

# Matrix Theory (EE5609) Assignment 14

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**Abstract**—This document proves the existence of inverse of Hilbert Matrix.

All the codes for the figure in this document can be found at

[https://github.com/Arko98/EE5609/blob/master/Assignment\\_14](https://github.com/Arko98/EE5609/blob/master/Assignment_14)

## 1 PROBLEM

Prove that the following matrix is invertible and  $\mathbf{A}^{-1}$  has integer entries.

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

## 2 SOLUTION

Let  $\mathbf{A}_3$  be  $3 \times 3$  matrix i.e

$$\mathbf{A}_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \quad (2.0.1)$$

Now we find the inverse of the matrix  $\mathbf{A}_3$  as follows,

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{pmatrix} \quad (2.0.2)$$

$$\begin{matrix} R_2 = R_2 - \frac{1}{2}R_1 \\ R_3 = R_3 - \frac{1}{3}R_1 \end{matrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{1}{45} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \quad (2.0.3)$$

$$\begin{matrix} R_3 = R_3 - R_2 \end{matrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{pmatrix} \quad (2.0.4)$$

$$\begin{matrix} R_2 = 12R_2 \\ R_3 = 180R_3 \end{matrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (2.0.5)$$

$$\begin{matrix} R_2 = R_2 - R_3 \\ R_1 = R_1 - R_3 \end{matrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{2} & 0 & -9 & 60 & -60 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (2.0.6)$$

$$\begin{matrix} R_1 = R_1 - \frac{1}{2}R_2 \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (2.0.7)$$

Hence we see that  $\mathbf{A}_3$  is invertible and the inverse contains integer entries and  $\mathbf{A}_3^{-1}$  is given by,

$$\mathbf{A}_3^{-1} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \quad (2.0.8)$$

Let,  $\mathbf{A}_4$  be  $4 \times 4$  matrix as follows,

$$\mathbf{A}_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \quad (2.0.9)$$

Now, expressing  $\mathbf{A}_4$  using  $\mathbf{A}_3$  we get,

$$\mathbf{A}_4 = \begin{pmatrix} \mathbf{A}_3 & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (2.0.10)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \quad (2.0.11)$$

$$d = \frac{1}{7} \quad (2.0.12)$$

Now assuming  $\mathbf{A}_4$  has an inverse, then from (2.0.10), the inverse of  $\mathbf{A}_4$  can be written using block matrix inversion as follows,

$$\mathbf{A}_4^{-1} = \begin{pmatrix} \mathbf{A}_3^{-1} + \mathbf{A}_3^{-1}\mathbf{u}x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} & -\mathbf{A}_3^{-1}\mathbf{u}x^{-1} \\ -x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} & x^{-1} \end{pmatrix} \quad (2.0.13)$$

where,

$$x = d - \mathbf{u}^T\mathbf{A}_3^{-1}\mathbf{u} \quad (2.0.14)$$

Now, the assumption of  $\mathbf{A}_4$  being invertible will hold if and only if  $\mathbf{A}_3$  is invertible, which has been proved in (2.0.8) and  $x$  from (2.0.14) is invertible or  $x$  is a nonzero scalar. We now prove that  $x$  is invertible as follows,

$$x = \frac{1}{7} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \quad (2.0.15)$$

$$\Rightarrow x = \frac{1}{2800} \quad (2.0.16)$$

Hence,  $x$  is a scalar, hence  $x^{-1}$  exists and is given by,

$$x^{-1} = 2800 \quad (2.0.17)$$

Hence,  $\mathbf{A}_4$  is invertible. Now putting the values of  $\mathbf{A}_3^{-1}$ ,  $x^{-1}$  and  $\mathbf{u}$  we get,

$$\mathbf{A}_3^{-1} + \mathbf{A}_3^{-1}\mathbf{u}x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} = \begin{pmatrix} 16 & -120 & 240 \\ -120 & 1200 & -2700 \\ 240 & -2700 & 6480 \end{pmatrix} \quad (2.0.18)$$

$$-\mathbf{A}_3^{-1}\mathbf{u}x^{-1} = \begin{pmatrix} -140 \\ 1680 \\ -4200 \end{pmatrix} \quad (2.0.19)$$

$$x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} = \begin{pmatrix} -140 & 1680 & -4200 \end{pmatrix} \quad (2.0.20)$$

$$x^{-1} = 2800 \quad (2.0.21)$$

Putting values from (2.0.18), (2.0.19), (2.0.20) and (2.0.21) into (2.0.13) we get,

$$\mathbf{A}_4^{-1} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix} \quad (2.0.22)$$

Hence, from (2.0.22) we prove that,  $\mathbf{A}_4$  is invertible and has integer entries.

Let  $\mathbf{A}_{n-1}$  be invertible with integer entries. Then we can represent  $\mathbf{A}_n$  as follows,

$$\mathbf{A}_n = \begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (2.0.23)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (2.0.24)$$

$$d = \frac{1}{2n-1} \quad (2.0.25)$$

Now assuming  $\mathbf{A}_n$  has an inverse, then from (2.0.23), the inverse of  $\mathbf{A}_n$  can be written using block matrix inversion as follows,

$$\mathbf{A}_n^{-1} = \begin{pmatrix} \mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1}\mathbf{u}x^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & -\mathbf{A}_{n-1}^{-1}\mathbf{u}x^{-1} \\ -x^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & x^{-1} \end{pmatrix} \quad (2.0.26)$$

$$= x^{-1} \begin{pmatrix} x\mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1} - \mathbf{u}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & -\mathbf{A}_{n-1}^{-1}\mathbf{u} \\ -\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & 1 \end{pmatrix} \quad (2.0.27)$$

where,

$$x = d - \mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u} \quad (2.0.28)$$

Now, the assumption of  $\mathbf{A}_n$  being invertible will hold if and only if  $\mathbf{A}_{n-1}$  is invertible, which has been assumed and  $x$  from (2.0.28) is invertible or  $x$  is a nonzero scalar. We now prove that  $x$  is invertible as follows,

$$x = \frac{1}{2n-1} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{2n-2} \end{pmatrix} \mathbf{A}_{n-1}^{-1} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (2.0.29)$$

In equation (2.0.29)  $\mathbf{u}$  contains no negative or zero entries, again  $\mathbf{A}_{n-1}^{-1}$  has non zero integer entries, hence  $\mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u}$  is a non zero scalar. Moreover  $d$  is

not equal to  $\mathbf{u}^T \mathbf{A}_{n-1}^{-1} \mathbf{u}$  hence in (2.0.29)  $x$  is non-zero scalar and invertible and hence it has an inverse. Hence  $\mathbf{A}_n$  is invertible, proved.