LINEAR ALGEBRAS

1. Chapter: Integers and Polynomials

1、证明：
反证法. 假设$a \mid b+c$,又因为$a \mid b$, 所以$a \mid b+c-b=c$,与$a \nmid c$矛盾. 假设不成立,即$a \nmid b+c$. $\square$

2、解：
(1) $222=187\times 1+35$, $187=35\times 5+12$, $35=12\times 2+11$, 12=11+1, $\therefore (222,187)=1$.
(2) $11137=851\times 13+74$, $851=74\times 11+37$, $74=37\times 2$, $\therefore (11137,851)=37$.
(3) $691=159\times 4+55$, $159=55\times 2+49$, $55=49+6$, $49=6\times 8+1$, $\therefore (691,159)=1$.

3、解：
(1) $(314,159)=1,\ 1=-40\times 314+79\times 159;$
(2) $(11137,559)=43,\ 43=-11137+20\times 559;$
(3) $(2046,2008)=2,\ 2=-317\times 2046+323\times 2008.$

4、解：$57=(111001)_2$, $128=(10000000)_2$, $567=(1000110111)_2$, $2001=(11111010001)_2$.

5、证明：
(1)当$n<m$时, 因为$m>2$,我们有$2^m-1>2^n+1$,因此$2^m-1\nmid 2^n+1$.
(2)当$n\geqslant m$时, 由带余除法有$n=m*q+r, q\in\mathbb{Z},\ 0\leqslant r<m$. 易知$2^m-1\mid 2^{m*q}-1$,假设$2^m-1\mid 2^n+1$, 则$2^m-1\mid 2^{mq}-1+2^n+1=2^{mq}(2^r+1)$. 但是$(2^m-1,2^{mq})=1$,所以$2^m-1\mid 2^r+1$, 与(1)矛盾. 所以假设不成立, 即$2^m-1\nmid 2^n+1$. $\square$

6、证明：
$'\Rightarrow'$: 假设方程$ax+by=c$有整数解, 则存在$m,n\in\mathbb{Z}$使得$am+bn=c$. 我们有$(a,b)\mid a,\ (a,b)\mid b,\ \Rightarrow (a,b)\mid am+bn=c$.
$'\Leftarrow'$: 记$d=(a,b)$, 由带余除法知存在$u,v\in\mathbb{Z}$,使得$d=au+bv$. 因为$d=(a,b)\mid c$,所以$c=d\times q,\ q\in\mathbb{Z}$. 则$c=d\times q=a\times qu+b\times qv$. 特别地，方程$ax+by=c$有整数解. $\square$

7、解：
设1分、5分、10分的数量分别为x,y,z. 由题意有$x+y+z=13,\ x+5y+10z=83$.由带余除法知$83=16\times 5+3$, 推出$x=5\times q+3$,又因为$x\leqslant 13$, 所以$x=3,8,13$. 分别带入方程可得$x=3,y=4,z=6$.

8、证明：$(d\neq 0)$
$d=(a,b)$, 存在$u,v\in\mathbb{Z}$ 使得$d=au+bv=da'u+db'v$, 推出$1=a'u+b'v$, 所以$(a',b')=1$. $\square$

9、证明：(1)(考虑$d\neq 0$)
方法1：设$m=(da_1,da_2,\cdots,da_n),\ l=(a_1,a_2,\cdots,a_n)$, 由定义可知存在

$$k_1,k_2,\cdots,k_n,p_1,p_2,\cdots,p_n\in\mathbb{Z}$$

使得$da_i = mk_i, a_i = lp_i, i = 1, 2, \ldots, n$. 则$da_i = dlp_i$, 即$dl \mid da_i, \therefore dl \mid m$. 不妨假设$m = dl \times q, q \in \mathbb{Z}$, 带入$da_i = m \times k_i = dlq \times k_i$, 因为$d \neq 0$, 所以$a_i = lq * k_i$, 特别的, $lq \mid a_i$, 所以$lq \mid l$ (l 为最大公因子), 推出$\mid q \mid = 1$, 即$m = \mid d \mid l$.
方法2：设$m = (da_1, da_2, \cdots, da_n)$, $l = (a_1, a_2, \cdots, a_n)$, 则存在$k_i \in \mathbb{Z}$, 使得$l = a_1k_1 + a_2k_2 + \cdots + a_nk_n$. 两边同时乘以$d$有, $dl = da_1k_1 + da_2k_2 + \cdots + da_nk_n$, 易知, $dl \mid m$. 又因$m \mid da_i$, 所以$m \mid da_1k_1 + da_2k_2 + \cdots + da_nk_n = dl$, 所以$m = \mid d \mid l$.
(2) $b_i = ca_i$, 由(1)知，$(b_1, b_2, \cdots, b_n) = \mid c \mid (a_1, a_2, \cdots, a_n) = 1$, 推出$c \mid 1, \therefore \mid c \mid = 1$.
□

10、证明；
设$(u+v, u-v) = d$, 则存在$k_1, k_2 \in \mathbb{Z}$使得$u+v = dk_1, u-v = dk_2$, 推出$2u = d(k_1+k_2), 2v = d(k_1-k_2)$,$2 = (2u, 2v) = d(k_1+k_2, k_1-k_2)$,
$\therefore d \mid 2,\ i.e. d = 1\ or\ 2$. □

11、证明：
$(a, a+k) \mid a,\ (a, a+k) \mid a+k, \Rightarrow (a, a+k) \mid a+k-a = k$. □

12、证明：
由算术基本定理, $n = p_1^{r_1}p_2^{r_2}\cdots p_m^{r_m}$, 其中$p_i$为互不相同的素数, 则存在$k_1, k_2, \cdots, k_m$使得$r_i = 2k_i,\ or\ r_i = 2k_i + 1$. 令$b = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}, a = n/b^2$则满足题目要求. □

13、证明：
由算术学基本定理有, $u = p_1^{r_1}p_2^{r_2}\cdots p_n^{r_n}$, $v = q_1^{s_1}q_2^{s_2}\cdots q_m^{s_m}$, p_i, q_j 为互不相同的素数. 因为$(u, v) = 1$, 则任意的$(p_i,\ q_j) = 1$. 但是$uv = a^2 = \prod_{i=1..n,\ j=1..m} p_i^{r_i}q_j^{s_j}$, 推出$2 \mid r_i, 2 \mid s_j$, 特别的, u, v为平方数. □

14、证明：
$a = \pm p_1^{r_1}p_2^{r_2}\cdots p_n^{r_n}, b = \pm p_1^{s_1}p_2^{s_2}\cdots p_n^{s_n}$. 记$t_i = min(r_i, s_i)$, 则$d = p_1^{t_1}p_2^{t_2}\cdots p_n^{t_n} \mid a,\ d \mid b,\ \therefore d \mid (a, b)$. 假设$(a, b) = d * k, k > 1$, 由$k \mid a, k \mid b$以及算术学基本定理知, 存在$p_i \mid k$. 不妨假设$p_1 \mid k$, 则$p_1^{t_1+1} \mid (a, b)$, 特别的, $p_1^{t_1+1} \mid a$, $p_1^{t_1+1} \mid b$, 矛盾. 所以$(a, b) = d$. □

15、证明：
$(1)' \Rightarrow'$: $a \equiv b (mod\ m) \Rightarrow m \mid (a-b), a = m \times q_1 + r_1, b = m \times q_2 + r_2, a - b = m(q_1 - q_2) + (r_1 - r_2)$, 由$m \mid (a-b)$有, $m \mid (r_1 - r_2)$, 但是$0 \leq r_i < m$, 所以$r_1 = r_2$.
$' \Leftarrow'$: $a = m \times q_1 + r_1, b = m * q_2 + r_2, r_1 = r_2$, 推出$a - b = m(q_1 - q_2)$, 特别的, $m \mid (a-b)$.
(2) $a \equiv 0 (mod\ 0) \Leftrightarrow m \mid (a - 0) = 0$.
(3) $a \equiv a_1 (mod\ m), b \equiv b \equiv b_1 (mod\ m) \Rightarrow m \mid (a - a_1),\ m \mid (b - b_1) \Rightarrow m \mid (a - a_1) + (b - b_1) = (a+b) - (a_1 + b_1) \Rightarrow a + b \equiv a_1 + b_1 (mod\ m)$.
$m \mid (a - a_1), m \mid (b - b_1) \Rightarrow m \mid (a - a_1)b, m \mid (b - b_1)a_1 \Rightarrow m \mid (a - a_1)b + (b - b_1)a_1 = ab - a_1b_1 \Rightarrow ab \equiv a_1b_1 (mod\ m)$.
(4)$(a, m) = 1$, 由带余除法可知, 存在$u, v \in \mathbb{Z}$使得$1 = ua + vm$, 推出$ua - 1 = vm \Rightarrow m \mid ua - 1 \Rightarrow ua \equiv 1 (mod\ m)$.
(5)$(3, 2006) = 1 \Rightarrow 1 = 669 * 3 - 2006 \Rightarrow 3 * 669 \equiv 1 (mod\ 2006)$. □

16、解：
由题意可设$x\equiv 2(mod\ 7)$, 求$x+276\equiv ?(mod\ 7)$即可. 易知$x+276\equiv 5(mod\ 7)$.

17、解：
(1) 设$N=x_0+10x_1+10^2x_2+\cdots+10^kx_k$, $9\mid N\Leftrightarrow N\equiv 0(mod\ 9)$, 易知$10^i\equiv 1(mod\ 9), i\geq 0$. 所以$N\equiv x_0+x_1+\cdots+x_k(mod\ 9)$, 特别的, $N\equiv 0(mod\ 9)\Leftrightarrow x_0+x_1+\cdots+x_k\equiv 0(mod 9)$.
(2) $10^i\equiv(-1)^i(mod\ 11)\Rightarrow N\equiv x_0-x_1+\cdots+(-1)^kx_k(mod 11)$. □

18、证明：
(1) 因为p为素数, 所以对于任意的$0\leq i<p$, $i\nmid p$. 因此$p\mid C_p^i$. 特别地, $C_p^i\equiv 0(mod\ p)$.
(2) $(a+b)^p=a^p+C_p^1a^{p-1}b+\cdots+C_p^{p-1}ab^{p-1}+b^p$. 由(1)知, $C_p^i\equiv 0(mod\ p)$, 所以$(a+b)^p\equiv a^p+b^p(mod\ p)$.
(3)由(2)知$a^p=(a-1+1)^p\equiv(a-1)^p+1^p$, 如果$a>0$, 那么$a^p=1^p+\cdots+1^p=a$. 同理可知当$a\leq 0$时也成立. □

19、证明：$(n>1)$
$a^n-1=(a-1)(a^{n-1}+a^{n-2}+\cdots+a+1)$, 如果$a\neq 2$, 则$a^n-1$为合数,与题意矛盾.因此$a=2$. 又如果$n$不是素数,则可设$n=p*q$, $2^n-1=(2^p-1)((2^p)^{q-1}+(2^p)^{q-2}+\cdots+1)$.此时$2^n-1$不为合数,与题意矛盾. 所以$n$为素数. □

20、证明：
假设a为奇数，a^n为奇数，a^n+1为偶数与题意矛盾.所以a为偶数。又如果n不是2的方幂，则存在素数$p\neq 2$,使得$p\mid n$. 令$p*q=n$,$a^n+1=(a^q+1)((a^q)^{p-1}+(a^q)^{p-2}+\cdots+1)$与题意矛盾. □

21、证明：
(1)$1=1+0*\sqrt{-1}\in\mathbb{Q}(\sqrt{-1})$,
(2)$\forall a_1+b_1\sqrt{-1}, a_2+b_2\sqrt{-1}\in\mathbb{Q}(\sqrt{-1})$,则

$$a_1+b_1\sqrt{-1}+a_2+b_2\sqrt{-1}=(a_1+a_2)+(b_1+b_2)\sqrt{-1}\in\mathbb{Q}(\sqrt{-1}),$$

$$a_1+b_1\sqrt{-1}-a_2-b_2\sqrt{-1}=(a_1-a_2)+(b_1-b_2)\sqrt{-1}\in\mathbb{Q}(\sqrt{-1}),$$

$$(a_1+b_1\sqrt{-1})*(a_2+b_2\sqrt{-1})=(a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)\sqrt{-1}\in\mathbb{Q}(\sqrt{-1}),$$

(3) 如果$a_2+b_2\sqrt{-1}\neq 0$, 则

$$(a_1+b_1\sqrt{-1})/(a_2+b_2\sqrt{-1})=(a_1a_2+b_1b_2)/(a_2^2+b_2^2)+(a_2b_1-a_1b_2)\sqrt{-1}/(a_2^2+b_2^2)\in\mathbb{Q}(\sqrt{-1}).$$

所以$\mathbb{Q}(\sqrt{-1})$为数域. □

22、证明：
设F为任意包含$\sqrt{5}$的数域,则$\mathbb{Q}\subset F$, $\sqrt{5}\in F$,由数域的定义可知$b\sqrt{5}\in F,\forall b\in\mathbb{Q}$, $a+b\sqrt{5}\in F,\forall a\in\mathbb{Q}$,所以$\mathbb{Q}(\sqrt{5})\subset F$. □

23、证明：
(1)$\zeta=e^{2\pi i/n}$, $\zeta^n=1$.

如果$n \mid k, \zeta^{ik}=1, \forall i=0,1,\cdots,n-1$.因此$\sum\limits_{i=0}^{n-1}\zeta^{ik}=n$.
如果$n \nmid k$,那么$\zeta^k-1\neq 0$. 我们有$\Sigma_{i=0}^{n-1}\zeta^{ik}=(\zeta^{nk}-1)/(\zeta^k-1)=0$.
(2)由带余除法易知，连续的n个自然数有且仅有一个数恰被n整除。因此在$-k,1-k,\cdots,n-1-k$中只有一个j使得$n\mid j-k$,不妨设j_0使得$n\mid j_0-k$,则
$s=1/n\sum\limits_{i=0}^{n-1}\sum\limits_{j=0}^{n-1}\zeta^{i(j-k)}=1/n\sum\limits_{i=0}^{n-1}\zeta^{i(j_0-k)}+1/n\sum\limits_{i=0}^{n-1}\sum\limits_{j\neq j_0}\zeta^{i(j-k)}=1.\ (by(1))$ □

24、证明：

$$LHS=\sum_{i+r=t}a_i\sum_{j=0}^{r}b_jc_{r-k}=\sum_{i=0}^{t}a_i\sum_{j=0}^{t-i}b_jc_{t-i-k}$$
$$RHS=\sum_{i=0}^{t}a_i\sum_{j+k=t-i}b_jc_k=\sum_{i=0}^{t}a_i\sum_{j=0}^{t-i}b_jc_{t-i-k}$$

$\therefore LHS=RHS.$ □

25、解：
(1) $f(x)=2x^5+x^3-2x^2-7x+2,\ g(x)=x^3-x^2-2,$
$$f(x)=(2x^2+2x+3)g(x)+5x^2-3x+8;$$
(2) $f(x)=x^4+2x,\ g(x)=x^2-1,$
$$f(x)=(x^2+1)g(x)+2x+1.$$
□

26、证明：
因为$f(x)\mid g(x),\ f(x)\mid h(x)$, 不妨设$g(x)=f(x)k_1(x),\ h(x)=f(x)k_2(x)$.
$$g(x)u(x)+h(x)v(x)=f(x)(k_1(x)u(x)+k_2(x)v(x))$$
因此$f(x)\mid g(x)u(x)+h(x)v(x)$. □

27、证明：
由带余除法可知，$x^2-m+1\mid x^4+px^2-q \Leftrightarrow\ r(x)=-q-(p+m-1)(1-m)=0$
所以$(m-1)^2+p(m-1)-q=0$. □

28、解：
(1) $x^4=1+4(x-1)+6(x-1)^2+4(x-1)^3+(x-1)^4$;
(2) $2x^3-7x^2+x-11=-15-7(x-1)-(x-1)^2+2(x-1)^3$. □

29、解：
(1) $f(x)=2x^3+5x+1,\ g(x)=x^3-x^2+2x+6,$
$f(x)=2g(x)+2x^2+x-11;$
$g(x)=(2x^2+x-11)(1/2x-3/4)+33/4x-9/4;$
$2x^2+x-11=(33/4x-9/4)(8/33x+68/363)-1280/121;$
$\therefore (f(x),g(x))=1,\ 1=(-11/960x^2+x/120-13/160)f(x)+(11/480x^2+x/160+$

$173/960)g(x)$;

$(2)f(x)=x^6-x^4+2x^3+x^2+1,\ g(x)=x^4+x$;
$f(x)=g(x)(x^2-1)+x^3+x^2+x+1$;
$g(x)=(x^3+x^2+x+1)(x-1)+x+1$;
$x^3+x^2+x+1=(x^2+1)(x+1)$;
$\therefore (f(x),g(x))=(x+1)$;
$x+1=(1-x)f(x)+(x^3-x^2-x+2)g(x)$;

$(3)f(x)=x^4-8x^2+1,\ g(x)=x^3+\sqrt{3}x^2-2\sqrt{3}$;
$f(x)=g(x)(x-\sqrt{3})-5x^2+2\sqrt{3}x-5$;
$(f(x),g(x))=1.$ □

30、证明：
记$d(x)=(f(x),g(x))$, 对任意的公因子$h(x)\mid f(x),\ h(x)\mid g(x)$, 由最大公因子的定义可知$h(x)\mid d(x)$, 特别的,$\exists u(x),\ s.t.\ d(x)=h(x)u(x), \Rightarrow deg\ h(x)\le deg\ d(x)$. □

31、证明：
$d(x)=f(x)u(x)+g(x)v(x),\ f(x)=d(x)f_1(x),\ g(x)=d(x)f_2(x), \Rightarrow d(x)=d(x)u(x)f_1(x)+d(x)v(x)g_1(x)\ \Rightarrow 1=u(x)f_1(x)+v(x)g_1(x)(since\ d(x)\neq 0)\ \Rightarrow (f_1(x),g_1(x))=1,\ (u(x),v(x))=1.$ □

32、证明：
(1)记$d(x)=(f(x),g(x)),\ f(x)=d(x)f_1(x),\ g(x)=d(x)g_1(x)$,

$$f_1(x)=\frac{f(x)}{(f(x),g(x))}, g_1(x)=\frac{g(x)}{(f(x),g(x))}.$$

由带余除法知存在,$u(x),v(x),s.t.\ d(x)=u(x)f(x)+v(x)g(x)$,

$$\Rightarrow 1=u(x)f_1(x)+v(x)g_1(x),\ (f_1(x),g_1(x))=1.$$

如果$deg\ u(x)\ge deg\ g_1(x)$, 由带余除法知,$u(x)=g_1(x)q(x)+r(x),\ deg\ r(x)<deg\ g_(x)\ or\ r(x)=0$. 如果$r(x)=0$, $(v(x)+f(x)q(x))g_1(x)=1$,因为$f_1(x),g_1(x)$不为常数,所以矛盾. 即$deg\ r(x)<deg\ g_(x)$, 则$r(x)f_1(x)+(v(x)+q(x))g_1(x)=1$, $deg\ r(x)f_1(x)<deg\ f_1(x)+deg\ g_1(x), \Rightarrow deg\ v(x)+q(x)<deg\ f_1(x)$, 令$u_0(x)=r(x),v_0(x)=v(x)+q(x)$即满足题意.
(2)设$u(x)f(x)+v(x)g(x)=d(x)$,则有$u(x)f_1(x)+v(x)g_1(x)=1$,
由(1)知,$u_0(x)f_1(x)+v_0(x)g_1(x)=1$. $(u(x)-u_0(x))f_1(x)=-(v(x)-v_0(x))g_1(x)$, $\because (f_1(x),g_1(x))=1, \therefore f_1(x)\mid v(x)-v_0(x), g_1(x)\mid u(x)-u_0(x)$. 设$v(x)-v_0(x)=f_1(x)h_1(x),\ u(x)-u_0(x)=g_1(x)h_2(x), \Rightarrow g_1(x)h_2(x)f_1(x)=-f_1(x)h_1(x)g_1(x)$, $\therefore h_1(x)=-h_2(x)$, 令$h(x)=h_2(x)$即可. □

33、证明：
$d(x)\mid f(x),\ d(x)\mid g(x),\ d(x)=u(x)f(x)+v(x)g(x)$. 如果$h(x)\mid f(x),\ h(x)\mid g(x)$, 那么$h(x)\mid u(x)f(x)+v(x)g(x)=d(x)$, 所以$d(x)$是$f(x),g(x)$的一个最大公因子. □

34、证明：
(1)由带余除法可知存在$u_0(x)$, $v_0(x), s.t.$ $(f(x),g(x)) = u_0(x)f(x) + v_0(x)g(x)$, 因此$(f(x),g(x)) \in I$. 因为$(f(x),g(x)) \mid f(x)$, $(f(x),g(x)) \mid g(x)$,所以$(f(x),g(x)) \mid u(x)f(x)+v(x)g(x), \forall\, u(x),v(x) \in \mathbb{F}[x]$. 特别地,$(f(x),g(x)) \mid d(x)$, $deg\ (f(x),g(x)) \leq deg\ d(x)$.因为$d(x)$是I中首项系数为1次数最低的，所以$(f(x),g(x)) = d(x)$.
(2)由(1)知对任意的$u(x)f(x) + v(x)g(x) \in I$, $d(x) \mid u(x)f(x) + v(x)g(x)$. 因此存在$h(x), s.t.\ u(x)f(x)+v(x)g(x) = d(x)h(x), \therefore I \subset \{d(x)h(x)|h(x) \in \mathbb{F}[x]\}$.又对任意的$h(x)d(x) = h(x)u_0(x)f(x) + h(x)v_0(x)g(x) \in I, \therefore \{d(x)h(x)|h(x) \in \mathbb{F}[x]\} \subset I$.所以$I = \{d(x)h(x)|h(x) \in \mathbb{F}[x]\}$.
(3)$d(x) = u_0(x)f(x) + v_0(x)g(x), \therefore d(x)h(x) = h(x)u_0(x)f(x) + h(x)v_0(x)g(x)$,易知$d(x)h(x) \mid f(x)h(x), d(x)h(x) \mid g(x)h(x)$,且$h(x)$为首项系数为一，所以$d(x)h(x) = (f(x)h(x), g(x)h(x))$. □

35、证明：
$x - c_1, x - c_2$为不可约多项式,且$(x - c_1.x - c_2) = 1$,所以$c_1 \neq c_2$.所以$((x - c_1)^n, (x - c_2)^n) = 1$. （注：一般结论，如果$(f(x), g(x)) = 1$,那么$(f(x)^n, g(x)^m) = 1$.） □

36、证明：
$f(x)^n \mid g(x)^n \Rightarrow g(x)^n = f(x)^n k(x)$.设$d(x) = (f(x), g(x))$, $f(x) = d(x)f_1(x), g(x) = d(x)g_1(x), (f_1(x), g_1(x)) = 1. \Rightarrow f(x)^n = d(x)^n f_1(x)^n$, $g(x)^n = d(x)^n g_1(x)^n \Rightarrow d(x)^n f_1(x)^n k(x) = d(x)^n g_1(x)^n \Rightarrow f_1(x)^n k(x) = g_1(x)^n \Rightarrow f_1(x)^n \mid g_1(x)^n$,但是

$$(f_1(x), g_1(x)) = 1 \Rightarrow (f_1(x)^n, g_1(x)^n) = 1.$$

所以$f_1(x) = c \in \mathbb{F}$.特别的,$f(x) \mid g(x)$. □

37、证明：
(1)记$d(x) = (f(x), g(x)), f(x) = d(x)f_1(x), g(x) = d(x)g_1(x), (f_1(x), g_1(x)) = 1$.
$\frac{f(x)g(x)}{d(x)} = f_1(x)d(x)g_1(x) \Rightarrow f(x) \mid \frac{f(x)g(x)}{d(x)}$, $g(x) \mid \frac{f(x)g(x)}{d(x)}$.即$\frac{f(x)g(x)}{d(x)}$为$f(x), g(x)$的公倍数,下证$\frac{f(x)g(x)}{d(x)}$是最小的公倍数. 设$m(x)$是$f(x), g(x)$的任意的公倍数, 则$m(x) = f(x)s(x) = g(x)t(x) \Rightarrow f_1(x)d(x)s(x) = g_1(x)d(x)t(x) \Rightarrow f_1(x)s(x) = g_1(x)t(x) \Rightarrow f_1(x) \mid g_1(x)t(x)$,因为$(f_1(x), g_1(x)) = 1, \therefore f_1(x) \mid t(x)$, $i.e.\ t(x) = f_1(x)t_1(x) \Rightarrow m(x) = g_1(x)d(x)f_1(x)t_1(x)$, $i.e. \frac{f(x)g(x)}{d(x)} \mid m(x)$.所以$[f(x), g(x)] = \frac{f(x)g(x)}{d(x)}$.
(2)$f_i(x) \mid m(x) \Rightarrow [f_1(x), f_2(x)] \mid m(x) \Rightarrow [[f_1(x), f_2(x)], f_3(x)] \mid m(x) \cdots \Rightarrow [f_1(x), f_2(x), \cdots f_n(x)] \mid m(x)$.
(3)$(f(x), g(x)) = x + 1, [f(x), g(x)] = \frac{f(x)g(x)}{(f(x),g(x))} = (x^2 + 1)(x + 1)(x^2 - 2)$. □

38、证明：
因为$\mathbb{Q} \subset \mathbb{F}, \forall c \in \mathbb{Q}, f(x) = x-c$为不可约多项式,且若$c_1 \neq c_2$,则$(x-c_1, x-c_2) = 1$.所以$\mathbb{F}[x]$中有无限多不可约多项式. □

39、证明；
$p(x)$为不可约多项式,所以对任意的$f(x) \in \mathbb{F}[x], (p(x), f(x)) = 1$,或者$p(x) \mid f(x)$.假设对任意的$f_i(x), p(x) \nmid f_i(x) \Rightarrow (p(x), f_i(x)) = 1 \Rightarrow (p(x), f_1(x)f_2(x) \cdots f_n(x)) = 1$,与$p(x) \mid f_1(x)f_2(x) \cdots f_n(x)$矛盾,所以假设不成立,即存在$f_i(x)$, $s.t.\ p(x) \mid f_i(x)$.(Remark:也

可以用归纳法证明) □

40、解：
(1) $f'(x)=3px^{p-1}+3px^2$;
(2)$f'(x)=4x^3+12x^2+12x+4$;
(3)$f'(x)=(x^2-2)^2(7x^2-600x-2)$;
(4)$f'(x)=8x^3-6x^2-40x+66$. □

41、证明；
(1)$g(x)=(x-a_1)(x-a_2)\cdots(x-a_n),\ g'(x)=\sum\limits_{i=1}^{n}\frac{g(x)}{x-a_i}$.
$F(x)=\sum\limits_{i=1}^{n}\frac{g(x)}{(x-a_i)g'(a_i)}$.
考虑多项式$G(x)=F(x)-1$,则易知$G(a_i)=0,\forall i=1,2,...n$.即$a_1,a_2,\cdots,a_n$为$G(x)$的n个根，但是$deg\ G(x)\leq n-1$,由代数学基本定理知$G(x)=0,\ i.e.\ F(x)=1$.
(2)令$r(x)=\sum\limits_{i=1}^{n}\frac{f(a_i)g(x)}{(x-a_i)g'(a_i)}$.
$K(x)=f(x)-r(x)$,易知$K(a_i)=0,\forall i=1,2,...n$.特别的$a_1,a_2,\cdots,a_n$为$K(x)$的n个根, 所以$(x-a_i)\mid K(x)$,又因为$a_i$互不相同,所以$g(x)\mid K(x)$. $\Rightarrow K(x)=g(x)h(x)\Rightarrow f(x)=g(x)h(x)+r(x)$,而$deg\ r(x)<deg\ g(x)$所以$r(x)$为$f(x)$除以$g(x)$的余项.
(3)$L(x)=\sum\limits_{i=1}^{n}\frac{b_ig(x)}{(x-a_i)g'(a_i)}$.容易验证$L(a_i)=b_i$. 若$deg\ f(x)<n$,且满足$f(a_i)=b_i$,考虑$r(x)=L(x)-f(x)$,则$r(a_i)=0$, 特别的$a_i$为多项式的$r(x)$的n个根,但是$deg\ r(x)<n$,由代数基本定理知$r(x)=0,\ i.e.\ f(x)=L(x)$.
(4)设$g(x)=(x-1)(x-2)(x-3)(x-4)$,则$L(x)=-\frac{1}{2}(x-1)(x-4)$. □

42、解：
$f(x)=(x-3)(x^5-2x^4-3x^2+6x+1)$

43、解：
$f(x)=x^4+ax+b,f'(x)=4x^3+a$. $f(x)$有重根当且仅当$(f(x),f'(x))\neq 1$.由带余除法知当$27a^4=2^8b^3$时$f(x)$有重根.

44、证明：
$f(x)=\frac{x^n}{n!}+\frac{x^{n-1}}{(n-1)!}+\cdots+x+1,f'(x)=\frac{x^{n-1}}{(n-1)!}+\cdots+x+1$. 假设$f(x)$有重根$\alpha$,则$f(\alpha)=0,f'(\alpha)=0$.推出$f(\alpha)-f'(\alpha)=\frac{\alpha^n}{n!}=0$.则有$\alpha=0$,但是$f(0)\neq 0$.矛盾,所以$f(x)$没有重根. □

45、证明：(也可以用唯一分解定理来证明)
(1)易知$\frac{f(x)}{(f(x),f'(x))}$的根必为$f(x)$的根.设$x-c$为$f(x)$的k重因式,则$(x-c)^{k-1}\mid f'(x),(x-c)^k\nmid f'(x)$,推出$(x-c)^{k-1}\mid(f(x),f'(x)),(x-c)^k\nmid(f(x),f'(x))$.推出$(x-c)\mid\frac{f(x)}{(f(x),f'(x))},(x-c)^2\nmid\frac{f(x)}{(f(x),f'(x))}$.所以$\frac{f(x)}{(f(x),f'(x))}$没有重根.
(2)由(1)可知,对任意的复根c,$x-c\mid f(x)\Rightarrow x-c\mid\frac{f(x)}{(f(x),f'(x))}$,所以$f(x)$的根皆为$\frac{f(x)}{(f(x),f'(x))}$的根. 又$\frac{f(x)}{(f(x),f'(x))}$的根必为$f(x)$的根.所以他们的根在不计重数的情况下

是一致的. □

46、解：
$(f(x), f'(x)) = x^8 + x^6 + 4x^5 + 4x^3 + 4x^2 + 4, h(x) = \frac{f(x)}{(f(x), f'(x))} = x^5 + x^3 + 2x^2 + 2.$

47、证明：
We should prove $f'(x) \mid f(x) \Leftrightarrow f(x) = a(x-b)^n$.
" $\Leftarrow$ " : is trivial.
" $\Rightarrow$ " : $f'(x) \mid f(x) \Rightarrow (f(x),\ f'(x)) = cf'(x)$ for some $c \in \mathbb{F}$. By (45), f(x) and $\frac{f(x)}{(f(x), f'(x))}$ have the same roots over $\mathbb{C}$, but deg $\frac{f(x)}{(f(x), f'(x))} = 1$ and $\frac{f(x)}{(f(x), f'(x))} \in \mathbb{F}[x]$, which implies $\frac{f(x)}{(f(x), f'(x))} = a(x-b)$ for some $a,\ b \in \mathbb{F}$. Therefore, $f(x) = a(x-b)^n$.
□

48、证明：
设$f(x), g(x), f_1(x), g_1(x) \in \mathbb{F}[x] \subset \mathbb{C}[x]$. 在$\mathbb{F}[x]$中$(f(x), g(x)) = 1$, 则存在$u(x), v(x) \in \mathbb{F}[x] \subset \mathbb{C}[x], s.t.\ u(x)f(x) + v(x)g(x) = 1$, 则推出在$\mathbb{C}[x]$中$(f(x), g(x)) = 1$.因此对任意的$x-c \mid f(x), c \in \mathbb{C}, x-c \nmid g(x)$. 因为$f_1(x)$的根都是$f(x)$的根,所以$(f_1(x), g(x)) = 1$.同理可知$(f(x), g_1(x)) = 1$.由$f(x)g(x) = f_1(x)g_1(x)$知,$f(x) \mid f_1(x), f_1(x) \mid f(x)$,又因为$f(x), g(x), f_1(x), g_1(x)$为首1的多项式,所以$f(x) = f_1(x), g(x) = g_1(x)$. □

49、证明：
设α为$f(x)$的根, 则由$f(x) \mid f(x^n)$知α^n也是$f(x)$的根,所以$\alpha^{n^2}, \alpha^{n^3}, \cdots$皆是$f(x)$的根.但是$f(x)$为有限次多项式, 由代数学基本定理知必定存在$s, t$使得$\alpha^{n^s} = \alpha^{n^t}$,特别的,$f(x)$的根为0或者是n次单位根. □

50、证明：
(1)$f(x) \mid g(x) \Rightarrow g(x) = f(x)h(x) \Rightarrow \overline{g(x)} = \overline{f(x)} \cdot \overline{h(x)} \Rightarrow \overline{f(x)} \mid \overline{g(x)}$.
(2)考虑多项式$f(x) = x^2 + 1, \overline{f(x)} = f(x)$,但是$f(x)$的根为虚数. □

51、解：
Over $\mathbb{C}$:

$$f(x) = \prod_{k=1}^{n} (x - e^{i\frac{2k\pi}{n}})$$

Over $\mathbb{R}$:
if $n = 2m$

$$f(x) = (x-1)(x+1) \prod_{k=1}^{m-1} (x^2 - 2cos\frac{k\pi}{m}x + 1),$$

if $n = 2m+1$

$$f(x) = (x-1) \prod_{k=1}^{m} (x^2 - 2cos\frac{2k\pi}{n}x + 1);$$

□

52、解：
Over $\mathbb{R}$:

if $n = 2m$

$$f(x) = (x - \sqrt[n]{2})(x + \sqrt[n]{2}) \prod_{k=1}^{m-1} (x^2 - 2\sqrt[n]{2} cos\frac{2k\pi}{n} x + 1),$$

if $n = 2m+1$

$$f(x) = (x - \sqrt[n]{2}) \prod_{k=1}^{m} (x^2 - 2\sqrt[n]{2} cos\frac{2k\pi}{n} x + 1),$$

$f(x)$ is irreducible over $\mathbb{Q}$. □

53、证明：
因为$f(x) \in \mathbb{R}[x]$,所以$x - c \mid f(x) \Rightarrow x - \bar{c} \mid \overline{f(x)} = f(x)$. 假设$f(x)$没有实根,则$deg\ f(x)$必定为偶数,与$deg\ f(x)$为奇数矛盾,所以$f(x)$至少有一个实根. □

54、证明：
设$f(x) = a\prod_{i=1}^{k}(x-c_i)^{t_i} \cdot \prod_{j=1}^{l} p_j(x)^{s_j}$, $a \in \mathbb{R}$, 其中$p_j(x)$为$\mathbb{R}[x]$上的首项系数为1 的2次不可约多项式. 易知$a \geq 0$.
记$g(x) = \prod_{i=1}^{k}(x-c_i)^{t_i}$, $h(x) = \prod_{j=1}^{l} p_j(x)^{s_j}$.第一步我们证明$h(x)$可以写成两个实系数多项式的平方和.

(1)由假设可知,$h(x)$的根皆是虚数,故不妨设根为$\alpha_1, \alpha_2, \cdots, \alpha_l, \bar{\alpha_1}, \bar{\alpha_2}, \cdots, \bar{\alpha_l}$, $h(x) = \prod_{i=1}^{l}((x-\alpha_i)(x-\bar{\alpha_i}))^{s_j} = \prod_{i=1}^{l}(x-\alpha_i)^{s_j}(x-\bar{\alpha_i})^{s_j}$.令$h_1(x) = \prod_{i=1}^{l}(x-\alpha_i)^{s_j}$, 则$h(x) = h_1(x)\overline{h_1(x)}$.因为$\alpha_i$为复数,将$h_1(x)$展开可得$h_1(x) = u_1(x) + iv_1(x)$,其中$u_1(x), v_1(x) \in \mathbb{R}[x]$.于是$h(x) = (u_1(x) + iv_1(x))\overline{(u_1(x) + iv_1(x))} = u_1(x)^2 + v_1(x)^2$.

因为$f(x) \geq 0$,由(1)知$h(x) \geq 0$,所以$g(x) \geq 0$.下面证明$g(x)$中不可约因式的重数皆为偶数,$i.e. t_i$为偶数.

(2)假设$g(x)$中含有t_i不为偶数,重新编号设因式$x - c_i$, $i = 1, 2, ...n$为g(x)中出现的次数为奇数的所有的不可约因式,且满足$c_1 < c_2 < \cdots < c_n$.记相应的重数为t_i.取$\beta_1 < c_1$和$c_1 < \beta_2 < c_2$,计算$g(\beta_1)$和$g(\beta_2)$可知他们异号,与$g(x) \geq 0$矛盾.所以对任意的t_i皆为偶数.特别的,$g(x) = u_0(x)^2$.

由(1)(2)可知$f(x) = (u_0(x)u_1(x))^2 + (u_0(x)v_1(x))^2$. □

55、证明：
" $\Rightarrow$: " 假设$f(x)$有有理根$\alpha \in \mathbb{Q}$ 则$f(x) = (x-\alpha)g(x)$.易知$g(x) \in \mathbb{Q}[x]$,所以$f(x)$可约,与题目假设$f(x)$ 不可约矛盾,所以$f(x)$没有有理根.
" $\Leftarrow$: "假设$f(x)$可约,$f(x) = (ax+b)(\alpha x^2 + \beta x + \gamma)$,易知$x = \frac{-b}{a}$为$f(x)$的有理根,与题目假设$f(x)$无有理根矛盾,所以$f(x)$不可约.
当$deg\ f(x) > 3$时此结论不成立,如$f(x) = (x^2+1)^2$在$\mathbb{Q}$可约,但无有理根. □

56、解：
(1)$x = 1$为有理根;

(2)没有有理根;
(3)$x = 1, 2$为有理根;
(4)没有有理根. $\square$

57、解:
(1)错误,如$g(x) = x,\ f(x) = x^m$.
(2)错误,如$f(x) = x^2 + 1$. $\square$

58、证明:
(1)取$\phi(x)$为$I(\alpha)$中次数最低的首项系数为1的多项式.下面证明$\phi(x)$不可约.
假设$\phi(x)$可约,设$\phi(x) = g(x)h(x)$,则$\phi(\alpha) = g(\alpha)h(\alpha) = 0$,推出$g(\alpha) = 0$或者$h(\alpha) = 0$.不妨设$g(\alpha) = 0$,则由$I(\alpha) = 0$的定义知$g(x) \in I(\alpha)$,但是$deg\, g(x) < deg\, \phi(x)$与$\phi(x)$的取法矛盾.所以$\phi(x)$不可约.
(2)由(1)可知$\phi(x)$的次数最小,所以对任意的$f(x) \in I(\alpha)$,由带余除法知$f(x) = \phi(x)q(x) + r(x)$,$r(x) = 0$或者$deg\, r(x) < deg\, \phi(x)$.如果$r(x) \neq 0$,则$r(x) = f(x) - \phi(x)q(x), r(\alpha) = f(\alpha) - \phi(\alpha)q(\alpha) = 0$,推出$r(x) \in I(\alpha)$.但是$deg\ r(x) < deg\ \phi(x)$矛盾.所以$r(x) = 0$,特别的$\phi(x) \mid f(x)$.所以$I(\alpha) \subset \{\phi(x)h(x) | h(x) \in \mathbb{Q}[x]\}$.易知$\{\phi(x)h(x) | h(x) \in \mathbb{Q}[x]\} \subset I(\alpha)$,所以$I(\alpha) = \{\phi(x)h(x) | h(x) \in \mathbb{Q}[x]\}$.
(3)$(x-1)^3 + 2$以$1 + (-2)^{1/3}$为根. $\square$

59、证明:
"$\Rightarrow$":假设$f(ax+b)$为可约的,则不妨设$f(ax+b) = g(x)h(x)$.因为$a \neq 0$,令$y = ax + b$,有$f(y) = g(\frac{y-b}{a})h(\frac{y-b}{a})$,所以$f(x)$为可约的,矛盾.所以假设不成立.
"$\Leftarrow$:"假设$f(x)$可约,$f(x) = g(x)h(x)$,则$f(ax+b) = g(ax+b)h(ax+b)$,推出$f(ax+b)$可约,矛盾.所以$f(x)$不可约. $\square$

60、证明:
(1)$f(x) = 2x^3 + 3x^2 - x - 1, f(-\frac{1}{2}) = 0$,为$f(x)$的有理根,所以$f(x)$可约.
(2)$f(x) = x^5 + 5x^2 - 1, f(x+1) = x^5 + 5x^4 + 10x^3 + 15x^2 + 15x + 5$,取$p = 5$,由Eisenstein Criterion知$f(x)$不可约.
(3)$f(x) = x^3 - 2x + 4, f(-2) = 0$,所以$f(x)$可约.
(4)$f(x) = x^4 - 8x^3 + 2x^2 + 14x - 6$,取$p = 2$,由Eisenstein Criterion知$f(x)$不可约.
(5)$f(x) = x^4 - x^3 - 3x^2 + 5x - 2, f(1) = 0$,所以$f(x)$可约.
(6)$f(x) = x^6 - x^3 + 1, f(x-1) = (x-1)^6 - (x-1)^3 + 1$,取$p = 3$,由Eisenstein Criterion知$f(x)$不可约.
(7)$f(x) = x^p + px + 2p - 1, f(x+1) = (x+1)^p + p(x+1) + 2p - 1$,
如果$p \neq 3$,由Eisenstein Criterion知$f(x)$不可约.
如果$p = 3$,由$f(x)$无有理根知$f(x)$不可约. $\square$

61、证明:
反证法.p为素数.
假设$f(x) = a_n x^n + \cdots + a_1 x + a_0$不存在次数大于等于k的不可约因子.
$f(x) = \prod_{i=1}^{m} p_i(x)^{r_i}$,其中$p_i(x)$为不可约因子.
设

$$p_i(x) = a_{i,l_i} x^{l_i} + \cdots + a_{i,1} x + a_{i,0}.$$

易知

$$a_n = \prod_{i=1}^{m} (a_{i,l_i})^{r_i}, a_0 = \prod_{i=1}^{m} (a_{i,o})^{r_i}$$

由$p \nmid a_n, p \mid a_0, p^2 \nmid a_0$知$p \nmid a_{i,l_i}$,且只存在一个$i$使得$p \mid a_{i,0}, p^2 \nmid a_{i,0}, r_i = 1$. 不妨设$p_i(x) = b_l x^l + \cdots + b_1 x + b_0$,则$f(x) = p_i(x)g(x), g(x) = \prod\limits_{j \neq i} p_j(x)^{r_j}$. 令$g(x) = c_{n-l}x^{n-l} + \cdots + c_1 x + c_0$,则$p \nmid c_{n-l}, p \nmid c_0$.设$p \nmid b_t, p \mid b_j, j < t$,则$t \leq l < k$, 考虑$a_t = b_0 c_t + b_1 c_{t-1} + \cdots + b_{t-1}c_1 + b_t c_0$,则$p \nmid a_t$, 与题目中$p \mid a_i, i = 0, 1, \ldots k-1$矛盾.所以假设不成立,特别的,$f(x)$有次数大于等于于k的不可约因子. □

62、证明：
设$m = \prod\limits_{i=1}^{k} p_i^{r_i}, \exists r_i,\ \ s.t\ \ n \nmid r_i$.假设$\sqrt[n]{m} = \frac{a}{b} \in \mathbb{Q}, (a,b) = 1$. 则有$b^n m = a^n$,推出$b^n \mid a^n$,但是$(a,b) = 1 \Rightarrow (a^n, b^n) = 1$.所以$b = 1$,特别地,$a^n = m$与题意矛盾. □

63、证明：
假设$f(x)$有整数根$\alpha \in \mathbb{Z}$. $f(x) = (x-\alpha)g(x)$,易知$g(x) \in \mathbb{Z}[x]$. $f(0) = -\alpha g(0) \in \mathbb{Z}, g(1) = (1-\alpha)g(1) \in \mathbb{Z}$.因为$f(0), f(1)$都为奇数,所以$\alpha, 1-\alpha$都为奇数,矛盾. □

64、证明：
(注意：零多项式的定义：每个单项式的系数为0)
对n 作归纳假设。
当$n = 1$时，$f(x_1)$为一元多项式，由代数学基本定理，至多存在有限多根。所以存在c使得$f(x) \neq 0$。
假设当$n = k-1$时，命题成立，特别地，对任意$k-1$元多项式$f(x_1, \cdots, x_{k-1})$存在$c = (c_1, \cdots, c_{k_1})$使得$f(c_1, \cdots, c_{k-1}) \neq 0$。
当$n = k$时，

$$f(x_1, \cdots, x_k) = f_0(x_1, \cdots, x_{k-1}) + x_k f_1(x_1, \cdots, x_{k-1}) + \cdots + x_k^m f_m(x_1, \cdots, x_{k-1}).$$

因为$f(x_1, x_2, \cdots, x_n)$非零多项式，所以存在$f_i(x_1, \cdots, x_{k-1}) \neq 0$，由归纳假设知，存在$c = (c_1, \cdots, c_{k-1})$ 使得$f_i(c) \neq 0$。令

$$g(x_k) = f(c_1, \cdots, c_{k-1}, x_k),$$

则$g(x_k)$为非零的一元多项式，所以存在$c_k \in \mathbb{C}$ 使得$g(c_k) \neq 0$. 综上所述，原命题成立。 □

65、解：
(1)(3,2,1), (2,2,2);
(2)(4,3,0), (4,2,1),(3,3,1),(3,2,2);
(3)(6,4,2), (6,3,3),(5,5,2),(5,4,3),(4,4,4);
(4)(3,0,0), (2,1,0),(1,1,1);
(5)(5,5,1), (5,4,2),(5,3,3),(4,4,3);
(6)(7,5,3), (7,4,4),(6,6,3), (6,5,4),(5,5,5). □

66、证明：
设$\alpha = a x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$为$f(x)$的最后一项,$\beta = b x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$是$g(x)$ 的最后一项.对任

意的$f(x)$的项γ,$g(x)$的项σ,我们有$\gamma \geq \alpha, \sigma \geq \beta$. 则易知$\gamma\sigma \geq \alpha\beta$.特别地,$\alpha\beta$为$f(x)g(x)$的最后一项. □

67、解：
(1)$f(x) = x_1^3x_2 + x_1^3x_3 + x_2^3x_1 + x_2^3x_3 + x_3^3x_1 + x_3^3x_2$;
(2)$f(x) = 2(x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2) - (x_1x_2 + x_1x_3 + x_2x_3) - 3(x_1^2x_2^3 + x_1^2x_3^3 + x_2^2x_1^3 + x_2^2x_3^3 + x_3^2x_1^3 + x_3^2x_2^3)$;
(3)?? □

68、解：
(1)$f(x) = \sigma_1\sigma_2 - 3\sigma_3$;
(2)$f(x) = \sigma_1\sigma_2 - \sigma_3$;
(3)$f(x) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 + \sigma_1^2 - 2\sigma_2$;
(4)$f(x) = \sigma_1^2 - 2\sigma_2$. □

69、解：
(1)$f(x) = \sigma_1^2\sigma_2^2 - \sigma_1^3\sigma_3 - \sigma_2^3 = -10$; (2)$f(x) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 = -24$. □

70、证明：
设x_1, x_2, x_3为$f(x)$的三个根,且$x_1^2 = x_2^2 + x_3^2$. $\sigma_1 = -a, \sigma_2 = b, \sigma_3 = -c$.
$a^2 - 2b = 2x_1^2$.易知$a^2 + 2ax_1 - 2x_2x_3 = 0$.
$a^2 = -2ax_1 + 2x_2x_3 \Rightarrow a^4 = (-2ax_1 + 2x_2x_3)^2$,
$a^4(a^2 - 2b) = 2x_1^2a^4 = 2(2ax_1^2 - 2x_1x_2x_3)^2 = 2(a(a^2 - 2b) + 2c) = 2(a^3 - 2ab + 2c)^2$.□

71、证明：
(1)$n = 2, D(f) = (x_1 - x_2)^2 = \sigma_1^2 - 4\sigma_2$;
(2)$n = 3, D((f) = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 = \sigma_1^2\sigma_2^2 - 4\sigma_1^3\sigma_3 - 4\sigma_2^3 + 18\sigma_1\sigma_2\sigma_3 - 27\sigma_3^3$;
(3)$D((f) = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 > 0 \Rightarrow x_1 \neq x_2, x_1 \neq x_3, x_2 \neq x_3$. 假设$x_1 = a + ib, b \neq 0$为$f(x)$的根,则$= a - ib$也为$f(x)$的根，记$x_2 = a - ib$，则$f(x)$的根$x_3$为实数. $D(f) = -b^2(a - x_3 + ib)^2(a - x_3 - ib)^2 = -b^2((a - x_3 + ib)(a - x_3 - ib))^2 \leq 0$.矛盾，所以$f(x)$的根全为实数.
(4)$D(f) = -4a^3 - 27b^2 = 0$. □

72、解：
no ?,yes ?

73、证明：

(1)$f'(x) = nx^{n-1} - (n-1)\sigma_1x^{n-2} + \cdots + (-1)^{n-2}2\sigma_{n-2}x + (-1)^{n-1}\sigma_{n-1} = \sum_{i=1}^{n} \frac{f(x)}{x - x_i}$.

$$\begin{aligned} x^{k+1}f'(x) &= \sum_{i=1}^{n}\frac{x^{k+1}f(x)}{x-x_i} \\ &= \sum_{i=1}^{n}\frac{(x^{k+1}-x_i^{k+1}+x_i^{k+1})f(x)}{x-x_i} \\ &= \sum_{i=1}^{n}\frac{(x^{k+1}-x_i^{k+1})f(x)}{x-x_i}+\sum_{i=1}^{n}\frac{x_i^{k+1}f(x)}{x-x_i} \end{aligned}$$

$$\because \frac{x^{k+1}-x_i^{k+1}}{x-x_i}=x^k+x^{k-1}x_i+x^{k-2}x_i^2+\cdots+x_i^k.$$

$$\therefore \sum_{i=1}^{n}\frac{x^{k+1}-x_i^{k+1}}{x-x_i}=nx^k+s_1x^{k-1}+\cdots+s_k$$

$\therefore x^{k+1}f'(x)=(s_0x^k+s_1x^{k-1}+\cdots+s_k)f(x)+g(x), g(x)=\sum\limits_{i=1}^{n}\frac{x_i^{k+1}f(x)}{x-x_i}$
(2)$x^{k+1}f'(x)=nx^{n+k}-(n-1)\sigma_1x^{n+k-1}+\cdots+(-1)^{n-2}2\sigma_{n-2}x^{k+2}+(-1)^{n-1}x^{k+1}$
考虑上述多项式中x^n项的系数a
当$1\le k\le n$,$a=(-1)^k(n-k)\sigma_k$,又由(1)有$a=s_k-\sigma_1s_{k-1}+\cdots+(-1)^{k-1}\sigma_{k-1}s_1+(-1)^ks_0\sigma_k$. 所以$(-1)^k(n-k)\sigma_k=s_k-\sigma_1s_{k-1}+\cdots+(-1)^{k-1}\sigma_{k-1}s_1+(-1)^ks_0\sigma_k$,推出$s_k-\sigma_1s_{k-1}+\cdots+(-1)^{k-1}\sigma_{k-1}s_1+(-1)^kk\sigma_k=0$.
当$k>n$，则上述多项式中x^n的系数为0。由(1)知x^n的系数又等于$s_k-\sigma_1s_{k-1}+\cdots+(-1)^n\sigma_ns_{k-n}$. 所以$s_k-\sigma_1s_{k-1}+\cdots+(-1)^n\sigma_ns_{k-n}=0$.
(3)$s_1=s_2=\cdots=s_{n-1}=0, s_n=1$.假设$f(x)=x^n-\sigma_1x^{n-1}+\cdots+(-1)^n\sigma_n$. 则由newton公式知$\sigma_1=\sigma_2=\cdots=\sigma_{n-1}=0, s_n+(-1)^nn\sigma_n=0$,推出$\sigma_n=\frac{(-1)^{n+1}}{n}$, 所以$f(x)=x^n-\frac{1}{n}$.

74、证明：
对k作归纳.
当$k=1$显然成立。
假设对小于或等于$k-1$都成立。下证k时成立。
当$k\le n$时，由newton公式知$s_k=\sigma_1s_{k-1}-\sigma_2s_{k-2}+\cdots+(-1)^{k+1}k\sigma_k$.
我们只需要对每个单项证明即可，特别的对$\sigma_1^{j_1}\sigma_2^{j_2}\cdots\sigma_n^{j_n}$,我们需证明此项系数为

$$a_{j_1j_2\cdots j_n}=(-1)^{j_2+j_4+\cdots}\frac{(j_1+j_2+\cdots+j_n-1)!k}{j_1!j_2!\cdots j_n!}$$

设$j_1+2j_2+\cdots+nj_n=k$,则$(j_1-1)+2j_2+\cdots+nj_n=k-1, j_1+2(j_2-1)+3j_3+\cdots+nj_n=k-2,\cdots$,所以

$$\begin{aligned} a_{j_1j_2\cdots j_n} &= (-1)^{j_2+j_4+\cdots}\frac{((j_1-1)+j_2+\cdots+j_n-1)!(k-1)}{(j_1-1)!j_2!\cdots j_n!} \\ &\quad -(-1)^{j_2-1+j_4+\cdots}\frac{(j_1+j_2-1+\cdots+j_n-1)!(k-2)}{j_1!(j_2-1)!\cdots j_n!}+\cdots \end{aligned}$$

令$\gamma=j_1+j_2+\cdots+j_n-2$则

$$\begin{aligned} a_{j_1j_2\cdots j_n} &= (-1)^{j_2+j_4+\cdots}\frac{\gamma!j_1(k-1)+\gamma!j_2(k-2)+\cdots}{j_1!j_2!\cdots j_n!} \\ & (-1)^{j_2+j_4\cdots}\frac{\gamma!(j_1+j_2+\cdots+j_n)k-(j_1+2j_2+\cdots+nj_n)}{j_1!j_2!\cdots j_n!} \\ & (-1)^{j_2+j_4\cdots}\frac{(\gamma+1)!k}{j_1!j_2!\cdots j_n!} \end{aligned}$$

（注意：上述过程中$k\le n$,$j_{k+1}=\cdots j_n=0$）
当$k>n$同理可以证明. □

75、证明：
(1)设(x,y,z)为$a^2+b^2=c^2$的解且$(x,y,z)=1$.则$x^2+y^2=z^2$

$$(\frac{x}{z})^2+(\frac{y}{z})^2=1$$

且$\frac{y}{z},\frac{x}{z}\in\mathbb{Q}$.

若(x_1,y_1)为$x^2+y^2=1$的解且$x_1,y_1\in\mathbb{Q}$ 令$x_1=\frac{q_1}{p_1},y_1=\frac{q_2}{p_2},(p_1,q_1)=(p_2,q_2)=1$.则有

$$\begin{aligned} &(\frac{q_1}{p_1})^2+(\frac{q_2}{p_2})^2=1 \\ &(\frac{\frac{[p_1,p_2]q_1}{p_1}}{[p_1,p_2]})^2+(\frac{\frac{[p_1,p_2]q_2}{p_2}}{[p_1,p_2]})=1 \\ &(\frac{[p_1,p_2]q_1}{p_1})^2+(\frac{[p_1,p_2]q_2}{p_2})^2=([p_1,p_2])^2 \end{aligned}$$

且$(\frac{[p_1,p_2]q_1}{p_1},\frac{[p_1,p_2]q_2}{p_2},[p_1,p_2])=1$. It's easy to show that they are 1-1 correspondence.

2. Chapter: Systems of Linear Equations

1、
(1)$x=(2,-2,3)$
(2)$x=(3,1,1)$
(3)$x=(-1,-1,0,1)$
(4)$x=(2,0,0,0)$
(5)$x=(0,0,0,0)$
(6)$x=(1,-\frac{4}{5},\frac{3}{5},-\frac{4}{5},1)$
(7)$x=(2,0,-2,-2,1)$.

2、

$$(1)x=(0,0,0,0,0); \qquad (2)\begin{cases} x_1 = 11x_2 \\ x_3 = -7x_2 \end{cases}$$

$$(3)\ \begin{cases} x_1 = \frac{1}{17}(23x_3 - 16x_4) \\ x_2 = \frac{1}{17}(19x_3 - 20x_4) \end{cases}; \qquad (4)\ \begin{cases} x_1 = -\frac{1}{2}x_4 \\ x_2 = -\frac{1}{2}x_4 \\ x_3 = \frac{1}{2}x_4 \\ x_5 = 0 \end{cases}$$

$(5)x=(0,0,0,0,0)$
$(6)x=(-\frac{3}{5},-\frac{1}{5},\frac{1}{5},-\frac{2}{5},-\frac{12}{5})$
$(7)x=(1,2,3)$
(8)

$$\begin{cases} x_1 = -8 \\ x_2 = x_4 + 3 \\ x_3 = 2x_4 + 6 \end{cases}$$

3、解：

$$\begin{cases} x_1 + x_2 + x_3 = r \\ x_1 + 6x_2 + 3x_3 = s \\ 3x_1 - 2x_2 + x_3 = t \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + x_3 = r \\ 5x_2 + 2x_3 = s - r \\ -5x_2 - 2x_3 = t - 3r \end{cases}$$

方程组有解$\Leftrightarrow s-r=3r-t \Leftrightarrow s+t=4r.$ □

4、解：

$$(1)\ \begin{cases} a = 4 - c \\ b = -2, \end{cases}; (2)a = 2 - b - c.$$

5.解：

$$\begin{cases} x - y = 0 \\ y - z = 0 \\ x - z = 0. \end{cases}$$

6.Answer: Suppose (c_1,c_2,c_3) and (d_1,d_2,d_3) are two different solutions. We have

$$\begin{cases} a_{11}(c_1-d_1)t + a_{12}(c_2-d_2)t + a_{13}(c_3-d_3)t = 0 \\ a_{21}(c_1-d_1)t + a_{22}(c_2-d_2)t + a_{23}(c_3-d_3)t = 0 \\ a_{31}(c_1-d_1)t + a_{32}(c_2-d_2)t + a_{33}(c_3-d_3)t = 0. \end{cases}$$

Thus, $(c_1+(c_1-d_1)t, c_2+(c-2-d_2)t, c_3+(c_3-d_3)t)$ is a solution of the system of equations. □

7. Answer: $\mathcal{S}$ is linearly dependent $\Leftrightarrow \exists k_1,\ k_2$ which are not all zero such that

$$k_1\alpha + k_2\beta = 0.$$

Suppose that $k_1 \neq 0$, we have $\alpha = -\frac{k_2}{k_1}\beta$. □

8. Answer: (1) $\alpha_1+\alpha_2,\ \alpha_2+\alpha_3$ are linearly independent;
(2) linearly dependent;

(3) linearly dependent;
(4) linearly independent. □

9. Answer: Suppose that $\alpha_1,\ \alpha_2,\cdots,\ \alpha_r$ are linearly dependent. We have $k_1,\ k_2,\cdots,\ k_r$, such that

$$k_1\alpha_1+k_2\alpha_2+\cdots+k_r\alpha_r=0.$$

We have

$$\begin{cases} k_1a_{11}=0\\ k_1a_{12}+k_2a_{22}=0\\ \cdots\\ k_1a_{1r}+\cdots+k_ra_{rr}=0.\end{cases}$$

Then $a_{ii}\neq 0$ implies $k_1=k_2=\cdots=k_r=0$, contradiction. □

10. Answer:Suppose that $\alpha_{i1},\cdots,\alpha_{it}$ are linearly dependent, we have $k_{i1},\cdots,k_{it}$ which are not all zero such that

$$k_{i1}\alpha_{i1}+\cdots+k_{it}\alpha_{it}=0.$$

Let $k_j=0$ for j does not belong to $\{i1,i2,\cdots,it\}$. Thus, we have

$$\sum_{i=1}^{n}k_i\alpha_i=0$$

which contradicts to that $\alpha_1,\cdots\alpha_n$ are linearly independent. □

11. Answer: Suppose that $\beta_1,\cdots,\beta_n$ are linearly dependent. There exists $k_1,\cdots,k_n$ which are not all zero such that $\sum k_i\beta_i=0$. We have

$$\begin{cases} k_1a_{11}+\cdots+k_na_{1n}=0\\ \cdots\\ k_1a_{m1}+\cdots+k_na_{mn}=0\\ k_1b_1+\cdots+k_nb_n=0,\end{cases}$$

which implies that $k_1\alpha_1+\cdots+k_n\alpha_n=0$. Thus $\alpha_1,\cdots,\alpha_n$ are linearly dependent, contradiction. □

12. Answer: "$\Rightarrow$" "设$\beta=\sum k_i\alpha_i$ 表示唯一."
若$\alpha_1,\cdots,\alpha_n$ 线性相关, 则存在不全为零的l_i 使得$\sum l_i\alpha_i=0$. 我们有$\beta=\sum(k_i+l_i)\alpha_i$ 与β 的表示唯一性矛盾. 所以$\alpha_1,\cdots,\alpha_n$ 线性无关.
"$\Leftarrow$" 已知$\alpha_1,\cdots,\alpha_n$ 线性无关. 假设β 的表示不唯一, $\beta=\sum k_i\alpha_i=\sum l_i\alpha_i$, 其中至少存在$1\leq i\leq n$ 使得$k_i\neq l_i$. 推出$\sum(k_i-l_i)\alpha_i=0$, 其中至少存在i 使得$k_i-l_i\neq 0$. 矛盾. 所以β的表示唯一. □

13. 证明：由题意知$\alpha_1,\cdots,\alpha_n$ 线性相关，则存在不全为零的γ_i，$1\leq i\leq n$使得$\sum\gamma_i\alpha_i=0$。因为$\alpha_1\neq 0$，所以存在$k>1$ 使得$\gamma_k\neq 0$ 且$\gamma_i=0$，$i>k$。因此

$$\alpha_k=-\frac{1}{\gamma_k}(\gamma_1\alpha_1+\cdots+\gamma_{k_1}\alpha_{k-1}).$$

□

14. 证明：假设$\alpha_1+\alpha,\cdots,\alpha_n+\alpha$ 线性相关，则存在不全为零的γ_i, $1\leq i\leq n$ 使得

$$\gamma_1(\alpha_1+\alpha)+\cdots+\gamma_n(\alpha_n+\alpha)=0.$$

我们有

$$\gamma_1\alpha_1 + \cdots + \gamma_n\alpha_n + \sum_{i=1}^{n}\gamma_i\alpha = 0.$$

如果$\sum_{i=1}^{n}\gamma_i = 0$，那么$\sum_{i=1}^{n}\alpha_i = 0$ 与α_i 线性无关矛盾。所以$\sum_{i=1}^{n}\gamma_i \neq 0$。因此

$$\alpha = -\frac{1}{\gamma_1 + \cdots \gamma_n}\sum_{i=1}^{n}\alpha_i$$

与α 不能有$\alpha_1, \cdots, \alpha_n$线性表出矛盾。所以$\alpha_1 + \alpha, \cdots, \alpha_n + \alpha$线性无关。 □

15. 解:

(1)

$$\begin{cases} 2x_1 + x_2 + 2x_3 + x_4 = 11 \\ x_1 - x_3 + x_4 = 4 \\ 11x_1 + 4x_2 + 5x_3 + 7x_4 = 56 \\ 2x_1 - x_2 + 6x_3 + 3x_4 = 5 \end{cases} \Rightarrow \begin{cases} x_1 = 4 - x_4 \\ x_2 = 3 + x_4 \\ x_3 = 0 \end{cases}$$

$\Rightarrow \beta = 4\alpha_1 + 3\alpha_2.$

(2)

$$\begin{cases} x_1 - x_3 + x_4 = 4 \\ x_2 + 2x_4 = 5 \\ x_3 + 3x_4 = 6 \\ x_1 + 2x_2 + 2x_3 + 14x_4 = 32 \\ 4x_1 = 5x_2 + 2x_3 + 32x_4 = 77 \end{cases} \Rightarrow \begin{cases} x_1 = 10 - 4x_4 \\ x_2 = 5 - 2x_4 \\ x_3 = 6 - 3x_4 \end{cases}$$

$\Rightarrow \beta = 10\alpha_1 + 5\alpha_2 + 6\alpha_3.$ □

16. 解: 不。

$$S_1 = \{(1,0) \in \mathbb{R}\}, S_2 = \{(0,1) \in \mathbb{R}\}.$$

rank S_1 =rank S_2=1。 □

17. 证明：因为$\alpha_1, \cdots, \alpha_s$ 的线性无关子集仍为$\alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_t$的线性无关子集。所以$r_1 \leq r_3$。同理$r_2 \leq r_3$。特别的，$\max(r_1, r_2) \leq r_3$。不失一般性，假设$\alpha_1, \cdots, \alpha_{r_1}$ 为$\alpha_1, \cdots, \alpha_s$的一个极大线性无关组，$\beta_1, \cdots, \beta_{r_2}$ 为$\beta_1, \cdots, \beta_t$ 的一个极大线性无关组。对任意的α属于$\alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_t$皆可由$\alpha_1, \cdots, \alpha_{r_1}, \beta_1, \cdots, \beta_{r_2}$线性表出。所以$r_3 \leq r_1 + r_2$。 □

18. 解：NO.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

19. 证明：设$\alpha_1, \cdots, \alpha_r$为矩阵$A$的$r$个非零列向量，则$\alpha_1, \cdots, \alpha_r$线性无关。 □

20. 解：$r_1 = 3$, $r_2 = 2$, $r_3 = 2$, $r_4 = 4$, $r_5 = 4$, $r_6 = 4$. □

21. 解：

$$(1)\ A^T = \begin{pmatrix} 8 & 1 & 2 \\ 7 & -2 & 0 \\ 6 & 3 & 0 \\ 5 & -4 & 1 \end{pmatrix}; \quad (2)\ A^T = \begin{pmatrix} 14 & 2 & -1 \\ 3 & 3 & 3 \\ 7 & 3 & 0 \end{pmatrix}.$$

22. 证明：方法一：$\mathrm{rank}A=\mathrm{rank}A^T$. 令

$$A^T=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

由题意可知

$$\sum_{i=1}^n a_{ij}>0,\forall j.$$

对n做归纳。当$n=1$时显然成立。假设对$n-1$阶矩阵都成立。对A^T做行变换

$$A_1=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ & \cdots & \cdots & \\ 0 & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

记

$$B=\begin{pmatrix} b_{22} & \cdots & b_{2n} \\ & \cdots & \cdots \\ b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

其中$b_{ij}=-\frac{a_{i1}}{a_{11}}a_{1j}+a_{ij}$, $2\le i,j\le n$.

$$\sum_{i=2}^n b_{ik}=\sum_{i=2}^n -\frac{a_{i1}}{a_{11}}a_{1k}+a_{ik}=-\frac{a_{21}+\cdots+a_{n1}}{a_{11}}a_{1k}+(a_{2k}+\cdots+a_{nk})$$

因为$a_{11}+\cdots+a_{n1}>0$，所以

$$-\frac{a_{21}+\cdots+a_{n1}}{a_{11}}<1.$$

因为$a_{ij}<0$, $i\neq j$, 所以

$$-\frac{a_{21}+\cdots+a_{n1}}{a_{11}}a_{1k}+(a_{2k}+\cdots+a_{nk})>a_{1k}+\cdots+a_{nk}>0.$$

易知

$$b_{ii}=-\frac{a_{i1}}{a_{11}}a_{1i}+a_{ii}>0\text{且}b_{ij}<0,i\neq j.$$

由归纳假设知$\mathrm{rank}\ B=n-1$。所以$\mathrm{rank}\ A^T=\mathrm{rank}\ A_1=n$。
方法二：假设$\mathrm{rank}\ A<n$，则方程组$AX=0$有非零解。设$(k_1,\cdots,k_n)$为一组非零解。不妨设$|k_1|=\max\{|k_1|,|k_2|,\cdots,|k_n|\}$。因为$a_{11}k_1+a_{12}k_2+\cdots+a_{1n}k_n=0$，所以

$$|a_{11}k_1|=|-a_{12}k_2-\cdots-a_{1n}k_n|\le|a_{12}k_2|+\cdots+|a_{1n}k_n|$$

因为$a_{ij}<0,i\neq j$，所以

$$|a_{11}k_1|\le -a_{12}|k_1|-\cdots-a_{1n}|k_n|\le -a_{12}|k_1|-\cdots-a_{1n}|k_1|$$

特别的，$a_{11}\le -a_{12}-\cdots-a_{1n}$。矛盾。 □

23. 证明：设

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

$$B=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} \\ & \cdots & \cdots & \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(n-1)} \end{pmatrix}.$$

由$\forall i, \sum_{j=1}^{n} a_{ij} = 0$知rank $A \leq n-1$。又因为$a_{ij} < 0,\ i \neq j$，所以对任意$1 \leq i \leq n-1, \sum_{j=1}^{n-1} a_{ij} > 0$。由22题知，rank $B = n-1$。所以rank $A = n-1$。 □

24. 考虑方程组，对方程组的系数矩阵作行变换不改变方程组的解。

25. 解：(1)

$$A=\begin{pmatrix} 0 & 4 & 10 & 2 \\ 4 & 8 & 18 & 2 \\ 10 & 18 & 40 & 4 \\ 1 & 7 & 17 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 7 & 17 & 3 \\ 0 & 4 & 10 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

rank $A = 2$, $S = \{\alpha_1, \alpha_2\}$.

(2) rank $A = 4$, $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. □

26. 证明：已知

$$\alpha_i = a_{1i}\eta_1 + a_{2i}\eta_2 + \cdots + a_{mi}\eta_m, 1 \leq i \leq n.$$

$A = (a_{ij})_{m\times n}$. 设$k_1, \cdots, k_r \in \mathbb{F}$,

$$k_1\alpha_{i_1} + \cdots + k_r\alpha_{i_r} = 0.$$

$\Leftrightarrow$

$$(k_1 a_{1i_1} + \cdots + k_r a_{1i_r})\eta_1 + \cdots + (k_1 a_{mi_1} + \cdots + k_r a_{mi_r})\eta_m = 0$$

$\Leftrightarrow$

$$\begin{cases} k_1 a_{1i_1} + \cdots + k_r a_{1i_r} = 0 \\ \cdots \\ k_1 a_{mi_1} + \cdots + k_r a_{mi_r} = 0 \end{cases}$$

$\Leftrightarrow$

$$k_1\beta_{i_1} + \cdots + k_r\beta_{i_r} = 0$$

所以$\alpha_{i_1}, \cdots, \alpha_{i_r}$线性无关当且仅当$\beta_{i_1}, \cdots, \beta_{i_r}$线性无关。 □

27. 证明：设$S = \{\alpha_1, \cdots, \alpha_n\}$ 为A的列向量组，$S_1 = \{\alpha_1, \cdots, \alpha_n, \beta\}$为$A_{aug}$的列向量组。

$$\text{rank } A = \text{rank } S, \text{rank } A_{aug} = \text{rank } S_1.$$

所以rank $A =$ rank A_{aug} 或者rank $A + 1 =$ rank A_{aug}。 □

28. 解：

$$(1)\ \begin{cases} x_1 = \frac{24+19x_4}{12} \\ x_2 = \frac{-24-5x_4}{12} \\ x_3 = \frac{36-23x_4}{12} \end{cases}; \qquad (2) \begin{cases} x_1 = 3 \\ x_2 = 1 \\ x_3 = 1 \end{cases};$$

$$(3)\left\{\begin{array}{l} x_1=-\frac{1}{4}\\ x_2=\frac{1}{12}\\ x_3=\frac{31}{48}\\ x_4=\frac{17}{48}\end{array}\right. ;\qquad (4)\left\{\begin{array}{l} x_1=2\\ x_2=0\\ x_3=0\\ x_4=0\end{array}\right. .$$

29. 解：(1)

$$A=\begin{pmatrix} a & 1\\ 1 & a\end{pmatrix}.$$

$$A_{aug}=\begin{pmatrix} a & 1 & a^2\\ 1 & a & 1\end{pmatrix}\Rightarrow\begin{pmatrix} 1 & a & 1\\ 0 & 1-a^2 & a^2-a\end{pmatrix}.$$

方程组无解等价于rank $A\neq$ rank A_{aug}。所以rank $A=1$, rank $A_{aug}=2$。

$$\left\{\begin{array}{l} 1-a^2=0\\ a^2-a\neq 0\end{array}\right.$$

所以$a=-1$。
方程组有无穷多组解等价于rank $A=$ rank $A_{aug}=1$。

$$\left\{\begin{array}{l} 1-a^2=0\\ a^2-a=0\end{array}\right.$$

所以$a=1$。

(2)

$$A_{aug}=\begin{pmatrix} 1 & a & 1\\ a & 1 & a^3\end{pmatrix}\Rightarrow\begin{pmatrix} 1 & a & 1\\ 0 & 1-a^2 & a^3-a\end{pmatrix}.$$

方程组无解等价于

$$\left\{\begin{array}{l} 1-a^2=0\\ a^3-a\neq 0\end{array}\right.$$

所以方程组一定有解。
方程组有无穷多解等价于

$$\left\{\begin{array}{l} 1-a^2=0\\ a^3-a=0\end{array}\right.$$

所以$a=\pm1$。 □

31. 解：

$$A=\begin{pmatrix} \lambda & 1 & 2\lambda & 2\\ 1 & \lambda & \lambda+1 & 2\lambda\\ 2\lambda-1 & 1 & 3\lambda-1 & \lambda+1\end{pmatrix}\Rightarrow\begin{pmatrix} \lambda & 1 & 2\lambda & 2\\ 1 & \lambda & \lambda+1 & 2\lambda\\ -1 & -1 & -\lambda-1 & \lambda-3\end{pmatrix}.$$

$$\Rightarrow\begin{pmatrix} 1 & 1 & \lambda+1 & 3-\lambda\\ 1 & \lambda & \lambda+1 & 2\lambda\\ \lambda & 1 & 2\lambda & 2\end{pmatrix}\Rightarrow\begin{pmatrix} 1 & 1 & \lambda+1 & 3-\lambda\\ 0 & \lambda-1 & 0 & 3(\lambda-1)\\ 0 & 1-\lambda & \lambda(1-\lambda) & (\lambda-2)(\lambda-1)\end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & \lambda+1 & 3-\lambda \\ 0 & \lambda-1 & 0 & 3(\lambda-1) \\ 0 & 0 & \lambda(1-\lambda) & (\lambda+1)(\lambda-1) \end{pmatrix}.$$

方程组有解等价于rank $A=$ rank A_{aug}。
1)rank $A=$ rank $A_{aug}=3 \Rightarrow \lambda(1-\lambda)\neq 0$

$$\begin{cases} x_1=\frac{2\lambda+1}{\lambda} \\ x_2=3 \\ x_3=\frac{-\lambda-1}{\lambda} \end{cases}$$

2)rank $A=2$

$$\begin{cases} \lambda-1\neq 0 \\ \lambda(1-\lambda)=0 \end{cases}$$

所以$\lambda=0$，但是此时rank $A_{aug}=3$。
3)rank $A=1$

$$\begin{cases} \lambda-1=0 \\ \lambda(1-\lambda)=0 \end{cases}$$

所以$\lambda=1$，此时rank $A_{aug}=1$。

$$x_1+x_2+2x_3=2.$$

32、证明：
因为$r(A_{aug})\geq r(A)$，而$r(A_{aug})\leq r(B)$ (A_{aug}的行向量为B的行向量)。所以$r(A)=r(B)$推出$r(A)=r(A_{aug})$，方程组(*)有解。
反之不成立。因为$r(A_{aug})$可以小于$r(B)$。如

$$B=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

□

33、证明：
(1)" $\Leftarrow$ " trivial.
" $\Rightarrow$ "假设A_{mn}为满行秩且满列秩,

$$\text{rank}_{row}(A_{mn})=m,\ \text{rank}_{col}(A_{mn})=n$$

但是矩阵的行秩和列秩相等，所以$m=n$。
(2)" $\Leftarrow$ " A_{mn}满行秩，rank $(A_{mn})=m$，特别的$\text{rank}_{col}A_{mn}=m$。设$A=(\alpha_1,\cdots,\alpha_n)$。因为$\{\alpha_1,\cdots,\alpha_n\}$的秩为$m$且$\alpha_i\in\mathbb{R}^m$。所以对任意的$\beta=(d_1,\cdots,d_m)$可以由$\alpha_1,\cdots,\alpha_n$线性表示，特别的存在$k_1,\cdots,k_n$使得$\sum k_i\alpha_i=\beta$。
" $\Rightarrow$ " 取$\beta_i=(0,\cdots,1,0,\cdots,0)$为$\mathbb{R}^m$的标准基。由题意知$\beta_i$可由$\alpha_1,\cdots,\alpha_n$线性表示。所以rank$\{\beta_1,\cdots,\beta_m\}\leq$ rank$\{\alpha_1,\cdots,\alpha_n\}$, 推出rank$(A)=m$。
(3)" $\Leftarrow$ " $A=(\alpha_1,\cdots,\alpha_n)$满列秩。所以$\alpha_1,\cdots,\alpha_n$线性无关，即方程组只有零解。
" $\Rightarrow$ "假设$r(A)<n$，则$\alpha_1,\cdots,\alpha_n$线性相关。推出存在不全为零的$K=(k_1,\cdots,k_n)$使得$AK^t=0$。所以此时方程组至少有两个解K和0，矛盾。
(4)A满列秩，$r(A)=n$。推出$AX=0$只有零解。若$AX=b$有两个不同的解X_1,X_2,则$A(X_1-$

$X_2) = 0$，特别的$X_1 - X_2$为$AX = 0$的解。矛盾。所以方程组$AX = b$有解则只有一个解。
$"\Rightarrow"$ $AX = 0$有解则只有零解，所以$r(A) = n$。 □

34、证明：
$AX = 0$对任意$X \in F^n$成立。取$x_1 = (1, 0, \cdots, 0), \cdots, x_n = (0, \cdots, 0, 1)$。则可得$a_{ij} = 0$。 □

35、证明：
因为解空间的秩为$n - r$，取任意$n - r$个线性无关的解$\alpha_1, \cdots, \alpha_{n-r}$，则对任意解$\beta$有$\beta, \alpha_1, \cdots, \alpha_n$线性相关，所以$\beta$可以被$\alpha_1, \cdots, \alpha_n$线性表示。因此$\alpha_1, \cdots, \alpha_{n-r}$为一组基。 □

36、证明：
设α为方程组$AX = b$的解, 则存在$l_1, \cdots, l_s$使得$\alpha = \gamma_0 + l_1\gamma_1 + \cdots + l_s\gamma_s$。

$$\alpha = \gamma_0 + l_1(\gamma_0 + \gamma_1) + \cdots + l_s(\gamma_0 + \gamma_s) - (l_1 + \cdots + l_s)\gamma_0 = (1 - \sum l_i)\gamma_0 + l_1(\gamma_0 + \gamma_1) + \cdots + l_s(\gamma_0 + \gamma_s)$$

令$k_1 = 1 - \sum l_i, k_2 = l_1, \cdots, k_{s+1} = l_s$, 则$\alpha = \sum k_i\alpha_i, \sum k_i = 1$。 □

37、解：
因为$n = 4, n - r = 2$，所以$r = 2$。

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = 0 \end{cases}$$

代入α_1, α_2得

$$\begin{cases} a_{11} - 4a_{12} + 3a_{13} = 0 \\ -a_{21} - a_{22} + a_{24} = 0 \end{cases}$$

不唯一。 □

39、解:

40、解:
let $S = \{\alpha_1, \cdots, \alpha_r\}$, consider the homogeneous system of linear equations $AX = 0$, where $A = (\alpha_1, \cdots, \alpha_r)^t$. We know that the solution space has rank $n - r$. Let $\beta_1, \cdots, \beta_{n-r}$ be a basis of the solution space. Consider $BX = 0$, where $B = (\beta_1, \cdots, \beta_{n-r})^t$, then S is exactly a basis for the solution space. □

3. Chapter Linear Maps, Matrices and Determinants

1. 证明：$f(k_1x_1 + k_2x_2, k_1y_1 + k_2y_2) = k_1f(x_1, y_1) + k_2f(x_2, y_2)$.

$$A = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}.$$

□

2.解：

$$B=\begin{pmatrix} n & 0 & \cdots & 0 & 0 \\ 0 & n-1 & 0 & \cdots & 0 \\ \cdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}_{n\times(n+1)}.$$

□

3.解：

$$B=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \cdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{n\times n}.$$

$b_{kk}=1.$ □

4. 解：
1)2) 为线性映射，3)4)不是线性映射。 □

5.解：
齐次线性方程组的解为线性空间。 □
6.解：
(1)由线性映射的定义知，$f(x)=f(x\times 1)=xf(1)$。令$r=f(1)$.
(2)$f(-2046)=-1023f(2)=5\times 1023=5115.$ □

7.解：
(1)$T(x,y)=T(x,0)+T(0,y)=xT(1,0)+yT(0,1)$. 令$a=T(1,0),b=T(0,1)$。$T=(a,b)$。
(2) $a+b=0,2a+3b=4\Rightarrow a=-4,b=4,T=(-4,4),T(2,1)=-4.$ □
8.解：
9.解：
(1)
$$(TS)(x,y,z)=(x-y-z,0,-x+y+z);$$
(2)
$$(S-T)(x,y,z)=(-z,z-y,x-z);$$
(3)
$$(ST)(1,0,1)=(2,0,0);$$
(4)
$$(S+T)(1,1,0)=(0,1,-1);$$
(5)
$$S(S+T)(1,1,0)=(0,0,0).$$

$\square$

10.解：

$$B = \begin{pmatrix} 1 & 0 \\ 24 & 34 \\ -6 & 2 \end{pmatrix}.$$

11.解：

$$(1) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};\ (2)\ \begin{pmatrix} 0 & 0 \\ -3 & 1 \\ 10 & 3 \end{pmatrix}; (3)\ \begin{pmatrix} 0 & 0 & -3 \\ -15 & -12 & -1 \\ -6 & 10 & -2 \end{pmatrix};$$

$$(4) \begin{pmatrix} -2 & 7 \\ -2 & 10 \end{pmatrix};\ (5)\ \begin{pmatrix} x^2+y^2+z^2 & xz+xy+zy & x+y+z \\ xy+xz+yz & x^2+y^2+z^2 & x+y+z \\ x+y+z & x+y+z & 3 \end{pmatrix};$$

$$(6)\ \begin{pmatrix} 2 & 11 \\ -11 & 2 \end{pmatrix}; (7)\ \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}; (8)\ \begin{pmatrix} 1 & n\lambda \\ 0 & 1 \end{pmatrix};$$

$$(9)\ 4; (10)\ \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ -3 & -2 & -1 \end{pmatrix};$$

(11) $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$;

(12) $\sum\limits_{i,j} a_{ij}x_ix_j$.

12. 证明：

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}; B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & & \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}.$$

其中$0 \le a_{ij} \le 1, 0 \le b_{ij} \le 1, \sum_{j=1}^n a_{ij} = 1 = \sum_{j=1}^n b_{ij}$. $(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$. 设$m \le 1$为$\{b_{kj} | k = 1, \cdots, n\}$中最大的数。$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj} \le \sum_{k=1}^n a_{ik}m = m \le 1$.

$$\sum_{j=1}^n (AB)_{ij} = \sum_{k=1}^n a_{ik}b_{k1} + \cdots + \sum_{k=1}^n a_{ik}b_{kn} = \sum_{j=1}^n a_{ij} = 1.$$

所以AB 为Markov 矩阵。 $\square$

13. 略 $\square$

14. 解：

$$AB = \begin{pmatrix} -5 & 14 & -5 \\ -2 & 3 & -3 \\ -1 & -3 & 7 \end{pmatrix}; \quad BA = \begin{pmatrix} -3 & -3 & 8 \\ -2 & 1 & -3 \\ -4 & -8 & 7 \end{pmatrix}.$$

$$A^2 - 2AB + B^2 = \begin{pmatrix} 7 & -17 & 16 \\ 1 & -1 & -1 \\ -2 & -6 & 4 \end{pmatrix}.$$

□

15. 略

16. 解：

$$X = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}.$$

其中$c \in \mathbb{R}$. □

17. 证明:
设$A = (a_{ij}),\ B = (b_{ij})$, 其中$a_{ij} = a_{ji},\ b_{ij} = -b_{ji}$. 记$D = A^2$, $C = B^2$, 则

$$d_{ii} = \sum_{k=1}^{n} a_{ik}a_{ki} = \sum_{k=1}^{n} a_{ik}^2,\ c_{ii} = \sum_{k=1}^{n} b_{ik}b_{ki} = -\sum_{k=1}^{n} b_{ik}^2.$$

由$D = C$知$d_{ii} = c_{ii}$, 特别的,

$$\sum_{k=1}^{n} a_{ik}^2 = -\sum_{k=1}^{n} b_{ik}^2.$$

所以$a_{ik} = 0 = b_{ik}$ 对任意的$i,\ k$. 所以$A = 0 = B$. □

18. 解:
(1)

$$f(x) = (x-2)(x-1),$$

$$f(A) = (A-2)(A-1) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(2) $f(A) = 0$.
(3)$A^3 = 0$, 所以

$$f(A) = A - 3 = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix}.$$

19. 解:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; BA = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

20. 略

21. 证明:
由题意知$A\eta_i = \beta$,

$$A(\sum_{i=1}^{m} k_i\eta_i) = \sum_{i=1}^{m} k_i A\eta_i = \sum_{i=1}^{m} k_i\beta = \beta.$$

□

22. 解：
由上题可知$\frac{1}{2}\eta_1 + \frac{1}{2}\eta_2$，$\frac{1}{2}\eta_2 + \frac{1}{2}\eta_3$，$2\eta_3 - \eta_4$ 为方程组$AX = \beta$的解. 而

$$\alpha_1 = \eta_1 - \eta_3 = (0, 1, 1-2),$$

$$\alpha_2 = 2(2\eta_2 - \eta_4) - 2(\frac{1}{2}\eta_2 + \frac{1}{2}\eta_3) = (3, 2, -3, 7)$$

为导出组$AX = 0$的解. 因为$\mathrm{rank}A = 2$, 且α_1, α_2线性无关，所以$AX = \beta$的解为$(2\eta_3 - \eta_4) + k_1\alpha_1 + k_2\alpha_2$. □

23. 解:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$(AB)^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; (BA)^T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

24. 证明:

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2;$$
$$(A+B)(A-B) = A^2 - AB + BA - B^2 = A^2 - B^2.$$

□

25. 解：
设

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c & b-d \\ c & d \end{pmatrix};$$

$$BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b-a \\ c & d-c \end{pmatrix};$$

$AB = BA \Rightarrow ac = a, b-d = b-a, d-c = d \Rightarrow a = d, c = 0.$

$$C(A) = \{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{R}\}.$$

□

26. 证明:

(1)

$$AE_{ij} = \begin{pmatrix} 0 & \cdots & a_{1i} & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & a_{ni} & \cdots & 0 \end{pmatrix}, where\ (a_{1i}, \cdots, a_{ni})\ is\ the\ j-th\ column\ of\ AE_{ij}.$$

$$E_{ij}A = \begin{pmatrix} 0 & \cdots & 0 \\ \cdots & & \\ a_{j1} & \cdots & a_{jn} \\ 0 & \cdots & 0 \end{pmatrix}, where\ (a_{j1}, \cdots, a_{jn})\ is\ the\ i-th\ row\ of\ E_{ij}A.$$

$$AE_{ij} = E_{ij}A \Rightarrow a_{ii} = a_{jj}, a_{ki} = 0\ for\ k \neq i, a_{jk} = 0\ for\ k \neq j.$$

(2) A 与所有矩阵交换，则A与E_{ij}交换，对任意的i, j. 所以$a_{11} = \cdots = a_{nn}, a_{ij} = 0, i \neq j$, 特别的，$A = cE, c \in \mathbb{R}$. □

27. 解:

$$(AB)^T = B^T A^T = BA.$$

若AB对称，则$BA = (AB)^T = AB$. □

28. 证明:

设$S = (s_{ij}), U = (u_{ij})$, $s_{ij} = s_{ji}, u_{ij} = -u_{ji}$. 令$s_{ij} + u_{ij} = a_{ij}, s_{ij} - u_{ij} = a_{ji}$, 则$s_{ij} = \frac{a_{ij}+a_{ji}}{2}, u_{ij} = \frac{a_{ij}-a_{ji}}{2}$. 此时$A = S + U$. 唯一性显然. □

29. 证明：

(1) 设$\alpha_1, \cdots, \alpha_n$为$A$的$n$个行向量，因为$r(A) = 1$, 所以存在$\alpha_i \neq 0$，使得$\alpha_j$可以由$\alpha_i$线性表示, 特别地，$\alpha_j = k_j \alpha_i$。不妨设$\alpha_i = (b_1, \cdots, b_n)$, 则

$$A = \begin{pmatrix} k_1 \\ \cdots \\ k_n \end{pmatrix} (b_1, \cdots, b_n).$$

(2)

$$A^2 = \begin{pmatrix} a_1 \\ \cdots \\ a_n \end{pmatrix} (b_1, \cdots, b_n) \begin{pmatrix} a_1 \\ \cdots \\ a_n \end{pmatrix} (b_1, \cdots, b_n) = \begin{pmatrix} a_1 \\ \cdots \\ a_n \end{pmatrix} \sum_{i=1}^{n} a_i b_i (b_1, \cdots, b_n) = (\sum_{i=1}^{n} a_i b_i) A.$$

令$c=\sum_{i=1}^n a_ib_i$, 由归纳假设知

$$A^k=AA^{k-1}=Ac^{k-2}A=c^{k-1}A.$$

□

30. 证明：

设$A=(a_{ij})_{n\times n}, B=(b_{ij})_{n\times n}$. 设

$$AB=\begin{pmatrix}\alpha_1\\ \cdots\\ \alpha_n\end{pmatrix}; B=\begin{pmatrix}\beta_1\\ \cdots\\ \beta_n\end{pmatrix}$$

为矩阵AB和B 的行向量. 易知

$$\alpha_i=a_{i1}\beta_1+\cdots+a_{in}\beta_n, \text{对任意的 } i.$$

即$\alpha_1,\cdots,\alpha_n$可以由$\beta_1,\cdots,\beta_n$线性表示. 所以

$$\text{rank }\{\alpha_1,\cdots,\alpha_n\}\leq \text{rank }\{\beta_1,\cdots,\beta_n\}$$

特别地rank $AB\leq$ rank B. □

31. 证明：

(1) 设$v\in Im(f+g)$, 则存在v_1使得$(f+g)(v_1)=v$, $(f+g)(v_1)=f(v_1)+g(v_1)=v$, 所以$v\in Im(f)+Im(g)$.

设$v\in Ker\ f\cap Ker\ g$, 则$f(v)=0=g(v)$, $f(v)+g(v)=(f+g)(v)=0$, 所以$v\in Ker\ (f+g)$.

(2)$\forall x\in Im\ f$, 存在$v_1\in V_1$使得$f(v_1)=x$. 有$g(x)=gf(v_1)=0$, 所以$x\in Ker\ g$. □

32. 证明：

设$f: F^n\to F^m$的矩阵为$A_{m\times n}$.则$r(Im\ f)=\text{rank}_{col}(A), r(ker\ f)=n-\text{rank}_{col}(A)$. 所以$r(Im\ f)+r(Ker\ f)=n$. □

33. 证明：

设$A=(\alpha_1,\cdots,\alpha_n), B=(\beta_1,\cdots,\beta_n)$.

$$A+B=(\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n).$$

则$\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n$可以由$\alpha_1,\cdots,\alpha_n,\beta_1,\cdots,\beta_n$线性表示. 所以$r(A+B)\leq r(A)+r(B)$. □

34. 解：

$$(1)\begin{pmatrix}0 & E_n\\ E_n & 0\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}C & D\\ A & B\end{pmatrix};\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}0 & E_n\\ E_n & 0\end{pmatrix}=\begin{pmatrix}B & A\\ D & C\end{pmatrix}$$

$$(2)\begin{pmatrix}c_1A & c_1B\\ c_2C & c_2D\end{pmatrix};\begin{pmatrix}c_1A & c_2B\\ c_1C & c_2D\end{pmatrix};(3)\begin{pmatrix}A+c_1C & B+c_1D\\ C & D\end{pmatrix};\begin{pmatrix}A & c_1A+c_2B\\ C & c_1C+c_2D\end{pmatrix}$$

$$(4)\begin{pmatrix} E_n & 0 \\ -A & E_n \end{pmatrix}\begin{pmatrix} E_n & B \\ A & E_n \end{pmatrix}=\begin{pmatrix} E_n & B \\ 0 & -AB+E_n \end{pmatrix};$$
$$\begin{pmatrix} E_n & B \\ A & E_n \end{pmatrix}\begin{pmatrix} E_n & -B \\ 0 & E_n \end{pmatrix}=\begin{pmatrix} E_n & 0 \\ A & -AB+E_n \end{pmatrix}.$$

35. 证明：
设$B=(\beta_1,\cdots,\beta_l)$. $AB=(A\beta_1,\cdots,A\beta_n)=0$, 所以$A\beta_i=0$，特别的$\beta_i$为方程组$AX=0$的解。所以

$$\operatorname{rank}(\beta_1,\cdots,\beta_l)\le n-r(A)\Rightarrow r(A)+r(B)\le n.$$

□

36. 证明：
$A^2=E\Rightarrow(A-E)(A+E)=0$, 由上题可知$r(A-E)+r(A+E)\le n$. 又因为$r(A+E)+r(A-E)\ge r(A+E-(A+E))=r(2E)=n$. 即$r(A-E)+r(A+E)=n$.
□

37. 证明：
考虑

$$X=\begin{pmatrix} E_n & B \\ A & 0 \end{pmatrix}.$$

$$C=\begin{pmatrix} E_n & 0 \\ -A & E_m \end{pmatrix}\begin{pmatrix} E_n & B \\ A & 0 \end{pmatrix}\begin{pmatrix} E_n & -B \\ 0 & E_l \end{pmatrix}=\begin{pmatrix} E_n & 0 \\ 0 & -AB \end{pmatrix}$$

$r(X)=r(C)=n+r(AB)$, $r(A)+r(B)\le r(X)=n+r(AB)$, 即$r(AB)\ge r(A)+r(B)-n$. □

38. 证明：
设

$$f(x)=a_nx^n+\cdots+a_1x+a_0,$$

则

$$f(A)=a_nA^n+\cdots+a_1A+a_0.$$

$$A^2=\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \cdots & \\ & & & A_m \end{pmatrix}\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \cdots & \\ & & & A_m \end{pmatrix}=\begin{pmatrix} A_1^2 & & & \\ & A_2^2 & & \\ & & \cdots & \\ & & & A_m^2 \end{pmatrix}.$$

由归纳法易知

$$A^n=\begin{pmatrix} A_1^n & & & \\ & A_2^n & & \\ & & \cdots & \\ & & & A_m^n \end{pmatrix}.$$

所以

$$\begin{aligned} f(A) &= a_n \begin{pmatrix} A_1^n & & & \\ & A_2^n & & \\ & & \cdots & \\ & & & A_m^n \end{pmatrix} + \cdots + a_1 \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \cdots & \\ & & & A_m \end{pmatrix} + a_0 E \\ &= \begin{pmatrix} a_n A_1^n + \cdots + a_0 & & & \\ & a_n A_2^n + \cdots + a_0 & & \\ & & \cdots & \\ & & & a_n A_m^n + \cdots + a_0 \end{pmatrix} \\ &= \begin{pmatrix} f(A_1) & & & \\ & f(A_2) & & \\ & & \cdots & \\ & & & f(A_m) \end{pmatrix}. \end{aligned}$$

39. 略

40. 略

41. 略

42. 略

43. 解：

(1)

$$A \bigotimes B = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & -3 & 1 & 2 & 3 \\ -3 & -4 & -5 & 3 & 4 & 5 \\ 2 & 0 & -1 & -2 & 0 & 1 \end{pmatrix}.$$

(2)

$$kA \bigotimes B = \begin{pmatrix} ka_{11}B & \cdots & ka_{1n}B \\ & \cdots & \\ ka_{m1}B & \cdots & ka_{mn}B \end{pmatrix} = k \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ & \cdots & \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} = k(A \bigotimes B);$$

$$A \bigotimes kB = \begin{pmatrix} a_{11}kB & \cdots & a_{1n}kB \\ & \cdots & \\ a_{m1}kB & \cdots & a_{mn}kB \end{pmatrix} = k(A \bigotimes B).$$

(3)

$$(A \bigotimes B)^T = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ & \cdots & \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}^T = k \begin{pmatrix} a_{11}B^T & \cdots & a_{m1}B^T \\ & \cdots & \\ a_{1n}B^T & \cdots & a_{mn}B^T \end{pmatrix} = A^T \bigotimes B^T;$$

取

$$A=\begin{pmatrix}1&0\\-1&1\end{pmatrix};B=\begin{pmatrix}2&3\\3&4\end{pmatrix}.$$

则

$$A\bigotimes B=\begin{pmatrix}2&3&0&0\\3&4&&\\-2&-3&2&3\\-3&-4&3&4\end{pmatrix};B\bigotimes A=\begin{pmatrix}2&0&3&0\\-2&2&-3&3\\3&0&4&0\\-3&3&-4&4\end{pmatrix}.$$

(4)

$$\begin{aligned}(A\bigotimes B)(C\bigotimes D)&=\begin{pmatrix}a_{11}B&\cdots&a_{1n}B\\&\cdots&\\a_{m1}B&\cdots&a_{mn}B\end{pmatrix}\begin{pmatrix}c_{11}D&\cdots&c_{1s}D\\&\cdots&\\c_{n1}D&\cdots&c_{ns}D\end{pmatrix}\\&=\begin{pmatrix}\sum a_{1i}c_{i1}BD&\cdots&\sum a_{1i}c_{is}BD\\&\cdots&\\\sum a_{mi}b_{i1}BD&\cdots&\sum a_{mi}c_{is}BD\end{pmatrix}\\&=AC\bigotimes BD.\end{aligned}$$

44. 略

45.证明：
直接验证乘积为单位矩阵.

46. 证明：
由上题可知

$$A_\theta^{-1}=\begin{pmatrix}cos\theta&sin\theta\\-sin\theta&cos\theta\end{pmatrix}=A_{-\theta}.$$

$$\begin{aligned}A_{\theta_1}A_{\theta_2}&=\begin{pmatrix}cos\theta_1&-sin\theta_1\\sin\theta_1&cos\theta_1\end{pmatrix}\begin{pmatrix}cos\theta_2&-sin\theta_2\\sin\theta_2&cos\theta_2\end{pmatrix}\\&=\begin{pmatrix}cos\theta_1cos\theta_2-sin\theta_1sin\theta_2&-(cos\theta_1sin\theta_2+sin\theta_1cos\theta_2)\\sin\theta_1cos\theta_2+cos\theta_1sin\theta_2&cos\theta_1cos\theta_2-sin\theta_1sin\theta_2\end{pmatrix}\\&=\begin{pmatrix}cos(\theta_1+\theta_2)&-sin(\theta_1+\theta_2)\\sin(\theta_1+\theta_2)&cos(\theta_1+\theta_2)\end{pmatrix}\\&=A_{\theta_1+\theta_2}.\end{aligned}$$

47. 略

48. 证明：

$$(A\bigotimes B)(A^{-1}\bigotimes B^{-1})=AA^{-1}\bigotimes BB^{-1}=E^n\bigotimes E^n=E_{n^2}$$

同理 $(A^{-1}\otimes B^{-1})(A\otimes B)=E$, 所以$A\otimes B$可逆. □

49. 解：
(1)

$$(A+E)(A+3E) = A^2 + 4A + 3E = 2E,$$

所以

$$(A+E)(\frac{1}{2}A + \frac{3}{2}E) = E.$$

(2)

$$(A-2E)(A+3E) = A^2 + A - 6E = -3E.$$

所以

$$(A-2E)^{-1} = \frac{-1}{3}(A+3E).$$

□

50. 解：

$$\begin{aligned}(E+BA)(E-B(E+AB^{-1}A)) &= E - B(E+AB)^{-1}A + BA - BAB(E+AB)^{-1}A \\ &= E + BA - B(E+AB)(E+AB)^{-1}A \\ &= E + BA - BA = E\end{aligned}$$

51. 解：

$$(A-B^{-1})B(AB-E)^{-1} = (AB-E)(AB-E)^{-1} = E;$$

$$((A-B^{-1})^{-1} - A^{-1})(ABA-A) = (B(AB-E)^{-1} - A^{-1})(AB-E)A = BA - A^{-1}(ABA-A) = E.$$

□

52. 证明：
由题意知存可逆矩阵P 使得

$$\begin{pmatrix} A \\ E \end{pmatrix} P = \begin{pmatrix} E \\ B \end{pmatrix} \Rightarrow \begin{pmatrix} AP \\ EP \end{pmatrix} = \begin{pmatrix} E \\ B \end{pmatrix};$$

所以$B = P, A^{-1} = P, B = A^{-1}$. □

53. 解:

$$(1)\begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}; \qquad (2)\begin{pmatrix} -161 & 138 & 11 & -38 \\ 29 & -25 & -2 & 7 \\ -8 & 7 & 1 & -2 \\ 5 & -4 & 0 & 1 \end{pmatrix};$$

$$(3)\begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}; \quad (4)\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix};$$

$$(5)\begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 31 & -19 & 3 & -4 \\ -23 & 14 & -2 & 3 \end{pmatrix}; (6)\begin{pmatrix} -2 & -7 & -2 & 9 \\ -2 & -6 & -1 & 7 \\ 0.8 & 3 & 0.8 & -3.6 \\ 1 & 3 & 1 & -4 \end{pmatrix};$$

54. 解：

(1)

$$P\begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} PA & PB \end{pmatrix} = (E\ C)$$

所以$P = A^{-1}, A^{-1}B = C \Rightarrow B = AC.$

(2)

$$\begin{pmatrix} A \\ B \end{pmatrix} P = \begin{pmatrix} AP \\ BP \end{pmatrix} = \begin{pmatrix} E \\ C \end{pmatrix};$$

所以$A^{-1} = P, BP = C \Rightarrow B = CA.$ □

55. 解:

$$(1)\begin{pmatrix} -9 & 42 & 9 \\ 2 & -8 & -1 \end{pmatrix}; (2)\begin{pmatrix} -1.5 & 1 & 0.5 \\ -1 & 2 & -2 \\ -17 & 11 & -4 \end{pmatrix}; (3)\begin{pmatrix} 3 & -8.5 \\ -10 & 22.5 \\ 6 & -12.5 \end{pmatrix}$$

56. 解：

$$(A-E)B = A^2 - E$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 4 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 4 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

57. 解:
(1)1; (2)1; (3)−1; (4)$(-1)^{\frac{n(n-1)}{2}}$.

58. 证明:
设σ_1为 σ经过一次对换所得，则

$$s(\sigma_1) = (-1)s(\sigma).$$

由归纳法知 $s(\sigma) = (-1)^t s(1, 2, \cdots, n) = (-1)^t$.

59. 略

60. 解：由行列式的定义

$$d = det(a_{ij}) = \sum s(i_1, \cdots, i_n) a_{1i_1} \cdots a_{ni_n} \in \mathbb{F}.$$

61. 解：设$\vec{a} = O\vec{P}_1$, $\vec{b} = O\vec{P}_2$. 则此平行四边形的面积为

$$\begin{aligned} S &= \vec{a} \times \vec{b} \\ &= (a_1 e_1 + a_2 e_2) \times (b_1 e_1 + b_2 e_2) \\ &= a_1 b_1 e_1 \times e_2 + a_2 b_1 e_2 \times e_1 \\ &= (a_1 b_2 - a_2 b_1) e_1 \times e_2 \end{aligned}$$

因为$e_1 \times e_2 = 1$, 所以$S = a_1 b_2 - a_2 b_1$. □

62. 证明：$(A - A^T)^T = -(A - A^T)$, 令$B = A - A^T$, 则B为奇数阶反对称矩阵. 设B的阶为n, 则

$$det(B) = (-1)^n det(B^T) = -det(B) \Rightarrow det(B) = 0.$$

□

63. 解：

$$\begin{aligned} det(B) &= \begin{vmatrix} a-3g & b-3h & c-3i \\ g & h & i \\ 2d & 2e & 2f \end{vmatrix} \\ &= \begin{vmatrix} a & b & c \\ g & h & i \\ 2d & 2e & 2f \end{vmatrix} + \begin{vmatrix} -3g & -3h & -3i \\ g & h & i \\ 2d & 2e & 2f \end{vmatrix} \\ &= 2 \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = 10. \end{aligned}$$

64. 解：(1) −1; (2) $(-1)^{\frac{n(n+1)}{2}} n!$; (3) 28; (4) n!; (5) 0.

65. 证明：

$$det(A) = \sum s(i_1, \cdots, i_n) 1 = 0,$$

所以奇置换和偶置换一样多。

66. 解：(1) −2; (2) $a^2+b^2+c^2-2ab-2bc-2ac$; (3) −5314; (4) 74; (5) $-2(a^3+b^3)$; (6) 1; (7) $4(b-a)(b-c)(c-a)$; (8) 224.

67. 证明：”(1) ⇒ (2)(3)” 显然。
(2) ⇒ (1): $AB = E_n \Rightarrow det(A)det(B) = 1 \Rightarrow det(A) \neq 0 \Rightarrow A$ 可逆.

68. 证明：(1)

$$(E+A)(E+B)=E+A+B+AB=E,$$

所以$(E+A)$可逆.

(2)

$$(E+A)(E+B)=E=(E+B)(E+A)\Rightarrow AB=BA.$$

69．略

70. 证明： 因为$A^k=0$，所以$rankA<2$. 如果$rankA=0$, 则$A=0\Rightarrow A^2=0$; 如果$rankA=1$, 则

$$A=\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}(b_1\ b_2).$$

$$A^k=(a_1b_1+a_2b_2)^{k-1}A=0$$

因为 $A\neq 0$, 所以 $a_1b_1+a_2b_2=0\Rightarrow A^2=(a_1b_1+a_2b_2)A=0.$

71. 略

72. 证明： 注意到

$$A=\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ & \cdots & \cdots & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ & \cdots & \cdots & \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix}$$

$$\therefore det(A)=(\prod_{1\leq j<i\leq n}(x_i-x_j))(\prod_{1\leq j<i\leq n}(x_i-x_j))=\prod_{1\leq i<j\leq n}(x_i-x_j)^2.$$

73. 解：(1)

$A_{11}=8,A_{12}=-16,A_{13}=8,A_{21}=4,A_{22}=-8,A_{23}=4,A_{31}=-4,A_{32}=8,A_{33}=-4.$

(2)

$$A_{11}=-32,A_{12}=0,A_{13}=0,A_{14}=0,A_{21}=0,A_{22}=32,A_{23}=0,A_{24}=0;$$

$$A_{31}=0,A_{32}=0,A_{33}=32,A_{34}=0,A_{41}=0,A_{42}=0,A_{43}=0,A_{44}=-32.$$

74. 解： (1)

$$A_{11}=1,A_{12}=-52,A_{13}=-9,$$
$$A_{21}=-12,A_{22}=-12,A_{23}=-36,$$
$$A_{31}=-42,A_{32}=4,A_{33}=-15.$$

(2)

$$A_{11}=x^3,A_{12}=x,A_{13}=-8-9x,$$
$$A_{21}=x-8,A_{22}=x(x^2-9),A_{23}=0,$$
$$A_{31}=-9x,A_{32}=-8,A_{33}=x^3+x.$$

75. 证明：

设

$$A=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

$S=45,s=\sum a_i=\sum b_i\sum c_i=15=a_i+b_i+c_i=a_1+b_2+c_3=c_1+b_2+a_3.$

$$
\begin{aligned}
det(A) &= \begin{vmatrix} s & b_1 & c_1 \\ s & b_2 & c_2 \\ s & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} s & b_1 & c_1 \\ s & b_2 & c_2 \\ 3s & s & s \end{vmatrix} \\
&= s\begin{vmatrix} 1 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 3 & s & s \end{vmatrix} = 3s\begin{vmatrix} s & b_1 & c_1 \\ s & b_2 & c_2 \\ 1 & 5 & 5 \end{vmatrix}
\end{aligned}
$$

$\therefore det(A)/S \in \mathbb{Z}$.

76. 解： (1) -480; (2) $\alpha\beta\gamma(ad-bc)$.

77. 证明：

(1)按最后一行展开

$$P_n(x) = A_{nn}x^{n-1} + A_{n(n-1)}x^{n-2} + \cdots + A_{n1},$$

其中A_{ni}为a_{ni}的代数余子式．所以$degf(x) \leq n-1$.

(2)

$$A_{nn} = \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-2} \\ & \cdots & \cdots & \\ 1 & a_{n-1} & \cdots & a_{n-1}^{n-2} \end{vmatrix} = \prod_{1\leq i<j\leq n-1}(a_j - a_i)$$

因为a_i互不相同， 所以$A_{nn} \neq 0$, $degf(x) = n-1$.

$$P_n(a_i) = \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ & \cdots & \cdots & \\ 1 & a_{n-1} & \cdots & a_{n-1}^{n-1} \\ 1 & a_i & \cdots & a_i^{n-1} \end{vmatrix} = 0$$

所以$a_1, \cdots, a_{n-1}$为$P_n(x)$的$n-1$个根, $P_n(x) = A_{nn}(x-a_1)\cdots(x-a_{n-1})$.

(3)$P_n(a_n) = A_{nn}(a_n-a_1)\cdots(a_n-a_{n-1})$.

78. 证明：

设

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & \cdots & \cdots & \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}.$$

如果A是可逆的, 则$det(A) \neq 0, i.e. a_{11}a_{22}\cdots a_{nn} \neq 0$. $A^{-1} = \frac{A^*}{detA}$, A^* 为A的伴随矩阵。由定义$A^*_{ij} = A_{ji} = 0,\ i > j$, A_{ji}为代数余子式。所以A^{-1} 为上三角矩阵.

79. 证明：(1) 设

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & \cdots & \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{pmatrix}.$$

则

$$AA^T = \begin{pmatrix} \sum a_{1i}^2 & \sum a_{1i}a_{2i} & \cdots & \sum a_{1i}a_{ni} \\ & \cdots & \cdots & \\ & \cdots & \cdots & \\ \sum a_{ni}a_{1i} & \cdots & \cdots & \sum a_{ni}^2 \end{pmatrix}.$$

因为$c_{ij} = ca_{ij}$, 所以

$$\sum a_{ki}a_{li} = \sum a_{ki}\frac{1}{c}c_{li} = \frac{1}{c}\sum a_{ki}c_{li} = 0, \text{if } k \neq l,$$

$$\sum a_{ki}a_{li} = \frac{1}{c}|A|, \text{if } k = l.$$

$\therefore AA^T = \frac{1}{c}|A|E_n = A^TA.$

(2)$\because$ $AA^* = det(A)E_n, A^* = (c_{ij})^T = c^3A^T = A^T, \therefore det(AA^*) = det(A)^2 = det(A)$。如果$det(A) = 0$, 那么$AA^T = 0 \Rightarrow A = 0$, 所以$det(A) \neq 0 \Rightarrow det(A) = 1$, 所以$AA^T = A^TA = E_3$.

80. 解：

$$(1)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}; (2)\begin{pmatrix} 11 & 0 & 0 \\ 44 & 33 & 0 \\ 2 & 3 & 3 \end{pmatrix}; (3)\begin{pmatrix} -14 & 4 & 25 \\ 0 & -6 & -27 \\ 0 & 0 & 21 \end{pmatrix}.$$

81. 证明：设A为一可逆矩阵且每行的和为k, 证明其逆矩阵每行的和为$\frac{1}{k}$.
设$A = (a_{ij})_{n\times n}$, $\sum_{j=1}^n a_{ij} = k$. 易知$k \neq 0$.

$$|A| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} k & a_{12} & \cdots & a_{1n} \\ & \cdots & \cdots & \\ k & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= k\begin{vmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & \cdots & \cdots & \\ 1 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k(c_{11} + \cdots + c_{n1})$$

其中c_{i1}为相应的代数余子式。同理（将所有列加到第j列），可得

$$|A| = k(c_{1j} + c_{2j} + \cdots + c_{nj}).$$

又

$$A^{-1} = \frac{A^*}{det(A)}.$$

所以

$$\sum_{j=1}^n (A^{-1})_{ij} = \sum_{j=1}^n \frac{1}{det(A)}c_{ji} = \frac{1}{c}.$$

82. 证明：令$A = (a_{ij})_{n\times n}, B = kA = (ka_{ij})_{n\times n}$. 记$b_{ij}$为$B$的相应的代数余子式，$c_{ij}$为$A$的相应的代数余子式. 则$b_{ij} = k^{n-1}c_{ij}$. 而$A^* = (b_{ji})$, 所以$(kA)^* = k^{n-1}A^*$.

83. 证明：当$r(A) = n$时, $AA^* = |A|E_n \Rightarrow det(A)det(A^*) = det(A)^n \Rightarrow det(A^*) = det(A)^{n-1} \neq 0$. 所以$r(A^*) = n$.
当$r(A) = n-1$时, $AA^* = 0$，A^*的列向量为$AX = 0$的解，所以$r(A^*) \leq 1$. 易知存

在$c_{ij}\neq 0$，所以$r(A^*)=1$.
当$r(A)<n-1$时，易知$A^*=0$, 所以$r(A^*)=0$.

84. 解：(1)$n=2, D_2=7, n\geq 3, D_n=(n-3)!6$; (2) $(-1)^{\frac{n(n+1)}{2}}\frac{n^n+n^{n-1}}{2}$;

85. 将F_{n+2}按最后以行展开得到$F_{n+2}=F_{n+1}-X$, 交换最后两列可知$X=-F_n$。

86. 证明：(1)由Laplace 定理可得.

(2)

$$\begin{pmatrix} E_n & 0 \\ -C & A \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\begin{pmatrix} A & B \\ 0 & AD-CB \end{pmatrix}.$$

$$\therefore \begin{vmatrix} E_n & 0 \\ -C & A \end{vmatrix}\begin{vmatrix} A & B \\ C & D \end{vmatrix}=\begin{vmatrix} A & B \\ 0 & AD-CB \end{vmatrix}$$

$$\Rightarrow |A|\begin{vmatrix} A & B \\ C & D \end{vmatrix}=|A||AD-CB|.$$

因为A可逆，所以)

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}=|AD-CB|.$$

(3)$\lambda=0$显然成立，下面假设$\lambda\neq 0$,

$$\begin{vmatrix} E & B \\ A & \lambda E \end{vmatrix}=\begin{vmatrix} E-\frac{1}{\lambda}AB & 0 \\ A & \lambda E \end{vmatrix}=|\lambda E-AB|;$$

$$\begin{vmatrix} E & A \\ B & \lambda E \end{vmatrix}=\begin{vmatrix} E-\frac{1}{\lambda}AB & A \\ 0 & \lambda E \end{vmatrix}=|\lambda E-AB|;$$

$$\begin{pmatrix} E_n & 0 \\ -A & E_n \end{pmatrix}\begin{pmatrix} E & B \\ A & \lambda E \end{pmatrix}=\begin{pmatrix} E & B \\ 0 & \lambda E-AB \end{pmatrix};$$

$$\begin{pmatrix} E_n & A \\ B & \lambda E_n \end{pmatrix}\begin{pmatrix} E & -A \\ 0 & E \end{pmatrix}=\begin{pmatrix} E & 0 \\ B & \lambda E-BA \end{pmatrix};$$

所以$|\lambda E-AB|=|\lambda E-BA|$.

88. 证明：易知如果$|A|=0$, 那么$|A\bigotimes B|=0$. 下面假设$|A|\neq 0$，特别的A可逆.

$$(A^{-1}\bigotimes E_m)(A\bigotimes B)=E_n\bigotimes B,$$

$$\therefore |A^{-1}\bigotimes E_m)(A\bigotimes B)|=|E_n\bigotimes B|=|B|^n.$$

$$\therefore det(A^{-1})^m det(A\bigotimes B)=det(B)^n\Rightarrow det(A\bigotimes B)=det(A)^m det(B)^n.$$

Chapter 4. Linear Spaces and Linear Maps

1. (1) Yes; (2) Yes; (3) No(homogenous system is); (4) Yes; (5) No($1\alpha)\neq\alpha$); (6) No(if α and β are linearly dependent); (7) Yes.

2.

Proof. 设$\mathcal{F}$为数域, 由定义可知$\mathcal{Q}\in\mathcal{F}$. 容易验证$\mathcal{F}$满足线性空间的八条定义. □

3.

Proof. 因为$\mathcal{L}$是$\mathcal{F}$上的线性空间, 所以$0 \in \mathcal{L}, \exists x \in \mathcal{L}, s.t. \beta + x = 0$. 记$x = -\beta$, 则$k(\alpha - \beta) = k(\alpha + x) = k\alpha + kx = k\alpha - k\beta$. $\alpha + \beta = \alpha + \gamma \Rightarrow -\alpha + \alpha + \beta = -\alpha + \alpha + \gamma \Rightarrow \beta = \gamma$. □

4.

Proof. $\alpha + x = \beta \Rightarrow -\alpha + \alpha + x = -\alpha + \beta \Rightarrow x = -\alpha + \beta$. 如果$\exists y \in \mathcal{L}, s.t. \alpha + y = \beta$, 则$-\alpha + \alpha + y = -\alpha + \beta = x \Rightarrow x = y$. □

5.

Proof. 因为$\beta_1, \cdots, \beta_n$是$\mathcal{L}$的一组基, $\forall \alpha_i \in \mathcal{L}$, 所以$\exists a_{ij}, s.t. \alpha_i = a_{1i}\beta_1 \cdots + a_{ni}\beta_n$. 记$A = (a_{ij})_{n\times n}$, 则$(\alpha_1, \cdots, \alpha_n) = (\beta_1, \cdots, \beta_n)A$. 又因为$\alpha_1, \cdots, \alpha_n$为$\mathcal{L}$的基, 所以存在矩阵$B_{n\times n}$, 使得$(\beta_1, \cdots, \beta_n) = (\alpha_1, \cdots, \alpha_n)B$. 于是

$$(\beta_1, \cdots, \beta_n) = (\beta_1, \cdots, \beta_n)AB.$$

因为$\beta_1, \cdots, \beta_n$线性无关, 所以$AB = E$,特别的A可逆. □

6.

Proof. 假设$\gamma_1, \cdots, \gamma_m$线性相关, 则存在不全为零的$k_1, \cdots, k_m$ 使得$\sum k_i\gamma_i = 0$.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_m \\ & \cdots & \cdots & \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_m^{n-1} \end{pmatrix} \begin{pmatrix} k_1 \\ \cdots \\ k_m \end{pmatrix} = 0.$$

记A为系数矩阵, 则取B为A的前面m行, 易知B为范德蒙矩阵,因为$a_1, \cdots, a_m$互不相同,所以$r(B) = m$,所以$r(A) = m$，特别的方程组只有零解, 矛盾. 所以$\gamma_1, \cdots, \gamma_m$线性无关. □

7. (1)线性无关; (2) 线性无关; (3)线性相关.

8.解：设

$$k_1(\alpha_1 + \alpha_2) + k_2(\alpha_2 + \alpha_3) + \cdots + k_n(\alpha_n + \alpha_1) = 0,$$

等价于

$$(k_1 + k_n)\alpha_1 + (k_1 + k_2)\alpha_2 + \cdots + (k_{n-1} + k_n)\alpha_n = 0.$$

因为$\alpha_1, \cdots, \alpha_n$线性无关, 所以

$$k_1 + k_n = 0, k_1 + k_2 = 0, \cdots, k_{n-1} + k_n = 0$$

当$n = 2t$时, 存在非零解; 当$n = 2t + 1$时, $k_1 = k_2 = \cdots = k_n = 0$, 所以当$n$为偶数时$\alpha_1 + \alpha_2, \cdots, \alpha_n + \alpha_1$线性相关; 当$n$为奇数时, 线性无关.

9. (1)yes;
(2) $S_1 = \{(1,0),(0,1)\}, S_2 = \{(1,1)\}$,则$S_1 \cup S_2$线性相关;
(3)$S_1 \cup S_2$线性无关当且仅当$V(S_1) \cap V(S_2) = 0$, 其中$V(S_i)$表示S_i张成的线性空间.

10. 由定义.

11.

Proof. 因为$\alpha_1, \alpha_2, \cdots, \alpha_n$为线性空间$L$的一组基, 对任意的$v \in L$, 存在$b_1, b_2, \cdots, b_n$使得

$$v = \sum_{i=1}^{n} b_i\alpha_i.$$

如果

$$v=\sum_{i=1}^{n+1}a_i\alpha_i=\sum_{i=1}^{n}(a_i-a_{n+1})\alpha_i,$$

那么$b_i=a_i-a_{n+1},1\le i\le n$. 又因为$\sum_{i=1}^{n+1}a_i=0$, 所以

$$a_{n+1}=\frac{-\sum_{i=1}^{n}b_i}{n+1},a_i=b_i-\frac{-\sum_{i=1}^{n}b_i}{n+1}.$$

特别的，存在a_i使得$v=\sum_{i=1}^{n+1}a_i\alpha_i$且$\sum a_i=0$. 易知这种表示方法唯一(由$b_i$的唯一性确定). □

12.

Proof. (1)(2)显然.下证(3). 易知对任意的$v\in\mathcal{Q}(\sqrt[n]{2})$, v可以被$1,\sqrt[n]{2},\cdots,(\sqrt[n]{2})^{n-1}$线性表示. 只需要证明$1,\sqrt[n]{2},\cdots,(\sqrt[n]{2})^{n-1}$线性无关. 假设线性相关, 存在不全为零的$k_1,k_2,\cdots,k_n$使得

$$k_1+k_2\sqrt[n]{2}+\cdots+k_n(\sqrt[n]{2})^{n-1}=0.$$

记$f(x)=k_1+k_2x+\cdots+k_nx^{n-1}\in\mathcal{Q}[x]$, 则$f(\sqrt[n]{2})=0$. 易知$g(x)=x^n-2$在$\mathcal{Q}$上不可约, 且$g(\sqrt[2]{2})=0$. 我们有$(f(x),g(x))=g(x)$或者$(f(x),g(x))=1$, 矛盾. 所以$1,\sqrt[n]{2},\cdots,(\sqrt[n]{2})^{n-1}$在$\mathcal{Q}$上线性无关.

(4)由(3)知，对任意的n，存在$\mathcal{R}$的子空间$\mathcal{Q}(\sqrt[n]{2})$维数为n，所以$\dim\mathcal{R}$不可能有限. □

13.

Proof. (1)由定义.

(2，3)取S_1为S的任意有限子集, 则分两种情况讨论(1) $(x)\bar{\in}S_1$; (2)$g(x)\in S_1$. 对于第一种情形S_1显然线性无关.

当$g(x)\in S_1$时, 假设S_1线性相关, 则

$$g(x)=\sum_{i=1}^{n}k_ix^{i_m},$$

因为S_1中除$g(x)$外的元线性无关. 由代数学基本定理知上式不可能成立, 因为左边没有零点，右边有零点，或者n_m+1阶导数后左边不为零，右式为零. 矛盾.

(4) 易知$\{1,x,x^2,\cdots,\}$在$\mathcal{R}$上线性无关, 所以$\dim W$为无穷. □

14.

Proof. 设$f_0(x)=1,f_1(x)=x,\cdots$, 易知$\{f_i(x)\}$在$\mathcal{R}$上线性无关,且$f_i(x)\in\mathcal{C}[a,b]$. 所以$\dim\mathcal{C}[a,b]=\infty$ □

15.

Proof. 因为$\dim L=m$, 所以我们只需要证明$\eta_1,\eta_1+\eta_2,\cdots,\eta_1+\cdots+\eta_m$线性无关，则$\eta_1,\eta_1+\eta_2,\cdots,\eta_1+\cdots+\eta_m$为$L$的一组基. 假设他们线性相关, 即存在不全为零的$k_1,k_2,\cdots,k_m$使得

$$k_1\eta_1+k_2(\eta_1+\eta_2)+\cdots+k_m(\eta_1+\cdots+\eta_m)=0,$$

即

$$(k_1+\cdots+k_m)\eta_1+(k_2+\cdots+k_m)\eta_2+\cdots+k_m\eta_m=0.$$

因为$\eta_1,\cdots,\eta_m$为L的基, 所以

$$k_1+k_2+\cdots+k_m=0,\cdots,k_m=0,$$

特别的$k_1=k_2=\cdots=k_m=0$. 假设不成立. □

16. (1)$\{E_{ij}|1\leq i,j\leq n\},\mathrm{dim}M_n(\mathcal{R})=n^2$;
(2)$\{E_{ii},E_{ij}+Eji|1\leq i\leq n,i<j\},\mathrm{dim}V=\frac{n(n+1)}{2}$;
(3)$\{E_{ij}|1\leq i\leq j\leq n\},\mathrm{dim}V=\frac{n(n+1)}{2}$;
(4)$\{E,A\},\mathrm{dim}V=2$; (5)$\{a\neq 1\},\dim\,V=1$.
17. (1)(0, 1, 1); (2)(1,-2,1); (3)(0, -1, 2).
19.

Proof. (1)

$$A=\begin{pmatrix}1&4&3\\1&5&3\\-1&-6&-4\end{pmatrix},$$

$\mathrm{rank}(A)=3$, 所以$\mathrm{rank}\{\eta_1,\eta_2,\eta_3\}=3$. 因为$\mathrm{dim}R^3=3$, 所以$\eta_1,\eta_2,\eta_3$为$\mathcal{R}^3$的一组基.

(2)

$$(\eta_1,\eta_2,\eta_3)=(\epsilon_1,\epsilon_2,\epsilon_3)\begin{pmatrix}1&4&3\\1&5&3\\-1&-6&-4\end{pmatrix},(\epsilon_1,\epsilon_2,\epsilon_3)=(\eta_1,\eta_2,\eta_3)\begin{pmatrix}2&2&3\\-1&1&0\\1&-2&-1\end{pmatrix}.$$

□

20.

Proof.

$$(1)A=\begin{pmatrix}5/2&1&9/2&-5\\-1&-1&-2&3\\3/2&1&7/2&-4\\-1&0&-2&2\end{pmatrix};(2)A=\frac{1}{13}\begin{pmatrix}5&6&4&-15\\17&10&24&14\\1&9&6&-3\\-5&7&9&-11\end{pmatrix},$$

$\tau=(3/13,5/13,-2/13,-3/13)$;

$$(3)A=\frac{-1}{13}\begin{pmatrix}-4&-6&10&-12\\-11&3&-5&-7\\-6&4&2&8\\4&6&16&-14\end{pmatrix},\tau=(5/2,-1,-1/2,0).$$

□

21.

Proof.

$$\begin{aligned}1&=1+0\times(x+1)+0\times(x+1)^2;\\x&=-1+(x+1);\\x^2&=1-2(x+1)+(x+1)^2.\end{aligned}$$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix},$$

$\therefore \alpha = (c-b+a, b-2a, a)$ □

22.

Proof. 首先证明$E_{11}, E_{22}, E_{33}, E_{12}+E_{21}, E_{13}+E_{31}, E_{23}+E_{32}, E_{12}-E_{21}, E_{13}-E_{31}, E_{23}-E_{32}$在$\mathcal{R}$上线性无关, 又因为$\dim\ M_3\mathcal{R} = 9$, 所以为一组基.

$x = (a_{11}, a_{22}, a_{33}, \frac{a_{12}+a_{21}}{2}, \frac{a_{13}+a_{31}}{2}, \frac{a_{23}+a_{32}}{2}, \frac{a_{12}-a_{21}}{2}, \frac{a_{13}-a_{31}}{2}, \frac{a_{23}-a_{32}}{2})$ □

23.

Proof.

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)A = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1, \end{pmatrix}$$

其中$\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$)为标准基. 易知$\mathrm{rank}A = 4$, 特别的$\alpha_1, \cdots, \alpha_4$线性无关, 又因为$\dim\mathcal{F}^4 = 4$, 所以$\alpha_1, \alpha_2, \alpha_3, \alpha_4$为$\mathcal{F}^4$的一组基.

$x = Ax \Rightarrow (A-E)x = 0 \Rightarrow x_1 = x_2 = 0, x_3, x_4 \in \mathcal{R}$. □

24.

Proof. (1)$\mathrm{rank}(\alpha_1, \alpha_2) = 2$;

(2)$\alpha_3 = 2\alpha_1 - \alpha_2$;

(3)$\beta = (1,0,0), \alpha_3 = 2\alpha_1 - \alpha_2 + 0 \times \beta$. □

25.

Proof. (1)yes, (2)no, (3)yes, (4)yes, (5)yes, (6)yes, (7)yes, (8)no. □

26.

Proof. (1)当$\mathcal{C}$看成$\mathcal{R}$上的线性空间时，f是线性映射；(2)当$\mathcal{C}$看成$\mathcal{C}$上的线性空间时，f不是线性映射,$f(k\alpha) = \overline{k\alpha} = \overline{kf\alpha} \neq kf(\alpha)$. □

27.

Proof. (1)题目：证明线性映射h将线段映成线段.

$h: \mathcal{R}^n \to \mathcal{R}^n$, $\forall x \in \mathcal{L} = \{tu + (1-t)v | t \in [0,1]\}, x = t_1u + (1-t_1)v$, $h(x) = h(= t_1u + (1-t_1)v) = h(t_1u) + h((1-t_1)v) = t_1h(u) + (1-t_1)h(v)$.

(2)题目：证明线性映射保凸性. 设$V \subset \mathcal{R}^n$为凸集, 则$\forall x, y \in V, tx + (1-t)y \in V, t \in [0,1]$. 对任意的$h(x), h(y)$, $th(x) + (1-t)h(y) = h(tx + (1-t)y) \in h(V)$，所以$h(V)$为凸集. □

28.

Proof.

$$f = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, g = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, fg = 0, gf = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \neq 0.$$

□

29.

Proof. (1)$f(x,y)=xf(1,0)+yf(0,1)$;
(2)$f(\epsilon_1)+f(\epsilon_2)=2, 2f(\epsilon_1)+3f(\epsilon_2)=7$, $f(\epsilon_1)=-1, f(\epsilon_2)=3$, $f(3,2)=3$. □

31.

Proof. (1)injective: T_1, T_2, T_5;
(2)surjective: T_1, T_2, T_5, T_6, T_8;
(3)bijective: T_1, T_2, T_5;
(4)linear map: T_1, T_3, T_5, T_8;
(5)isomorphism: T_1, T_5. □

32.

Proof. $(f-1)(f^4+f^3+f^2+f+1)=f^5-1 \Rightarrow (f-1)(-f^4-f^3-f^2-f-1)=1$, $(f-1)^{-1}=-f^4-f^3-f^2-f-1$. □

33.

Proof. (1)$f(X_1+X_2)=A(X_1+X_2)=AX_1+AX_2=f(X_1)+f(X_2)$; $f(kX)=AkX=kAX=kf(X)$.
(2)"$\Rightarrow$" f为同构，则对任意的$Y\in M_n(\mathcal{F})$, 都存在X使得$f(X)=Y=AX$,特别的，取$Y=E$, 则存在X, $AX=E$，所以A可逆.
"$\Leftarrow$" A可逆, 令$g(X)=A^{-1}X$，易知g为$M_n(\mathcal{F})$上的线性变换, $g\cdot f=1=g\cdot f$, 所以f为同构. □

34.

Proof. (1)$k_1f(\alpha_1)+\cdots+k_mf(\alpha_m)=0 \Rightarrow f(k_1\alpha_1+\cdots+k_m\alpha_m)=0$. 因为$f$为单射, 所以

$$k_1\alpha_1+\cdots+k_m\alpha_m.$$

因为$\alpha_1,\cdots,\alpha_m$线性无关, 所以$k_1=\cdots=k_m=0$. 特别的，$f(\alpha_1),\cdots,f(\alpha_m)$线性无关.
(2) f不是单射, 则存在$x\in\mathcal{L}$，$f(x)=0$, 对任意的包含x的线性无关组$x,\alpha_1,\cdots,\alpha_{m-1}$, 则$f(x), f(\alpha_1),\cdots,f(\alpha_{m-1})$线性相关. □

36.

Proof. (1) $\Rightarrow$ (2)(3) 平凡.
(2) $\Rightarrow$ (3): 设$\alpha_1,\cdots,\alpha_n$为$\mathcal{L}$的一组基, 因为f单射, 所以$f(\alpha_1),\cdots,f(\alpha_n)$线性无关. 又因为$\dim\mathcal{L}=\dim\mathcal{M}$, 所以$f(\alpha_1),\cdots,f(\alpha_n)$为$\mathcal{M}$的一组基，特别的，$f$为满射.
(3) $\Rightarrow$ (1): 设$\beta_1,\cdots,\beta_n$为$\mathcal{M}$的一组基，因为f为满射, 则存在$\alpha_1,\cdots,\alpha_n$使得$f(\alpha_i)=\beta_i$. 易知$\alpha_1,\cdots,\alpha_n$线性无关. 又因为$\dim\mathcal{L}=\dim\mathcal{M}$, 所以$\alpha_1,\cdots,\alpha_n$为$\mathcal{L}$的一组基. 所以$f(\alpha)=0\Rightarrow\alpha=0$, f为单射(线性映射)，所以f为同构. □

37.

Proof. $a\neq 1$

$$f:\mathcal{R}^+\longrightarrow\mathcal{R} \qquad g:\mathcal{R}\rightarrow\mathcal{R}^+$$

$$k\longrightarrow log_a k \qquad x\longrightarrow a^x$$

$\square$

38.

Proof.

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 9 & -7 \end{pmatrix}.$$

$\square$

39.

Proof.

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

$\square$

40.

Proof.

$$A = \begin{pmatrix} 4 & -2 & -11 \\ 0 & 2 & 7 \end{pmatrix}.$$

$\square$

41.

Proof.

$$(1)A = \begin{pmatrix} a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \\ a_{12} & a_{23} & a_{11} \end{pmatrix}; (2)A = \begin{pmatrix} a_{11} & \frac{a_{12}}{k} & \frac{a_{13}}{k} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{pmatrix};$$
$$(3)A = \begin{pmatrix} \frac{a_{11}+a_{21}+a_{12}+a_{22}}{2} & \frac{a_{11}+a_{21}-a_{12}-a_{22}}{2} & \frac{a_{13}+a_{23}}{2} \\ \frac{a_{11}-a_{21}+a_{12}-a_{22}}{2} & \frac{a_{11}-a_{21}-a_{12}+a_{22}}{2} & \frac{a_{13}-a_{23}}{2} \\ a_{31}+a_{32} & a_{31}-a_{32} & a_{33} \end{pmatrix}.$$

$\square$

42.

Proof. (1) $\forall A, B \in \mathcal{S}, (A+B)^t = A^t + B^t = A + B, (kA)^t = kA.$

(3) $\dim\mathcal{S} = \frac{n(n+1)}{2}.$

(4)$\forall A, B \in \mathcal{S}, T(A)^t = (X^tAX)^t = X^tA^tX = X^tAX \in \mathcal{S};$

$T(k_1A + k_2B) = X^t(k_1A + k_2B)X = k_1X^tAX + k_2X^tBX.$

(5)

$$A = \begin{pmatrix} 1 & 9 & 6 \\ 4 & 16 & 16 \\ 2 & 12 & 10 \end{pmatrix}.$$

$\square$

43.

Proof.

$$A_1=\begin{pmatrix} a & 0 & b & 0\\ 0 & a & 0 & b\\ c & 0 & d & 0\\ 0 & c & 0 & d\end{pmatrix};A_2=\begin{pmatrix} a & c & 0 & 0\\ b & d & 0 & 0\\ 0 & 0 & a & c\\ 0 & 0 & b & d\end{pmatrix}.$$

□

45.

Proof. 由题意知

$$(\alpha_1,\cdots,\alpha_n)=(\eta_1,\cdots,\eta_n)A;(\beta_1,\cdots,\beta_n)=(\eta_1,\cdots,\eta_n)B.$$

$$f(\eta_1,\cdots,\eta_n)=f(\alpha_1,\cdots,\alpha_n)A^{-1}=(\beta_1,\cdots,\beta_n)A^{-1}=(\eta_1,\cdots,\eta_n)BA^{-1}.$$

□

46.

Proof. (1)

$$a_0\eta+a_1f(\eta)+\cdots+a_{k-1}f^{k-1}(\eta)=0,$$

作用f^{k-1},

$$f^{k-1}(\sum_{i=0}^{k-1}a_if^i\eta)=a_0f^{k-1}(\eta)=0,$$

因为$f^{k-1}(\eta)\neq 0$, 所以$a_0=0$. 同理可知$a_i=0$. 所以$\eta,\cdots,f^{k-1}(\eta)$线性无关.

(2)由(1)知$\eta,f(\eta),\cdots,f^{n-1}(\eta)$线性无关, $\dim\mathcal{L}=n$, 所以$\eta,f(\eta),\cdots,f^{n-1}(\eta)$为$\mathcal{L}$的一组基. 易知此基即为所求. □

47.

Proof. (1) 如果$\mathcal{V}$包含不止一个元素，那么存在$0\neq v\in\mathcal{V}$, 于是$kv\in\mathcal{V},k\in\mathcal{F}$.

(2) 设$v_1,\cdots,v_m$是$\mathcal{V}$的一组基，则$v_1,\cdots,v_m$在$\mathcal{L}$中线性无关，所以$\dim\mathcal{L}\geq m$.

(3)$\dim\mathcal{L}=\dim\mathcal{V}\leq\infty$. 设$v_1,\cdots,v_n$是$\mathcal{V}$的一组基，则$v_1,\cdots,v_n$也是$\mathcal{L}$的一组基，所以$\mathcal{V}=\mathcal{L}$. □

48.

Proof. (1)$2(\alpha+\beta)\in\alpha+\mathcal{V}\Rightarrow 2\alpha+2\beta=\alpha+\beta'$, 其中$\beta'\in\mathcal{V}$. 所以$\alpha=\beta-\beta'\in\mathcal{V}$. 此时$\alpha+\mathcal{V}=\mathcal{V}$.

(2)$\alpha_1+\mathcal{V}=\alpha_2+\mathcal{V}\Rightarrow\alpha_1+\beta=\alpha_2+\beta',\beta,\beta'\in\mathcal{V}$, $\alpha_1-\alpha_2=\beta'-\beta\in\mathcal{V}$.

$\alpha_1-\alpha_2\in\mathcal{V}$. 对任意$\beta\in\mathcal{V}$, $\alpha_1+\beta=\alpha_2+(\alpha_1-\alpha_2+\beta)$, $\alpha_1-\alpha_2+\beta\in\mathcal{V}$, 所以$\alpha_1+\mathcal{V}\subseteq\alpha_2+\mathcal{V}$. 同理可知$\alpha_2+\mathcal{V}\subseteq\alpha_1+\mathcal{V}$. □

51.

Proof. $\ker\mathbb{D}=\mathcal{F},\dim\ker\mathbb{D}=1$; $\mathrm{im}\,\mathbb{D}=\mathcal{F}[x]_{n-1},\dim\mathrm{im}\,\mathbb{D}=n-1$. □

52.

Proof.

$$f(\epsilon_1,\cdots,\epsilon_4)=(\varepsilon_1,\varepsilon_2,\varepsilon_3)A=(\varepsilon_1,\varepsilon_2,\varepsilon_3)\begin{pmatrix}1&1&0&0\\1&2&1&-1\\1&0&-1&1\end{pmatrix}$$

$\operatorname{rank}f=\operatorname{rank}A=2$, $\ker f=\operatorname{span}\{\alpha_1=(1,-1,10),\alpha_2=(-1,1,0,1)\}$; $\operatorname{im} f=\operatorname{span}\{\beta_1=(1,2,0),\beta_2=(0,1,-1)\}$. □

53.

Proof. (1)显然.

(2)设$A=(\alpha_1,\cdots,\alpha_n)$. $\operatorname{im} f=\mathcal{L}\{\alpha_1,\cdots,\alpha_n\}$, 所以$\dim\operatorname{im} f=\operatorname{rank}\{\alpha_1,\cdots,\alpha_n\}=r(A)\Rightarrow\dim\mathcal{S}=n-r(A)$. □

54.

Proof. $f:\mathcal{V}\to\mathcal{V}$, $\forall x\in\operatorname{im} f^r$, $\exists v\in\mathcal{V}$, s.t.

$$x=f^r(v)=f^{r-1}(f(v))\Rightarrow x\in\operatorname{im} f^{r-1},$$

因此$\operatorname{im} f^{r-1}\supseteq\operatorname{im} f^r$.

$\forall x\in\ker f^{r-1}, f^{r-1}(x)=0,\ f^r(x)=f(f^{r-1}(x))=0,$

$\Rightarrow x\in\ker f^r, i.e.\ \ker f^{r-1}\subseteq\ker f^r$. □

55.

Proof.

$$\begin{aligned}A:\qquad \mathcal{L}(\alpha_1,\cdots,\alpha_s)&\longrightarrow\mathcal{F}^m\\ X&\longrightarrow AX\end{aligned}$$

所以$\dim\mathcal{L}(A\alpha_1,\cdots,A\alpha_s)=\dim\mathcal{L}(\alpha_1,\cdots,\alpha_s)-\dim\ker A=r-\dim\ker A$.

而$\dim\ker A\le n-r(A)$, 所以$\dim\mathcal{L}(A\alpha_1,\cdots,A\alpha_s)\ge r+r(A)-n$. □

56.

Proof. 注意一般$(\mathcal{V}_1+\mathcal{V}_2)\cap\mathcal{V}_3\ne(\mathcal{V}_1\cap\mathcal{V}_3)+(\mathcal{V}_2+\mathcal{V}_3)$.

我们需要证明$\mathcal{V}_1+(\mathcal{V}_2+\mathcal{V}_3)$和$(\mathcal{V}_1+\mathcal{V}_2)\cap\mathcal{V}_3$相互包含.

$\forall x=v_1+y\in\mathcal{V}_1+(\mathcal{V}_2\cap\mathcal{V}_3)$, $v_1\in\mathcal{V}_1, y\in\mathcal{V}_2\cap\mathcal{V}_3$, 因为$\mathcal{V}_1\subset\mathcal{V}_3$,

$$\therefore v_1\in\mathcal{V}_3,\Rightarrow v_1+y\in\mathcal{V}_3, v_1+y\in\mathcal{V}_1+\mathcal{V}_2,$$

$$\Rightarrow x\in(\mathcal{V}_1+\mathcal{V}_2)\cap\mathcal{V}_3, i.e.\mathcal{V}_1+(\mathcal{V}_2\cap\mathcal{V}_3)\subseteq(\mathcal{V}_1+\mathcal{V}_2)\cap\mathcal{V}_3.$$

同理可证$(\mathcal{V}_1+\mathcal{V}_2)\cap\mathcal{V}_3\subseteq\mathcal{V}_1+(\mathcal{V}_2\cap\mathcal{V}_3)$. □

57.

Proof. (1)$\mathcal{V}1+\mathcal{V}_2=\mathcal{L}(\alpha_1,\alpha_2,\beta_1,\beta_2)$, 易知$\alpha_1,\beta_1,\beta_2$线性无关. 所以$\mathcal{V}_1+\mathcal{V}_2=\mathcal{L}(\alpha_1,\beta_1,\beta_2)$. $\dim\mathcal{V}_1\cap\mathcal{V}_2=4-3=1,\beta_2=\alpha_1-\alpha_2,\Rightarrow\mathcal{V}_1\cap\mathcal{V}_2=\mathcal{L}(\beta_2)$.

(2) $\dim\mathcal{V}_1+\mathcal{V}_2=4$, $\dim\mathcal{V}_1\cap\mathcal{V}_2=1$. □

58.

Proof. Since $\mathcal{V}_1, \mathcal{V}_2$ are two non-trivial subspaces of $\mathcal{L}$, there exists $\alpha_1 \in \mathcal{L}, \alpha_1 \overline{\in} \mathcal{V}_1$ and $\alpha_2 \in \mathcal{L}, \alpha_2 \overline{\in} \mathcal{V}_2$. Consider $\alpha_1 + k\alpha_2$, where $k \in \mathcal{F}$, we claim that there must be one k such that $\alpha_1 + k\alpha_2 \overline{\in} \mathcal{V}_1$ and $\alpha_1 + k\alpha_2 \overline{\in} \mathcal{V}_2$. Suppose not, then there are at least k_1, k_2 such that $\alpha_1 + k_1\alpha_2, \alpha_1 + k_2\alpha_2 \in \mathcal{V}_1$(or $\mathcal{V}_2$). Consider $\alpha_1 + k_1\alpha_2 - (\alpha_1 + k_2\alpha_2) = (k_1 - k_2)\alpha_2 \in \mathcal{V}_1$. In particular, $\alpha_2 \in \mathcal{V}_1$ which implies $\alpha_1 \in \mathcal{V}_1$. Contradiction.

(1) is trivial, since we have find a $\alpha_1 + k\alpha_2 \overline{\in} \mathcal{V}_1$ and $\alpha_1 + k\alpha_2 \overline{\in} \mathcal{V}_2$.

(2)$\mathcal{V}_1 \not\subseteq \mathcal{V}_2, \mathcal{V}_2 \not\subseteq \mathcal{V}_1$, we have $v_1 \in \mathcal{V}_1, v_1 \overline{\in} \mathcal{V}_2$ and $v_2 \in \mathcal{V}_2, v_2 \overline{\in} \mathcal{V}_1$, then $v_1 + v_2 \overline{\in} \mathcal{V}_1 \cup \mathcal{V}_2$ which implies $\mathcal{V}_1 \cup \mathcal{V}_2$ is not a subspace of $\mathcal{L}$. □

60.

Proof. (1)$\dim\mathcal{L}(\alpha_1, \alpha_2, \alpha_3) = \mathrm{rank}(\alpha_1, \alpha_2, \alpha_3)$.

(2)$\mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2, \alpha_3) = \mathcal{L}(\alpha_1) \oplus \mathcal{L}(\alpha_2, \alpha_3) \Leftrightarrow \mathcal{L}(\alpha_1) \cap \mathcal{L}(\alpha_2, \alpha_3) \Leftrightarrow \alpha_1$ can not be represented by α_2, α_3. □

61.

Proof. 易知$\mathcal{L}(\alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_t) = \mathcal{L}(\alpha_1, \cdots, \alpha_s) + \mathcal{L}(\beta_1, \cdots, \beta_t)$. 我们只需证明

$$\mathcal{L}(\alpha_1, \cdots, \alpha_s) \cap \mathcal{L}(\beta_1, \cdots, \beta_t) = 0.$$

设$0 \neq x \in \mathcal{L}(\alpha_1, \cdots, \alpha_s) \cap \mathcal{L}(\beta_1, \cdots, \beta_t)$,则存在不全为零的$k_i, l_j$

$$x = \sum_{i=1}^{s} k_i\alpha_i = \sum_{j=i}^{t} l_j\beta_j,$$

$$\sum_{i=1}^{s} k_i\alpha_i - \sum_{j=i}^{t} l_j\beta_j = 0$$

与$\alpha_1, \cdots, \alpha_s, \beta_1, \cdots, \beta_t$线性相关矛盾. □

62.

Proof. (1) 错误. 考虑$\mathcal{L} = \mathcal{R}^2$, $\mathcal{V}_1 = \{x = 0\}, \mathcal{V}_2 = \{y = 0\}, \mathcal{V}_3 = \{y = x\}$.

(2) 错误. 同上.

(3) 错误. 考虑$A : \mathcal{R}^3 \to \mathcal{R}^3$.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

则$(1, 0, 0) \in \mathsf{ker}\, A$ 且$(1, 0, 0) \in \mathsf{im}\, A$. □

63.

Proof. (1) 对任意的$f(x), g(x) \in \mathcal{V}$, $f(x_0) + g(x_0) = 0 \Rightarrow f(x) + g(x) \in \mathcal{V}$, $kf(x_0) = 0 \Rightarrow kf(x) \in \mathcal{V}$. 所以$\mathcal{V}$ 为$\mathcal{R}_n[x]$的子空间.

(2)$f(x_0) = 0 \Rightarrow (x - x_0)|f(x) \Rightarrow f(x) = g(x)(x - x_0), g(x) \in \mathbb{R}_{n-1}[x]$. 所以

$$\mathcal{V} = \{g(x)(x - x_0)|g(x \in \mathbb{R}_{n-1}[x]\}.$$

$\dim\mathcal{V} = n-1, \dim\mathcal{W} = 1$. 易知$\mathcal{W} = \mathbb{R}, \mathcal{W} \cap \mathcal{V} = 0$. $p : \mathbb{R}_n[x] \to \mathcal{W}, p(f(x)) = f(x_0)$. 因为$f(x) = (f(x) - f(x_0)) + f(x_0)$.

(3)设$\mathcal{V}=\{f\in\mathcal{C}[a,b]|f(x_0)=0\}$. 假设$g\in\mathcal{V}\cap\mathbb{R}$. $g(x_0)=0\Rightarrow g=0$, 特别的g为$[a,b]$上的零函数. 对任意$f\in\mathcal{C}[a,b]$, $f=(f-f(x_0))+f(x_0)$, 其中$f-f(x_0)\in\mathcal{V}$, 所以$\mathcal{C}[a,b]\subseteq\mathcal{V}+\mathbb{R}\subseteq\mathcal{C}[a,b]$. 因此$\mathcal{C}[a,b]=\mathcal{V}\oplus\mathbb{R}$. □

64.

Proof. 错误. $\mathcal{L}=\mathbb{R}^2$, $\mathcal{V}_1=\{x=0\},\mathcal{V}_2=\{y=0\},\mathcal{V}_3=\{y=x\}$. □

66.

Proof. 设$\beta_1,\cdots,\beta_t$ 为$\mathsf{im}\,f$ 的一组基，$\alpha_1,\cdots,\alpha_t\in\mathcal{L}$ 使得$f(\alpha_i)=\beta_i,1\leq i\leq t$. 易知$\alpha_1,\cdots,\alpha_t$ 线性无关. 令$\mathcal{U}=\mathcal{L}(\alpha_1,\cdots,\alpha_t)$，则$f|_{\mathcal{U}}:\mathcal{U}\to\mathsf{im}\,f$ 为同构. 对任意的$v\in\mathcal{L}$, 设

$$f(v)=\sum_{i=1}^{t}k_i\beta_i.$$

考虑$\alpha=\sum_{i=1}^{t}k_i\alpha_i$, 则$f(v-\alpha)=0$, $v-\alpha\in\ker f$, $v=(v-\alpha)+\alpha\in\ker f+\mathcal{U}$. 下证$\ker f\cap\mathcal{U}=0$. 假设$x\in\ker f\cap\mathcal{U}$, $x=\sum l_i\alpha_i$, $f(x)=\sum l_i\beta_i=0$ 推出$l_i=0$，特别的, $x=0$. 所以

$$\mathcal{L}=\mathcal{U}\oplus\ker f.$$

□