# MA2K4 Numerical Methods and Computing
# Assignment 3

**Submission Details**

This assignment is due to be handed in before

**Thursday 29th February 2024, 12 noon**

to be submitted via the submission point on the module moodle page.

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|---|---|
| Student ID | 2103459 |
| Student ID of partner | 2100988 |

Please fill in your ID and that of your working partner above. You need to submit your work even when working with a partner.

Please read carefully all instructions on this page to make sure you understand all rules and restrictions. Every submission consists of two parts: A theoretical part, which can be handwritten or computer-written, and a numerical part, which is to be submitted as jupyter notebook `.ipynb`-file. Further rules and restrictions:

- All handwritten parts must be legible to receive full marks.
- Make sure to provide explanations and show your work. The correct answer alone and without explanation will not receive full marks.
- Numerical work should come in an `.ipynb`-file with comments, explanations, code, and figures. The submission file needs to be able to run through from top to bottom to reproduce all the results of your submission.
- Python is the only computer language to be used in this course.
- You are allowed to use pre-defined mathematical functions (such as `exp`, `linspace`, `max`, ...), but you are NOT allowed to use high-level numerical functions (such as `interpolate`, `polyfit`, ...).
- All work must be explained, commented on, and all work must be shown. Implementation should contain comments and explanations in markdown surrounding them. Figures should have appropriate axis scaling, labels, legends, etc.
- The assignments can be worked on and submitted in pairs. If you choose to do so, indicate your own and your partner's student ID in the fields above. Each partner must upload all submission files to moodle. In this case,both partners receive the same mark.
- Late panelties apply automatically to late submissions. Even if you submit together with a partner, and your partner submits in time, you will be penalised for a late submission if your own submission is not in time.
- To reiterate the above: You will only earn marks if you submit work, and do so before the deadline.
- The use of generative AI in this assignment is strictly forbidden.

# MA2K4 Assignment 3.

3.1) Let ~~be~~ $n$ be even. And consider the integration interval $x \in [a,b]$. Suppose ~~the~~ the NQ calculates exactly up to $n$,

The midpoint of $[a,b]$ is $\frac{a+b}{2}$ and note that the polynomial $x^n$ is an even function.

$\therefore$ Notice that $\int_a^b (x - \frac{1}{2}(a+b))^{n+1}\,dx = 0$ as it ~~has~~ has middle point $\frac{a+b}{2}$ and it is an odd function. We can see that the Newton-quadrature also gives zero - we can observe this by noticing $f(x_i) = -f(x_{n-i})$ for $i \in [0, \frac{a+b}{2}]$ so all strips cancel out giving zero. $\therefore$ It has degree of precision $n+1$. //D.

3.2) $\therefore$ We have $x_0 = -1$, $x_1 = -1/3$, $x_2 = 1/3$, $x_3 = 1$.

$\therefore$ The integral we are trying to approximate is:

$$\int_{-1}^{1} f(x)\,dx.$$

$\therefore$ Since it is exact for all polynomials of degree 3 we have: it must be exact for $1, x, x^2, x^3$.

$$2 = \int_{-1}^{1} 1\,dx = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$$

$$0 = \int_{-1}^{1} x\,dx = -\alpha_0 - \frac{1}{3}\alpha_1 + \frac{1}{3}\alpha_2 + \alpha_3$$

$$\frac{2}{3} = \int_{-1}^{1} x^2\,dx = \alpha_0 + \frac{1}{9}\alpha_1 + \frac{1}{9}\alpha_2 + \alpha_3$$

$$0 = \int_{-1}^{1} x^3\,dx = -\alpha_0 - \frac{1}{27}\alpha_1 + \frac{1}{27}\alpha_2 + \alpha_3$$

$\therefore$ Solving

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1/3 & 1/3 & 1 \\ 1 & 1/9 & 1/9 & 1 \\ -1 & -1/27 & 1/27 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2/3 \\ 0 \end{pmatrix}$$

We get

$$\alpha_0 = \frac{4}{16} \quad \alpha_1 = \frac{3}{4} \quad \alpha_2 = \frac{3}{4} \quad \alpha_3 = \frac{4}{16}.$$

3.3) $\therefore I_{1,n}(f) = \frac{h}{2}\left\{ f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b) \right\}$.

$$I_{1,2n}(f) = \frac{h}{4}\left\{ f(a) + 2\sum_{i=1}^{2n-1} f\left(a+\frac{ih}{2}\right) + f(b) \right\}$$

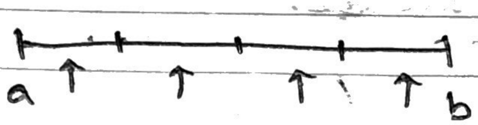
$$I_{2,n}(f) = \frac{h}{3}\left\{ f(a) + 4\sum_{\text{odd } i}^{2n-1} f(a+ih) + 2\sum_{\text{even } i}^{2n-1} f(a+ih) + f(b) \right\}$$

$\therefore$ Calculating $\frac{4}{3}I_{1,2n}(f) - \frac{1}{3}I_{1,n}(f)$.

$$\therefore \frac{4}{3}\left[\frac{h}{4}\left\{f(a) + 2\sum_{i=1}^{2n-1} f\left(a+i\frac{h}{2}\right) + f(b)\right\}\right] - \frac{1}{3}\left[\frac{h}{2}\left\{f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right\}\right]$$

$$(*) \ldots = \frac{1}{3}h\left\{f(a) + 2\sum_{i=1}^{2n-1} f\left(a+i\frac{h}{2}\right) + f(b)\right\} - \frac{h}{6}\left\{f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right\}$$

Now finding this algebraically is very hard, so let's look at its geometry:
Modelling it with a $n=4$ case:




The arrows represent extra points added for the $I_{1,2n}$ case. And the lines on the horizontal are $I_{1,n}$ points $\therefore$ Let's take 4/3 times all points as in $I_{1,2n}$. Notice that when $1/3\, I_{1,n}$ is subtracted, it is every other point that is subtracted, namely all even points. Thus using this to help $(*)$ becomes:

$$\frac{1}{6}h f(a) + \frac{4h}{6}\sum_{\text{odd } i}^{2n-1} f(a+ih) + \frac{2h}{6}\sum_{\text{even } i}^{2n-1} f(a+ih) + \frac{1}{6}h f(b)$$

Notice that we need to multiply by 2 for the missing deducted points $\therefore$ $*$ becomes:

$$\frac{1}{3}h\left\{f(a) + 4\sum_{\text{odd } i}^{2n-1} f(a+ih) + 2\sum_{\text{even } i}^{2n-1} f(a+ih) + f(b)\right\} = I_{2,n}(f)$$ //D.

$$x^5 - \frac{11}{10}x^4$$

3.4) $\therefore$ Consider $f(x) =$ ~~$x^5 \ldots x^4$~~ on $[0,1]$.

$\therefore$ Approximating $\int_0^1$ ~~$(\ldots)\,dx$~~ $(x^5 - \frac{11}{10}x^4)\,dx$

Trapezium rule:

~~$\left[\frac{x^6}{6} - \frac{1}{5}x^5\right]_0^1$~~

$$\frac{1}{2}f(1) = 1 - \frac{11}{10} = -\frac{1}{10}.$$

Simpson's rule:

$$\frac{1}{6}\left(4f\left(\tfrac{1}{2}\right) - 1 - \frac{11}{10}\right) = \frac{1}{6}\left(4\left(\frac{1}{32} - \frac{11}{10}\cdot\frac{1}{16}\right) - 1 - \frac{11}{10}\right) = -\frac{4}{24}$$

$$\int_0^1 (x^5 - \tfrac{11}{10}x^4)\,dx = \left[\frac{x^6}{6} - \frac{11}{10}\cdot\frac{1}{5}x^5\right]_0^1 = -\frac{4}{75}$$

$$\therefore \left|-\frac{1}{10} + \frac{4}{75}\right| \text{ ~~[illegible]~~ } < \left|-\frac{1}{24} - \left(-\frac{4}{75}\right)\right|$$

$\therefore$ Trapezium rule is <u>more</u> accurate.

$$\int_0^1 (x^5 - \lambda x^4)\,dx$$

T: $\frac{1}{2}f(1) = \frac{1}{2} - \frac{\lambda}{2}$

S: $\frac{1}{6}(4f(\frac{1}{2}) + 1 - \lambda) = \frac{1}{6}\left\{\left(\frac{1}{32} - \frac{\lambda}{16}\right)\cdot 4 + 1 - \lambda\right\}$.

$$= \frac{3}{16} - \frac{5\lambda}{24}$$

$$\int_0^1 (x^5 - \lambda x^4)\,dx = \frac{1}{6} - \frac{\lambda}{5}$$

$\therefore$ we need $\left|\frac{1}{6} - \frac{\lambda}{5} - \frac{1}{2} + \frac{\lambda}{2}\right| < \left|\frac{1}{6} - \frac{\lambda}{5} - \frac{3}{16} + \frac{5\lambda}{24}\right|$

$$\therefore \left|\frac{3\lambda}{10} - \frac{1}{3}\right| < \left|-\frac{1}{48} + \frac{\lambda}{120}\right|$$

Case (i) both +ve

~~$\ldots$~~

$$-\frac{7}{24}\lambda < -\frac{5}{16}$$

$$\therefore \lambda > 15/14$$

Case (ii) One +ve One −ve

$$\frac{3}{10}\lambda - \frac{1}{3} < \frac{1}{48} - \frac{\lambda}{120}$$

$$\therefore \lambda < \frac{85}{74}$$

$$\therefore \boxed{\frac{15}{14} < \lambda < \frac{85}{74}}$$ T more accurate than S for $\lambda$ in this.

3.5) $G_2(1) = \int_{-1}^{1} 1\,dx = 2 \qquad G_2(x) = \int_{-1}^{1} x\,dx = 0 \qquad G_2(x^2) = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$

$G_2(x^3) = \int_{-1}^{1} x^3\,dx = 0 \qquad G_2(x^4) = \int_{-1}^{1} x^4\,dx = \frac{2}{5} \qquad G_2(x^5) = \int_{-1}^{1} x^5\,dx = 0$

$\therefore G_2(f) = \alpha_0 f(x_0) + \alpha_1 f(x_1) + \alpha_2 f(x_2)$

$$\left.\begin{aligned}
\alpha_0 + \alpha_1 + \alpha_2 &= 2 \\
\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 &= 0 \\
\alpha_0 x_0^2 + \alpha_1 x_1^2 + \alpha_2 x_2^2 &= 2/3 \\
\alpha_0 x_0^3 + \alpha_1 x_1^3 + \alpha_2 x_2^3 &= 0 \\
\alpha_0 x_0^4 + \alpha_1 x_1^4 + \alpha_2 x_2^4 &= 2/5 \\
\alpha_0 x_0^5 + \alpha_1 x_1^5 + \alpha_2 x_2^5 &= 0
\end{aligned}\right\}$$

By Symmetry:

$x_1 = -x_2$ and $\alpha_1 = \alpha_2$

$\therefore \alpha_0 + 2\alpha_1 = 2$

$\alpha_0 x_0 = 0 \quad \therefore \alpha_0 = 0$ or $x_0 = 0$

$\alpha_0 \neq 0$ as it is a 3-node $\therefore$

$x_0 = 0$

$$\left.\begin{aligned}
\therefore\ \alpha_1 x_1^2 &= 1/3 \\
\alpha_1 x_1^4 &= 1/5
\end{aligned}\right\} \quad \therefore x_1^2 = 3/5 \quad \therefore x_1 = \sqrt{3/5}$$

$\therefore x_2 = -\sqrt{3/5}$

$\therefore \alpha_1 x_1^2 = 1/3 \quad \therefore \Rightarrow \alpha_1 = \frac{5}{9} = \alpha_2$

$\alpha_0 + \alpha_1 + \alpha_2 = 2 \quad \therefore \quad \alpha_0 = 8/9$

$\therefore$ We have

$$G_2(f) = \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) + \frac{5}{9} f\left(-\sqrt{3/5}\right)$$

$\therefore$ For the interval $[a,b]$:

$$G_n(f) = \frac{1}{2}(b-a) \sum_{i=0}^{n} \alpha_i f\left(\frac{1}{2}(a+b) + x_i \frac{1}{2}(b-a)\right)$$

where $\alpha_i = \begin{cases} 8/9 & i=0 \\ 5/9 & i=1,2 \end{cases}$

$$x_i = \begin{cases} 0 & i=0 \\ \sqrt{3/5} & i=1 \\ -\sqrt{3/5} & i=2 \end{cases}.$$

3.6) a) $y'(t) = (y(t))^{1/5}$, $y(0)=0$

$\therefore \frac{dy}{dt} = y^{1/5}$ $\quad \therefore \int y^{-1/5} \frac{dy}{dt} . dt = \int 1 dt$

$\therefore y^{-1/5} \frac{dy}{dt} = 1$ $\quad \therefore \frac{5}{4} y^{4/5} = t + C$

$y(0)=0 \therefore C=0$

$\therefore y(t) = \frac{4}{5} t^{5/4}$ //.

b) $u_{n+1} = u_n + h u_n^{1/5}$

$u_0 = y(0)$

$u_1 = u_0 + h u_0^{1/5}$ $\therefore$ note that $u_0^{1/5} = 0$ $\therefore u_1 = u_0$

$u_2 = u_1 + h u_1^{1/5}$. $\therefore u_2 = u_0 + h u_0^{1/5} = u_0$.

$\therefore$ we see that $u_n = u_0 \ \forall n \in \mathbb{N}$. $\therefore$ It doesn't work.

It doesn't work due to the condition $y(0)=0$ //.