Analysis I

Home Assignments

Arkaraj Mukherjee
B.Math., First Year, ISI Bangalore

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1 Home Assignment I (Due: Aug 19, 2025)

Here, the set of all natural numbers $\mathbb{N}$ *contains zero as an element. It was confirmed with the professor that "countable" means "either finite or countably infinite."*

Lemma : A set A is countable if there exists a surjection $f : \mathbb{N} \to A$.

Proof. If A is finite then we are done as its countable by definition. Now assume that A is infinite. For each $a \in A$ define $m(a) = \min\{n \in \mathbb{N} : f(n) = a\}$, which exists since f is surjective and the minimum of a nonempty subset of $\mathbb{N}$ exists. Then the function $g : \{m(a) : a \in A\} \to A$ with $g(m(a)) = a$, is well defined and injective because each $m(a)$ is unique for each a as previously shown and surjective because every $a \in A$ is clearly reached. The domain of g is an infinite subset of $\mathbb{N}$, hence equipotent with $\mathbb{N}$ as shown in class and there exists a bijection between this and $\mathbb{N}$. Now composing these two gives us a bijection between $\mathbb{N}$ and A thus proving that $\mathbb{N}$ and A are equipotent implying that A is countablely infinite hence countable. $\square$

Problem 1.1. Let C, D be sets with 4 and 5 elements respectively. Find the number of functions from C to D which are: (i) injective; (ii) surjective.
Similarly, find the number of functions from D to C which are: (iii) injective; (iv) surjective.

Solution. (i) Enumerate the sets as $C = \{c_1, \ldots, c_4\}$ and $D = \{d_1, \ldots, d_5\}$. Now to count the number of injections we can first choose which 4 elements from D will be in the range of f, there are ${}^5C_4$ ways to do this and as the order matters the number of such injections will be

$$4! \times \binom{5}{4} = 5!$$

(ii) There are no surjections from C to D as C and D are finite sets where $|C| < |D|$ i.e. D has strictly more values and we can not have all of this in the range of f as the range has atmost as many elements as the domain i.e. $4 < 5$. (iii) A function from D to C can not be an injection as we have 5 elements in D but atmost 4 values their image can be thus we ought to have some element in C thats the image of two distinct elements in C. (iv) for it to be surjective we see that all the elements in C must be in its range thus exactly one element in C must be the image of some two distict elements in D and there are ${}^5C_2$ ways choose these two elements and 4 ways to choose the element in C which will be their image whereas the 3 elements that are left in D will be mapped to 3 distinct elements in C and there are 3! ways to do this. Now the multiplicative principle in combinatorics there are

$$4 \times \binom{5}{2} \times 3!$$

surjections. $\square$

Problem 1.2. Suppose X is a non-empty set and $f : X \to X$ is a function. Prove or disprove the following: (i) f injective $\Leftrightarrow$ $f \circ f$ injective; (ii) f surjective $\Leftrightarrow$ $f \circ f$ surjective; (iii) f bijective $\Leftrightarrow$ $f \circ f$ bijective.

Solution. (i) This is true. If $f : X \to X$ is injective then so is $f \circ f$ as for $a, b \in X, f(f(a)) = f(f(b)) \implies f(a) = f(b) \implies a = b$ using f's injectivity twice, this proves the only if part. Now for the if part, when $f \circ f : X \to X$ is injective,for some $a, b \in X, f(a) = f(b) \implies f(f(a)) = f(f(b)) \implies a = b$ using the fact that f is a function and then the fact that its an injection, thus f is also an injection. (ii) This is true. Using the notation

$$f(S) := \{f(x) : x \in S\} \text{ for subsets } S \text{ of } X$$

we see that f being a surjection is equivalent to the equality $f(X) = X$ being true. Now for the if part, if $f \circ f$ is a surjection, we see that as $f : X \to X$ we have that $f(X) \subseteq X$ and thus $X = f(f(X)) \subseteq f(X) \subseteq X$ giving us $f(X) = X$. For the only if part we can clearly see that $f(X) = X \implies f(f(X)) = f(X) = X$. (iii) This is true as a function is bijecitve iff its injective and its also surjective and we have already seen that f is injective iff $f \circ f$ is and the same goes for surjectivity. □

Problem 1.3. Find three functions u, v, w from $\mathbb{N}$ to $\mathbb{N}$, which are injective and have disjoint ranges.

Solution. Let $u, v, w : \mathbb{N} \to \mathbb{N}$ be the functions: $u : k \mapsto 3k, v : k \mapsto 3k+1, w : k \mapsto 3k+2$ these are clearly injective and have disjoint ranges. □

Problem 1.4. Let R, S be two non-empty sets. Suppose there exists an injective function $g : R \to S$. Show that there exists a surjective function $h : S \to R$.

Solution. As R is nonempty, fix some $x_o \in R$. Now if g is injective then we see that for all $y \in g(R) \subseteq S$, there exists an unique $x \in R$ such that $g(x) = y$ and for these y we set $h(y) = x$ and for $y \in S \setminus g(R)$(if nonempty, otherwise g was a bijection and we can set $h = g^{-1}$ in that case,) we set $h(y) = x_o$. This is well defined as g is an injection and its a surjection as every element in the domain of g is mapped to some element in the range. □

Problem 1.5. Suppose A and B are countable sets. Show that $A \cup B$ is countable.

Solution. As A, B are countable there exist surjections f, g from $\mathbb{N}$ to A and B respectively. Define $h : \mathbb{N} \to A \cup B$ as

$$h(n) := \begin{cases} f(k) \text{ if } n = 2k+1 \text{ for some } k \in \mathbb{N} \\ g(k) \text{ if } n = 2k \text{ for some } k \in \mathbb{N} \end{cases}$$

This is clearly a surjection as $A, B \subseteq f(\mathbb{N}) \cup g(\mathbb{N}) = h(\mathbb{N})$ and thus $A \cup B \subseteq h(\mathbb{N})$ and by definititon $A \cup B$ is countable. □

Problem 1.6. Suppose $A_1, A_2, \ldots$ is a sequence of countable sets. Show that

$$\bigcup_{n=1}^{\infty} A_n = \{a : a \in A_n \text{ for some } n \in \mathbb{N}\}$$

is countable. (In other words, a countable union of countable sets is countable.)

Solution. We re index the sets as $A_0, A_1, \ldots$. Now as each A_n is countable, there exists a surjection $f_n : \mathbb{N} \to A_n$ for all $n \in \mathbb{N}$. Define $f : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} A_n$ by

$$f(X) := \begin{cases} f_{\nu_2(X)}\left(\dfrac{\frac{X}{2^{\nu_2(X)}} - 1}{2}\right) & \text{if } X \neq 0, \\ f_0(0) & \text{if } X = 0, \end{cases}$$

where $\nu_2(X)$ denotes the largest integer m such that $2^m \mid X$. Every $X > 0$ can be written uniquely in the form $X = 2^m(2k+1)$ with $m, k \in \mathbb{N}$, and then $f(X) = f_m(k)$. To check surjectivity, let $a \in A_m$ for some m. Since f_m is surjective, there exists $k \in \mathbb{N}$ with $f_m(k) = a$. Setting $X = 2^m(2k+1)$ gives $f(X) = a$. Thus f is surjective, and it follows that $\bigcup_{n\in\mathbb{N}} A_n$ is countable.

□

Problem 1.7. Let X be a non-empty set. Show that the set of all functions from X to $\{0,1\}$ is in bijective correspondence with the power set of X. (Here X need not be a finite set.)

Solution. We define a function $M : \{0,1\}^X \to \mathcal{P}(X)$ as,

$$M(f) := f^{-1}(\{1\}) = \{x \in X : f(x) = 1\}$$

This is injective as $M(f) = M(g)$ implies that for all $x \in X, f(x) = 1$ iff $g(x) = 1$ and as the only other values these could have is 0 we see that for all $x \in X$ we also have that $f(x) = 0 \iff g(x) = 0$ thus $f = g$. This is surjective as for any subset $S \subseteq X$ we can find $\{0,1\}^X \ni f : X \to \{0,1\}$ defined as

$$f(x) := \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ otherwise} \end{cases}$$

and we see that $M(f) = S$. As it is both a surjection and an injection we see that its a bijection. □

Problem 1.8. ⋆Let Y be a non-empty set. What is the maximum possible number of distinct sets we can form using n-subsets $A_1, A_2, \ldots, A_n$ of Y, using set theoretic operations of union, intersection, complement in Y?

For instance, when $n = 1$, the answer is 4: A_1, A_1^c, $\varnothing = A_1 \cap A_1^c$, $Y = A_1 \cup A_1^c$.

For $n = 2$, the answer is 16, where the list goes on something like A_1, A_2, $A_1 \cap A_2$, $A_1 \cup A_2$, $A_1 \cap A_2^c$, $A_1^c \cap A_2$, $A_1^c \cup A_2^c$, etc.

Guess the answer for general n and prove it. (Hint: Think of the Venn diagram.)

Solution. disjoint For a set S and a collection of subsets, X we define $\mathfrak{G}(X)$ to be the collection of all sets that are formed with the sets in X via the set theoretic operations of union, intersection and complement in S. We claim that the maximum possible number of such sets is 2^{2^n} i.e. $|\mathfrak{G}(\{A_1, \ldots, A_n\})| \leq 2^{2^n}$ and a case where this is achieved is for subsets $A_i = \{(x_1, \ldots, x_n) : (\forall j \neq i) x_j \in \{0,1\} \text{ and } x_i = 1\}$ for all $i = 1, \ldots, n$ and $Y = \{(x_1, \ldots, x_n) : (\forall i) x_i \in \{0,1\}\}$. For a function $f : \{1, \ldots, n\} \to \{0,1\}$ we define

$$\mathfrak{G}(\{A_1, \ldots, A_n\}) \ni A(f) := \bigcap_{i=1}^{n} A_i^{f(i)} \text{ where } A^1 := A \text{ and } A^0 := A^c$$

In this case we see that $A(f) = \{(f(1), \ldots, f(n))\}$ are all disjoint sets for different such functions and as there are 2^n of these as there are 2^n such functions and we can make 2^{2^n} distinct sets using these by choosing which ones to include in the union; formally this collection of sets can be written as,

$$\left\{ \bigcup_{f \in S} A(f) \middle| S \subseteq \{f : \{0, \ldots, n\} \to \{0, 1\}\} \right\}$$

In this case we also see that this collection is precisely the powerset of Y itself and thus $|\mathfrak{G}(\{A_1, \ldots, A_n\})| = 2^{\text{number of subsets of } Y} = 2^{2^n}$. Now we will prove the inequality. Say Y is a set and $A_1, \ldots, A_n$ are subsets, then we claim that, with $A(f)$ defined in the same way, $\mathfrak{G}(\{A_1, \ldots, A_n\}) = \mathfrak{G}(\{A(f) | f : \{1, \ldots, n\} \to \{0, 1\}\})$. This is true because, for all $f, A(f) \in \mathfrak{G}(\{A_1, \ldots, A_n\})$ and this proves one direction of the set inequality, $\mathfrak{G}(\{A(f) | f : \{1, \ldots, n\} \to \{0, 1\}\}) \subseteq \mathfrak{G}(\{A_1, \ldots, A_n\})$ as anything on the left can have its individual sets $A(f)$ be written in terms of elements on the right with set theoretic operations and clearly $\mathfrak{G}(\text{anything})$ is closed under all the set theoretic operations and thus it contains these sets. We also see that,

$$(\forall i) A_i = \bigcup_{\substack{f : \{1, \ldots, n\} \to \{0,1\} \\ f(i) = 1}} A(f) \in \mathfrak{G}(\{A(f) | f : \{1, \ldots, n\} \to \{0, 1\}\})$$

and similarly we get the other direction of the set equality. Thus it suffices to show that $|\mathfrak{G}(\{A(f) | f : \{1, \ldots, n\} \to \{0, 1\}\})| \leq 2^{2^n}$. We first see that the collection in question $\{A(f) | f : \{1, \ldots, n\} \to \{0, 1\}\}$ is closed under complements in Y and the intersection of any sets in it is empty as all the sets in it are disjoint. Also the complement of the union of some sets in this is again an union of some sets in this as these sets partition Y(they are disjoint as for any $f \neq g$ there must exist some i such that $f(i) \neq g(i)$ and we would have that $A(f) \cap A(g) \in A_i^{f(i)} \cap A_i^{g(i)} = A^1 \cap A^0 = \emptyset$ and also we see that the union of all of these $A(f)$ is Y as any $x \in Y$ is, for all i, either in A_i or A_i^c and we can take the intersection of the ones which contain x to get a elemenent in our collection containing x.), thus any expression with unions and intersections and complements in Y can be reduced to a union of some sets in this collection and we have the choices of whether to include some set $A(f)$ in the union and as there are atmost(some of the intersections may be empty) 2^n elements in $\{A(f) | f : \{1, \ldots, n\} \to \{0, 1\}\}$ there can be atmost 2^{2^n} such unions giving us the final inequality,

$$|\mathfrak{G}(\{A_1, \ldots, A_n\})| = |\mathfrak{G}(\{A(f) | f : \{1, \ldots, n\} \to \{0, 1\}\})| \leq 2^{2^n}$$

□

Problem 1.9. Let $K = \{0, 1\}$ and $L = \{0, 1, 2, 3\}$. Consider Cartesian products of countably many copies of K and L:

$$M = K \times K \times \cdots, \quad N = L \times L \times \cdots$$

Show that M and N are equipotent.

Solution. We define a function $f : N \to M$ which maps $(a_0, a_1, \ldots)$ to $(b_0, b_1, \ldots)$ where for all $n \in \mathbb{N}$, the representation of the a_n in binary using two digits (a redundant leading zero is allowed)is $b_{2n} b_{2n+1}$ viewing this as a digit and not the product. For example,

$$(1, 0, 2, 3, \ldots) \longmapsto (0, 1, 0, 0, 1, 0, 1, 1, \ldots)$$

where 1 in binary is written as 01, 2 as 10, 3 as 11 and 0 as 00. This function is a surjection as given any $b = (b_0, b_1, \ldots) \in M$ we see that $(2b_0 + b_1, 2b_2 + b_3, \ldots) \in N$ is mapped to b as $2^1 \cdot b_{2n} + 2^0 \cdot b_{2n+1}$ is the decimal representation of $b_{2n}b_{2n+1}$(as a number binary following pteviously stated convention), in the decimal system. The function is also injective because each $b \in M$ uniquely determines the $a \in N$ such that $f(a) = b$ as seen above. As it is an injection and also a surjection, it is a bijection and the two sets are equipotent. □

Problem 1.10. A real number x is said to be a rational number if $x = \frac{p}{q}$, for some integers p, q with $q \neq 0$. Let $\mathbb{Q}$ be the set of rational numbers. Show that $\mathbb{Q}$ is countable.

Solution. The sets $D_q := \{p/q : p \in \mathbb{Z}\}$ for $q \in \mathbb{N} \setminus \{0\}$ are all clearly countable and so is $N \setminus \{0\}$. And as,

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} D_n$$

its countable as its a countable union of countable sets by Problem 1.6. □

Problem 1.11. Read about "Proof by infinite descent" and write down one such proof.

We will prove a result due to fermat: The only integer solutions (x, y, z) to the diophantine equation $x^3 + 2y^3 + 4z^3 = 0$ is $(0, 0, 0)$.
For the sake of contradiction assume that we have some solution (x, y, z) in the integers such that $(x, y, z) \neq (0, 0, 0)$. Now we see that, $x^3 + 2y^3 + 4z^3 = 0 \implies x^2 = -2(y^3 + 2z^3) \implies 2|x^3 \implies 2|x$ and we can write $x = 2x_*$ for an integer x_*. Now substituing this in the equation and dividing by two we get $y^3 + 2z^3 + 4x_*^3 = 0$ and this has the exact same structure as the original equation! So if we have a solution (x, y, z) then we can find another integer solution $(y, z, x_*) = (y, z, x/2)$. We can keep doing this as follows,

$$(x, y, z) \to (y, z, x/2) \to (z, x/2, y/2) \to (x/2, y/2, z/2) \to \ldots \to (x/2^n, y/2^n m z/2^n) \to \ldots$$

and all of these must be integer solutions by the construction. But this implies that for all $n \in \mathbb{N}, 2^n | x, y, z$ which is a contradiction unless all of x, y, z are zero as a nonzero integer can only have a finite exponent of 2 in it Thus the only solution is $(0, 0, 0)$.

Problem 1.12. Suppose a rabbit moves along a straight line on the lattice points of the plane, making identical jumps every minute (the initial position and the jump vector are unknown). If we can place a trap once every hour at any lattice point of the plane, and the trap captures the rabbit if it is at that point at that moment, can we guarantee capturing the rabbit in a finite amount of time?

Solution. Each rabbit path is determined by an initial position $X \in \mathbb{Z}^2$ and a jump vector $Y \in \mathbb{Z}^2$, and can be written as

$$w(X, Y) = \{X + Yt : t \in \mathbb{N} \cup \{0\}\}$$

This gives an injective map from the set of paths to $(\mathbb{Z}^2)^2$, so the set of possible paths is countable. Hence we can enumerate them as $\{w(n) : n \in \mathbb{N}\}$. We place traps as follows. At stage 1, place a trap at the point $X + Y \cdot 2^1$ on path $w(1)$. At stage 2, place traps at $X + Y \cdot 2^2$ for both $w(1)$ and $w(2)$. At stage 3, place traps at $X + Y \cdot 2^3$ for $w(1), w(2), w(3)$, and so on. In general, at stage n we place traps at $X + Y \cdot 2^n$ for each of $w(1), \dots, w(n)$. Let $P > 0$ denote the number of rabbit jumps that occur in the time it takes us to place one trap. In the problem $P = 60$, but the argument works for any positive P. By stage n, we have placed traps at 2^n jumps along each of $w(1), \dots, w(n)$. The total time elapsed is

$$T(n) = P \cdot \frac{n(n+1)}{2}$$

Suppose the rabbit is traveling along path $w(h)$ for some $h \in \mathbb{N}$. By time $T(m)$, it has made at most $Pm(m+1)/2$ jumps. If $m \geq h$, then a trap has been placed at 2^m jumps along $w(h)$. If in addition

$$2^m > \frac{Pm(m+1)}{2}$$

then the trap lies ahead of the rabbit on its path and the rabbit will eventually reach it. Since 2^m grows exponentially while $\frac{Pm(m+1)}{2}$ grows quadratically, this inequality holds for all sufficiently large m. Therefore for large enough $m \geq h$, the rabbit is guaranteed to be caught. Thus the rabbit will always be captured in finite time. $\square$

2 Home Assignment II (Due: Sep 04, 2025)

Problem 2.1. Take

$$C = \left\{2 - \frac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{5 - \frac{1}{n} : n \in \mathbb{N}\right\}$$

Show that every nonempty subset of C has a minimal element. Determine as to whether the same property holds for D where,

$$D = \left\{3 - \frac{1}{m} - \frac{1}{n^2} : m, n \in \mathbb{N}\right\}$$

Solution. Firstly we note that every element in $\{2 - 1/n : n \in \mathbb{N}\}$ is less than every element in $\{5 - 1/n : n \in \mathbb{N}\}$, so if the subset we choose has nonempty intersection with the first set then it suffices to prove existence of a minimal element for this. Assume that the subset we choose has nonempty intersection with the first set and consider their intersection which is a subset of the first element, it suffices to show that this has a minimal element. Consider the $n \in \mathbb{N}$ for which $2 - 1/n$ is in our subset, this being a subset of the natural numbers has a minimal element and we claim that if this is n_o then $2 - 1/n_o$ is the minimal element we are after, which is clearly true as $2 - 1/n \leq 2 - 1/m$ iff $n \leq m$. Now if our chosen subset was disjoint with the first set, we can repeat the same arguement on the second set using 5 in plcae of 2. For the second part let $S \subseteq D$ be nonempty. Define

$$m_0 := \min\{m \in \mathbb{N} : \exists n \in \mathbb{N} \text{ with } 3 - \frac{1}{m} - \frac{1}{n^2} \in S\} \text{ and } n_0 := \min\{n \in \mathbb{N} : \ 3 - \frac{1}{m_0} - \frac{1}{n^2} \in S\}.$$

If $3 - 1/m_0 - 1/n_0^2$ is minimal we are done. Otherwise there exists $3 - 1/m - 1/n^2 \in S$ strictly smaller and by the minimality of m_0 we have $m \geq m_0$, hence for such m necessarily $n < n_0$ (otherwise the element would not be strictly smaller). Among those m pick the least $m_1 \geq m_0$ and any corresponding n_1, then $n_1 < n_0$. If we keep repeating this then it produces a strictly decreasing sequence $n_0 > n_1 > n_2 > \dots$ of natural numbers which can't go on forever, so we encounter a minimal element eventually. Thus a minimal element exists.

$\square$

Problem 2.2. Find the infimum and supremum of the following subsets of the real line:

$$A_1 = \left\{3 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}, \quad A_2 = \{x^2 + 1 : 0 \le x \le 1\}.$$

Solution. First we prove a *lemma* : if a nonempty set $S \subseteq \mathbb{R}$ has a maximal (or, minimal) element then the supremum (or, infimum) of S is exactly that element. For a proof assume that $\max S = m$ then we see that for all $x \in S$ we have $x \le m$ thus, m is an upper bound and $\sup S \le m$. On the other hand we know that $\sup S$ is also an upper bound and as $m = \max S \in S, m \le \sup S$. This implies that $m = \sup S$. For the infimum and minimal element its very similar. Now we see that in A_1, a maximal element exists which is $3.5 = 3 + (-1)^2/2$ as $3 + (-1)^1/1 = 2 < 3.5$ and for all $n > 2$,

$$3 + \frac{(-1)^n}{n} \le 3 + \frac{1}{n} < 3 + \frac{1}{2}$$

Thus the supremum of A_1 is 3.5. We similarly see that the minimal element of A_1 is $3 + (-1)^1/1 = 2$ as for all $n > 1$,

$$3 + \frac{(-1)^1}{1} < 3 + \frac{(-1)}{n} \le 3 + \frac{(-1)^n}{n}$$

and thus this must be the infimum. It was shown in class that for $a, b \ge 0$ the inequalities $a \ge b$ and $a^2 \ge b^2$ are equivalent, we can add 1 to both sides of the second inequality to get $a \ge b \iff a^2 + 1 \ge b^2 + 1$ for all $a, b \ge 0$ and thus we clearly see that for all x such that $0 \le x \le 1$ we have, $1 = 0^2 + 1 \le x^2 + 1 \le 1^2 + 1 = 2$ and thus 1 and 2 are the minimal and maximal elements of A_2 respectively and as shown previously they must also be the infimum and supremum respectively. □

Problem 2.3. Let A, B be non-empty, bounded subsets of $\mathbb{R}$. Define

$$A+B = \{a+b : a \in A, b \in B\}, \quad A-B = \{a-b : a \in A, b \in B\}, \quad AB = \{ab : a \in A, b \in B\}.$$

Show that these sets are bounded. Determine which of the following statements are true and which are false in general (prove your claim):

(a) $\sup(A \cup B) = \max\{\sup A, \sup B\}$. (b) $\sup(A \cap B) = \min\{\sup A, \sup B\}$. (c) $\sup(A + B) = \sup A + \sup B$. (d) $\sup(A - B) = \sup A - \sup B$. (e) $\sup(AB) = (\sup A)(\sup B)$.

Solution. Since A, B are bounded pick $M, N > 0$ with $|a| \le M$ for all $a \in A$ and $|b| \le N$ for all $b \in B$. Then for all $a \in A, b \in B$ we have $|a + b| \le M + N$, $|a - b| \le M + N$, and $|ab| \le MN$. Hence $A + B, A - B, AB$ are bounded. The proofs for $A \cup B, A \cap B$ being bounded are shown below.

(a) The claim is true. For any $x \in A \cup B$ we see that x is either in A or in B and if $x \in A$ then $|x| \le M \le \max\{M, N\}$ and if $x \in B$ then $|x| \le N \le \max\{M, N\}$ thus for all $x \in A \cup B$ we have that $|x| \le \max\{M, N\}$ so these are bounded and a supremum exists. Let $m = \sup(A \cup B)$, then for all $x \in A, B$ we have that $x \le m$ thus $m \ge \sup A, \sup B$ which implies that $m \ge \max\{\sup A, \sup B\}$. Now if $x \in A \cup B$ then $x \in A$ or $x \in B$ and if $x \in A$ then $x \le \sup A \le m$ and if $x \in B$ then $x \le \sup B \le m$ thus we see that $\sup(A \cup B) \le m$. This implies that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

(b) The claim is false. For any $x \in A \cap B$ we see that $x \in A$ so $|x| \le M$ and thus its

bounded. Now consider $A = \{0, 2\}$ and $B = \{0, 1\}$, then we have that $\min\{\sup A, \sup B\} = \min\{1, 2\} = 1 \neq 0 = \sup\{0\} = \sup A \cap B$.
(c) This claim is true. Say $a = \sup A$ and $b = \sup B$ for brevity. Then for all $x+y \in A+B$ with $x \in A, y \in B$ we see that $x \leq a$ and $y \leq b$ which implies that $x+y \leq a+b$ thus $a+b$ is an upper bound for $A+B$. Now given any $\varepsilon > 0$ we can find $x_o + y_o \in [a+b-\varepsilon, a+b]$ by choosing x_o in A and y_o in B such that $x_o \in [a - \varepsilon/2, a]$ and $y_o \in [b - \varepsilon/2, b]$ (which exist as we defined a, b to be the supremums of A, B respectively) as

$$a + b - \varepsilon = (a - \varepsilon/2) + (b - \varepsilon/2) \leq x_o + y_o \leq a + b$$

This is enough to conclude that $a + b$ is the supremum of $A + B$. (It was shown in class that if for some nonempty set $S \subseteq \mathbb{R}$, s is an upper bound such that for all $\varepsilon > 0$ we have $S \cap [s - \varepsilon, s] \neq \emptyset$ then its the supremum.)
(d) This is false. Consider $A = \{0\}$ and $B = [-1, 1]$. Then we have that $A - B = \{0 - x : x \in [-1, 1]\} = [-1, 1]$ and thus $\sup(A - B) = 1$. But as $\sup A = 0$ and $\sup B = 1$, $\sup(A - B) = 1$ is not equal to $\sup A - \sup B = 0 - 1 = -1$ disproving the statement.
(e) This is false. Consider $A = \{-1\}$ and $B = [-1, 1]$. We see that $AB = \{-x : x \in [-1, 1]\} = [-1, 1]$ and thus $\sup AB = 1$ but this is not equal to $(\sup A)(\sup B) = (-1)(1) = -1$ disproving the statement. □

Problem 2.4. Let $\{t_n\}_{n \geq 1}$ be a sequence defined by

$$t_1 = 2, \quad t_{n+1} = \tfrac{1}{2}\left(t_n + \frac{2}{t_n}\right) \quad (n \geq 1).$$

Show that $\{t_n\}$ is a convergent sequence, converging to $\sqrt{2}$.

Solution. Firstly by AM-GM inequality we see that for all $n \geq 1$,

$$t_{n+1} = \frac{t_n + 2/t_n}{2} \geq \sqrt{t_n \cdot \frac{2}{t_n}} = \sqrt{2}$$

and $t_1 = 2 \geq \sqrt{2}$ as well so the sequence is bounded below by $\sqrt{2}$. From $t_n \geq \sqrt{2}$ we have, $t_n^2 \geq 2$ which implies that $t_n \geq 2/t_n$. Using this, for all $n \geq 1$ we have,

$$t_{n+1} = \frac{t_n + 2/t_n}{2} \leq \frac{t_n + t_n}{2} = t_n$$

i.e. $\{t_n\}_{n \geq 1}$ is a decreasing sequence. As this is also bounded below, by a theorem proved in class (all decreasing sequences that are bounded above are convergent) this must converge to some limit L and we must also have that $L \neq 0$ as $L \geq \sqrt{2}$ by another theorem proved in class (if $\{t_n\}_{n \geq 1}$ is bounded below by m and it converges, then $\lim_{n\to\infty} x_n \geq m$ as well.) Taking the limit on both sides and using the algebra of limits of sequences as shown in class we have,

$$L = \lim_{n\to\infty} t_n = \lim_{n\to\infty} t_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(t_n + \frac{2}{t_n}\right) = \frac{1}{2}\left(\lim_{n\to\infty} t_n + \frac{2}{\lim_{n\to\infty} t_n}\right) = \frac{1}{2}\left(L + \frac{2}{L}\right)$$

which is a quadratic in L with solutions $\pm\sqrt{2}$ and as $L \geq \sqrt{2}$ it must be $\sqrt{2}$. We have thus shown that it converges to $\sqrt{2}$. □

Problem 2.5. Show that
$$\lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n$$
exists. (Hint: Prove that it is a monotone bounded sequence.)

Solution. For all $n \geq 1$, by the binomial theorem we have that,
$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} = \sum_{k=0}^{n}\frac{n(n-1)\dots(n-k+1)}{k!}\frac{1}{n^k} \leq \sum_{k=0}^{n}\frac{n^k}{k!}\cdot\frac{1}{n^k} \leq \sum_{k=0}^{n}\frac{1}{k!}$$
$$= 1+\frac{1}{1}+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\dots \leq 1+\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{2\cdot 2}+\frac{1}{2\cdot 2\cdot 2}+\dots\right) = 1+2 = 3$$
Thus the sequence is bounded above by 3. Using the AM-GM inequality we see that,
$$\frac{1+\underbrace{\left(1+\frac{1}{n}\right)+\dots+\left(1+\frac{1}{n}\right)}_{\text{n times}}}{n+1} \geq \left(1\cdot\underbrace{\left(1+\frac{1}{n}\right)\cdot\dots\cdot\left(1+\frac{1}{n}\right)}_{\text{n times}}\right)^{\frac{1}{n+1}}$$
simplifying this we see that its just the inequality,
$$1+\frac{1}{n+1} \geq \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}$$
and now raising both sides to the exponent of $n+1$ this reads as
$$\left(1+\frac{1}{n+1}\right)^{n+1} \geq \left(1+\frac{1}{n}\right)^n$$
so the sequence is increasing. Now as shown in class we can conclude that its convergent from the fact that its increasing and bounded above. $\square$

Problem 2.6. Show that there exists a unique positive real number x such that $x^3 = 2$.

Solution. Let $S := \{x \in \mathbb{R} : x^3 < 2\}$. The set S is nonempty because $1^3 = 1 < 2$ so $1 \in S$. It is bounded above because if $x \geq 2$ then $x^3 \geq 8 > 2$, hence $x \notin S$, so every $x \in S$ satisfies $x < 2$ and thus 2 is an upper bound. By the completeness axiom, $s := \sup S$ exists. We claim $s^3 = 2$.

Suppose first that $s^3 < 2$. For any $\varepsilon > 0$ with $\varepsilon \leq 1$ we expand
$$(s+\varepsilon)^3 = s^3 + 3s^2\varepsilon + 3s\varepsilon^2 + \varepsilon^3 \leq s^3 + (3s^2+3s+1)\varepsilon.$$
Since $s^3 < 2$, choose $\varepsilon > 0$ so small that $(3s^2+3s+1)\varepsilon < 2 - s^3$. Then $(s+\varepsilon)^3 < 2$, so $s+\varepsilon \in S$. But $s+\varepsilon > s$, contradicting the definition of s as the least upper bound of S. Therefore $s^3 \geq 2$.

Suppose instead that $s^3 > 2$. For any $\varepsilon > 0$ with $\varepsilon \leq 1$ we expand
$$(s-\varepsilon)^3 = s^3 - 3s^2\varepsilon + 3s\varepsilon^2 - \varepsilon^3 \geq s^3 - (3s^2+1)\varepsilon.$$
Since $s^3 > 2$, choose $\varepsilon > 0$ so small that $(3s^2+1)\varepsilon < s^3 - 2$. Then $(s-\varepsilon)^3 > 2$ and as s is the supremum there exists some $x \in S$ such that $s - \varepsilon \leq x \leq s$ which implies that $2 < (s-\varepsilon)^3 \leq x^3 < 2 \implies 2 < 2$, a contradiction, thus $s^3 \leq 2$. Combining both

inequalities we conclude $s^3 = 2$, so such a real number exists. For uniqueness, suppose $a^3 = b^3 = 2$ with $a, b > 0$. Then $(a-b)(a^2+ab+b^2) = a^3 - b^3 = 0$. Since $a^2+ab+b^2 > 0$, we must have $a - b = 0$, so $a = b$. Hence there exists exactly one positive real s with $s^3 = 2$. □

Problem 2.7. Prove that the following sequences are convergent:

$$a_n = \prod_{k=1}^{n}\Big(1 - \frac{1}{k+1}\Big), \quad n \in \mathbb{N};$$

$$b_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}, \quad n \in \mathbb{N}.$$

Solution. For all $n \in \mathbb{N}$,

$$a_n = \prod_{k=1}^{n}\left(1 - \frac{1}{k+1}\right) = \prod_{k=1}^{n}\frac{k}{k+1} = \frac{1}{2}\cdot\frac{2}{3}\cdot\ldots\cdot\frac{n}{n+1} = \frac{1}{n+1}$$

Thus $a_n = 1/(n+1)$ and it converges to $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/(n+1) = 0$ so its convergent. We will show that b_n is bounded above and increasing, this is sufficient as we can conclude convergence from this as discussed in class. For boundedness,

$$b_n = \sum_{k=1}^{n}\frac{1}{n+k} \leq \sum_{k=1}^{n}\frac{1}{n+1} = \frac{n}{n+1} < 1$$

so its bounded above by 1. To prove that its increasing its sufficnent to show that $b_{n+1} - b_n > 0$ for all $n \in \mathbb{N}$, if we write these out we have,

$$b_{n+1} - b_n = \left(\frac{1}{n+2} + \frac{1}{n+3}\ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$$

$$= \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1} > \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1} = 0$$

so we are done. □

Problem 2.8. Prove that the sequence

$$c_n = 5 + (-1)^n\Big(2 + \frac{1}{n}\Big)$$

is not convergent.

Solution. Consider the subsequences $\{c_{2k}\}_{k\geq 1}$ and $\{c_{2k-1}\}_{k\geq 1}$. We have,

$$c_{2k-1} = 5 + (-1)^{2k-1}(1 + 1/(2k-1)) = 5 - 1 - 1/(2k-1) = 4 - 1/(2k-1)$$

taking the limits,

$$\lim_{k\to\infty} c_{2k-1} = \lim_{k\to\infty}(4 - 1/(2k-1)) = \lim_{k\to\infty} 4 + (-1)\cdot\lim_{k\to\infty} 1/(2k-1) = 4 - 0 = 4$$

and similarly for the other subsequence we have

$$c_{2k} = 5 + (-1)^{2k}(1 + 1/2k) = 5 + 1 + 1/2k = 6 + 1/2k$$

again taking the limits

$$\lim_{k\to\infty} c_{2k} = \lim_{k\to\infty}(6+1/2k) = \lim_{k\to\infty} 6 + \lim_{k\to\infty} 1/2k = 6+0=6$$

So we have two subsequences with different limits which implies that the sequence can't converge as then they'd have the same limits by the theorem: all subsequences of a converging sequence also converge to the same limit as that sequence (this was shown in class). □

Problem 2.9. Suppose $\{x_n\}$ is a real sequence. For $n \geq 1$ define the averages

$$y_n = \frac{1}{n}\sum_{k=1}^{n} x_k.$$

Show that if $\{x_n\}$ converges, then $\{y_n\}$ also converges. However, the converse is not true.

Solution. Assume that x_n converges to x. Then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|x - x_n| < \varepsilon$. Now for $M > N$, using the triangle inequality we have,

$$\left| x - \frac{1}{M}\sum_{k=1}^{M} x_k \right| = \frac{1}{M}\left|\sum_{k=1}^{M}(x-x_k)\right| \leq \frac{1}{M}\left|\sum_{k=1}^{N}(x-x_k)\right| + \frac{1}{M}\sum_{k=N+1}^{M}|x-x_k|$$

$$\leq \frac{1}{M}\left|\sum_{k=1}^{N}(x-x_k)\right| + \frac{(M-N)\varepsilon}{M} \leq \frac{1}{M}\left|\sum_{k=1}^{N}(x-x_k)\right| + \varepsilon$$

As the quantity $G = |\Sigma_{k=1}^{N}(x-x_k)|$ is fixed we can find $M' \in \mathbb{N}$ such that $G/M' < \varepsilon$ and clearly for all $K > M'$ we have that $G/K \leq G/M' < \varepsilon$. Thus for all $K > M'$ we have,

$$\left| x - \sum_{k=1}^{K} x_k \right| \leq \frac{G}{K} + \varepsilon \leq \frac{G}{M'} + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon$$

which can be made arbitrarily small and thus we see that for all $\varepsilon > 0$ there exists $M' \in \mathbb{N}$ such that for all $K > M'$ we have $|y_n - x| < \varepsilon$ so by the definition of convergence, y_n also converges to x. We provide a counterexample for the converse i.e. a sequence $\{x_n\}$ that does not converge but for which $\{y_n\}$ does. Such an example is the sequence $\{x_n\}$ with $x_{2n} := 1, x_{2n-1} := 0$ for $n \in \mathbb{N}$, which clearly does not converge but the average,

$$y_n = \frac{1}{n}\sum_{k=1}^{n} x_k = \frac{\lfloor n/2 \rfloor}{n}$$

does converge which we can conclude from the fact that its bounded on both sides by convergent sequences that converge to the same limit as follows,

$$\text{(has limit half as well clearly) } 1/2 - 1/n = \frac{\frac{n}{2}-1}{n} \leq y_n \leq \frac{n/2}{n} = 1/2 \text{ (constant sequence)}$$

where the inequalities are true as for any real x, the integer part $\lfloor x \rfloor \in (x-1, 1]$. As we provided a counterexample to the converse it can't be true. □

Problem 2.10. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(x) + f(y), \quad f(xy) = f(x)f(y)$$

for all $x, y \in \mathbb{R}$. (Hint: You may need order properties and completeness of $\mathbb{R}$.)

Solution. The only solutions are the identity function and the identically zero function. The proof works solely because of completeness axiom and the order defined on the reals which directly or indirectly implies the existence of roots down the line which is crucial to this proof.

Firstly we see that $f(1) = f(1 \cdot 1) = f(1)^2 \implies f(1) = 0$ or, 1 and if its 0 then everything is i.e. for all x we would have $f(x) = f(x \cdot 1) = f(x) \cdot f(1) = f(x) \cdot 0 = 0$ so f would be identically zero. We work with the case where $f(1) = 1$ now. From the definition, $f(0) = f(0+0) = f(0) + f(0) \implies f(0) = 0$ and this is the only element that is mapped to zero as otherwise if we had some nonzero v being mapped to zero, this would imply that, for all $f(1) = f(v \cdot (1/v)) \cdot f(v) = f(1/v) \cdot 0 = 0$ which is not possible as we assumed $f(1)$ to be 1. We also see that $f(-x) = f((-1) \cdot x) = f(-1) \cdot f(x)$ and as $f(-1) + f(1) = f(-1+1) = f(0) = 0 \implies f(-1) = -1$ we see that $f(-x) = -f(x)$, this shows us that f is uniquely determined by its values on the positive reals. Now for positive integers n,

$$f(n) = f\left(\underbrace{1 + \ldots + 1}_{\text{n times}}\right) = \underbrace{f(1) + \ldots + f(1)}_{\text{n times}} = \underbrace{1 + \ldots + 1}_{\text{n times}} = n$$

and we can generalise this to negative integers too using the fact that $f(-n) = -f(n) = -n$. As $f(0) = 0$ too we see that f is the identity on the integers. Now for nonzero integers n and any integer m we see that,

$$f(m/n) \cdot n = f(m/n) \cdot f(n) = f(m) = m \implies f(m/n) = m/n$$

thus f acts as the identity function on the rationals as well. Now for positive x we can see that $f(x) = f(\sqrt{x} \cdot \sqrt{x}) = f(\sqrt{x})^2 > 0$ (the existence of these $\sqrt{x}$ can be proven just like in Problem 2.6 or, by using the intermediate value theorem on the continous function $x \mapsto x^2$ which was all discussed in class. The strict inequality is due to $\sqrt{x}$ being nonzero for nonzero x.) We will now show that $f(x) = x$ for all $x \in \mathbb{R}$. For the sake of contradiction assume that $f(k) < k$ for some $k \in \mathbb{R}$ i.e. $f(k) = k - \varepsilon$ for some $\varepsilon > 0$. Now for any rationals $r < k$ i.e. $k - r > 0$ we must have that $0 < f(k - r) = f(k) - f(r) = f(k) - r = k - r - \varepsilon \implies \varepsilon < k - r$ which can't be true as we can choose a rational r in $(k - \varepsilon/2, k)$ (we have shown in class that any open interval in the reals contains atleast one rational) for which we would have $k - r < \varepsilon/2$ but this implies that $\varepsilon < \varepsilon/2$ which is false for positive ε and we have a contradiction so $f(k) \geq k$ for all reals k. Now again, with the goal of showing that $f(k) \leq k$ for all reals k, for the sake of contradiction assume that for some real k we have $f(k) > k$ i.e. $f(k) = k + \varepsilon$ for some $\varepsilon > 0$. This time consider rationals $r > k$ i.e. $r - k > 0$. For these we have that, $0 < f(r - k) = f(r) - f(k) = r - f(k) = r - k - \varepsilon \implies r - k > \varepsilon$ which again can't be true as we can similarly choose some rational $r \in (k, k + \varepsilon/2)$ for which we would have $r - k < \varepsilon/2$ implying that $\varepsilon/2 < \varepsilon$ which is false for positive ε and thus we have a contradiction so $f(k) \leq k$ for all reals k. As we showed that $k \leq f(k) \leq k$ for all reals k we have proved that $f(k) = k$ for all reals k and we are done. So the functions f satisfying the conditions of the problem are exactly the identity function i.e $x \mapsto x$ and the identically zero function i.e. $x \mapsto 0$. $\square$

3 Home Assignment III (Due: Oct 10, 2025)

Problem 3.1. *A binary expansion $0.b_1b_2b_3\dots$ is said to be periodic if there exist natural numbers M and p such that $b_{k+p} = b_k$ for all $k \geq M$. Show that a binary expansion of a real number in the interval $[0, 1]$ is periodic if and only if it is a rational number. Use this to show that the set of all rational numbers in $[0, 1]$ is countable.

Solution. For one direction, let $x \in [0, 1]$ have a periodic binary expansion $0.b_1b_2b_3\dots$. By definition, there exist natural numbers M and p such that $b_{k+p} = b_k$ for all $k \geq M$. We can express x as the sum of its non-periodic and periodic components as follows,

$$x = \sum_{k=1}^{M-1} \frac{b_k}{2^k} + \sum_{k=M}^{\infty} \frac{b_k}{2^k}$$

The first sum is a finite sum of rational numbers and thus its rational itself. Now for the second sum, the periodic part can be rearranged and summed in this manner

$$\frac{b_M}{2^M} + \frac{b_{M+1}}{2^{M+1}} + \dots = \left(\sum_{j=1}^{p} \frac{b_{M+j-1}}{2^{M+j-1}}\right) + \left(\sum_{j=1}^{p} \frac{b_{M+j-1}}{2^{M+j+p-1}}\right) + \dots$$

$$= \left(\sum_{j=1}^{p} \frac{b_{M+j-1}}{2^{M+j-1}}\right)\left(1 + \frac{1}{2^p} + \frac{1}{2^{2p}} + \dots\right)$$

The first factor is rational, being a finite sum of rationals and the second factor is a geometric series converging to $1/(1-2^{-p})$ so the whole thing is rational, being a product of two rationals. As the periodic and nonperiodic parts are both rational, the original number being a sum of those two is also rational.

For the second part, let $x = p/q$ be our rational, here $p \in \mathbb{Z}_{\geq 0}$ and $q \in \mathbb{N}$. Suppose this has binary expansion,

$$x = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots$$

Where $b \in \{0, 1\}$. multiplying by two and taking the floor,

$$\lfloor 2x \rfloor = \left\lfloor b_1 + \frac{b_2}{2} + \frac{b_3}{2^2} + \dots \right\rfloor = b_1 + \Bigg\lfloor \underbrace{\frac{b_2}{2} + \frac{b_3}{2^2} + \frac{b_4}{2^3} + \dots}_{\star\star} \Bigg\rfloor = b_1 + 0 = b_1$$

($\star\star$) This is because the sum $\frac{b_2}{2} + \frac{b_3}{4} + \dots$ satisfies $0 \leq S \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ and the case where the sum equals 1 (e.g., $0.0111\ldots = 0.1$) corresponds to a number that has a terminating binary expansion. The algorithm, by its nature, produces the terminating expansion, ensuring the remainder is always strictly less than 1. Therefore, the sum is in the interval $[0, 1)$, and its floor is 0. Now look at $\{2x\} = 2x - \lfloor 2x \rfloor$, the fractional part. This is,

$$\{2x\} = 2x - \lfloor 2x \rfloor = \left(b_1 + \frac{b_2}{2} + \frac{b_3}{4} + \dots\right) - b_1 = \frac{b_2}{2} + \frac{b_3}{4} + \frac{b_4}{8} + \dots$$

And we can keep repeating what we did to x initially to get the bits b_i one by one following this recurrence,

$$b_1 = \lfloor 2x \rfloor, r_1 = \{2x\} \text{ and } r_k = \{2r_{k-1}\}, b_k = \lfloor 2r_{k-1} \rfloor \text{ for } k \geq 2$$

Writing $x = p/q$ again we see that,

$$b_1 = \left\lfloor \frac{2p}{q} \right\rfloor, r_1 = \frac{2p}{q} - \left\lfloor \frac{2p}{q} \right\rfloor = \frac{2p \pmod q \text{(nonnegative remainder upon division by } q)}{q}$$

Again,

$$r_2 = \{2 \cdot r_1\} = \frac{2 \times (2p \pmod q) \pmod q}{q}$$

this keeps going on and on. We can see that all the r_k are fractions of form d/q where $0 \leq d < q$. But as there are only finitely many such d, there must exist some two r_k and $r_{k+M}(M > 0)$ which are equal as their numerator must be the same (we can not keep getting unique numerators as there are only a finite amount of them, infact only q of them.) Now we see that, $r_k = r_{k+M} \implies b_{k+1} = \lfloor 2 \cdot r_k \rfloor = \lfloor 2 \cdot r_{k+M} \rfloor = b_{k+M+1}$ and also $r_{k+1} = \{2 \cdot r_k\} = \{2 \cdot r_{k+M}\} = r_{k+M+1}$ which in turn again implies $b_{k+2} = b_{k+M+2}$ and this keeps going on and on. We thus see that for all $n \geq k$ we have $r_n = r_{n+M}$ i.e. its periodic and we are done.

Now let S be the set of all periodic binary expansions. Every $x \in \mathbb{Q} \cap [0, 1]$ corresponds to some periodic binary expansion as we showed above, thus there is a surjection from S to $\mathbb{Q} \cap [0, 1]$ and to prove that $\mathbb{Q} \cap [0, 1]$ is countable, its sufficient to show that S is countable.

An element of S is uniquely defined by a finite amount of information: a pre-period of length $M - 1$ and a period of length p. Let $S_{M,p}$ be the set of all such expansions for fixed $M, p \in \mathbb{N}$. Any expansion in $S_{M,p}$ is determined by its first $M + p - 1$ bits. There are exactly 2^{M+p-1} such sequences, so $S_{M,p}$ is a finite set.

The total set of all periodic expansions is the union over all possible lengths:

$$S = \bigcup_{(M,p) \in \mathbb{N} \times \mathbb{N}} S_{M,p}$$

This is a countable union of finite sets as the index set $\mathbb{N} \times \mathbb{N}$ is countable. By Problem 1.6 we can say that S is countable.

Since there is a surjection from the countable set S onto $\mathbb{Q} \cap [0, 1]$, we conclude that $\mathbb{Q} \cap [0, 1]$ must be countable. $\square$

Problem 3.2. Determine lim sup and lim inf of the following sequences of real numbers:

(i) $\{(-1)^n 5 + (-\frac{1}{2})^n 7 : n \in \mathbb{N}\}$.

(ii) $\{\frac{n}{2^n} - \frac{n}{3^n} : n \in \mathbb{N}\}$.

(iii) $\{\frac{n+6}{n^2-2n-8} : n \in \mathbb{N}\}$.

Solution. In 1, let the terms of the sequence be a_n, now partition the whole sequence $\{a_n\}$ into two subsequences $\{a_{2n}\}$ and $\{a_{2n-1}\}$. Clearly one converges to 5 whereas the other to -5. Now as this is a partition, any convergent subsequence of this sequence $\{a_n\}$ must be eventually always a subsequence of $\{a_{2n}\}$ or $\{a_{2n-1}\}$ as alternating between these two infinitely often will not allow the subsequence in question to converge as both these parts have different limits. Thus the only limit points are these two i.e. 5 and -5 because every convergent subsequence of the sequence is eventually always a subsequence of one of the parts stated above, which converge and thus so do these subsequences, to the same limit as these parts. Thus,

$$\limsup_{n\to\infty} a_n = \sup\{+5, -5\} = +5 \text{ and, } \liminf_{n\to\infty} a_n = \inf\{+5, -5\} = -5$$

Using the fact proven in class that the limit supremum is the supremum of the subsequential limits and the limit infimum, the infimum of the same.

Both the sequences in 2 and 3 converge to 0 as exponential growth dominates polynomial growth as shown in class and n^2 dominates n as well. As these are convergent sequences, they have the same limit extremums and limits all of which are 0. $\square$

Problem 3.3. Suppose $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ are two bounded sequences of real numbers. Show that $\{a_n + b_n\}_{n\in\mathbb{N}}$ is a bounded sequence of real numbers and

(roman) $\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(a_n + b_n)$;

(roman) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

Give examples to show that equality may not hold in (i) or (ii).

Solution. If a_n and b_n are bounded sequence, there exists $A, B \geq 0$ such that $|a_n| \leq A$ and $|b_n| \leq B$ for all n. Now by the triangle inequality we have that for all n,

$$|a_n + b_n| \leq |a_n| + |b_n| \leq A + B$$

so $a_n + b_n$ is also bounded. For 1, by definition,

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n = \lim_{n\to\infty} \inf_{m\geq n} a_m + \lim_{n\to\infty} \inf_{m\geq n} b_m = \lim_{n\to\infty} \left(\inf_{m\geq n} a_m + \inf_{m\geq n} b_m \right)$$

Now we see that, $\inf_{m\geq n} a_m + \inf_{m\geq n} b_m \leq a_m + b_m$ for all $m \geq n$ which by the definition of the infimum implies that, for all n,

$$\inf_{m\geq n} a_m + \inf_{m\geq n} b_m \leq \inf_{m\geq n} (a_m + b_m)$$

Combined with the previous equality we get,

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n = \lim_{n\to\infty} \left(\inf_{m\geq n} a_m + \inf_{m\geq n} b_m \right) \leq \lim_{n\to\infty} \inf_{m\geq n} (a_m + b_m) = \liminf_{n\to\infty}(a_n + b_n)$$

This proves the desired inequality. To prove the statement for limit supremums it suffices to consider sequences $-a_n, -b_n$ as follows,

$$\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = -\left(\liminf_{n\to\infty}(-a_n) + \liminf_{n\to\infty}(-b_n) \right)$$

$$\geq -\liminf_{n\to\infty}(-(a_n + b_n)) = \limsup_{n\to\infty}(a_n + b_n)$$

Where we used the above proven inequality and also this fact proven in class: For bounded sequences of real numbers $\{a_n\}_{n\in\mathbb{N}}$ we have,

$$\limsup_{n\to\infty} a_n = -\liminf_{n\to\infty}(-a_n)$$

Here equality might not occur in either case, for example when $a_n = (-1)^n$ and $b_n = -a_n$, in this case we see that

$$-2 = -1 + -1 = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n < \liminf_{n\to\infty} (a_n + b_n) = 0$$

$$2 = 1 + 1 = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n > \limsup_{n\to\infty} (a_n + b_n) = 0$$

□

Problem 3.4. Show that a continuous function $g : [0, 1] \to (0, 1)$ can not be surjective. Give an example of a surjective continuous function $h : (0, 1) \to [0, 1]$.

Solution. Functions g as in the question, by the extreme value theorem will furnish some point $m \in [0, 1]$ such that $f(m)$ is the maximum value f reaches. Now as per the question we must have that $0 < f(m) < 1$ but this means that f misses all the values in the nonempty interval $(f(m), 1)$ by the maximality of $f(m)$, so it can't be surjective. An example of a continous surjection $h : (0, 1) \to [0, 1]$ is,

$$h(x) := \begin{cases} 0 \text{ if } x < 1/3 \\ \frac{x-1/3}{2/3-1/3} \text{ if } 1/3 \leq x \leq 2/3 \\ 1 \text{ otherwise} \end{cases}$$

□

Problem 3.5. Let $k : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists $t \in [0, 1]$ such that $k(t) = 1 - t^2$.

Solution. Consider the continuous (this is continous as its the sum of two continuous functions) $f(x) := k(x) - (1 - x^2)$. By the definition of k, for all $x \in [0, 1]$ we must have $0 \leq k(x) \leq 1$ thus,

$$f(0) = k(0) - (1 - 0^2) = k(0) - 1 \leq 0 \text{ and } f(1) = k(1) - (1 - 1^2) = k(1) \geq 0$$

Thus by the intermediate value theorem done in class, for all $c \in [f(0), f(1)] \supseteq [0, 0] = \{0\}$ there exists some $t \in [0, 1]$ such that $f(t) = c$. Choosing $c = 0$, there exists some t such that $f(t) = 0$ i.e. $k(t) - (1 - t^2) = 0$ which upon rearrangement becomes $k(t) = 1 - t^2$ and we are done. □

Problem 3.6. Prove or disprove: Suppose A is a non-empty subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ has the property that $\{f(a_n)\}_{n\in\mathbb{N}}$ is Cauchy whenever $\{a_n\}_{n\in\mathbb{N}}$ is Cauchy in A. Then f is uniformly continuous.

Solution. This property does not hold. For a counterexample consider $f : x \mapsto x^2$ in $\mathbb{R}$, this was shown to not be uniformly continuous in class. If $\{x_n\}$ is a cauchy sequence then it is convergent in $\mathbb{R}$ (this was shown in class) and as $x \mapsto x^2$ is continous we see that $\{f(x_n)\}$ also converges to $f(\lim_{n\to\infty} x_n)$. But as all convergent sequences are cauchy we see that $\{f(x_n)\}$ is cauchy too and we are done as we found a non uniformly continous function that has said property. □

Problem 3.7. Let A be a non-empty subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ be uniformly continuous functions. Prove or disprove:

(i) $af + bg$ is uniformly continuous for every $a, b \in \mathbb{R}$.

(ii) fg is uniformly continuous on A.

(iii) If $g(x) \neq 0$ for every $x \in A$, then $\frac{f}{g}$ is uniformly continuous on A.

Solution. Only 1 is true and the rest are false, we prove our claim regarding the rest first. For 2 consider f, g to be the identity on $A = \mathbb{R}$. Here f, g are clearly uniformly continuous whereas $(f \cdot g)(x) = x^2$ is not uniformly continuous. For 3 we can take $A = (0, +\infty)$ and set $f(x) = 1$ on A, a constant function and then set $g(x) = x$ on A. Here again f, g are both uniformly continuous and $g \neq 0$ but $(f/g)(x) = 1/x$ is not uniformly continous as shown in class. Now we will prove 1. Given any $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $x, y \in A$ and $|x - y| < \delta_1 \implies |f(x) - f(y)| < \varepsilon$ and $|x - y| < \delta_2 \implies |g(x) - g(y)| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$ then we see that, $x, y \in A$ and $|x - y| < \delta$ implies,

$$|af(x) + bg(x) - (af(y) + bg(y)| \leq |af(x) - af(y)| + |bg(x) - bg(y)| < |a| \cdot \varepsilon + |b| \cdot \varepsilon$$

using the triangle inequality. If at first, instead of choosing $\varepsilon > 0$ we had chosen $\varepsilon/(|a| + |b|) > 0$ (assuming that not both of a, b are zero, in that case it would be trivial as it'd be the zero function which is constant and thus clearly uniformly continous.) then we would have gotten

$$|(af(x) - bg(x)) - (af(y) + bg(y))| < \varepsilon$$

which suffices to prove uniform continuity as $\varepsilon > 0$ was arbitrary and we found some $\delta > 0$ such that $x, y \in A$ and $|x - y| < \delta$ implies the above inequality. □

Problem 3.8. Prove or disprove: Let A, B be non-empty subsets of $\mathbb{R}$. Suppose $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are uniformly continuous functions such that $f(A) \subseteq B$. Then $h = g \circ f$ is uniformly continuous.

Solution. By definiton, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in B$ and $|x - y| < \delta \implies |g(x) - g(y)| < \varepsilon$ and again there exists $\delta_* > 0$ such that $x, y \in A$ and $|x - y| < \delta_* \implies |f(x) - f(y)| < \delta$. Now when $x, y \in A$ and $|x - y| < \delta_*$ we have that $|f(x) - f(y)| < \delta \implies |g(f(x)) - g(f(y))| < \varepsilon$. Thus we have proven that $g \circ f$ is also uniformly continuous. □

Problem 3.9. Show that $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x^3 - 5x$, $x \in \mathbb{R}$ is not uniformly continuous.

Solution. We have to show that there exists some $\varepsilon > 0$ such that for all $\delta > 0$ there exists x, y with $|x - y| < \delta$ such that $|f(x) - f(y)| \geq \varepsilon$. Choose $\varepsilon = 1$ and look at $|p(x+\delta) - p(x)| = |(x+\delta/2)^3 - x^3 - 5(x+\delta/2) + x| = |1.5x^2\delta + \text{lower order terms}|$, this is clearly dominated by the $1.5\delta x^2$ term for large x and is thus unbounded so there must be some x for which its atleast 1, no matter what $\delta > 0$ we start with. As $|x - (x+\delta/2)| < \delta$ we could clearly find such points given any δ so we are done. □

Problem 3.10. Give an example of a bijection $f : \mathbb{R} \to \mathbb{R}$ where f is differentiable at every point in $\mathbb{R}$ but f^{-1} is not differentiable at some point. (Hint: Think of polynomials). Prove your claim.

Solution. Such a function is $f(x) = x^3$, its well known to be a bijection and its differentiable as all polynomials are. Its inverse is $g(x) = x^{1/3}$ and it is not differentiable at 0 as the limit

$$\lim_{x\to 0} \frac{x^{1/3} - 0^{1/3}}{x - 0} = \lim_{x\to 0} x^{-2/3}$$

clearly doesn't exist. $\square$

4 Home Assignment IV (Due: Oct 26, 2025)

Problem 4.1. Let $g : [0,1] \to \mathbb{R}$ be a function such that g is strictly increasing and satisfies the intermediate value property, that is, for any y with $g(0) < y < g(1)$, there exists $0 < x < 1$ such that $g(x) = y$. Show that g is continuous.

Solution. Fix some $\varepsilon > 0$ and $x \in (0,1)$ (continuity at the ends can be proved in the exact same manner as below by considering points on only 'one side' of the interval, depending on which end we look at) as in the usual definition of continuity. By the problem statement $g(0) < g(x) < g(1)$ so clearly there exists some $0 < \varepsilon_o < \varepsilon$ such that

$$g(0) < g(x) - \varepsilon_o < g(x) < g(x) + \varepsilon_o < g(1)$$

Again by the problem statement, there exists x_l, y_l such that,

$$g(x_l) = g(x) - \varepsilon_o, g(x_r) = g(x) + \varepsilon_o$$

and by strict increasing(ness) of $g, x_l < x < x_r$. Now let $\delta > 0$ be such that,

$$x_l < x - \delta < x < x + \delta < x_r$$

We will show that this δ works as in the definition of continuity. Given any $y \in (x-\delta, x+\delta)$ (which is just $|y - x| < \delta$ rephrased) by the above inequality we see that

$$x_l < x - \delta < y < x + \delta \implies g(x) - \varepsilon_o < g(x_l) < g(y) < g(x_r) = g(x) + \varepsilon_o$$

which implies that $|g(x) - g(y)| < \varepsilon_o < \varepsilon$ and we are done. $\square$

Problem 4.2. Let $h : [0,1] \to [0,1]$ be a continuous function. A real number $x \in [0,1]$ is said to be a fixed point of h, if $h(x) = x$. Suppose h is a strict contraction that is: $|h(x) - h(y)| < c|x - y|$ for all $x \neq y$ in $[0,1]$ for some $0 < c < 1$. Given $a \in [0,1]$ define a sequence $\{a_n\}_{n\geq 1}$ by $a_1 = a$ and $a_{n+1} = h(a_n)$. Show that $\{a_n\}_{n\geq 1}$ is a convergent sequence and the limit is a fixed point of h.

Solution. This is a case of the banach fixed point theorem. We will prove that the sequence $\{a_n\}$ is cauchy which will imply that its convergent in $\mathbb{R}$, with limit being a fixed point as

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} h(a_n) = h\left(\lim_{n\to\infty} a_n\right) = h(L)$$

where we used the continuity of h (can be easily shown, take $\delta = \varepsilon/c > 0$ given $\varepsilon > 0$) in the fourth equality. We ease the strict equality $|h(x) - h(y)| < c|x - y|$ for $x \neq y$ to

$|h(x)-h(y)| \leq c|x-y|$ which allows us to extend it into including the case of $x = y$. For any $n \geq 2$ we see that,

$$|a_{n+1} - a_n| = |h(a_n) - h(a_{n-1})| \leq c|a_n - a_{n-1}| \leq \dots$$

proceeding inductively we have, for all $n \geq 2$

$$|a_{n+1} - a_n| \leq c^{n-2}|a_2 - a_1|$$

For some large $N \in \mathbb{N}$ we see that for distinct $m, n > N$ (wlog $m > n$),

$$|a_m - a_n| = \left|\sum_{k=0}^{m-n-1} (a_{m-k} - a_{m-k-1})\right| \leq \sum_{k=0}^{m-n-1} |a_{m-k} - a_{m-k-1}| \leq \sum_{k=0}^{m-n-1} c^{m-k-3}|a_2 - a_1|$$

$$\implies |a_m - a_n| \leq |a_2 - a_1| \sum_{k=n-2}^{m-3} c^k \leq |a_2 - a_1| \sum_{k=n-2}^{\infty} c^k = \left(\frac{|a_2 - a_1|}{1-c}\right) \cdot c^{n-2}$$

As $n > N$ and $0 < c < 1$ we also see that,

$$|a_m - a_n| < \left(\frac{|a_2 - a_1|}{1-c}\right) \cdot c^{N-2}$$

because $c^{N-2} > c^{n-2}$ as $c \in (0,1)$. Now given any $\varepsilon > 0$, as the right hand side clearly tends to 0 as N tends to infinity, we can find some $M > 0$ such that (that constant)$c^M < \varepsilon$ and by the previous inequality, for all $m, n > M$ we must have

$$|a_m - a_n| < \varepsilon$$

so its cauchy and we are done. □

Problem 4.3. For any two continuous functions f, g on $[0,1]$ define

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}.$$

Show the triangle inequality:

$$d(f,g) \leq d(f,h) + d(h,g),$$

for any three continuous functions f, g, h on $[0,1]$.

Solution. We know that continuous functions over closed sets are bounded as proven in class so f, g, h are all bounded. For continuous functions ν on $[0,1]$ define,

$$\|\nu\|_\infty := \sup\{|\nu(x)| : x \in [0,1]\} < +\infty \text{ (as its bounded)}$$

As $|\nu(x)| \leq \|\nu\|_\infty$ for all $x \in [0,1]$ and similarly $|\mu| \leq \|\mu\|_\infty$ for another continuous μ on $[0,1]$ we see that

$$|\nu(x) + \mu(x)| \leq |\nu(x)| + |\mu(x)| \leq \|\nu\|_\infty + \|\mu\|_\infty$$

Now by the definition of the supremum we must have,

$$\|\nu + \mu\|_\infty \leq \|\nu\|_\infty + \|\mu\|_\infty$$

and the problem statement is deduced from the case where

$$\mu = f - h, \nu = h - g \text{ are continuous functions on } [0, 1]$$

using the fact that for continous ν, μ on $[0, 1]$ we have $d(\mu, \nu) = \|\nu - \mu\|_\infty$ as follows,

$$d(f, g) = \|f - g\|_\infty = \|(f - h) + (h - g)\|_\infty \leq \|f - h\|_\infty + \|h - g\|_\infty = d(f, h) + d(h, g)$$

$\square$

Problem 4.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$. Show that there exists $d \in \mathbb{R}$ such that $f(x) = dx$ for all $x \in \mathbb{R}$.

Solution. Just as in Problem 2.10. we can show that for any rational r we have $f(r) = f(1) \cdot r$ and as given any real number x there exists a sequence of rationals $\{x_n\}$ converging to it we see that,

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(1) \cdot x_n = f(1) \cdot \lim_{n\to\infty} x_n = f(1) \cdot x$$

where we used the continuity of f in the second inequality and a basic fact from the algebra of limits in the fourth. Thus we see that $d = f(1)$ and $f(x) = f(1) \cdot x$ for all $x \in \mathbb{R}$ $\square$

Problem 4.5. Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(x) = h(5x)$ for all $x \in \mathbb{R}$. Show that h is a constant function.

As $5 \cdot (x/5) = x$ for all reals x, we have that $f(x/5) = f(x)$. Using this equation repeatedly we have, for any $n \in \mathbb{N}$

$$f(x) = f(x/5) = \ldots = f(x/5^n)$$

We can thus take the limits on both sides to get,

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} f(x/5^n) = f\left(\lim_{n\to\infty} x/5^n\right) = f(0)$$

where we used the continuity of f in the third equality and the fact that $x/5^n$ clearly converges to 0 as n tends to infinity, for any real x, in the last equality. Thus $f(x) = f(0)$ for all $x \in \mathbb{R}$ proving that its constant.

Problem 4.6. Show that there is no continuous function u on $\mathbb{R}$ such that $u(x)$ is irrational whenever x is rational and $u(x)$ is rational whenever x is irrational.

Solution. As there are no reals that are simultaneously rational and irrational, the function can't be constant. Thus there exists reals x, y with $f(x) \neq f(y)$ and as there are uncountably many irrational numbers between $f(x)$ and $f(y)$ we see that f, if continuous must achieve all of these values at some point in between x and y which it can not possibly do as there are only countably many rationals in between x, y which are the only points where it achieves irrational values. $\square$

Problem 4.7. Let B be a nonempty subset of $\mathbb{R}$. Define a function $k : \mathbb{R} \to \mathbb{R}$ by

$$k(x) = \inf\{|x - b| : b \in B\}.$$

Show that k is a continuous function.

Solution. For any $x, y \in \mathbb{R}$ we see that,

$$k(x) = \inf_{b\in B} |x - b| \leq \inf_{b\in B} (|x - y| + |y - b|) = k(y) + |x - y|$$

where the second inequality follows from the definition of the infimum after considering how $k(x) \leq |x - b| \leq |x - y| + |y - b|$ for all $b \in B$ by the definition of the infumum and triangle inequality respectively, and the second equality can be easily proven. Similarly we can show that, $k(y) \leq k(x) + |x - y|$. Combining the two inequalities,

$$k(x) - k(y) \leq |x - y|, k(y) - k(x) \leq |x - y| \implies |k(x) - k(y)| \leq |x - y|$$

This proves that k is continous as we can take the delta and epsilon in the definition to be equal. □

Problem 4.8. Show that the function $m : \mathbb{R} \to \mathbb{R}$ defined by $m(x) = \frac{5}{1+x^2}$ is uniformly continuous.

For any $x, y \in \mathbb{R}$,

$$|m(x) - m(y)| = \left|\frac{5(y^2 - x^2)}{(1 + x^2)(1 + y^2)}\right| = \left(\frac{5|x + y|}{(1 + x^2)(1 + y^2)}\right) \cdot |x - y|$$

Now, by the AM-GM inequality we see that,

$$\frac{|x + y|}{(1 + x^2)(1 + y^2)} \leq \frac{|x|}{(1 + x^2)(1 + y^2)} + \frac{|y|}{(1 + x^2)(1 + y^2)} \leq \frac{|x|}{1 + x^2} + \frac{|y|}{1 + y^2} \leq 1/2+1/2 = 1$$

where we used the triangle inequality for the first inequality, the fact that $1+x^2, 1+y^2 \geq 1$ for the second and AM-GM on the third as $2|x| = 2\sqrt{1 \cdot x^2} \leq 1 + x^2$, similarly for y. Thus, refining the original inequality,

$$|m(x) - m(y)| \leq 5|x - y|$$

this proves that m is continuous as we can set $\delta = \varepsilon/5$ with ε, δ being the usual variables used to define continuity.

Problem 4.9. Suppose C is a bounded subset of $\mathbb{R}$ and $f : C \to \mathbb{R}$ is uniformly continuous. Show that $f(C)$ is bounded.

Solution. Say $M \geq 0$ is such that $C \subseteq [-M, M]$ which exists as B is bounded. As f is uniformly continuous there exists some $\delta > 0$ such that $|x-y| < \delta \implies |f(x)-f(y)| < 1$. Now consider the partition of $\mathbb{R}$ into the intervals $I_m := [m\delta/2, (m+1)\delta/2)$ where $m \in \mathbb{Z}$. There also exists some n such that,

$$[-M, M] \subseteq \bigcup_{k=-n}^{n} I_k$$

by the archimidean principle, Now consider those I_k from these which have nonempty intersection with C, let these be $I_{k_1}, \ldots, I_{k_m}$. Now choose some $x_m \in I_{k_m} \cap C$. By construction, $x, y \in I_j \implies |x - y| < \delta$ thus for any $x \in B \cap I_{k_m}$ we have $|x - x_m| < \delta$ and thus $|f(x) - f(x_m)| < \delta \implies |f(x)| < |f(x_m)| + 1$. Let,

$$\Lambda := \max_{1\leq i\leq m} |f(x_m)| + 1$$

Now as,

$$C = \bigcup_{i=1}^{m} (I_{k_i} \cap C)$$

we see that $x \in C$ implies its in some $I_{k_j} \cap C$ and thus $|f(x)| < |f(x_j)| + 1 \leq \Lambda$. So for all $x \in C$ we must have $|f(x)| \leq \Lambda$ making this function is bounded. $\square$

Problem 4.10. Let k be a function on $\mathbb{R}$ defined by

$$k(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that k is a differentiable function, however the derivative is not continuous.

Solution. Clearly k is differentiable everywhere except at 0 following the algebra of derivatives and composition law, after some algebraic manipulation, for $x \neq 0$,

$$k'(x) = 2x \sin(1/x) - \cos(1/x)$$

For $x = 0$, we take the limit,

$$\lim_{t \to 0} \frac{t^2 \sin(1/t) - f(0)}{t - 0} = \lim_{t \to 0} t \sin(1/t) = 0$$

where the last limit is 0 as sin is a bounded function and the identity function tends to 0 in this cthe identity function tends to 0 in this case. Thus the function is infact differentiable with derivative,

$$k'(x) = \begin{cases} 0 \text{ if, } x = 0 \\ 2x \sin(1/x) - \cos(1/x) \text{ otherwise} \end{cases}$$

The derivative is not continuous at zero because if it were then for any sequence $\{x_n\}$ with limit zero, the sequence $\{k'(x_n)\}$ would also have limit $k'(0) = 0$ which is not the case here. This can be seen by considering the sequence $1/n\pi$ which has limit 0 but the sequence $\{k'(1/n\pi)\}$ does not tend to $k'(0)$. This can be seen by examining the terms,

$$k'(1/n\pi) = \frac{2\sin(n\pi)}{n\pi} - \cos(n\pi) = 0 - (-1)^n = (-1)^{n+1}$$

which does not tend to $0 = k'(0)$ at all. $\square$

Problem 4.11. Let f be n times differentiable on $\mathbb{R}$. Let $x_1, x_2, \ldots, x_{n-1}$ be $(n+1)$ distinct real numbers for some $n \geq 2$. Suppose there exists a real polynomial p of degree $(n-1)$ satisfying,

$$f(x_j) = p(x_j), \forall j \in \{1, 2, \ldots, n-1\}.$$

Show that there exists some $c \in \mathbb{R}$ such that $f^{(n)}(c) = 0$.

Solution. Consider $h(x) := f(x) - p(x)$, as polynomials are infinitely differentiable and f is n times differentiable, h is also n times differentiable. We are given $(n+1)$ distinct reals x_i, wlog assume $x_1 < x_2 < \ldots < x_{n+1}$ on which h is 0. Now by applying rolle's (or, mid value theorem as the endpoints are both 0, hence equal.) on the intervals

$(x_i, x_{i+1}); 1 \le i \le n$, we get points $y_i \in (x_i, x_{i+1})$ where $0 = h'(y_i) = f'(y_i) - p'(y_i)$. We also see that the y_i are distinct and form a increasing finite sequence as well. Repeat the same procedure with $y_i's$ serving the role of $x_i's$ and h', which is now $n-1$ times differentiable, serving the role of h until we get down to one point c such that $0 = h^{(n)}(c) = f^{(n)}(c) + p^{(n)}(c)$. As p is a polynomial of degree $n-1$, its n-th derivative is identically zero thus, $0 = f^{(n)}(c) + 0 \implies f^{(n)}(c) = 0$ and we are done. □

Problem 4.12. Find the maximum and the minimum points of the polynomial

$$q(x) = x(x-1)(x-2),$$

in the interval $[0, 3]$. (Justify your claim).

Solution. Extrema occur at points where the derivative is zero, or on the edges in this case. The derivative $p'(x) = 3x^2 - 6x + 2$ has roots $1 \pm 3^{-0.5}$ and the values of p on these are $\mp 2 \cdot 3^{-1.5}$. The values on the edge are 0 and 6 so these are the only candidate for extrema and thus the minimum point is at $1 + 3^{-0.5}$ and the maximum is at 3. □

Problem 4.13. Use Taylor's theorem to prove the Binomial theorem:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots + x^n$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

$f(x) = (1+x)^n$ is infinitely differentiable but all the derivaitives of order $n+1$ or more are 0 as its a polynomial of degree n. Using taylor's theorem centered around 0 we have,

$$(1+x)^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)(x-0)^k}{k!}$$

Also differentiating f, k times for $k \le n$ gives,

$$f^{(k)}(x) = n(n-1)\ldots(n-k+1)(1+x)^{n-k} \implies f^{(k)}(0) = \frac{n!}{(n-k)!}$$

Combining these two, the coeffecient of x^k in $f(x)$ must be,

$$\frac{n!}{(n-k)!} \cdot \frac{1}{k!} = \binom{n}{k}$$

thus we have shown that

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Problem 4.14. Use Taylor's theorem around the point $x_0 = 4$ to get a good approximation of $\sqrt{5}$ (You may consider the function $f(x) = \sqrt{x}$ on $[0, \infty)$ and take $n = 3$).

Solution. Let $f(x) = \sqrt{x}$. First we compute the derivatives,

$$f^{(0)}(x) = x^{1/2}, f^{(1)}(x) = (1/2)x^{-1/2}, f^{(2)}(x) = (-1/4)x^{-3/2}$$

$$f^{(3)}(x) = (3/8)x^{-5/2}, f^{(4)}(x) = (-15/16)x^{-7/2}$$

Now using taylor's theorem we can say that there exists some $\zeta \in (4,5)$ such that,

$$f(5) = \sum_{k=0}^{3} \frac{f^{(k)}(4)(5-4)^k}{k!} + \frac{f^{(4)}(\zeta)}{4!} = \frac{1145}{512} - \frac{15}{16} \cdot \frac{\zeta^{-7/2}}{4!}$$

And as $\zeta \in (4,5)$ we can say that,

$$\sqrt{5} \in \left(\frac{1145}{512} - \frac{15 \cdot 4^{-7/2}}{4! \cdot 16}, \frac{1145}{512} - \frac{15 \cdot 9^{-7/2}}{4! \cdot 16} \right) = \left(\frac{36635}{16384}, \frac{2504095}{1119744} \right)$$

upon long division we see that,

$$\sqrt{5} \in (2.23602294921875, 2.2363102637746)$$

So we can say that $\sqrt{5} \approx 2.236$ is a good approximation. $\square$

Problem 4.15. Fix $n \in \mathbb{N}$. Prove that the function v defined on $[0, \infty)$ by

$$v(x) = (x+1)^{\frac{1}{n}} - x^{\frac{1}{n}}$$

is decreasing.

Solution. We can calculate the derivative,

$$v'(x) = \frac{(1+x)^{-\frac{n-1}{n}} - x^{-\frac{n-1}{n}}}{n}$$

Let $\beta = -(n-1)/n \leq 0$, then $f(x) = x^{\beta}$ is decreasing on our domain and as $1 + x > x$ we must have that $f(x+1) - f(x) \leq 0$ for all x, thus, $v'(x) = (f(1+x) - f(x))/n \leq 0$ for all $x \in [0, \infty)$ so the function is decreasing. $\square$

5 Home Assignment V (Due: Dec 4, 2025)

Problem 5.1. (Discrete L'Hospital): Let $\{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1}$ be sequences of non-zero real numbers converging to 0. Suppose $b_n > b_{n+1}$ for all n and $v := \lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$ exists as a real number. Show that $\lim_{n \to \infty} \frac{a_n}{b_n}$ exists and equals v. Give one such example.

Solution. Given any $\varepsilon > 0$ there exists N such that $\forall n > N$,

$$v - \varepsilon < \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < v + \varepsilon \implies (v - \varepsilon)(b_{n+1} - b_n) > a_{n+1} - a_n > (v + \varepsilon)(b_{n+1} - b_n)$$

Where the signs flipped because $b_{n+1} - b_n < 0$ as stated in the problem. We sum both sides of this inequality for $n, n+1, \ldots, m$ to get,

$$(v - \varepsilon) \sum_{k=n}^{m} (b_{k+1} - b_k) > \sum_{k=n}^{m} (a_{k+1} - a_k) > (v + \varepsilon) \sum_{k=n}^{m} (b_{k+1} - b_k)$$

$$\implies (v - \varepsilon)(b_{m+1} - b_n) > a_{m+1} - a_n > (v + \varepsilon)(b_{m+1} - b_n)$$

Thus this holds for all $m \geq n$ and taking the limits as $m \to \infty$ on all of the terms in this inequality we get,

$$(v - \varepsilon)(-b_n) \geq -a_n \geq (v + \varepsilon)(-b_n) \implies v - \varepsilon \leq \frac{a_n}{b_n} \leq v + \varepsilon$$

which is enough to conclude that $a_n/b_n \longrightarrow v$ as well. For an example consider $a_n = 1/n^2$ and $b_n = 1/n$, both of these tend to 0 and b_n decreases strictly. Now, $a_{n+1} - a_n = -(2n+1)/(n^4 + 2n^3 + n^2)$ and $b_{n+1} - b_n = -1/(n^2 + n)$. Thus,

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{(2n+1)(n^2+n)}{n^4 + 2n^3 + n^2} = 0$$

and we can verify,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n^2}{1/n} = 0$$

clearly as well. $\square$

Problem 5.2. Suppose $k \in \mathbb{N}$ and $b_1, b_2, \ldots, b_k$ are strictly positive real numbers. Show that (i) $\lim_{n\to\infty} b_j^{\frac{1}{n}} = 1$. (ii) $\lim_{n\to\infty}(b_1^n + b_2^n + \cdots + b_k^n)^{\frac{1}{n}} = b$ where $b = \max\{b_j : 1 \leq j \leq k\}$.

Solution. For (i), let $x > 0$ and consider the two cases : $x \geq 1$ and $x < 1$. For the first case, by bernoulli's inequality we see that,

$$x = (x^{1/n})^n = (1 + (x^{1/n} - 1))^n \geq 1 + n(x^{1/n} - 1) \implies x^{1/n} - 1 \leq \frac{x-1}{n}$$

Here we have used the fact that $x^p \geq 1$ for any positive p, making $x^{1/n} - 1$ nonnegative. Thus we can use the squeeze theorem on the following inequality to conclude,

$$0 \leq x^{1/n} - 1 \leq \frac{x-1}{n}$$

And when $0 < x < 1$ we can just consider $1/x > 1$ and get that $1/x^{1/n} \xrightarrow{n\to\infty} 1$ and using algebra of limits we get that $1/\lim x^{1/n}$ is also 1 which makes $\lim x^{1/n} = 1$. To prove (ii) we will also use squeeze theorem. As b is the maximum of these b_i we have,

$$b^n < b_1^n + \ldots + b_k^n \leq b^n + \ldots + b^n = kb^n$$

$$\implies b < (b_1^n + \ldots + b_k^n)^{1/n} \leq k^{1/n} b$$

Using the fact $k^{1/n} \longrightarrow 1$, which is proven in the immediate next exercise, we can conclude using squeeze theorem. $\square$

Problem 5.3. Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Solution. For large enough($n \geq 3$) we can say that $n^{1/n} \geq 1$ and then using the binomial theorem

$$n = (1 + (n^{1/n} - 1))^n = \sum_{k=0}^{n} \binom{n}{k} (n^{1/n} - 1)^k \geq \frac{n(n-1)}{2}(n^{1/n} - 1)^2$$

$$\implies 1 \leq n^{1/n} \leq 1 + \sqrt{\frac{2}{n-1}}$$

And yet again we conclude using the squeeze theorem. $\square$

Problem 5.4. Let $f : \mathbb{R} \to \mathbb{R}$, be a continuous function satisfying $f(3x) = f(x), \forall x \in \mathbb{R}$. Show that f is a constant function.

Solution. For all real x, by the condition we have $f(x) = f(3 \cdot (x/3)) = f(x/3)$ and continuing for n steps we have $f(x) = f(x/3^n)$. Now from the continuity of f,

$$f(x) = \lim_{n\to\infty} f(x/3^n) = f\left(\lim_{n\to\infty} x/3^n\right) = f(0)$$

which makes f a constant function as $\forall x \in \mathbb{R}, f(x) = f(0)$. $\square$

Problem 5.5. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $g(x + y) = g(x) + g(y), \forall x, y \in \mathbb{R}$. Show that $g = cx$ for some $c \in \mathbb{R}$.

Solution. Just as in Problem 2.10. (problem 10 in the 2nd assignment) we can show that for any rational r we have $g(r) = g(1) \cdot r$ and as given any real number x there exists a sequence of rationals $\{x_n\}$ converging to it we see that,

$$g(x) = g\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} g(1) \cdot x_n = g(1) \cdot \lim_{n\to\infty} x_n = g(1) \cdot x$$

where we used the continuity of g in the second inequality and a basic fact from the algebra of limits in the fourth. Thus we see that $c = g(1)$ and $g(x) = g(1) \cdot x$ for all $x \in \mathbb{R}$ $\square$

Problem 5.6. Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$, is said to be convex if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \forall x < y$ in $I, 0 < \lambda < 1$. (i) Show that if $f : (0, 1) \to \mathbb{R}$ is convex then for $0 < s < t < u < 1$,

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

(ii) Show that if $f : (0, 1) \to \mathbb{R}$, is convex then it is continuous. (iii) Show that a convex function $g : [0, 1] \to \mathbb{R}$, need not be continuous.

Solution. Rearranging the first inequality we get the following equivalent inequality,

$$\frac{f(t)}{t - s} - \frac{f(u)}{u - s} \leq \frac{f(s)}{t - s} - \frac{f(s)}{u - s} = \left(\frac{u - t}{(t - s)(u - s)}\right) f(s)$$

$$\iff f(t) \leq \frac{u - t}{u - s} \cdot f(s) + \frac{t - s}{u - s} \cdot f(u)$$

Which is true from the definiton of convexity via $\lambda = (u - t)/(u - s) \in (0, 1)$. The second inequality is proven similarly. For (ii), we will show that for any $x \in (0, 1)$, the right and left limits of $f(y)$ as y tends to x is zero which will imply that the limit exists and is zero as well, proving continuity. Without loss of generality assume that $x < y$, then choose $u, v, w \in (0, 1)$ such that $u < x < y < v$. Now by the previous result we see that,

$$\frac{f(x) - f(u)}{x - u} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(v) - f(y)}{v - y} \leq \frac{f(w) - f(v)}{w - v}$$

$$\implies \frac{f(x) - f(u)}{x - u} \cdot (y - x) \leq f(y) - f(x) \leq (y - x) \cdot \frac{f(v) - f(y)}{v - y} \leq (y - x) \cdot \frac{f(w) - f(v)}{w - v}$$

Now, using the squeeze theorem, as y tends to x from the right we get,

$$\lim_{y\to x; y>x} (f(y) - f(x)) = 0$$

and the same can be said for the left limit using a very similar arguement and we are done. For (iii) we can consider the following function,

$$f(x) := \begin{cases} 0 \text{ if } 0 < x \leq 1 \\ 1 \text{ if } x = 0 \end{cases}$$

To show that this is convex we will consider the inequality as in the definition on all points $x, y \in [0, 1]$. The inequality is clearly always true for $0 < x, y$ as the function is constant on $(0, 1]$ and when some x, y is 0, wlog say $x = 0$, then,

$$f(\lambda x + (1-\lambda)y) = f((1-\lambda)y) \leq \underbrace{\lambda f(x)}_{=1} + (1-\lambda)f(y)$$

for both $y = 0$ and $y > 0$ as in the first case its $1 \leq 1$ and in the second, as $1 - \lambda > 0$ its $0 \leq 1$. □

Problem 5.7. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$, such that for every $y \in \mathbb{R}$, there are exactly two real numbers x_1, x_2, such that $f(x_1) = f(x_2) = y$.

Solution. For the sake of contradiction assume that there is some f that is continuous and also satisfies said criterion. Let $x < y$ be the reals such that $f(x) = f(y) = 0$. Without loss of generality we can asssume $x < y$. Now consider the interval $[x, y]$, as f is continuous it must have some it must achieve its extrema in this interval at some points in the interval. Unless there is an extrema which is achieved only in (x, y) we can easily see that f is constant in this interval which gives us much more than two, infact infinitely many reals where f is zero which is not allowed as per the assumption. Again without loss of generality assume that we reach a maxima at $u \in (x, y)$. We claim that u is the unique such point in $[x, y]$. For a proof of this claim, for the sake of contradiction asssume without loss of generality that we also reach a maxima at $v \in (u, y]$. f can not be constant on the interval $[u, v]$ so there is some $w \in (u, v)$ such that $0 < f(w) < f(u)$ where the first inequality can be achieved due to continuity because by construction $f(u) = f(v) > 0$ and if we did not have some w it would produce jump discontinuities in (u, v). Now for any $\lambda \in (f(w), f(u))$ we can find a real z with $f(z) = \lambda$ in all three of the intervals : $(x, u), (u, v), (v, y)$ by IVT, which contradicts the assumption as there should be exactly two such z and the claim is proved. By the condition on f, there must be another $g \notin [x, y]$ such that $f(u) = f(g)$. Without loss of generality assume that $g > y$, then for any $\lambda \in (0, f(u))$ we can find some z with $f(z) = \lambda$ from all of these three intervals : $(x, u), (u, y), (y, g)$ by IVT which is a contradiction and we are done. □

Problem 5.8. Fix $n \in \mathbb{N}$. Let $x_1, x_2, \ldots, x_n$ be n distinct real numbers and let $y_1, y_2, \ldots, y_n$ be another n-tuple of not-necessarily distinct real numbers. Show that there is a unique polynomial p of degree $(n-1)$ such that $p(x_k) = y_k$ for $1 \leq k \leq n$. Hint: Consider

$$p(x) = \sum_{j=1}^{n} y_j \frac{\prod_{i \neq j}(x - x_i)}{\prod_{i \neq j}(x_j - x_i)}.$$

Solution. The polynomial p provided in the hint is clearly of degree atmost $n - 1$ and when evaluated at x_i, all but the i'th term in the sum vanish and in this term the numerator and denominator of the fraction cancel out leaving only y_i and thus $p(x_i) = y_i$. Assume p, q both have degree atmost $n - 1$ and they both are polynomials which satisfy the condition in the problem, then we see that $p-q$ has n zeroes. $p-q$ clearly has degree atmost $n-1$ and thus for it to have $n > n-1$ zeroes, by the fundamental theorem of algebra we see that $p-q$ must be identically zero i.e. $p = q$ and hence such a polynomial is unique. $\square$

Problem 5.9. Let $a < b$ be real numbers. Let $f : (a, b) \to \mathbb{R}$ be a twice differentiable function. Show that for any $x_0 \in (a, b)$

$$f''(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}.$$

Solution. f being twice differentiable makes it continuous and thus the numerator tends to 0 as $h \to 0$ and the denominator also tends to zero as $h \to 0$ and also the derivative of the denominator is nonzero near 0, except at 0. Let the numerator be $X(h)$ and denominator $Y(h)$, then,

$$\frac{X'(h)}{Y'(h)} = \frac{f'(x_0 + h) - f'(x_0 - h)}{2h}$$

$$= \frac{f'(x_0 + h) - f'(x_0)}{2h} - \frac{f'(x_0) - f'(x_0 - h)}{2h} \xrightarrow{h\to 0} \frac{f''(x_0)}{2} + \frac{f''(x_0)}{2} = f''(x_0)$$

and we can thus conclude via LHopital's rule. $\square$

Problem 5.10. (i) For $x \in \mathbb{R}$, show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely. (ii) Define $e : \mathbb{R} \to \mathbb{R}$ by $e(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Show that $e(x + y) = e(x)e(y), \forall x, y \in \mathbb{R}$.

Solution. For (i), we will show that for any $x \in \mathbb{R}$ this series is absolutely convergent using the ratio test,

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$

so we are done. For (ii), using the cauchy product theorem for product of sums, atleast one of which is absolutely convegrent we have,

$$e(x)e(y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = e(x+y)$$

$\square$

Problem 5.11. Let $h : \mathbb{R} \to \mathbb{R}$ be the function $h(x) = \sin(x)$ (Here we are assuming your familiarity with trigonometric functions). Show that the remainder term in Taylor's theorem converges to 0 as $n \to \infty$ for every x_0 and x.

Solution. If we keep taking the derivative if h we can easily see that $|h^{(n)}(x)|$ is either $|\sin x|$ or $|\cos x|$, eitherways its bounded by 1. Now given any x, x_0 the $n-th$ remainder term, say R_n tends to zero as follows,

$$|R_n| \le \frac{|x_0 - x|^n}{n!} \cdot \sup_{x\in\mathbb{R}} |h^{(n)}(x)| = \frac{|x - x_0|^n}{n!} \xrightarrow{n\to\infty} 0$$

just as we did in Problem 5.10.. $\square$

Problem 5.12. Consider the function $h : [0, \infty) \to \mathbb{R}$ defined by $h(x) = \frac{x}{1+x^2}$. Determine the set $\{h(x) : x \in [0, \infty)\}$.

Solution. Clearly h is strictly decreasing with $h(0) = 1$ and $\lim_{x\to\infty} h(x) = 0$ so the set is $(0, 1]$ which we get by using IVT as we can find arbitrary small positive values of h at very large inputs and $h(0) = 0$. $\square$

Problem 5.13. Let $v : \mathbb{R} \to \mathbb{R}$ be the function $v(x) = x^3 - 6x^2 + 9x$, for all x. Determine the set: $\{x : v(x) > 0\}$.

Solution. We can factor this into $x(x - 3)^2$, so it is zero at 0 and 3. For $x \neq 0, 3$ it has the same sign as x as $(x - 3)^2$ is positive. Thus the set is $(0, \infty) \setminus \{3\}$. $\square$

Problem 5.14. Let $f : [0, 1] \to [0, 1]$ be a continuous strictly increasing bijection. Assume that $x < f(x)$ for all $x \in (0, 1)$. Fix $x_0 \in (0, 1)$. Define x_n for $n \in \mathbb{Z}$ by $x_n = f^n(x_0)$. Show that: (i) $0 < x_m < x_n < 1$ for $m < n$ in $\mathbb{Z}$; (ii) $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to-\infty} x_n = 0$; (iii) For every $n \in \mathbb{Z}$, f maps $[x_n, x_{n+1}]$ bijectively to $[x_{n+1}, x_{n+2}]$.

Solution. As f is strictly increasing we see that we must have $f(0) = 0$ and $f(1) = 1$ and as its also a bijection, 0 or 1 never appear in f orbits i.e. $\{f^n(x) : n \in \mathbb{Z}\}$ of any points in $(0, 1)$. Thus we can see that for any $m < n$,

$$f^n(x_0) = \underbrace{f(f^{n-1}(x_0)) > f^{n-1}(x_0) > \ldots > f^m(x_0)}_{n-m \text{ inequalities}}$$

and as 1 and 0 never appear in the orbit of x_0 we see that $0 < f^n(x_0) < 1$ for all $n \in \mathbb{Z}$ and together it reads $0 < x_m < x_n < 1$, we have thus proved (i). For (ii), as $n \to \infty$, $\{x_n\}$ must converge to a limit as its increasing and bounded above, say it converges to x, then $0 < x \leq 1$. Using the continuity of f we have can show that this is a fixed point as follows

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$$

so it has to be 1. We can similarly show that $x_n \xrightarrow{n\to-\infty} 0$ as well. For every $n \in \mathbb{Z}$ we see that $x_n < x_{n+1}$ and thus $x_{n+1} = f(x_n) < f(x_{n+1}) = x_{n+2}$ so we have $f([x_n, x_{n+1}]) \subseteq [x_{n+1}, x_{n+2}]$ using the fact that f is increasing. Using IVT and the fact that $x_{n+1} = f(x_n), x_{n+2} = f(x_{n+1})$ we can say that we have equality here i.e. $f([x_n, x_{n+1}]) = [x_{n+1}, x_{n+2}]$ and the f is clearly still a bijection here, this proves (iii). $\square$

Problem 5.15. Consider the set up of the previous question. Fix $y_0 \in (x_0, x_1)$. Set $y_n = f^n(y_0)$ for $n \in \mathbb{Z}$. Let $h : [x_0, y_0] \to [y_0, x_1]$ be a continuous strictly increasing bijection. Define $g : [0, 1] \to [0, 1]$ by $g(0) = 0, g(1) = 1$,

$$g(t) = h(t) \quad \forall t \in [x_0, y_0];$$

$$g(t) = f \circ h^{-1}(t) \quad \forall t \in [y_0, x_1];$$

and more generally, for $n \in \mathbb{Z}$, define

$$g(t) = f^n \circ h \circ f^{-n}(t) \quad \forall t \in [x_n, y_n]$$

and

$$g(t) = f^{n+1} \circ h^{-1} \circ f^{-n}(t) \quad \forall t \in [y_n, x_{n+1}].$$

Then show that: (i) For every n, g maps $[x_n, y_n]$ bijectively to $[y_n, x_{n+1}]$ and it maps $[y_n, x_{n+1}]$ bijectively to $[x_{n+1}, y_{n+1}]$. (ii) g is a strictly increasing continuous bijection. (iii) $f = g \circ g$. This shows that f has infinitely many square roots.

Solution. h being a continuous bijection forces $h(x_0) = y_0$ and $h(y_0) = x_1$, this will be used here on. We know that inverses of continuous bijections in $\mathbb{R}$ are also continuous bijections and both of these are strictly increasing. From the previous problem we can see that f^{-n} bijectively maps $[x_n, y_n]$ to $[x_0, y_0]$, after this h maps $[x_0, y_0]$ to $[y_0, x_1]$ bijectively and at the end f^n maps this bijectively to $[y_n, x_{n+1}]$, thus g maps $[x_n, y_n]$ bijectively to $[y_n, x_{n+1}]$. Using similar arguements we can show that g also bijectively maps $[y_n, x_{n+1}]$ to $[x_{n+1}, y_{n+1}]$ and (i) is proven. We can see that

$$\mathfrak{I} := \{[x_n, y_n), [y_n, x_{n+1}) : n \in \mathbb{Z}\} \cup \{\{0\}, \{1\}\}$$

is a partition of $[0, 1]$, this is because $\{[x_n, x_{n+1}) : n \in \mathbb{Z}\} \cup \{\{0\}, \{1\}\}$ is a partition (this can be deduced from parts (i) and (ii) of Problem 5.14.) and we just partition each of the intervals into two intervals. If we were to write these out in order(visually speaking) then $[0, 1]$ would look like this,

$$\ldots [y_{-2}, x_{-1}) \cup [x_{-1}, y_{-1}) \cup [y_{-1}, x_0) \cup [x_0, y_0) \cup [y_0, x_1) \cup [x_1, y_1) \cup [y_1, x_2) \ldots$$

As $g(0) = 0, g(1) = 1$ and,

$$g([x_n, y_n)) = [y_n, x_{n+1}] \setminus \{g^{-1}(y_n)\} = [y_n, x_{n+1})$$

As, $g(y_n)$= $f^n(h(f^{-n}(y_n))) = f^n(h(y_0)) = f^n(x_1) = x_{n+1}$. Similarly, for the other type of intervals, as, $g(x_{n+1}) = f^{n+1}(h^{-1}(f^{-n}(x_{n+1})) = f^{n+1}(h^{-1}(x_1)) = f^{n+1}(y_0) = y_{n+1}$ we get,

$$g([y_n, x_{n+1})) = [x_{n+1}, y_{n+1})$$

Now, looking at g as a set function defined by $g(S) := \{g(x) : x \in S\}$ we can clearly see that g is a bijection on $\mathfrak{I}$ and as its a bijection on each set in $\mathfrak{I}$ which is a partition of $[0, 1]$, its clearly a bijection on $[0, 1]$ as a whole. Thus we have shown that g is a bijection. But we still have to show that it is continous, which it is on the smaller intervals we partitioned the larger interval $[0, 1]$ into, thus for it to be continuous on the larger interval we need to check continuity at the points where one of these smaller intervals end and the other begins i.e. x_n and y_n for $n \in \mathbb{Z}$. At x_n, we arrive at the left limit for g i.e. $g(x_n-)$ by using its definition on $[y_{n-1}, x_n)$ so due to continuity on this interval this is,

$$g(x_n-) = f^{(n-1)+1}(h^{-1}(f^{-(n-1)}(x_n))) = f^n(h^{-1}(x_1) = f^n(y_0) = y_n$$

And now the right limit will use the definition of g on $[x_n, y_n)$ and thus similarly,

$$g(x_n+) = f^n(h(f^{-n}(x_n))) = f^n(h(x_0)) = f^n(y_0) = y_n$$

So its continuous at the points x_n. The continuity at the points y_n can be proved very similarly. Thus it must be strictly increasing as well hence proving (ii). For (iii), firstly $(g \circ g)(0) = 0 = f(0)$ and $(g \circ g)(1) = 1 = f(1)$, now if some $x \in (0, 1)$ is in an interval of form $[x_n, y_n]$ then when g is first applied to it, it acts as $f^n \circ h \circ f^{-n}$ and sends it to the interval $[y_n, x_{n+1}]$ as seen in (ii) and thus on the second application g acts as $f^{n+1} \circ h^{-1} \circ f^{-n}$ as in the definition of g, thus,

$$(g \circ g)(x) = (f^{n+1} \circ h^{-1} \circ f^{-n} \circ f^n \circ h \circ f^{-n})(x) = (f^{n+1} \circ h^{-1} \circ (f^{-n} \circ f^n) \circ h \circ f^{-n})(x)$$

$$= (f^{n+1} \circ h^{-1} \circ \mathrm{Id} \circ h \circ f^{-n})(x) = (f^{n+1} \circ h^{-1} \circ (\mathrm{Id} \circ h) \circ f^{-n})(x) = (f^{n+1} \circ h^{-1} \circ h \circ f^{-n})(x)$$
$$= (f^{n+1} \circ (h^{-1} \circ h) \circ f^{-n})(x) = (f^{n+1} \circ \mathrm{Id} \circ f^{-n})(x) = (f^{n+1} \circ (\mathrm{Id} \circ f^{-n}))(x) = (f^{n+1} \circ f^{-n})(x)$$
$$= f(x)$$

and when x is in an interval of form $[y_n, x_{n+1}]$ using similar arguements we have that,

$$(g \circ g)(x) = (\underbrace{f^{n+1} \circ h \circ f^{-(n+1)}}_{\text{as } g(x) \in [x_{n+1}, y_{n+1}]} \circ f^{n+1} \circ h^{-1} \circ f^{-n})(x) = (f^{n+1} \circ f^{-n})(x) = f(x)$$

Thus we have shown that $g \circ g = f$ on $[0, 1]$ and we are done. $\square$