NP and NP-Complete

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Some slides are courtesy of Dr. Mahini.

Overview

- Introduction
- P vs. NP
- Reduction
- NP-Complete
- NP-Complete Examples
- Sample Problems

Introduction

- What has this semester been about?
  - We've taken problems you probably knew how to solve slowly, and we figured out how to solve them faster.
- In some sense, that's the job of a computer scientist. Figure out how to take our problems and make the computer do the hard work for us.
- Let's take a big step back, and try to break problems into three types:
  1. Those for which a computer might be able to help.
  2. Those which would take so long to solve even on a computer we wouldn't expect to solve them.
  3. Those which a computer cannot solve regardless of how long we wait.
- There are problems we could solve in finite time...but we'll all be long dead before our computer tells us the answer.

Efficient (کارا یا کارآمد)

- We'll consider a problem *efficiently solvable*, if it has a polynomial time algorithm.
  - In other words, there's an algorithm which runs in $O(n^k)$, where k is a constant.
  - Are these algorithms always actually *optimum* (بهینه)?
    - Not necessarily!

- Your n^{10000} algorithm or even your $2^{2^{2^{2^2}}} . n^3$ algorithm probably aren't going to finish anytime soon.
  - But these edge cases are rare, and polynomial time is good as a low bar.
  - If we can't even find an n^{10000} algorithm, we should probably rethink our strategy.

Decision Problem vs. Optimization Problem

Optimization Problem

- Try to optimize
- More complex
- Examples
  - **Max flow**: Find the maximum flow
  - **Shortest-path**: Find the shortest path
  - **Knapsack**: Find the maximum possible value.

Decision Problem

- Output is *yes* or *no*
- Simple
- Examples
  - **Max flow**: Is there any feasible flow of size k?
  - **Shortest-path**: Is there any path of length less than or equal to k?
  - **Knapsack**: Is there any solution with the value of at least k?

Class P

A decision problem Q is in **P** if there is a **polynomial-time algorithm** A **called decider** such that for all inputs x:

- if $x \in Q$ then $A(x) = YES$,
- if $x \notin Q$ then $A(x) = NO$,

$Q \in \mathbf{P} \leftrightarrow [\exists A \text{ such that } \forall x: x \in Q \leftrightarrow A(x) = yes]$

Examples:

- Connectivity problem
- Shortest path problem
- Summation

Input: Undirected graph G
Output: Is G a connected graph?

Input: Directed weighted graph G and two vertices s and t and a value of k
Output: Is there any path between s and t with the length of at most k?

Input: Three numbers x, y, and z
Output: Is $x + y = z$?

Class **NP** (Non-deterministic Polynomial)

A decision problem Q is in **NP** if there is a **polynomial-time algorithm** V called **verifier** such that for all inputs x:

- if $x \in Q$ then there is a **certificate** y such that $V(x, y) = YES$,
- if $x \notin Q$ then for all certificates y we have $V(x, y) = NO$,

$Q \in \mathbf{NP} \leftrightarrow [\exists V \text{ such that } \forall x: x \in Q \leftrightarrow (\exists y \text{ such that } V(x, y) = yes)]$

Size of y should be a polynomial of size of x

Examples:

- Traveling Sales Man
- Clique
- Longest path

Input: Directed weighted graph G and a value of k
Output: Is there any tour of length at most k?

Input: Undirected graph G and a value of k
Output: Can we find k vertices in graph G such that they are all adjacent to each other?

Input: Directed weighted graph G and two vertices s and t and a value of k
Output: Is there any path between s and t with the length of at least k?

What if the $|y|$ is not polynomial in $|x|$?

- Note that the runtime of a verifier (or generally any algorithm) is defined in terms of the *input size*.
- If $|y|$ is arbitrarily large, then there is an algorithm that can verify a given NP problem, in polynomial time w.r.t. $|y|$, but not necessarily polynomial in $|x|$.
- Does a given program P **halts** (finishes) in less than or equal to 2^n steps?
  - This problem can be shown not to be in NP.
  - If we could use an arbitrarily large certificate, one can pass this certificate:

    $\langle c_0, c_1, c_2, \dots, c_m \rangle$, where c_i is the configuration of the program after each step.
  - A simple verifier can walk through these certificates and check the following properties:
    1. All c_i's are legit configurations.
    2. Any $c_i \rightarrow c_{i+1}$ is a legit step.
    3. $m \leq 2^n$

The $1M Question

- **The Clay Mathematics Institute:** [Millennium Prize Problems]

1. Birch and Swinnerton-Dyer Conjecture
2. Hodge Conjecture
3. Navier-Stokes Equations
4. **P vs. NP**
5. Poincaré Conjecture ← **Solved in 2002 by Grigori Perelman**
6. Riemann Hypothesis
7. Yang-Mills Theory

The P versus NP problem

- Is one of the biggest open problems in computer science (and mathematics) today.
- It's currently unknown whether there exist polynomial time algorithms for NP-complete problems
  - That is, does P = NP?
  - People generally believe P ≠ NP, but no proof yet.
- But what is the P-NP problem?

$P \subseteq NP$ Proof

A decision problem Q is in **P** if there is an **polynomial-time algorithm** A **called decider** such that for all inputs x:

- if $x \in Q$ then $A(x) = YES$,
- if $x \notin Q$ then $A(x) = NO$,

$Q \in \mathbf{P} \leftrightarrow [\exists A \text{ such that } \forall x: x \in Q \leftrightarrow A(x) = yes]$

$$V(x, y) = A(x)$$

A decision problem Q is in **NP** if there is an **polynomial-time algorithm** V called **verifier** such that for all inputs x:

- if $x \in Q$ then *<u>there is</u>* a **certificate** y such that $V(x, y) = YES$,
- if $x \notin Q$ then for all certificates y we have $V(x, y) = NO$,

$Q \in \mathbf{NP} \leftrightarrow [\exists V \text{ such that } \forall x: x \in Q \leftrightarrow (\exists y \text{ such that } V(x, y) = yes)]$

Is **P** a proper subset of **NP**?

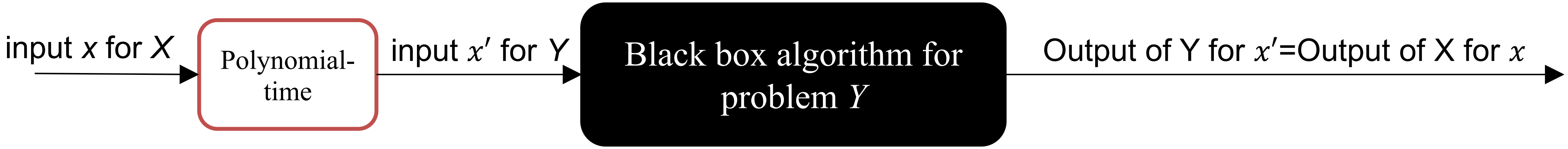
Polynomial-Time Reductions (تقلیل یا تحویل)

- The purpose of a reduction is to show that *some problem is at least as hard as some other problem*.
- If problem X reduces to problem Y, then solving Y implies solving X.
  - Y is at least as hard as X, denoted $X \leq Y$.
- Reduction types:
  - Karp reduction
    - We use it in this course to prove NP-hardness of problems.
  - Cook reduction
    - We don't talk about it in this course.
  - Levin reduction
    - We don't talk about it in this course.

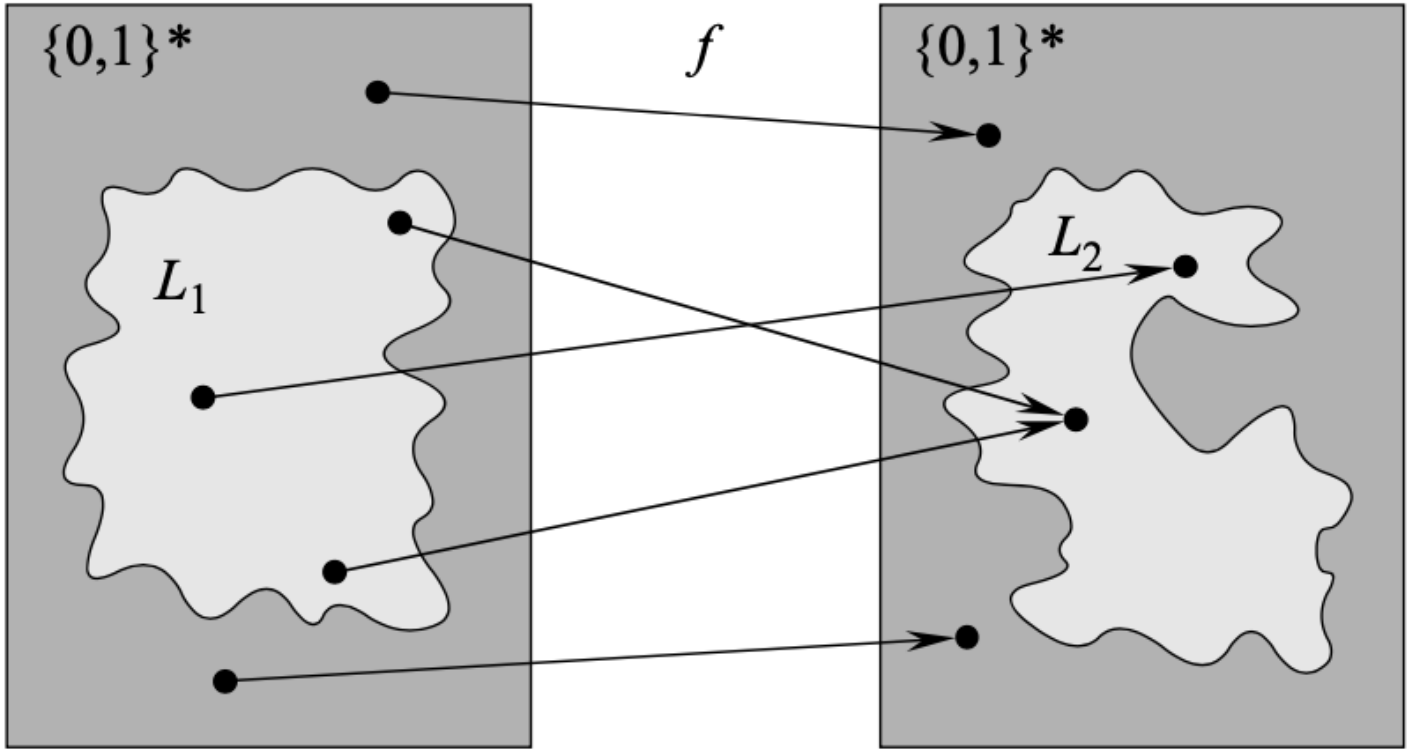
Karp Reduction

- A polynomial time reduction from a decision problem X to a decision problem Y is an algorithm f that has the following properties:
  - Given an instance I_X of X, f produces an instance I_Y of Y.
  - f runs in polynomial time w.r.t. $|I_X|$. This implies that $|I_Y|$ (size of I_Y) is polynomial in $|I_X|$
  - Answer to I_X YES iff answer to I_Y is YES. In other words, $x \in X \Longleftrightarrow f(x) \in Y$.
- Karp reduction is also called many-one reduction and polynomial transformations.
- **Notation:** $X \leq_P Y$ (or $X \leq_m^P Y$) if X reduces to Y.
- **Proposition:** If $X \leq_P Y$, then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

Richard M. Karp



Karp Reduction (cont'd)



Karp Reduction: Example 1

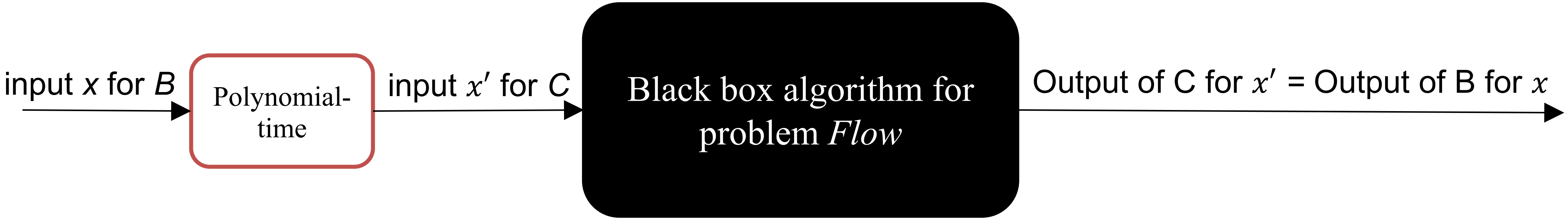
Matching

Input: Undirected bipartite graph G and a value of k
Output: Does G have a matching of size at least k?

Matching $\leq_p$ *Flow*

Flow

Input: Network flow G and a value of k
Output: Does G has a feasible flow of at least k?



Karp Reduction: Example 2

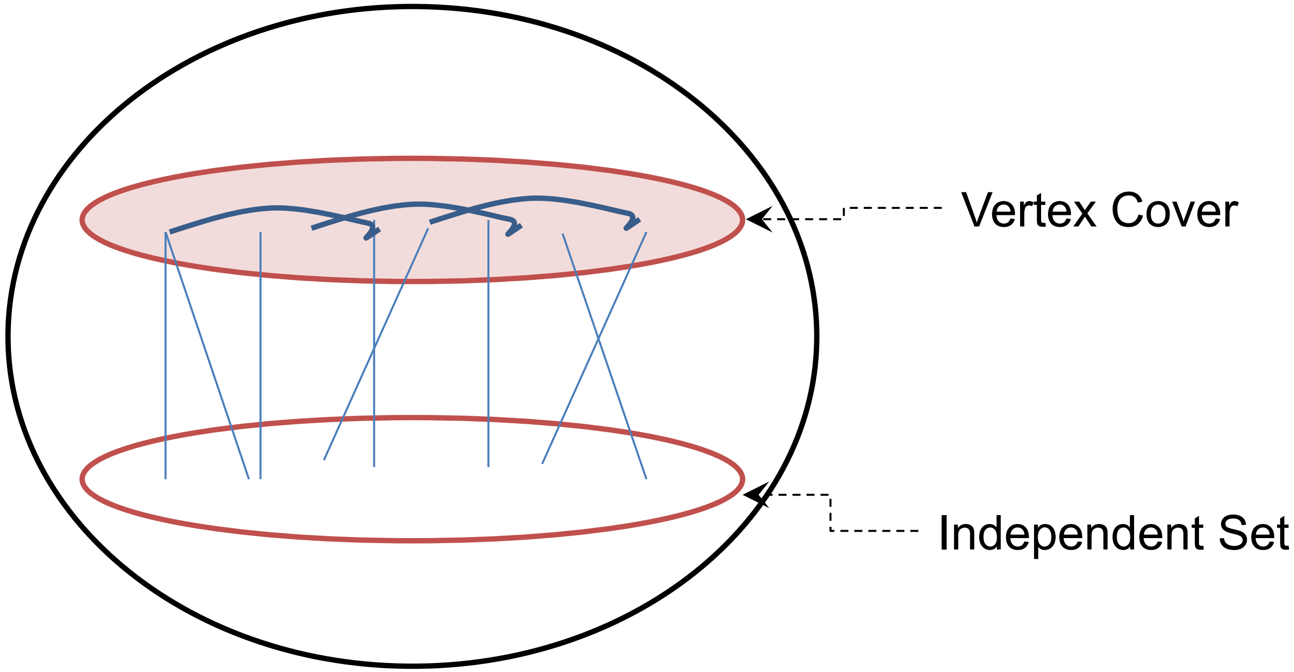
Independent Set

Input: Undirected graph G and a value of k
Output: Is there any set of vertices of size k in G that none of them are adjacent?

Independent Set $\leq_P$ Vertex Cover

Vertex Cover

Input: Undirected graph G and a value of k
Output: Can we color k vertices of G such that for each edge one of its endpoints is colored?



Karp Reduction: Example 2 (cont'd)

Independent Set

Input: Undirected graph G and a value of k
Output: Is there any set of vertices of size k in G that none of them are adjacent?

Independent Set $\leq_P$ Vertex Cover

Vertex Cover

Input: Undirected graph G and a value of k
Output: Can we color k vertices of G such that for each edge one of its endpoints is colored?

Polynomial-Time Karp Reduction Transitivity

- **Theorem:** If $A \leq_p B$ and $B \leq_p C$ then $A \leq_p C$.
- **Proof:**
  - Per definition, $\exists f, g$ such that $x \in A \Leftrightarrow f(x) \in B$ and y $\in B \Leftrightarrow g(y) \in C$.
  - $x \in A \Leftrightarrow f(x) \in B \Leftrightarrow g\big(f(x)\big) \in C$.
  - $g\big(f(.)\big)$ is polynomial because $f(x)$ is polynomial in x.

NP-Complete and NP-Hard

NP-Complete

- The most difficult problems in NP to solve
- If we can solve an NP-complete problem in polynomial time, we can solve all NP problems in polynomial time.

$Q \in \mathbf{NP-Complete}$:

- $Q \in \mathbf{NP}$
- $\forall Q' \in \mathbf{NP}$, we have $Q' \leq_p Q$

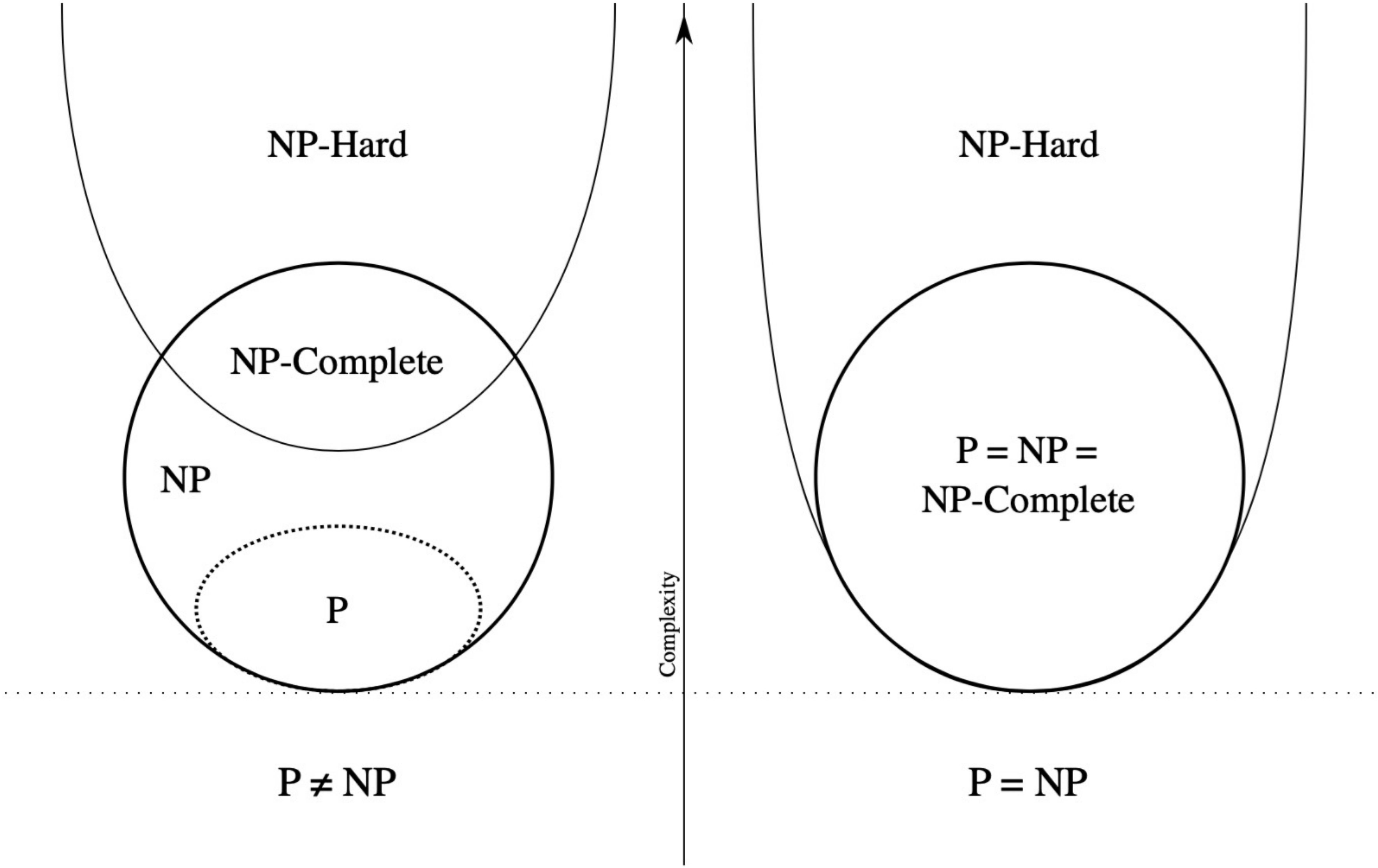
NP-Hard

- If we can solve an NP-hard problem in polynomial time, we can solve all NP problems in polynomial time.
- They are not necessarily in NP.

$Q \in \mathbf{NP-hard}$:

- $\forall Q' \in \mathbf{NP}$, we have $Q' \leq_p Q$

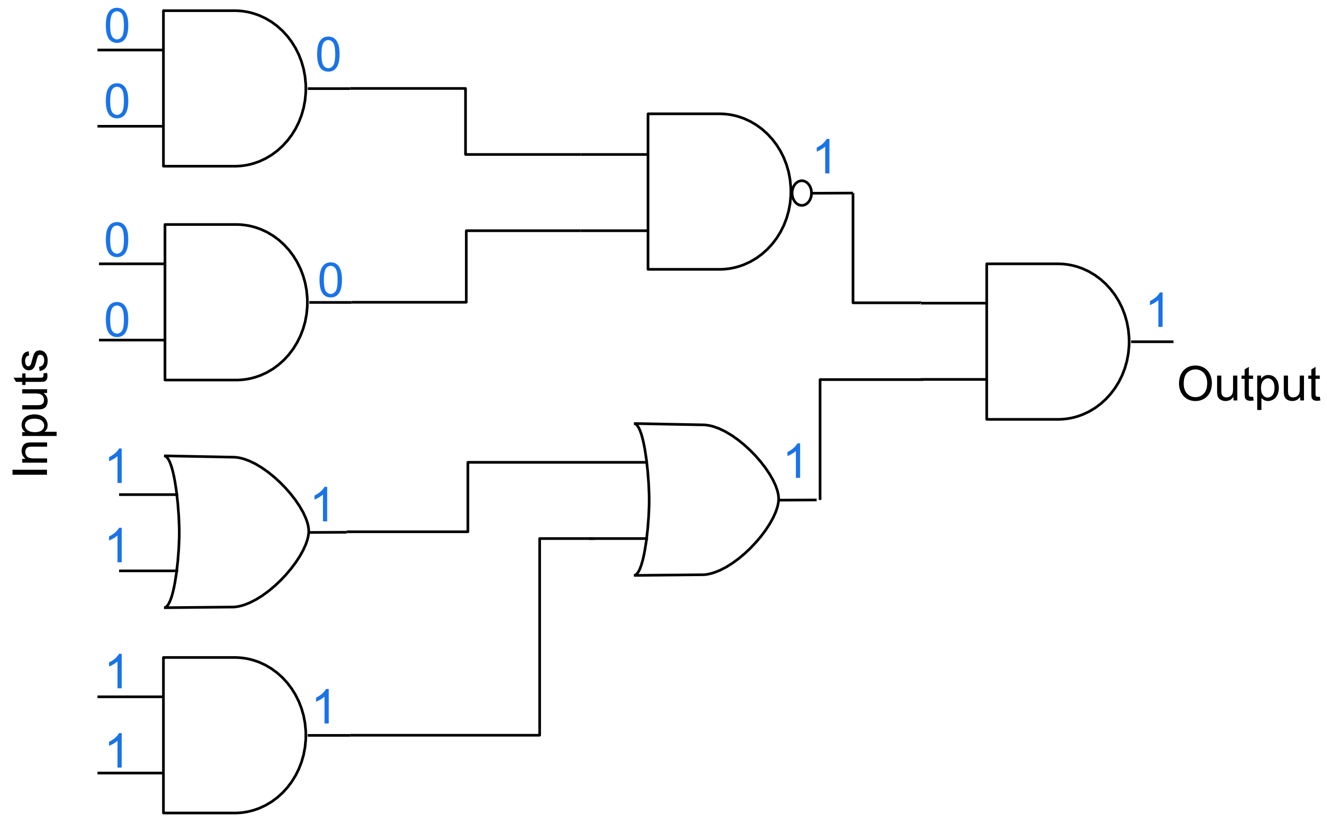
NP-Complete and NP-Hard



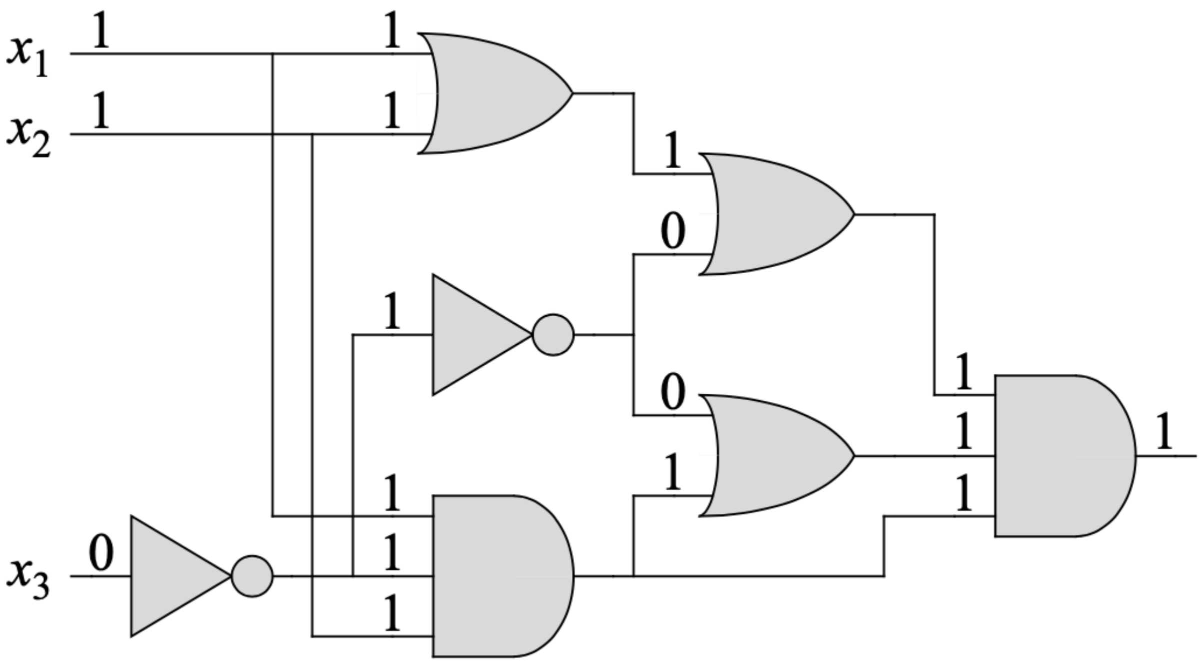
The First NP-Complete Problem

Circuit Satisfiability (CS)

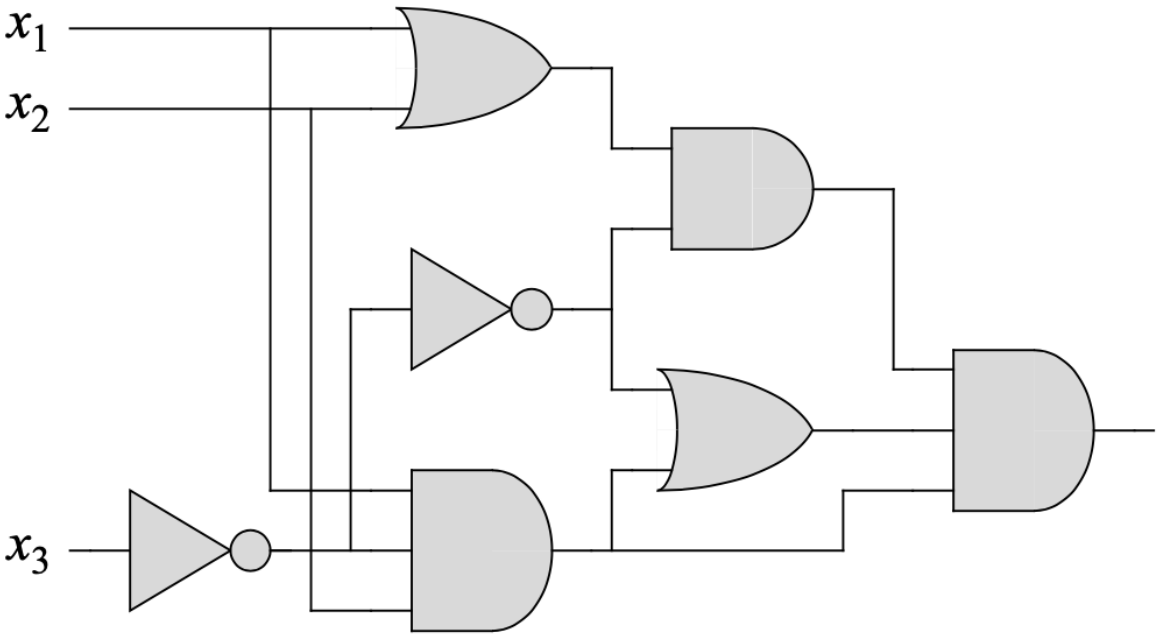
Input: Logical circuit with AND, OR, and NOT gates with n inputs, m gates and one output
Output: Can we set n inputs such that the output becomes 1?



CS Problem: Example



Satisfiable



Unsatisfiable

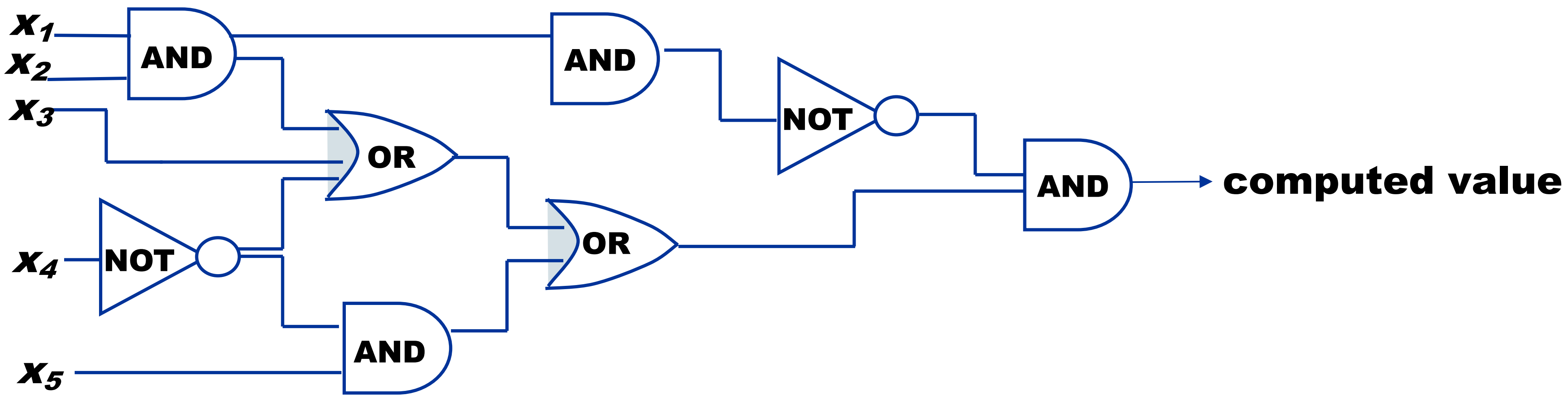
NP-Completeness of CS Problem: Proof

- This is the first NP-complete problem we prove.
- Two steps are required to prove the theorem:
  - Circuit-SAT ∈ **NP**
  - $\forall Q \in NP,$ we have $Q \leq_p$Circuit-SAT

$$Q \in \mathbf{NP} \leftrightarrow [\exists A \text{ such that } \forall x: x \in Q \leftrightarrow (\exists y \text{ such that } A(x, y) = yes)]$$

CS Problem is NP

- Lemma 1: Circuit-SAT is in NP
- Proof:
  - Must show that there exists a polynomial-time verifier.
  - We can easily check in polynomial time if truth assignment produces TRUE.



CS Problem is NP-Hard (1)

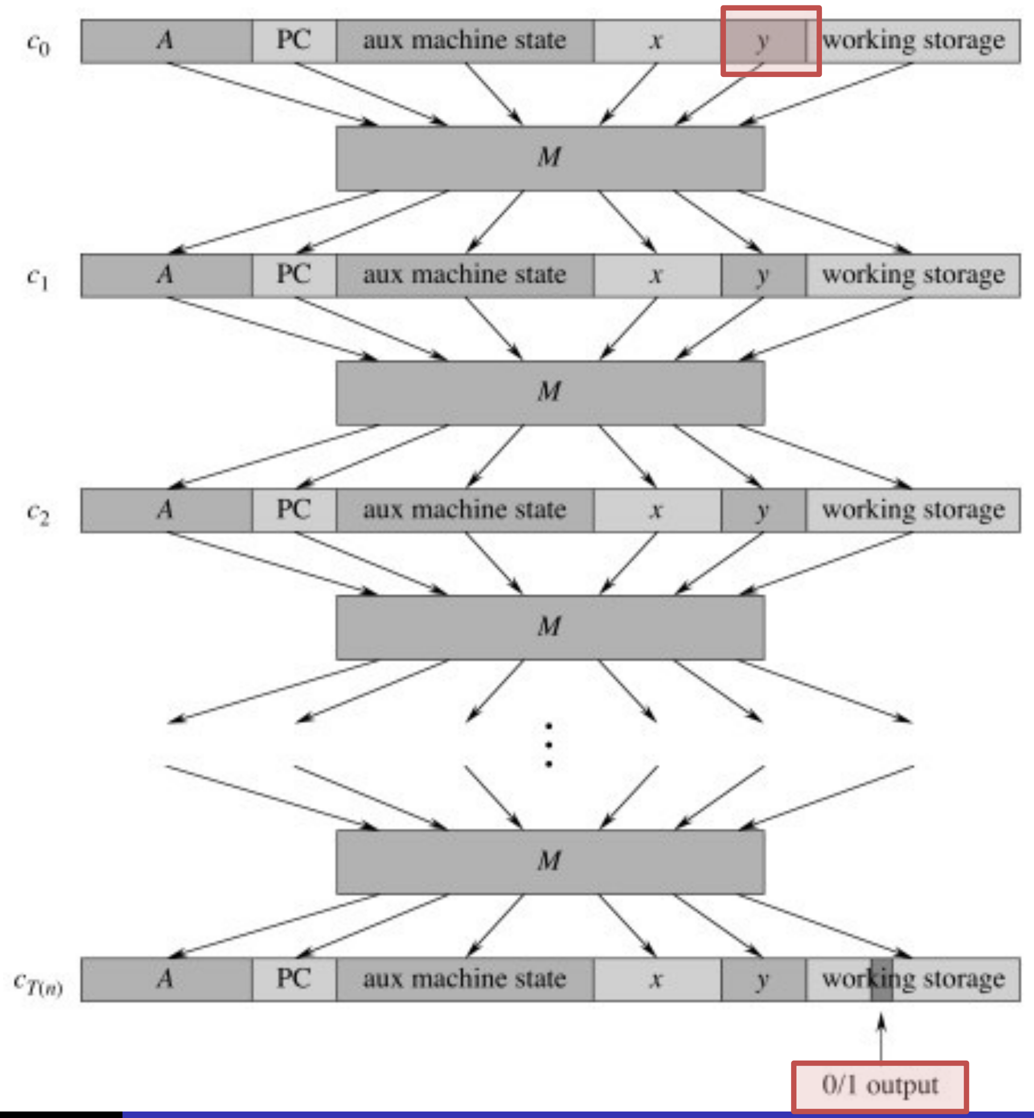
- Lemma 2: For every problem Q in NP, we have: $Q \leq_P$ Circuit-SAT
  - Or Circuit-SAT is NP-hard.
- Proof: (sketch of the proof; for full sketch, see CLRS 34.3)
  - If an algorithm runs in polynomial time, then there is a polynomial-size Boolean circuit that "implements" the algorithm, and such a circuit can be constructed in polynomial time.
  - Idea: Algorithm runs on computer that is essentially a Boolean circuit.
  - To complete the proof, we should create a reduction function f such that for any $x \in Q$, we have $\mathrm{f}(x) \in \mathrm{Circuit-SAT}$, and vice versa.

CS Problem is NP-Hard (2)

- Step 2: Describe algorithm that performs reduction:
  - Q is in NP, so Q has a polynomial-time verification algorithm $A(x,y)$ that checks if x is a "yes"-instance using certificate y.
  - Construct circuit implementing A.
    - Circuit runs in polynomial time.
      - Input has size polynomial.
      - A combinational circuit implementing a mapping on the polynomial-size input has size polynomial.
      - A polynomial-sized circuit can run in polynomial time.
  - "Fix" the variables corresponding to x according to the given input
  - Run Circuit-SAT:
    - Circuit-SAT returns "yes" iff x is a "yes" instance for problem Q.

CS Problem is NP-Hard (3)

Fix input x, we want to decide whether $x \in Q$



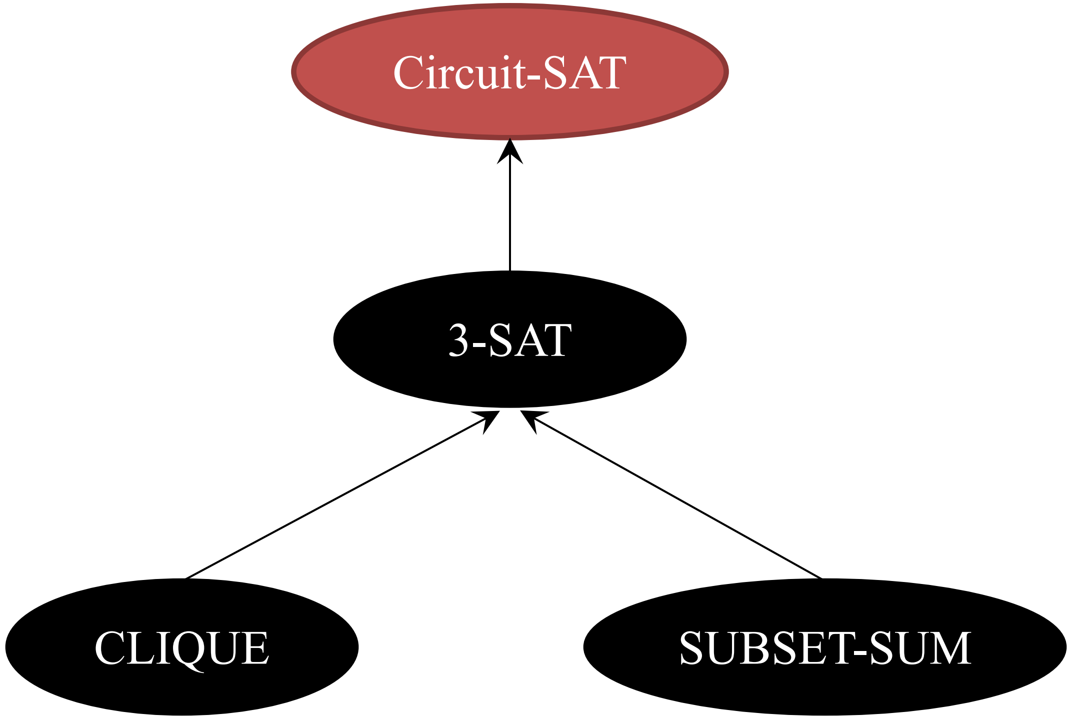
If runtime of Algorithm A for x is $T(n)$, then the size of our logical circuit is polynomial of $T(n)$.

NP-Complete Examples

Now, we have a strong tool to prove a new problem Q is NP-Complete

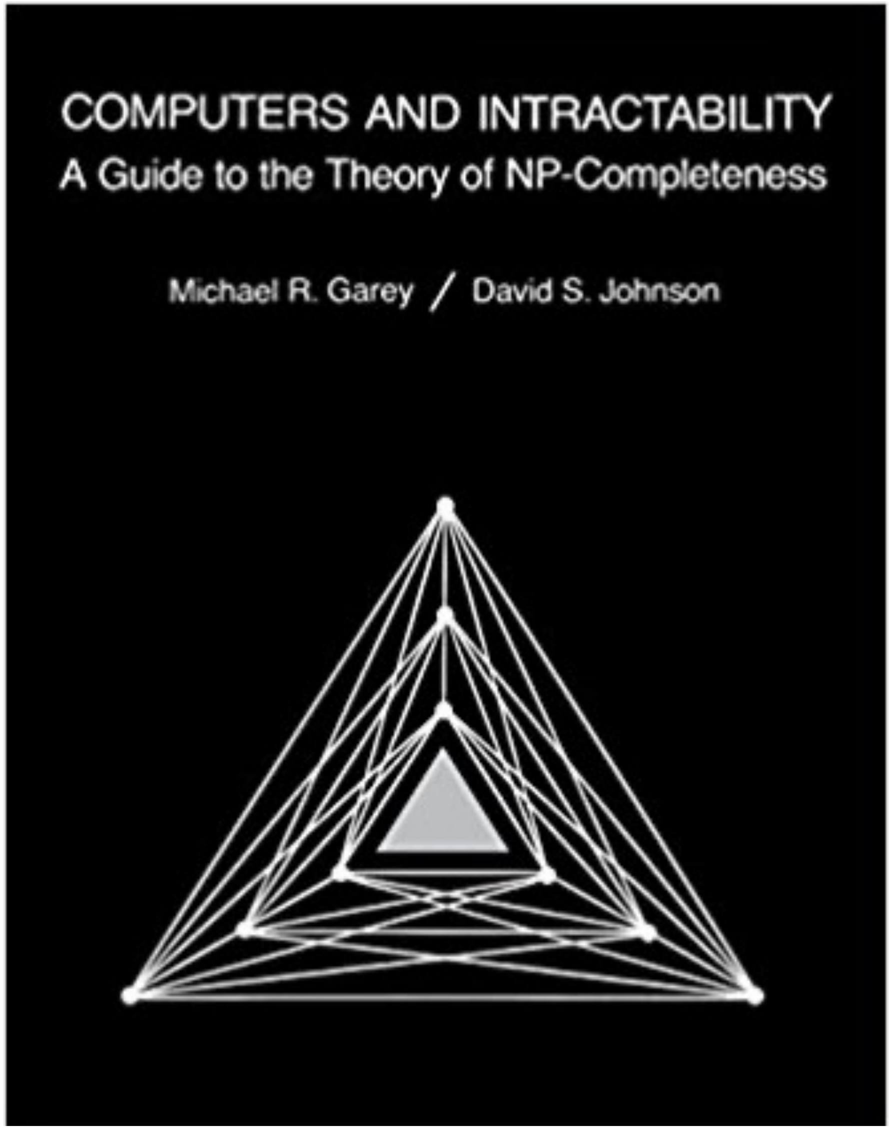
We just need to prove:

- $Q \in \mathbf{NP}$
- $CS \leq_p Q$



NP-Complete Problems

- By 1979, at least 300 problems had been proven NP-complete.
- Garey and Johnson put a list of all the NP-complete problems they could find at the time in this textbook.
- Took them almost 100 pages to just list them all.
- No one has made a comprehensive list since.



3-SAT Problem is NP-Complete

3-SAT (3-Conjunctive Normal Form or 3-CNF) Problem

- $x_1 \lor \overline{x_2} \lor x_3$
- $\overline{x_1} \lor \overline{x_2} \lor x_3$
- $\overline{x_1} \lor x_2 \lor \overline{x_3}$
- $x_1 \lor x_2 \lor \overline{x_3}$
- $\overline{x_1} \lor \overline{x_2} \lor x_3$

- $x_1 = 1$
- $x_2 = 0$
- $x_3 = 0$

True, Satisfiable

Satisfiability (SAT) and 3-SAT

Input: m Boolean clauses and n Boolean variable. Each clause is like $x_1 \lor \overline{x_2} \lor x_4$ (with only $\lor$ and **NOT** operators)
Output: Can we find an assignment such that all Boolean clauses become true?

- $\overline{x_1} \lor \overline{x_2}$
- $\overline{x_1} \lor x_2$
- $x_1 \lor x_2$
- $x_1 \lor \overline{x_2}$

Not Satisfiable

- $x_1 = 1$
- $x_2 = 0$
- $x_3 = 1$

3-SAT Example

$(x_1 \lor \overline{x_2} \lor x_3) \land (x_1 \lor x_2 \lor \overline{x_3}) \land (\overline{x_1} \lor \overline{x_2} \lor x_3)$

True, Satisfiable

3-SAT is NP

Two steps:

- 3-SAT $\in$ **NP**
- $CS \leq_p$ 3-SAT(it means $\forall Q \in NP$ *we have* $Q \leq_p$ 3-SAT)

How to design a poly-time verifier for 3-SAT to prove 3-SAT $\in$ **NP**?

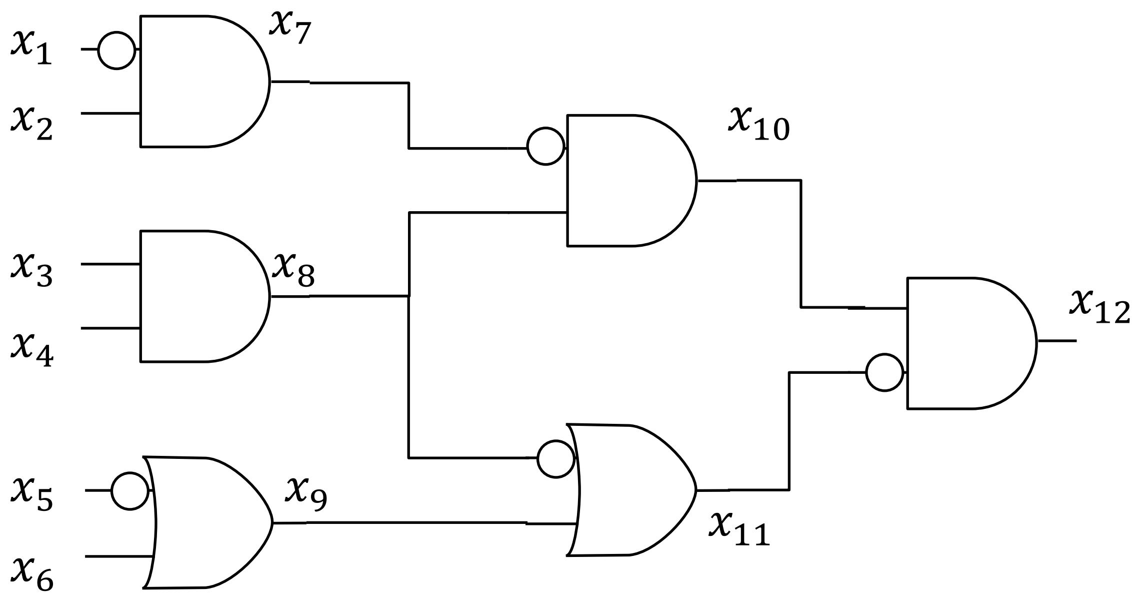
3-SAT is NP-Hard (1)

Two steps:

- 3-SAT$\in$ **NP**
- $CS \leq_p$3-SAT (it means $\forall Q \in NP$ we have $Q \leq_p$3-SAT)

For any Boolean circuit, one can:

1. Break each Boolean gates into 2-input gates.
2. Each of the intermediate results are stored in a variable.
3. Boolean equation equivalent of the circuit is written.
4. The formula is satisfiable when all of intermediate equations are satisfied. Hence, we can AND them together.



- $x_7 \leftrightarrow \overline{x_1} \land x_2$
- $x_8 \leftrightarrow x_3 \land x_4$
- $x_9 \leftrightarrow \overline{x_5} \lor x_6$
- $x_{10} \leftrightarrow \overline{x_7} \land x_8$
- $x_{11} \leftrightarrow \overline{x_8} \lor x_9$
- $x_{12} \leftrightarrow \overline{x_{11}} \land x_{10}$
- x_{12}

$$(x_7 \leftrightarrow \overline{x_1} \land x_2) \land (x_8 \leftrightarrow x_3 \land x_4) \land (x_9 \leftrightarrow \overline{x_5} \lor x_6) \land (x_{10} \leftrightarrow \overline{x_7} \land x_8) \land (x_{11} \leftrightarrow \overline{x_8} \lor x_9) \land (x_{12} \leftrightarrow \overline{x_{11}} \land x_{10}) \land x_{12}$$

3-SAT is NP-Hard (2)

Two steps:

- 3-SAT$\in$ **NP**
- $CS \leq_p$ 3-SAT (it means $\forall Q \in NP$ *we have* $Q \leq_p$3-SAT)

$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	$\phi = x_3 \leftrightarrow x_1 \wedge \overline{x_2}$
1	1	1	0
1	0	1	1
0	1	1	0
0	0	1	0
1	1	0	1
1	0	0	0
0	1	0	1
0	0	0	1

$$\bar{\phi} = (x_1 \wedge x_2 \wedge x_3) \vee (\overline{x_1} \wedge x_2 \wedge x_3) \vee (\overline{x_1} \wedge \overline{x_2} \wedge x_3) \vee (x_1 \wedge \overline{x_2} \wedge \overline{x_3})$$

DeMorgan's law:

- $\overline{(a \wedge b)} = \bar{a} \vee \bar{b}$
- $\overline{(a \vee b)} = \bar{a} \wedge \bar{b}$

$$\phi = (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3)$$

3-SAT is NP-Hard (4)

Two steps:

- 3-SAT$\in$ **NP**
- $CS \leq_p$3-SAT(it means $\forall Q \in NP$ *we have* $Q \leq_p$3-SAT)

Last step: How to convert everything to clauses with 3 variables?

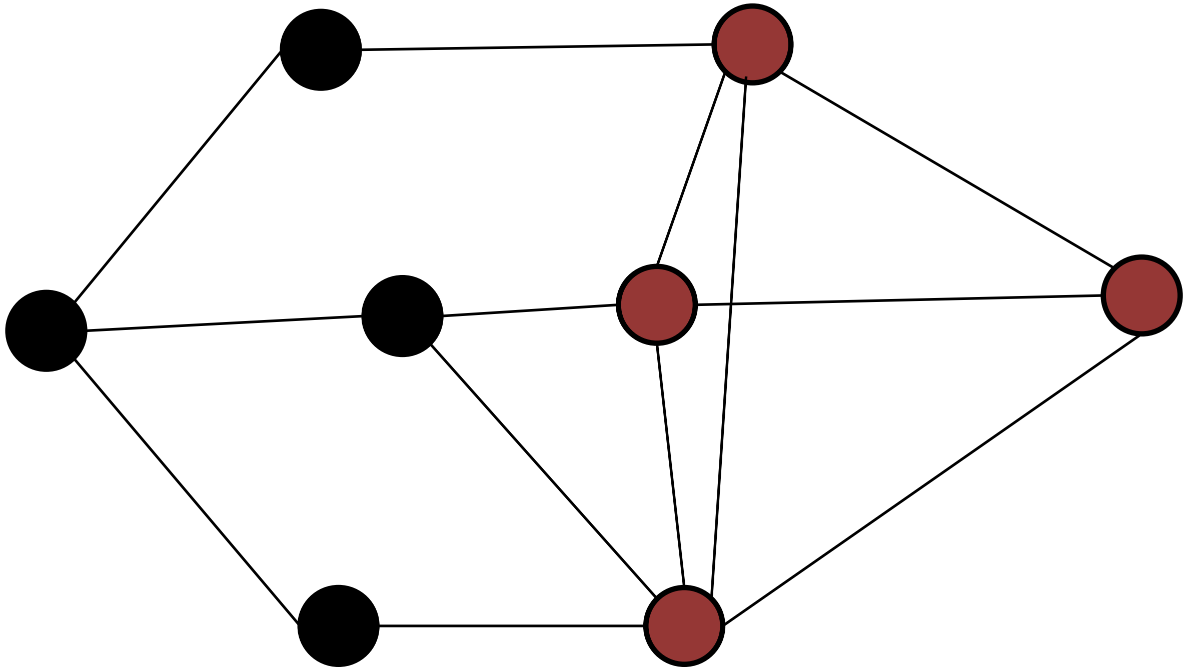
- If a clause has 2 literals, it can be converted to 3 literals as follows:
  - $(l_1 \vee l_2) \rightarrow (l_1 \vee l_2 \vee p) \wedge (l_1 \vee l_2 \vee \bar{p})$
- If a clause has 1 literal, it can be converted to 3 literals as follows:
  - $l \rightarrow (l \vee p \vee q) \wedge (l \vee p \vee \bar{q}) \wedge (l \vee \bar{p} \vee q) \wedge (l \vee \bar{p} \vee \bar{q})$

CLIQUE Problem is NP-Complete

CLIQUE problem

CLIQUE

Input: Undirected graph G and a value of k
Output: Can we find k vertices in graph G such that there are all adjacent to each other?



Has a clique of size 4 but not 5

CLIQUE is NP

Two steps:

- CLIQUE $\in$ **NP**
- 3-SAT$\leq_p$CLIQUE (it means $\forall Q \in NP$ we have $Q \leq_p$ CLIQUE)

How to design a verifier to prove CLIQUE $\in$ **NP**?

CLIQUE is NP-Hard

Two steps:

- CLIQUE$\in$ **NP**
- 3-SAT$\leq_p$ CLIQUE (it means $\forall Q \in NP$ we have $Q \leq_p$ CLIQUE)

- Construct graph $G = (V, E)$ as follows:
  - Introduce a node for each *literal* in each clause.
  - Put edge between each pair of nodes such that
    - Nodes are in different clauses.
    - Nodes are not each other's opposite.

CLIQUE Problem Reduction Example

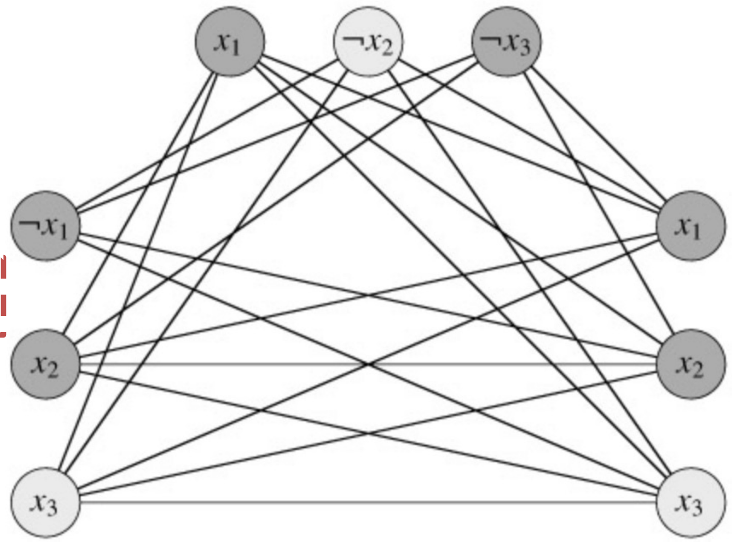
Two steps:

- CLIQUE$\in$ **NP**
- 3-SAT$\leq_p$CLIQUE (it means $\forall Q \in NP$ we have $Q \leq_p$CLIQUE)

$(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$

$x_1 \vee \overline{x_2} \vee \overline{x_3}$

$\overline{x_1} \vee x_2 \vee x_3$



$x_1 \vee x_2 \vee x_3$

Does it have a clique of size k?

- Should select exactly one vertex from each clause.
- Set the value of selected vertex to 1 in 3-SAT instance

Formula Is Satisfiable ⇔ A Clique of Size *k* Exists

- Formula is satisfiable ⇒ A clique of size *k* exists
  - Assume the formula with *k* clauses is satisfiable.
  - For each clause, select the TRUE node.
  - Then these nodes must form a clique.
- A clique of size *k* exists ⇒ Formula is satisfiable
  - Assume G has a clique of size at least *k*.
  - Set variables such that these nodes evaluate to TRUE.
  - Must be a consistent setting that makes formula satisfiable true.
- It suffices to show that CLIQUE problem is NP-hard in this special case. Why?
  - If we had a polynomial-time algorithm that solved clique on general graphs, it would also solve CLIQUE on restricted graphs.

SUBSET-SUM Problem is NP-Complete

Subset-Sum problem

SUBSET-SUM

Input: Set $S = \{x_1, x_2, \dots, x_n\}$ with integer values and a value of t
Output: Is there any subset of S such that sum of its elements is equal to t?

- $S = \{1, 2, 5, 10, 11\}$ and $t = 17$
- Answer is *yes* because of $\{2, 5, 10\}$

- $S = \{1, 2, 5, 10, 11\}$ and $t = 19$
- Answer is *yes* because of $\{1, 2, 5, 11\}$

- $S = \{1, 2, 5, 10, 11\}$ and $t = 20$
- Answer is *no*

Subset-Sum is NP

Two steps:

- SUBSET-SUM ∈ **NP**
- 3-SAT$\leq_p$SUBSET-SUM (it means $\forall Q \in NP$ we have $Q \leq_p$SUBSET-SUM)

How to design a verifier to prove SUBSET-SUM ∈ **NP**?

SUBSET-SUM is NP-Hard (1)

Two steps:

- SUBSET-SUM$\in$ **NP**
- 3-SAT$\leq_p$SUBSET-SUM (it means $\forall Q \in NP$ we have $Q \leq_p$SUBSET-SUM)

- n variables x_i and m clauses C_j
- For each variable x_i , construct numbers v_i and v_i' of $n + m$ digits:
  - The i-th digit of v_i and v_i' is equal to 1.
  - For $n + 1 \leq j \leq n + m$, the j-th digit of v_i is equal to 1 if x_i is in clause C_{j-n}
  - For $n + 1 \leq j \leq n + m$, the j-th digit of v_i' is equal to 1 if $\overline{x_i}$ is in clause C_{j-n}
- All other digits of v_i and v_i' are 0.
- Example:
  - $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$

	x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	1	0	0	1	0	0	1
v_1'	1	0	0	0	1	1	0
v_2	0	1	0	0	0	0	1
v_2'	0	1	0	1	1	1	0
v_3	0	0	1	0	0	1	1
v_3'	0	0	1	1	1	0	0

Subset-Sum is NP-Hard (2)

Two steps:

- SUBSET-SUM∈ **NP**
- 3-SAT$\leq_p$SUBSET-SUM (it means $\forall Q \in NP$ we have $Q \leq_p$SUBSET-SUM)

- For each clause C_j, construct *slack variables* s_j and s_j' of $n + m$ digits:
  - The $(n+j)$-th digit of s_j is equal to 1.
  - The $(n+j)$-th digit of s_j' is equal to 2.
  - All other digits of s_j and s_j' are 0.
- Finally, construct a sum number t of n + m digits:
  - For $1 \leq j \leq n$, the j-th digit of t is equal to 1.
  - For $n + 1 \leq j \leq n + m$, the j-th digit of t is equal to 4.
- Example:
  - $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$

	x_1	x_2	x_3	C_1	C_2	C_3	C_4
s_1	0	0	0	1	0	0	0
s_1'	0	0	0	2	0	0	0
s_2	0	0	0	0	1	0	0
s_2'	0	0	0	0	2	0	0
s_3	0	0	0	0	0	1	0
s_3'	0	0	0	0	0	2	0
s_4	0	0	0	0	0	0	1
s_4'	0	0	0	0	0	0	2
t	1	1	1	4	4	4	4

Subset-Sum is NP-Hard (3)

- $C_1 = x_1 \vee \overline{x_2} \vee \overline{x_3}$
- $C_2 = \overline{x_1} \vee \overline{x_2} \vee \overline{x_3}$
- $C_3 = \overline{x_1} \vee \overline{x_2} \vee x_3$
- $C_4 = x_1 \vee x_2 \vee x_3$

Two numbers for each variable

Two numbers for each clause

	x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	1	0	0	1	0	0	1
v'_1	1	0	0	0	1	1	0
v_2	0	1	0	0	0	0	1
v'_2	0	1	0	1	1	1	0
v_3	0	0	1	0	0	1	1
v'_3	0	0	1	1	1	0	0
s_1	0	0	0	1	0	0	0
s'_1	0	0	0	2	0	0	0
s_2	0	0	0	0	1	0	0
s'_2	0	0	0	0	2	0	0
s_3	0	0	0	0	0	1	0
s'_3	0	0	0	0	0	2	0
s_4	0	0	0	0	0	0	1
s'_4	0	0	0	0	0	0	2
t	1	1	1	4	4	4	4

$$\text{Subset} = \{v'_1, v'_2, v_3, s_1, s'_1, s'_2, s_3, s_4, s'_4\}$$
$$= \{1000110, 101110, 10011, 1000, 2000, 200, 10, 1, 2\}$$
$$t = 1114444$$

Formula Satisfiable ⇒ Subset Exists

- Take v_i if x_i is true.
- Take v_i' if x_i is false.
- Take both s_j and s_j' if number of true literals in C_j is 1.
- Take s_j' if number of true literals in C_j is 2.
- Take s_j if number of true literals in C_j is 3.
- Example:
  - $(x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_2} \vee x_3)$
  - $x_1 = x_2 = x_3 = \text{TRUE}$
  - Subset $= \{v_1, v_2, v_3, s_1, s_2, s_2', s_3, s_3', s_4'\}$

	x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	1	0	0	1	0	0	1
v_2	0	1	0	1	0	1	0
v_3	0	0	1	1	1	0	1
s_1	0	0	0	1	0	0	0
s_2	0	0	0	0	1	0	0
s_2'	0	0	0	0	2	0	0
s_3	0	0	0	0	0	1	0
s_3'	0	0	0	0	0	2	0
s_4'	0	0	0	0	0	0	2
t	1	1	1	4	4	4	4

Subset Exists ⇒ Formula Satisfiable

- Assign value TRUE to x_i if v_i is in subset.
- Assign value FALSE to x_i if v_i' is in subset.
- Exactly one number per variable must be in the subset .
  - Otherwise one of first n digits of the sum is not equal to 1.
- At least one variable number corresponding to a literal in a clause must be in the subset.
  - Otherwise one of next m digits of the sum is smaller than 4.
- Each clause is satisfied.

An Undecidable Problem: HALTING Problem

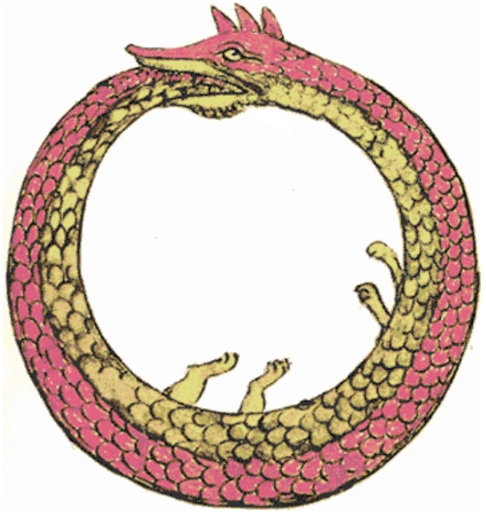
HALTING Problem

Input: Program P and input I
Output: Returns **yes** if program P halts on input I and **no** otherwise

Assume program $H(P, I)$ decides the **HALTING Problem.**

```
G(x) {
    if H(x,x) = yes
        Loop forever;
    else
        Halt;
}
```

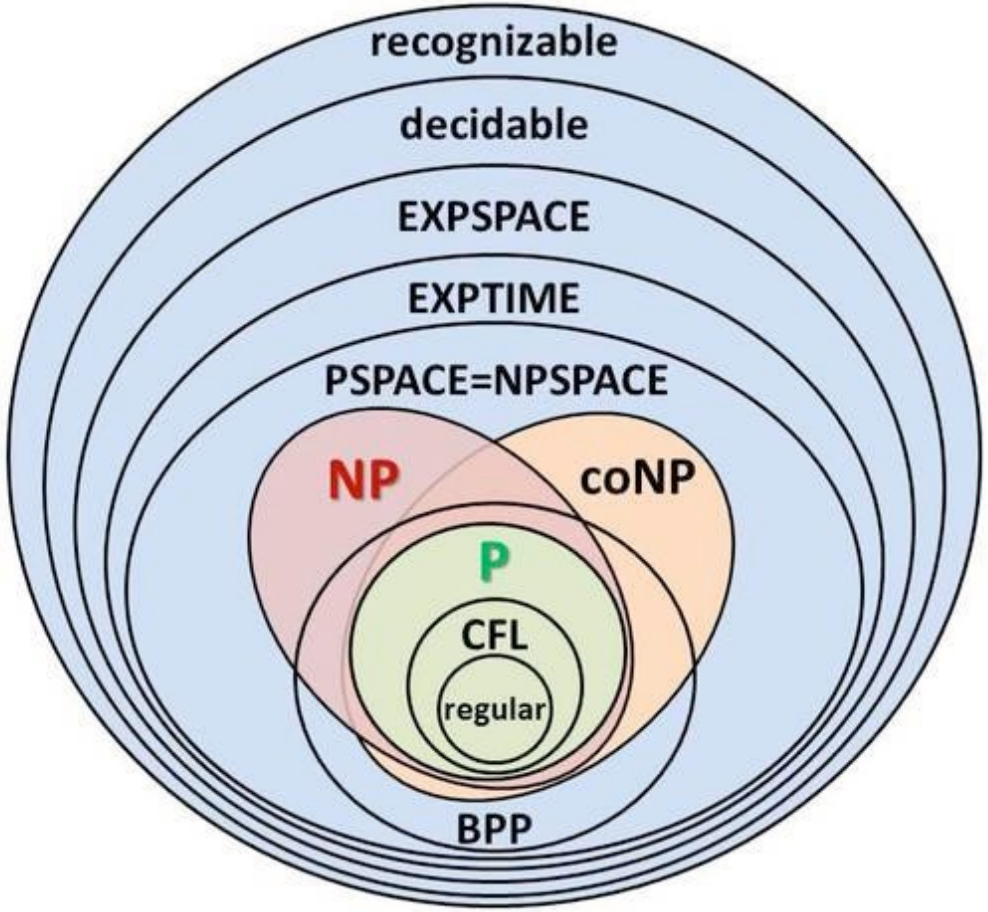
Contradiction:
What is the result of $G(G)$?



Ouroboros: a dragon that continually consumes itself

Why did we study about complexity classes?

- As a scientist:
  - You need to understand complexity classes.
  - If you you establish a problem as **NP-complete**, it's a good evidence for its intractability.
  - There are MANY MANY more classes we didn't discuss in the class.
    - In the CS theory field, many researchers are actively working on this subject.
- As an engineer:
  - Find an approximate algorithm instead of trying to solve the problem exactly.
  - Solve a tractable special case.



Sample Problems

True or False?

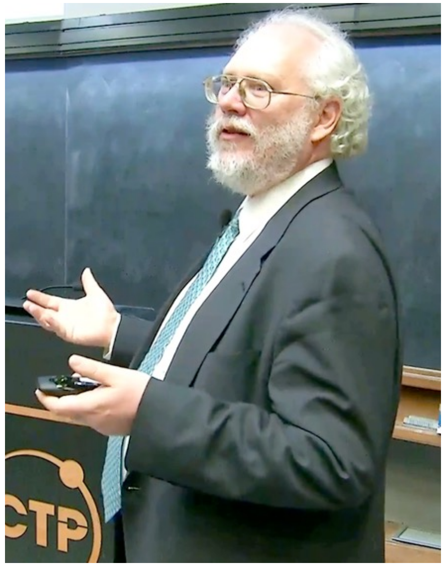
1. **NP** is the class of problems that are verifiable in polynomial time.
2. It is not known whether **P** $\neq$ **NP** or **P** = **NP**.
3. If a problem is not in **P**, it should be in **NP-complete**.
4. If a problem is in **NP**, it must also be in **P**.
5. If a problem is **NP-complete**, it must not be in **P**.
6. **NP-complete** problems cannot be decided efficiently.
7. **NP-complete** problems are the hardest decision problems.

True or False? (cont'd)

8. Assume $\mathbf{P} \neq \mathbf{NP}$. Let A and B be decision problems. If A is in **NP-complete** and $A \leq_{\mathrm{P}} B$, then B is not in **P**.
9. There exists a decision problem X such that for all Y in **NP**, Y is polynomial-time reducible to X.
10. If $\mathbf{P} = \mathbf{NP}$, then $\mathbf{NP} = \mathbf{NP\text{-}complete}$.
11. If a problem is not in **P**, then it must be in **NP**.
12. **NP** is the class of problems that are not decidable in polynomial time.

Integer Factorization Problem

- *Integer factorization* is the decomposition of a composite number into a product of smaller integers greater than 1.
  - If these factors are further restricted to prime numbers, the process is called *prime factorization*.
- No efficient (*non-quantum*) integer factorization algorithm is known.
  - However, it has not been proven that no efficient algorithm exists.
  - The presumed difficulty of this problem is at the heart of widely used algorithms in cryptography such as RSA.
    - Take a course on computer security or cryptography to learn more about it.
  - *Peter Shor* came up with an algorithm in 1994 which could factorize integers in polynomial-time on quantum computers.
    - Take a course on quantum computing/information processing to learn more about it.



Peter Shor

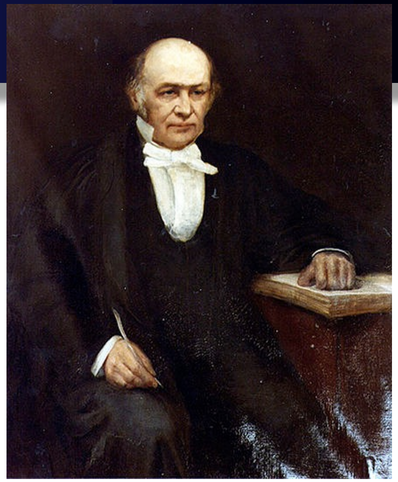
Polynomial-Time Solution for Integer Factorization!

- We have learned that no algorithm has been published that can factor any integer in polynomial time.
- I claim that I can come up with a polynomial-time algorithm though!

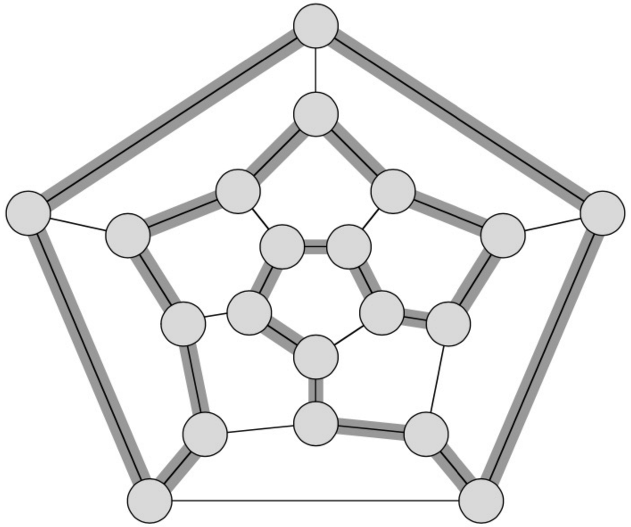
```
factorize(n) {
    for i = 2 to n - 1 {
        if n % i == 0 {
            return i, n / i
        }
    }
    return n + " is prime."
}
```

Prove or disprove whether this algorithm factorizes *n* in polynomial time.

Traveling Salesman Problem (TSP)


William Rowan Hamilton

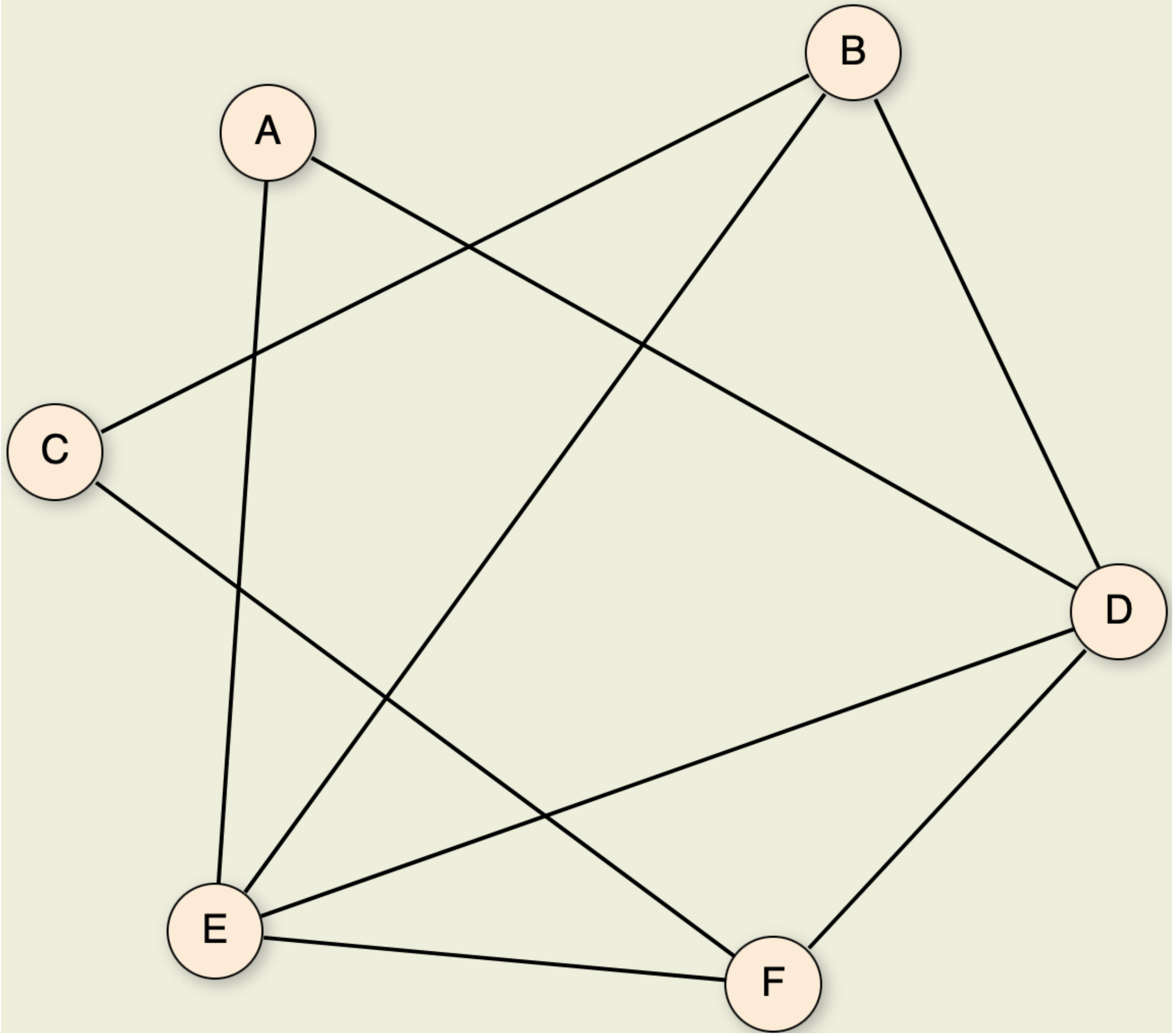
- *Hamiltonian cycle* is a cycle which passes through all the vertices of the graph exactly once.
  - Assume that deciding whether a graph has a Hamiltonian cycle (HAM-CYCLE) is **NP-complete**.
  - See the NP-completeness proof in CLRS 34.5.3.
- **Traveling salesman problem (TSP):**
  - Given a weighted complete graph G with non-negative edges and integer k, decide whether the graph G contains a ***tour*** (or Hamiltonian cycle) of cost k or smaller.
  - Prove that TSP is NP-complete.
  - **Hint:** Show HAM-CYCLE $\leq_p$ TSP


Hamiltonian cycle

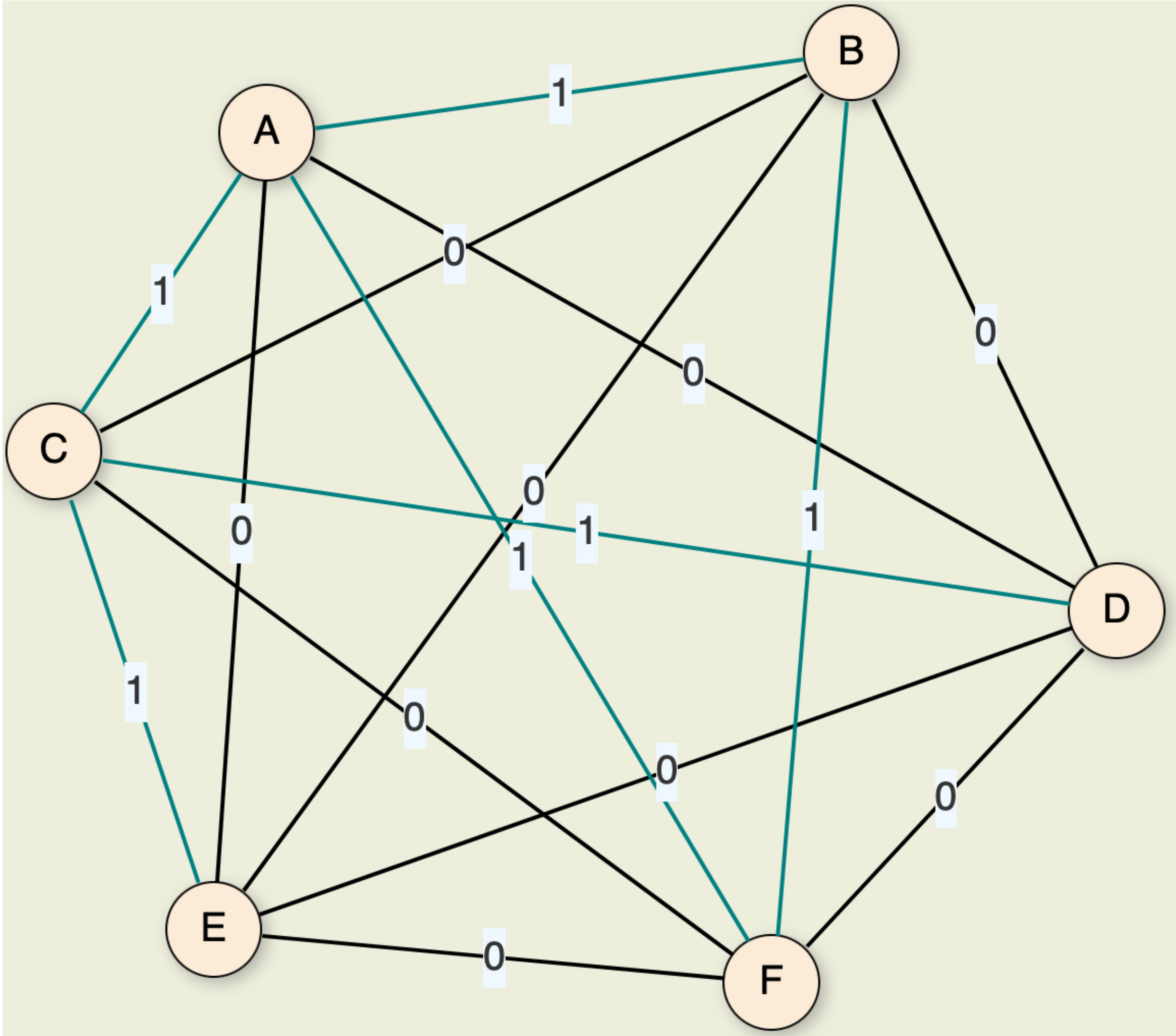
TSP is NP

- How to design a verifier to prove that TSP ∈ **NP**?

Graph G



Graph G'



TSP is NP-Hard (cont'd)

- G has a Hamiltonian cycle if and only if G' has a tour of cost at most 0.
  - If G has a Hamiltonian cycle, G' has a tour of cost at most 0.
  - If G' has a tour of cost at most 0, G has a Hamiltonian cycle.

Recommended Website

- See Chapter 28 slides of this website for nice proves of different NP-complete problems:
  - https://opendsa-server.cs.vt.edu/ODSA/Books/Everything/html/index.html
    - For instance, circuit satisfiability problem is detailed here:
      - https://opendsa-server.cs.vt.edu/ODSA/Books/Everything/html/circuitSAT.html