Linear Algebra I - Homework problems

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If problem **n** *from* **exercise_x.pdf** *is assigned as homework then it will be referred to as* **x.n**.

1.8 It was already shown in class that $\mathbb{C}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{C}\}$ forms a vectorspace over $\mathbb{R}$ with the same operations defined here. We show that V is a subspace and hence a vectorspace itself. For all $\alpha \in \mathbb{R}$ and $f, g \in V$ then for all $t \in \mathbb{R}$,

$$\overline{(\alpha f + g)(t)} = \overline{\alpha f(t) + g(t)} = \overline{\alpha} \cdot \overline{f(t)} + \overline{g(t)} = \alpha \cdot f(-t) + g(-t) = (\alpha f + g)(-t)$$

and thus $\alpha f + g \in V$ and we are done as according to what was discussed in class, V is a subspace if for all $\alpha \in \mathbb{R}$ and $f, g \in V$ its the case that $\alpha f + g \in V$ too. An example of a function with some non real outputs in V is $f(t) = it$.

2.11 We have seen that if $A \subseteq B$ are subsets of V then $Sp(A) \subseteq Sp(B)$ and using this if A, B are any subsets of V (not the ones used to state the result earlier) that as $A, B \subseteq A \cup B$ we have that $Sp(A), Sp(B) \subseteq Sp(A \cup B)$ and thus $Sp(A) \cup Sp(B) \subseteq Sp(A \cup B)$ and the second bulleted claim follows similarly from the fact that $A \cap B \subseteq A, B$. The last claim is false, we can take disjoint sets A, B such that $Sp(A) = Sp(B) \neq \{0\}$, for example in $V = \mathbb{R}^2$ take $A = \{(0,1), (1,0)\}$ and $B = \{(0,-1), (-1,0)\}$.

2.17 By definition, if $W_1 + W_2 = V$ then for all $v \in V$ there are some $w_1 \in W_1$ and $w_2 \in W_2$ such that $w_1 + w_2 = W$. We claim that these are unique. Say there are $(w_1, w_2), (u_1, u_2) \in W_1 \times W_2$ such that $w_1 + w_2 = v = u_1 + u_2$. Then we see that $w_1 + w_2 = u_1 + u_2$ and thus $W_1 \ni w_1 - u_1 = u_2 - w_2 \in W_2$ as these are vectorspaces, thus $w_1 - u_1, u_2 - w_2 \in W_1 \cap W_2$ as both of these are in both of these subsets. Now as $\{0\} = W_1 \cap W_2$ we see that $w_1 - u_1 = 0 = w_2 - u_2$ thus $w_1 = u_1$ and $w_1 = u_2$ thus the w_1, w_2 we get are unique.

3.9 As per the condition on U we see that all $(x_1, x_2, x_3, x_4, x_5) \in U$ can be written in the form $(3x_2, x_2, 7x_4, x_4, x_5)$ where $x_2, x_4, x_5 \in \mathbb{R}$. It can be easily seen that U is a subspace via the definiton and it is also spanned by the linearly independant set $\{(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1)\}$ and hence this is a basis for U.

3.11 If $\alpha \in \mathbb{R}$ then $\{x_1, x_2 + \alpha x_1, \ldots, x_n + \alpha x_1\}$ is linearly independant and hence a basis iff for $\beta_1, \ldots, \beta_n \in \mathbb{R}$,

$$\beta_1 x_1 + \beta_2(x_2 + \alpha x_1) + \ldots + \beta_n(x_n + \alpha x_1) = 0 \implies \forall k, \beta_k = 0$$

we can rewrite this as,

$$(\beta_1 + \alpha(\beta_2 + \ldots + \beta_n))x_1 + \beta_2 x_2 + \ldots + \beta_n x_n = 0$$

But as $\{x_1, \ldots, x_n\}$ was already a basis and hence linearly independant, we see that, $\beta_1 + \alpha(\beta_2 + \ldots + \beta_n) = \beta_2 = \ldots = \beta_n = 0$ and thus $\beta_1 = -\alpha(0 + \ldots + 0) = 0$ as well and thus this new set is linearly independant and thus also a basis being of size n. Now we can clearly take α large enough and get a basis where all the vectors have all positive coordinates.

4.5 For the only if part we see that if $S \oplus T_1 = V = S \oplus T_2$ then $\dim(T_1) = \dim(T_2) = \dim(V) - \dim(S)$ by the modular law. For the sake of contradiction (we will contradict the fact that $T_1 \cap T_2 = \{0\}$) assume that $\dim S < \dim V/2$, this implies that $\dim T_1 = \dim T_2 > \dim V/2$. Now take some basis $\{a_1, \ldots, a_m\}$ for T_1 and another basis $\{b_1, \ldots, b_m\}$ for T_2, we now see that as the set $\{a_1, \ldots, a_m, b_1, \ldots, b_m\}$ has size strictly greater than $2 \times \dim V/2 = \dim V$ and thus it must be linearly dependant because the size of some linearly independant set is atmost that of some spanning set and in this case the basis with $\dim V$ elements spans V so they have size atmost $\dim V$. We can thus find scalars $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$ not all zero such that

$$\sum_{i=1}^{m} \alpha_i a_i + \sum_{i=1}^{m} \beta_i b_i = 0 \implies T_1 \ni \sum_{i=1}^{m} \alpha_i a_i = \sum_{i=1}^{m} (-\beta_i) b_i \in T_2.$$

Here not all the scalars are zero so wlog say $\alpha_1 \neq 0$, now this implies that the left side isn't zero itself as otherwise we would contradict the linear independance of elements of the basis of T_1 i.e. $\{a_1, \ldots, a_m\}$. Thus we found a nonzero vector $v = \sum_{i=1}^{m} \alpha_i a_i \in T_1 \cap T_2$ which contradicts the fact that $T_1 \cap T_2 = \{0\}$. From this we have that $\dim T_1 = \dim T_2 \leq \dim V/2$ which implies that $\dim S = \dim V - \dim T_1 \geq \dim V/2$.

For the if part we will provide a construction of such T_1 and T_2. First take some basis $\{v_1, \ldots, v_n\}$ for S and extend it to a basis $\{v_1, \ldots, v_n, \ldots, v_{n+m}\}$ where $m (\leq n)$ might be zero, in which case taking $T_1 = T_2 = \{0\}$ suffices as S is the whole space anyways and all the conditions hold. Now assume $m \geq 1$ and let T_1 be the subspace spanned by $\{v_{n+1}, \ldots, v_{n+m}\}$. We can see that $T_1 + S$ is clearly V. Now we prove that $T_1 \cap S = \{0\}$. If we had some vector $v = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=n+1}^{n+m} \beta_i v_i \in S \cap T_1$ then

$$\sum_{i=1}^{n} \alpha_i v_i + \sum_{j=n+1}^{n+m} (-\beta_j) v_j = 0 \implies \alpha_1 = \ldots = \alpha_n = -\beta_{n+1} = \ldots = -\beta_{n+m} = 0$$

by the linear independance of $\{v_1, \ldots, v_{n+m}\}$ as its a basis. This implies that $v = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} 0 \cdot v_i = \sum_{i=1}^{n} 0 = 0$ and thus $S \cap T_1 = \{0\}$ so by the definition of a direct sum, $S \oplus T_1 = V$.

Now we claim that, the subspace T_2 spanned by $\{v_{n+1} + v_1, \ldots, v_{n+m} + v_m\}$ is also a complement of S and that $T_1 \cap T_2 = \{0\}$. First we check that $S + T_2 = V$. Take any vector $v = \sum_{k=1}^{n+m} \alpha_k v_k \in V$. Then we can rewrite this vector as

$$\sum_{k=1}^{m} (\alpha_k - \alpha_{n+k}) v_k + \sum_{m<k\leq n} \alpha_k v_k + \sum_{k=n+1}^{n+m} \alpha_k (v_k + v_{k-n}).$$

The first two sums are in S and the last sum is in T_2, so every vector of V can be expressed as an element of $S + T_2$, which shows that $S + T_2 = V$.

Now we check $S \cap T_2 = \{0\}$. Suppose $x \in S \cap T_2$. Then we can write x in two ways, once as $x = \sum_{i=1}^{n} \alpha_i v_i$ since $x \in S$ and again as $x = \sum_{k=1}^{m} \gamma_k (v_{n+k} + v_k)$ since $x \in T_2$. Comparing the coefficients in the basis $\{v_1, \ldots, v_{n+m}\}$ we see that for v_{n+k} we must have $\gamma_k = 0$ for all $k = 1, \dots, m$. Putting these back into the second expression we get $x = 0$. Hence $S \cap T_2 = \{0\}$.

Finally we check $T_1 \cap T_2 = \{0\}$. Suppose $y \in T_1 \cap T_2$. Then $y = \sum_{k=1}^{m} \beta_k v_{n+k}$ since $y \in T_1$ and also $y = \sum_{k=1}^{m} \gamma_k (v_{n+k} + v_k)$ since $y \in T_2$. Comparing coefficients of v_k gives $\gamma_k = 0$ for all $k = 1, \dots, m$, and thus the right hand side becomes $y = \sum_{k=1}^{m} 0 \cdot (v_{n+k} + v_k) = 0$. Therefore $y = 0$ and $T_1 \cap T_2 = \{0\}$.

Thus we have shown that $S \oplus T_2 = V$ and $T_1 \cap T_2 = \{0\}$ as required.