# TD2: WiFi Performance Analysis

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## 1 System Model

### Question 1

In our model, the $n$ stations are symmetric. They have the same back-off parameters and use the same algorithm. We conclude that the expected value of the time spent in back-off is the same for all stations.

## 2 Back-off Analysis

### Question 2

**Computing $\mathbb{E}[R]$**

To compute the expected value, we shall use the two series below

$$\begin{cases} \sum_{k=0}^{K} x^k & = \frac{1-x^{K+1}}{1-x} \\ \sum_{k=0}^{K} kx^k & = \frac{x(1-(K+1)x^K+Kx^{K+1})}{(1-x)^2} \end{cases} \tag{1}$$

$$\begin{aligned}
\mathbb{E}[R] &= \left(\sum_{k=0}^{K} k\mathbb{P}[R=k]\right) + (K+1)\mathbb{P}[R=K+1] && (2)\\
&= \left(\sum_{k=0}^{K} k(1-\gamma)\gamma^{k-1}\right) + (K+1)\gamma^K && (3)\\
&= (1-\gamma)\left(\sum_{k=0}^{K-1} k\gamma^k + \sum_{k=0}^{K-1}\gamma^k\right) + (K+1)\gamma^K \quad \text{now we use the two series sums to simplify} && (4)\\
&= (1-\gamma)\left(\frac{\gamma(1-K\gamma^{K-1}+(K-1)\gamma^K)}{(1-\gamma)^2} + \frac{1-\gamma^K}{1-\gamma}\right) + (K+1)\gamma^K \quad \text{now we expand the terms to simplify} && (5)\\
&= \frac{1-\gamma^{K+1}}{1-\gamma} \quad \text{and here we identify a series} && (6)\\
&= \sum_{k=0}^{K}\gamma^k && (7)
\end{aligned}$$

**Computing $\mathbb{E}[X]$**

Here we will use the sum

$$\sum_{i=k}^{K}\gamma^i = \frac{\gamma^i - \gamma^{K+1}}{1-\gamma} \approx \frac{\gamma^i}{1-\gamma} \tag{8}$$

Let $t_i = \sum_{k=0}^{i} b_k \quad \forall i \in \{0,..,K\}$ denote the possible values of X. We have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{i=0}^{K} t_i\mathbb{P}[X=t_i] && (9)\\
&= \sum_{i=0}^{K}\left(\sum_{k=0}^{i} b_k\right)\gamma^i(1-\gamma) \quad \text{now we can inverse the sums} && (10)\\
&= \sum_{k=0}^{K} b_k(1-\gamma)\left(\sum_{i=k}^{K}\gamma^i\right) \quad \text{we use the series sum to simplify} && (11)\\
&= \sum_{k=0}^{K} b_k\gamma^k && (12)
\end{aligned}$$

### Question 3

Let $\phi = \{t_n, n \geq 1\}$ denote the timestamps of successful transmissions. If the reward is equal to the number of attempts until a success, we observe that $R$ and $X$ are defined exactly like in Question 2.

The ratio $\frac{R(t)}{t}$ is the attempt rate, which we know is $\beta$. We use the *Renewal reward* theorem to conclude

$$\beta = \frac{\mathbb{E}[R]}{\mathbb{E}[X]} \implies \beta = \frac{\sum_{k=0}^{K} \gamma^k}{\sum_{k=0}^{K} k\gamma^k} = G(\gamma) \tag{13}$$

### Question 4

So we start with $b_0 = CW_{\min}$ and double the back-off duration until $b_m$ where it becomes constant.

$$b_k = \begin{cases} 2^k \cdot CW_{\min} & \forall 0 \leq k \leq m \\ 2^m \cdot CW_{\min} & \forall k > m \end{cases} \tag{14}$$

## 3 Fixed Point

### Question 5

There is a collision $\iff$ one or more of the other $n-1$ stations attempts a transmission.

$$\begin{aligned} \Gamma(\beta) &= \sum_{k=1}^{n-1} \mathbb{P}[k \text{ attempts}] && (15) \\ &= 1 - \mathbb{P}[0 \text{ attempts}] && (16) \\ &= 1 - (1-\beta)^{n-1} && (17) \end{aligned}$$

### Question 6

We use the fact that the interval $[0,1] \in \mathbb{R}$ is convex and compact in $\mathbb{R}$.

We know that $\Gamma : [0,1] \to [0,1]$ and $G : [0,1] \to [0,1]$ are both continuous real functions $\implies \Gamma \circ G : [0,1] \to [0,1]$ is continuous. Thus, by *Brouwer's theorem*, $\Gamma \circ G$ has a fixed point.

### Question 7

Suppose that $\{b_k\}_{0 \leq k \leq K}$ is an increasing sequence.

We observe that $\frac{d\Gamma(\beta)}{d\beta} = (n-1)(1-\beta)^{n-2} \geq 0 \quad \forall \beta \in [0,1]$. So $\Gamma$ is an increasing function.

Given that $G$ is decreasing, $\Gamma \circ G$ is decreasing.

The identity function is increasing so $\gamma = \Gamma \circ G(\gamma)$ can have at most one fixed point.

In Question 6, we proved there is at least one fixed point $\implies$ there must exist exactly one fixed point.

## 4 Throughput Calculation

### Question 8

We observe that $1-(1-\beta)^n$ = probability no one attempts to transmit in a slot.

Thus, $\frac{1}{1-(1-\beta)^n}$ = expected number of slots until an attempt is made.

The only scenario where the reward $R_j > 0$ is when only one of the station transmits during these slots.

In this case, a packet of length $L$ is transmitted in the network. We take the average reward over this number of slots

$$\frac{L\mathbb{P}[R_j = L]}{1-(1-\beta)^n} = \frac{n\beta(1-\beta)^{n-1}L}{1-(1-\beta)^n} \tag{18}$$

### Question 9

Using Question 8, we know that $\frac{1}{1-(1-\beta)^n}$ = expected number of slots until an attempt is made.

Since each slot has duration $\sigma$, the average duration of this period is

$$\frac{\sigma}{1-(1-\beta)^n}$$

## Question 10

First we have the duration until at least one station make an attempt and then the time spent either in a successful transmission or a collision (two disjoint possibilities).

- we spend time $T_s$ in successful transmission with probability $(1-\beta)^{n-1}$
- we spend time $T_c$ in collision with probability $1-(1-\beta)^{n-1}$

We conclude that average duration of activity period is

$$\frac{\sigma}{1-(1-\beta)^n} + T_s(1-\beta)^{n-1} + T_c\left(1-(1-\beta)^{n-1}\right) \tag{19}$$

## Question 11

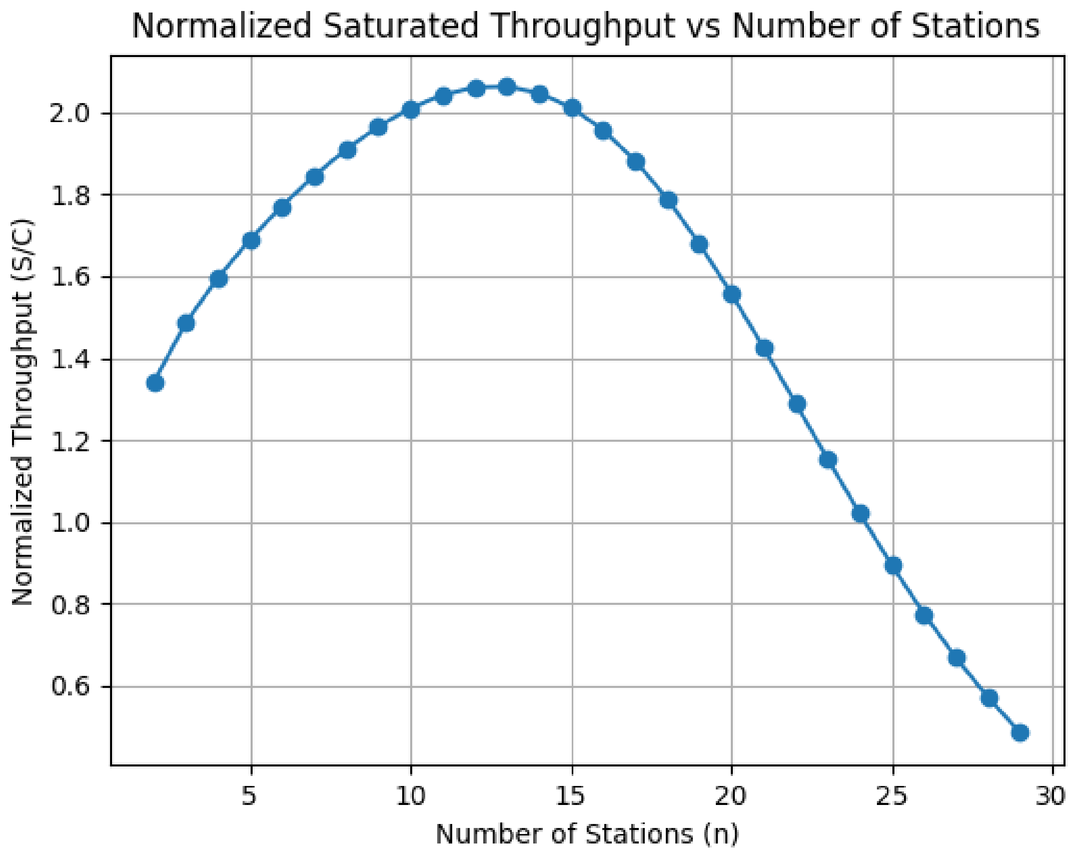
We calculate the throughput $S = \frac{R(t)}{t}$ using the renewal reward theorem.

$$S = \frac{\mathbb{E}[\mathbb{R}]}{\mathbb{E}[X]} \quad \text{we use the results of questions 8 and 10} \tag{20}$$

$$= \frac{\left(\frac{n\beta(1-\beta)^{n-1}L}{1-(1-\beta)^n}\right)}{\left(\frac{\sigma}{1-(1-\beta)^n} + T_s(1-\beta)^{n-1} + T_c\left(1-(1-\beta)^{n-1}\right)\right)} \tag{21}$$

## Question 12

The `fixed_point` function from scipy was used to compute $\gamma$ and the function $G$ was used to compute $\beta$ in our script. The plot obtained is




We observe that for $n < 12$ the throughput is increasing! Indicating that at first the number of successful transmissions is not upset by collisions between the stations. However for $n > 12$ collisions get more and more important.