Exercise sheet 5

Only the exercises marked [HW] need to be submitted.

Vectors will be denoted in bold unlike scalars.

1. Is there a linear transformation T from $\mathbb{R}^3$ into $\mathbb{R}^2$ such that $T(1,-1,1) = (1,0)$ and $T(1,1,1) = (0,1)$?

2. Describe explicitly a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ which has as its range the subspace spanned by $(1,0,-1)$ and $(1,2,2)$.

3. Let V be an n-dimensional vector space over F and let $T : V \to V$ be a linear transformation such that the null space and range of T are identical. Prove that n is even. Can you give an example of such a linear transformation?

4. Let V be a vector space and $T : V \to V$ a linear transformation. Prove that the following two statements about T are equivalent.

   - The intersection of the range of T and the null space of T is the zero subspace of V.
   - If $T(T\mathbf{x}) = \mathbf{0}$, then $T\mathbf{x} = \mathbf{0}$.

5. Let $T : V \to W$ be a linear transformation.

   - Show that T is 1-1 if and only if the null space of T is $\{\mathbf{0}\}$.
   - Show that T is 1-1 if and only if T carries each linearly independent subset of V onto a linear independent subset of W.

6. Let V and W be finite dimensonal vector spaces over F with the same dimension n. If $T : V \to W$ is a linear transformation show that the following are equivalent:

   - T is invertible.
   - T is 1-1
   - T is onto, that is the range of T is W.
   - If $\{\mathbf{x}_1, \cdots, \mathbf{x}_n$ is a basis of V then $\{T\mathbf{x}_1, \cdots, T\mathbf{x}_n\}$ is a basis of W.
   - There is some basis $\{\mathbf{x}_1, \cdots, \mathbf{x}_n$ of V such that $\{T\mathbf{x}_1, \cdots, T\mathbf{x}_n\}$ is a basis of W

7. Let V be the space of polynomials over $\mathbb{R}$ of degree less than or equal to 3. Let $D : V \to V$ be the differentiation operator. Find the matrix of D with respect to the basis $\{x^j,\ j = 0, 1, 2, 3\}$.

8. Let V be the vector space of all $n \times n$ real matrices over $\mathbb{R}$. Let B be a fixed $n \times n$ matrix. If $T(A) = AB - BA$ verify that T is a linear operator on V.

9. Let T be a linear operator on $\mathbb{R}^3$ defined by
$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$
   Is T invertible? If so, find a rule for T^{-1} like the one that defines T. Prove that
$$(T^2 - I)(T - 3I) = 0.$$

10. Let T be a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^2$, and let U be a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$. Prove that the transformation UT is not invertible. Generalize the theorem.

11. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.

12. Let V be a vector space and T a linear operator on V. If $T^2 = 0$ what can you say about the relation of the range of T to the null space of T? Give an example of a linear operator T on $\mathbb{R}^2$ such that $T^2 = 0$ but $T \neq 0$.

13. Let T be a linear operator on the finite dimensional vector space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when V is not finite dimensional (Hint: let $T = D$, the differentiation operator on the space of polynomial functions)

14. Let V be a finite dimensional vector space and let T be a linear operator on V. Suppose that $\mathrm{rank}(T^2) = \mathrm{rank}(T)$. Prove that the range and null space of T are disjoint, i.e. have only the zero vector in common.

15. Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$

- What is the matrix of T in the standard ordered basis for $\mathbb{R}^3$?
- What is the matrix of T in the ordered basis $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, where $\mathbf{x} = (1, 0, 1), \mathbf{y} = (-1, 2, 1)$ and $\mathbf{z} = (2, 1, 1)$?
- Prove that T is invertible and find T^{-1}?

16. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1).$$

- What is the matrix of T in the standard ordered basis for $\mathbb{R}^3$ and $\mathbb{R}^2$?
- What is the matrix of T in the ordered basis $\mathcal{B} = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ and $\mathcal{B}' = \{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{x} = (1, 0, -1), \mathbf{y} = (1, 1, 1)$, $\mathbf{z} = (1, 0, 0)$, $\mathbf{u} = (0, 1), \mathbf{v} = (1, 0)$?