EC504 Algorithms and Data Structure

Rich Brower Tu- Tr 11:00- 12:45

EC504 Course Organization

- Why Algorithm ==> Data Strutures
- CRSL text with Slide Summaries
- Scaling, Math and Empirical Analysis on Simple Cases.
- Use GitHub (EC405), Slack and CCS and Unix Tools
- HW’s pencil and paper handed at CCS
- Software delivered CCS — Must run from Makefile.
- Basic Unix environment — useful for computer engineers to know!

Course Organization

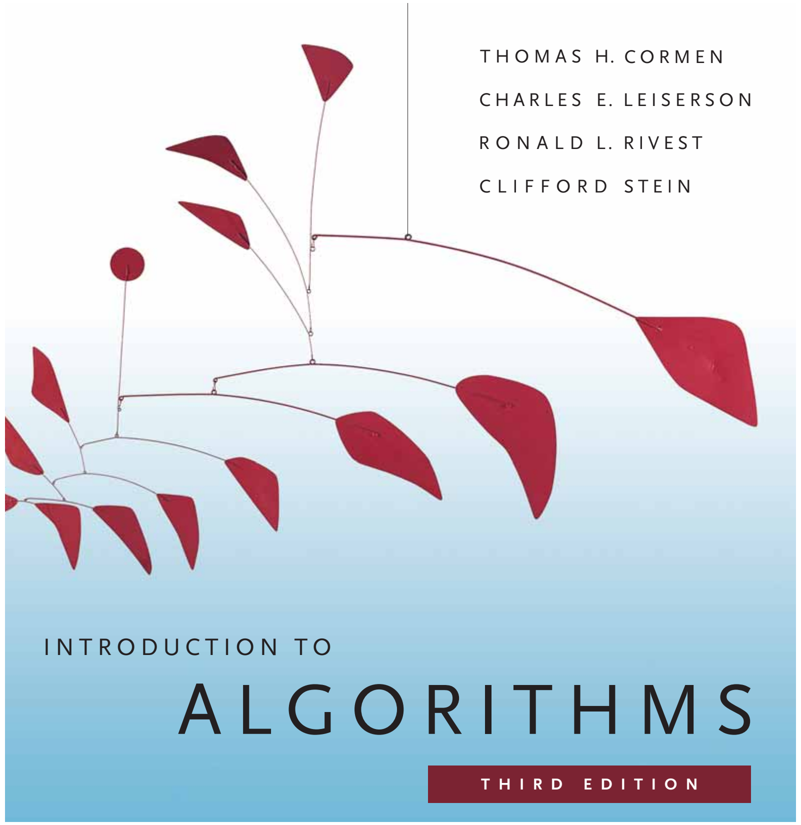
- Text:
  - Cormen, Leiserson, Rivest & Stein (CLRS), Fundamental text!

    "Introduction to Algorithms" 3rd Edition MIT Pres
- Keynote Slides guide to CLRS
- Reference:

- Wikepedia!
- Mark Allen Weiss "Data Structure and Algorithms in C".
- UNIX, Makefiles, very basic C/C++ and gnuplot:

- Grading:

– HW with Coding Programs:	30%
– Class participation	10%++
– Project	25%
– Midterm (up to trees)	15%
– Final (comprehensive)	20%



C vs C++ Advice — C++ compiler is C

1. To understand what is under the hood and how algorithms work go to C (The C++ compiler generates C!)

3. Bottom up: Simpler the better to see how the computer executes an algorithm and to optimized performance.

5. (KISS) Avoid C++ sugar. Adopt C++ism only when that add value.

7. Of course you can always use C++ libraries!

9. Standards are always better Avoid re-inventing the wheel!

Course Outline

- Algorithms Analysis CRSL 2-4 (5) (HW1?)
  - Definition of Problem Class of Size N
  - Math for large N Asymptotics:
- I. 1-D Data Structures CRSL 6,7,8,9 (HW2?)
  - Arrays, Lists, Stacks, Queues CRSL 10
  - Searching, Sorting, String Matching, Scheduling
- II. 1.5 D Trees CRLS 12 -`14 (HW3?)
  - BST, AVL,
  - Coding, Union/Join CRLS 18-21, midterm (HW4?)
- III. 2D Graphs CRLS 22,23,24,25, (HW 5?)
  - Traversal, Min Spanning Tree, Shortest Path, Capacity, Min Flow CRLS 26, (HW6?)
- IV Selected Advanced Topics & Projects
  - Spatial Data Structures, FFT's, Complexity, Approx. Solutions, Quantum Computing etc

INTRODUCTORY READING IN CRLS

- CRLS I.2

- CRLS I.3

- CRLS I.4

- CRLS 1.5 Just a bit of averaging!

What is an algorithm? An unambiguous list of steps (program) to transform some input into some output.

- Pick a Problem (set)
- Find method to solve
  1. Correct for all cases (elements of set)
  2. Each step is finite ($\Delta\, t_{step}$ < max time)
  - Next step is unambiguous
  - Terminate in finite number of steps
- You know many examples:

  GCD, Multiply 2 N bit integers, …

Abu Ja'far Muhammid ibn Musa Al-Khwarizmi
Bagdad (Iraq) 780-850

Searching Sorted List :

- int a[0], a[1],a[2],a[3],.... a[m],....   a[2],a[N-1]

Three Algorithms:

- *Linear Search* ➔ O(N)

  (after Sorting)

- *Bisection Search* ➔ O(log(N)).
- *Dictionary Search* ➔ O(log[log[N]])

Euclid's Algorithm GCD

(325-265 BC in Egypt)

*The Greatest Common Divisor (gcd) of positive integers **p** and **q** is the largest integer which divides **p** and **q** evenly.*
Can assume $p > q$

If $p = n\, q + r$, then
$\gcd(p,q) = \gcd(q,r)$ →

```
int qcd(int p, int q)
{  int r;
while(q!=0){
  r = p%q;
   p=q; q=r;}
Return p; }
```

gcd(22,8): $22 = 2*8 + 6$

gcd(8,6): $8 = 1*6 + 2$

gcd(6,2): $6 = 3*2 + 0$

Answer = 2

Complexity: q = N: T(N) = calls to gcd
Worst case: $T(N) < 1.44 \log_2(N)$
Average: $T(N)$ » $(12 \ln 2/\pi^2) \ln N$

Proof of Euclid's Algorithms

- With r = p%q, p = n q + r and k = gcd(p, q) then
- Therefore p = k p' and q = k q'

) k p' = n k q' + r

) r = k & q has k as a factor so

) gcd(q, r) = K >= k
- BUT K can't be bigger then k since

) p = n q + r = n (K q '' + K r)

) k = n K >= K

) gcd(q,r) = gcd(p,q)

Also note p mod q < p/2 so T(N) < 2 $\log_2$(N)

Halting Problem:
is this an algorithm?

The 3 x + 1 Problem by L. Collatz (1937)

```
ENDS?(x):
    while x > 1:
        print x
        if x is even
            then x gets x/2
            else x gets 3x+1
    print x
    halt!
```

Examples:

x=1 → 1
x=2 → 2,1
x=3 → 3,10,5,16,8,4,2,1
...
x=27→27,82,41,124,62,31,94,47,142,71,214,107,322,161,484,
242,121,364,182,91,274,137,412,206,103,310,155,466,233,
700,350,175,526,263,790,395,1186,593,1780,890,445,1336,618,
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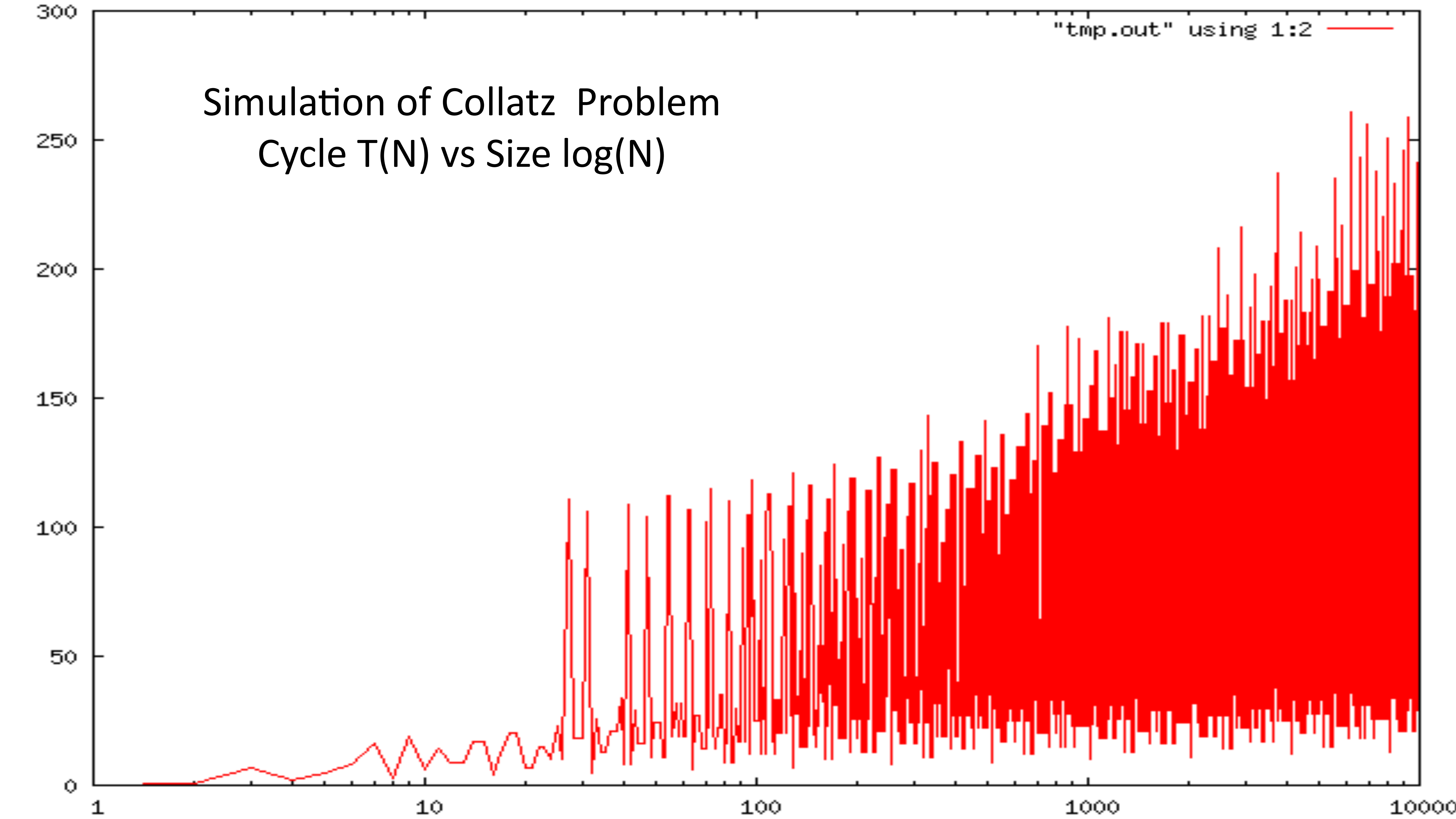
5,16,8,4,2,1

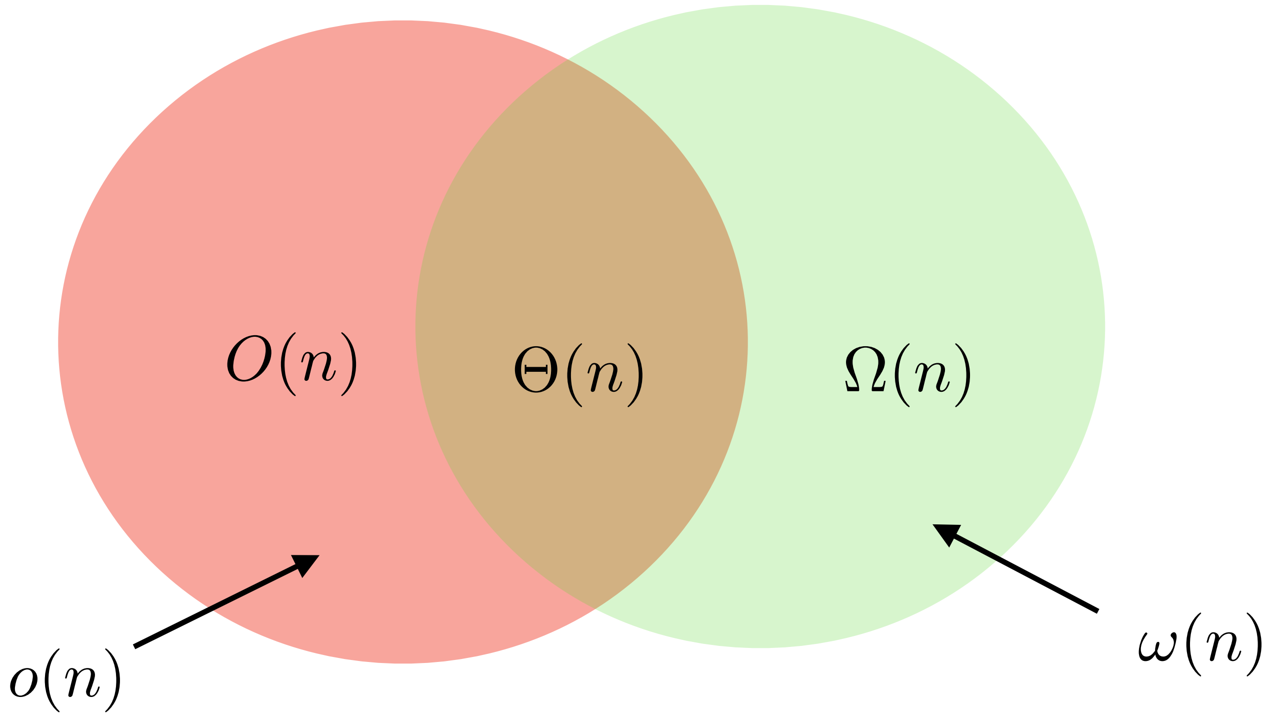


Simulation of Collatz Problem
Cycle T(N) vs Size log(N)
"tmp.out" using 1:2
300
250
200
150
100
50
0
1
10
100
1000
10000

Growth of Algorithm with Size n

$$T(n) = O(g(n)) \quad \text{or} \quad T(n) \in O(g(n))$$

- T(n) in set O(g(n))
  - like T(n) <= g(n) for large
  - e.g n^a log(n) exp[n] etc.



Why is big-O important?

(processor doing ~1,000,000 steps per second)

time / input size						
N	**10**	**20**	**30**	**40**	**50**	**60**
log n	3.3µsec	4.4µsec	5µsec	5.3µsec	5.6µsec	5.9µsec
n	10µsec	20µsec	30µsec	40µsec	50µsec	60µsec
n^2	100µsec	400µsec	900µsec	1.5msec	2.5msec	3.6msec
n^5	0.1sec	3.2sec	24.3sec	1.7min	5.2min	13min
3^n	59msec	48min	6.5yrs	385,500yrs	2×10^8 centuries...	
n!	3sec	7.8×10^8 millennia ……				

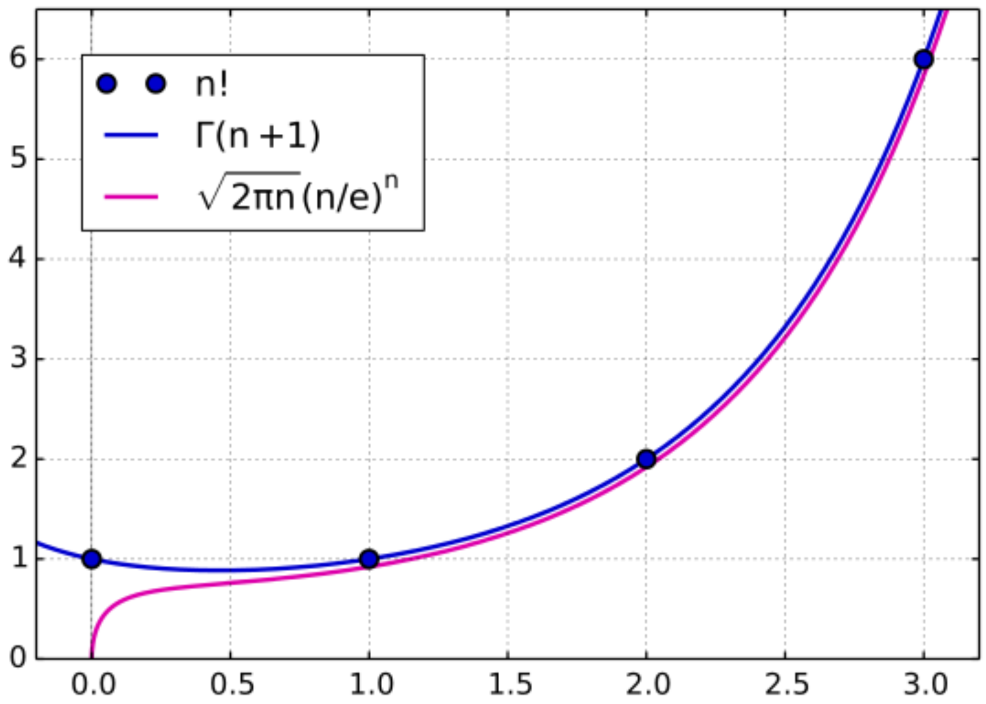
Non polynomial algorithms are terrible!
Logs are great!

All Logarithms are the "same"

$$N = b^{\log_b(N)}$$

Therefore $$\log_a(N) = log_a(b^{\log_b(N)}) = \log_a(b)\log_b(N)$$

Factorial: Worse than Exponential



$$\text{Decisions } 2^D > N! \text{ implies}$$

$$D > Log_2(N!) \simeq NlogN \cdots$$

Rules of thumb

- For polynomials, only the largest term matters.

$$a_0 + a_1 N + a_2 N^2 + \cdots + a_k N^k \in O(N^k)$$

- log N is in o(N)

  Proof: As N ➔ 1 the ratio log(N)/N ➔ 0

- Some common functions in increasing order:

1 log N √N N NlogN N^2 N^3 N^{100} 2^N 3^N N! N^N

Insertion Sort --- Deck of Cards

- Insertion Sort(a[0:N-1]):

```
for (i=1;  i < n;  i ++)
        for (j = i;  (j>0) && (a[j]<a[j-1]);  j--)
                swap a[j] and a[j-1] ;
```

Worst case $\Theta(N^2)$ number of "swaps" (i.e. time)

Outer loop trace for Insertion Sort: O(n^2)

- a[0] a[1] a[2] a[3] a[4] a[5] a[6] a[7]

(Swaps)

- 6 | 5 2 8 3 4 7 1 (1)

5 ← → 6

- 5 6 | 2 8 3 4 7 1

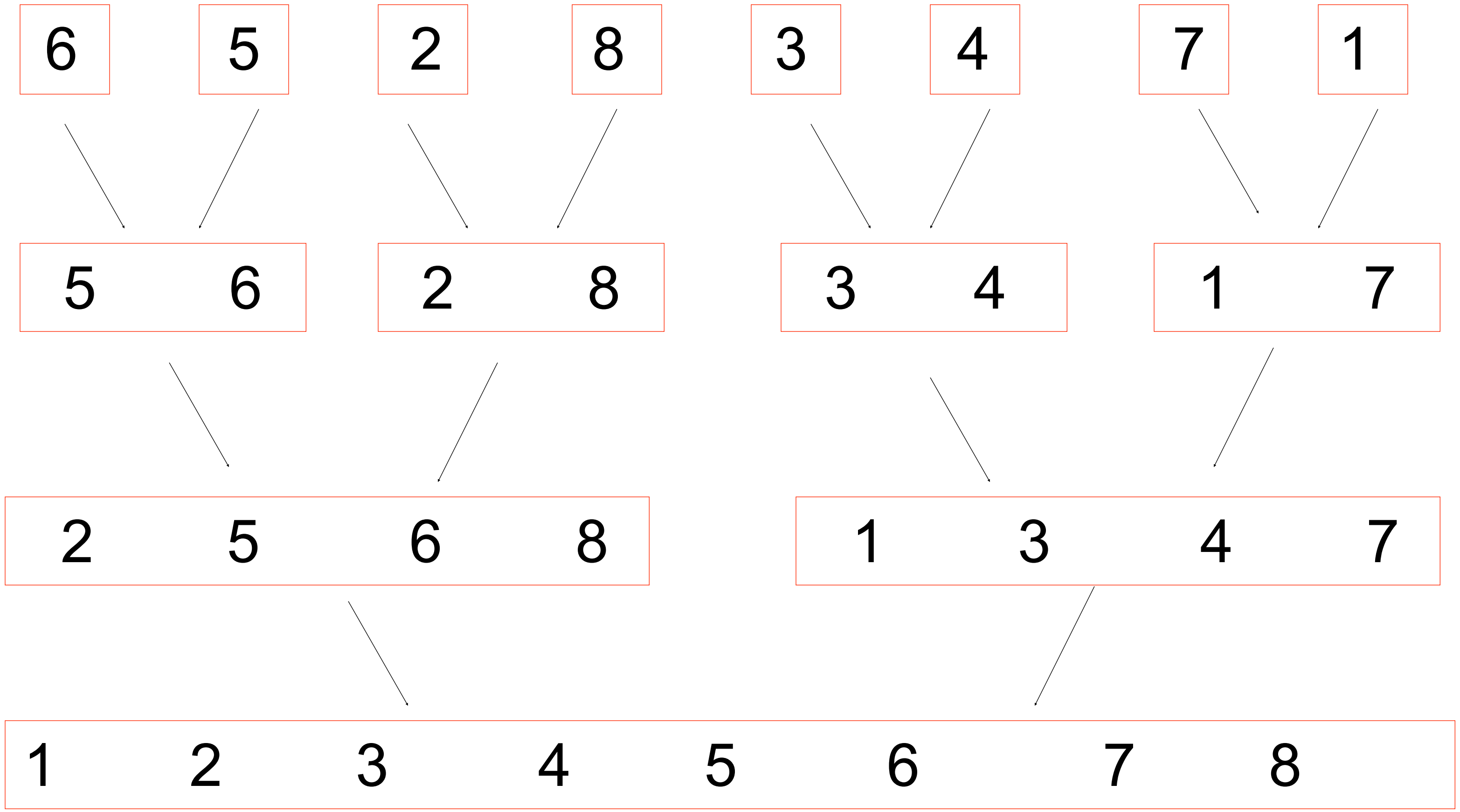
(2)

2 ← → 6

2 ← → 5

- 2 5 6 | 8 3 4 7 1 (0)
- 2 5 6 8 | 3 4 7 1 (3)
- 2 3 5 6 8 | 4 7 1 (3)
- 2 3 4 5 6 8 | 7 1 (1)
- 2 3 4 5 6 7 8 | 1 (7)

Merge Sort - Recursive O(n log(n))

- a[0] a[1] a[2] a[3] a[4] a[5] a[6] a[7]
- 

How do we find T(n)? What is big Oh ?

- Count the number of steps:
  - What is a step? RAM serial model.
  - Iterative loops: Sum series like

$$\sum_{i=0}^{N} i^k = 1 + 2^k + 3^k + \cdots + N^k \sim O(N^{k+1})$$

but k= -1 ➔ O(log(n))

  - Solve Recursive Relations:

T(n) = a T(n/b) + O(f(n))

Sums

- Cases:

$$\sum_{i=1}^{N} 1 = N \approx \frac{1}{1} N$$

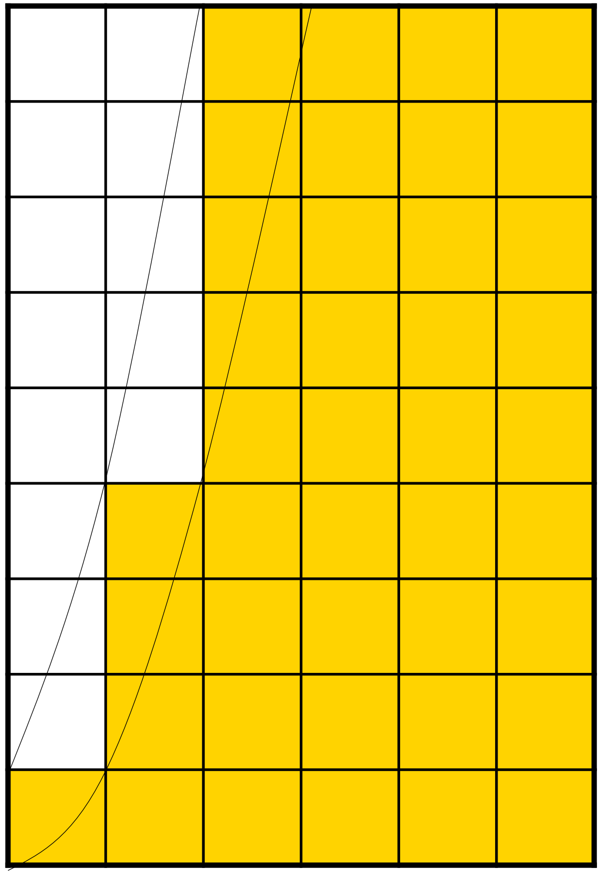
$$\sum_{i=1}^{N} i = \frac{1}{2} N(N+1) \approx \frac{1}{2} N^2$$

$$\sum_{i=1}^{N} i^2 = \frac{1}{6} N(N+1)(2N+1) \approx \frac{1}{3} N^3$$

$$\sum_{i=1}^{N} i^k \approx \frac{1}{k+1} N^{k+1}$$

Prove this by Integration:

Estimating Sums



- Integral Bounds:

$$S_k = \sum_{i=1}^{N} i^k$$

Estimate by integrating $S_k(x) = x^k$

$$\int_0^N x^k dx \le S_k = \sum_{i=1}^{N} i^k \le \int_0^N (x+1)^k dx$$

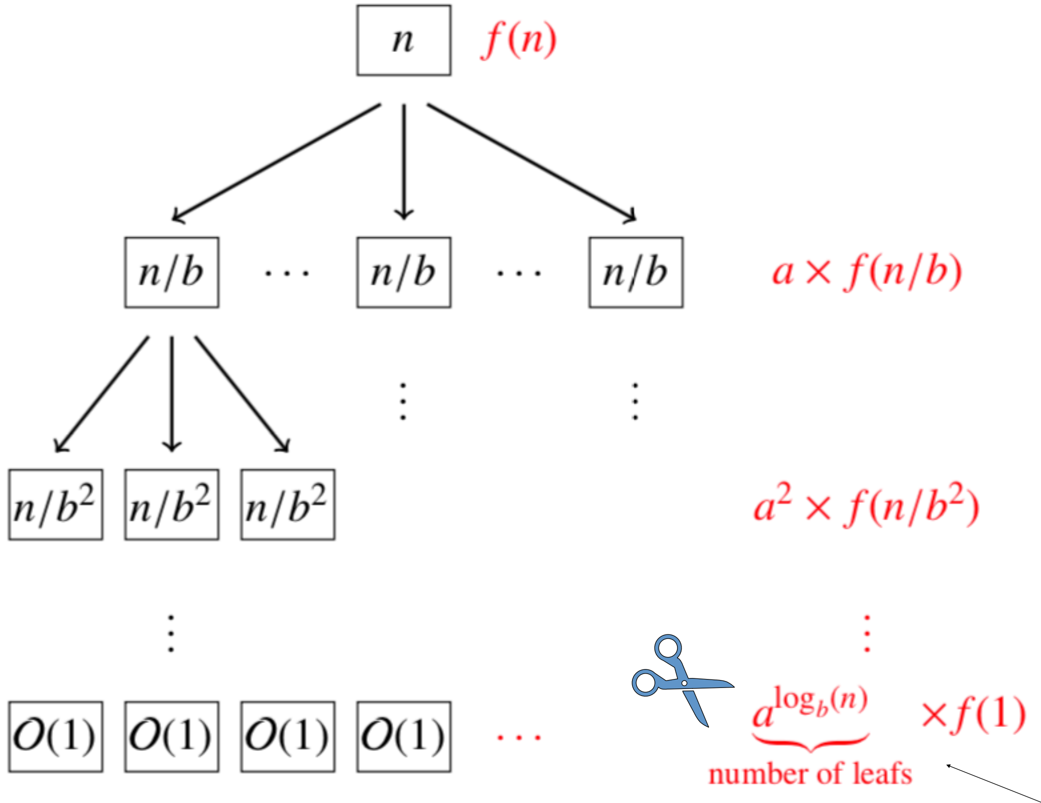
$$\frac{1}{k+1} N^{k+1} \le S_k \le \frac{1}{k+1}((N+1)^{k+1} - 1)$$

Build Tree to Solve

$$T(n) = aT(n/b) + f(n)$$

$$\boxed{n/b^h = 1 \implies h = \log_b(n)}$$

Height



Breadth

$$\boxed{T(n) = f(n) + af(n/b) + \cdots + a^{log_b(n)-1} f(b^2) + a^h T(1)}$$

Master Equation (brute force): T(n) = aT(n/b) + f(n)

$$
\begin{aligned}
T(n) &= aT(n/b) + f(n) \\
aT(n/b) &= a^2T(n/b^2) + af(n/b) \\
a^2T(n/b^2) &= a^3T(n/b^3) + a^2f(n/b^2) \\
\ldots \qquad & \ldots \\
a^{h-2}T(b^2) &= a^{h-1}T(b) + a^{h-2}f(b^2) \\
a^{h-1}T(b) &= a^hT(1) + a^{h-1}f(b)
\end{aligned}
$$

$$T(n) = a^hT(1) + f(n) + af(n/b) + a^2f(n/b^2) + \cdots + a^{h-1}f(b)$$

$a^h = n^\gamma$ **using:** $n/b^h = 1 \implies h = \log_b(n)$

Let's be very careful for $f(n) = cn^k$

$$T(n) = aT(n/b) + c\, n^k$$

$$aT(n/b) = a^2T(n/b^2) + c\; an^k/b^k$$

$$a^2T(n/b^2) = a^3T(n/b^3) + c\; a^2n^k/b^{2k}$$

$$\cdots \qquad \cdots$$

$$a^{h-2}T(b^2) = a^{h-1}T(b) + c\; a^{h-2}n^k/b^{(h-2)k}$$

$$a^{h-1}T(b) = a^hT(1) + c\; a^{h-1}n^k/b^{(h-1)k}$$

Therefore $$T(n) = a^hT(1) + c\; n^k \frac{(a/b^k)^h - 1}{a/b^k - 1}$$

$$a^h = n^\gamma \longrightarrow \quad = n^\gamma T(1) + c\, \frac{n^\gamma - n^k}{a/b^k - 1}$$

$$\boxed{\text{since} \quad 1 + a/b^k + (a/b^k)^2 + (a/b^k)^3 + \cdots + (a/b^k)^{h-1} = \frac{(a/b^k)^h - 1}{a/b^k - 1}}$$

Master Equation:

$T(N) = a\,T(N/b) + \Theta(g(N))$

Theorem: The asymptotic Solution is:

- $T(N) \in \Theta(N^\gamma)$ if $g(N) \in O(N^{\gamma-\epsilon})\ \forall\epsilon > 0$
- $T(N) \in \Theta(g(N))$ if $g(N) \in \Omega(N^{\gamma+\epsilon})\ \forall\epsilon > 0$
- $T(N) \in \Theta(N^\gamma \log(N))$ if $g(N) \in \Theta(N^\gamma)$

where $a = b^\gamma$ or $\gamma = \log(a)/\log(b)$

L'Hospital's Rule

Limit for ratio is same as for ratio of derivatives!

$$\lim_{N\to\infty} \frac{f(N)}{g(N)} = \lim_{N\to\infty} \frac{\frac{df(N)}{dN}}{\frac{dg(N)}{dN}}$$

e.g.

$$\lim_{N\to\infty} \frac{\log^2(N)}{N} =$$

$$\lim_{N\to\infty} \frac{2\log(N)/N}{1} = \lim_{N\to\infty} \frac{2/N}{1} = 0$$

$$\gamma - k \to 0, \quad where \quad a = b^\gamma$$

$$T(N) = N^\gamma T(1) + c_0(N^\gamma - N^k)/(a/b^k - 1)$$

$$T(N) = N^\gamma T(1) + c_0 N^k \frac{N^{\gamma-k} - 1}{b^{\gamma-k} - 1}$$

Take derivative with respect to $x = \gamma - k$

$$T(N) = N^\gamma T(1) + c_0 N^k \log(N)/\log(b)$$

More useful stuff

- Logarithmic sum (Harmonic Series):

$$H_N = \sum_{n=1}^{N} \frac{1}{n} = \ln(N) + \gamma_{Euler} + \Theta(1/N)$$

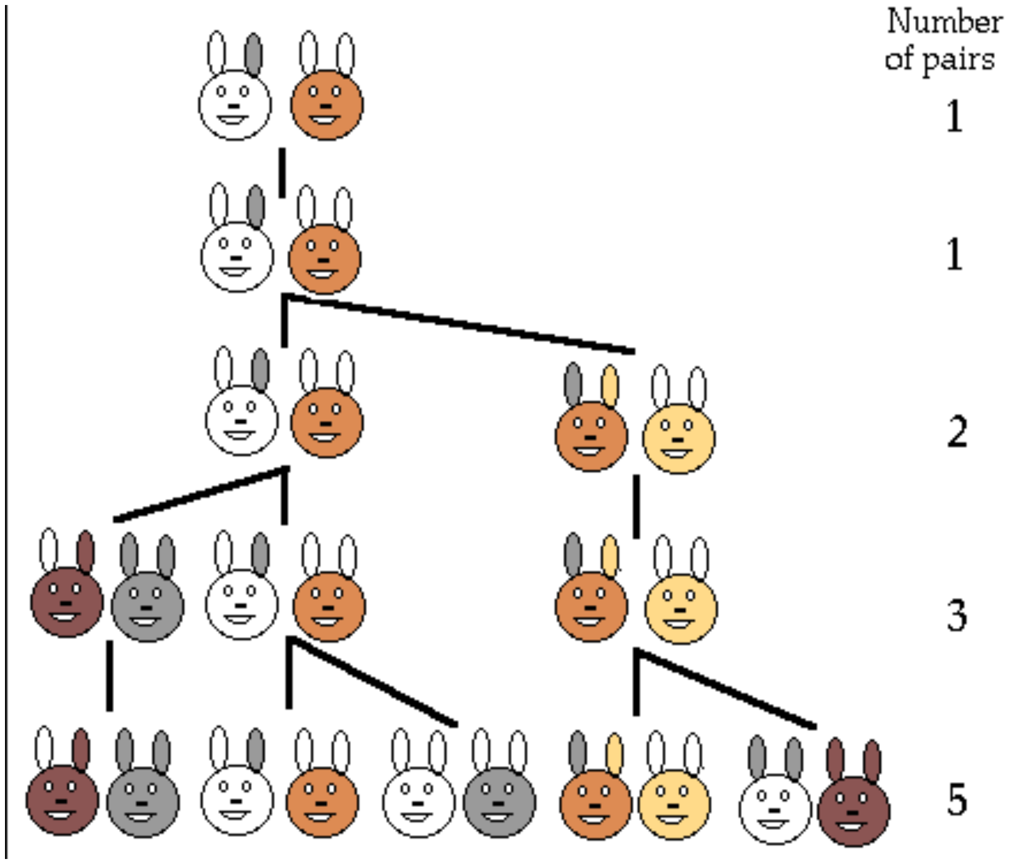
$$\gamma_{Euler} = 0.57721566490153286060651209008 2$$

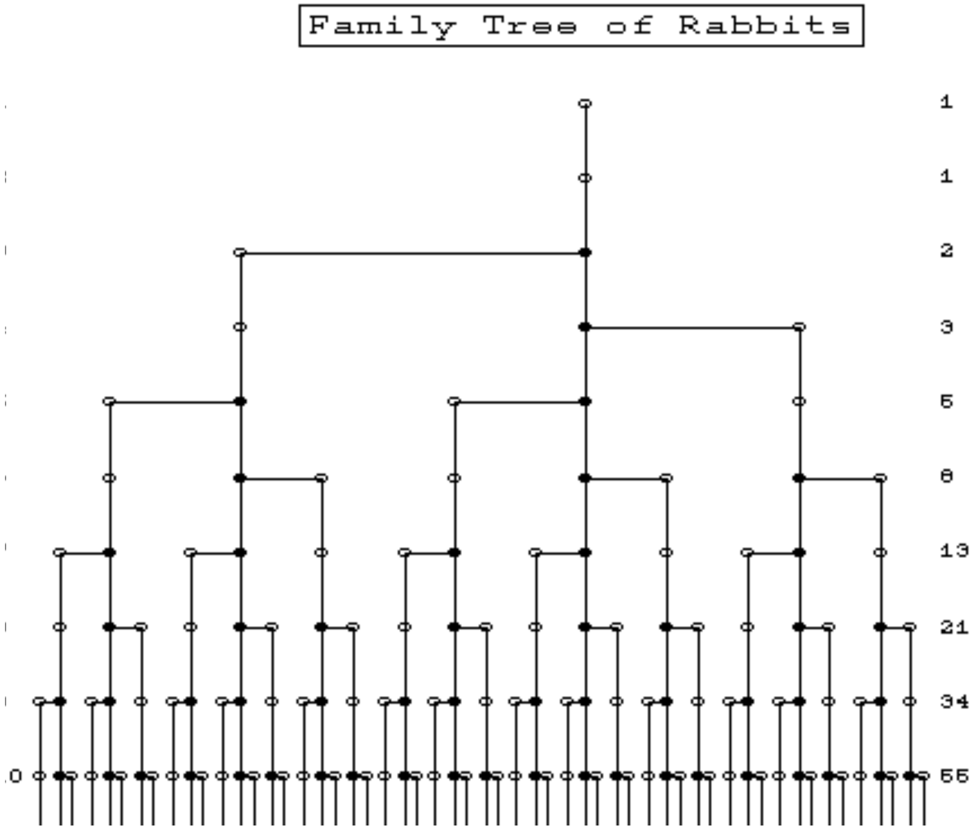
- Stirling's Approx: $N! \simeq \sqrt{2\pi N} N^N e^{-N}(1 + O(1/N))$

$$\log(N!) = N\log(N) - N\log(e) + \frac{\log(2\pi N)}{2} + \Omega(1/N)$$

Rabbits

Fibonacci investigated (in the year 1202) was about how fast rabbits could breed in ideal circumstances.
Females take one month to mature: Pairs mate and produce a male and female in a month





Fibonacci: $F_k = F_{k-1} + F_{k-2}$ => *0, 1, 1, 2, 3, 5, 8 , ...*

Characteristic equation, try:

$$F_k = \phi^k \quad \Longrightarrow \quad \phi^k = \phi^{k-1} + \phi^{k-2}$$

$$\phi^2 - \phi - 1 = 0 \qquad \phi = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$$

$$F_k = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^k + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^k$$

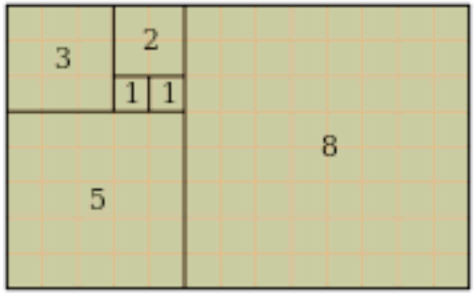
$$F_k = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \right]$$

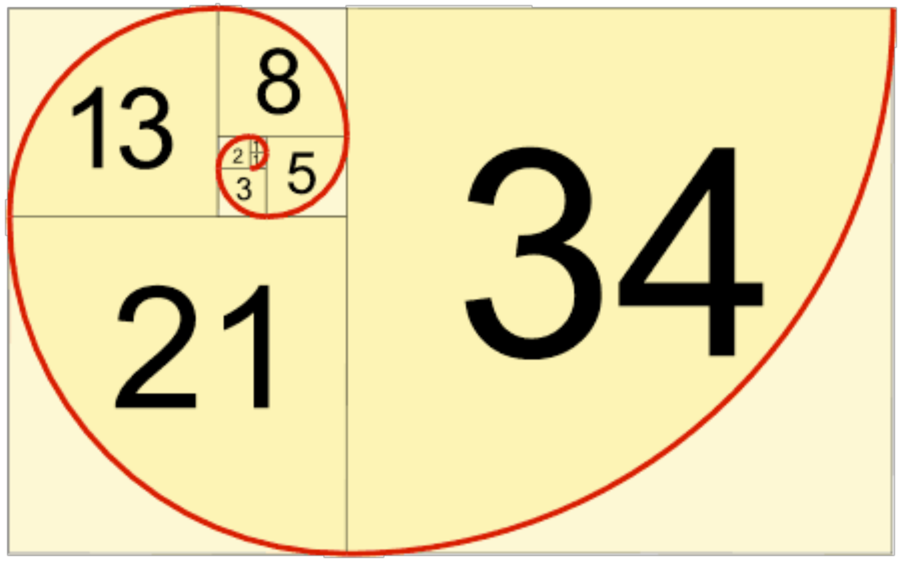
$$ax^2 + bx + c = 0 \quad \Longrightarrow \quad x = -b/2a \pm \sqrt{(b/2a)^2 - c/a}$$

Fibonacci: F(N) = F(N-1) + F(N-2) ➔

0,1,1,2,3,5,8 , ... for N = 0,1,2,3,....

- Many examples in nature!





Rabits, Bees and Double Window Panes

Maximum Subsequence Sum: CLRS 4.1

- Given a[0], a[1] ,..., a[N-1] find max $\boxed{\sum_{k=i}^{j} a[k]}$
  - Dumbest $\boxed{O(N^3)}$
  - Dumb $\boxed{O(N^2)}$
  - Smart $\boxed{O(N \log(N))}$
  - Smartest $\boxed{O(N)}$

```
for ( i = 0; i< N; i++)
    for(j = i; j < N; j++)
      { Sum = 0;
        for(k=i; k<j+1; k++)
              Sum += a[k];
              if(Sum > MaxSum)
               MaxSum = Sum;
}
```

$$O(N^3)$$

$$\begin{aligned}
\Sigma_{k=i}^{j} 1 &= j - i + 1 \\
\Sigma_{j=i}^{N-1} (j - i + 1) &= \tfrac{1}{2}(N - i)(N - i + 1) \\
\tfrac{1}{2}\Sigma_{i=0}^{N-1} i(i + 1) &= \tfrac{1}{6}(N^3 + 3N + 2N)
\end{aligned}$$

```
for ( i = 0; i< N; i++)
      { Sum = 0;
        for(j=i; j<N; j++)
              Sum += a[j];
              if(Sum > MaxSum)
               MaxSum = Sum;
}
```

$$O(N^2)$$

$$\begin{aligned}
\Sigma_{j=i}^{N-1} 1 &= N - 1 - i + 1 = N - i \\
\Sigma_{i=0}^{N-1} (N - i) &= N^2 - \left(\frac{N(N-1)}{2}\right) \\
&= \tfrac{1}{2}(N^2 + N)
\end{aligned}$$

Recursion versus Single Pass

- T(N) = 2 T(N/2) + c N
  - Large left/right + sum to left and right for split screen.

$\boxed{O(N\ \log(N))}$

$\boxed{O(N)}$

- On line:
  - Quit when you are in debt and start over.

```
Sum = 0;
     for(j=0; j<N; j++){
          Sum += a[j];
          if(Sum > MaxSum)
               MaxSum = Sum
          else if (Sum < 0)
               Sum = 0;
               }
```