Optimization et al Algorithms

- Scheduling
  - Integer/fraction Knapsack — CLRS 16.2
  - Job Schedule — Ex. 16-2 (a)
  - Deadline Schedule — CLRS 16.5
- KMP String matching — CLRS 32
- Union/Find Algorithm — CLRS 21
  - Union by height/size
  - Path compression
    - $O(M \log^*(N))$ explained
  - Binomial Queue: Union/Find

Scheduling & Linear Programming

- *Knapsack*
- *Queuing*
- *Schedule with Deadlines*

Fractional Knapsack

- Given set of task that take "time" $t_1, t_2, t_3, \ldots, t_N$ and value $v_1, v_2, v_3, \ldots, v_N$

Allocate to maximize objective

$$S(x_i) = \Sigma_i x_i V_i$$

with limited resource T:

$$\Sigma_i x\_i \; t_i <= T$$

Problem is to find assignment fractions

$$0 <= x_i <= 1$$

(Put most valuable parts of N objects in to your sack)

Solution:

- Sort in increasing value per time v_i/t_i and take them in order as far as possible:
- All: $x_1 = 1$, $x_2 = 1$, ..., $x_{m-1} = 1$, Some of $0 \cdot x_m \cdot 1$, None: $x_{i+m} = 0$, ..., $x_N = 0$
- So that $t_1 + t_2 + \ldots t_{m-1} + x_m t_m = T$ exactly $0 \cdot x_m = (T - t_1 - t_2 - \ldots - t_{m-1})/t_m \cdot 1$

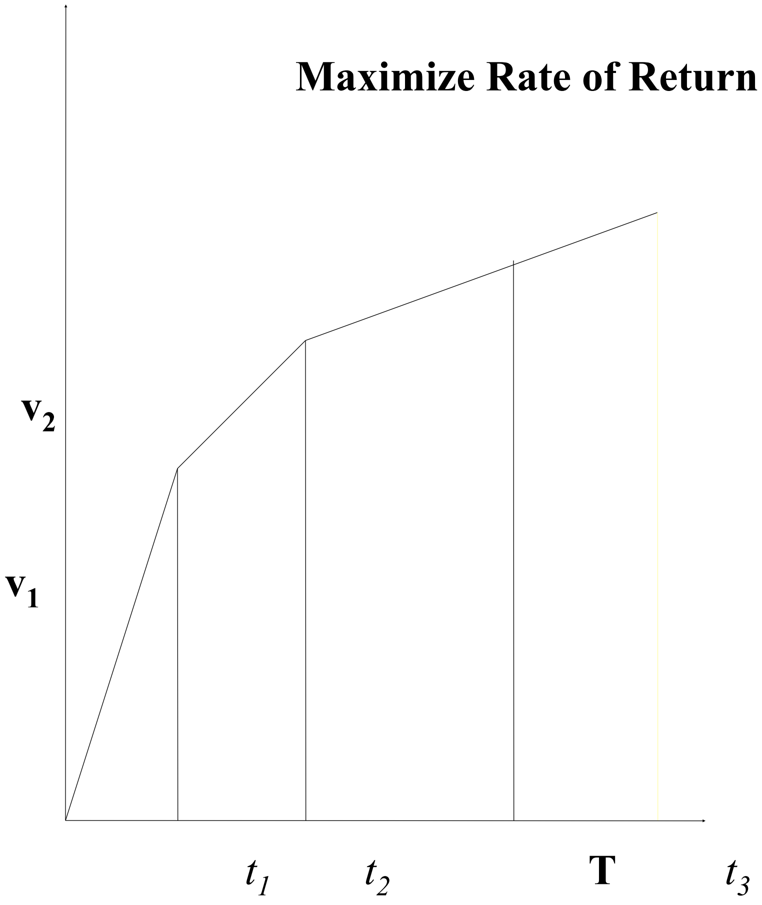
Integer Knapsack: x_i = 0 or 1 is VERY HARD to SOLVE!

Example: T = 50

Objects: v/t = \$60 / 10, \$100/20, \$120/30

Greedy \$60 + \$100 = \$160
Try it: Best integer is \$100 + \$120 = \$220
Best fractional: \$240

See CRLS Fig. 16.2



Queuing:

- GOAL: Minimize Waiting in Line
- Jobs in queue take time t_i
- Minimize total waiting time:

$$t_1 + (t_1 + t_2) + (t_1+t_2+t_3) + \ldots + (t_1 + \ldots + t_N)$$

$$W = N\, t_1 + (N-1)\, t_2 + \ldots + t_N$$

- Solution: Min W by sorting t's in ascending order (shorter jobs first)!
- Recall sorting is by "swap theorem"

$$\text{Sort} \;\equiv\; \text{Max over permutation} \sum_i i a[i]$$

Scheduling with deadlines

- GOAL: Maximize Value

Δ

- Task all take same time: T but they have different values and deadlines

- Sort in descending values (or penalties!):

  $v_1, \quad v_2, \ldots, v_N$

  $d_1, \quad d_2, \quad \ldots, \quad d_N$

- Feasible Solutions can always be in DFF!
- So this is the construction:
  - Select them one at a time from ordered list of v's

CRLS Fig 16.7

a	1	2	3	4	5	6	7
d	4	2	4	3	1	4	6
v	70	60	50	40	30	20	10

a4 a2 a3 a1 a7

Select a1, a2,a3 and a4

Reject a5 and a6

Select a7

Knuth-Morris-Pratt string matching

- Text: T(1), T(2),…, T(N)
- Pattern: P(1), P(2),…, P(M)
- Match if
  - T(s+1) =P(1), T(s+2) = P(2),.., T(M) = P(M)
  - or T(s+1:s+M) = P(1:M)
- Trivial scan O(M (N-M+1))
- KMP algorithm O(N+M) by amortized analysis

Prefix function:

- Given a partial match P(1:q) = T(s+1:s+q) what is the smallest shift that matches end of T(s+1:s+q)
- It is q – pi(q) where pi(q) prefix function for the max match of prefix to suffix of P(1:q)

- pi(q) = Max{k<q s.t. P(1:k) = P(1+q- k:q}

i.e. q matches become pi(q) ds = q – pi(q)

- Strategy pre-compute pi(q) and use it to advance the match.

KMP Algorithm

q matchs out

q matchs in

- Compute pi(q) set q = 0
- For i = 1, N

```
{   // count c_i at while c_i + q(i) – q(i-1) < =1
    while q>0 and P(q+1) \= T(i) set q =pi(q) ;
     if P(q+1) = T(i) then q = q+1 ;
     if q = M {
         report pattern found T(i – M;i);
         q = pi(q);
                }
 }
```

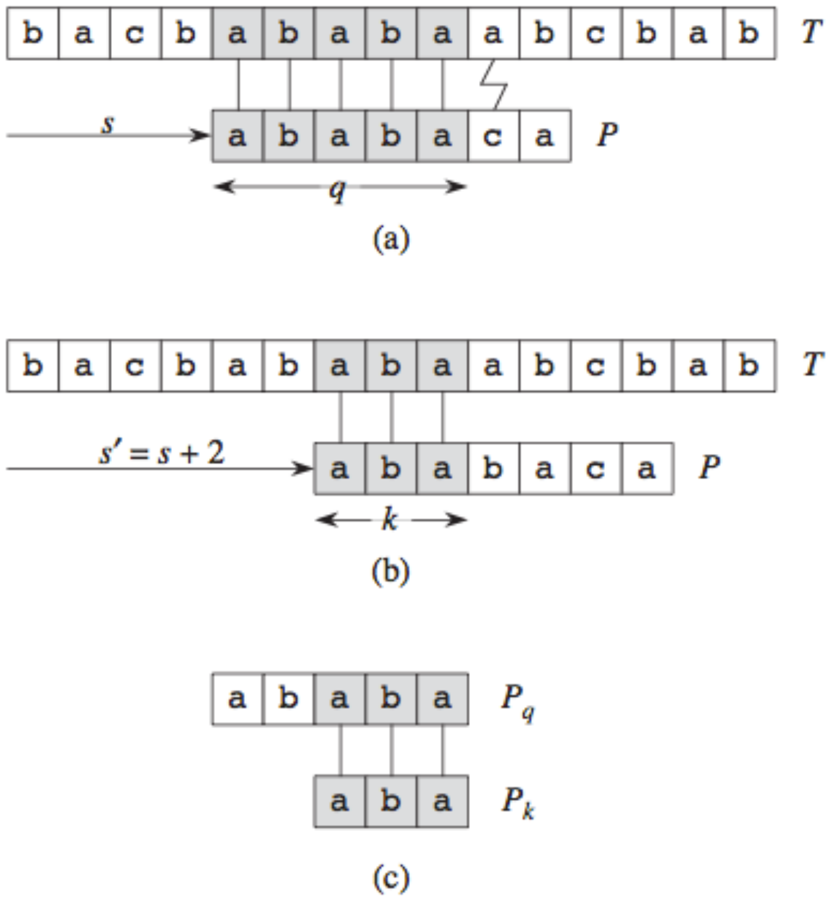


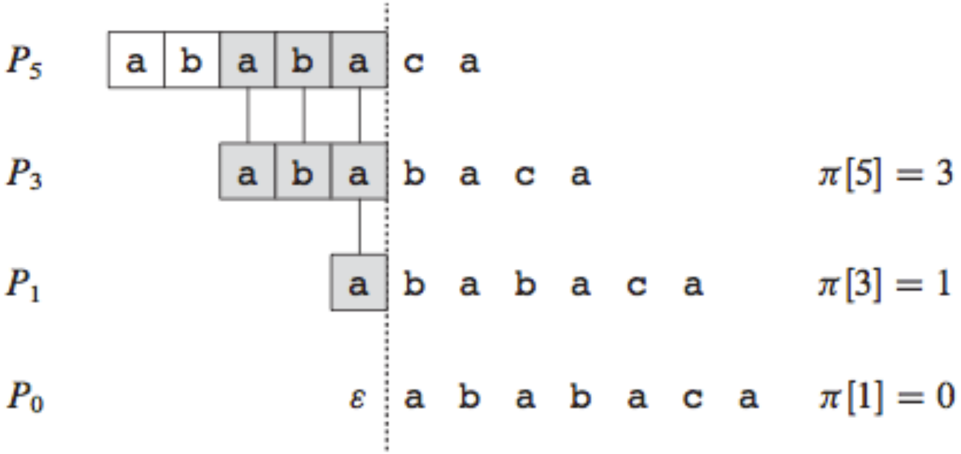
Figure 32.10 The prefix function π. **(a)** The pattern P = `ababaca` aligns with a text T so that the first $q = 5$ characters match. Matching characters, shown shaded, are connected by vertical lines. **(b)** Using only our knowledge of the 5 matched characters, we can deduce that a shift of $s + 1$ is invalid, but that a shift of $s' = s+2$ is consistent with everything we know about the text and therefore is potentially valid. **(c)** We can precompute useful information for such deductions by comparing the pattern with itself. Here, we see that the longest prefix of P that is also a proper suffix of P_5 is P_3. We represent this precomputed information in the array π, so that $\pi[5] = 3$. Given that q characters have matched successfully at shift s, the next potentially valid shift is at $s' = s + (q - \pi[q])$ as shown in part (b).

KMP Amortize analysis

- At While statement
- All sets are O(N) except at the while statement dq is called c(i) times
- c(i) + q(i) – q(i-1) <= 1 worst case.
- sum_i (c(i) + q(i) – q(i-1) <=N
- sum_i c(i) <= N + q(0) – q(N) <= N
- Therefore is O(N)
- Construction of pi(M) is O(M)

i	1	2	3	4	5	6	7
$P[i]$	a	b	a	b	a	c	a
$\pi[i]$	0	0	1	2	3	0	1

(a)



(b)

Computing pi(q)

- Set pi(1) = 0; q = 0
- For i= 2,.., M

```
{
    while q>0 and P(q+1) \= P(i) set q =pi(q) ;
     if P(q+1) = P(i)  { q = q+1;   pi(i) = q; }
 }
```

Relations: *Boolean valued Matrix R[a,b]*

- Set: S = {a,b,c,....}
- Relation (a,b) 2 S x S: a R b is True?
- Properties:
  - Reflexive: a R a is True
  - Anti-symmetric: a R b and b R a $\rightarrow$ a = b
  - Transitive: a R b and b R c $\rightarrow$ a R c
  - Total Ordering: a R b or b R a (inclusive or)
  - Self dual: a R b $\leftrightarrow$ b R a
  - Transpose: a R b $\leftrightarrow$ b R^T a
- RAT is partial ordering: e.g. descendants in a tree!

(e.g. <= is total ordering for int but g(N) = O(f(N)) is partial ordering!)

- Equivalence class is Reflexive, Transitive and Symmetric
  - Symmetric a R b if and only if b R a

Union/Find

- Equivalence class and Sets.
- O(1) Find
- O(1) Union
  - Union by Height/Size
  - Path Compressions
  - $\text{Log}^*(N)$ function
- Binomial Queue

Dynamic Equivalence

- Object a[i] can be numbered 0,...,N-1 (like nodes)
- Sequence of new equivalence a ~ b
- Two operations:
- FIND a in Si ?
  - Must return T/F for find(a) ~find(b)
- UNION Sk = Si U Sj
  - Operation of find(a) not~ find(b)

- Want M finds and upto N unions: O(M+N)?
- Almost but actually impossible!

Determine if a ~ b

- O(1) answer is a ~`b if use array of Set #
- Set up a 2-d are for SxS and look up T/F
- Alternatively "partition" S into equivalence classes (like connected component) is C_i for all a's ~ b's
- S = C1 U C2 U ...U Cn and C's are disjiont: Ci ^ Ci = 0 (null set).

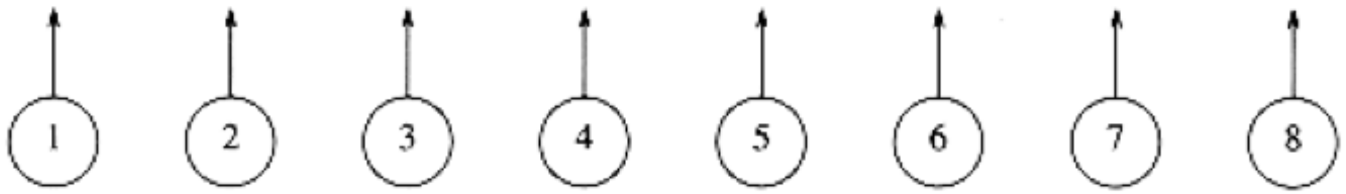


Figure 8.1 Eight elements, initially in different sets

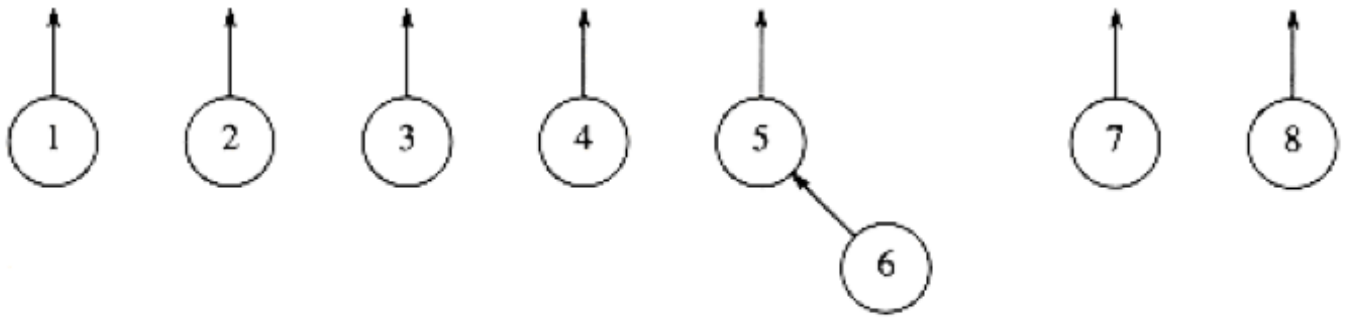


Figure 8.2 After union (5, 6)

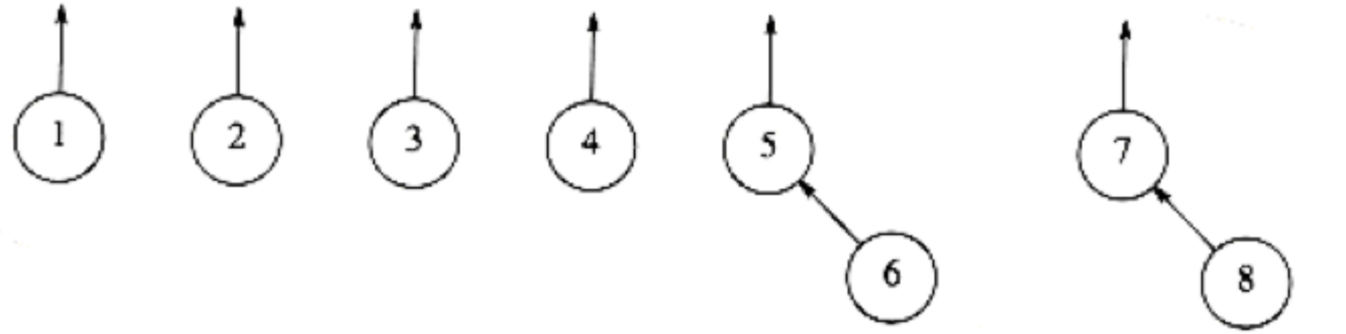


Figure 8.3 After union (7, 8)

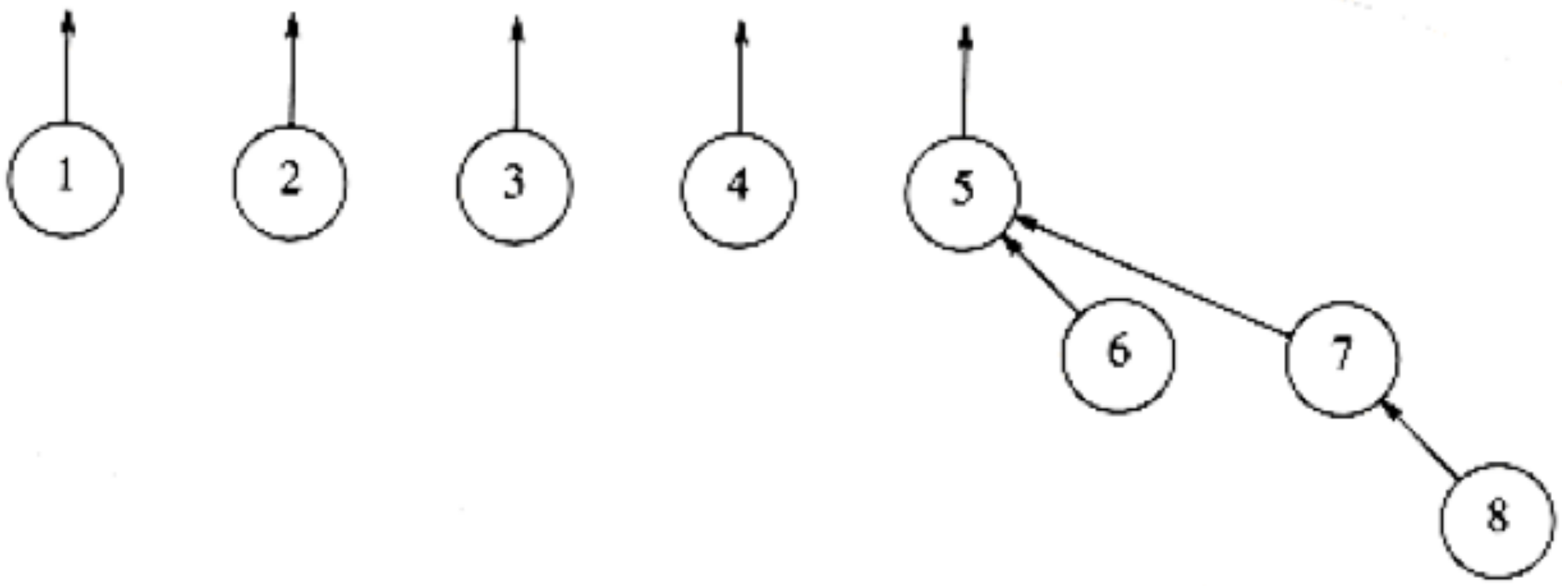


Figure 8.4 After union (5, 7)

0	0	0	0	0	5	5	7
1	2	3	4	5	6	7	8

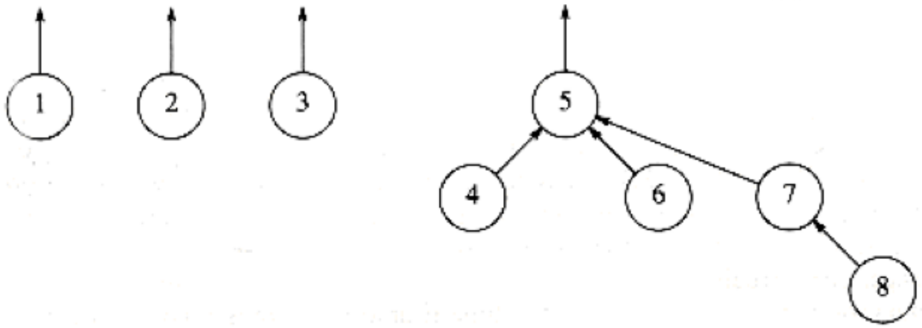


Figure 8.10 Result of union-by-size

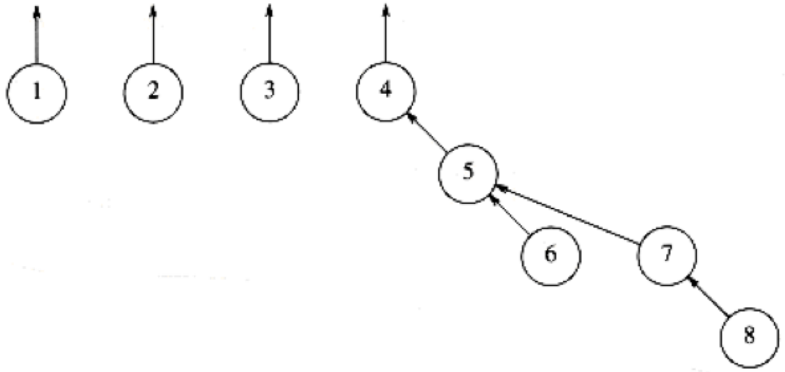


Figure 8.11 Result of an arbitrary union

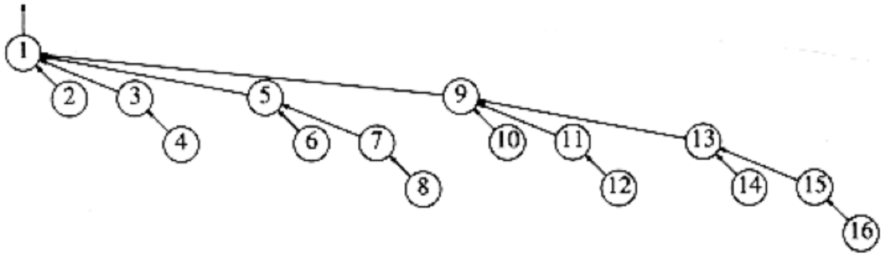
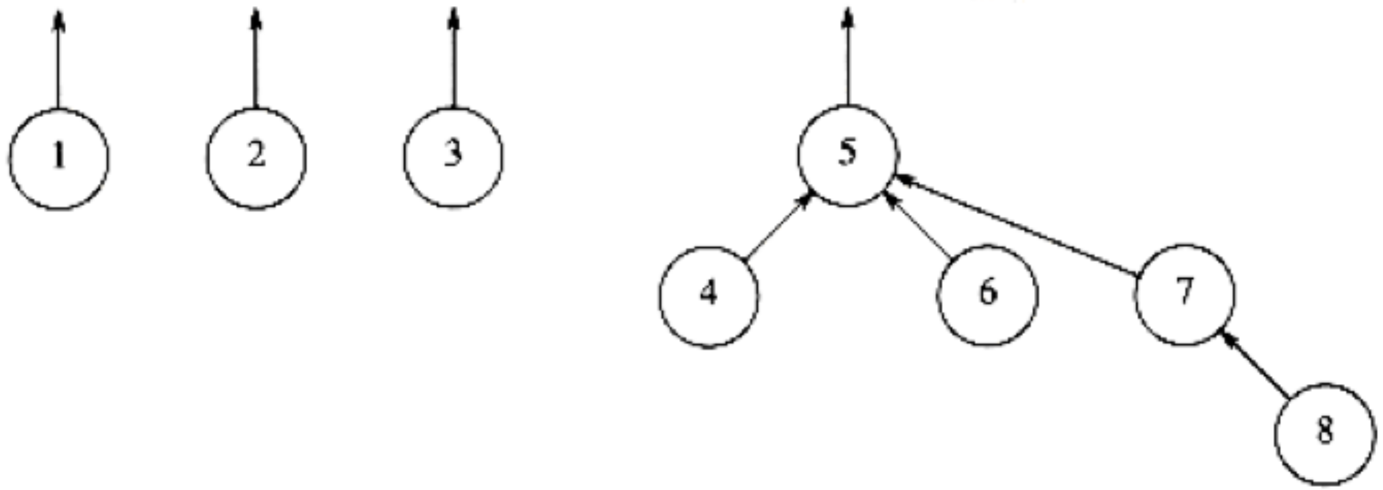


Figure 8.12 Worst-case tree for n = 16

The following figures show a tree and its implicit representation for both union-by-size and union-by-height. The code in Figure 8.13 implements union-by-height.



−1	−1	−1	5	−5	5	5	7
1	2	3	4	5	6	7	8

- size

0	0	0	5	−2	5	5	7
1	2	3	4	5	6	7	8

- height

$Log^*(N)$ inverse Ackermann function:

- *Ackerman Function:*
  - $A(i) = 2^{A(i-1)}$; $A(0) = 1$
  - $A(1) = 2$, $A(2) = 4$, $A(3) = 16$, $A(4) = 2^{16} = 65536$, $A(5) = 2^{65636}$,
  - $A(6)$ = *VERY VERY VERY BIG!*
- *Inverse Ackerman:*
  - *i= Log*(N) = min number times you take $\log_2$ to get equal or smaller than below 1.*
  - *Worst case Union-by-Rank with path compression is O(M log^*(N) for M unions.*

Binomial Queue

- Combine: Priority Queue and Union/Find: log(N) Forest of trees:

  $H = n_0 B_0 + n_1 B_1 + n_2 B_2 + \boxed{?} + n_p B_p$

  with $n_i = 0,1$

- B_0 = root, $B_{k+1} = B_k + B_k$ attached to root of first. Since B_k has 2^k nodes this is just at binary bit representation N = $(n_p \ldots, n_0)$ for nodes of size $N \cdot 2^p$
- Build so that min key is in root of B_k.
- Adding H_1 to H_2 is binary arithmetic O(p = log(N))
- Always combing B_k + B_k \rightarrow B_{k+1} with min at root.

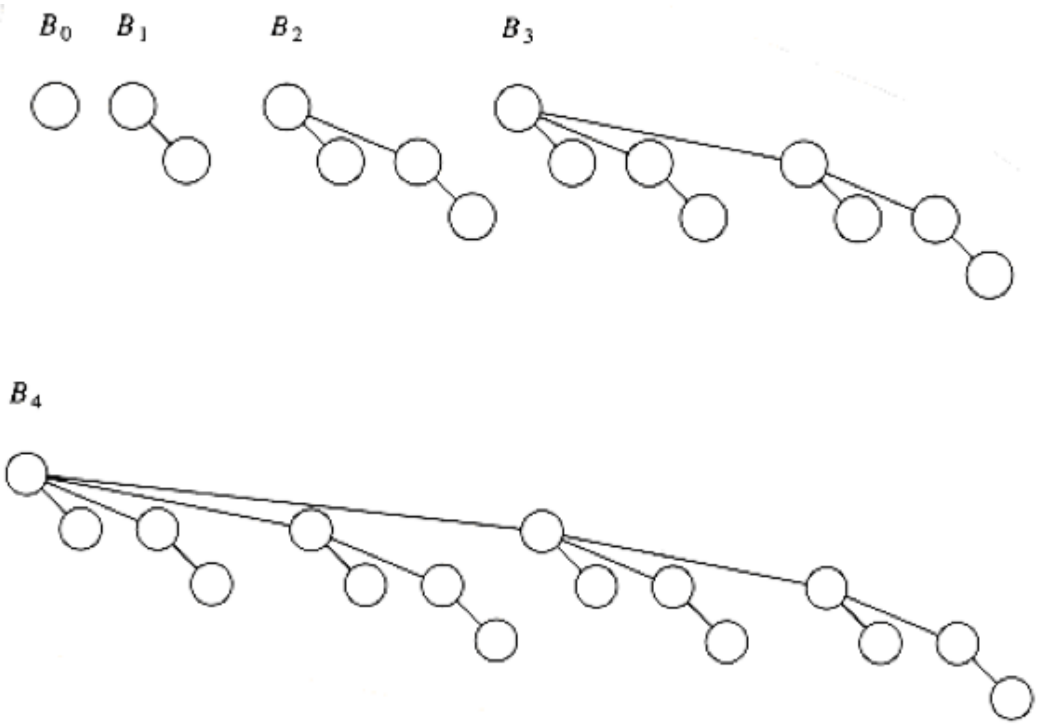
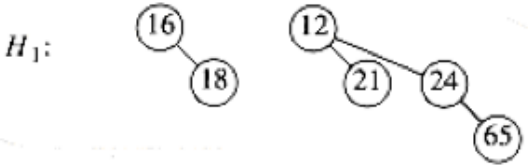


Figure 6.34 Binomial trees B_0, B_1, B_2, B_3, and B_4



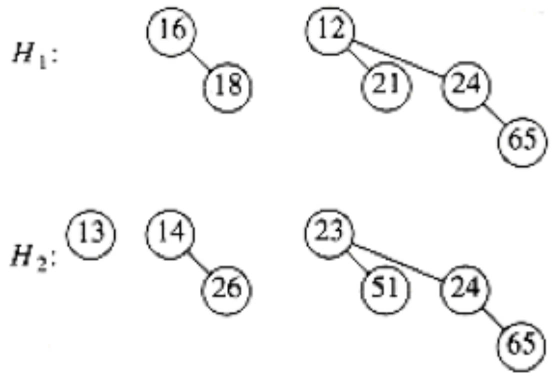


Figure 6.36 Two binomial queues H_1 and H_2

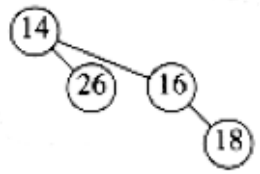


Figure 6.37 Merge of the two B_1 trees in H_1 and H_2