Matrix Theory (EE5609) Challenging Problem 1

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***Abstract*—This document explains the concept of finding the closest points on two skew lines in 3-Dimensions.**

Download latex codes from

https://github.com/Arko98/EE5609/tree/master/Challenge_2

1 Problem

Find the closest points on two skew lines where the distance between the lines is shortest.

2 Theory

Let the two lines be $\mathbf{L_1}$ and $\mathbf{L_2}$ defined as follows

$$\mathbf{L_1} : \mathbf{r_1} = \mathbf{x_1} + k_1\mathbf{v_1} \tag{2.0.1}$$

$$\mathbf{L_2} : \mathbf{r_2} = \mathbf{x_2} + k_2\mathbf{v_2} \tag{2.0.2}$$

Let $\mathbf{P}$ and $\mathbf{Q}$ be two points on the lines $\mathbf{L_1}$ and $\mathbf{L_2}$ respectively, where the distance between the two skew lines i.e $\|\mathbf{P} - \mathbf{Q}\|$ is shortest.
The direction vector of the line joining along the two points $\mathbf{P}$ and $\mathbf{Q}$ i.e $\mathbf{Q} - \mathbf{P}$ is parallel to $\mathbf{v_1} \times \mathbf{v_2}$.
Now as, $\mathbf{P}$ and $\mathbf{Q}$ are two points on the lines $\mathbf{L_1}$ and $\mathbf{L_2}$ then,

$$\mathbf{P} = \mathbf{x_1} + k_1\mathbf{v_1} \tag{2.0.3}$$

$$\mathbf{Q} = \mathbf{x_2} + k_2\mathbf{v_2} \tag{2.0.4}$$

Hence the direction vector of the line $\mathbf{Q} - \mathbf{P}$ is,

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} \mathbf{x_1} & \mathbf{v_1} \end{pmatrix} \begin{pmatrix} 1 \\ k_1 \end{pmatrix} - \begin{pmatrix} \mathbf{x_2} & \mathbf{v_2} \end{pmatrix} \begin{pmatrix} 1 \\ k_2 \end{pmatrix} \tag{2.0.5}$$

3 Explanation

The vectors $\mathbf{v_1}$,$\mathbf{v_2}$ are perpendicular to the line $\mathbf{Q} - \mathbf{P}$. So the dot product of $\mathbf{v_1}$,$\mathbf{v_2}$ with the direction vector $\mathbf{Q} - \mathbf{P}$ is zero. Hence,

$$\mathbf{v_1^T} \begin{pmatrix} \mathbf{x_1} & \mathbf{v_1} \end{pmatrix} \begin{pmatrix} 1 \\ k_1 \end{pmatrix} - \mathbf{v_1^T} \begin{pmatrix} \mathbf{x_2} & \mathbf{v_2} \end{pmatrix} \begin{pmatrix} 1 \\ k_2 \end{pmatrix} = 0 \tag{3.0.1}$$

$$\mathbf{v_2^T} \begin{pmatrix} \mathbf{x_1} & \mathbf{v_1} \end{pmatrix} \begin{pmatrix} 1 \\ k_1 \end{pmatrix} - \mathbf{v_2^T} \begin{pmatrix} \mathbf{x_2} & \mathbf{v_2} \end{pmatrix} \begin{pmatrix} 1 \\ k_2 \end{pmatrix} = 0 \tag{3.0.2}$$

Rearranging (3.0.1) and (3.0.2) in matrix form we get,

$$\begin{pmatrix} \mathbf{v_1^T x_1} & \mathbf{v_1^T v_1} & -\mathbf{v_1^T x_2} & -\mathbf{v_1^T v_2} \\ \mathbf{v_2^T x_1} & \mathbf{v_2^T v_1} & -\mathbf{v_2^T x_2} & -\mathbf{v_2^T v_2} \end{pmatrix} \begin{pmatrix} 1 \\ k_1 \\ 1 \\ k_2 \end{pmatrix} = 0 \tag{3.0.3}$$

Solving ((3.0.3)) we get the values of k_1 and k_2. Substituting the values of k_1 and k_2 in ((2.0.3)) and ((2.0.4)), we get the values of the points $\mathbf{P}$ and $\mathbf{Q}$ accordingly.

4 Example

We explain the theory with the following example.
Find the points where the distance is shortest between the lines

$$\mathbf{L_1} : \mathbf{r_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{4.0.1}$$

$$\mathbf{L_2} : \mathbf{r_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{4.0.2}$$

From theory, using ((3.0.3)), we get the points on skew lines $\mathbf{L_1}$ and $\mathbf{L2}$ as follows,

$$\mathbf{P} = \begin{pmatrix} \frac{29}{19} \\ \frac{8}{19} \\ \frac{77}{19} \end{pmatrix} = \begin{pmatrix} 1.52 \\ 0.42 \\ 4.05 \end{pmatrix} \tag{4.0.3}$$

$$\mathbf{Q} = \begin{pmatrix} \frac{20}{19} \\ \frac{11}{19} \\ \frac{86}{19} \end{pmatrix} = \begin{pmatrix} 1.05 \\ 0.57 \\ 4.52 \end{pmatrix} \tag{4.0.4}$$

The figure 1 is the illustration of the above problem.

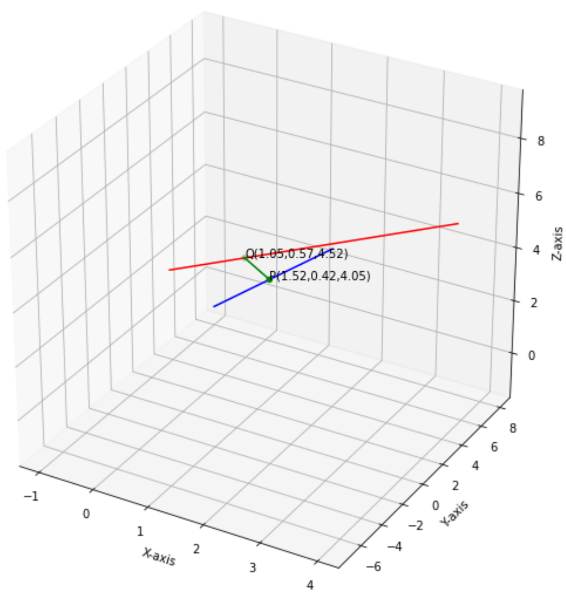

Fig. 1: Closest Points on Skew Lines