Analysis I

Home Assignment V

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October 31, 2025

Contents

1 Home Assignment V (Due: Dec 4, 2025)

Problem 1.1. (Discrete L'Hospital): Let $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$ be sequences of non-zero real numbers converging to 0. Suppose $b_n > b_{n+1}$ for all n and $v := \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ exists as a real number. Show that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and equals v. Give one such example.

Solution. Given any $\varepsilon > 0$ there exists N such that $\forall n > N$,

$$v - \varepsilon < \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < v + \varepsilon \implies (v - \varepsilon)(b_{n+1} - b_n) > a_{n+1} - a_n > (v + \varepsilon)(b_{n+1} - b_n)$$

Where the signs flipped because $b_{n+1} - b_n < 0$ as stated in the problem. We sum both sides of this inequality for $n, n+1, \ldots, m$ to get,

$$(v - \varepsilon)\sum_{k=n}^{m}(b_{k+1} - b_k) > \sum_{k=n}^{m}(a_{k+1} - a_k) > (v + \varepsilon)\sum_{k=n}^{m}(b_{k+1} - b_k)$$

$$\implies (v - \varepsilon)(b_{m+1} - b_n) > a_{m+1} - a_n > (v + \varepsilon)(b_{m+1} - b_n)$$

Thus this holds for all $m \geq n$ and taking the limits as $m \to \infty$ on all of the terms in this inequality we get,

$$(v - \varepsilon) \geq (-b_n) \geq -a_n \geq (v + \varepsilon)(-b_n) \implies v - \varepsilon \leq \frac{a_n}{b_n} \leq v + \varepsilon$$

which is enough to conclude that $a_n/b_n \longrightarrow v$ as well. □

Problem 1.2. Suppose $k \in \mathbb{N}$ and $b_1, b_2, \ldots, b_k$ are strictly positive real numbers. Show that (i) $\lim_{n\to\infty} b_j^{\frac{1}{n}} = 1$. (ii) $\lim_{n\to\infty}(b_1^n + b_2^n + \cdots + b_k^n)^{\frac{1}{n}} = b$ where $b = \max\{b_j : 1 \leq j \leq k\}$.

Solution. For (i), let $x > 0$ and consider the two cases : $x \geq 1$ and $x < 1$. For the first case, by bernoulli's inequality we see that,

$$x = (x^{1/n})^n = (1 + (x^{1/n} - 1))^n \geq 1 + n(x^{1/n} - 1) \implies x^{1/n} - 1 \leq \frac{x - 1}{n}$$

Here we have used the fact that $x^p \geq 1$ for any positive p, making $x^{1/n} - 1$ nonnegative. Thus we can use the squeeze theorem on the following inequality to conclude,

$$0 \leq x^{1/n} - 1 \leq \frac{x - 1}{n}$$

For the second case, we use another one of bernoulli's inequality again to get

$$x = (1 - (1 - x^{1/n}))^n \geq 1 - n(1 - x^{1/n}) \implies 1 - x^{1/n} \leq \frac{1 - x}{n}$$

where we used the fact that $0 < x^p < 1$ for any positive p, making $1 - x^{1/n}$ positive. Again we use the squeeze theorem on the following expression to conclud,e

$$0 \leq 1 - x^{1/n} \leq \frac{1 - x}{n}$$

To prove (ii) we will also use squeeze theorem. As b is the maximum of these b_i we have,

$$b^n < b_1^n + \ldots + b_k^n \leq b^n + \ldots + b^n = nb^n$$

$$\implies b < (b_1^n + \ldots + b_k^n)^{1/n} \le n^{1/n} b$$

Using the fact $n^{1/n} \longrightarrow 1$, which is proven in the immediate next exercise, we can conclude using squeeze theorem. $\square$

Problem 1.3. Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Solution. For large enough($n \ge 3$) we can say that $n^{1/n} \ge 1$ and then using the binomial theorem

$$n = (1 + (n^{1/n} - 1))^n = \sum_{k=0}^{n} \binom{n}{k} (n^{1/n} - 1)^k \ge \frac{n(n-1)}{2}(n^{1/n} - 1)^2$$

$$\implies 1 \le n^{1/n} \le 1 + \sqrt{\frac{2}{n-1}}$$

And yet again we conclude using the squeeze theorem. $\square$

Problem 1.4. Let $f : \mathbb{R} \to \mathbb{R}$, be a continuous function satisfying $f(3x) = f(x), \forall x \in \mathbb{R}$. Show that f is a constant function.

Solution. For all real x, by the condition we have $f(x) = f(3 \cdot (x/3)) = f(x/3)$ and continuing for n steps we have $f(x) = f(x/3^n)$. Now from the continuity of f,

$$f(x) = \lim_{n\to\infty} f(x/3^n) = f\left(\lim_{n\to\infty} x/3^n\right) = f(0)$$

which makes f a constant function as $\forall x \in \mathbb{R}, f(x) = f(0)$. $\square$

Problem 1.5. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $g(x + y) = g(x) + g(y), \forall x, y \in \mathbb{R}$. Show that $g = cx$ for some $c \in \mathbb{R}$.

Solution. Just as in Problem 2.10. (problem 10 in the 2nd assignment) we can show that for any rational r we have $f(r) = f(1) \cdot r$ and as given any real number x there exists a sequence of rationals $\{x_n\}$ converging to it we see that,

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(1) \cdot x_n = f(1) \cdot \lim_{n\to\infty} x_n = f(1) \cdot x$$

where we used the continuity of f in the second inequality and a basic fact from the algebra of limits in the fourth. Thus we see that $c = f(1)$ and $f(x) = f(1) \cdot x$ for all $x \in \mathbb{R}$ $\square$

Problem 1.6. Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$, is said to be convex if $f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y), \forall x < y$ in $I, 0 < \lambda < 1$. (i) Show that if $f : (0, 1) \to \mathbb{R}$ is convex then for $0 < s < t < u < 1$,

$$\frac{f(t) - f(s)}{t - s} \le \frac{f(u) - f(s)}{u - s} \le \frac{f(u) - f(t)}{u - t}.$$

(ii) Show that if $f : (0, 1) \to \mathbb{R}$, is convex then it is continuous. (iii) Show that a convex function $g : [0, 1] \to \mathbb{R}$, need not be continuous.

Solution. Rearranging the first inequality we get the following equivalent inequality,

$$\frac{f(t)}{t-s} - \frac{f(u)}{u-s} \leq \frac{f(s)}{t-s} - \frac{f(s)}{u-s} = \left(\frac{u-t}{(t-s)(u-s)}\right) f(s)$$

$$\iff f(t) \leq \frac{u-t}{u-s} \cdot f(s) + \frac{t-s}{u-s} \cdot f(u)$$

Which is true from the definiton of convexity via $\lambda = (u-t)/(u-s) \in (0,1)$. The second inequality is proven similarly. For (ii), we will show that for any $x \in (0,1)$, the right and left limits of $f(yi)$ as y tends to x is zero which will imply that the limit exists and is zero as well, proving continuity. Without loss of generality assume that $x < y$, then choose $u, v, w \in (0,1)$ such that $u < x < y < v$. Now by the previous result we see that,

$$\frac{f(x) - f(u)}{x-u} \leq \frac{f(y) - f(x)}{y-x} \leq \frac{f(v) - f(y)}{v-y} \leq \frac{f(w) - f(v)}{w-v}$$

$$\implies \frac{f(x) - f(u)}{x-u} \cdot (y-x) \leq f(y) - f(x) \leq (y-x) \cdot \frac{f(v) - f(y)}{v-y} \leq (y-x) \cdot \frac{f(w) - f(v)}{w-v}$$

Now, using the squeeze theorem, as y tends to x from the right we get,

$$\lim_{y \to x; y > x} (f(y) - f(x)) = 0$$

and the same can be said for the left limit using a very similar arguement and we are done. For (iii) we can consider the following function,

$$f(x) := \begin{cases} 1 \text{ if } 0 < x \leq 1 \\ 0 \text{ if } x = 0 \end{cases}$$

To show that this is convex we will consider the inequality as in the definition on all points $x, y \in [0,1]$. The inequality is clearly always true for $0 < x, y$ as the function is constant on $(0,1]$ and when some x, y is 0, wlog say $x = 0$, then,

$$f(\lambda x + (1-\lambda)y) = f((1-\lambda)y) \leq \underbrace{\lambda f(x)}_{=0} + (1-\lambda) f(y)$$

for both $y = 0$ and $y > 0$ as in the first case we have $0 \leq 0$ and in the second, as $1 - \lambda > 0, 1 = 1$. □

Problem 1.7. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$, such that for every $y \in \mathbb{R}$, there are exactly two real numbers x_1, x_2, such that $f(x_1) = f(x_2) = y$.

Solution. For the sake of contradiction assume that there is some f that is continuous and also satisfies said criterion. Let $x < y$ be the reals such that $f(x) = f(y) = 0$. Without loss of generality we can asssume $x < y$. Now consider the interval $[x, y]$, as f is continuous it must have some it must achieve its extrema in this interval at some points in the interval. Unless there is an extrema which is achieved only in (x, y) we can easily see that f is constant in this interval which gives us much more than two, infact infinitely many reals where f is zero which is not allowed as per the assumption. Again without loss of generality assume that we reach a maxima at $u \in (x, y)$. We claim that u is the unique such point in $[x, y]$. For a proof of this claim, for the sake of contradiction asssume without loss of generality that we also reach a maxima at $v \in (u, y]$. Now f can not be continous on the interval $[u, v]$ so there is some $w \in (u, v)$ such that $0 < f(w) < f(u)$ where the first inequality can be achieved due to continuity because by construction $f(u) = f(v) > 0$ and if we did not have some w it would produce jump discontinuities in (u, v). Now for any $\lambda \in (f(w), f(u))$ we can find a real z with $f(z) = \lambda$ in all three of the intervals : $(x, u), (u, v), (v, y)$ by IVT, which contradicts the assumption as there should be exactly two such z and the claim is proved. By the condition on f, there must be another $g \notin [x, y]$ such that $f(u) = f(g)$. Without loss of generality assume that $g > y$, then for any $\lambda \in (0, f(u))$ we can find some z with $f(z) = \lambda$ from all of these three intervals : $(x, u), (u, y), (y, g)$ by IVT which is a contradiction and we are done. $\square$

Problem 1.8. Fix $n \in \mathbb{N}$. Let $x_1, x_2, \ldots, x_n$ be n distinct real numbers and let $y_1, y_2, \ldots, y_n$ be another n-tuple of not-necessarily distinct real numbers. Show that there is a unique polynomial p of degree $(n-1)$ such that $p(x_k) = y_k$ for $1 \leq k \leq n$. Hint: Consider

$$p(x) = \sum_{j=1}^{n} y_j \frac{\prod_{i \neq j}(x - x_i)}{\prod_{i \neq j}(x_j - x_i)}.$$

Solution. The polynomial p provided in the hint is clearly of degree atmost $n-1$ and when evaluated at x_i, all but the i'th term in the sum vanish and in this term the numerator and denominator of the fraction cancel out leaving only y_i and thus $p(x_i) = y_i$. Assume p, q both have degree atmost $n-1$ and they both are polynomials which satisfy the condition in the problem, then we see that $p - q$ has n zeroes. $p - q$ clearly has degree atmost $n-1$ and thus for it to have $n > n-1$ zeroes, by the fundamental theorem of algebra we see that $p - q$ must be identically zero i.e. $p = q$ and hence such a polynomial is unique. $\square$

Problem 1.9. Let $a < b$ be real numbers. Let $f : (a, b) \to \mathbb{R}$ be a twice differentiable function. Show that for any $x_0 \in (a, b)$

$$f''(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}.$$

Solution. f being twice differentiable makes it continuous and thus the numerator tends to 0 as $h \to 0$ and the denominator also tends to zero as $h \to 0$ and also the derivative of the denominator is nonzero near 0,except at 0. Let the numerator be $X(h)$ and denominator $Y(h)$, then,

$$\frac{X'(h)}{Y'(h)} = \frac{f'(x_0 + h) - f'(x_0 - h)}{2h}$$

$$= \frac{f'(x_0 + h) - f'(x_0)}{2h} - \frac{f'(x_0) - f(x_0 - h)}{2h} \xrightarrow{h \to 0} \frac{f''(x_0)}{2} + \frac{f''(x_0)}{2} = f''(x_0)$$

and we can thus conclude via LHopital's rule. $\square$

Problem 1.10. (i) For $x \in \mathbb{R}$, show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely. (ii) Define $e : \mathbb{R} \to \mathbb{R}$ by $e(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Show that $e(x + y) = e(x)e(y), \forall x, y \in \mathbb{R}$.

Solution. We can say, using Problem 2.9. that for a sequence of positive numbers x_n that converge to x, that $\sqrt[n]{x_1 x_2 \ldots x_n} \xrightarrow{n \to \infty} x$ using that result on $\log(x_n)$ and using the continuity of log on its domain. We know from Problem 2.5. that the limit $(1+1/n)^n$ as n tends to infinity exists and using the binomial theorem we clearly see that $(1+1/n)^n \geq 1 + n(1/n) = 2$ so the limit is positive, call this $\mathtt{not_e}$. Now consider the product,

$$\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^k = \frac{2^1}{1^1}\cdot\frac{3^2}{2^2}\cdots\frac{(n+1)^n}{n^n} = \frac{(n+1)^n}{n!}$$

Using the results mentioned above we have,

$$\frac{n+1}{(n!)^{1/n}} = \sqrt[n]{\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^k} \xrightarrow{n\to\infty} \mathtt{not_e}$$

For (i), we will show that for any $x \in \mathbb{R}$ this series is absolutely convergent using the root test,

$$\lim_{n\to\infty}\sqrt[n]{|x^n/n!|} = \lim_{n\to\infty}\frac{|x|}{(n!)^{1/n}} = \lim_{n\to\infty}\frac{|x|}{n+1}\cdot\frac{n+1}{(n!)^{1/n}}$$
$$= \left(\lim_{n\to\infty}\frac{|x|}{n+1}\right)\cdot\lim_{n\to\infty}\frac{n+1}{(n!)^{1/n}} = 0\cdot\mathtt{not_e} = \mathtt{0} < \mathtt{1}$$

so we are done. For (ii), using the cauchy product theorem for product of sums, atleast one of which is absolutely convegrent we have,

$$e(x)e(y) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} = \sum_{n=0}^{\infty}\frac{(x+y)^n}{n!} = e(x+y)$$

□

Problem 1.11. Let $h:\mathbb{R}\to\mathbb{R}$ be the function $h(x)=\sin(x)$ (Here we are assuming your familiarity with trigonometric functions). Show that the remainder term in Taylor's theorem converges to 0 as $n\to\infty$ for every x_0 and x.

Solution. If we keep taking the derivative if h we can easily see that $|h^{(n)}(x)|$ is either $|\sin x|$ or $|\cos x|$, eitherways its bounded by 1. Now given any x, x_0 the $n-th$ remainder term, say R_n tends to zero as follow,

$$|R_n| \leq \frac{|x_0-x|^n}{n!}\cdot\sup_{x\in\mathbb{R}}|h^{(n)}(x)| = \frac{|x-x_0|^n}{n!} \xrightarrow{n\to\infty} 0$$

just as we did in Problem 5.10.. □

Problem 1.12. Consider the function $h:[0,\infty)\to\mathbb{R}$ defined by $h(x)=\frac{x}{1+x^2}$. Determine the set $\{h(x): x\in[0,\infty)\}$.

Solution. Clearly $h(x)\geq 0$ with equality at 0 and using AM-GM, $h(x)\leq 1/2$ with equality at $x=1$. By IVT we can say that it also achieves every value in between and hence the set is $[0,1/2]$ □

Problem 1.13. Let $v:\mathbb{R}\to\mathbb{R}$ be the function $v(x)=x^3-6x^2+9x$, for all x. Determine the set: $\{x: v(x)>0\}$.

Solution. We can factor this into $x(x-3)^2$, so it is zero at 0 and 2. For $x\neq 0,2$ it has the same sign as x as $(x-3)^2$ is positive. Thus the set is $(0,\infty)\setminus\{2\}$. □

Problem 1.14. Let $f : [0,1] \to [0,1]$ be a continuous strictly increasing bijection. Assume that $x < f(x)$ for all $x \in (0,1)$. Fix $x_0 \in (0,1)$. Define x_n for $n \in \mathbb{Z}$ by $x_n = f^n(x_0)$. Show that: (i) $0 < x_m < x_n < 1$ for $m < n$ in $\mathbb{Z}$; (ii) $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to-\infty} x_n = 0$; (iii) For every $n \in \mathbb{Z}$, f maps $[x_n, x_{n+1}]$ bijectively to $[x_{n+1}, x_{n+2}]$.

Solution. As f is strictly increasing we see that we must have $f(0) = 0$ and $f(1) = 1$ and as its also a bijection, 0 or 1 never appear in f orbits i.e. $\{f^n(x) : n \in \mathbb{Z}\}$ of any points in $(0,1)$. Thus we can see that for any $m < n$,

$$f^n(x_0) = \underbrace{f(f^{n-1}(x_0) > f^{n-1}(x_0) > \ldots > f^m(x_0)}_{n-m \text{ inequalities}}$$

and as 1 and 0 never appear in the orbit of x_0 we see that $0 < f^n(x_0) < 1$ for all $n \in \mathbb{Z}$ and together it reads $0 < x_m < x_n < 1$, we have thus proved (i). For (ii), as $n \to \infty$, $\{x_n\}$ must converge to a limit as its increasing and bounded above, say it converges to x, then $0 < x \leq 1$. Using the continuity of f we have can show that this is a fixed point as follows

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$$

so it has to be 1. We can similarly show that $x_n \xrightarrow{n\to-\infty} x_n = 0$ as well. For every $n \in \mathbb{Z}$ we see that $x_n < x_{n+1}$ and thus $x_{n+1} = f(x_n) < f(x_{n+1}) = x_{n+2}$ thus we have $f([x_n, x_{n+1}]) \subseteq [x_{n+1}, x_{n+2}]$ using the fact that f is increasing. Using IVT we can say that we have equality here i.e. $f([x_n, x_{n+1}]) = [x_{n+1}, x_{n+2}]$ and the f is clearly still a bijection here, this proves (iii). $\square$

Problem 1.15. Consider the set up of the previous question. Fix $y_0 \in (x_0, x_1)$. Set $y_n = f^n(y_0)$ for $n \in \mathbb{Z}$. Let $h : [x_0, y_0] \to [y_0, x_1]$ be a continuous strictly increasing bijection. Define $g : [0,1] \to [0,1]$ by $g(0) = 0, g(1) = 1$,

$$g(t) = h(t) \quad \forall t \in [x_0, y_0];$$

$$g(t) = f \circ h^{-1}(t) \quad \forall t \in [y_0, x_1];$$

and more generally, for $n \in \mathbb{Z}$, define

$$g(t) = f^n \circ h \circ f^{-n}(t) \quad \forall t \in [x_n, y_n]$$

and

$$g(t) = f^{n+1} \circ h^{-1} \circ f^{-n}(t) \quad \forall t \in [y_n, x_{n+1}].$$

Then show that: (i) For every n, g maps $[x_n, y_n]$ bijectively to $[y_n, x_{n+1}]$ and it maps $[y_n, x_{n+1}]$ bijectively to $[x_{n+1}, y_{n+1}]$. (ii) g is a strictly increasing continuous bijection. (iii) $f = g \circ g$. This shows that f has infinitely many square roots.