MA2K4 Numerical Methods and Computing
Assignment 1

Submission Details

This assignment is due to be handed in before

Thursday 1st February 2024, 12 noon

to be submitted via the submission point on the module moodle page.

Student ID	2103459
Student ID of partner	2100988

Please fill in your ID and that of your working partner above. You need to submit your work even when working with a partner.

Please read carefully all instructions on this page to make sure you understand all rules and restrictions. Every submission consists of two parts: A theoretical part, which can be handwritten or computer-written, and a numerical part, which is to be submitted as jupyter notebook `.ipynb`-file. Further rules and restrictions:

- All handwritten parts must be legible to receive full marks.
- Make sure to provide explanations and show your work. The correct answer alone and without explanation will not receive full marks.
- Numerical work should come in an `.ipynb`-file with comments, explanations, code, and figures. The submission file needs to be able to run through from top to bottom to reproduce all the results of your submission.
- Python is the only computer language to be used in this course.
- You are allowed to use pre-defined mathematical functions (such as `exp`, `sin`, `max`, ...), but you are NOT allowed to use high-level numerical functions (such as `diff`, `polyfit`, ...).
- All work must be explained, commented on, and all work must be shown. Implementation should contain comments and explanations in markdown surrounding them. Figures should have approprate axis scaling, labels, legends, etc.
- The assignments can be worked on and submitted in pairs. If you choose to do so, indicate your own and your partner's student ID in the fields above. Each partner must upload all submission files to moodle. In this case,both partners receive the same mark.
- Late panelties apply automatically to late submissions. Even if you submit together with a partner, and your partner submits in time, you will be penalised for a late submission if your own submission is not in time.
- To reiterate the above: You will only earn marks if you submit work, and do so before the deadline.
- The use of generative AI in this assignment is strictly forbidden.

1.1) (i) $\|Ax\| > 0$, $\|x\| > 0$ Since $\|\cdot\|$ is a norm on $\mathbb{R}^n$, $\therefore \frac{\|Ax\|}{\|x\|} > 0$ $\therefore$

$|||A||| > 0 \ \forall A \in \mathbb{R}^{n \times n}$

(ii) $|||\alpha A||| = \frac{\|\alpha A x\|}{\|x\|} = \frac{|\alpha| \|Ax\|}{\|x\|} = |\alpha| \cdot |||A||| \quad \forall \alpha \in \mathbb{R}, \forall A \in \mathbb{R}^{n \times n}$

(Since $\|\cdot\|$ is a norm on $\mathbb{R}^n$)

(iii) $|||A+B||| = \frac{\|(A+B)x\|}{\|x\|} = \frac{\|Ax + Bx\|}{\|x\|} \leq \frac{\|Ax\| + \|Bx\|}{\|x\|} = |||A||| + |||B|||$

$\therefore$ $|||A|||$ is a norm on $\mathbb{R}^{n \times n}$

1.2) a) $k(d) \approx \|G'(d)\| \frac{\|d\|}{\|G(d)\|}$

$\therefore x = a^d \quad \therefore G(d) = a^d$

$G'(d) = a^d \ln a$

$\therefore k(d) = \frac{\|a^d \ln a\| \cdot \|d\|}{\|a^d\|}$

$= \frac{|a^d| \ln a| \cdot |d|}{|a^d|} = |d \ln a|$. //

b) $x = d+1 \quad \therefore G(d) = d+1$

$G'(d) = 1$

$\therefore k(d) = \frac{\|1\| \cdot \|d\|}{\|d+1\|} = \frac{|1| \cdot |d|}{|d+1|}$

$= \left| \frac{d}{d+1} \right|$. //

1.3 a) $\begin{cases} x + dy = 1 & \dots ① \\ dx + y = 0 & \dots ② \end{cases}$

$d \cdot ① - ②$: $d^2 y - y = d$

$\therefore y = \frac{d}{d^2 - 1} \quad \dots ③$

$\therefore ③ \to ②$

$dx = -\frac{d}{d^2-1}$

$\therefore x = -\frac{1}{d^2-1} = \frac{1}{1-d^2}$

$\therefore$ The problem is well-posed for $\{d \in \mathbb{R} \mid d \neq 1\}$

b) $x = \frac{1}{1-d^2}$, $\therefore G(d) = (1-d^2)^{-1}$

$\therefore G'(d) = -(-2d)(1-d^2)^{-2}$

$= \frac{2d}{(1-d^2)^2}$

$\therefore k(d) = \left\| \frac{2d}{(1-d^2)^2} \right\| \cdot \frac{\|d\|}{\left\| \frac{1}{1-d^2} \right\|}$

$= \left| \frac{2d}{(1-d^2)^2} \right| |d| \, |1-d^2|$

$= \left| \frac{2d^2}{1-d^2} \right|$. //..

c) $y = \frac{d}{d^2-1}$ $\therefore G(d) = \frac{d}{d^2-1}$

$\therefore G'(d) = \frac{d^2-1-2d^2}{(d^2-1)^2} = \frac{-d^2-1}{(d^2-1)^2} = -\frac{(d^2+1)}{(d^2-1)^2}$

$\therefore k(d) = \left\| \frac{(d^2+1)}{(d^2-1)^2} \right\| \frac{\|d\|}{\left\| \frac{d}{d^2-1} \right\|} = \left| \frac{(d^2+1)}{(d^2-1)^2} \right| \cdot |d^2-1| = \left| \frac{d^2+1}{d^2-1} \right|$ //-

1.4)

a) $G(p) = -p + \sqrt{p^2+q}$ $\therefore G'(p) = -1 + \frac{1}{2} \cdot 2p(p^2+q)^{-1/2} = \frac{p-\sqrt{p^2+q}}{\sqrt{p^2+q}}$

$\therefore k(p) = \left\| \frac{p-\sqrt{p^2+q}}{\sqrt{p^2+q}} \right\| \frac{\|p\|}{\|-p+\sqrt{p^2+q}\|} = \frac{|p|}{|\sqrt{p^2+q}|} = \left| \frac{p}{\sqrt{p^2+q}} \right|$

$\therefore$ From above $k(p) = \left\| \frac{-G(d)}{\sqrt{p^2+q}} \right\| \cdot \frac{\|p\|}{\|G(d)\|}$ since

$G'(p) = \frac{-G(p)}{\sqrt{p^2+q}}$ Hence it does not depend whether x_+ or x_- is considered.

b) $G(q) = -p \pm \sqrt{p^2+q}$ $\therefore G'(q) = \pm \frac{1}{2}(p^2+q)^{-1/2}$

$\therefore k(d) = \| \pm \frac{1}{2}(p^2+q)^{-1/2} \| \cdot \frac{\|q\|}{\|-p \pm \sqrt{p^2+q}\|}$

$= \frac{1}{2} \left| \frac{1}{\sqrt{p^2+q}} \right| \frac{|q|}{|-p \pm \sqrt{p^2+q}|} = \frac{1}{2} \left| \frac{q}{\sqrt{p^2+q}} \right| \left| \frac{1}{-p \pm \sqrt{p^2+q}} \right|$ //-

1.5) $\therefore P_1(x) = f(x_0) + (x - x_0)\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$\therefore P_1(x) = 0^3 + 0 + (x - 0)\cdot\dfrac{a^3 - a - 0}{a - 0} = x(a^2 - 1)$

$\therefore e(x) = f(x) - P_1(x) = x^3 - x - x(a^2 - 1)$ $\quad \cancel{= x^3 - (a^2+1)x + x}$

$= x^3 - x - xa^2 + x$ $\quad \cancel{= x^3 - (a^2+2)x}$

$= x^3 - xa^2$ //

$\therefore f''(x) = 6x \qquad \therefore f''(\xi) = 2(x + a)$

$\therefore e(x) = \frac{1}{2}\cdot 2(x + a)x(x - a) = x(x^2 - a) = x^3 - ax$

which is the same as above //.

1.6) $\therefore$ using formula from 1.5):

we have:

$P_1(x) = (-a)^4 + (x - 0)\dfrac{0^4 - (-a)^4}{a - 0} = a^4 \mp a^3 x$

$\therefore e(x) = (x - a)^4 - a^4 \pm a^3 x = x^4 - 4x^3 a + 6x^2 a^2 \mp 3xa^3$

$\therefore f''(x) = 12(x - a)^2$

$\therefore e(x) = \frac{1}{2} f''(\xi)\, x(x - a)$

$$
\therefore \quad x^2 - ax \;\overline{\big)\, x^4 - 4x^3a + 6x^2a^2 - 3xa^3}^{\;x^2 - 3xa + 3a^2}
$$

$-3x^3a \quad 3x^2a^2$

$\therefore \frac{1}{2} f''(\xi) = x^2 - 3xa + 3a^2$

$\therefore 6(\xi - a)^2 = x^2 - 3xa + 3a^2$

$\therefore 6\xi^2 - 12a\xi + 6a^2 = x^2 - 3xa + 3a^2$

$\therefore (\xi - a)^2 = \frac{1}{6}x^2 - \frac{1}{2}xa + \frac{3}{6}a^2$

$\therefore \xi - a = \pm\sqrt{\frac{1}{6}x^2 - \frac{1}{2}xa + \frac{1}{2}a^2}$

$\therefore \xi = a \pm \sqrt{\frac{1}{6}x^2 - \frac{1}{2}xa + \frac{1}{2}a^2}$

$\therefore$ 2 possibilities for ξ.