SPARKCHARTS™

GEOMETRY

SPARK CHARTS

POINTS, LINES, PLANES

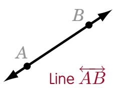
Technically, the words "point," "line," "plane" cannot be defined without using circular reasoning.

DIMENSION 0: POINTS

Point: A specific location in space with no width, length, or height. Point P is denoted by a dot: $\bullet P$ but the dot is thick and only a crude representation of the actual point.

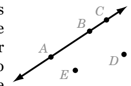
DIMENSION 1: LINES, RAYS, SEGMENTS

Line: A line is an infinite collection of points arrayed in a straight formation with only one dimension: length.



Line $\overleftrightarrow{AB}$

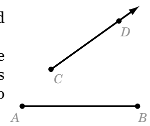
- Any two different points define a unique line. The line containing points A and B is denoted $\overleftrightarrow{AB}$.
- Three or more points that lie on a line are **collinear**. Three or more points that do not lie on the same line are **non-collinear.** Any two points are collinear.



Points A, B, C are collinear. B, D, E are not. Also, A—B—C.

- If three points are collinear, one of them is **between** the other two. Notation: A—B—C (or C—B—A) means that point B is between points A and C.

Line segment: A portion of a line with two endpoints and finite length. The segment with endpoints A and B is denoted $\overline{AB}$.
- A segment is the set of all points *between* the two endpoints.



Line segment $\overline{AB}$; Ray $\overrightarrow{CD}$

Ray: A "half-line"—a portion of a line that has an endpoint and extends without end in one direction. The ray with endpoint A extending through B is denoted $\overrightarrow{AB}$.
- $\overrightarrow{AB}$ is the set of all points X such that X is between A and B or B is between A and X.
- Two rays that have a common endpoint but extend in opposite directions are called **opposite rays**.

DIMENSION 2: PLANES

Plane: A flat, boundless surface in space with two dimensions: length and width. A plane has no height.
- Any three noncollinear points define a unique plane.
- Any line and a point not on the line define a unique plane.
- A set of points or lines are **coplanar** if they lie in the same plane; **non-coplanar** if they do not.

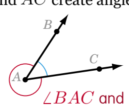
DIMENSION 3: SPACE

Space: The infinite set of all points. Space has three dimensions: length, width, and height.

ANGLES

Two rays that share a common endpoint create an **angle**. The common endpoint is called the **vertex** of the angle.
- We can think of an angle as being formed by rotating a ray **clockwise** or **counterclockwise**, and distinguish between the angle's **initial side** (starting position of the ray) and **terminal side** (end position of the ray).
- **Notation:** Rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ create angle $\angle BAC$ (or $\angle CAB$).

A pair of rays actually create *two* angles (one is bigger than half a revolution). To avoid confusion, indicate the one you mean with an arc.



$\angle BAC$ and $\angle BAC$, both with vertex A.

MEASURING ANGLES; CONGRUENT ANGLES

We measure angles to see how wide or narrow they are. The measure of $\angle A$ is denoted $m\angle A$.
- In geometry, angles are usually measured in **degrees** (°). A degree is a unit of angle measure in which a complete revolution is $360°$; each degree is subdivided into 60 **minutes** ($'$), and each minute is subdivided into 60 **seconds** ($''$).
- Angles can also be measured in **radians** (**rad**). A complete revolution measures 2π radians.

Radian measure is technically unitless: the radian measure of an angle is the length of an arc of a circle of radius 1 cut off by that angle.

Converting between degrees and radians:

$1 \text{ revolution} = 360° = 2\pi \text{ rad}$

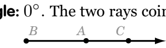
$1 \text{ rad} = \frac{180}{\pi}° = \frac{1}{2\pi} \text{ revolution}$

$1° = \frac{\pi}{180} \text{ rad} = \frac{1}{360} \text{ revolution}$

Congruent angles: If $\angle A$ and $\angle B$ have the same measure, they are **congruent.** Write $\angle A \cong \angle B$ or $m\angle A = m\angle B$. In diagrams, congruent angles are noted by crossing the arc of the angle with an equal number of strokes. See diagram in *Angle pairs: Vertical angles*, below.

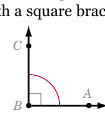
TYPES OF ANGLES

Zero angle: $0°$. The two rays coincide.



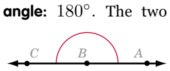
zero angle
$m\angle ABC = 0°$

Right angle: $90°$. The two rays are perpendicular. In diagrams, a right angle is indicated with a square bracket.



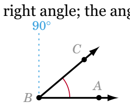
right angle
$m\angle ABC = 90°$

Straight angle: $180°$. The two rays are opposite.



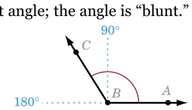
straight angle
$m\angle ABC = 180°$

Acute angle: Less than $90°$. Between a zero angle and a right angle; the angle is "sharp."



acute angle

Obtuse angle: Greater than $90°$ and less than $180°$. Between a right angle and a straight angle; the angle is "blunt."



obtuse angle

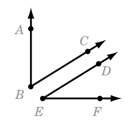
Oblique angle: Either acute or obtuse (not zero, right, or straight).

ANGLE PAIRS

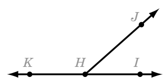
Adjacent angles: Two angles that have a common vertex and a common side (but no common interior points).

Complementary angles: A pair of angles the sum of whose measure is $90°$.

Supplementary angles: A pair of angles the sum of whose measure is $180°$.

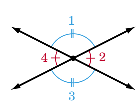


$\angle ABC$ and $\angle DEF$ are complementary.

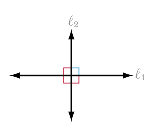


$\angle KHJ$ and $\angle JHI$ are supplementary; they are also adjacent.

Vertical angles: Two intersecting lines (or segments) form four angles: two pairs of opposing vertical angles. *The vertical angles in each pair are congruent.* Any two adjacent angles will be supplementary.
- If one of the angles in the intersection of two lines is a right angle, then all four angles are right angles.



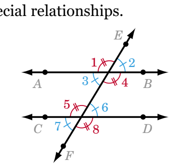
$\angle 1$ and $\angle 3$ are vertical angles; $\angle 2$ and $\angle 4$ are vertical angles.



Lines ℓ_1 and ℓ_2 intersect in four right angles.

PARALLEL LINES CUT BY TRANSVERSAL

Two parallel lines interesected by a **transversal**—a line not parallel to either—create 8 angles (four sets of vertical angles) with special relationships.



Parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are cut by the transversal $\overleftrightarrow{EF}$. Here $\angle 1 \cong \angle 4 \cong \angle 5 \cong \angle 8$ and $\angle 2 \cong \angle 3 \cong \angle 6 \cong \angle 7$.

Corresponding angle pairs: Four pairs; on the same side of the transversal and separated by one of the parallel lines. Corresponding angles are congruent. Here $\angle 1 \cong \angle 5$; $\angle 2 \cong \angle 6$; $\angle 3 \cong \angle 7$; and $\angle 4 \cong \angle 8$.

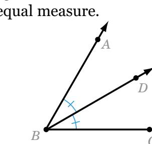
Alternate interior angle pairs: Two pairs; between the parallel lines, on opposite sides of the transversal. Alternate interior angles are congruent. Here, $\angle 3 \cong \angle 6$ and $\angle 4 \cong \angle 5$.

Alternate exterior angle pairs: Two pairs; outside the parallel lines, on opposite sides of the transversal. Alternate exterior angles are congruent. Here, $\angle 1 \cong \angle 8$ and $\angle 2 \cong \angle 7$.

Interior angle pairs: Two pairs; on the same side of the transversal. Interior angles are supplementary. Here, $\angle 3$ and $\angle 5$; $\angle 4$ and $\angle 6$.

ANGLE BISECTOR

The ray that lies within the interior of an angle with its endpoint on the vertex of that angle is the **angle bisector** if it creates two new angles of equal measure.



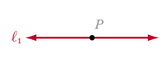
Ray $\overrightarrow{BD}$ bisects $\angle ABC$.

PARALLEL AND PERPENDICULAR LINES

PARALLEL LINES

If two lines are in the same plane, then they either intesect or are parallel. Parallel lines never **intersect.** If two lines ℓ_1 and ℓ_2 are **parallel,** we can write $\ell_1 \parallel \ell_2$.

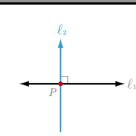
Parallel Postulate: Given a line and a point not on the line, there is exactly one line through that point parallel to the given line.

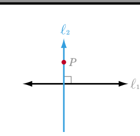


ℓ_1 and ℓ_2 are parallel. Line ℓ_1 is the unique line through point P parallel to ℓ_2.

PERPENDICULAR LINES

Two interesecting lines are **perpendicular** if their intersection forms a right angle (equivalently, four right angles). If ℓ_1 and ℓ_2 are perpendicular, we write $\ell_1 \perp \ell_2$.
- Given a line and any point (whether on the line or not), there is exactly one perpendicular to the line that passes through the point.





In both diagrams, ℓ_2 is the unique line perpendicular to ℓ_1 and passing through P.

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"IN SO FAR AS THE STATEMENTS OF GEOMETRY SPEAK ABOUT REALITY, THEY ARE NOT CERTAIN, AND IN SO FAR AS THEY ARE CERTAIN, THEY DO NOT SPEAK ABOUT REALITY."

ALBERT EINSTEIN

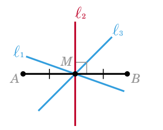
DISTANCE IN THE PLANE

A line segment has a specific length. If the length of $\overline{AB}$ is 5, we write $AB = 5$.

- Two segments that have the same length are **congruent**; we write $\overline{AB} \cong \overline{CD}$ or $AB = CD$.

BISECTING A SEGMENT

- **Midpoint of a segment:** The point on a segment that lies exactly halfway between the two endpoints.
- **Bisector of a segment:** Any line passing through the midpoint of a segment.
- **Perpendicular bisector:** A line bisecting and perpendicular to a segment. Every segment has a unique perpendicular bisector.



Segment $\overline{AB}$ with midpoint M and bisectors ℓ_1, ℓ_2, and ℓ_3. Line ℓ_2 is the perpendicular bisector. Here $AM = MB$.

- The points along the perpendicular bisector are equidistant from the endpoints of the bisected segment.

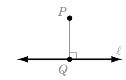
MEASURING DISTANCE

Between points:
The distance between two points is the length of the segment connecting them.

From a point to a line:
The distance from a point and a line is the shortest distance from the point to any point on the line.

- The shortest distance between a point and a line is always along a perpendicular.
- A point that is the same distance away from two points (from two lines, from a line and a point) is said to be **equidistant** from them.

Between two parallel lines:
The distance between two parallel lines is the shortest distance between any two points on those lines. Drop a perpendicular from any point on one line to the other line, and measure the distance.



The distance between line ℓ and point P is the length of the segment $\overline{PQ}$.

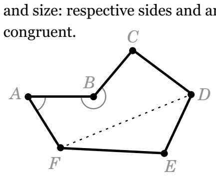
POLYGONS IN THE PLANE

A **polygon** is a plane figure composed of segments joined at endpoints. These endpoints are the **vertices** (sing. **vertex**) of a polygon. The segments are called **sides.** The number of sides and vertices is equal.

- A polygon is named by listing its vertices in order, often (but not always) starting with the lower left and proceeding clockwise.
- The **angles** of a polygon are understood to mean the "inside" angles.
- Two sides that share a vertex are called **adjacent**. Two endpoints of a single polygon side are consecutive vertices.

Diagonal of a polygon: A segment whose endpoints are non-consecutive vertices.

Congruent polygons: Two polygons are congruent if they are the same shape and size: respective sides and angles are congruent.



Polygon $ABCDEF$ with diagonal $\overline{DF}$ is a concave hexagon. Also, $m\angle A = 58°$; $m\angle B = 230°$.

Similar polygons: Two polygons are similar if they have the same shape; one is an enlarged version of the other. Corresponding angles are congruent; corresponding side lengths are proportional to each other. See *Triangle Congruence and Similarity.*

Perimeter: The perimeter of a polygon is the sum of the lengths of all of its sides.

Area: The area of a polygon is a measure of how much plane space it encloses, measured in terms of how many unit squares (basic unit of area) can fit inside.

Classifying polygons by number of sides

3	*triangle*	8	*octagon*
4	*quadrilateral*	9	*nonagon*
5	*pentagon*	10	*decagon*
6	*hexagon*	12	*dodecagon*
7	*septagon*	n	*n-gon*

Classifying polygons by shape

Convex polygon: A polygon whose angles are all less than $180°$. In common usage, "polygon" means "convex polygon."

Concave polygon: A polygon with at least one angle measuring more than $180°$. Polygon $ABCDEF$, left, is concave.

Equilateral polygon: A polygon all of whose sides are congruent.

- **Perimeter:**
  $P = (\text{number of sides}) \times (\text{side length})$

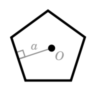
Equilateral pentagon

Equiangular polygon: A polygon all of whose angles are congruent. Necessarily convex.

Equiangular pentagon

Regular polygon: A polygon all of whose sides are congruent and all of whose angles are congruent. Necessarily convex.

- **Center of a regular polygon:** The point in the middle that is equidistant from all the vertices (also, from the midpoint of every side).
- **Apothem:** Segment connecting the center and the midpoint of a side in a regular polygon.
- **Perimeter:**
  $P = (\text{number of sides}) \times (\text{side length})$
- **Area:**
  $A = \frac{1}{2}(P) \times (\text{apothem length})$



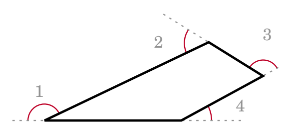
Regular pentagon with center O and apothem a

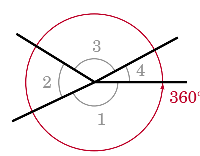
Number of Diagonals

A polygon with n sides has exactly $\frac{n(n-3)}{2}$ diagonals.

Angle Sums

- The sum of the n interior angles of any polygon is $(n-2)180°$.
- **Exterior angle:** Any angle less than $180°$ has a corresponding exterior angle (see diagram). In a convex polygon, the sum of all the exterior angles is $360°$.

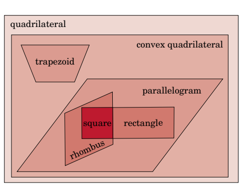




Quadrilateral with exterior angles 1, 2, 3, 4. The sum of the four exterior angle measures is $360°$.

QUADRILATERALS: 4 SIDES

Quadrilaterals have four sides, four vertices, and two diagonals. All quadrilaterals discussed below are convex.



Types of quadrilaterals

PARALLELOGRAM

A **parallelogram** is a quadrilateral with two pairs of parallel sides.

Properties

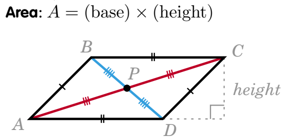
- Opposite sides are parallel and congruent.
- Opposite angles are congruent.
- Diagonals bisect each other.
- Sum of interior angles is $360°$. Any pair of unequal angles is supplementary.

Altitude: Perpendicular dropped between a pair of opposite sides, often from a vertex of the top side. Depends on point of view; in diagrams, we usually draw the altitude between the sides that are horizontal in the page.

- The length of the altitude is called the **height** of a parallelogram.
- The side to which the altitude is dropped is called the **base** of the parallelogram.

Perimeter:
$P = 2(\text{sum of lengths of two unequal sides})$

Area: $A = (\text{base}) \times (\text{height})$



Parallelogram $ABCD$

Diagonals: $\overline{AC}$ and $\overline{BD}$

Parallel sides: $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$

Congruent sides: $AB = CD$, $BC = AD$,

Bisected diagonals: $AP = PC$, $BP = PD$

Congruent angles: $m\angle A = m\angle C$ and $m\angle B = m\angle D$

Supplementary angles: $m\angle A + m\angle B = m\angle C + m\angle B = m\angle C + m\angle D = m\angle A + m\angle D = 180°$

Special types of Parallelograms

- **Rectangle:** All four angles are right angles. Equiangular quadrilateral.
- **Rhombus:** All four sides are congruent. Equilateral parallelogram.
- **Square:** Four congruent sides and four congruent angles. Regular quadrilateral. Both a square and a rhombus.

RECTANGLE

A **rectangle** is a quadrilateral with four right angles. It is necessarily a parallelogram.

Properties

- Opposite sides are parallel and congruent.
- Diagonals are congruent and bisect each other.
- All four angles are equal and measure $90°$.

Perimeter: $P = 2\,((\text{base}) + (\text{height}))$

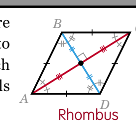
Area: $A = (\text{base}) \times (\text{height})$

RHOMBUS

A **rhombus** is a parallelogram with four congruent sides.

Properties

- Opposite sides are parallel; all sides are congruent.
- Opposite angles are congruent.
- Diagonals are perpendicular to and bisect each other; diagonals bisect angles.



Rhombus $ABCD$

Perimeter:
$P = 4(\text{side length})$

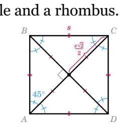
Area:
$A = \frac{1}{2}(\text{long diagonal}) \times (\text{short diagonal})$

SQUARE

A **square** has four right angles and four equal sides. It is both a rectangle and a rhombus.

Properties

- Opposite sides are parallel; all sides are congruent.
- All angles are equal and measure $90°$.
- **Diagonals** are perpendicular and congruent; they bisect each other and divide the square into four $45°$–$45°$–$90°$ right triangles. The length of a diagonal is $\sqrt{2}(\text{side length})$.



Square $ABCD$

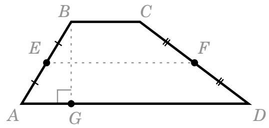
Perimeter: $P = 4(\text{side length})$

Area: $A = (\text{side length})^2$

QUADRILATERALS (CONTINUED)

TRAPEZOID

A **trapezoid** is a quadrilateral with exactly one pair of parallel sides. A parallelogram is not a trapezoid.



Trapezoid $ABCD$ with bases $\overline{AD} \parallel \overline{BC}$, legs $\overline{AB}$ and $\overline{CD}$, median $\overline{EF}$, and altitude $\overline{BG}$.
Also, $EF = \frac{1}{2}(AD + BC)$

Anatomy of a trapezoid

Base: Either of two parallel sides.

Leg: Either of the other two (non-parallel) sides.

Median: Segment joining the midpoints of the two legs. Parallel to the bases; its length is the average of the lengths of the two bases.

Altitude: Segment perpendicular to the bases; joins a point on one base to the line that contains the other base.

Perimeter: $P =$ sum of four side lengths.

Area: $A = \frac{1}{2}(\text{base}_1 + \text{base}_2) \times (\text{altitude}) = (\text{median}) \times (\text{altitude})$

ISOSCELES TRAPEZOID

An **isosceles trapezoid** is a trapezoid with congruent legs (one pair of parallel sides and one pair of opposite congruent sides).

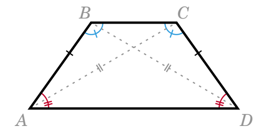
Properties:

- Legs are congruent.
- Median is parallel to the bases; its length is the average of the two base lengths.
- Diagonals are congruent.
- Lower base angles (two angles adjacent to the lower base) are congruent.
- Upper base angles (two angles adjacent to the upper base) are congruent.
- The two angles adjacent to a leg are supplementary.

Perimeter:

$$P = (\text{base}_1) + (\text{base}_2) + 2(\text{leg})$$

Area: $A = \frac{1}{2}(\text{base}_1 + \text{base}_2) \times (\text{altitude})$



Isosceles trapezoid $ABCD$

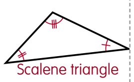
TRIANGLES: 3 SIDES

A **triangle** has three sides, three vertices, and three angles.

TYPES OF TRIANGLES

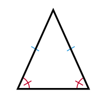
Classification by side length

Scalene triangle: Triangle with three different side lengths (and three different angle measures).



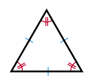
Scalene triangle

Isosceles triangle: Triangle with (at least) two congruent sides and two congruent angles (opposite the congruent sides).



Isosceles triangle

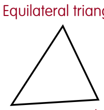
Equilateral triangle: Triangle with three congruent sides and three congruent angles, each measuring $60°$.



Equilateral triangle

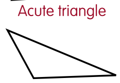
Classification by angle

Acute triangle: Triangle with three acute angles.



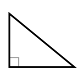
Acute triangle

Obtuse triangle: Triangle with one obtuse angle.



Obtuse triangle

Right triangle: Triangle with one right $(90°)$ angle. The other angles are complementary.



Right triangle

TRIANGLE DEFINITIONS

Base: One side of a triangle; which side it is depends on perspective. Usually, the base is the side that is oriented horizontally.

Altitude: Perpendicular line segment from one vertex to the line that contains the opposite side. Altitudes are often drawn to the base. The length of the altitude is often called the **height**.

Median: Line drawn from vertex to midpoint of opposite side.

Area: $A = \frac{1}{2}(\text{base}) \times (\text{height})$.

Heron's Area Formula: The area of a triangle with sides a, b, c, is given by

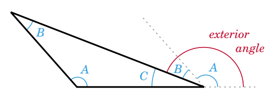
$$A = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$$

Median Area Fact: A median divides the triangle into two triangles of equal area.

Triangle Inequality: The length of a side of a triangle is always less than the sum of the lengths of the other two sides.

Largest angle opposite longest side: In a triangle, two angles are equal if and only if the sides opposite them are equal. If angles are unequal, the longer side is opposite the larger angle. If sides are unequal, the larger angle is opposite the longer side.

Exterior angle equality: For any vertex, the measure of the exterior angle is equal to the sum of the measures of the other two ("remote interior") angles.



$m\,(\text{exterior angle to } \angle C) = m\angle A + m\angle B$

Exterior angle inequality: For any vertex, the measure of the exterior angle is always greater than the measure of either remote interior angle.

ISOSCELES TRIANGLE FACTS

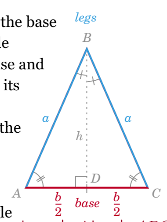
The two equal sides of an isosceles triangle are called the **legs**; the third side is the **base**.

Angle measure: If $\angle A$ and $\angle C$ are the base angles, and $\angle B$ is the vertex angle (as in the diagram below), then

$$m\angle A = m\angle C = \tfrac{1}{2}(180° - m\,\angle B)$$
$$m\angle B = 180° - 2m\,\angle A$$
$$m\angle B = 180° - 2m\,\angle C$$

Properties

- The altitude to the base bisects the angle opposite the base and hits the base at its midpoint. The altitude is also the median.
- The altitude splits the isosceles triangle into two congruent right triangles.



Isosceles triangle ABC with altitude $\overline{BD}$. D is the midpoint of $\overline{AC}$

See *Congruence and Similarity,* below.

Side lengths: If b is the length of the base, a is the length of the legs, and h is the altitude, then

$$a = \sqrt{\frac{b^2}{4} + h^2}, \qquad b = 2\sqrt{a^2 - h^2},$$
$$\text{and } h = \sqrt{a^2 - \frac{b^2}{4}}.$$

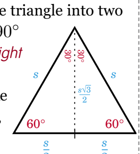
EQUILATERAL TRIANGLE FACTS

An **equilateral** triangle is also isosceles; everything that is true about isosceles triangles is also true about equilateral triangles. Also:

- Every equilateral triangle is congruent—has the same shape.
- The altitude splits the triangle into two congruent 30°-60°-90° right triangles. See *Right Triangle Facts,* below.

Height: If s is the side length of the triangle, then the altitude has length $\frac{s\sqrt{3}}{2}$.

Area: $A = \frac{s^2\sqrt{3}}{4}$



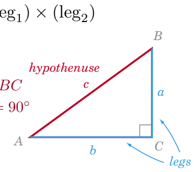
Equilateral triangle

RIGHT TRIANGLE FACTS

The side opposite the right angle (the longest side) is called the **hypotenuse**; the other two sides are the **legs**.

Angle measure: The two angles adjacent to the hypotenuse are complementary. In right triangle ABC whose right angle is at C, $m\angle A = 90° - m\angle B$.

Area: $A = \frac{1}{2}(\text{leg}_1) \times (\text{leg}_2)$



Right triangle ABC
$m\angle A + m\angle B = 90°$

Pythagorean Theorem: The length of the hypotenuse squared is equal to the sum of the squares of the lengths of the legs:

$$a^2 + b^2 = c^2$$

Common integer right triangle side lengths:

- 3–4–5 (and its multiples: **Ex:** 6–8–10)
- 5–12–13 (and multiples)
- 7–24–25 (and multiples)

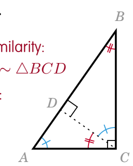
Right triangle similarity

Median to hypotenuse: The median to the hypotenuse of a right triangle is half the length of the hypotenuse.

Altitude to the hypotenuse: The altitude to the hypotenuse of a right triangle splits the triangle up into two smaller right triangles, congruent to each other and to the original triangle.

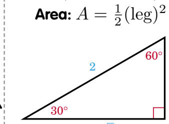
Right triangle similarity:
$\triangle BAC \sim \triangle CAD \sim \triangle BCD$

Length proportions give:
$(CD)^2 = (AD)(BD)$
$(BC)^2 = (AB)(BD)$
$(AC)^2 = (AB)(AD)$

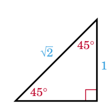


Special right triangles

1. $30°$–$60°$–$90°$: Half of an equilateral triangle.
   Side lengths:
   $(\text{hypotenuse}) = 2(\text{shorter leg})$
   $(\text{longer leg}) = \sqrt{3}(\text{shorter leg})$
   Area: $A = \frac{\sqrt{3}}{2}(\text{shorter side})^2$
2. $45°$–$45°$–$90°$: Half of a square; isosceles right triangle.
   Side lengths: The legs are equal.
   $(\text{hypotenuse}) = \sqrt{2}(\text{leg})$
   Area: $A = \frac{1}{2}(\text{leg})^2$



$30°$–$60°$–$90°$ triangle



$45°$–$45°$–$90°$ triangle

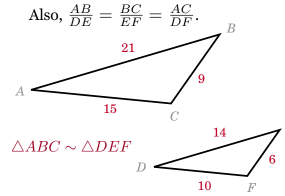
CONGRUENCE AND SIMILARITY

Triangles are **congruent** when they are the same shape and size; that is, when corresponding angles and sides are all congruent.

- **Notation:** Write $\triangle ABC \cong \triangle DEF$ with corresponding vertices in order: So $AB = DE$, $BC = EF$, and $AC = DF$. Also, $m\angle A = m\angle D$, $m\angle B = m\angle E$, and $m\angle C = m\angle F$.

Triangles are **similar** when they are the same shape; corresponding angles are congruent; corresponding sides are proportional.

- **Notation:** Write $\triangle ABC \sim \triangle DEF$ to mean that $m\angle A = m\angle D$, $m\angle B = m\angle E$, and $m\angle C = m\angle F$.
  Also, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.



$\triangle ABC \sim \triangle DEF$

TESTING TRIANGLE CONGRUENCE

A triangle has six attributes: three angles and three sides. We need to prove equivalence of three corresponding pairs of attributes (one of which must be a side) to establish congruence.

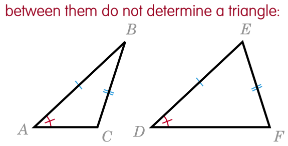
SSS: If all three sides of two triangles are congruent, the triangles are congruent. Three sides determine a triangle.

SAS: If two sides and the angle between them are congruent, the two triangles are congruent. Two sides and an included angle determine a triangle.

ASA: If two angles and the side between them are congruent, the triangles are congruent. Two angles and the side between them determine a triangle.

AAS: Find the third angle (sum of angles in triangle is $180°$) and use ASA. Two angles and any side determine a triangle.

WARNING: **ASS** (a.k.a. **SSA**) is not a congruence test! Two sides and an angle not between them do not determine a triangle:



Triangles ABC and DEF have a pair of congruent angles: $m\angle A = m\angle D$, and two pairs of congruent sides: $AB = DE$ and $BC = EF$; but they are not congruent.

CONTINUED ON OTHER SIDE

TRIANGLES (CONTINUED)

TESTING TRIANGLE SIMILARITY

Use these tests to determine whether two triangles are similar:

AA: If two angles of one triangle are congruent to two angles of another triangle, the two triangles are similar.

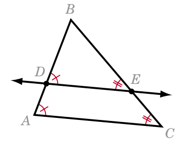
SSS: If all three sides of one triangle are proportional to the sides of another triangle, the two triangles are similar.

SAS: If two sides of one triangle are proportional to two sides of another triangle, and the angles between these sides are congruent, the triangles are similar.

ANOTHER WARNING: Again, **ASS** (or **SSA**) is not a similarity test! The test angle must be included between the proportional sides.

CUTTING SIMILAR TRIANGLES

A line parallel to one of the sides of a triangle sections off a similar triangle.



$\overleftrightarrow{DE} \parallel \overline{AC}$, so $\triangle ABC \sim \triangle DBE$

MEASUREMENTS OF SIMILARITY

Suppose that $\triangle ABC \sim \triangle DEF$ and that c is the proportionality constant (i.e., $c = \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$).

Perimeter: (perimeter of $\triangle ABC$) $= c$(perimeter of $\triangle DEF$)

Area: (area of $\triangle ABC$) $= c^2$(area of $\triangle DEF$)

Altitudes and medians in similar triangles are proportional to the sides.

CIRCLES

A **circle** is the set of all points equidistant from a given point called the **center**. The distance from any point to the center is the **radius** (pl. **radii**) of the circle. Circles are usually named by their center.

Diameter: Segment containing the center of a circle both of whose endpoints are on the circle. Any diameter has length twice the radius.

The words "radius" and "diameter" can refer either to a specific segments or to its length.

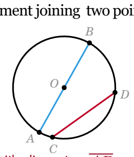
Circumference: The perimeter of a circle. The circumference of a circle with radius r is $2\pi r$, where π is a special real number (approximately 3.14159).

Area: The area of a circle with radius r is πr^2.

— Two circles are **concentric** if they lie in the same plane and have the same center.

CHORDS, SECANTS, TANGENTS

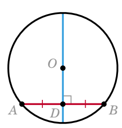
Chord: Segment joining two points on a circle.



Circle O with diameter $\overline{AB}$ and chord $\overline{CD}$

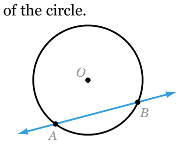
— The diameter perpendicular to a chord bisects that chord.

— In fact, any diameter bisects a chord if and only if it is perpendicular to the chord.



Circle O with chord $\overline{AB}$ and the diameter that bisects and is perpendicular to the chord.

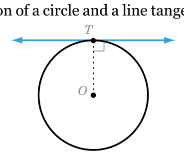
Secant: Line that interesects the circle at two points; a secant line contains a chord of the circle.



Secant line $\overleftrightarrow{AB}$ containing chord $\overline{AB}$

Tangent to a circle: Line that intersects a circle at exactly one point.

— **Point of tangency:** The point of intersection of a circle and a line tangent to it.



Line is tangent to circle O at T. Radius $\overline{OT}$ is perpendicular to the line.

— A tangent line is always perpendicular to the radius drawn to the point of tangency.

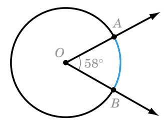
ARCS AND CENTRAL ANGLES

Arc: A continuous section of a circle.

— Any two points on a circle define *two* arcs. Any chord defines two arcs. The smaller arc with endpoints A and B is denoted as $\widehat{AB}$.

— Arcs are measured in degrees (°) by the central angle that they **subtend.** A circle is an arc of measure $360°$.

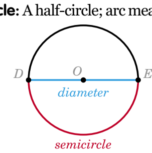
Central angle: An angle whose vertex is the center of a circle. Any central angle intercepts the circle at two points, defining an arc.



Central angle $\angle AOB$ defines arc $\widehat{AB}$.

Types of arcs:

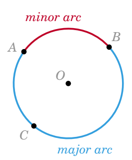
Semicircle: A half-circle; arc measuring $180°$.



Diameter $\overline{DE}$ cuts circle O into two semicircles.

Major arc: Larger than a semicircle; an arc whose measure is greater than $180°$.

Minor arc: Smaller than a semicircle; an arc whose measure is less than $180°$.



Minor arc $\widehat{AB}$; major arc $\widehat{ACB}$

Arc length: In a circle of radius r, an arc measuring $\theta°$ has length $\frac{\theta}{180}\pi r$.

SEGMENTS AND SECTORS

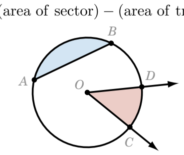
Sector: Region inside a circle bounded by a central angle and the arc it defines.

— **Area of sector:** A sector whose arc has measure $\theta°$ has area $\frac{\theta}{360}\pi r^2$.

Segment: Region inside a circle bounded by a chord and the arc it defines.

— **Area of segment:**

$A =$ (area of sector) $-$ (area of triangle)

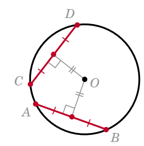


Segment defined by chord $\overline{AB}$; Sector defined by central $\angle COD$

CHORD AND ARC THEOREMS

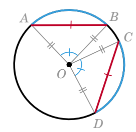
1. The triangle defined by a chord and the central angle of the arc it cuts is isosceles.

2. Congruent chords are equidistant from the center of a circle.



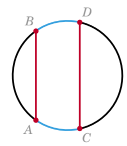
Chords $\overline{AB}$ and $\overline{CD}$ are congruent and equidistant from center O.

3. Congruent chords in the same circle cut congruent arcs.



$AB = CD$, so $m\,\widehat{AB} = m\,\widehat{CD}$.

4. Parallel chords in the same circle create congruent arcs between them.



Chords $\overline{AB}$ and $\overline{CD}$ are parallel, so $m\,\widehat{AC} = m\,\widehat{BD}$.

ANGLE AND ARC MEASURE

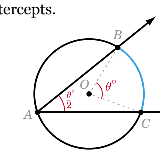
These theorems are about measures of angles formed by secant or tangent lines to circles.

MORAL: If the angle is inside the circle, add the measures of intercepted arcs; if outside, subtract.

VERTEX ON CIRCLE

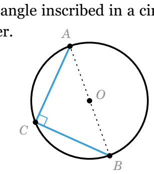
2 CHORDS

Inscribed Angle: The measure of an inscribed angle is half the measure of the arc it intercepts.



$m\angle BAC = \frac{1}{2}m\,\widehat{BC} = \frac{1}{2}m\angle BOC$

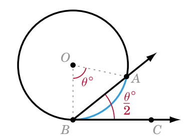
A right angle inscribed in a circle cuts off a diameter.



$m\angle ACB = 90°$ and $\overline{AB}$ is a diameter of circle O.

CHORD AND TANGENT

The measure of an angle formed by a chord and a tangent is half the measure of the arc it intercepts.

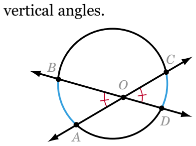


$m\angle ABC = \frac{1}{2}m\,\widehat{BA} = \frac{1}{2}m\angle AOB$

VERTEX INSIDE CIRCLE

2 CHORDS

The measure of an angle formed by two secants intersecting inside the circle is half the sum of measures of the arcs intercepted by the vertical angles.



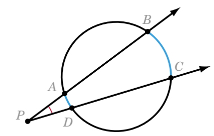
$m\angle AOB = \frac{1}{2}\left(m\,\widehat{AB} + m\,\widehat{CD}\right)$;

$m\angle BOC = \frac{1}{2}\left(m\,\widehat{BC} + m\,\widehat{AD}\right)$

VERTEX OUTSIDE CIRCLE

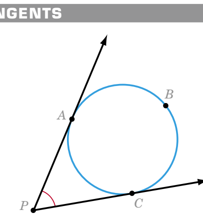
The measure of an angle formed by secant and/or tangent lines such that its vertex is outside the circle is half the difference of the measures of the arcs intercepted by the secant and/or tangent lines.

2 SECANTS



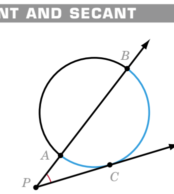
$m\angle P = \frac{1}{2}\left(m\,\widehat{BC} - m\,\widehat{AD}\right)$

2 TANGENTS



$m\angle P = \frac{1}{2}\left(m\,\widehat{ABC} - m\,\widehat{AC}\right)$

TANGENT AND SECANT



$m\angle P = \frac{1}{2}\left(m\,\widehat{BC} - m\,\widehat{AC}\right)$

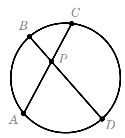
CIRCLES (CONTINUED)

POWER OF A POINT THEOREMS

These theorems are about segments formed by intersections of chords, secants, or tangents with a circle and with one another.

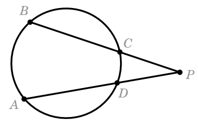
MORAL: Relative to a circle, every point P has a constant "power"—the product of the distances between P and the two intersection points with the circle along any line.

Intersection of two chords: The product of the lengths of the segments of one chord is equal to the product of the lengths of the segment of the other chord.



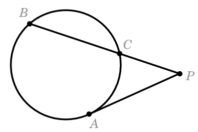
$(AP)(PC) = (BP)(PD)$

Intersection of two secants: When two secant segments share an endpoint outside the circle, the products of the lengths of each secant segment with its external part are equal.



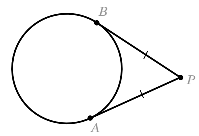
$(AP)(DP) = (BP)(CP)$

Interesection of secant and tangent: When a secant segment and a tangent segment share an endpoint not on the circle, the length of the tangent segment squared is equal to the product of the secant segment and its external part.



$(AP)^2 = (BP)(CP)$

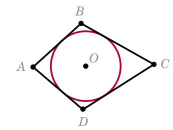
Intersection of two tangents: Tangent segments that share an endpoint outside the circle are congruent.



$AP = BP$

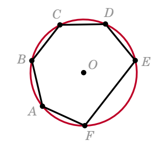
CIRCLES AND POLYGONS

A circle is said to be "**inscribed** in a polygon" if all of the sides of the polygon are tangent to the circle. (We can also say that the polygon is "circumscribed about the circle.")



Circle O is inscribed in quadrilateral $ABCD$

A circle is said to be "**circumscribed** about a polygon" if all of the vertices of the polygon are on the circle. (We can also say that the polygon is "inscribed in the circle.")



Circle O is circumscribed about hexagon $ABCDEF$

LINES AND PLANES IN SPACE

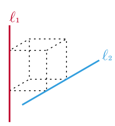
TWO LINES

Lines either do or do not lie in the same plane.

Two lines that lie in the same plane either **intersect** or are **parallel**. Contrariwise...

- If two lines in space intersect, then they lie in the same plane.
- If two lines in space are parallel, then they lie in the same plane.

Two lines that do not lie in the same plane (equivalently, neither intersect nor are parallel) are called **skew** lines.

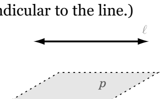


Skew lines ℓ_1 and ℓ_2

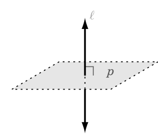
LINE AND PLANE

A line and a plane in space always either **intersect** or are **parallel**.

- If they are parallel, they never intersect.
- A line is perpendicular to a plane if and only if it is perpendicular to every line in the plane that goes through their point of intersection.
- For every point in the plane, there is a unique line perpendicular to the plane that goes through that point. (Similarly, a line and a point on the line determine a unique plane containing the point and perpendicular to the line.)



Line ℓ is parallel to plane p.

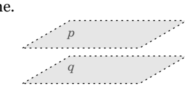


Line ℓ is perpendicular to plane p.

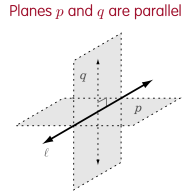
TWO PLANES

Two planes either **intersect** or are **parallel**.

- Two planes are parallel if and only if they never intersect. Any line perpendicular to one is perpendicular to both; the distance between the two planes is the distance between the points of intersection along a perpendicular line.
- Two non-parallel planes always intersect in a line.
- Two intersecting planes are said to be **perpendicular** if one of the planes contains a line perpendicular to the other plane.



Planes p and q are parallel.



Planes p and q are perpendicular. Line ℓ is their intersection.

SOLIDS IN SPACE

SOLIDS IN SPACE

A **polyhedron** (pl. **polyhedra**) is a solid region formed by the intersection of several (at least four) **planes**. The planes intersect in polygonal **faces** whose sides are called **edges** and whose vertices are the **vertices** of the polyhedron.

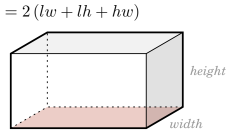
Surface Area: Total area of all of the faces of the polyhedron.

Volume: A measure of how much space fits inside a solid figure, calculated in cubic units.

SIMPLE 6-FACED SOLIDS: CUBE, RECTANGULAR SOLID, PARALLELEPIPED

Rectangular Solid: A polyhedron with six rectangular faces. Adjacent faces intersect at right angles. Has three measurements: length, width, height.

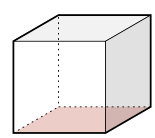
- **Volume:** $V = (\text{length}) \times (\text{width}) \times (\text{height})$
- **Surface Area:** $SA = 2\,(lw + lh + hw)$



Rectangular Solid

Cube: A rectangular solid with six congruent square faces. Has twelve congruent edges and eight vertices.

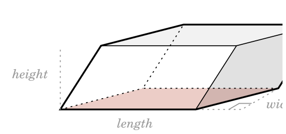
- **Volume:** $V = (\text{side length})^3$
- **Surface Area:** $SA = 6(\text{side length})^2$



Cube

Parallelepiped: A polyhedron whose six faces are parallelograms lying in pairs of parallel planes. Rectangular solids are parallelepipeds whose adjacent faces lie in perpendicular planes.

- **Volume:** $V = (\text{length}) \times (\text{width}) \times (\text{height})$

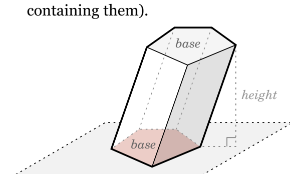


Parallelepiped

PRISM

A **prism** is a polyhedron two of whose faces (the **bases**) are congruent polygons lying in parallel planes; the other faces (the **lateral faces**) are parallelograms that join corresponding sides on the congruent polygons. The sides that join the lateral parallelograms to each other are called **lateral edges**.

- A prism is identified by the shape of its bases.
- **Height:** The (perpendicular) distance between the bases (or rather, the planes containing them).

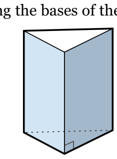


Pentagonal prism

- **Volume:** $V = (\text{Base area}) \times (\text{height})$
- **Lateral area:** The area of the lateral faces of a prism.

$(\text{Lateral Area}) + 2(\text{Base Area}) = \text{Surface Area}$

Right prism: A prism whose lateral edges are perpendicular to the planes containing the bases of the prism.



Right triangular prism.
Two of three lateral faces are shaded blue.

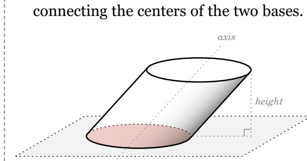
- **Lateral area (for right prism):** $LA = (\text{perimeter of Base}) \times (\text{height})$

Special types of prisms: Parallelepipeds are prisms with a parallelogram base. Rectangular solids and cubes are right prisms with a rectangular (or square) base.

CYLINDER

A **cylinder** is analogous to a prism, except its bases are circular. A cylinder is not a polyhedron (just like a circle is not a polygon). The lateral area, the height—and all of the computational formulas—are defined analogously.

- The **axis** of a cylinder is the line connecting the centers of the two bases.

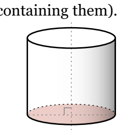


Cylinder

NOTE: The word *cylinder* actually refers to any solid shape with two congruent bases in parallel planes (including prisms and solids with, say, clover-leaf-shaped bases). The cylinders we mean here are called **circular cylinders.**

- **Volume:** $V = (\text{Base area}) \times (\text{height}) = \pi(\text{radius})^2 \times (\text{height})$

Right cylinder: A cylinder whose axis is perpendicular to its bases (rather, the planes containing them).



Right cylinder

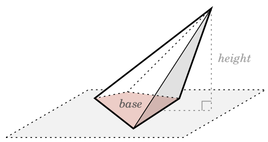
SOLIDS IN SPACE (CONTINUED)

— **Lateral area:**
$LA = (\text{perimeter of Base}) \times (\text{height})$
$= 2\pi(\text{radius}) \times (\text{height})$
— **Surface area:**
$SA = LA + 2(\text{Base area})$
$= 2\pi(\text{radius})^2 + 2\pi(\text{radius}) \times (\text{height})$

PYRAMID

A **pyramid** is the set of all points along segments that join a polygonal **base** with a **vertex** not in the plane of the base.

— If the base is a polygon with n edges, the pyramid has $n+1$ faces: one base and n triangular lateral faces.
— **Height:** The distance from the vertex to (the plane that contains) the base.
— **Volume:** $V = \frac{1}{3}(\text{Base area}) \times (\text{height})$

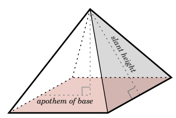


Pyramid with trapezoidal base

Regular pyramid: A pyramid with two properties:

1. The base is a regular polygon.
2. The line joining the vertex and the center of the base is perpendicular to (the plane of) the base. All lateral faces are congruent isosceles triangles.

— **Slant height:** The length of the altitude of one of the lateral faces. The Pythagorean Theorem tells us that
$(\text{apothem of base})^2 + (\text{height})^2 = (\text{slant height})^2$
— **Lateral area:**
$LA = \frac{1}{2}(\text{Base perimeter}) \times (\text{slant height})$
— **Surface area:** $SA = LA + (\text{Base area})$



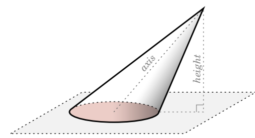
Square pyramid (regular pyramid)

Tetrahedron: A regular triangular pyramid; a tetrahedron has four triangular faces, four vertices, and six edges.

CONE

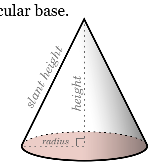
A **cone** is analogous to a pyramid, except the base is a circle.

— **Volume:** $V = \frac{1}{3}(\text{Base area}) \times (\text{height})$
$= \frac{1}{3}\pi(\text{radius})^2 \times (\text{height})$



Cone

Right cone: A cone whose axis is perpendicular to the plane containing the circular base.



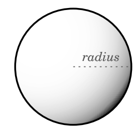
Right cone

— **Slant height:** The length of the shortest segment from the vertex to a point on the perimeter of the circular base. The segment is perpendicular to the tangent to the circle at the point where the segment hits the circular base. Pythagorean Theorem states that $(\text{radius})^2 + (\text{height})^2 = (\text{slant height})^2$
— **Lateral area:**
$LA = \frac{1}{2}(\text{base perimeter}) \times (\text{slant height})$
$= \pi(\text{radius}) \times (\text{slant height})$

SPHERE

A **sphere** is the three-dimensional equivalent of a circle: the set of all points in space equidistant from a fixed point called the **center**. The common distance is called the **radius**.

— **Hemisphere:** Half of a sphere.
— **Volume:** $V = \frac{4}{3}\pi(\text{radius})^3$
— **Surface area:** $SA = 4\pi(\text{radius})^2$



Sphere

FORMULAS

PLANE SHAPES: AREA AND PERIMETER

- The perimeter (P) of a plane region is the total length around its sides. It is measured in units of length: cm, m, km, in, ft. The perimeter of a circle is called the circumference (C).
- The area (A) of a plane region is the number of unit squares that can be fit inside it. It is measured in units of length^2: cm^2, m^2, km^2, in^2, ft^2.

SHAPE	PERIMETER	AREA
Square	$P = 4s$	$A = s^2$
Rectangle	$P = 2(b+h)$	$A = bh$
Parallelogram	$P = 2(b+s)$	$A = bh$
Rhombus	$P = 4s$	$A = \frac{1}{2}d_1d_2$
Trapezoid	$P = b_1 + b_2 + s_1 + s_2$	$A = \frac{1}{2}(b_1+b_2)h$
Triangle	$P = a + b + c$	$A = \frac{1}{2}bh$ $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$ is the semiperimeter.
Equilateral Triangle	$P = 3s$	$A = \frac{s^2\sqrt{3}}{4}$
Circle	$C = 2\pi r$	$A = \pi r^2$
Sector/Arc	$L = \frac{\theta}{180}\pi r$	$A = \frac{\theta}{360}\pi r^2$
Regular n-gon	$P = ns$	$A = \frac{1}{2}aP$

SOLID FIGURES: VOLUME AND SURFACE AREA

- **Surface area (SA):** Total area of all the surfaces of a solid. Measured in units of length^2: cm^2, m^2, km^2, in^2, ft^2.
- **Volume (V):** The number of cubic units that can be fit inside a solid figure. Measured in units of length^3: cm^3, m^3, km^3, in^3, ft^3.

For prisms, cylinders, pyramids, and cones only:

- **Lateral Area (LA):** Surface area exluding bases. Area around the sides, if the figure is upright.
- B: Base area.
- P: Perimeter of base.
- a: Apothem of regular polygonal base.
- s: Slant height.

SHAPE	SURFACE AREA	VOLUME
Cube	$SA = 6s^2$ $\text{Diagonal} = s\sqrt{3}$	$V = s^3$
Rectangular Solid	$SA = 2(lw + lh + hw)$ $\text{Diagonal} = \sqrt{l^2 + w^2 + h^2}$	$V = lwh$
Sphere	$SA = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$

SHAPE	LATERAL AREA	SURFACE AREA	VOLUME
Prism			$V = Bh$
Right Prism	$LA = Ph$	$SA = Ph + 2B$	$V = Bh$
Cylinder			$V = Bh$
Right Cylinder	$LA = Ph = 2\pi rh$	$SA = Ph + 2B$ $= 2\pi rh + 2\pi r^2$	$V = Bh$
Pyramid			$V = \frac{1}{3}Bh$
Regular Pyramid	$LA = \frac{1}{2}Ps$ $s^2 = a^2 + h^2$	$SA = \frac{1}{2}Ps + B$	$V = \frac{1}{3}Bh$
Cone			$V = \frac{1}{3}Bh$
Right Cone	$LA = \frac{1}{2}Ps = \pi rs$	$SA = \frac{1}{2}Ps + B$ $= \pi rs + \pi r^2$	$V = \frac{1}{3}Bh$ $= \frac{1}{3}\pi r^2 h$

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