Exercise sheet 1

Only the exercises marked [HW] need to be submitted.
Vectors will be denoted in bold unlike scalars.

1. Verify carefully that the examples of vector spaces given in class satisfy all the axioms of a vector space.

2. In each of the following, find precisely which axiomss in the definition of a vector space are violated. Take $V = \mathbb{R}^2$ and $F = \mathbb{R}$.

   (a) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, 0), \quad \alpha(x_1, x_2) = (\alpha x_1, 0)$.

   (b) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad \alpha(x_1, x_2) = (\alpha x_1, 0)$.

   (c) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad \alpha(x_1, x_2) = (\alpha x_1, 2\alpha x_2)$.

   (d) $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \quad \alpha(x_1, x_2) = (\alpha + x_1, \alpha + x_2)$.

3. Show that the set of all positive real numbers forms a vector space over $\mathbb{R}$ if the sum of $\mathbf{x}$ and $\mathbf{y}$ is defined as the usual product $\mathbf{xy}$ and $\alpha\mathbf{x} := \mathbf{x}^\alpha$.

4. Shwo that the set of polynomials (with no upper bound on the degree) in a variable t, with coefficients from F, forms a vector space over F under the usual operations of addition and scalar multiplication.

5. We denote the vector $\mathbf{u} + (-\mathbf{v})$ as $\mathbf{u} - \mathbf{v}$. Prove the following

   (a) $\mathbf{u} - \mathbf{v}$ is the unique solution of $\mathbf{v} + \mathbf{x} = \mathbf{u}$.

   (b) $(\mathbf{u} - \mathbf{v}) + \mathbf{w} = (\mathbf{u} + \mathbf{w}) - \mathbf{v}$

   (c) $\alpha(\mathbf{u} - \mathbf{v}) = \alpha\mathbf{u} - \alpha\mathbf{v}$.

6. One can talk about vector spaces over a general *field* F and not just $\mathbb{R}, \mathbb{C}, \mathbb{Q}$. A field F is a set of objects (called *scalars*) along with two operations: *scalar addition* and *scalar multiplication* with the following properties.

   - $\alpha + \beta = \beta + \alpha$
   - $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
   - There is a unique element 0 (zero) in F such that $\alpha + 0 = x$, for every $\alpha \in F$.
   - To each $\alpha \in F$ there is a unique element $(-\alpha)$ in F such that $\alpha + (-\alpha) = 0$.
   - Multiplication is commutative: $\alpha\beta = \beta\alpha$
   - Mulitiplcation is associative: $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
   - There is a unique non-zero element 1 (one) in F such that $\alpha 1 = \alpha$ for every $\alpha \in F$.
   - To each non-zero $\alpha \in F$ there is a unique element α^{-1} such that $\alpha\alpha^{-1} = 1$
   - Mulitplication distributes over addition $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ for all $\alpha, \beta, \gamma \in F$.

   (a) $\mathbb{R}, \mathbb{C}, \mathbb{Q}$ are fields, however the set of integers $\mathbf{Z}$ is not (Show).

   (b) Here is another example of a field (show!) useful in computer science. Let $F = \{0, 1\}$ and define $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, $1 + 1 = 0$, and $0.0 = 0$, $0.1 = 1.0 = 0$, $1.1 = 1$.

   (c) In general, consider $\mathbf{Z}_p = \{0, 1, \cdots, p - 1\}$ with addition and multiplication (mod) p. Show that this is a field if and only if p is a prime.

7. Let Ω be a fixed non-empty set and let V be the set of all subsets of Ω (*power set of* Ω); the vectors in V are the subsets of Ω. Let $F = \mathbf{Z}_2$ defined in the above exercise. Define the sum of two vectors $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{A} + \mathbf{B} = (\mathbf{A} - \mathbf{B}) \cup (\mathbf{B} - \mathbf{A}).$$

   Define $1 \cdot \mathbf{A} = \mathbf{A}$ and $0 \cdot A = \emptyset$. Show that V is a vector space over F.

8. [HW 1, due August 1 in class] Let V be the set of all complex-valued functions f on the real line such that (for all $t \in \mathbb{R}$)

$$f(-t) = \overline{f(t)}.$$

The bar denotes complex conjugation. Show that V, with the operations

$$(f+g)(t) = f(t) + g(t)$$
$$(cf)(t) = cf(t)$$

is a vector space over the field of real numbers. Give an example of a function in it which is *not* real-valued.