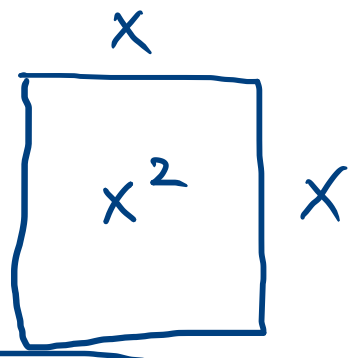
Function of a Random Variable

$X \sim U(0, 10)$

$Y = g(X)$

$Y = X^2$



$f_Y(y) = ?$

$I = \frac{V}{R}$

$R \sim N(10\Omega, 1)$

تابع یک متغیر تصادفی

تابعی از یک متغیر تصادفی گسسته

x_i	0	1	-1
P_i	1/3	1/3	1/3

$$y = x^2$$

$$P(y=0) = \frac{1}{3}$$

$$P(y=1) = \frac{2}{3}$$

$$P_Y(y) = \sum_{i:\, g(x_i)=y} P_X(x_i)$$

تابع یک متغیر تصادفی پیوسته

$f_Y(y) = ?$

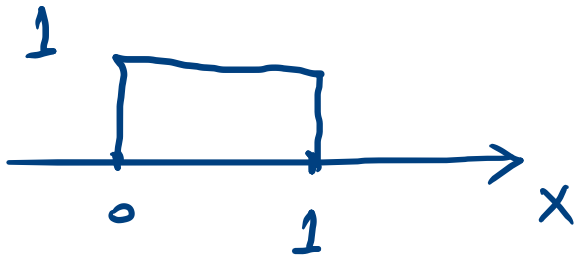
- مثال: $Y = aX + b$

$$F_Y(y) = P(Y \le y) = P(aX + b \le y)$$

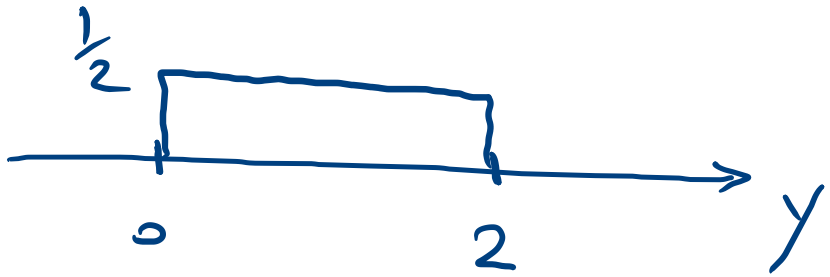
$$= \begin{cases} P(X \le \frac{y-b}{a}) & a > 0 \\ P(X \ge \frac{y-b}{a}) & a < 0 \end{cases} = \begin{cases} F_X(\frac{y-b}{a}) & a > 0 \\ 1 - F_X(\frac{y-b}{a}) & a < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{a} f_X(\frac{y-b}{a}) & a > 0 \\ \frac{-1}{a} f_X(\frac{y-b}{a}) & a < 0 \end{cases} = \frac{1}{|a|} f_X(\frac{y-b}{a})$$

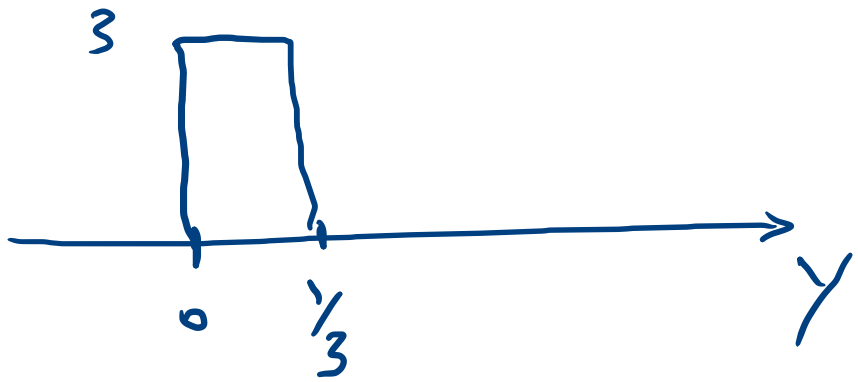
$X \sim U(0,1)$



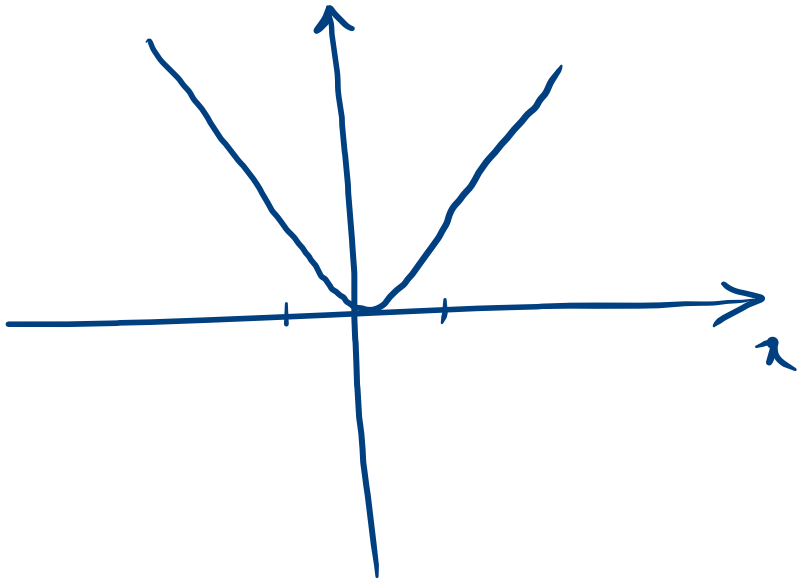
$Y = 2X$



$Y = \frac{1}{3}X$



$y = x^2$ $\boxed{y = g(x)}$



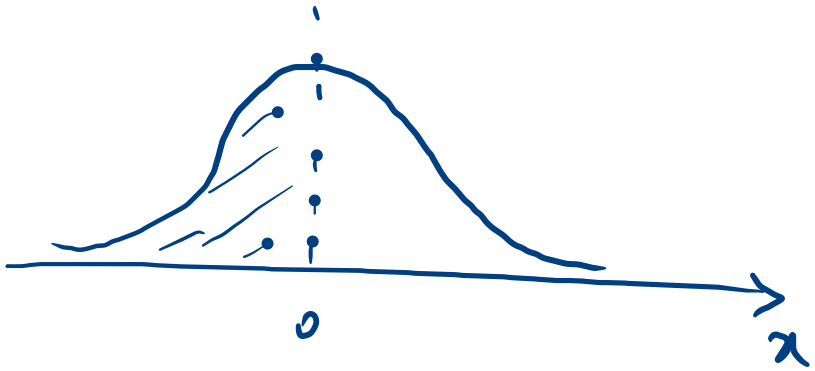
$2x$

$$\boxed{f_y(y) = \frac{f_x}{|g'(\cdot)|}}$$

argument

مثال ۲: $Y = |X|$

$f_Y(y) = ?$



$$f_Y(y) = \begin{cases} 0 & y < 0 \\ & y > 0 \end{cases}$$

$$F_Y(y) = P(Y \leq y) = P(|X| \leq y) = P(-y \leq X \leq y) = F_X(y) - F_X(-y)$$

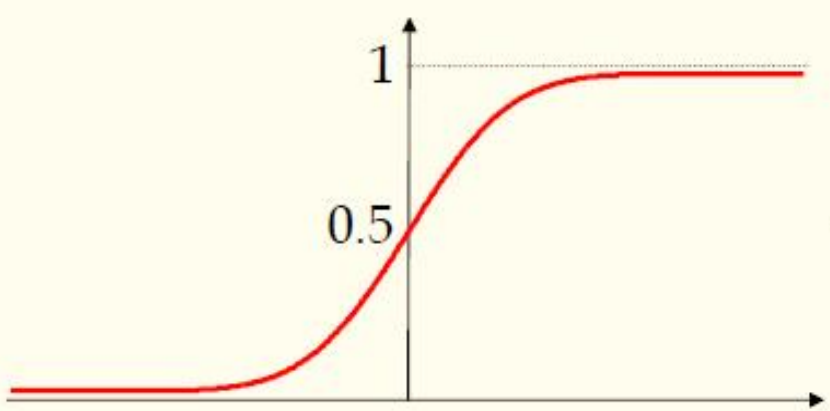
$$f_Y(y) = f_X(y) + f_X(-y)$$

$F(x) = 1 - e^{-x}$

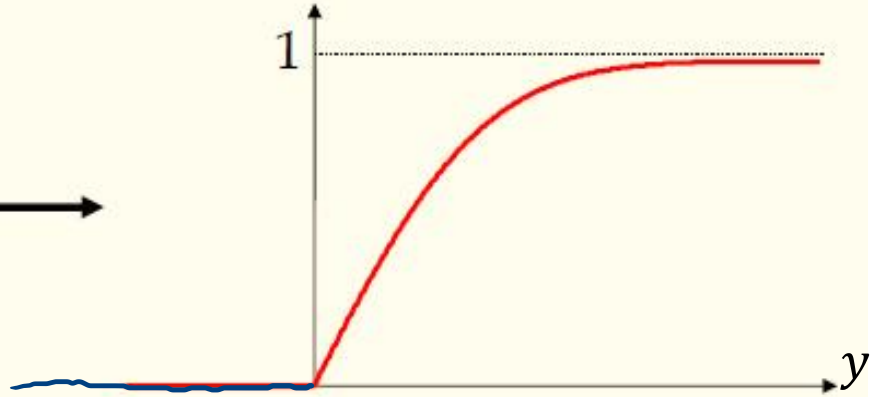
مثال ۲ برای توزیع نرمال

$1 - e^{-y} + (1 - e^{y})$

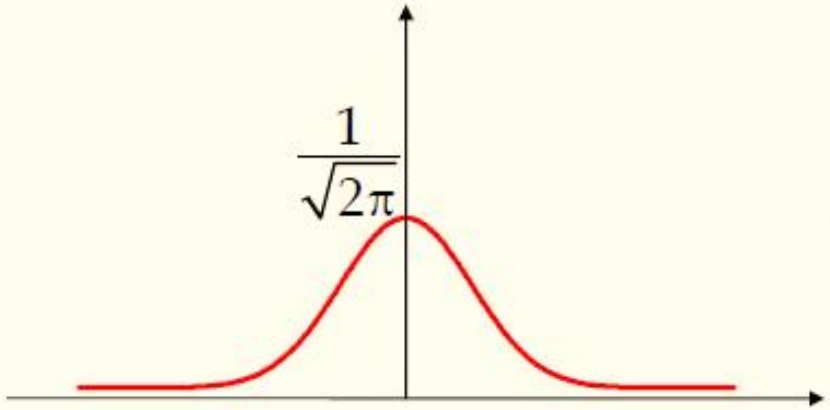
$F_X(x)$



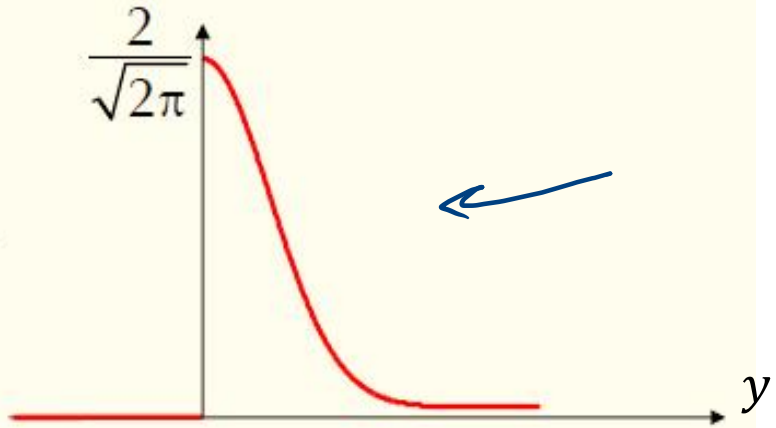
$F_Y(y) = F_X(y) - F_X(-y) : y > 0$



$f_X(x)$

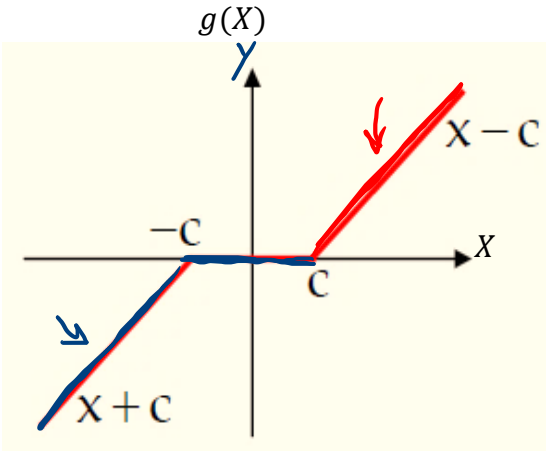


$f_Y(y) = f_X(y) + f_X(-y) : y > 0$



مثال ۳

$$Y = g(X) = \begin{cases} X + c & X < -c \\ 0 & -c \le X \le c \\ X - c & X > c \end{cases}$$



$$P(Y=0) = P(-c \le X \le c)$$

$$= F_X(c) - F_X(-c)$$

$$= \int_{-c}^{c} f_X(x)\,dx$$

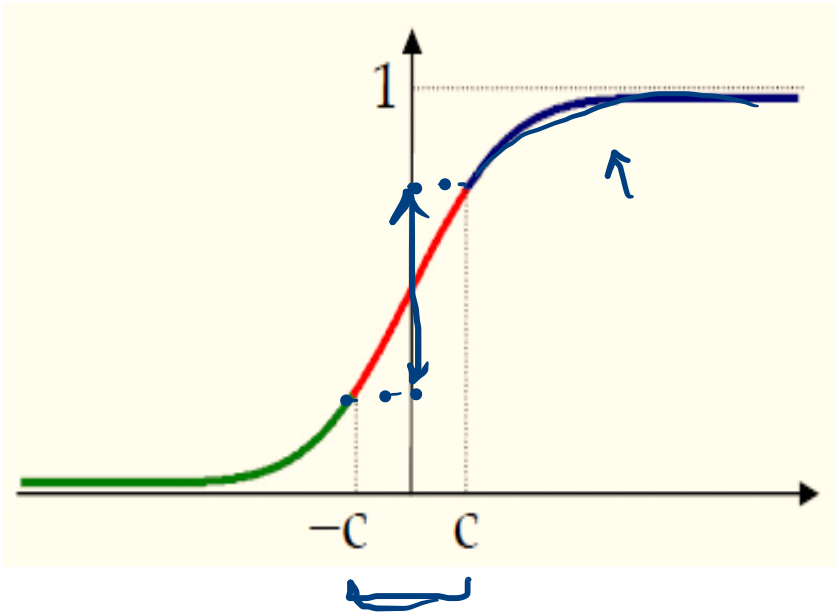
$$F_Y(y) = P(Y \le y) = \begin{cases} P(X - c \le y) & y > 0 \\ P(X + c \le y) & y < 0 \end{cases} = \begin{cases} P(X \le y + c) & y > 0 \\ P(X < y - c) & y < 0 \end{cases}$$

$$= \begin{cases} F_X(y + c) \\ F_X(y - c) \end{cases} \longrightarrow f_Y(y) = \begin{cases} f_X(y + c) & y > 0 \\ f_X(y - c) & y < 0 \end{cases}$$

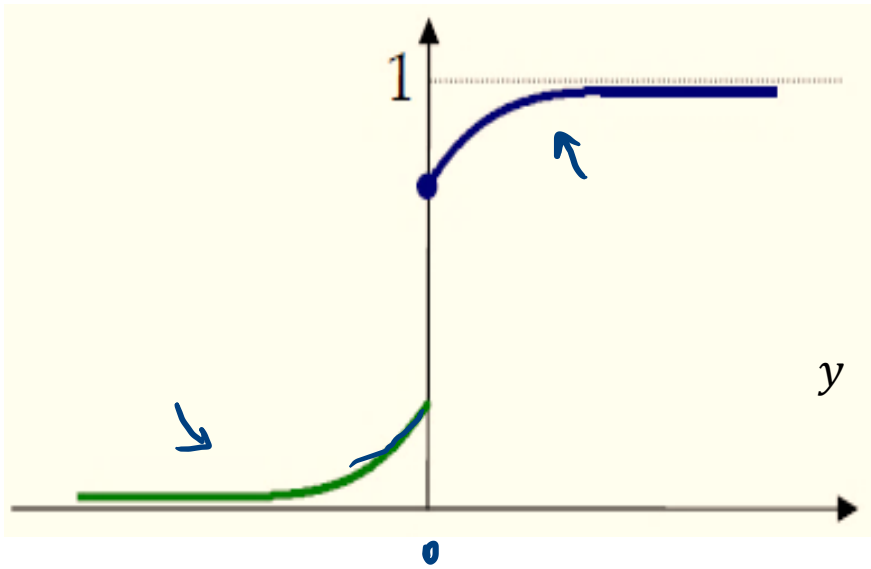
ادامه مثال ۳

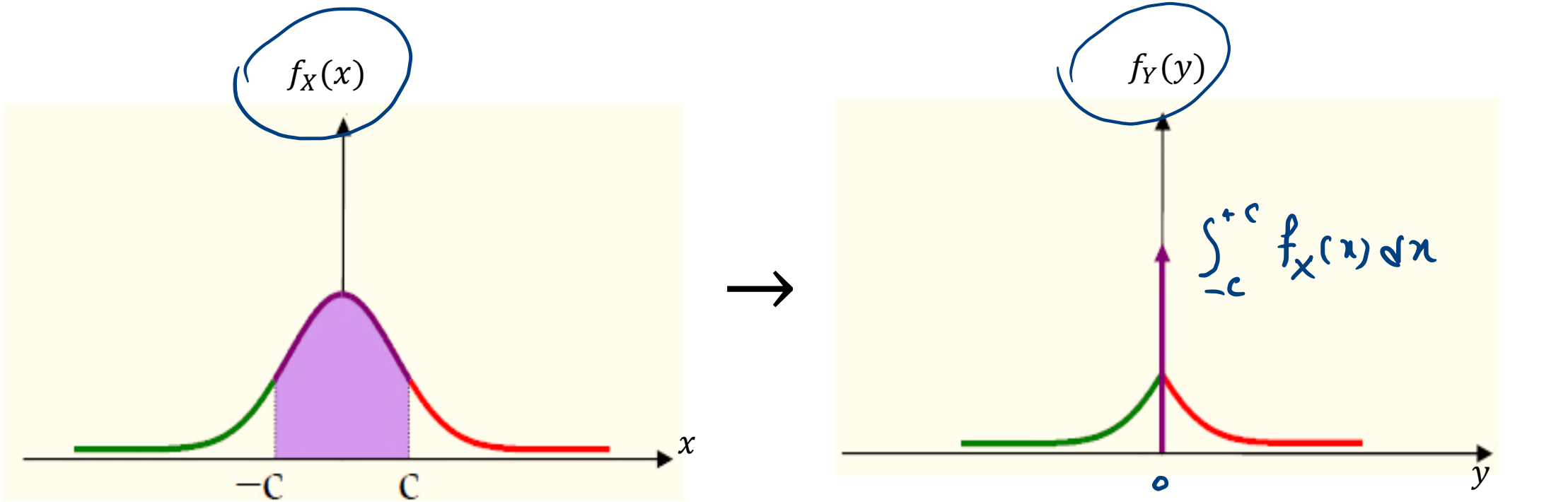
$X \sim N(\mu, 1)$

$F_X(x)$



$F_Y(y)$




$f_X(x)$
x
$-c$
c
$f_Y(y)$
$\int_{-c}^{+c} f_X(x)\,dx$
y
0

مثال: $Y = X^3$

$$F_Y(y) = P(X^3 \le y) = P(X \le y^{1/3}) = F_X(y^{1/3})$$

$$f_Y(y) = \frac{1}{3} y^{-2/3} f_X(y^{1/3})$$

$$f_X(x) \longleftrightarrow f_Y(y)$$

مثال: $Y = X^2$

$$y = g(x) = x^2$$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

محاسبه مستقیم f_Y از f_X

- فرض کنید تابع g معکوس پذیر باشد.

$Y = g(X)$

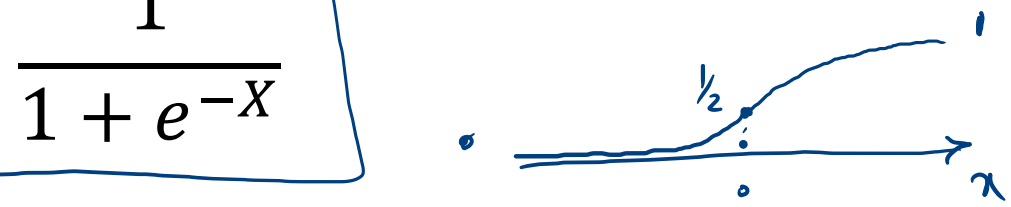
$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y))$$

$$= F_X(g^{-1}(y))$$

$$f_Y(y) = |(g^{-1}(y))'| f_X(g^{-1}(y)) = \frac{1}{|g'(y)|} f_X(g^{-1}(y))$$

مثال: تابع سیگموید

$$Y = \frac{1}{1+e^{-X}}$$



$$\rightarrow X \sim U(-1,1) \longrightarrow f_X(x) = \begin{cases} \frac{1}{2} & (-1,1) \\ 0 & \text{o.w.} \end{cases}$$

$$f_Y(y) = \begin{cases} 0 & y \le 0 \text{ or } y \ge 1 \\ ? & 0 \le y \le 1 \end{cases}$$

$$y = g(x) = \frac{1}{1+e^{-x}}$$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|} = \frac{\frac{1}{2}}{y-y^2}$$

$$g'(x) = \frac{e^{-x}+1-1}{(1+e^{-x})^2} = \underbrace{\frac{1}{1+e^{-x}}}_{y} - \underbrace{\left(\frac{1}{1+e^{-x}}\right)^2}_{y^2}$$

$$= y - y^2$$

محاسبه مستقیم f_Y از f_X

- قضیه: برای y داده شده، اگر معادله $g(x) = y$ دارای جواب‌های x_1، x_2، و ... باشد، خواهیم داشت:

$$f_Y(y) = \sum_i \frac{f_X(x_i)}{|g'(x_i)|}$$

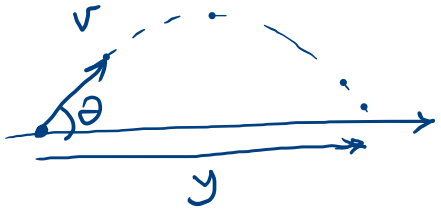
که در آن:

$$f_X(x_i) = f_X(x)\Big|_{x=x_i(y)}$$

$$g'(x_i) = \frac{d}{dx} g(x)\Big|_{x=x_i(y)}$$

مشروط بر این که برای y داده شده، تعداد نقاط x_i قابل شمارش باشد و $g(x)$ در نقاط x_i مشتق‌پذیر باشد.

Papoullis Example 5-13 مثال: پرتابه



$\theta \sim U(0, \pi/2)$

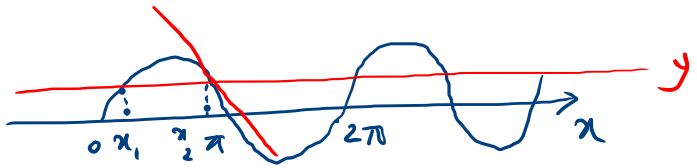
$f_Y(y) = ?$

$$y = \frac{v^2}{g} \sin 2\theta \qquad (x = 2\theta)$$

$$y = a \sin x$$

$$x \sim U(0, \pi)$$

$|y| \geq a \rightarrow f_Y(y) = 0$



$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|}$$

$y = g(x) \rightarrow x = h(y)$

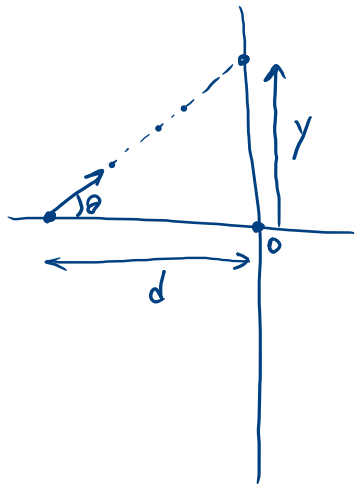
$$g'(x) = a\cos x = \pm a\sqrt{1-\sin^2 x} = \pm\sqrt{a^2 - a^2\sin^2 x} = \pm\sqrt{a^2 - y^2}$$

(منفی)

$$f_Y(y) = \frac{f_X(x_1)}{\sqrt{a^2-y^2}} + \frac{f_X(x_2)}{\sqrt{a^2-y^2}} = \frac{\frac{2}{\pi}}{\sqrt{a^2-y^2}}$$

$$f_Y(y) = \begin{cases} \dfrac{2}{\pi\sqrt{a^2 - y^2}} & |y| < a \\ 0 & |y| > a \end{cases}$$

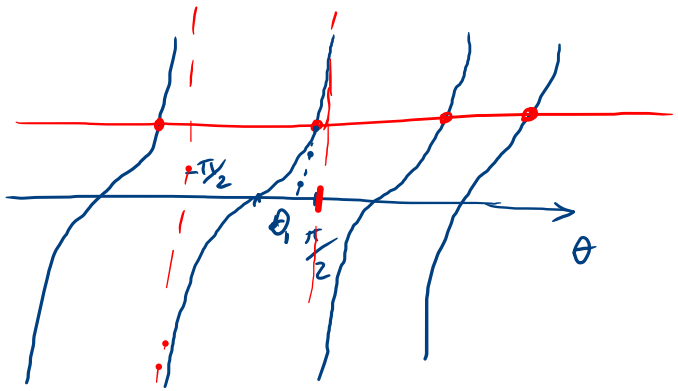
مثال: پرتابه بدون اثر گرانش



$$\theta \sim U\left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

$$y \in \mathbb{R}$$

$$\tan\theta = \frac{y}{d} \Rightarrow \boxed{y = d\tan\theta}$$



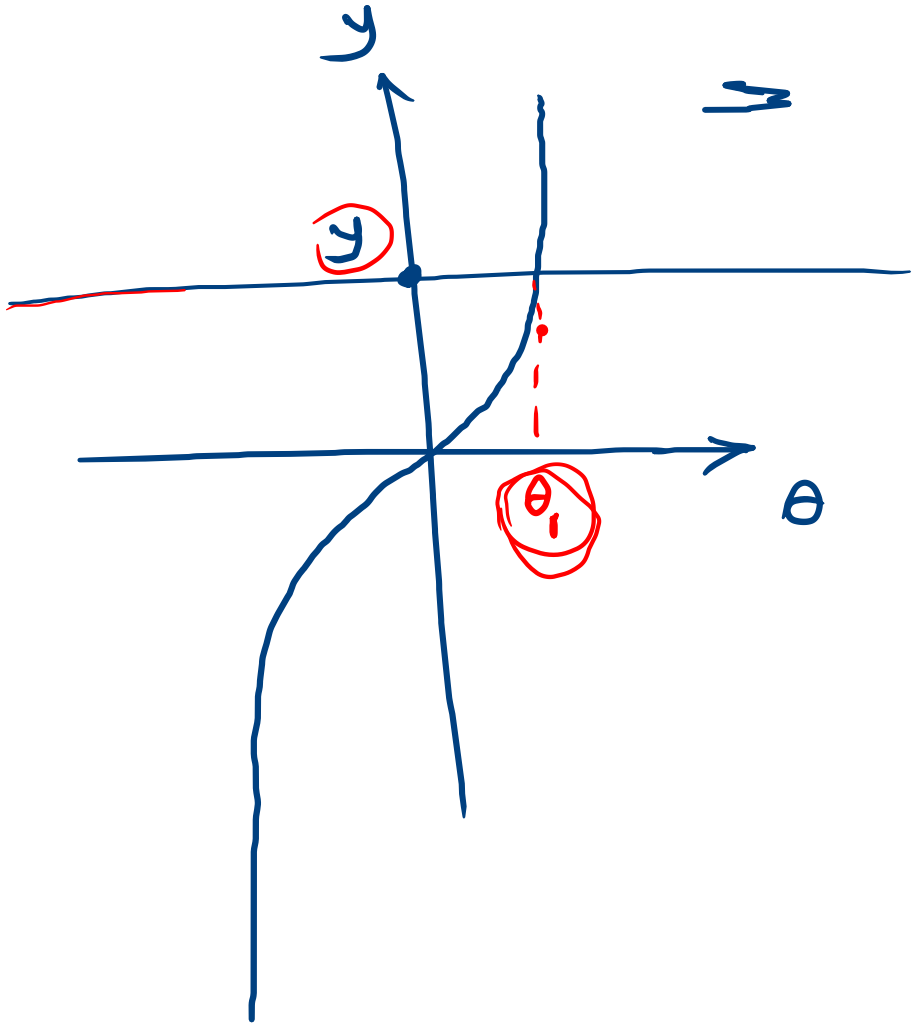
$$f_y(y) = \frac{f_\theta(\theta_1)}{|g'(\theta_1)|}$$

$$g(\theta) = d\tan\theta$$

$$g'(\theta) = d(1+\tan^2\theta) = d + \frac{d^2}{d}\tan^2\theta$$

$$= d + \frac{y^2}{d} > 0$$

$$\boxed{f_y(y) = \frac{1/\pi}{d + y^2/d} \qquad y \in \mathbb{R}}$$



$$\theta_1 = h(y)$$

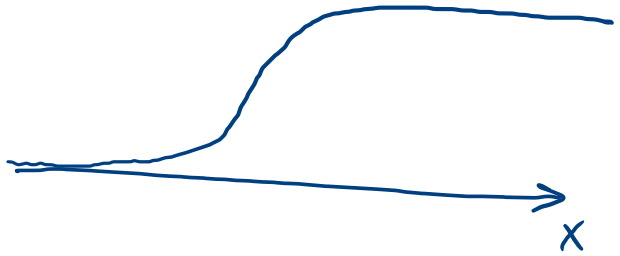
تمرین:

$$f_X(x) = \frac{2x}{\pi^2} \qquad 0 < x < \pi$$

$$Y = \sin X$$

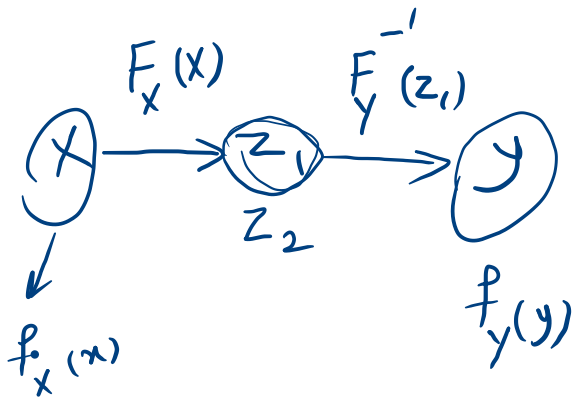
$$f_Y(y) = ?$$

$f_x(x)$ $f_y(y)$ $Y = g(X)$



$Z_1 = F_X(X)$ $0 < Z \leq 1$

$Z_1 \sim U(0,1)$



$Z_2 = F_Y(y)$ $Z_2 \sim U(0,1)$

$Y = F_Y^{-1}(Z_2)$

$$Y = F_Y^{-1}(F_X(X))$$

$Y = F_X(x)$ $\quad 0 < y \leq 1$

$$P(Y \leq y) = P(F_X(x) \leq y) = P(X \leq F^{-1}(y)) = F_X(F_X^{-1}(y))$$

$$= y$$

$f_Y(y) = 1$

$X \sim N(\text{---})$

$Y \sim N(\text{---})$

$P(X - Y \geq 0)$

Z

$Z \sim N(\mu, \sigma^2)$

$P(Z \leq 0)$

$E[Z] = E[X - Y] = \mu_x - \mu_y$

$E[Z^2] = E[X^2 - 2XY + Y^2] =$