MA2K4 Numerical Methods and Computing Assignment 2

Submission Details

This assignment is due to be handed in before

Thursday 15th February 2024, 12 noon

to be submitted via the submission point on the module moodle page.

Student ID	2103459
Student ID of partner	2100988

Please fill in your ID and that of your working partner above. You need to submit your work even when working with a partner.

Please read carefully all instructions on this page to make sure you understand all rules and restrictions. Every submission consists of two parts: A theoretical part, which can be handwritten or computer-written, and a numerical part, which is to be submitted as jupyter notebook `.ipynb`-file. Further rules and restrictions:

- All handwritten parts must be legible to receive full marks.
- Make sure to provide explanations and show your work. The correct answer alone and without explanation will not receive full marks.
- Numerical work should come in an `.ipynb`-file with comments, explanations, code, and figures. The submission file needs to be able to run through from top to bottom to reproduce all the results of your submission.
- Python is the only computer language to be used in this course.
- You are allowed to use pre-defined mathematical functions (such as `exp`, `linspace`, `max`, ...), but you are NOT allowed to use high-level numerical functions (such as `interpolate`, `polyfit`, ...).
- All work must be explained, commented on, and all work must be shown. Implementation should contain comments and explanations in markdown surrounding them. Figures should have appropriate axis scaling, labels, legends, etc.
- The assignments can be worked on and submitted in pairs. If you choose to do so, indicate your own and your partner's student ID in the fields above. Each partner must upload all submission files to moodle. In this case,both partners receive the same mark.
- Late panelties apply automatically to late submissions. Even if you submit together with a partner, and your partner submits in time, you will be penalised for a late submission if your own submission is not in time.
- To reiterate the above: You will only earn marks if you submit work, and do so before the deadline.
- The use of generative AI in this assignment is strictly forbidden.

MA2K4 assignment 2:

2.1) $f(x) = \log(x)$ $\quad \therefore P_2(x) = \sum_{k=0}^{2} L_k(x)\log(x_k)$

$\therefore L_k(x) = \prod_{j \neq k}^{2} \frac{x - x_j}{x_k - x_j}$.

$\therefore L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-11)(x-12)}{(-1)(-2)} = \frac{1}{2}(x-11)(x-12)$

$L_1(x) = \frac{(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)} = \frac{(x-12)(x-10)}{(-1)(1)} = -(x-12)(x-10)$

$L_2(x) = \frac{(x-x_1)(x-x_0)}{(x_2-x_1)(x_2-x_0)} = \frac{(x-11)(x-10)}{(1)(2)} = \frac{1}{2}(x-11)(x-10)$

$\therefore P_2(x) = \frac{1}{2}\log(10)(x-11)(x-12) - \log(11)(x-12)(x-10) + \frac{1}{2}\log(12)(x-11)(x-10)$

$\therefore$ To 3 digits accuracy

$P_2(x) =$

2.2) $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$, $x \in [0,1]$.

$\therefore$ using $|e(x)| \leq \frac{\|f^{(m+1)}\|_\infty \, |\pi_2(x)|}{2!}$, $(m=1)$

we have $\pi_2(x) = (x-x_0)(x-x_1)$ $\therefore$ let $x = x_0 + \theta h$, $x_1 = x_0 + h$

$\therefore$ for $\theta \in [0,1]$ we have

$|\pi_2(x)| = \theta h(h - \theta h) = h^2\theta(1-\theta)$,

which has a maximum at $\theta = 1/2$ $\therefore$ we have

$|\pi_2(x)| \leq h^2 \cdot 1/2\,(1 - 1/2) = 1/4\, h^2$

Fundamental theorem of Calculus.

$\therefore d/dx\,(\operatorname{erf}(x)) = \frac{2}{\sqrt{\pi}} e^{-x^2}$ $\quad \frac{d^2}{dx^2}(\operatorname{erf}(x)) = -\frac{4x}{\sqrt{\pi}} e^{-x^2}$

$\therefore \operatorname{erf}''(x) = -4x\pi^{-1/2} e^{-x^2}$ which has a maximum of $2\sqrt{\frac{2}{e\pi}}$

at $x = -\frac{1}{\sqrt{2}}$

$\therefore \|e f''(x)\|_\infty \leq 2\sqrt{\frac{2}{e\pi}}$

Hence $|e(x)| \leq \frac{1}{8}h^2 \cdot 2 \cdot \sqrt{\frac{2}{e\pi}}$

$\therefore$ let $\frac{1}{4}h^2\sqrt{\frac{2}{e\pi}} \leq 10^{-6}$

$\therefore h^2\sqrt{\frac{2}{e\pi}} \leq 4\cdot 10^{-6}$

$h^2 \leq 4\cdot 10^{-6}\cdot\sqrt{\frac{e\pi}{2}}$

$h \leq \sqrt{4\cdot 10^{-6}\cdot\sqrt{\frac{e\pi}{2}}}$

$\Rightarrow h \leq 2\cdot 10^{-3}\cdot\left(\frac{e\pi}{2}\right)^{1/4}$

$\therefore$ To 3 s.f.

$h \leq$

2.3) $\therefore P_m(x) = \sum_{k=0}^{m} L_k(x) y_k$, $L_k(x) = \prod_{j\neq k}^{m} \frac{(x-x_j)}{(x_k-x_j)}$

$\therefore q(x) = P_n(x)_1 = \sum_{k=0}^{n} L_k(x) y_k$

$r(x) = P_n(x)_2 = \sum_{k=1}^{n+1} L_k(x) y_k$

$P(x)_3 = \sum_{k=0}^{n+1} L_k(x) y_k$.

$\cancel{\therefore p(x)} \quad \cancel{\sum_{k=0}^{n} L_k(x) y_k}$

$\therefore$ $r(x)$ needs to be multiplied by $(x-x_0)$ as that is not present in $L_k(x)$ and $q(x)$ needs to be multiplied by $(x-x_{n+1})$ as that's not present for it in $L_k(x)$.
~~But both don't have $(x_k - x_0)$ in the denominator of $L_k(x)$, which is present in $p(x)_3$.~~
But $r(x)$ doesn't have (x_k-x_0) in the denominator and $q(x)$ doesn't have $(x_k - x_{n+1})$ in the denominator and numerator of both are the same say A.

$$A = (x-x_0)(x-x_1)\dots(x-x_{n+1})$$

Note $r(x)' = (x-x_0)r(x)$ $\quad q(x)' = (x-x_{n+1})q(x)$

$\therefore$ Consider the expression for each $p(x)$ and $q(x)$ for $k=1$ as both have it :

$$\therefore r(x)' = \frac{Ay_1}{(x_1-x_2)\cdots(x_1-x_{n+1})}$$

$$q(x)' = \frac{Ay_1}{(x_1-x_0)\cdots(x_1-x_n)}$$

Consider $r(x)' - q(x)'$

$\Rightarrow \quad Ay_1(x_1-x_0) - Ay_1(x_1-x_{n+1})$

$= Ay_1(x_{n+1}-x_0)$.

$\therefore$ We get an extra factor of $(x_{n+1}-x_0)$ so we may divide by that. And this works perfectly as this expression is not present for $k=0$ for $q(x)$ and $k=n+1$ for $r(x)$ $\therefore$ it adds it to the denominator of $L_k(x)$ for them making the full lagrange interpolation polynomial of degree $n+1$.

$$\therefore p(x) = \frac{(x-x_0)r(x) - (x-x_{n+1})q(x)}{x_{n+1}-x_0}$$

is the lagrange interpolation polynomial of degree $n+1$ for the nodes and values $\{(x_i, y_i), i = 0, 1, \ldots, n+1\}$.

2.4) $\therefore p_M(x) = \sum_{k=0}^{M} L_k(x) y_k$

$\therefore P_M(x) = L_0(x) f(a_0) + L_1(x) f[a_0,a_1] + L_2(x) f[a_0,a_1,a_2] + \ldots$

$q_0(x) = f(x_0)$, $q_1(x) = f(x_0) + f[x_0, x_1](x-x_0)$

$q_2(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$.

$\therefore f[x_0, x_1]$

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

$\therefore$ For $q_1(x)$:

$$\Rightarrow f(x_0) + \frac{(f(x_1) - f(x_0))(x - x_0)}{(x_1 - x_0)}$$

$$\therefore = f(x_0)\left(1 - \left(\frac{x - x_0}{x_1 - x_0}\right)\right) + f(x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)$$

$$= f(x_0)\left(\frac{x_1 - x}{x_1 - x_0}\right) + f(x_1)\left(\frac{x - x_0}{x_1 - x_0}\right) = P_1(x).$$

$\therefore$ Consider $f[x_0, x_1, x_2]$.

$$\therefore \Rightarrow \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$\therefore$ From the above pattern, it equals:

$$\frac{f(x_2) - f(x_1)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_2 - x_0)(x_1 - x_0)}.$$

$\therefore$ Consider $f[x_0, x_1, x_2, x_3]$

$$\therefore \Rightarrow \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}.$$

$$= \frac{f(x_3) - f(x_2)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_0)} - \frac{f(x_2) - f(x_1)}{\underset{\downarrow\ (x_3 - x_1)}{(x_2 - x_1)}(x_3 - x_0)} - \frac{f(x_2) - f(x_1)}{(x_2 - x_1)\underset{\downarrow\ (x_2 - x_0)}{(x_3 - x_0)}} - \frac{f(x_1) - f(x_0)}{\underset{\downarrow\ (x_2 - x_0)}{(x_1 - x_0)}(x_3 - x_0)}.$$

Note that $f(x_3)$ has all combinations of $x_3 - x_j$ for $j = 0, 1, 2$ in the denominator

Using Strong Induction:

$k = 0$ is true

Assume true for k up to n

$\therefore P_k(x) = q_k(x)$. for $0 \le k \le n$

Observe that $f[x_0, \ldots, x_n] - f[x_1, \ldots, x_n] = (x_0 - x_{n+1}) f[x_0, \ldots, x_{n+1}]$.

$\therefore$ Let $k = n+1$

$$\therefore q_{n+1}(x) = P_n(x) + f[x_0, \ldots, x_{n+1}](x - x_0) \cdots (x - x_n)$$

$\therefore$ $f[x_0, \ldots, x_{n+1}]$ will be of the form:

$$\frac{f(x_{n+1})}{(x-x_0)\cdots(x-x_n)} + \frac{A_n f(x_n)}{B_n} + \frac{A_{n-1} f(x_{n-1})}{B_{n-1}} + \cdots + \frac{A_0 f(x_0)}{B_0}.$$

$\therefore$ Multiplying by $c = (x - x_0) \cdots (x - x_n)$

$$f(x_{n+1}) L_{n+1}(x) + \frac{A_n c}{B_n} f(x_n) + \frac{A_{n-1}}{B_{n-1}} c\, f(x_{n-1}) + \cdots + \frac{A_0 c}{B_0} f(x_0).$$

Claim:

$$L_m(x) + \frac{A_m c}{B_m} = \prod_{j \ne k} \frac{(x - x_j)}{(x_k - x_j)} + \frac{A_m c}{B_m} = \prod_{j \ne k}^{n+1} \frac{(x - x_j)}{(x_k - x_j)}.$$

This is true due to the assumtion we made earlier and by the symmetry of that assumption and the observation.

$$\text{Hence } q_{n+1}(x) = \sum_{k=0}^{n+1} \prod_{j \ne k}^{n+1} \frac{(x - x_j)}{(x_k - x_j)} y_k = P_{n+1}(x) \ // \square.$$

2.5) $\alpha_j = \int_a^b L_j(x)\,dx$ where $L_j(x) =$ Lagrange polynomial associated with nodes x_j. $\therefore L_j(x) = \sum_{j=0}^{n} L_j(x_j)\, y_j$

{equal spacing starting from b (same but reverse order).}

$$\therefore x_j = a + \frac{j}{n}(b-a), \quad x_{n-j} = a + \frac{n-j}{n}(b-a) = b - \frac{j}{n}(b-a)$$

$$\therefore a_j = \int_a^b L_j(x)\,dx \qquad a_{n-j} = \int_a^b L_{n-j}(x)\,dx.$$

$$L_{n-j}(x) = \sum_{j=0}^{n} L_{n-j}(x_{n-j})\, y_{n-j}$$

We see that $L_{n-j}(x) = L_j(x)$

$\therefore$ We see that $\int_a^b L_j(x)\,dx = \int_a^b L_{n-j}(x)\,dx$

Hence $\alpha_j = \alpha_{n-j}$. //$\square$