sum_{i=1}^n f(\mathbf{x})\mathbf{x}_i$.

2. If V is a vector space over F and S is a subset of V, the annihilator of S is the set S^0 of linear functionals f on V such that $f(\mathbf{x}) = 0$ for every $x \in S$. For a finite dimensional vector space V over F, and a subspace W of V, show that $\dim W + \dim W^0 = \dim V$

3. If W_1 and W_2 are subspaces of a finite dimensional vector space, then $W_1 = W_2$ if and only if $W_1^0 = W_2^0$.

4. A hyperspace of an n dimensonal vector space V is a subspace of dimension $n-1$. Show that a k dimensional subspace W of V is the intersection of $n-k$ hyperspaces in V.

5. Show that all linear functionals on F^n are of the form $\mathbf{x} \mapsto \mathbf{a}^T\mathbf{x}$ for some vector $\mathbf{a}$. In general show that all linear transformations from F^n to F^m are of the form $\mathbf{x} \mapsto A\mathbf{x}$ for some $m \times n$ matrix A.

6. Let W_1 and W_2 be subspaces of a finite dimensional vector space V.

   (a) Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$

   (b) Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

7. Give an example of two square matrices A, B such that $AB \neq BA$.

8. A square matrix A is upper triangular if all elements below the diagonal are 0, that is $a_{ij} = 0$ for $i > j$. Show that the product of upper triangular matrices is upper triangular.
   Similarly the product of lower triangular matrices (i.e. $a_{ij} = 0$ for $i < j$) is lower triangular.

9. [HW 5, due Sept 26] Prove that $\rho \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \geq \rho(A) + \rho(C)$. Show that strict inequality can occur. Deduce that the rank of an upper triangular matrix is not less than the number of non-zero diagonal elements.

10. Let A be an $m \times n$ matrix of rank r. Determine the possible values for the rank of the matrix obtained by (i) changing exactly one element and (ii) changin two elements.

11. If A is an $m \times n$ matrix of rank r, determine the possible values for the rank of the submatrix obtained by deleting a row and a column.

12. [HW 5, due Sept 26] If A is an $m \times n$ matrix with rank r, show that for every k such that $1 \leq k \leq r$, A has a $k \times k$ submatrix with rank k.

13. Show that an $m \times n$ matrix A has rank at most 1 if and only if $A = \mathbf{x}\mathbf{y}^T$ for some column vectors $\mathbf{x}$ and $\mathbf{y}$. Show further that $\rho(A) = 1$ if and only if both $\mathbf{x}$ and $\mathbf{y}$ are non-null.