Master Equation:

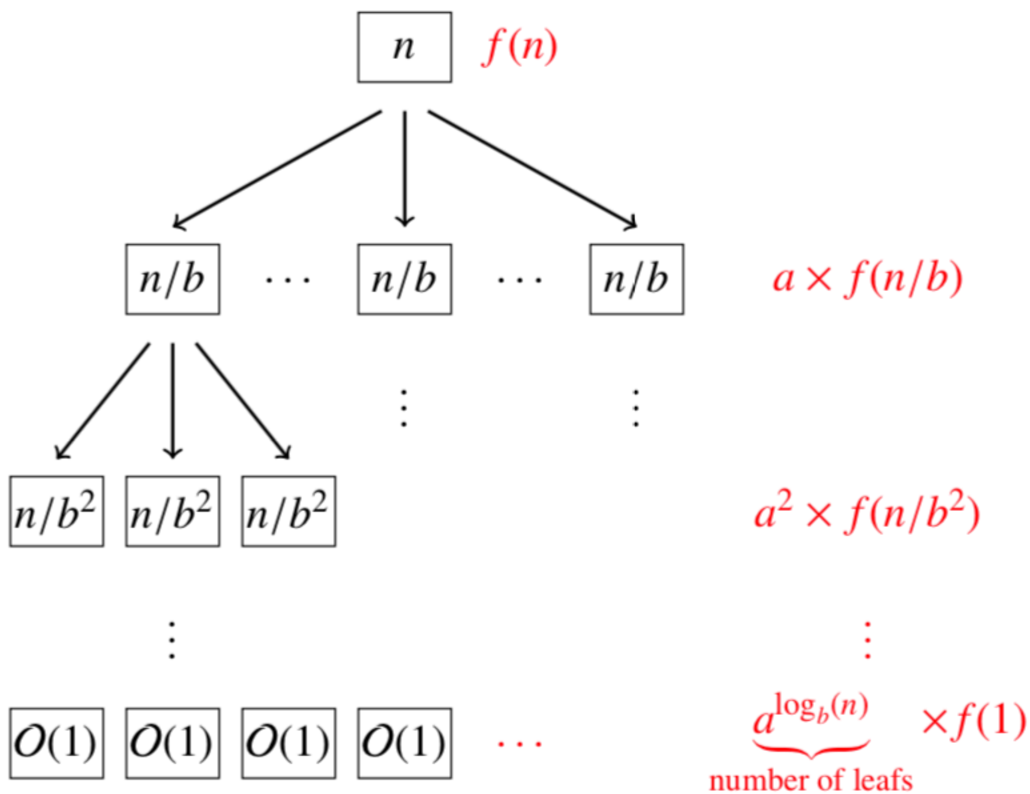
T(N) = a T(N/b) + Θ (g(N))

Theorem: The asymptotic Solution is:

- $T(N) \in \Theta(N^\gamma) \quad \text{if } g(N) \in O(N^{\gamma-\epsilon})\ \forall\epsilon > 0$
- $T(N) \in \Theta(g(N)) \quad \text{if } g(N) \in \Omega(N^{\gamma+\epsilon})\ \forall\epsilon > 0$
- $T(N) \in \Theta(N^\gamma \log(N)) \quad \text{if } g(N) \in \Theta(N^\gamma)$

where $a = b^\gamma$ or $\gamma = \log(a)/\log(b)$

Build Tree to Solve



$$n/b^h = 1 \implies h = \log_b(n)$$

MAGIC x-y SYMMETRY: x^log(y) = y^log(x) !

GUESS SOLUTION AND PLUG IN TO CHECK IT

$$T(N) = aT(N/b) + c_0 N^k$$

- General Solution:

Consider first the homogenous equation:

$$\begin{aligned} T(N) &= aT(N/b) \\ \text{try } T_1(N) &= cN^\gamma \Rightarrow N^\gamma = a\frac{N^\gamma}{b^\gamma} \\ &\Rightarrow \gamma = log(a)/log(b) \end{aligned}$$

$$\begin{aligned} \text{try } T_0(N) &= c_1 N^k \Rightarrow c_1 N^k = c_1 a\frac{N^k}{b^k} + c_0 N^k \\ \Rightarrow c_1 &= c_0/(1 - a/b^k) \end{aligned}$$

$$T(N) = cN^\gamma + c_1 N^k$$

Master Equation (brute force): T(n) = aT(n/b) + c n^k

Math Derivation if you care — probably not!

$$T(n) = aT(n/b) + c\, n^k$$
$$aT(n/b) = a^2T(n/b^2) + c\, an^k/b^k$$
$$a^2T(n/b^2) = a^3T(n/b^3) + c\, a^2n^k/b^{2k}$$
$$\cdots \qquad \cdots$$
$$a^{h-2}T(b^2) = a^{h-1}T(b) + c\, a^{h-2}n^k/b^{(h-2)k}$$
$$a^{h-1}T(b) = a^hT(1) + c\, a^{h-1}n^k/b^{(h-1)k}$$

Just do the darn sum explicity!

Therefore

$$T(n) = a^hT(1) + c\, n^k \frac{(a/b^k)^h - 1}{a/b^k - 1}$$

$$= n^\gamma T(1) + c\, \frac{n^\gamma - n^k}{a/b^k - 1}$$

$$a^h = n^\gamma$$

Extra Log comes from taking limit: k —> gamma

$$T(n) = a^h T(1) + c\, n^k \frac{(a/b^k)^h - 1}{a/b^k - 1}$$

$$a^h = n^\gamma$$

$$= n^\gamma T(1) + c\, \frac{n^\gamma - n^k}{a/b^k - 1}$$

$$b^h = n$$

$$\boxed{a = b^\gamma}$$

$$\Longrightarrow \; a^h = (b^h)^\gamma = n^\gamma$$

$$k = \gamma + \epsilon$$

k = gamma SINGULAR!

Expansion Rule:

$$x^\epsilon = e^{\epsilon\, \ln(x)} \simeq 1 + \epsilon\, \ln(x) + \epsilon^2\, \ln^2(x) + \cdots$$

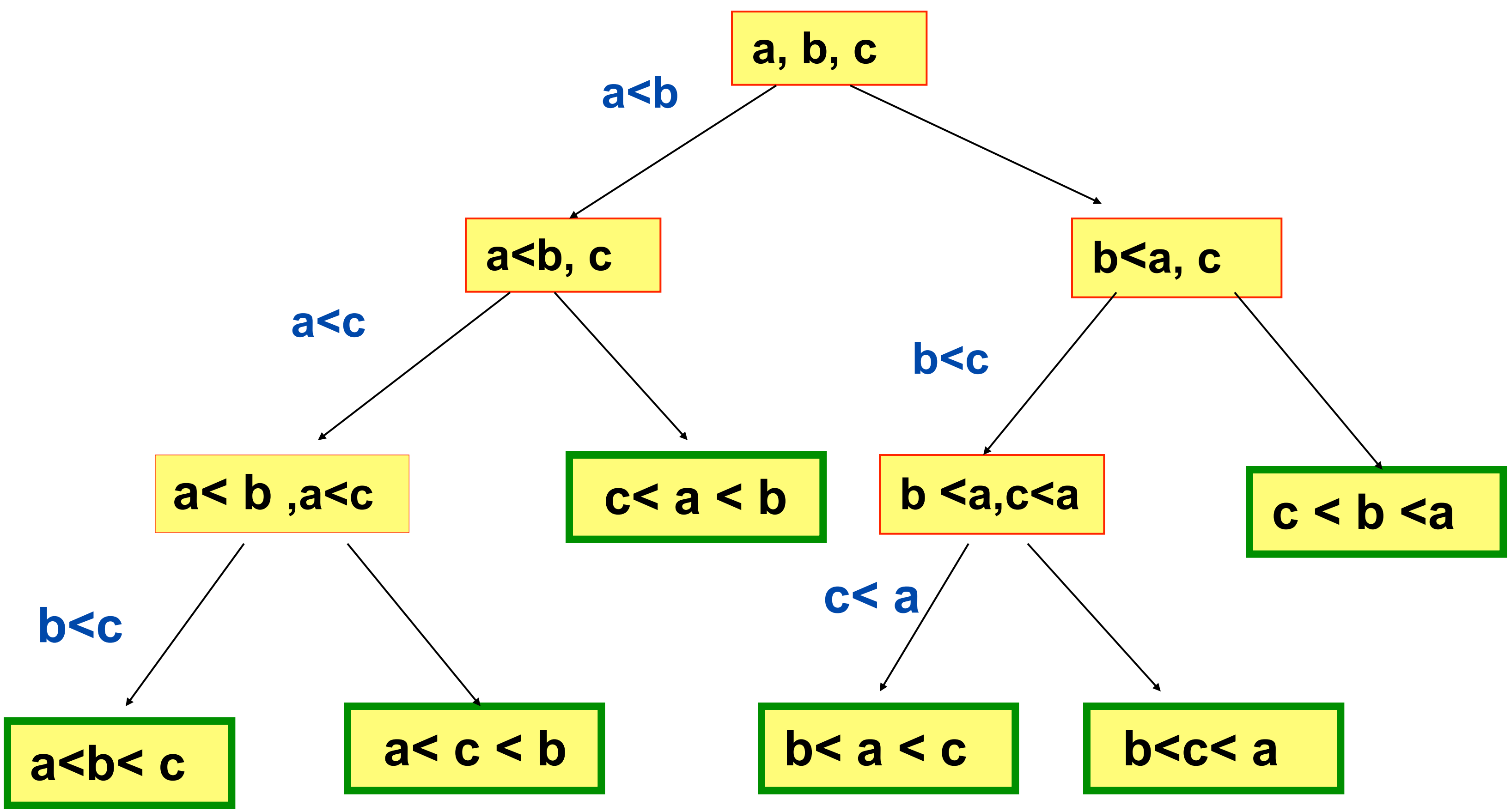
$$\gamma = h \log(a) log(n) = \log(a)/\log(b)$$

$$\text{L'Hospital's Rule} \;\Longrightarrow\; c\, n^\gamma \log(n)/(log(b) a/b^\gamma) = c n^\gamma \log(n)/log(b)$$

Which term dominates? If b^k = a got be careful!

Decision Tree

Proof of $\Omega(N\log(N))$



Binary decisions: 3! = 6 possible outcomes. Longest path: log(3!)

Lower Bound Theorem for Camparision Sort

Proof: Compute the maximum depth D of decision tree?

- Need N! leaves to get all possible outcomes of a sorting routine.
- Each level at most doubles: $1 ® 2 ® 4 ® 8 ® \bullet\bullet\bullet ® 2^D$
- Consequently for D levels: $N! \leq 2^D \Rightarrow D \geq log_2(N!)$

$$\Rightarrow T(N) = \Omega(D) = \Omega(\log_2(N!)) = \Omega(N \log_2(N))$$

$$Information = \log_2(N!) \cong N \log_2(N)$$

Number of bits to encode any (initial) state is information (- Entropy)