

# SUMMARY OF ALGORITHMS

❑ ADT – Abstract Data Type

**CLRS: 10**

❑ 1-d : Lists, Queue, Stack

❑ Array implementation

❑ Execution Stacks

❑ Heap (aka Priority Queue)

**CRLS: 16**

❑ Binary Trees

**See *appendix B.5 Trees*)**

❑ Traversals (pre-, in-, post-order)

❑ BST

**CRLS: 12**

❑ AVL and Red/Black

**CRLS: 13**

❑ Huffman Encoding

**CRLS: 16.3**

## I-D ADT'S:      ARRAYS, QUEUES, STACKS & LINKED LISTS.

- **Abstract Data Types (ADT):** data type (class) with ops (methods).
  - ◆ **Examples:** Int. (0,1,...,Maxint). All 2 by 2 real matrices. IEEE floats, etc.
  - ◆ **The implementation is not part of the ADT!**
  
- **Queue (or FIFO) is a list with methods:**
  - ◆ **Enqueue(item) & item = Dequeue**    *(relative to \*front/\*back respectively)*
  
- **STACK (or LIFO) is a list with methods:**
  - ◆ **push(item) & item = pop()**            *(relative to \*TOP)*
  
- **Linked List is a list with methods:**
  - ◆ **insert(item) & delete()**            *(relative to \*current)*
  - ◆ *current->next and current->last moves current*

## STACK IS FUNDAMENTAL

- Reverse Polish: 6 5 2 3 + 8 \* + 3 + \*

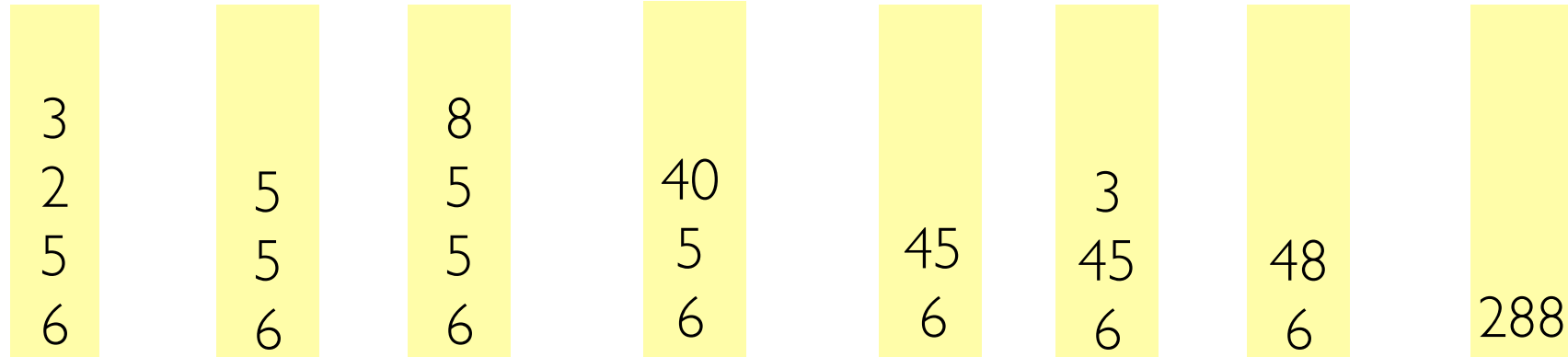
+

\*

+

+

\*



- See also conversion: infix  $\rightarrow$  postfix

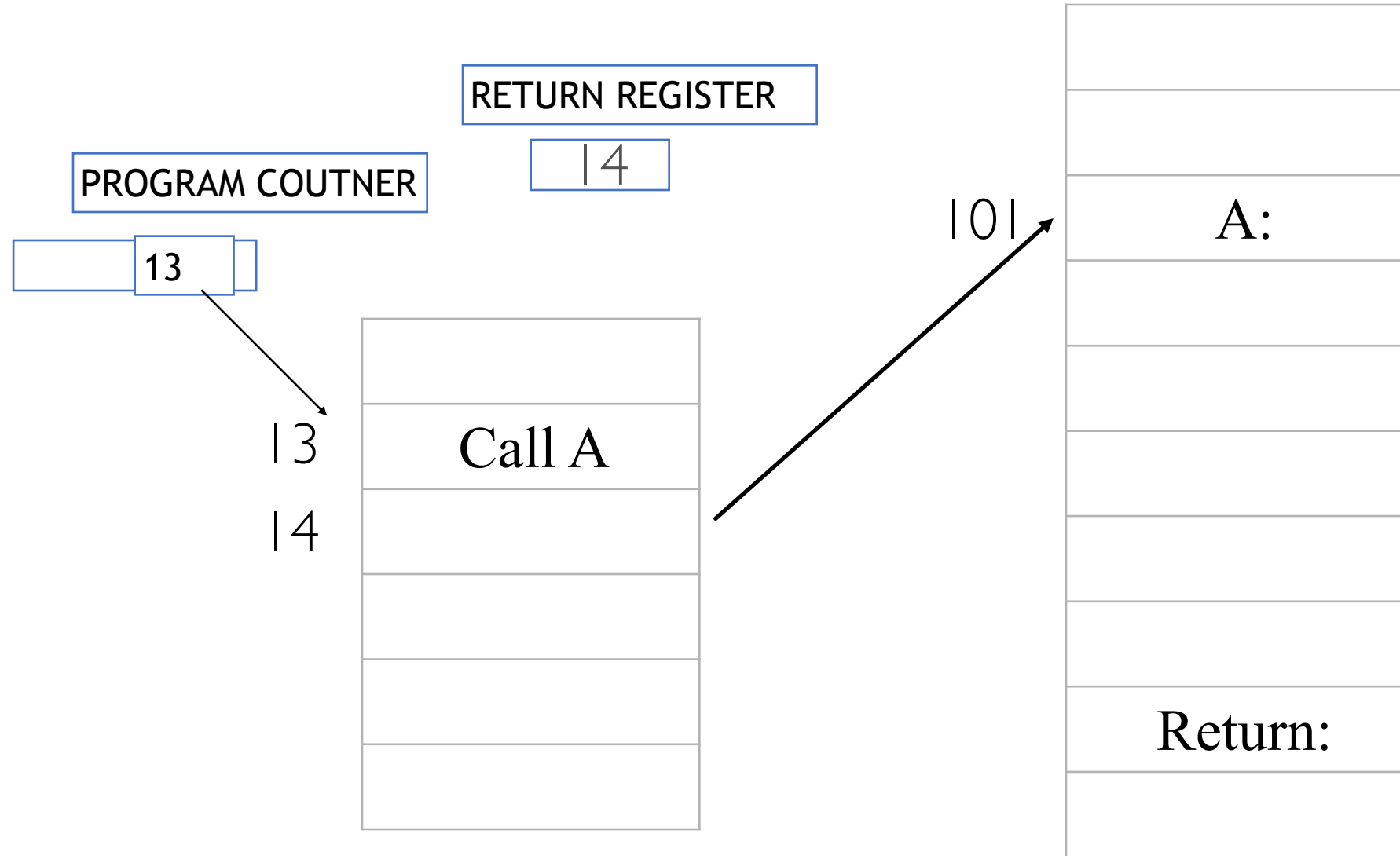
$(6 * (5 + ((2+3)*8 + 3))) = 6 \ 5 \ 2 \ 3 \ + \ 8 \ * \ + \ 3 \ + \ *$

- Execution Stacks for Function Calls:
  - Fixed return register
  - First line of subroutine (nested)
  - Execution stack (recursive)

# FORTTRAN'S EVOLUTION OF THE SUBROUTINE CALL

- Function Call & Return
- Version 1 -- Return Register
  - ◆ no nesting
- Version 2 --- Return to top of Function
  - ◆ nesting but no recursion
- Version 3 --- The stack frame AT LAST!
  - ◆ Call yourself (recursion)

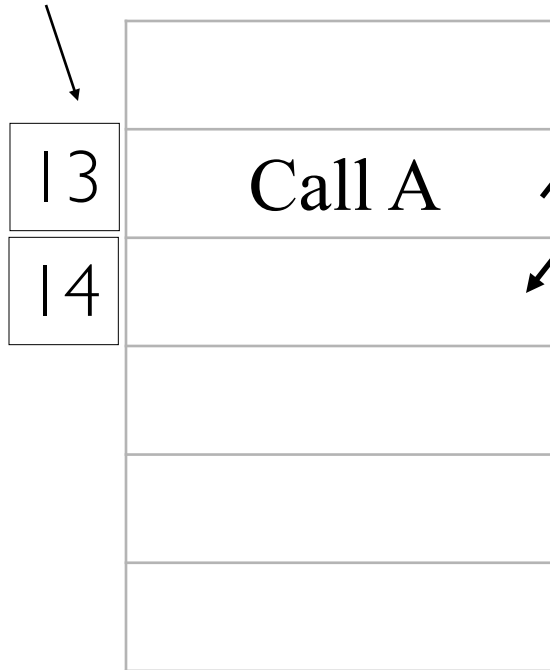
# VERSION I



# VERSION 2

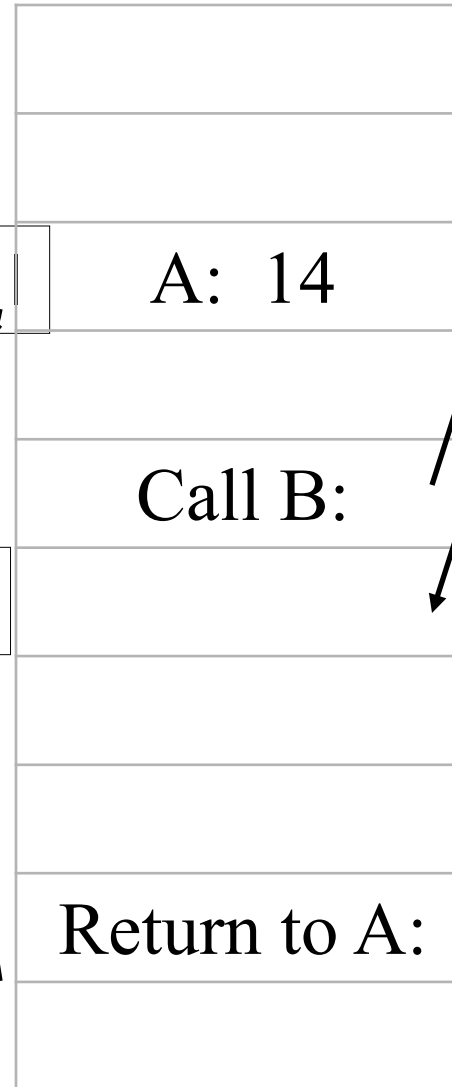
PROGRAM COUNTER

13



10

103



202

B: 103



# Version 3

PROGRAM COUNTER

13

13

14

Call A

101

103

A:

Call A:

Return to A:

STACK

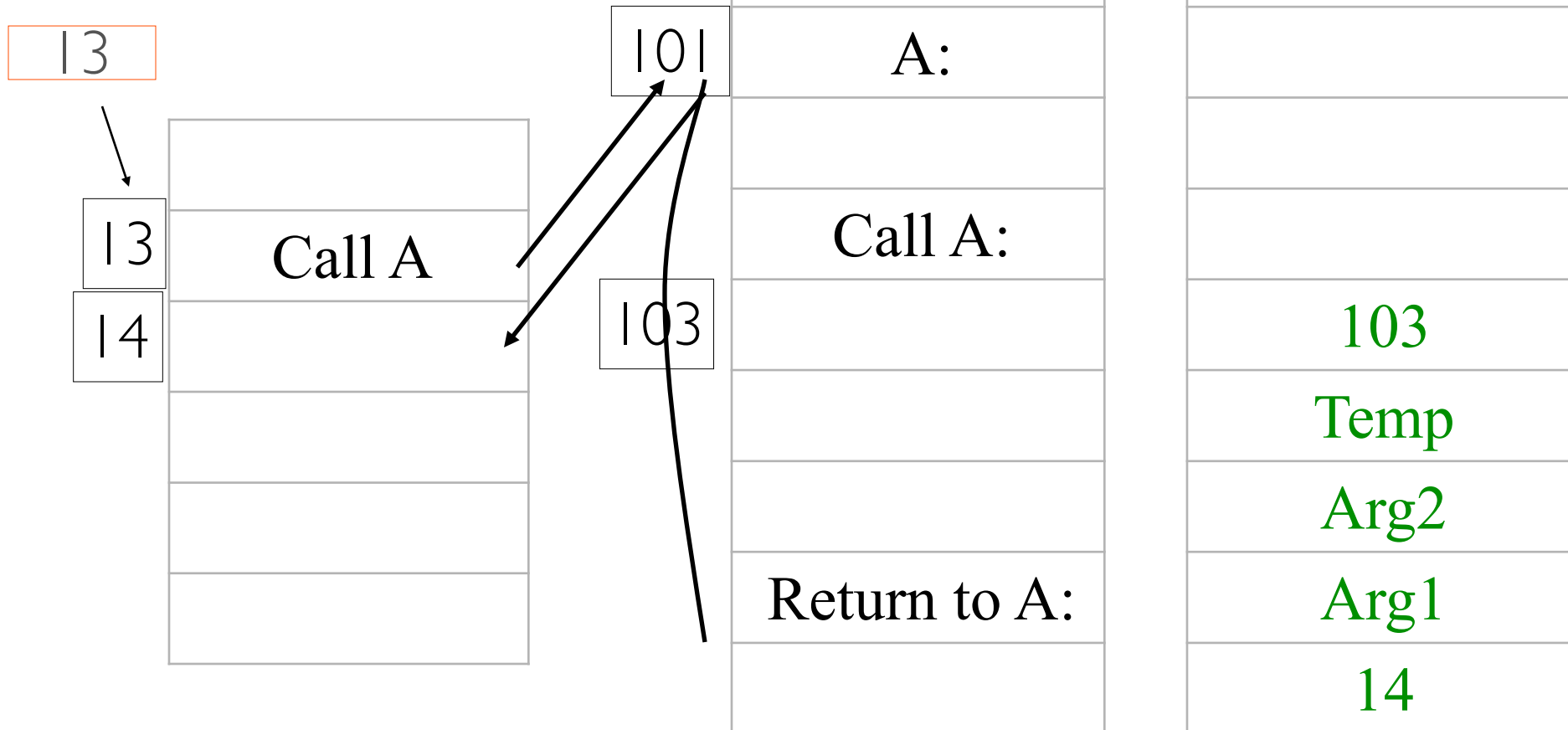
103

Temp

Arg2

Arg1

14



# Priority Heaps

- Basic Heap ADT:

- ◆ *Data is  $a[i]$ ,  $i = 1, \dots, N$  (ROOT = 1, LABEL 0)*
- ◆ *Methods: Insert (key), Delete(key), DeleteMin, Build and Sort*

- Q: When is a tree an array? A: **complete** tree

- ◆ *Parent  $a[i] \rightarrow a[2i] = \text{left child} \ \& \ a[2i+1] = \text{right child}$*
- ◆ *Child  $a[j]$ :  $\rightarrow a[j/2] = \text{parent}$  (integer division).*

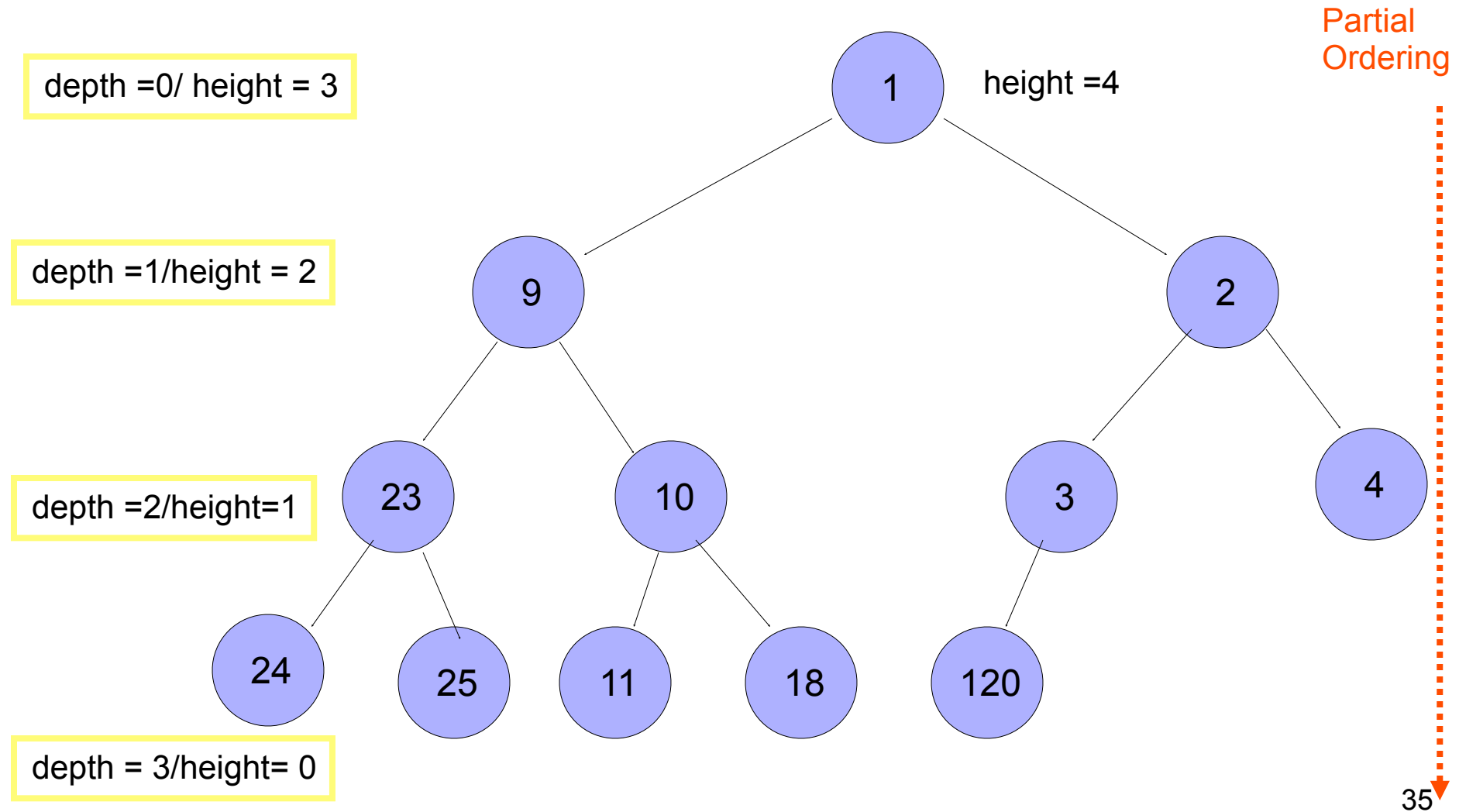
- Build Heap is  $O(N)$  by bottom up, DeleteMin is  $O(\log(N))$



- ◆ *Heap sort by deleting min over and over is  $O(N \log(N))$ .*

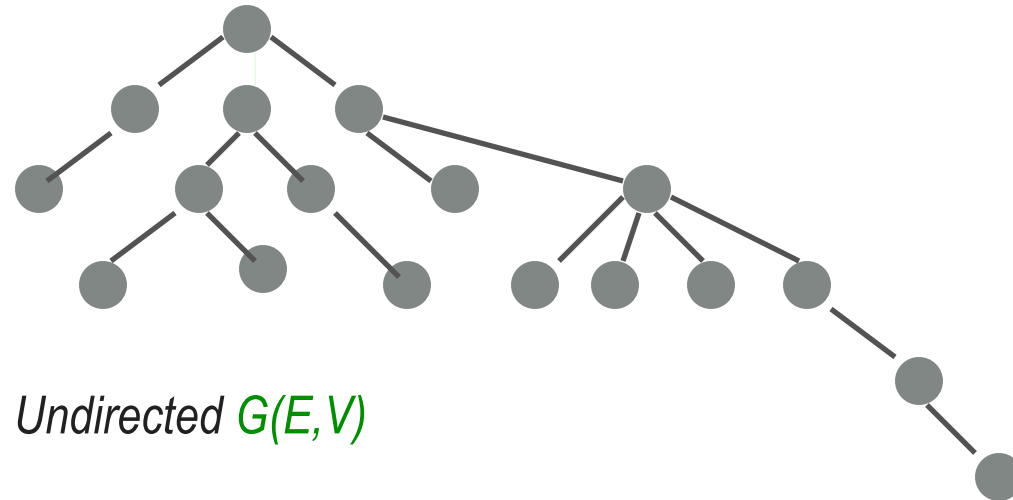


# Min Heap Order



# INTRODUCTION TO TREES

- *Trees: inheritance, partial ordering, execution graphs,*
- *A tree is a special kind of Graph  $G(E,V)$*
- *$E$  = “edges/arcs” connecting  $V$  = “vertices/nodes”*



- *A tree is **Connected**, **Acyclic**, **Undirected**  $G(E,V)$*
- *Binary Tree has 0,1,2 children (i.e. nodes have 1,2,3 edges)*

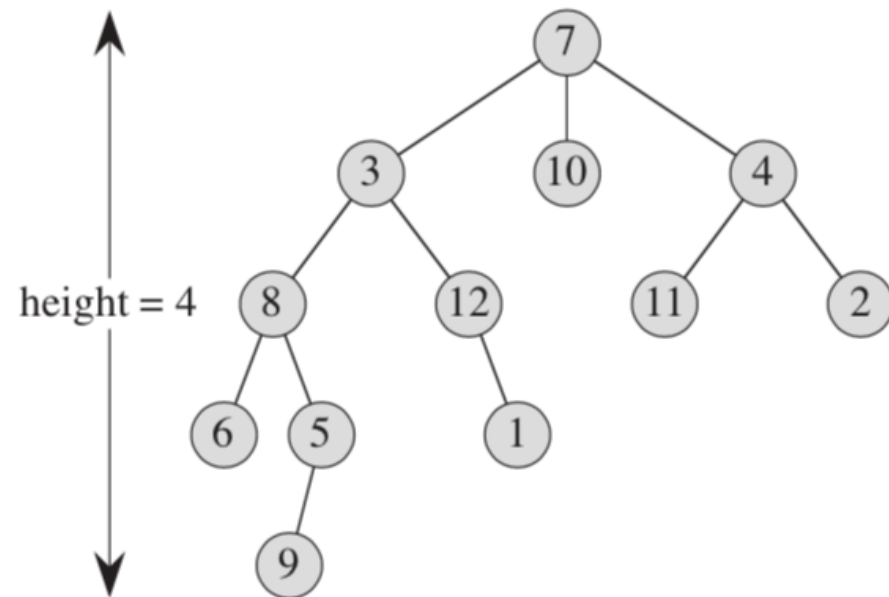
# DEFINITIONS FOR BINARY TREES

- ◆ Full Tree: 0, 2 children,
- ◆ Complete Tree: Consecutive nodes (aka Heap),
- ◆ Perfect Tree: Complete and full last row.
- Full Tree Theorem: # of leaves:  $L(N) = (N+1)/2$  for  $N$  nodes
- Perfect Tree with  $H$  levels (height or depth)
- Nodes in Perfect  $k$ -way tree :  $N(H) = (k^{H+1}-1)/(k-1) \Rightarrow 2^{H+1} - 1$
- Execution Tree
- Traversals: in-, pre-, post-order.

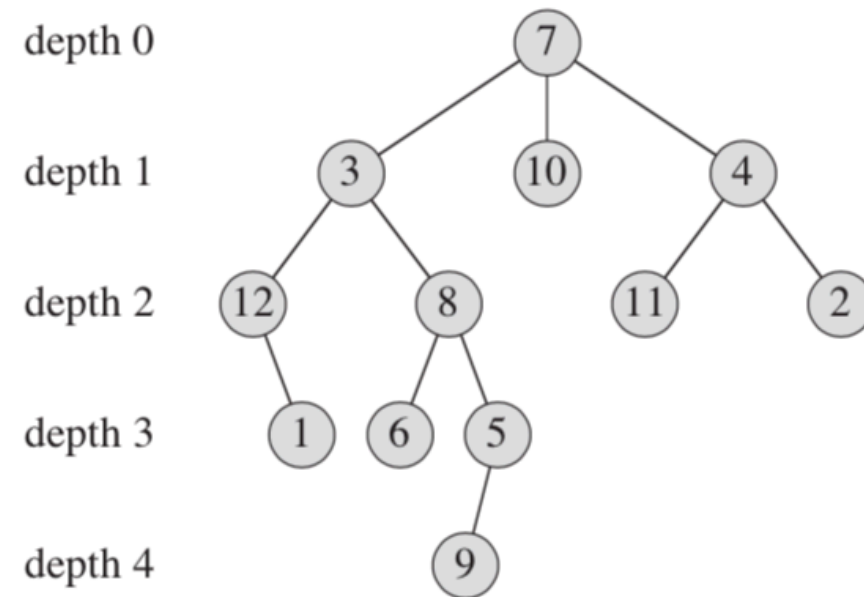
# Height vs Depth of “nodes”

B.5 Trees

1177



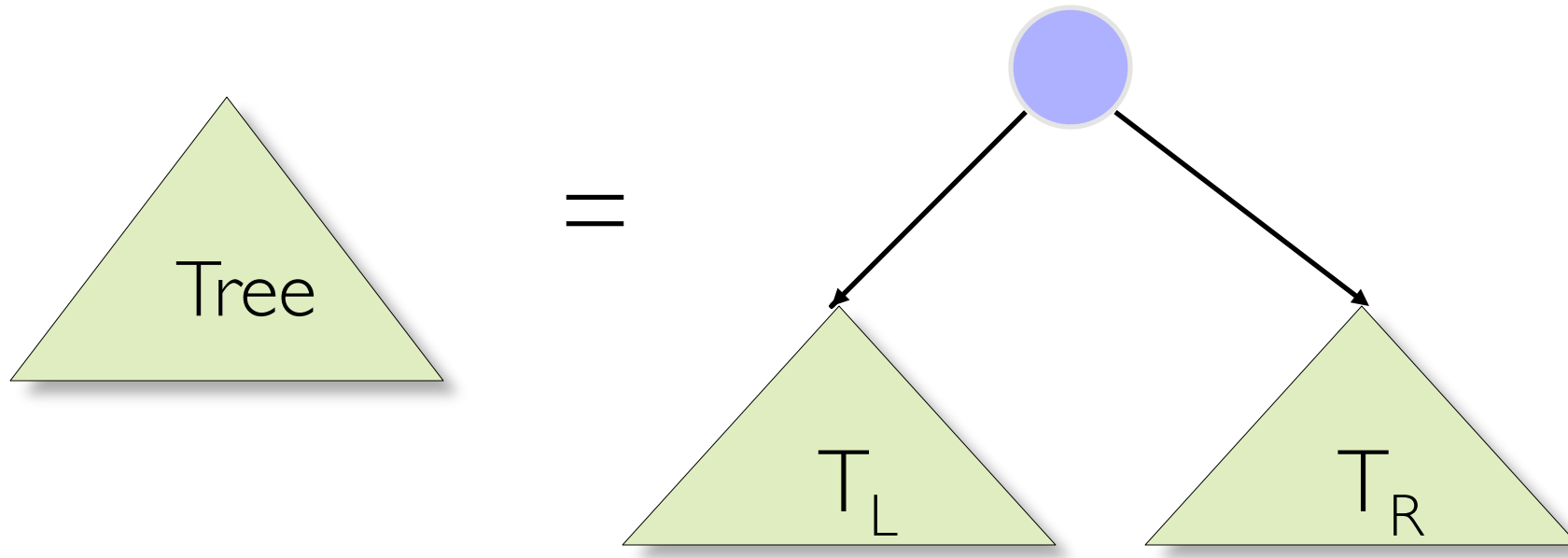
(a)



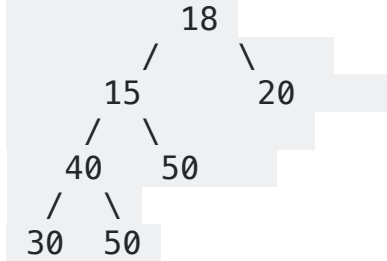
(b)

# *BINARY TREE: RECURSIVE DEFINITION*

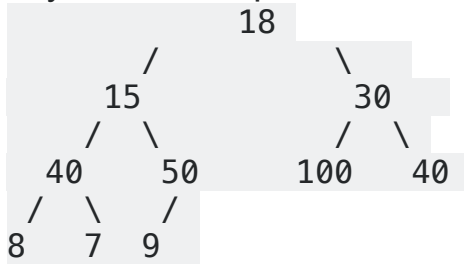
- A binary tree is null or a single node with a Right and Left Child that is a binary tree!  
(Useful for organizing recursive algorithms on binary trees.)



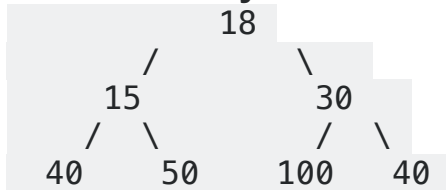
**Full Binary Tree:** A Binary Tree is full if every node has 0 or 2 children. Following are examples of a full binary tree.



**Complete Binary Tree:** A Binary Tree is complete Binary Tree if all levels are completely filled except possibly the last level and the last level has as many nodes as possible.



**Perfect Binary Tree:** A Binary tree is Perfect Binary Tree in which all internal nodes have two children and all leaves are at same level.

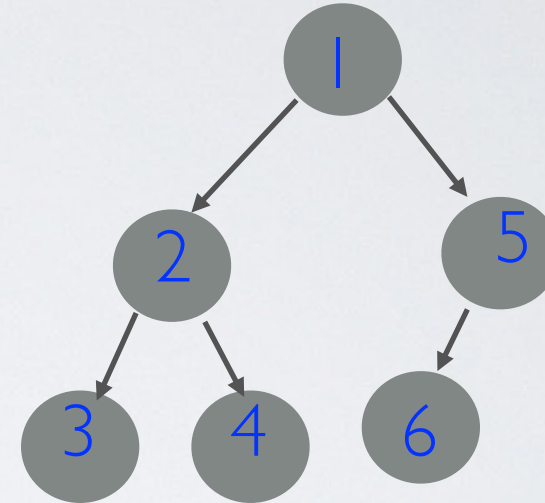


# TREE TRAVERSALS

■ Preorder:    Print [Tree]{  
                    Print root;  
                    Print Tree[LeftTree];  
                    Print Tree:[RightTree];  
                  }

■ Inorder:        Print [Tree]{  
                    Print Tree[LeftTree];  
                    Print root;  
                    Print Tree:[RightTree]  
                  }

■ Postorder:     Print [Tree]{  
                    Print Tree[LeftTree];  
                    Print Tree:[RightTree]  
                    Print root;  
                  }

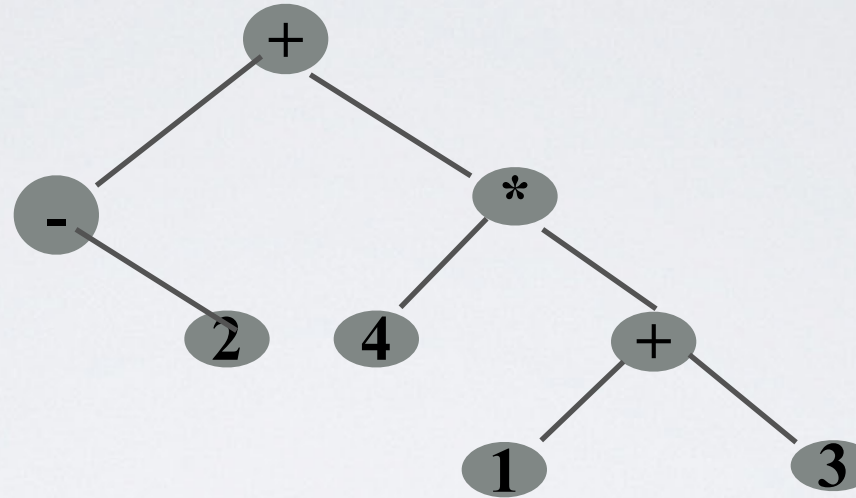


Pre:            1→2→3→4→5→6

In:            3→2→4→1→6→5 sort on BST

Post:          3→4→2→6→5→1

# Expression Trees



Preorder: + - 2 \* 4 + 1 3 (Lisp, Scheme) (+ (- 2) (\* 4 (+ 1 3)))

In order: -2 + 4 \* (1 + 3) (C, C++, Java) Standard precedence

Postorder: 2 - 4 1 3 + \* + (HP calculator, PS, Forth)

Binary Search Tree: left  $\leq$  root < right

- in order traversal gives sorted list
- easy to search

see [https://en.wikipedia.org/wiki/Binary\\_expression\\_tree](https://en.wikipedia.org/wiki/Binary_expression_tree)



# DIMENSIONS OF A PERFECT TREE

- Perfect Tree (all levels filled) with H levels:

(Height:  $H = \log_k(N)$  for  $k$ -array tree)

- # nodes:  $N(H) = 1 + k + k^2 + k^3 + \dots + k^H = (k^{H+1}-1)/(k-1)$

(binary tree:  $N = 1 + 2 + 2^2 + 2^3 + \dots + 2^H = 2^{H+1} - 1$ )

- total Depth:  $T_D(N) = k \, dN/dk = (H+1)k^{H+1}/(k-1) - k(k^{H+1}-1)/(k-1)^2$

→ (binary tree)  $2(H+1)2^H - 2(2^{H+1}-1) = (H-1)N + H + 1$

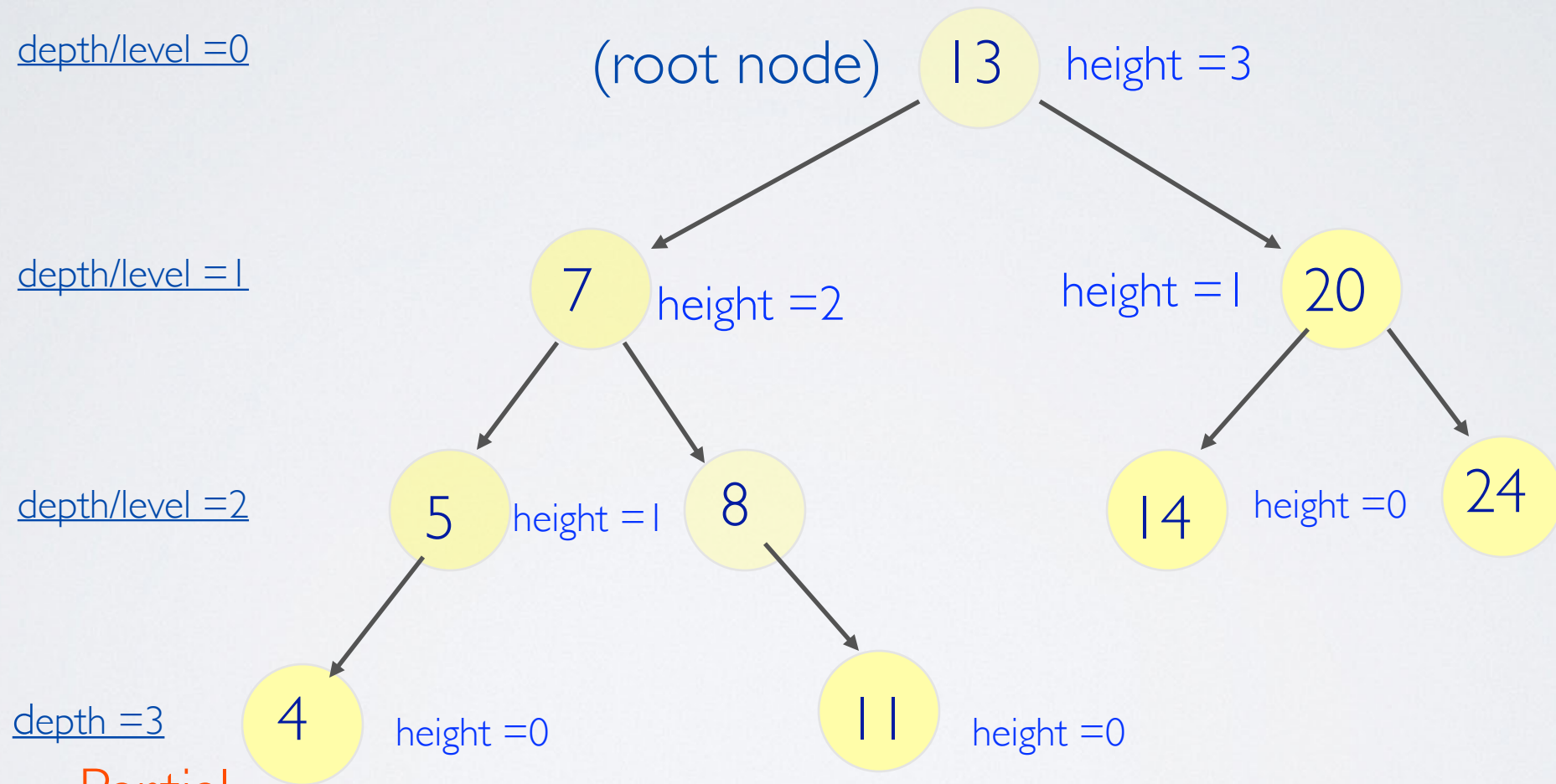
- total Height:  $T_H(N) + T_D(N) = H N$  (each  $h + d = H$ )

$$T_H = H N - T_D = N - H - 1 \quad (\text{binary tree})$$

# SEARCH TREES

- *BST tree Recursive definition*
  - ◆ *Insertion and Deletion*
- *AVL tree balance:*
  - ◆ *Insertions: single (zig-zig) and double (zig-zag) rotations.*
  - ◆ *Lazy Deletion*
- *Red/Black Tree*

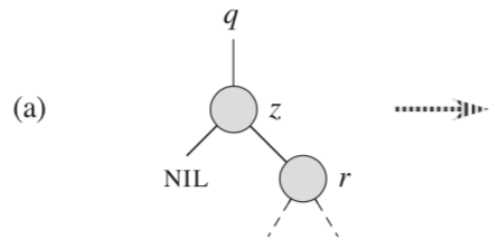
# BINARY SEARCH TREES



Partial  
Ordering

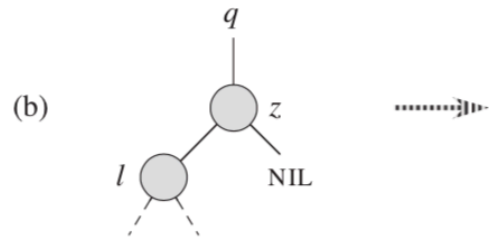
# BINARY SEARCH TREE: BST

1. BST is a Binary Tree with keys stored in each node.
2. The key ( $K_0$ ) in each node is: greater or equal to all keys in  $T_L$ , the Left subtree ( $K_{\text{left}} \leq K_0$ ) less than all keys in  $T_R$ , the Right subtree ( $K_0 < K_{\text{Right}}$ )
3. The BST defines a partial ordered set --- as you move down to the left/right the keys decrease/increase.
4. Insert new  $K_{\text{new}}$  push down to subtree Left/Right if  $K_{\text{new}} \leq / > K_0$ .
5. Delete  $K_0$  and replace by SMALLEST key in  $T_R$ , the Right subtree.

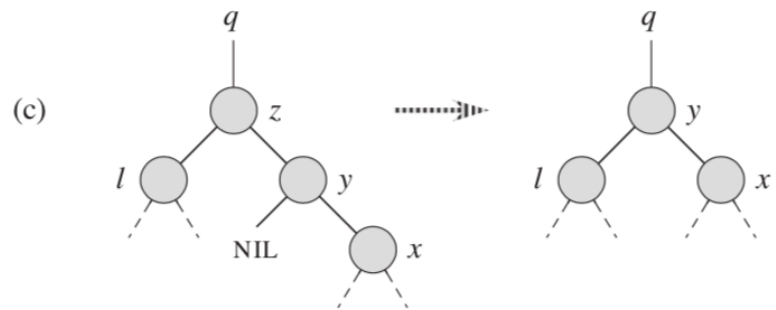


Right Only Child

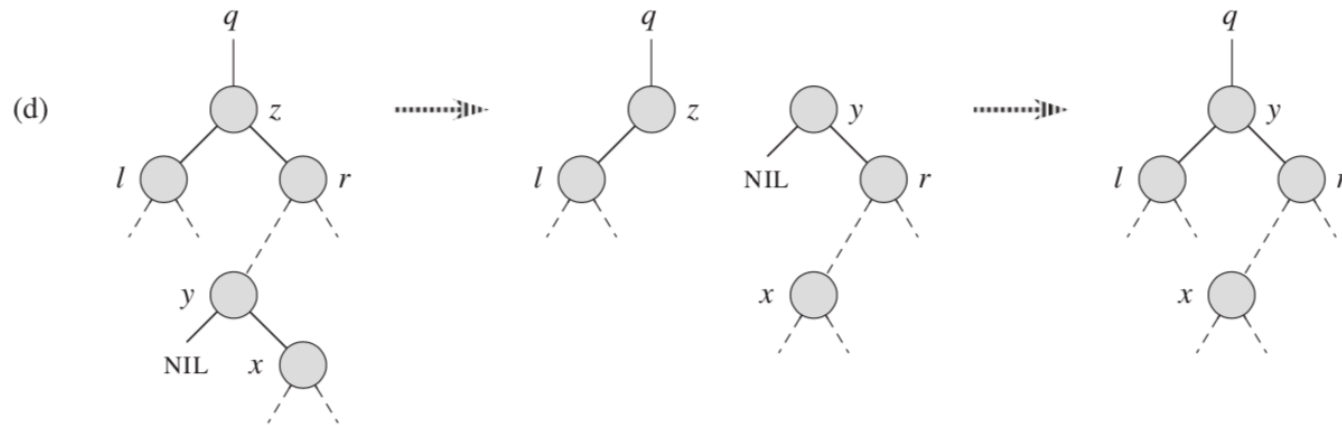
# BST DELETE



Left Only Child



R & L Twins



Right to Left Rotation



# AVERAGE TOTAL DEPTH OF BST

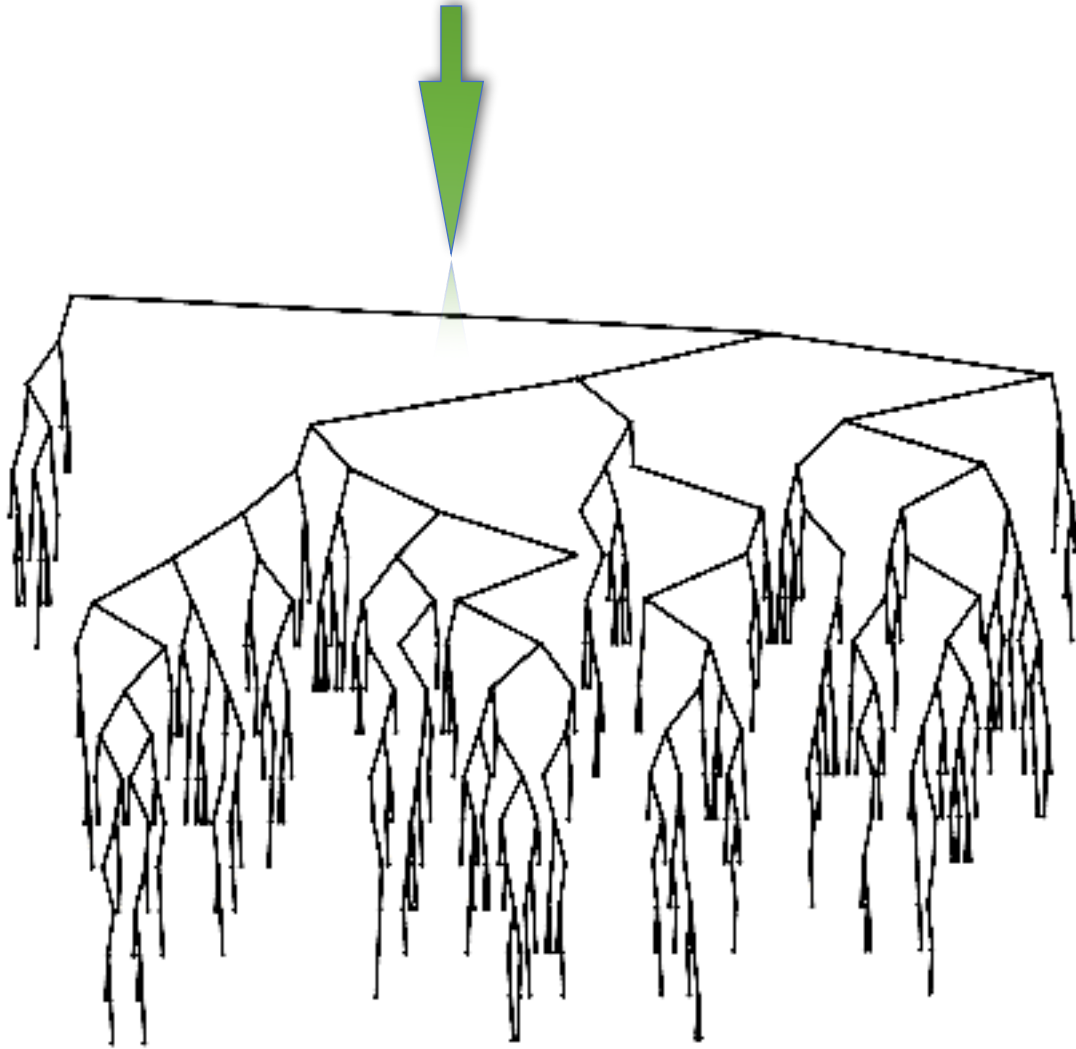
$$\begin{aligned} \blacksquare T_D(N) &= \frac{2}{N}[T_D(0) + T_D(1) + T_D(2) + \cdots + T_D(N-1)] + c(N-1) \\ T_D(x) &\simeq \frac{2}{x} \int_0^x T_D(x) + c(x-1) \\ xT_D(x) &\simeq 2 \int_0^x T_D(x) + c(x^2 - x) \\ &\Rightarrow T_D(x) + x \frac{dT_D(x)}{dx} = 2T_D(x) + c(2x-1) \\ \frac{dT_D(x)}{dx} &\simeq T_D(x)/x + 2c \\ &\Rightarrow T_D(x) = 2cx \log(x) \end{aligned}$$

SAME AS QUICK SORT!

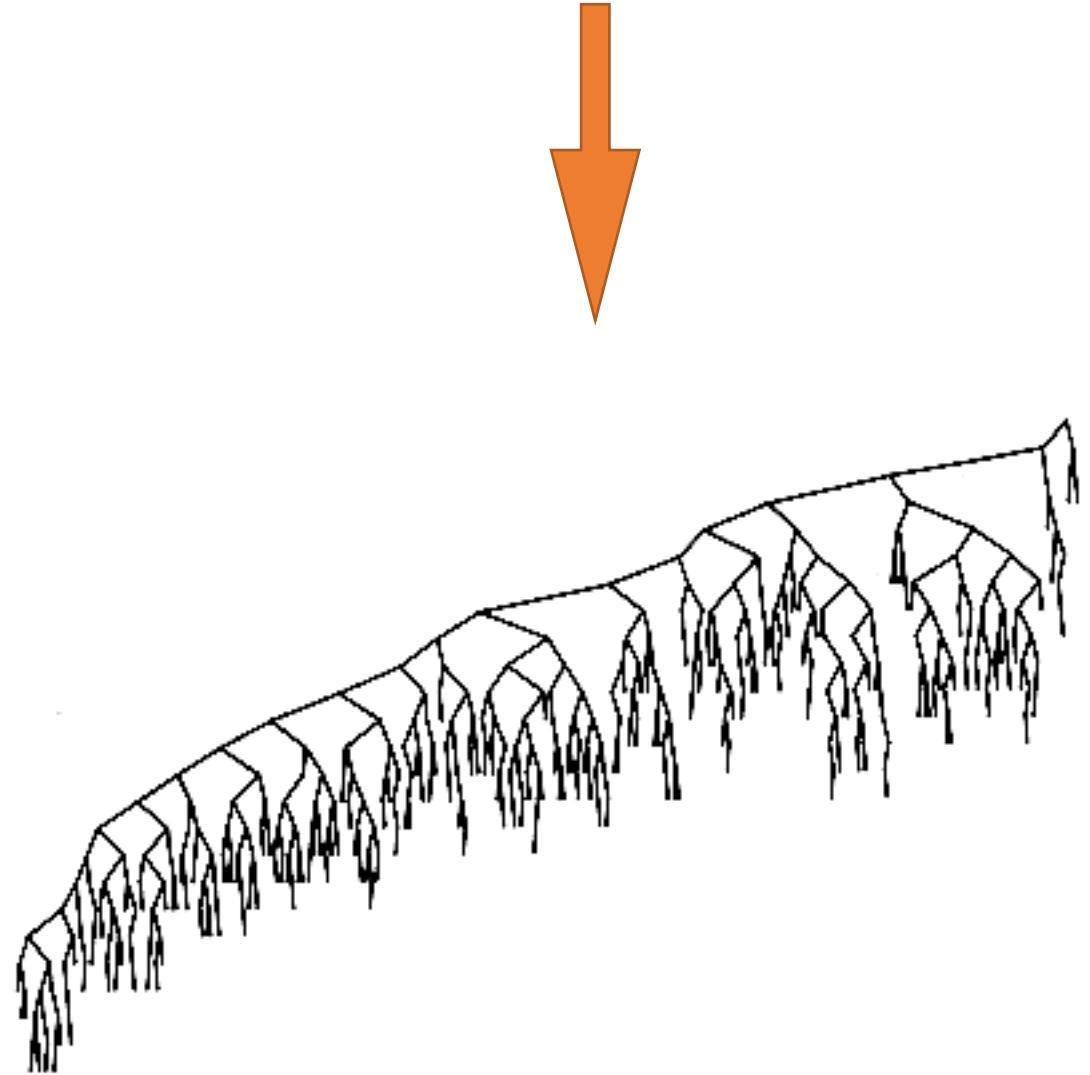
◆ Solution:  $T_D(N) = \Theta(N \log(N))$

# See Average of Quick Sort Sec 7.7.5 (p 278)<sup>20</sup>

# Average BST vs After $O(N^2)$ insert/delete



Depth  $O(\log N)$



Depth  $O(N)$

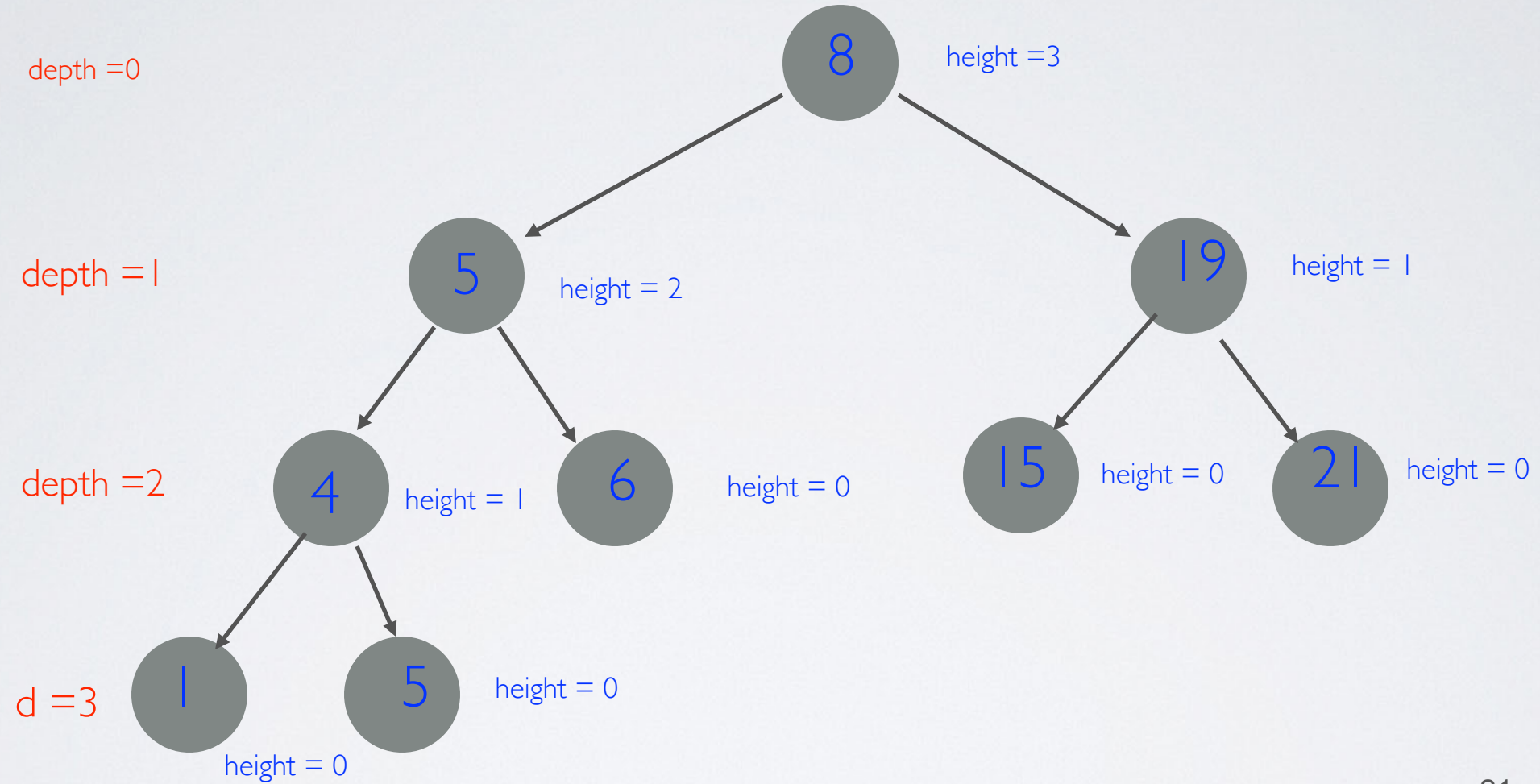
# RELATIONS: BOOLEAN VALUED MATRIX $R[A,B]$

- Set:  $S = \{a,b,c,\dots\}$
- Relation  $(a,b) \in S \times S$ :  $a R b$  is True?
- Properties:
  - ◆ Reflexive:  $a R a$  is True
  - ◆ Anti-symmetric:  $a R b$  and  $b R a \rightarrow a = b$
  - ◆ Transitive:  $a R b$  and  $b R c \rightarrow a R c$
  - ◆ Total Ordering:  $a R b$  or  $b R a$  (inclusive or)
  - ◆ Self dual:  $a R b \leftrightarrow b R a$
  - ◆ Transpose:  $a R b \leftrightarrow b R^T a$
- RAT is partial ordering: e.g. descendants in a tree!

(e.g.  $\leq$  is total ordering for int but  $g(N) = O(f(N))$  is partial ordering!)



# AVL: BST WITH $|H_L - H_R| = 0, 1$



# FIBONACCI SERIES & RABBITS

- 2 rabbit beget 2 rabbit every month after one month!
  - ◆  $F_k = (F_{k-1} - F_{k-2}) + 2 F_{k-2} = F_{k-1} + F_{k-2}$
  - ◆ with  $F_0 = F_1 = 1$
- 1, 1, 2, 3, 5, 8, 13, 21, .... Fibonacci:  $F_k = F_{k-1} + F_{k-2}$
- Bad Recursion:  $\Omega(C^N)$  with  $C = 1.61...$ 
  - ◆ Exponential  $T(N) = T(N-1) + T(N-2) + 1$
  - ◆  $T(N) > F_N > (2)^{N/2}$
- Iteration:  $\Theta(N)$
- Math  $\Theta(1)$ ! Try  $F_k = x^k$  find two homogeneous solutions to characteristic equation.

$$F_k = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{k+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{k+1} \right]$$

$$\frac{1 + \sqrt{5}}{2} \simeq 1.61803$$

# WORST CASE HEIGHT $H(N)$ FOR AVL

- Minimum # of Nodes (see Fig 4.33):

$$N(H) = N(H-1) + N(H-2) + 1 > N(H-1) + N(H-2)$$

- Almost Fibonacci:  $F_k = F_{k-1} + F_{k-2}$

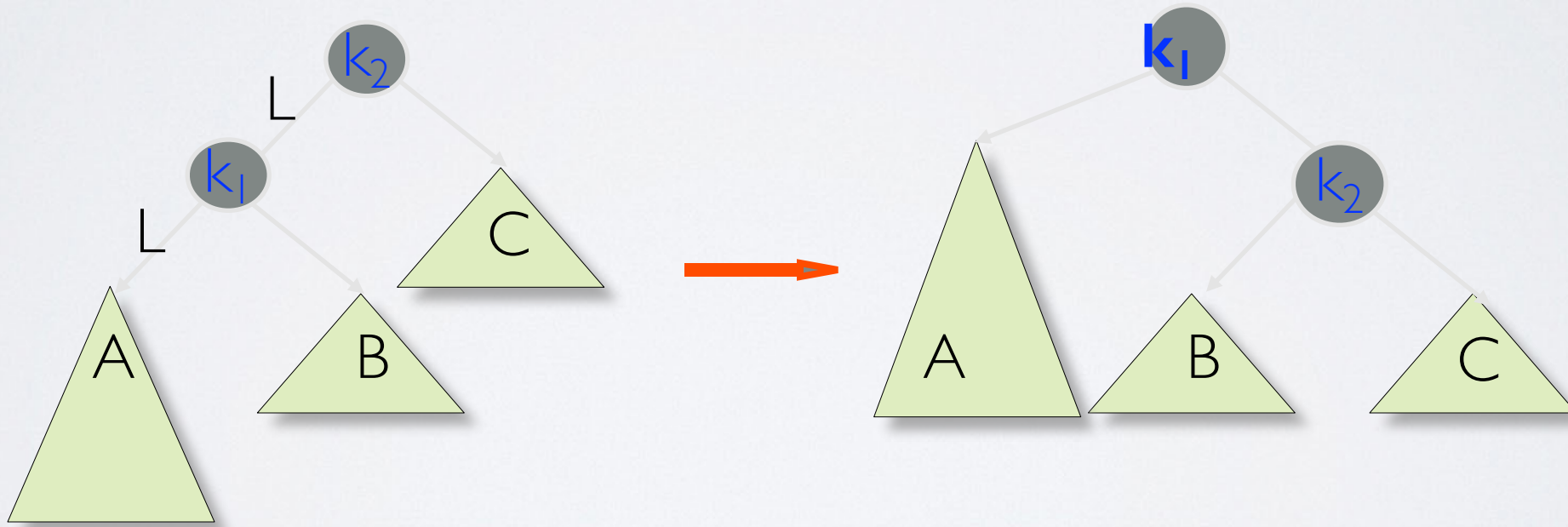
◆ So  $N(H) > F_H \sim c^H$  with  $c = (1 + 5^{1/2})/2 = 1.618034$

◆ Or  $H < \log(N)/\log(c) \quad 1.440420 \log_2(N) = 2.078 \ln(N)$   
 $= 4.784 \log_{10}(N)$

( Better estimate:  $H = 1.44 \log_2(N+2) - 0.328$

# ZIG-ZIG INSERTION FOR LL OR RR:

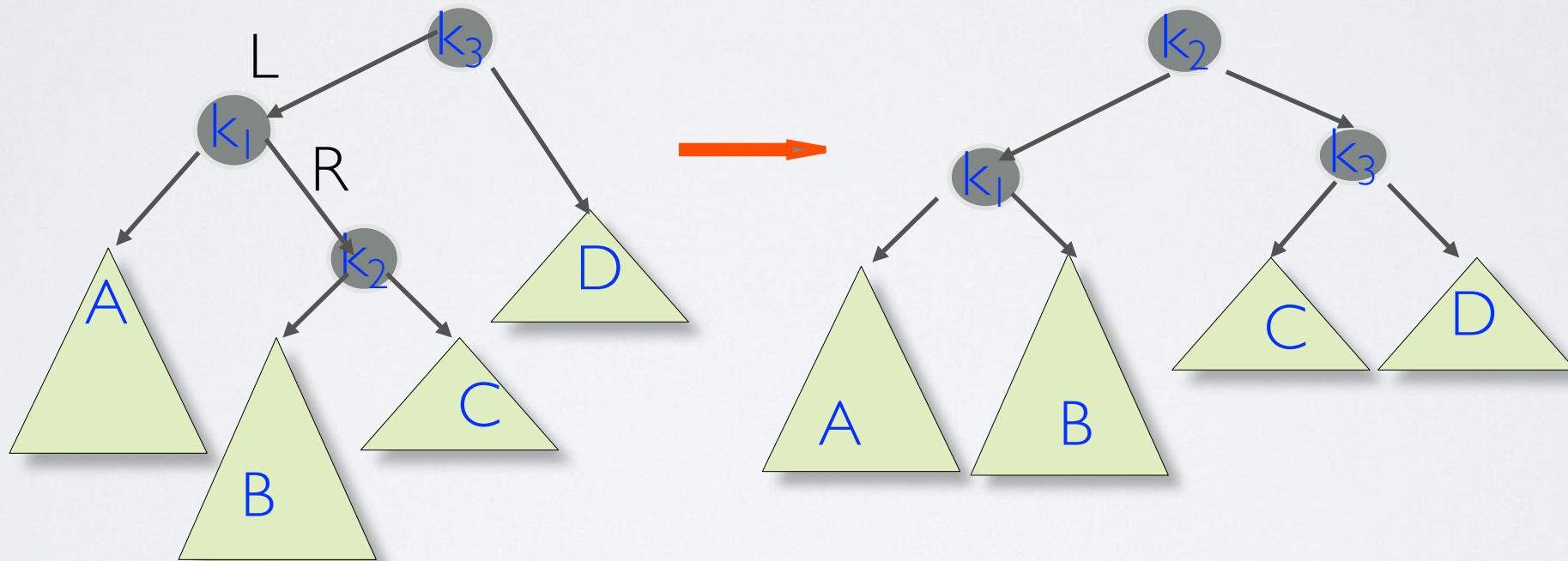
- Insert New Key along path going **L**eft and **L**eft again into A:
- This cause violation of AVL balance.
- $k_2$  is lowest node failing AVL balance.
- Single rotation of  $k_1 \rightarrow k_2$  restores AVL balance





# ZIG-ZAG INSERTION FOR LR

- Insert New Key along path going **L**eft and then **R**ight into B:
- This cause violation of AVL balance.
- $k_3$  is lowest node failing AVL balance.
- Double rotation of  $k_1 \rightarrow k_2 \rightarrow k_3$  restores AVL balance



# HUFFMAN CODING

- ❑ Place all letters at leaves of a binary tree
  - ❑ The code is path (i.e. address) of each leaf.
  - ❑ Binary code for each letter: e.g. "a" = 01001, "b" = 101, ..

$$\text{ext. depth} = \sum_i w_i d_i, \quad \text{average code length} = \frac{\sum_i w_i d_i}{\sum_i w_i} = \sum_i p_i d_i$$

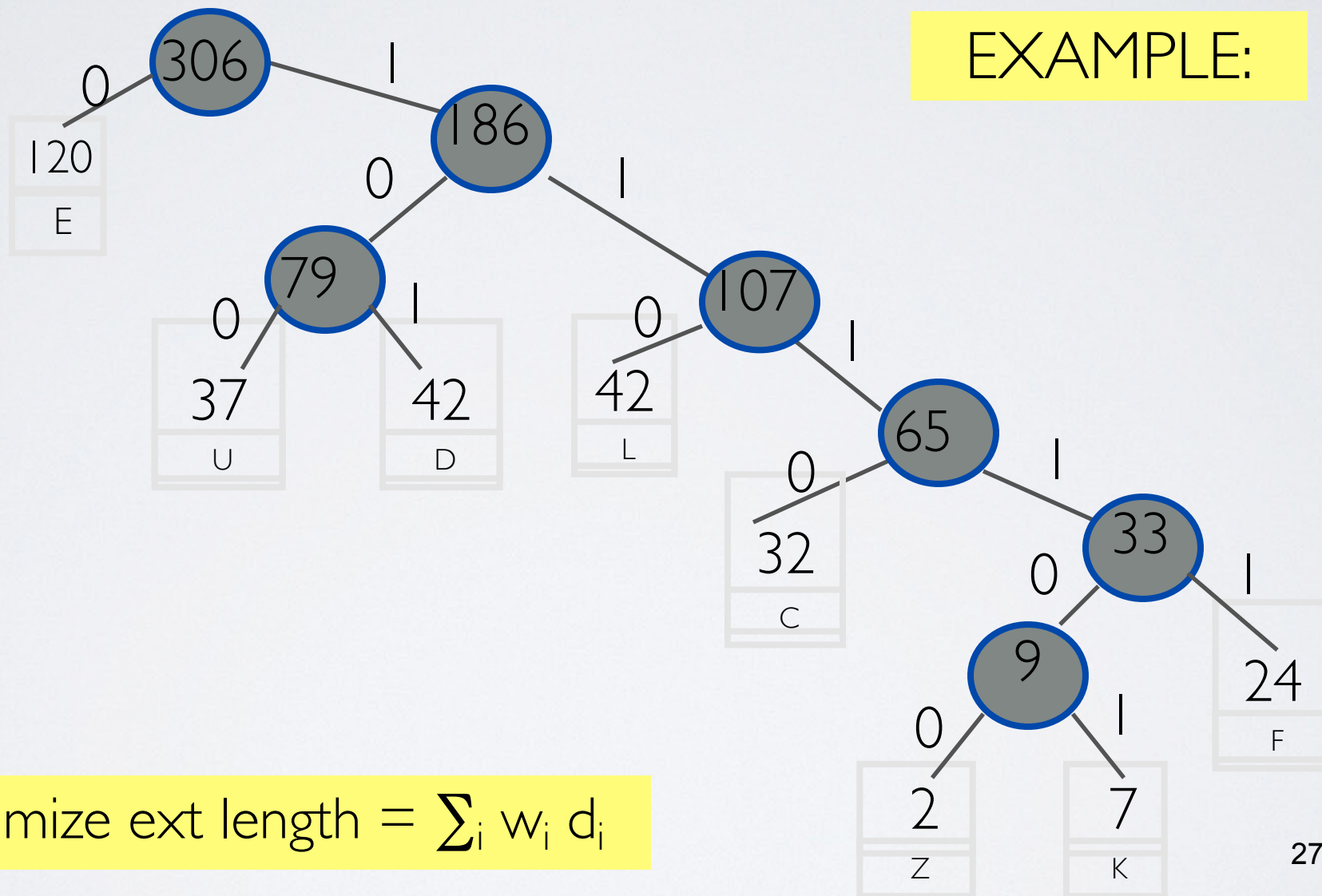
Build the Huffman tree:

- ❑ Sort symbol list:  $w_1 < w_2 < \dots < w_N$
- ❑ Remove  $w_1$  and  $w_2$  and place as left and right children of parent  $w_{(12)}$
- ❑ Place  $w_{(12)} = w_1 + w_2$  in symbol list and Repeat

$W_i$  →

|   |   |    |    |    |    |    |     |
|---|---|----|----|----|----|----|-----|
| 2 | 7 | 24 | 32 | 37 | 42 | 42 | 120 |
| Z | K | F  | C  | U  | D  | L  | E   |

EXAMPLE:



Minimize ext length =  $\sum_i w_i d_i$

RESULTING CODE:  $AVERAGE\ BITS/CHAR = 785/306 = 2.565$

|            | Letter | Weight | Code   | Bits | Count |
|------------|--------|--------|--------|------|-------|
| ■          | C      | 32     | 1110   | 4    | 128   |
| ■          | D      | 42     | 101    | 3    | 126   |
| ■          | E      | 120    | 0      | 1    | 120   |
| ■          | F      | 24     | 11111  | 5    | 120   |
| ■          | K      | 7      | 111101 | 6    | 42    |
| ■          | L      | 42     | 110    | 3    | 126   |
| ■          | U      | 37     | 100    | 3    | 111   |
| ■          | Z      | 2      | 111100 | 6    | 12    |
| Total: 306 |        |        |        |      | 785   |



# PROOF BY INDUCTION

- *Base case  $N=2$  has minimum with  $d_1 = d_2 = 1$*
- *Two smallest weights  $w_1$  &  $w_2$  are at max depth*
- *(If not swap with any other is smaller: See identity next )*
  - ◆ *Can swap to give same parent  $w_{12} = w_1 + w_2$*
- *Hence prove for  $N$ :*
- *$\text{Min}[(d_{12} + 1) (w_1 + w_2) + w_3 d_3 + \dots + w_N d_N]$  over all trees  $T$*   
$$= (w_1 + w_2) + \text{Min}[d_{12} w_{12} + w_3 d_3 + \dots + w_N d_N]$$

# “SCHWARTZ” PARING INEQUALITY!

Need for Huffman and Many Opt Algorithms

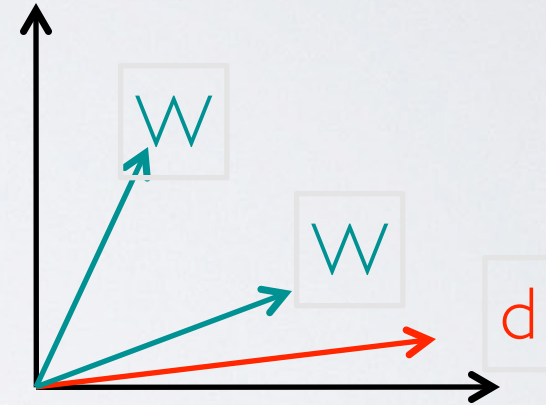
Prove: (more parallel is larger!)

$$w_S d_S + w_L d_L > w_L d_S + w_S d_L$$

because  $(w_L - w_S)(d_L - d_S) > 0$

scalar product is larger

when  $w$  and  $d$  are more nearly parallel!



# MORE OPTIMIZATION

- Object Function and elementary move

- Sorting  $S = \text{MIN}_{\pi} \sum_i I * a[\pi(I)]$

- swap minimize  $|a[I] - a[J]|$  if out of order

- Continuum vs Discrete:

- Bisection :  $\text{Log}(N)$  vs error  $\Rightarrow$  error/2

- Find zero:  $f(x) = 0$  or (continuous)

- Find key  $f[I] = (a[I] - \text{key})^2 = 0$  ( $a[I]$  sorted)

- Newton's, Secant, Regula falsi

- & Dictionary method (linear extrapolation)

- $\text{Log}(\text{Log}(N))$  vs error  $\Rightarrow$   $(\text{error})^\phi$

$\phi$  is 2 for Newton and the golden ratio 1.618 for secant.

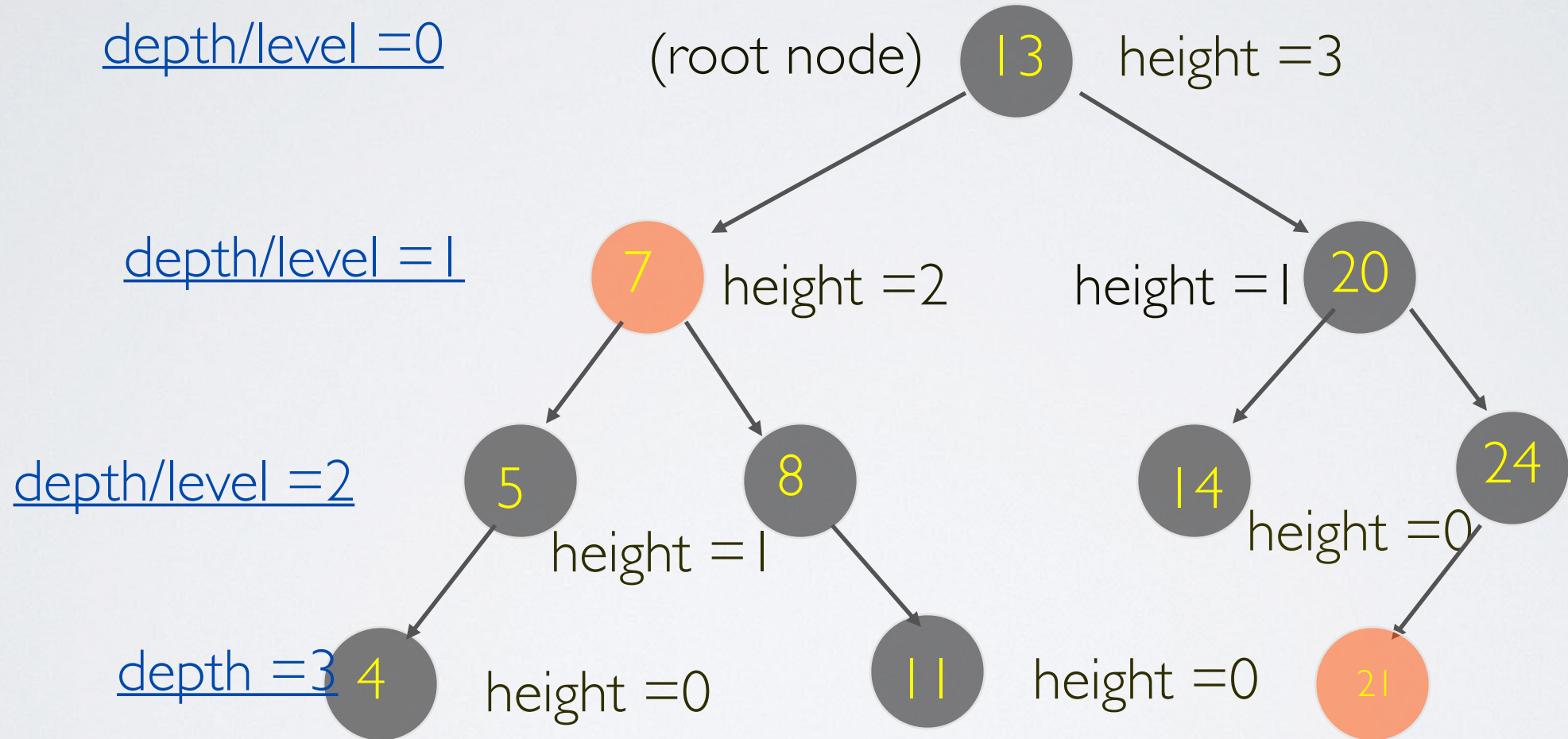
# RED-BLACK TREES

1. *Every node is red or black*
2. *The root is black*
3. *Every nil node (leaf) is black*
4. *A red node must have black children*
5. *Every path to a nil pointer must have same number of black nodes*

*Insertion: Place as new leaf in BST must color red ( due #5) If parent is red it violates #4. Must do rotations to fix it!*



# RED-BLACK TREES



Partial Ordering →