Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester

Analysis I — Instructor: B V Rajarama Bhat
Home Assignment IV — Due Date : Oct. 26, 2025

(1) Let $g : [0,1] \to \mathbb{R}$ be a function such that g is strictly increasing and satisfies the intermediate value property, that is, for any y with $g(0) < y < g(1)$, there exists $0 < x < 1$ such that $g(x) = y$. Show that g is continuous.

(2) Let $h : [0,1] \to [0,1]$ be a continuous function. A real number $x \in [0,1]$ is said to be a fixed point of h, if $h(x) = x$. Suppose h is a strict contraction that is: $|h(x) - h(y)| < c|x - y|$ for all $x \neq y$ in $[0,1]$ for some $0 < c < 1$. Given $a \in [0,1]$ define a sequence $\{a_n\}_{n\geq 1}$ by $a_1 = a$ and $a_{n+1} = h(a_n)$. Show that $\{a_n\}_{n\geq 1}$ is a convergent sequence and the limit is a fixed point of h.

(3) For any two continuous functions f, g on $[0,1]$ define

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}.$$

Show the triangle inequality:

$$d(f,g) \leq d(f,h) + d(h,g),$$

for any three continuous functions f, g, h on $[0,1]$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that there exists $d \in \mathbb{R}$ such that $f(x) = dx$ for all $x \in \mathbb{R}$.

(5) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $h(x) = h(5x)$ for all $x \in \mathbb{R}$. Show that h is a constant function.

(6) Show that there is no continuous function u on $\mathbb{R}$ such that $u(x)$ is irrational whenever x is rational and $u(x)$ is rational whenever x is irrational.

(7) Let B be a nonempty subset of $\mathbb{R}$. Define a function $k : \mathbb{R} \to \mathbb{R}$ by

$$k(x) = \inf\{|x - b| : b \in B\}.$$

Show that k is a continuous function.

(8) Show that the function $m : \mathbb{R} \to \mathbb{R}$ defined by $m(x) = \frac{5}{(1+x^2)}$ is uniformly continuous.

(9) Suppose C is a bounded subset of $\mathbb{R}$ and $f : C \to \mathbb{R}$ is uniformly continuous. Show that $f(C)$ is bounded.

(10) Let k be a function on $\mathbb{R}$ defined by

$$k(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } \; x \neq 0 \\ 0 & \text{if } \; x = 0. \end{cases}$$

Show that k is a differentiable function, however the derivative is not continuous.

(11) Let f be n times differentiable on $\mathbb{R}$. Let $x_1, x_2, \ldots, x_{n+1}$ be $(n+1)$ distinct real numbers for some $n \geq 2$. Suppose there exists a real polynomial p of degree $(n-1)$ satisfying,

$$f(x_j) = p(x_j), \; \forall j \in \{1, 2, \ldots, n+1\}.$$

Show that there exists some $c \in \mathbb{R}$ such that $f^{(n)}(c) = 0$.

P.T.O.

(12) Find the maximum and the minimum points of the polynomial

$$q(x) = x(x-1)(x-2),$$

in the interval $[0, 3]$. (Justify your claim).

(13) Use Taylor's theorem to prove the Binomial theorem:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots + x^n$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

(14) Use Taylor's theorem around the point $x_0 = 4$ to get a good approximation of $\sqrt{5}$ (You may consider the function $f(x) = \sqrt{x}$ on $[0, \infty)$ and take $n = 3$).

(15) Fix $n \in \mathbb{N}$. Prove that the function v defined on $[0, \infty)$ by

$$v(x) = (x+1)^{\frac{1}{n}} - x^{\frac{1}{n}}$$

is decreasing.