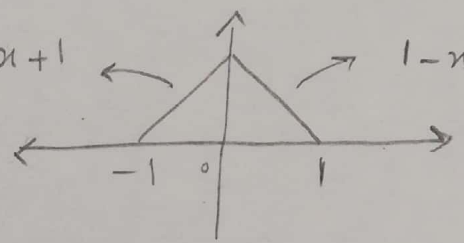
۸۱۰۱۰۱۴۹۲ علیرضا کرمی

۱) ابتدا برای یک دوره تناوب، ضریب سری فوریه را بدست می آوریم. در نتیجه خواهیم داشت:

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-\frac{in\pi}{L}x}\,dx \overset{L=1}{\Longrightarrow} C_n = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-in\pi x}\,dx$$

$$f(x) = \begin{cases} x+1 & -1<x<0 \\ 1-x & 0<x<1 \end{cases}$$



$$\Longrightarrow C_n = \frac{1}{2}\int_{-1}^{0}(1+x)\,e^{-in\pi x}\,dx + \frac{1}{2}\int_{0}^{1}(1-x)\,e^{-in\pi x}\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-in\pi x}\,dx + \frac{1}{2}\int_{-1}^{0} x e^{-in\pi x}\,dx - \frac{1}{2}\int_{0}^{1} x e^{-in\pi x}\,dx$$

$$\int_{\alpha}^{\beta} x e^{-in\pi x}\,dx = \left(\frac{xi}{n\pi} e^{-in\pi x}\right)\Big|_{\alpha}^{\beta} - \frac{i}{n\pi}\int_{\alpha}^{\beta} e^{-in\pi x}\,dx$$

$$= \left(\frac{xi}{n\pi} e^{-in\pi x}\right)\Big|_{\alpha}^{\beta} + \left(\frac{e^{-in\pi x}}{n^2\pi^2}\right)\Big|_{\alpha}^{\beta}$$

$$\Rightarrow C_n = \frac{i}{2n\pi}\left(e^{-in\pi} - e^{+in\pi}\right) + \frac{1}{2}\left(\left(\frac{xi}{n\pi} e^{-in\pi x}\right)\Big|_{-1}^{0} + \left(\frac{e^{-in\pi x}}{n^2\pi^2}\right)\Big|_{-1}^{0}\right) - \frac{1}{2}$$

$$\left(\left(\frac{xi}{n\pi} e^{-in\pi x}\right)\Big|_{0}^{1} + \left(\frac{e^{-in\pi x}}{n^2\pi^2}\right)\Big|_{0}^{1}\right) = \underline{\frac{i}{2n\pi}}\left(e^{-in\pi} \underline{\underline{- e^{+in\pi}}}\right) + \frac{1}{2}$$

$$\left(\underline{\underline{\frac{+i}{n\pi} e^{in\pi}}} + \frac{1}{n^2\pi^2} - \frac{e^{+in\pi}}{n^2\pi^2}\right) - \frac{1}{2}\left(\underline{\frac{ie^{-in\pi}}{n\pi}} + \frac{e^{-in\pi}}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right)$$

$$= \frac{1}{n^2\pi^2} - \frac{e^{in\pi} - e^{-in\pi}}{2n^2\pi^2} = \frac{1}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2} = \frac{1-(-1)^n}{n^2\pi^2}$$

$$\Rightarrow C_n = \frac{1-(-1)^n}{n^2\pi^2}$$

چون $n=0$ ریشه مخرج است، آن را به طور جداگانه حساب می‌کنیم و داریم:

$$C_0 = \frac{1}{2}\int_{-1}^{0}(1+x)\,dx + \frac{1}{2}\int_{0}^{1}(1-x)\,dx = \frac{1}{2}\left(x+\frac{x^2}{2}\right)\Big|_{-1}^{0} + \frac{1}{2}\left(x-\frac{x^2}{2}\right)\Big|_{0}^{1}$$

$$= \frac{1}{2}\left(1-\frac{1}{2}\right) + \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow C_n = \begin{cases} 0 & n=2k\ ,\ n\neq 0 \\ \dfrac{2}{n^2\pi^2} & n=2k+1 \\ \dfrac{1}{2} & n=0 \end{cases}$$

حال چون تابع ما متناوب است، از رابطه تبدیل فوریه تابع متناوب استفاده می‌کنیم: $F(\omega) = 2\pi\sum_{-\infty}^{\infty} C_n\,\delta(\omega - \overset{\frac{\pi}{L}}{\omega_0} n)$

$$\xrightarrow{L=1} F(\omega) = 2\pi\sum_{-\infty}^{\infty} C_n\,\delta(\omega - n\pi) = 2\pi\sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{n^2\pi^2}\,\delta(\omega - n\pi) + \underset{n=0}{\pi\,\delta(\omega)}$$

$$= 2\pi\sum_{-\infty}^{+\infty} \frac{2}{(2n+1)^2\pi^2}\,\delta(\omega \overset{-(2n+1)\pi}{- 2n\pi - \pi}) + \pi\,\delta(\omega)$$

2) طبق قضیه تقارن یا دوگانی داریم

$$f(x) \longrightarrow F(\omega)$$

$$\omega \to x$$

$$F(x) \longrightarrow 2\pi f(-\omega)$$

از طرفی می دانیم که اگر $f(x) = e^{-a|x|}$ خواهیم داشت :

$$F\{e^{-a|x|}\} = \frac{2a}{a^2+\omega^2}$$

$$\Longrightarrow \frac{1}{9+x^2} = \frac{1}{6} \cdot \frac{6}{9+x^2} \overset{x \to \omega}{=\!=} \frac{1}{6} \, \frac{2(3)}{9+\omega^2} = \frac{1}{6} \, F\{\overbrace{e^{-3|x|}}^{f(x)}\}$$

$$\Longrightarrow F\left\{\frac{1}{9+x^2}\right\} = F\left\{\frac{1}{6} \underbrace{F\{e^{-3|x|}\}}_{F(x)}\right\} = \frac{1}{6} F\{F(x)\}$$

$$= \frac{1}{6} \times 2\pi \times f(-\omega) = \frac{\pi}{3} \times e^{-3|-\omega|} = \frac{\pi}{3} e^{-3|\omega|}$$

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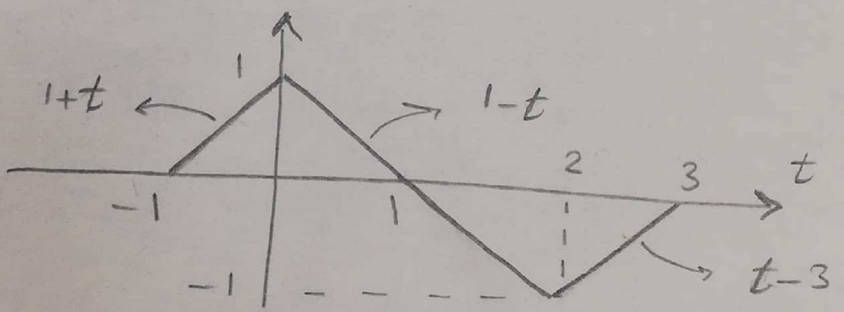

الف)

$$X(0) = F\{x(t)\}_{\omega=0} = \int_{\infty}^{+\infty} x(t) e^{-i\omega t} dt \overset{\omega=0}{=\!=} \int_{-\infty}^{+\infty} x(t)\, dt$$

این انتگرال معادل با مساحت زیر نمودار تابع $x(t)$ است که همانطور که مشاهده می کنید، مقدار آن صفر است. $X(0)=0 \Leftarrow$

راه کامل تر حل:

$$\Longrightarrow X(0) = \int_{-\infty}^{+\infty} x(t)\, dt = \int_{-1}^{0} (1+t)\, dt + \int_{0}^{2} (1-t)\, dt + \int_{2}^{3} (t-3)\, dt$$

$$= \left(t + \frac{t^2}{2}\right)_{-1}^{0} + \left(t - \frac{t^2}{2}\right)_{0}^{2} + \left(-3t + \frac{t^2}{2}\right)_{2}^{3} = -\left(-1+\frac{1}{2}\right) + (2-2) - \left(9 - \frac{9}{2} - (6-2)\right) = \frac{1}{2} - 4.5 + 4 = 0$$

مثل همان قسمت الف محاسبه می‌کنیم.

ب)
$$\int_{-\infty}^{+\infty} X(\omega)\,d\omega = \int_{-\infty}^{+\infty} X(t)\,dt = \mathcal{F}\{X(t)\}_{\omega=0}$$

از طرفی طبق خاصیت تقارن یا دوگانه داریم:
$$F(F(t)) = 2\pi f(-\omega) = 2\pi f(0)$$

طبق نمودار $f(0)=1 \Rightarrow$
$$\int_{-\infty}^{+\infty} X(\omega)\,d\omega = 2\pi f(0) = 2\pi$$

ج) از قضیه پارسوال استفاده می‌کنیم که داریم:
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

$$\Longrightarrow \int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

$$= 2\pi\left[\int_{-1}^{0} (1+t)^2 dt + \int_{0}^{2} (1-t)^2 dt + \int_{2}^{3} (t-3)^2 dt\right]$$

$$= 2\pi\left[\int_{-1}^{0} (1+2t+t^2) dt + \int_{0}^{2} (1-2t+t^2) dt + \int_{2}^{3} (t^2-6t+9) dt\right]$$

$$= 2\pi\left[\left(t+t^2+\frac{t^3}{3}\right)_{-1}^{0} + \left(t-t^2+\frac{t^3}{3}\right)_{0}^{2} + \left(\frac{t^3}{3}-3t^2+9t\right)_{2}^{3}\right]$$

$$= 2\pi\Big[\underbrace{\cancel{+1}\cancel{-1}+\frac{1}{3}+2-4+\cancel{\frac{8}{3}}+9\cancel{-27}\cancel{+27}\cancel{-\frac{8}{3}}+12-18}_{\frac{4}{3}}\Big] = \frac{8}{3}\pi$$

4) $y''(t) + y'(t) - 2y(t) = x(t) = e^{-3t}u(t)$; $\mathcal{F}\{e^{-3t}u(t)\} = \dfrac{1}{a+i\omega}$

; $\mathcal{F}\{f^{(n)}(t)\} = (i\omega)^n F(\omega)$

$$\xrightarrow[\text{می گیریم}]{\text{از دو طرف تبدیل فوریه}} (i\omega)^2 Y(\omega) + i\omega Y(\omega) - 2Y(\omega) = \frac{1}{3+i\omega}$$

$$\Longrightarrow Y(\omega) = \frac{1}{(3+i\omega)\left((i\omega)^2 + i\omega - 2\right)} = \frac{A}{3+i\omega} + \frac{B}{2+i\omega} + \frac{C}{-1+i\omega}$$

$$\Longrightarrow A = \frac{1}{4},\quad B = -\frac{1}{3},\quad C = \frac{1}{12}$$

$$\Longrightarrow Y(\omega) = \frac{1}{4(3+i\omega)} - \frac{1}{3(2+i\omega)} + \frac{1}{12(-1+i\omega)}$$

$$\xrightarrow[\text{فوریه معکوس می گیریم}]{\text{از دو طرف تبدیل}} y(t) = \frac{1}{4}\mathcal{F}^{-1}\left\{\frac{1}{3+i\omega}\right\} - \frac{1}{3}\mathcal{F}^{-1}\left\{\frac{1}{2+i\omega}\right\} + \frac{1}{12}\mathcal{F}^{-1}\left\{\frac{1}{-1+i\omega}\right\}$$

$$\Longrightarrow y(t) = \frac{1}{4}e^{-3t}u(t) - \frac{1}{3}e^{-2t}u(t) + \frac{1}{12}e^{t}u(t)$$

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الف) ما می دانیم که : $f^{-1}\left\{\frac{2a}{a^2+w^2}\right\}=e^{-a|t|}$

$$F(w)=\frac{1}{4+\frac{w^2}{2\pi}}\times\frac{2\pi}{2\pi}=\frac{2\pi}{8\pi+w^2}=\frac{2\pi}{(\sqrt{\text{۸}\pi})^2+w^2}=\frac{2\pi}{2\sqrt{\text{۸}\pi}}\times\frac{2\sqrt{\text{۸}\pi}}{(\sqrt{\text{۸}\pi})^2+w^2}$$

$$\Longrightarrow f^{-1}\left\{\frac{1}{4+\frac{w^2}{2\pi}}\right\}=f^{-1}\left\{\sqrt{\frac{\pi}{8}}\ \frac{2\sqrt{8\pi}}{(\sqrt{8\pi})^2+w^2}\right\}=\sqrt{\frac{\pi}{8}}\ f^{-1}\left\{\frac{2\sqrt{8\pi}}{(\sqrt{8\pi})^2+w^2}\right\}$$

$$=\sqrt{\frac{\pi}{8}}\ e^{-\sqrt{8\pi}\,|t|}$$

ب) $$F(w)=\frac{1}{(iw+4)(iw-4)}=\frac{1}{-w^2-16}=\frac{-1}{w^2+16}=\frac{-1}{8}\ \frac{8}{w^2+4^2}$$

$$\Longrightarrow f^{-1}\left\{\frac{1}{(iw+4)(iw-4)}\right\}=f^{-1}\left\{\frac{-1}{8}\cdot\frac{8}{w^2+4^2}\right\}=\frac{-1}{8}\ f^{-1}\left\{\frac{8}{w^2+4^2}\right\}$$

$$=\frac{-1}{8}\ e^{-4|t|}$$

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$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases} \qquad \text{I} \qquad\qquad (16$$

$$F(\omega) = \int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx = \int_{-1}^{1} e^{-i\omega x}\, dx = \frac{e^{-i\omega} - e^{+i\omega}}{(-i\omega)}$$

$$= \frac{2}{\omega}\, \frac{e^{i\omega} - e^{-i\omega}}{2i} = \frac{2\sin(\omega)}{\omega}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, e^{i\omega x}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\sin(\omega)}{\omega} \left(\cos(\omega x) + i \sin(\omega x) \right) d\omega$$

تابع فرد است

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\sin(\omega)}{\omega} \cos(\omega x)\, d\omega + \frac{1}{2\pi} \int_{-\infty}^{+\infty} i\, \frac{2\sin(\omega)}{\omega} \sin(\omega x)\, d\omega \to 0$$

تابع زوج است

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\sin(\omega)}{\omega} \cos(\omega x)\, d\omega = \frac{2}{\pi} \int_{0}^{+\infty} \frac{\sin(\omega)}{\omega} \cos(\omega x)\, d\omega$$

$$\xRightarrow{\text{I}} f(0) = 1 = \frac{2}{\pi} \int_{0}^{+\infty} \frac{\sin(\omega)}{\omega}\, d\omega \Longrightarrow \int_{0}^{+\infty} \frac{\sin(\omega)}{\omega} = \frac{\pi}{2}$$

$$\int_{0}^{+\infty} \frac{\sin^3(\omega)}{\omega}\, d\omega = \int_{0}^{+\infty} \frac{\sin(\omega)}{\omega} \left(\frac{1 - \cos 2\omega}{2} \right) d\omega =$$

$$\frac{1}{2} \int_{0}^{+\infty} \left[\frac{\sin\omega}{\omega} - \frac{1}{2} \left(\frac{\sin 3\omega - \sin\omega}{\omega} \right) \right] d\omega =$$

$$\int_{0}^{+\infty} \left[\frac{1}{2} \frac{\sin\omega}{\omega} + \frac{1}{4} \frac{\sin\omega}{\omega} - \frac{1}{4} \frac{\sin 3\omega}{\omega} \right] d\omega = \frac{3}{4} \int_{0}^{+\infty} \frac{\sin\omega}{\omega} - \frac{1}{4} \int_{0}^{+\infty} \frac{\sin 3\omega}{\omega}\, d\omega$$

$$\to \begin{array}{l} 3\omega = \omega' \\ 3d\omega = d\omega' \end{array}$$

$$= \frac{3}{4} \int_{0}^{+\infty} \frac{\sin\omega}{\omega} d\omega - \frac{1}{4} \int_{0}^{+\infty} \frac{\sin\omega'}{\frac{\omega'}{3}} \cdot \frac{d\omega'}{3} = \frac{3}{4} \int_{0}^{+\infty} \frac{\sin\omega}{\omega}\, d\omega - \frac{1}{4} \int_{0}^{\infty} \frac{\sin\omega'}{\omega'}\, d\omega' =$$

$$\left(\frac{3}{4} - \frac{1}{4} \right) \frac{\pi}{2} = \frac{\pi}{4}$$

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7) داریم: $\mathcal{F}\{e^{-b|x|}\} = \dfrac{2b}{b^2+w^2}$, $\mathcal{F}\{x^n f(x)\} = i^n F^{(n)}(w)$

$$\Rightarrow F\{x e^{-b|x|}\} = i\left(\frac{2b}{b^2+w^2}\right)' = \frac{i2b(-2w)}{(b^2+w^2)^2} = \frac{-4ibw}{(b^2+w^2)^2}$$

$$\xrightarrow{b=\frac{1}{2}} \mathcal{F}\{x e^{\frac{-|x|}{2}}\} = \frac{-2iw}{(0{,}5^2+w^2)^2} = \frac{-2iw}{(0{,}25+w^2)^2}$$

طبق قانون پارسوال داریم :

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(w)|^2 dw$$

$$\Rightarrow \int_{-\infty}^{+\infty} x^2 e^{-|x|} dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{4w^2}{(0{,}25+w^2)^4} dw \overset{w\to x}{\Longrightarrow}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-|x|} dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{4x^2}{(0{,}25+x^2)^4} dx$$

هر دو طرف تابع زوج در انتگرال خود دارند.

$$\Rightarrow \int_{0}^{+\infty} x^2 e^{-|x|} dx = \frac{1}{2\pi}\int_{0}^{+\infty} \frac{4x^2}{(0{,}25+x^2)^4} dx$$

$$\int_{0}^{+\infty} x^2 e^{-|x|} dx = \int_{0}^{+\infty} x^2 e^{-x} dx = \left(-x^2 e^{-x}\right)_0^{+\infty} + 2\int_0^{+\infty} x e^{-x} dx =$$

$$\cancelto{0}{\left(-x^2 e^{-x}\right)_0^{+\infty}} + \cancelto{0}{\left(-2x e^{-x}\right)_0^{+\infty}} + 2\int_0^{+\infty} e^{-x} dx = \left(-2e^{-x}\right)_0^{+\infty} = +2$$

$$\Rightarrow \int_0^{+\infty} \frac{x^2}{(0{,}25+x^2)^4} dx = \frac{2\pi}{4}\int_0^{+\infty} x^2 e^{-|x|} dx = \frac{\pi}{2}\times 2 = \pi$$

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