Parameter Estimation

Point Estimation → Method of Moments

Point Estimation → Maximum Likelihood

Bayesian

$$\boxed{x \sim f_X(x;\theta)} \longrightarrow \boxed{X \sim f_X(x|\theta)}$$

$$\theta \sim f(\theta) \longrightarrow \text{prior}$$ احتمال پیشین

$$D = \{x_1, \dots, x_n\}$$

$$f(\theta) \xrightarrow{D} f(\theta|D) \longrightarrow \text{posterior}$$ احتمال پسین

Bayesian Estimation

$\vec{X} = D = \{X_1, X_2, \ldots, X_n\}$ i.i.d.

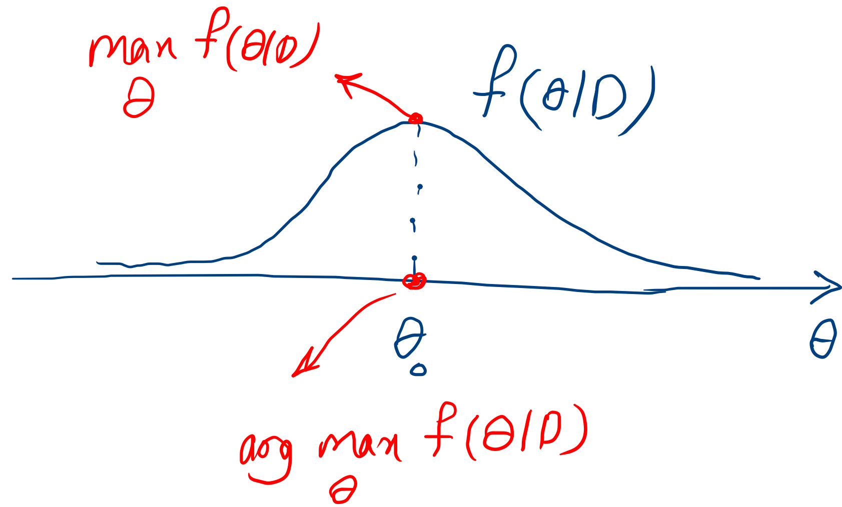
$f(\theta) \longrightarrow f(\theta | D)$

$$f(\theta | D) = \frac{f(D|\theta)\, f(\theta)}{f(D)} = \frac{1}{C} f(D|\theta) f(\theta)$$

$$= \frac{1}{C} f(x_1, \ldots, x_n | \theta)\, f(\theta) = \frac{1}{C} \prod_{i=1}^{n} f(x_i | \theta)\, f(\theta)$$

اگر توزیع پسین و پیشین از یک جنس باشند، آنگاه $f(\theta)$ را conjugate prior می‌گویند

$$\theta \longrightarrow f(\theta|D)$$



$$\hat{\theta}_{MAP} = \arg\max_\theta f(\theta|D)$$

$$\hat{\theta}_{MAP} = \arg\max_\theta \log f(\theta|D)$$

مثال ۱

$$P_X(x|\theta) = \theta^{x}(1-\theta)^{1-x} \qquad x \in \{0,1\}$$

$$0 \leq \theta \leq 1$$

$$\theta \sim Beta(\theta|\alpha,\beta) = \underbrace{c'\,\theta^{\alpha-1}(1-\theta)^{\beta-1}}_{f(\theta)}$$

$$D = \{X_1, \dots, X_n\}$$

$$f(\theta|D) = ?$$

$$f(\theta|D) = \frac{1}{c} f(\theta) \prod_{i=1}^{n} f(x_i|\theta) = \frac{1}{c} c'\,\theta^{\alpha-1}(1-\theta)^{\beta-1} \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$$

$$= c''\,\theta^{\alpha-1}(1-\theta)^{\beta-1}\;\theta^{\sum_{i=1}^{n} x_i}\;(1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

$$= c''\;\theta^{\overbrace{\alpha+\sum_{i=1}^{n} x_i-1}^{\alpha'}}\;(1-\theta)^{\overbrace{\beta+n-\sum_{i=1}^{n} x_i-1}^{\beta'}} = Beta(\theta|\alpha',\beta')$$

تمرین ۲

$$f_X(x|\lambda) = e^{-\lambda} \frac{\lambda^x}{x!}$$

$$\lambda \sim \mathrm{Gamma}(\lambda|\alpha, \beta) = \underbrace{c' \lambda^{\alpha-1} e^{-\beta\lambda}}_{f(\lambda)}$$

$$f(\lambda|D) = ?$$

$$f(\lambda|D) = \frac{1}{C} f(\lambda) \prod_{i=1}^{n} f(x_i|\lambda) = \frac{1}{C} c' \lambda^{\alpha-1} e^{-\beta\lambda} \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!}$$

$$= c'' \lambda^{\alpha + \sum_{i=1}^{n} x_i - 1} e^{-(\beta+n)\lambda} = \mathrm{Gamma}(\lambda | \underbrace{\alpha + \sum_{i=1}^{n} x_i}_{\alpha'}, \underbrace{\beta + n}_{\beta'})$$

$$\hat{\lambda}_{MAP} = ? \implies \hat{\lambda}_{MAP} = \arg\max_{\lambda} f(\lambda|D)$$

$$\mathrm{Gamma}(\lambda|\alpha, \beta) \longrightarrow \text{mode} = \frac{\alpha-1}{\beta}$$

$$\boxed{\hat{\lambda}_{MAP} = \frac{\alpha + \sum_{i=1}^{n} x_i - 1}{\beta + n}}$$

$$\hat{\lambda}_{MAP} \xrightarrow{n\to\infty} \hat{\lambda}_{ML} \qquad \lim_{n\to\infty} \hat{\lambda}_{MAP} = \lim_{n\to\infty} \underbrace{\frac{\sum_{i=1}^{n} x_i}{n}}_{\hat{\lambda}_{ML}}$$

$$\hat{\theta}_{MAP} = \arg\max_{\theta} f(\theta|D) = \arg\max_{\theta} \frac{1}{c} f(\theta) \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \arg\max_{\theta} \left(\underbrace{\log \frac{1}{c}}_{} + \underbrace{\log f(\theta)}_{} + \sum_{i=1}^{n} \log f(x_i|\theta) \right)$$

$$\overset{n\to\infty}{=\!=} \arg\max_{\theta} \sum_{i=1}^{n} \log f(x_i|\theta) = \hat{\theta}_{ML}$$

مثال ۳

$$D = \{x_1, \dots, x_n\}$$

$$x_i \sim N(\mu, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\mu)^2}$$

$$\boxed{f(\mu) = \frac{\mu}{\sigma^2} \exp\left(-\frac{\mu^2}{2\sigma^2}\right)} \qquad \mu > 0$$

$$\hat{\mu}_{MAP} = ? \longrightarrow \hat{\mu}_{MAP} = \arg\max_{\mu} \log f(\mu | D)$$

$$f(\mu|D) = \frac{1}{C} f(\mu) \prod_{i=1}^{n} f(x_i|\mu) = \frac{1}{C} \frac{\mu}{\sigma^2} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \prod_{i=1}^{n} \exp\left(\frac{-1}{2}(x_i-\mu)^2\right)$$

$$L(\mu) = \log f(\mu|D) = \log\frac{1}{C} + \log\mu - \log\sigma^2 - \frac{\mu^2}{2\sigma^2} + \sum_{i=1}^{n} \frac{-1}{2}(x_i-\mu)^2$$

$$\frac{dL}{d\mu} = 0 \Rightarrow \frac{1}{\mu} - \frac{\mu}{\sigma^2} + \sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\frac{1}{\mu} - \frac{\mu}{\sigma^2} + \underbrace{\sum_{i=1}^{n} x_i}_{Z} - n\mu = 0 \Rightarrow -\Big(\underbrace{\frac{1}{\sigma^2}+n}_{R}\Big)\mu^2 + Z\mu + 1 = 0$$

$$\Rightarrow R\mu^2 - Z\mu - 1 = 0 \Rightarrow \boxed{\hat{\mu}_{MAP} = \frac{Z}{2R}\left(1 + \sqrt{1+\frac{4R}{Z^2}}\right)}$$

MAP: Maximum Aposteriori