Matrix Theory (EE5609) Assignment 7

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***Abstract*—This finds whether a given second degree equation represents a pair of straight lines or not.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_7

1 Problem

Find the value of k so that the following equation may represent a pair of straight lines -

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0$$

2 Theory

The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

(2.0.1) can be written as,

$$\tag{2.0.2}$$

$$\mathbf{x^T V x} + 2\mathbf{u^T x} + f = 0 \tag{2.0.3}$$

where,

$$\mathbf{V} = \mathbf{V^T} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{2.0.5}$$

(2.0.3) represents a pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u^T} & f \end{vmatrix} = 0 \tag{2.0.6}$$

Otherwise, (2.0.3) represents a conic section.

3 Solution

The given second degree equation is,

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0 \tag{3.0.1}$$

Comparing coefficients of (3.0.1) with (2.0.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{u} = \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \tag{3.0.3}$$

$$f = -35 \tag{3.0.4}$$

From (2.0.6) the given second degree equation (3.0.1) will represent a pair of straight line if,

$$\begin{vmatrix} 6 & \frac{1}{2} & -\frac{11}{2} \\ \frac{1}{2} & k & \frac{43}{2} \\ -\frac{11}{2} & \frac{43}{2} & -35 \end{vmatrix} = 0 \tag{3.0.5}$$

Expanding the determinant,

$$k + 12 = 0 \tag{3.0.6}$$

$$\implies k = -12 \tag{3.0.7}$$

Hence, from (3.0.7) we find that for $k = -12$, the given second degree equation (3.0.1) represents pair of straight lines. For the appropriate value of k, (3.0.1) becomes,

$$6x^2 + xy - 12y^2 - 11x + 43y - 35 = 0 \tag{3.0.8}$$

4 Graphical Illustration

Let the pair of straight lines in vector form is given by

$$\mathbf{n_1}^T\mathbf{x} = c_1 \tag{4.0.1}$$

$$\mathbf{n_2}^T\mathbf{x} = c_2 \tag{4.0.2}$$

The pair of straight lines is given by,

$$(\mathbf{n_1}^T\mathbf{x} - c_1)(\mathbf{n_2}^T\mathbf{x} - c_2) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u^T}\mathbf{x} + f = 0 \tag{4.0.3}$$

Putting the values of $\mathbf{V}$ and $\mathbf{u}$ we get,

$$\mathbf{x}^T \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -12 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} -\frac{11}{2} & \frac{43}{2} \end{pmatrix} \mathbf{x} - 35 = 0 \tag{4.0.4}$$

Hence, from (4.0.4) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \tag{4.0.5}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \tag{4.0.6}$$

$$c_1c_2 = -35 \tag{4.0.7}$$

The slopes of the pair of straight lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \tag{4.0.8}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \tag{4.0.9}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{4.0.10}$$

Substituting the values in above equations (4.0.8) we get,

$$-12m^2 + m + 6 = 0 \tag{4.0.11}$$

$$\implies m_i = \frac{-\frac{1}{2} \pm \sqrt{-(-\frac{289}{4})}}{-12} \tag{4.0.12}$$

Solving equation (4.0.12) we get ,

$$m_1 = -\frac{2}{3} \tag{4.0.13}$$

$$m_2 = \frac{3}{4} \tag{4.0.14}$$

Hence putting the values of m_1 and m_2 in (4.0.10) we get

$$\mathbf{n_1} = k_1\begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \tag{4.0.15}$$

$$\mathbf{n_2} = k_2\begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \tag{4.0.16}$$

Putting values of $\mathbf{n_1}$ and $\mathbf{n_2}$ in (4.0.5) we get,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} -\frac{3k_2}{4} & 0 \\ k_2 & -\frac{3k_2}{4} \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} \frac{2k_1}{3} \\ k_1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \tag{4.0.17}$$

$$\implies \begin{pmatrix} -\frac{1}{2}k_1k_2 \\ -\frac{1}{12}k_1k_2 \\ k_1k_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \tag{4.0.18}$$

Thus, from (4.0.18), $k_1k_2 = -12$. Possible combinations of (k_1, k_2) are (6,-2), (-6,2), (3,-4), (-3,4) Lets assume $k_1 = 3$, $k_2 = -4$, then we get,

$$\mathbf{n_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{4.0.19}$$

$$\mathbf{n_2} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{4.0.20}$$

From equation (4.0.6) we get

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{4.0.21}$$

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \tag{4.0.22}$$

Hence we get the following equations,

$$2c_2 + 3c_1 = 11 \tag{4.0.23}$$

$$3c_2 - 4c_1 = -43 \tag{4.0.24}$$

The augmented matrix of (4.0.23) ,(4.0.24) is,

$$\begin{pmatrix} 2 & 3 & 11 \\ 3 & -4 & -43 \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 3 & -4 & -43 \end{pmatrix} \tag{4.0.25}$$

$$\xleftrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & -\frac{17}{2} & -\frac{119}{2} \end{pmatrix} \tag{4.0.26}$$

$$\xleftrightarrow{R_2 = -\frac{2}{17}} \begin{pmatrix} 1 & \frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 7 \end{pmatrix} \tag{4.0.27}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{3}{2}R_2} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 7 \end{pmatrix} \tag{4.0.28}$$

$$\tag{4.0.29}$$

Hence we get,

$$c_1 = -5 \tag{4.0.30}$$

$$c_2 = 7 \tag{4.0.31}$$

Hence (4.0.1), (4.0.2) can be modified as follows,

$$\begin{pmatrix} 2 & 3 \end{pmatrix}\mathbf{x} = -5 \tag{4.0.32}$$

$$\begin{pmatrix} 3 & -4 \end{pmatrix}\mathbf{x} = 7 \tag{4.0.33}$$

The figure below corresponds to the pair of straight lines represented by (4.0.32) and (4.0.33).

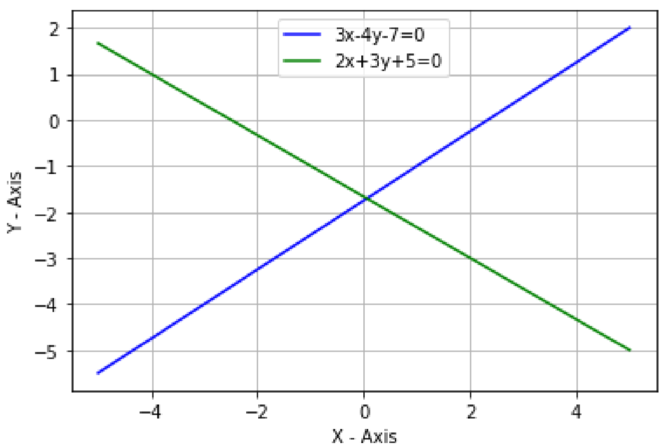

Fig. 1: Pair of Straight Lines