

# Matrix Theory (EE5609) Assignment 10

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**Abstract**—This finds the distance between a given point and a plane using Singular Value Decomposition.

All the codes for the figure in this document can be found at

[https://github.com/Arko98/EE5609/blob/master/Assignment\\_10](https://github.com/Arko98/EE5609/blob/master/Assignment_10)

Putting values in (2.0.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & \frac{3}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \quad (2.0.7)$$

Now, to solve (2.0.7), we perform Singular Value Decomposition on  $\mathbf{M}$  as follows,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (2.0.8)$$

Where the columns of  $\mathbf{V}$  are the eigen vectors of  $\mathbf{M}^T\mathbf{M}$ , the columns of  $\mathbf{U}$  are the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  and  $\mathbf{S}$  is diagonal matrix of singular value of eigenvalues of  $\mathbf{M}^T\mathbf{M}$ .

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix} \quad (2.0.9)$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & \frac{3}{2} \\ 3 & \frac{3}{2} & \frac{45}{4} \end{pmatrix} \quad (2.0.10)$$

From (2.0.6) putting (2.0.8) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (2.0.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (2.0.12)$$

Where  $\mathbf{S}_+$  is Moore-Penrose Pseudo-Inverse of  $\mathbf{S}$ . Now, calculating eigen value of  $\mathbf{M}\mathbf{M}^T$ ,

$$|\mathbf{M}\mathbf{M}^T - \lambda \mathbf{I}| = 0 \quad (2.0.13)$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & \frac{3}{2} \\ 3 & \frac{3}{2} & \frac{45}{4}-\lambda \end{vmatrix} = 0 \quad (2.0.14)$$

$$\Rightarrow \lambda^3 - \frac{53}{4}\lambda^2 + \frac{49}{4}\lambda = 0 \quad (2.0.15)$$

Hence eigen values of  $\mathbf{M}\mathbf{M}^T$  are,

$$\lambda_1 = \frac{49}{4} \quad (2.0.16)$$

$$\lambda_2 = 1 \quad (2.0.17)$$

$$\lambda_3 = 0 \quad (2.0.18)$$

## 1 PROBLEM

Find the distance of the point  $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$  from the plane

$$(6 \quad -3 \quad 2)\mathbf{x} = 4$$

## 2 SOLUTION

First we find orthogonal vectors  $\mathbf{m}_1$  and  $\mathbf{m}_2$  to the given normal vector  $\mathbf{n}$ . Let,  $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , then

$$\mathbf{m}^T \mathbf{n} = 0 \quad (2.0.1)$$

$$\Rightarrow (a, b, c) \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = 0 \quad (2.0.2)$$

$$\Rightarrow 6a - 3b + 2c = 0 \quad (2.0.3)$$

Putting  $a=1$  and  $b=0$  we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad (2.0.4)$$

Putting  $a=0$  and  $b=1$  we get,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \quad (2.0.5)$$

Now we solve the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (2.0.6)$$

Hence the eigen vectors of  $\mathbf{M}\mathbf{M}^T$  are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{4}{15} \\ \frac{2}{15} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -3 \\ -\frac{3}{2} \\ 1 \end{pmatrix} \quad (2.0.19)$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{4}{7\sqrt{5}} \\ \frac{2}{7\sqrt{5}} \\ \frac{3\sqrt{5}}{7} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{6}{7} \\ -\frac{3}{7} \\ \frac{2}{7} \end{pmatrix} \quad (2.0.20)$$

Hence we obtain  $\mathbf{U}$  of (2.0.8) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7} \\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7} \\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7} \end{pmatrix} \quad (2.0.21)$$

After computing the singular values from eigen values  $\lambda_1, \lambda_2, \lambda_3$  we get  $\mathbf{S}$  of (2.0.8) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.22)$$

Now, calculating eigen value of  $\mathbf{M}^T\mathbf{M}$ ,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (2.0.23)$$

$$\Rightarrow \begin{pmatrix} 10 - \lambda & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} - \lambda \end{pmatrix} = 0 \quad (2.0.24)$$

$$\Rightarrow \lambda^2 - \frac{53}{4}\lambda + \frac{49}{4} = 0 \quad (2.0.25)$$

Hence eigen values of  $\mathbf{M}^T\mathbf{M}$  are,

$$\lambda_4 = \frac{49}{4} \quad (2.0.26)$$

$$\lambda_5 = 1 \quad (2.0.27)$$

Hence the eigen vectors of  $\mathbf{M}^T\mathbf{M}$  are,

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \quad (2.0.28)$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \quad (2.0.29)$$

Hence we obtain  $\mathbf{V}$  of (2.0.8) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad (2.0.30)$$

Finally from (2.0.8) we get the Singular Value Decomposition of  $\mathbf{M}$  as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7} \\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7} \\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \quad (2.0.31)$$

Now, Moore-Penrose Pseudo inverse of  $\mathbf{S}$  is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.32)$$

From (2.0.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{27}{7\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \\ -\frac{33}{7} \end{pmatrix} \quad (2.0.33)$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{54}{49\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \end{pmatrix} \quad (2.0.34)$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{100}{146} \\ \frac{49}{49} \end{pmatrix} \quad (2.0.35)$$

Verifying the solution of (2.0.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \quad (2.0.36)$$

Evaluating the R.H.S in (2.0.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \quad (2.0.37)$$

$$\Rightarrow \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \quad (2.0.38)$$

Solving the augmented matrix of (2.0.38) we get,

$$\left( \begin{array}{ccc|c} 10 & \frac{9}{2} & -7 & \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} & \end{array} \right) \xrightarrow{R_1 = \frac{1}{10}R_1} \left( \begin{array}{ccc|c} 1 & \frac{9}{20} & -\frac{7}{10} & \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} & \end{array} \right) \quad (2.0.39)$$

$$\xrightarrow{R_2 = R_2 - \frac{9}{2}R_1} \left( \begin{array}{ccc|c} 1 & \frac{9}{20} & -\frac{7}{10} & \\ 0 & \frac{20}{49} & \frac{7}{20} & \end{array} \right) \quad (2.0.40)$$

$$\xrightarrow{R_2 = \frac{49}{20}R_2} \left( \begin{array}{ccc|c} 1 & \frac{9}{20} & -\frac{7}{10} & \\ 0 & 1 & \frac{146}{49} & \end{array} \right) \quad (2.0.41)$$

$$\xrightarrow{R_1 = R_1 - \frac{9}{20}R_2} \left( \begin{array}{ccc|c} 1 & 0 & -\frac{100}{146} & \\ 0 & 1 & \frac{146}{49} & \end{array} \right) \quad (2.0.42)$$

Hence, Solution of (2.0.36) is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{100}{146} \\ \frac{49}{49} \end{pmatrix} \quad (2.0.43)$$

Comparing results of  $\mathbf{x}$  from (2.0.35) and (2.0.43) we conclude that the solution is verified.