RELATIONSHIPS AMONG DIGITAL ONE/HALF BAND FILTERS, LOW/HIGH ORDER DIFFERENTIATORS AND DISCRETE/DIFFERENTIATING HILBERT TRANSFORMERS

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ABSTRACT

There exists a close relationships among digital one/half band filters, low/high order differentiators and discrete/differentiating Hilbert transformers. This paper is to present a complete picture of their interrelationships and conversions between each other. A useful table and some block diagrams have been developed for their impulse reponse's connections.

1. INTRODUCTION

Conventionally, we often use the well-known McClellan-Parks program [1][2] to design the FIR digital filters, differentiators and Hilbert transformers independently; Also recently the eigenfilter approach [3][4][5] has been developed to design these filters, differentiators and Hilbert transformers differently in each independent case. However, these designs can be closely related to each other, this paper presents their relationships and unifys them by means of a useful table and some block diagrams.

Jackson first pointed out that design of symmetric Hilbert transformers with odd lengths can be easily derivated from the corresponding designs for half-band filters and vice versa [6]. Vaidyanathan and Nguyen also found that there are some relationships between half-band filters and certain one-band filters [7]. This paper further exploits that the relations between one-band filters and low/high order differentiators exist, and the decimated versions of the odd symmetric Hilbert transformers become the Hilbert transformers with even lengths. Moreover, the relation between odd length symmetric Hilbert transformer and differentiating Hilbert transformer [8] is also discussed. Hence relationships among one/half band filters, low/high order differentiators and two Hilbert transformer types can be established and unified each other; Design of any one of the above filters except the high order differentiators will generate the others and vice versa.

2. INTERRELATIONS AMONG ONE/HALF BAND FILTERS, FIRST-ORDER DIFFERENTIATORS AND HILBERT TRANSFORMERS

The overall block diagram for illustrating the relationships among one-band filter (B_1), half-band filter ($B_{\frac{1}{2}}$), Case 3 symmetric Hilbert transformer (H_3), Case 4 Hilbert transformer (H_4), differentiating Hilbert transformer (H_d) and first-order type linear differentiator (D_1) are given in Fig.1(a) in which the transfer functions for each filter are characterized as below:

Assume N is an odd integer, then

One-band filter (even length: N+1):

$$B_1(Z)=\sum_{n=0}^{N} b_1(n)Z^{-n} \tag{1}$$

Half-band filter (odd length: 2N+1):

$$B_{\frac{1}{2}}(Z)=\sum_{n=0}^{2N} b_{\frac{1}{2}}(n)Z^{-n} \tag{2}$$

Case 3 symmetric Hilbert transformer (odd length: 2N+1):

$$H_3(Z) = \sum_{n=0}^{2N} h_3(n)Z^{-n} \tag{3}$$

Case 4 Hilbert transformer (even length: N+1):

$$H_4(Z) = \sum_{n=0}^{N} h_4(n)Z^{-n} \tag{4}$$

Differentiating Hilbert transformer (odd length: 2N+1):

$$H_d(Z) = \sum_{n=0}^{2N} h_d(n)Z^{-n} \tag{5}$$

First-order differentiator (even length: N+1):

$$D_1(Z) = \sum_{n=0}^{N} d_1(n)Z^{-n} \tag{6}$$

The representations of label numbers in Fig.1(a) are tabulated in TABLE I, and the derivation of which are described below. In [6], Jackson had shown that

$$h_3(n) = \begin{cases} (-1)^{\frac{n-N-1}{2}}\, 2b_{\frac{1}{2}}(n) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{7}$$

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and

$$b_{\frac{1}{2}}(n) = \begin{cases} (-1)^{\frac{n-N-1}{2}} \frac{1}{2} h_3(n) & n \text{ even} \\ 0 & n \text{ odd} \\ \frac{1}{2} & n=N \end{cases} \tag{8}$$

And Vaidyanathan and Nguyen [7] found that

$$b_{\frac{1}{2}}(n) = \begin{cases} \frac{1}{2} b_1(\frac{n}{2}) & n \text{ even} \\ 0 & n \text{ odd} \neq N \\ \frac{1}{2} & n = N \end{cases} \tag{9}$$

and

$$b_1(n) = 2b_{\frac{1}{2}}(2n) \qquad n=0,1,\ldots,N \tag{10}$$

In the following, we will further exploit the new relations between one-band filter and first-order differentiator, the relationship between differentiating Hilbert transformers and Case 3 symmetric Hilbert transformers, and the conversions of Case 3 and 4 Hilbert transformers, then unify these interrelations into a complete picture.

The amplitude responses of a one-band filter and a first-order differentiator can be represented by

$$\hat{B}_1(w) = \sum_{n=1}^{\frac{N+1}{2}} \hat{b}_1(n)\cos(n-\tfrac{1}{2})w \tag{11}$$

and

$$\hat{D}_1(w) = \sum_{n=1}^{\frac{N+1}{2}} \hat{d}_1(n)\sin(n-\tfrac{1}{2})w \tag{12}$$

where

$$\hat{b}_1(n) = 2b_1(\tfrac{N+1}{2}-n) \qquad n=1,2,\ldots,\tfrac{N+1}{2} \tag{13}$$

and

$$\hat{d}_1(n) = 2d_1(\tfrac{N+1}{2}-n) \qquad n=1,2,\ldots,\tfrac{N+1}{2} \tag{14}$$

The integration of Eq.(11) along the w-axis will lead to the amplitude response of the corresponding first-order differentiator (see Eq.(12)), then we can get

$$\hat{d}_1(n) = \frac{\hat{b}_1(n)}{n-\frac{1}{2}} \qquad n=1,2,\ldots,\tfrac{N+1}{2} \tag{15}$$

From Eqs.(13), (14), (15), it's easy to derive

$$b_1(n) = (\tfrac{N}{2}-n)d_1(n) \qquad n=0,1,\ldots,N \tag{16}$$

or

$$d_1(n) = \frac{b_1(n)}{\frac{N}{2}-n} \qquad n=0,1,\ldots,N \tag{17}$$

It's noted that if the filter coefficients of the differentiators are derived from the known impluse responses of one-band filters, its magnitude responses are only suitable for designing the nonfull-band differentiators due to the inherent zero-folding frequency constraint of the Case 2 one-band filters [9].

Recently Cizek proposed a new differentiating Hilbert transformer [8], the output of which is the derivative of the Hilbert transform of the input signal. This signal is useful for the evaluation of the instantaneous frequency by means of an analytic signal.

The amplitude responses of a Case 3 symmetric Hilbert transformer and differentiating Hilbert transformer can be represented by

$$\hat{H}_3(w) = \sum_{n=1}^{N} \hat{h}_3(n)\sin nw \tag{18}$$

and

$$\hat{H}_d(w) = \sum_{n=0}^{N} \hat{h}_d(n)\cos nw \simeq |w| = (jw)\cdot(-j) \tag{19}$$

respectively where

$$\hat{h}_3(n) = 2h_3(n) \qquad n=1,\ldots,N \tag{20}$$

and

$$\hat{h}_d(n) = \begin{cases} h_d(n), & n=0 \\ 2h_d(N-n), & n=1,\ldots,N \end{cases} \tag{21}$$

From Eq.(19), the differentiating Hilbert transformer can be implemented by cascade connection of a differentiator and a Hilbert transformer; However, the direct use of differentiating Hilbert transformer is more accurate and efficient than the above cascade scheme.

The integration of Eq.(18) along w-axis will approximately lead to the amplitude response $-\hat{H}_d(w)+\pi/2$, hence we can get the following conversions:

$$h_d(n) = \begin{cases} \frac{h_3(n)}{N-n} & n \text{ even} \\ 0 & n \text{ odd} \neq N \\ \frac{\pi}{2} & n=N \end{cases} \tag{22}$$

and

$$h_3(n) = \begin{cases} (N-n)h_d(n) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{23}$$

If we reduce the sampling rate on the Case 3 symmetric Hilbert transformer's impulse responses by a factor of 2, this decimation process [10] would correspond to extracting every 2nd sample of the discrete sequences. Since every other impulse response sample of the symmetric Hilbert transformer is equal to zero, by taking out these zero-valued impulse response samples, the decimated version of Case 3 symmetric Hilbert transformers will become the Case 4 Hilbert transformers with even length very interestingly. If we reverse the above process by inserting a zero-valued sample between each impulse response sequence of the Case 4 Hilbert transformer, then the Case 4 Hilbert transformers will become Case 3 symmetric Hilbert transformers after this interpolation process [10]. So there exists

$$h_3(n) = \begin{cases} h_4(\frac{n}{2}) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{24}$$

and

$$h_4(n) = h_3(2n) \qquad n=0,1,\ldots,N \tag{25}$$

The transition bandwidth of Case 4 Hilbert transformer is equal to twice of that of Case 3 symmetric Hilbert transformer, and the ripple is the same in these two type Hilbert transformers.

By the above description, it's easy to establish the relationships among one/half band filters, first-order differentiators and the Hilbert transformers; Fig.1(a) shows the block diagram for connecting these relationships, in which the exact formulation indicated by the label numbers are tabulated in TABLE I; Also the transition bandwidths and the passband ripples among all type filters are also included here in terms of one band filter's bandwidth Δ and ripple δ for comparison.

3. DERIVATIONS OF EVEN LENGTH HIGH-ORDER DIFFERENTIATORS FROM ONE-BAND FILTERS

High-order differentiators are very useful for calculation of geometric moments and for biological signal processing. Recently the modified McClellan-Parks program [2] and the eigen-approach [5] are proposed for designing these high-order differentiators. This paper is to point out that these designs can be easily derivated from the one-band filters without involving any time consuming optimization procedures.

Following Section II, the transfer function of even length i-th order differentiator can be written as

$$D_i(Z) = \sum_{i=0}^{N} d_i(n) Z^{-n} \tag{26}$$

where N is an odd integer.
For an odd-order Case 4 differentiator, the amplitude response of Eq.(26) is

$$\hat{D}_i(w) = \sum_{n=1}^{\frac{N+1}{2}} \hat{d}_i(n) \sin(n-\tfrac{1}{2})w \simeq (jw)^i, \quad i \text{ odd} \tag{27}$$

and for an even-order Case 2 differentiator, the amplitude response is

$$\hat{D}_i(w) = \sum_{n=1}^{\frac{N+1}{2}} \hat{d}_i(n) \cos(n-\tfrac{1}{2})w \simeq (jw)^i, \quad i \text{ even} \tag{28}$$

$$\text{where} \quad \hat{d}_i(w) = 2d_i(\tfrac{N+1}{2}-n) \qquad n=1,2,\ldots,\tfrac{N+1}{2} \tag{29}$$

Consider a second-order differentiator, its amplitude response can be derived by integrating the amplitude response of the first order differentiator (see Eq.(12)) along w-axis, i.e.

$$\hat{D}_2(w) = -2\int_0^w \hat{D}_1(w)dw = -2\int_0^w \{ \sum_{n=1}^{\frac{N+1}{2}} \hat{d}_1(n) \sin(n-\tfrac{1}{2})w \} dw$$

$$= \sum_{n=1}^{\frac{N+1}{2}} \frac{2\hat{d}_1(n)}{n-\frac{1}{2}} \cos(n-\tfrac{1}{2})w - \sum_{n=1}^{\frac{N+1}{2}} \frac{2\hat{d}_1(n)}{n-\frac{1}{2}} \cdot 1 \tag{30}$$

Substitute $\hat{d}_1(n) = \frac{\hat{b}_1(n)}{n-\frac{1}{2}}$ (15) and $\sum_{n=0}^{\frac{N+1}{2}} \hat{b}_1(n)\cos(n-\frac{1}{2})w \approx 1$ (one-band frequency response is approximately equal to 1 except near the folding frequency) in the above equation, we get

$$\hat{D}_2(w) \doteqdot \sum_{n=1}^{\frac{N+1}{2}} \frac{2\hat{b}_1(n)}{(n-\frac{1}{2})^2} \cos(n-\tfrac{1}{2})w - [\sum_{n=1}^{\frac{N+1}{2}} \frac{2\hat{b}_1(n)}{(n-\frac{1}{2})^2}][\sum_{n=0}^{\frac{N+1}{2}} \hat{b}_1(n)\cos(n-\tfrac{1}{2})w]$$

$$= \sum_{n=1}^{\frac{N+1}{2}} [\frac{2\hat{b}_1(n)}{(n-\frac{1}{2})^2} - (\sum_{n'=1}^{\frac{N+1}{2}} \frac{2\hat{b}_1(n')}{(n'-\frac{1}{2})^2} \hat{b}_1(n)] \cos(n-\tfrac{1}{2})w \tag{31}$$

Using Eqs.(28), (29) and (13), we can get the filter coefficients of the second-order differentiator form the impulse response of one-band filter by the following relation

$$d_2(n) = \frac{2b_1(n)}{(\frac{N}{2}-n)^2} + \sum_{n'=0}^{N} \frac{2b_1(n')}{(\frac{N}{2}-n')^2}] b_1(n), \quad n=0,1,\ldots,N \tag{32}$$

Similarly, for third-order differentiator, its amplitude response is

$$\hat{D}_3(w) = 3\int_0^w \hat{D}_2(w)dw$$

$$= \sum_{n=1}^{\frac{N+1}{2}} [\frac{3!\hat{b}_1(n)}{(n-\frac{1}{2})^3} - (\sum_{n'=1}^{\frac{N+1}{2}} \frac{3!\hat{b}_1(n')}{(n'-\frac{1}{2})^2}) \frac{\hat{b}_1(n)}{n-\frac{1}{2}}] \sin(n-\tfrac{1}{2})w \tag{33}$$

and its impulse response is

$$d_3(n) = \frac{-3!\hat{b}_1(n)}{(\frac{N}{2}-n)^3} + [\sum_{n'=0}^{N} \frac{3!\hat{b}_1(n')}{(\frac{N}{2}-n')^2}] \frac{b_1(n)}{\frac{N}{2}-n}, \quad n=0,1,\ldots,N \tag{34}$$

The relationships and the formulations between the one-band filters and these high-order differentiators are illustrated in Fig.1(b) and TABLE I (omitted for space saving).

For i>3, the impulse response coefficients of higher-order differentiator can be derived by similar procedures described above. Notice that the deviations of even-order differentiators (a one-band filter can be taken as a zero-order differentiators) is larger than that of odd-order differentiators, because we approximate directly a unit constant by the magnitude response of a one-band filter, for example as in Eq.(30). So the even-order differentiators from this method are generally inferior to the odd-order ones.

4. DESIGN EXAMPLE

A numerical example is given here for clear illustration. First we can design a length 30 one band filter with cutoff frequency 0.46 by the McClellan-Parks program [1]. From above we can easily get the filter coefficients of the half-band filter, first-order differentiator, discrete Hilbert transformers and first to third order differentiators by this one band filter's impulse response.

5. CONCLUSIONS

In this paper, digital one/half band filters, low/high order differentiators and discrete/differentiating Hilbert transformer have been shown in close relation to each other. A useful table and block diagrams have been developed for unifying their impulse's connections, and a design example is also given for illustration.

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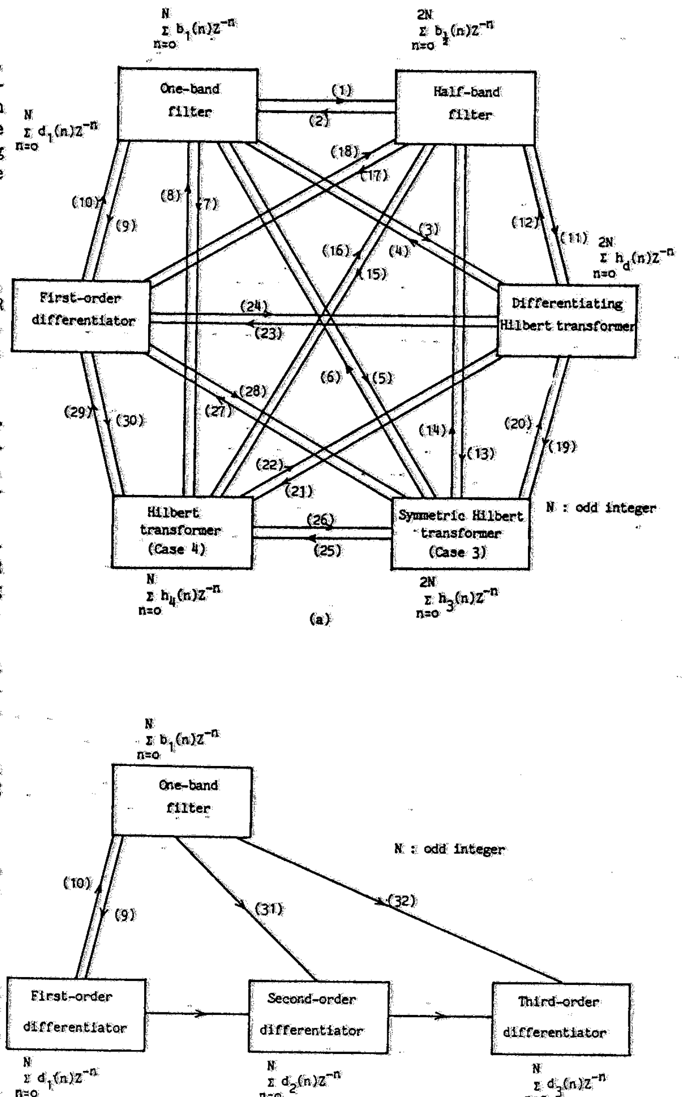

Fig. 1 Block diagram for illustrating the relationships among one/half band filters, lower/higher order differentiators, and discrete/differentiating Hilbert transformers, in which the