MA2K4 Numerical Methods and Computing
Assignment 4

Submission Details

This assignment is due to be handed in before

Thursday 14th March 2024, 12 noon

to be submitted via the submission point on the module moodle page.

Student ID	2103459
Student ID of partner	2100988

Please fill in your ID and that of your working partner above. You need to submit your work even when working with a partner.

Please read carefully all instructions on this page to make sure you understand all rules and restrictions. Every submission consists of two parts: A theoretical part, which can be handwritten or computer-written, and a numerical part, which is to be submitted as jupyter notebook `.ipynb`-file. Further rules and restrictions:

- All handwritten parts must be legible to receive full marks.
- Make sure to provide explanations and show your work. The correct answer alone and without explanation will not receive full marks.
- Numerical work should come in an `.ipynb`-file with comments, explanations, code, and figures. The submission file needs to be able to run through from top to bottom to reproduce all the results of your submission.
- Python is the only computer language to be used in this course.
- You are allowed to use pre-defined mathematical functions (such as `exp`, `linspace`, `max`, ...), but you are NOT allowed to use high-level numerical functions (such as `interpolate`, `polyfit`, ...).
- All work must be explained, commented on, and all work must be shown. Implementation should contain comments and explanations in markdown surrounding them. Figures should have approprate axis scaling, labels, legends, etc.
- The assignments can be worked on and submitted in pairs. If you choose to do so, indicate your own and your partner's student ID in the fields above. Each partner must upload all submission files to moodle. In this case,both partners receive the same mark.
- Late panelties apply automatically to late submissions. Even if you submit together with a partner, and your partner submits in time, you will be penalised for a late submission if your own submission is not in time.
- To reiterate the above: You will only earn marks if you submit work, and do so before the deadline.
- The use of generative AI in this assignment is strictly forbidden.

MA2K4 Assignment 4.

$\phi(t_n)$

1) $\therefore$ Taylor expanding u_{n+1} at time t_{n+1} i.e. the exact solution

$\phi(t_{n+1}) = $

$\therefore$ $\phi(t_n+h) = \phi(t_n) + h\phi'(t_n) + \frac{h^2}{2}\phi''(t_n) + O(h^3)$.

$\therefore \phi'(t_n) = f(t_n, \phi(t_n))$

$\therefore \phi''(t_n) = \frac{\partial f}{\partial t}(t_n, \phi(t_n)) + \frac{\partial f}{\partial u}(t_n, \phi(t_n)) \cdot \phi'(t_n)$.

$\therefore$ We are interested in $\phi(t_{n+1}) - u_{n+1}$

$\therefore u_{n+1} = \phi(t_n) + \alpha h \phi'(t_n) + \beta h(\phi'(t_n) + \gamma h \phi''(t_n))$

$u_{n+1} = \phi(t_n) + (\alpha+\beta) h \phi'(t_n) + \beta\gamma h^2 \phi''(t_n)$.

$\therefore \phi(t_{n+1}) - u_{n+1} = (\alpha+\beta-1) h \phi'(t_n) + O(h^2)$

Local Truncation error $= kh + O(h^2)$

where $k = (\alpha+\beta-1)\phi'(t_n)$.

$\therefore$ If this is method is consistent then we have

$\alpha+\beta-1 = 0 \quad \therefore \quad \alpha+\beta = 1$

If $\alpha+\beta=1$ then the h term doesn't exist $\therefore$

$\lim_{h\to 0} \tau(h) = 0 \quad \therefore$ Consistent //

b) $\therefore$ From ~~section~~ above we require $\beta\gamma$ to equal the coefficient of h^2 in $\phi(t_{n+1})$ hence $\beta\gamma = 1/2$.

c) u_{n+1} only goes up to order 2, $\therefore$ with the previous conditions, all powers above degree 2 go to zero hence cancel out if we calculate the local truncation error to the third order or above.

2) $u_{n+1} = u_n + \frac{1}{2}h\left(f_n + f(t_{n+1}, u_n + hf_n)\right)$

$\therefore$ Test problem $\begin{cases} y'(t) = \lambda y(t) \\ y(0) = 1 \end{cases}$, $t > 0$.

$\therefore$ Applying the test problem:

$u_{n+1} = u_n + \frac{1}{2}h\lambda u_n + \frac{1}{2}h(\lambda u_n + h\lambda^2 u_n)$

$\therefore u_{n+1} = u_n + h\lambda u_n + \frac{h^2\lambda^2 u_n}{2}$

$\therefore u_{n+1} = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right)u_n$

$\therefore$ Solving, we get

$u_n = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right)^n u_0$, $u_0 = 1$

$\therefore u_n \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right)^n$

$\therefore$ Region of ~~the~~ Absolute Stability: $\left|1 + h\lambda + \frac{h^2\lambda^2}{2}\right| < 1$ //.

3) $u_{n+1} = u_n + h((1-\theta)f_n + \theta f_{n+1})$ for $\theta \in [0, 1]$.

$\therefore$ Applying the test problem again:

Assuming the above is A-Stable.

$\therefore u_{n+1} = u_n + h((1-\theta)\lambda u_n + \theta\lambda u_{n+1})$

$\therefore u_{n+1} = u_n + h(1-\theta)\lambda u_n + h\lambda\theta u_{n+1}$

$\therefore (1 - h\lambda\theta)u_{n+1} = (1 + h\lambda - h\lambda\theta)u_n$

$\therefore u_{n+1} = \left(\frac{1 - h\lambda\theta}{1 - h\lambda\theta} + \frac{h\lambda}{1 - h\lambda\theta}\right)u_n$

$\therefore$ Since $u_0 = 1$, we have.

$u_n = \left(1 + \frac{h\lambda}{1 - h\lambda\theta}\right)^n$

∴ For absolute stability we require:

$$\left| 1 + \frac{h\lambda}{1 - h\lambda\theta} \right| < 1 \qquad \therefore \text{ let } z = h\lambda$$

$$\therefore \left| \frac{1 - z\theta + z}{1 - z\theta} \right| < 1$$

$\therefore |1 - z\theta + z| < |1 - z\theta|$

Note that it is also A-stable ∴ $Re(z) < 0$.

If $\theta = 0$ we have $|1 + z| < 1$ which is clearly not true i.e $z = -8$. ∴ let $Re(z) = a$

∴ we require:

$|1 - a\theta + a| < |1 - a\theta|$ ← $-ve < +ve$

∴ let $-1 + a\theta - a < 1 - a\theta$ $\{+ve < +ve\}$ always true

∴ ~~assume~~ $-\theta + 1 < \theta$ $\left(\begin{array}{c} 1 - z\theta + z < 1 - z\theta \\ 1 + z < 1 \\ \therefore z < 0 ✓ \end{array} \right)$

∴ $2\theta > 1$

∴ $\theta > 1/2$

Note that if $\theta = 1/2$ and $z = -8$ the inequality holds

∴ $\theta \geq 1/2$

∴ Assume $\theta \geq 1/2$.

∴ From before we have

$$\left| 1 + \frac{h\lambda}{1 - h\lambda\theta} \right| < 1 \qquad \text{let } z = h\lambda$$

$$\left| 1 + \frac{z}{1 - z\theta} \right| < 1$$

∴ From before

$|1 - z\theta + z| < |1 - z\theta|$

$z < 0$ is trivial

∴ let $-ve < +ve$ → $-1 + z\theta - z < 1 - z\theta$

∴ $2z\theta - z < 2$

$z(2\theta - 1) < 2$

∴ Since $\theta \geq 1/2$

we have $2\theta - 1 \geq 0$

∴ The above is true

Hence it is A stable //D.

4) a) Maximal order is 3 $\therefore$ Checking order conditions:

$$\sum_{i=1}^{3} b_i = \frac{2}{9} + \frac{1}{3} + \frac{4}{9} = \frac{2}{3} + \frac{1}{3} = 1 \checkmark$$

$$\sum_{i=1}^{3} b_i c_i = \left(\frac{2}{9}\right)(0) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{4}{9}\right)\left(\frac{3}{4}\right) = \frac{1}{2} \checkmark$$

$$\sum_{i=1}^{3} b_i c_i^2 = \left(\frac{2}{9}\right)(0^2) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{4}{9}\right)\left(\frac{3}{4}\right)^2 = \frac{1}{3} \checkmark$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} b_i a_{ij} c_j = \begin{pmatrix} \frac{2}{9} & \frac{1}{3} & \frac{4}{9} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 3/4 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/2 \\ 3/4 \end{pmatrix} = \frac{1}{6} \checkmark$$

Hence it is of maximal order. //o.

b) Consider $\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$ $\sum_{i=1}^{1} b_i = 1$ $\therefore$ it is of highest order 1

It is <u>not</u> unique as you can also have

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}.$$

c) $\therefore$ Consider $\begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}$

$\therefore$ $b_1 + b_2 = 1 \quad \cdots (1)$

$b_1 c_1 + b_2 c_2 = 1/2 \quad \cdots (2)$

From (1) : $b_2 = 1 - b_1 \quad \cdots (3)$

Substituting ③ into ②:

$b_1 c_1 + (1-b_1) c_2 = 1/2$

$\therefore\ b_1 (c_1 - c_2) = 1/2 - c_2 = \frac{1-2c_2}{2}$

$\therefore\ b_1 = \frac{1-2c_2}{2(c_1-c_2)}$ $\quad \therefore\ b_2 = 1 - \frac{1-2c_2}{2(c_1-c_2)} = \frac{2c_1-1}{2(c_1-c_2)}$.

$\therefore$ General Second order explicit 2-Stage Runge kutta:

$$\begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & \frac{1-2c_2}{2c_1-2c_2} & \frac{2c_2-1}{2c_1-2c_2} \end{array} = \begin{array}{c|cc} a_{11}+a_{12} & a_{11} & a_{12} \\ a_{21}+a_{22} & a_{21} & a_{22} \\ \hline & \frac{1-2(a_{21}+a_{22})}{2(a_{11}+a_{12})-2(a_{21}+a_{22})} & \frac{2(a_{21}+a_{22})-1}{2(a_{11}+a_{12})-2(a_{21}+a_{22})} \end{array}$$

since $c_i = \sum_{j=1}^{s} a_{ij}$

$\therefore\ k_1 = f(t_n + c_1 h,\ u_n + h(a_{11}k_1 + a_{12}k_2))$

$k_2 = f(t_n + c_2 h,\ u_n + h(a_{21}k_1 + a_{22}k_2))$

c_1 and c_2 can be expressed in terms of a's as in the tableau

$u_{n+1} = \left(\frac{1-2c_2}{2c_1-2c_2}\right) hk_1 + h\left(\frac{2c_1-1}{2c_1-2c_2}\right) k_2 + u_n$

$\therefore\ u_{n+1} = u_n + \frac{h}{2c_1-2c_2}\left\{(1-2c_2)k_1 + (2c_1-1)k_2\right\}$ //

5) $\therefore$ General 2 Stage Runge kutta:

$$\begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}$$

$\therefore$ Applying to the test problem:

$k_1 = \lambda u_n + \lambda h (a_{11}k_1 + a_{12}k_2)$

$k_2 = \lambda u_n + \lambda h (a_{21}k_1 + a_{22}k_2)$.

$\therefore$

∴ Expressing in matrix form as:

$$\begin{pmatrix} 1-h\lambda a_{11} & -\lambda h a_{12} \\ -\lambda h a_{21} & 1-\lambda h a_{22} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \lambda \begin{pmatrix} u_n \\ u_n \end{pmatrix}$$

∴ Finding Inverse of matrix

Let Δ denote the determinant of the 2x2 matrix

$$\therefore \text{Inverse} = \frac{1}{\Delta} \begin{pmatrix} 1-\lambda h a_{22} & \lambda h a_{12} \\ \lambda h a_{21} & 1-\lambda h a_{11} \end{pmatrix}$$

∴ Multiplying ~~the left~~ both sides by the inverse on the left.

$$\therefore \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{\lambda}{\Delta} \begin{pmatrix} 1-h\lambda a_{22} & \lambda h a_{12} \\ \lambda h a_{21} & 1-\lambda h a_{11} \end{pmatrix} \begin{pmatrix} u_n \\ u_n \end{pmatrix}$$

$$\therefore k_1 = (1-\lambda h a_{22} + \lambda h a_{12}) \cdot \lambda u_n / \Delta$$

$$k_2 = (\lambda h a_{21} + 1 - \lambda h a_{11}) \cdot \lambda u_n / \Delta$$

$$\therefore k_1 = (1+\lambda h(a_{12}-a_{22}))\, \lambda u_n / \Delta$$

$$k_2 = (1+\lambda h(a_{21}-a_{11}))\, \lambda u_n / \Delta.$$

Note $I - \lambda h A = \begin{pmatrix} 1-h\lambda a_{11} & -\lambda h a_{12} \\ -\lambda h a_{21} & 1-\lambda h a_{22} \end{pmatrix}$ //D.

6) $\therefore A = \begin{pmatrix} 1/4 & (3-2\sqrt{3})/12 \\ (3+2\sqrt{3})/12 & 1/4 \end{pmatrix}$

$\therefore \Delta = |I - \lambda h A| \rightarrow$ From the previous question (question 5)

$$\therefore \Delta = \left| \begin{pmatrix} 1-\lambda h/4 & \frac{-\lambda h(3-2\sqrt{3})}{12} \\ \frac{-\lambda h(3+2\sqrt{3})}{12} & 1-\frac{\lambda h}{4} \end{pmatrix} \right|$$

$$\therefore \Delta = \left(1-\frac{\lambda h}{4}\right)^2 - \frac{(\lambda h)^2(-3)}{144} = 1 - \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{16} + \frac{\lambda^2 h^2}{48}$$

$$\therefore \Delta = 1 - \frac{\lambda h}{2} + \frac{3\lambda^2 h^2}{48} = 1 - \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{12}$$

$$\text{So } A^{-1} = \frac{1}{\Delta} \begin{pmatrix} 1-\frac{\lambda h}{4} & \frac{\lambda h(3-2\sqrt{3})}{12} \\ \frac{\lambda h(3+2\sqrt{3})}{12} & 1-\frac{\lambda h}{4} \end{pmatrix}$$

$$\therefore \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \lambda A^{-1} \begin{pmatrix} u_n \\ u_n \end{pmatrix} = \lambda u_n A^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore u_{n+1} = u_n + \frac{h}{2}(k_1 + k_2).$$

$$\therefore k_1 + k_2 = \frac{\lambda}{\Delta}\left(2 + \frac{\lambda h}{6}\right)u_n = \left(2 + \frac{\lambda^2 h}{6}\right)\frac{u_n}{\Delta}$$
$$= \left(1 + \frac{\lambda h}{6}\right)\frac{\lambda u_n}{\Delta}$$

$$u_{n+1} = u_n + \frac{h}{2}\left(1 + \frac{\lambda h}{6}\right)\frac{\lambda u_n}{\Delta} = \frac{u_n}{\Delta}\left\{1 + \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{12}\right\}.$$

$$\therefore u_{n+1} = \left(\frac{1 + \frac{1}{2}\lambda h + \frac{1}{12}\lambda^2 h^2}{1 - \frac{1}{2}\lambda h + \frac{1}{12}\lambda^2 h^2}\right) u_n.$$

$$\therefore u_{n+1} = R(\lambda h) u_n.$$