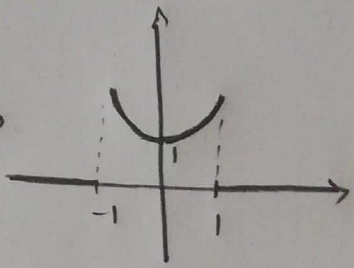
۸۱۰۱۰۱۴۹۲ علیرضا کریمی

1)

الف) $f(x) = \begin{cases} \cosh(x) & |x|<1 \\ 0 & |x|>1 \end{cases} \iff$



$\Rightarrow f(x) = f(-x)$

تابع زوج است $\Rightarrow B(w) = 0$

تابع زوج است:

$$A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos(wx)\,dx = \frac{2}{\pi}\int_0^1 \cosh(x)\cos(wx)\,dx = \frac{2}{\pi}\int_0^1 \underbrace{\cos\left(\frac{x}{i}\right)}_{=\cos(-ix)}\cos(wx)\,dx$$

$$= \frac{2}{\pi}\int_0^1 \cos(ix)\cos(wx)\,dx = \frac{1}{\pi}\int_0^1 \left[\cos(wx+ix) + (\cos wx - ix)\right] dx$$

$$= \frac{1}{\pi}\left(\frac{\sin(i+w)}{i+w} + \frac{\sin(w-i)}{w-i}\right) = \frac{1}{\pi}\,\frac{(w-i)\sin(i+w) + (i+w)\sin(w-i)}{1+w^2}$$

$$= \frac{1}{\pi}\,\frac{w(\sin(i+w) + \sin(w-i)) + i(\sin(w-i) - \sin(w+i))}{1+w^2}$$

$$= \frac{2}{\pi}\,\frac{w\sin w\cos i - i\sin i\cos w}{1+w^2}$$

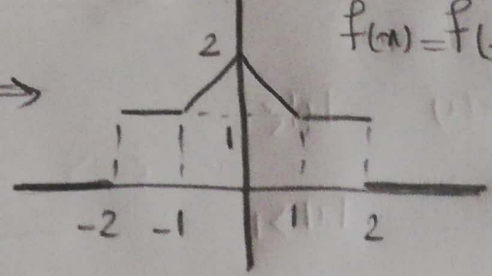
$\cosh(1) = \cos(i)$

$\sinh(1) = -i\sin(i)$

$$= \frac{2}{\pi}\,\frac{w\sin w\cosh(1) + \cos w\sinh(1)}{1+w^2}$$

$$\Longrightarrow f(x) = \int_0^\infty A(w)\cos wx\,dw = \frac{2}{\pi}\int_0^\infty \left(\frac{w\sin w\cosh(1) + \cos w\sinh(1)}{1+w^2}\right)\cos wx\,dw$$

ب) $f(x) = \begin{cases} 2-|x| & 0<|x|<1 \\ 1 & 1<|x|<2 \\ 0 & o.w \end{cases} \iff$



$f(x)=f(-x)$ تابع زوج است

$\Leftarrow \quad B(w)=0$

تابع زوج

$$\Longrightarrow A(w)=\frac{2}{\pi}\int_0^\infty f(x)\cos wx\,dx=\frac{2}{\pi}\int_0^1 (2-x)\cos wx\,dx+\frac{2}{\pi}\int_1^2 \cos wx\,dx=$$

$$\frac{4}{\pi}\int_0^1 \cos wx\,dx+\frac{2}{\pi}\int_1^2 \cos wx\,dx-\frac{2}{\pi}\int_0^1 x\cos wx\,dx$$

$$\int_0^1 \underset{u}{x}\,\underbrace{\cos wx\,dx}_{dv}=\left(\frac{x}{w}\sin wx\right)\Big|_0^1-\frac{1}{w}\underbrace{\int_0^1 \sin wx\,dx}_{-\frac{1}{w}\cos wx\big|_0^1}=\frac{\sin w}{w}+\frac{\cos w}{w^2}-\frac{1}{w^2}$$

$$\Longrightarrow A(w)=\frac{4}{\pi w}\sin w+\frac{2}{\pi w}(\sin 2w-\sin w)-\frac{2}{\pi}\left(\frac{\sin}{w}+\frac{\cos w}{w^2}-\frac{1}{w^2}\right)$$

$$=\frac{2w\sin w+2w\sin 2w-2w\sin w-2\cos w+2}{\pi w^2}=\frac{2(w\sin w-\cos w+1)}{\pi w^2}$$

$$\Longrightarrow f(x)=\int_0^\infty A(w)\cos wx\,dw=\int_0^\infty \frac{2(w\sin w-\cos w+1)}{\pi w^2}\cos wx\,dw$$

2)

$$\int_0^\infty Y(x)\sin(xt)\,dx = \int_0^\infty Y(\omega)\sin(\omega x)\,d\omega = \begin{cases} 1 & 0 \leq x < 1 \\ 2 & 1 \leq x < 2 \\ 0 & \text{o.w} \end{cases}$$

از آنجا که تنها هارمونی سینوسی داریم، می‌توانیم که گسترش فرد درنظر بگیریم و تابع را فرد درنظر بگیریم $\Longleftrightarrow$ $A(\omega) = 0$

$$\Rightarrow B(\omega) = Y(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)\,dx = \frac{2}{\pi}\int_0^1 \sin(\omega x)\,dx + \frac{4}{\pi}\int_1^2 \sin(\omega x)\,dx =$$

$$\frac{-2}{\pi\omega}(\cos\omega - 1) - \frac{4}{\pi\omega}(\cos 2\omega - \cos\omega) = \frac{2\cos\omega + 2 - 4\cos 2\omega}{\pi\omega}$$

$$\Longrightarrow \int_0^\infty Y(x)\sin(xt)\,dx = \frac{2}{\pi}\int_0^\infty \frac{\cos x + 1 - 2\cos 2x}{x}\sin xt\,dx$$

3) $$f(x) = \int_0^\infty \frac{\sin(\omega)}{\omega}\cos(\omega x)\,d\omega$$

تابع هارمونی سینوسی ندارد $\longleftarrow$ تابع زوج است $\Longleftarrow$ $B(\omega) = 0$ و $\alpha = -\beta$

$$A(\omega) = \frac{2}{\pi}\int_0^\beta c\cos(\omega x)\,dx = \frac{2c}{\pi\omega}\sin(\beta\omega) = \frac{\sin\omega}{\omega} \Longrightarrow \beta = 1,\ \alpha = -1,\ c = \frac{\pi}{2}$$

4) $f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & \text{o.w} \end{cases}$ $\qquad f(x) = \int_0^{\infty} \left[A(w)\cos(wx) + B(w)\sin(wx) \right] dw$

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x)\cos(wx)\,dx + \frac{1}{\pi} \int_0^{\pi} \sin(x)\cos(wx)\,dx =$$

$$\frac{1}{2\pi} \int_0^{\pi} \left[\sin(x+wx) + \sin(x-wx) \right] dx = \frac{-1}{2\pi} \left(\frac{1}{1+w}\cos(x+wx) + \frac{1}{1-w}\cos(x-wx) \right)_0^{\pi} =$$

$$\frac{-1}{2\pi} \left(\frac{\cos(x+wx) + \cos(x-wx) + w\left(\cos(x-wx) - \cos(x+wx)\right)}{1-w^2} \right)_0^{\pi} =$$

$$\frac{1}{\pi} \left(\frac{\cos(x)\cos(wx) + w\sin(x)\sin(wx)}{w^2-1} \right)_0^{\pi} = \frac{1}{\pi} \left(\frac{-\cos(\pi w) - 1}{w^2-1} \right) = \frac{1}{\pi} \left(\frac{1+\cos(w\pi)}{1-w^2} \right)$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x)\sin(wx)\,dx = \frac{1}{\pi} \int_0^{\pi} \sin(x)\sin(wx)\,dx = \frac{1}{2\pi} \int_0^{\pi} \left[\cos(x-wx) - \cos(x+wx) \right] dx$$

$$= \frac{1}{2\pi} \left(\frac{1}{1-w}\sin(x-wx) - \frac{1}{1+w}\sin(x+wx) \right)_0^{\pi} = \frac{1}{2\pi} \left(\frac{(1+w)\sin(x-wx) - (1-w)\sin(x-wx)}{1-w^2} \right)_0^{\pi}$$

$$= \frac{1}{2\pi} \, \frac{\overbrace{\sin(\pi - w\pi) - \sin(\pi + w\pi)}^{-2\cos\pi\sin w\pi} + w\overbrace{\left(\sin(\pi - \pi w) + \sin(\pi + \pi w)\right)}^{2\sin\pi\cos w\pi}}{1-w^2} = \frac{1}{\pi} \, \frac{\sin w\pi}{1-w^2}$$

$$\Longrightarrow f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{(1+\cos(w\pi))\cos(wx) + \sin(w\pi)\sin(wx)}{1-w^2}\,dw$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\cos(w\pi - wx) + \cos wx}{1-w^2}\,dw$$

$$\xrightarrow{x=\frac{\pi}{2}} f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 0 = \int_0^{\infty} \frac{\cos\left(w\pi - \frac{w\pi}{2}\right) + \cos\frac{w\pi}{2}}{1-w^2}\,dw$$

$$\overset{w=x}{\Longrightarrow} \int_0^{\infty} \frac{\cos^2\left(\frac{x\pi}{2}\right)}{1-x^2}\,dx = 0$$

5)

$$f(x) = \begin{cases} e^{-x} + e^{-2x} & x > 0 \\ e^{x} + e^{2x} & x < 0 \end{cases}$$

تابع $f(x)$ را به این صورت در نظر می گیریم :

(در واقع به نوعی گسترش زوج داریم)

در واقع چون فقط هارمون کسینوسی داریم ، تابع زوج در نظر گرفتیم و $B(w)=0$ و برابر $A(w)$ داریم :

$$A(w) = \frac{2}{\pi}\int_0^{+\infty} f(x)\cos(wx)\,dx = \frac{2}{\pi}\int_0^{+\infty}\left(e^{-x}+e^{-2x}\right)\cos wx\,dx =$$

$$\frac{2}{\pi}\int_0^{+\infty} e^{-x}\cos(wx)\,dx + \frac{2}{\pi}\int_0^{+\infty} e^{-2x}\cos(wx)\,dx = \frac{2}{\pi}\mathcal{L}\{\cos wx\}_{s=1} + \frac{2}{\pi}\mathcal{L}\{\cos wx\}_{s=2}$$

$$= \frac{2}{\pi}\left(\frac{1}{1+w^2} + \frac{2}{4+w^2}\right) = \frac{2}{\pi}\left(\frac{6+3w^2}{w^4+5w^2+4}\right) = \frac{6}{\pi}\,\frac{2+w^2}{w^4+5w^2+4}$$

$$\Longrightarrow f(x) = \int_0^{+\infty} A(w)\cos(wx)\,dw = \frac{6}{\pi}\int_0^{+\infty}\frac{2+w^2}{w^4+5w^2+4}\cos wx\,dw$$

6) $$f(x) = \int_{-\infty}^{+\infty} \left[\frac{1}{\omega^2+4}\cos(\omega x) + \frac{\omega}{\omega^2+4}\sin(\omega x)\right] d\omega \overset{\text{تابع زوج}}{=\!=\!=} \int_{0}^{+\infty}\left[\frac{2\cos(\omega x)}{\omega^2+4} + \frac{2\omega \sin(\omega x)}{\omega^2+4}\right]d\omega$$

$$\longrightarrow A(\omega) = \frac{2}{\omega^2+4}\ {}^{(*)}, \quad B(\omega) = \frac{2\omega}{\omega^2+4}\ {}^{(**)}$$

$$M = \int_{-\infty}^{+\infty} f(x)(2\cos^3 x + 3\sin^3 x)\,dx = \int_{-\infty}^{+\infty}\left[2f(x)\left(\frac{1+\cos 2x}{2}\right)\cos x + 3 f(x)\left(\frac{1-\cos 2x}{2}\right)\sin x\right]dx$$

$$= \int_{-\infty}^{+\infty} \left[f(x)\cos x + \frac{f(x)\cos 3x + f(x)\cos x}{2} + \frac{3}{2}\left(f(x)\sin x - \frac{f(x)\sin 3x - f(x)\sin x}{2}\right)\right]dx$$

$$= \frac{3\pi}{2}A(1) + \frac{\pi}{2}A(3) + \frac{9\pi}{4}B(1) - \frac{3\pi}{4}B(3) \qquad A(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\sin \omega x\, dx,\ B(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)\cos\omega x\, dx$$

$$\overset{(*),(**)}{=\!=\!=} \frac{3}{2}\pi \times \frac{2}{5} + \frac{\pi}{2}\times\frac{2}{13} + \frac{9}{4}\pi\times\frac{2}{5} - \frac{3}{4}\pi\times\frac{6}{13} = \frac{3}{5}\pi + \frac{\pi}{13} + \frac{9}{10}\pi - \frac{9}{26}\pi$$

$$= \frac{78+10+117-45}{130}\pi = \frac{165\pi}{130} = \frac{33}{26}\pi$$

7)

$$f(x) = \int_0^{+\infty} A(w)\cos(wx)\,dw$$

از آنجا که $f(x)$ تابعی زوج است ، داریم :

$$\longrightarrow A(w) = \frac{2}{\pi}\int_0^{+\infty} f(x)\cos(wx)\,dx \longrightarrow \frac{dA(w)}{dw} = \frac{2}{\pi}\int_0^{+\infty} f(x)\left(-x\sin(wx)\right)dx$$

$$\Longrightarrow 3\int_0^{+\infty} f(x)\cos(wx)\,dx - \int_0^{+\infty} x f(x)\sin(wx)\,dx = \frac{\pi}{2}\left(3A(w) + \frac{dA(w)}{dw}\right) = 0$$

$$\Longrightarrow 3A(w) + \frac{dA(w)}{dw} = 0 \Longrightarrow A(w) = Ke^{-3w}$$

$$\Longrightarrow f(x) = \int_0^{+\infty} A(w)\cos(wx)\,dw = \int_0^{+\infty} Ke^{-3w}\cos(wx)\,dw = K\,\mathcal{L}\{\cos wx\}_{s=3}$$

$$= \frac{3k}{x^2+9} \xrightarrow{x=0} f(0) = 1 = \frac{3k}{9} \Longrightarrow k = 3 \Longrightarrow f(x) = \frac{9}{x^2+9}$$