Matrix Theory (EE5609) Assignment 12

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Abstract—**This document proves that, each field of the characteristic zero contains a copy of the rational number field.**

All the codes for the figure in this document can be found at

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https://github.com/Arko98/EE5609/blob/master/
    Assignment_12
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1 Problem

Consider the system of equations $\mathbf{AX} = 0$ where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix over the field F. Prove the following -

- If every entry of $\mathbf{A}$ is 0, then every pair $(x1, x2)$ is a solution of $\mathbf{AX} = 0$.
- If $ad - bc \neq 0$, then the system $\mathbf{AX} = 0$ has only the trivial solution $x1 = x2 = 0$
- If $ad - bc = 0$ and some entry of $\mathbf{A}$ is different from 0, then there is a solution (x_1^0, x_2^0) such that $(x1, x2)$ is a solution if and only if there is a scalar y such that $x_1 = yx_1^0$ and $x_2 = yx_2^0$

2 Solution

2.1 Solution 1

If every entry of $\mathbf{A}$ is 0 then the equation $\mathbf{AX} = 0$ becomes,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{2.1.1}$$

$$\implies 0.x_1 + 0.x_2 = 0 \qquad \forall x_1, x_2 \in F \tag{2.1.2}$$

Hence proved, every pair (x_1, x_2) is a solution for the equation $\mathbf{AX} = 0$.

2.2 Solution 2

Case 1: Let $a = 0$. Since $ad - bc \neq 0$. As $bc \neq 0$ therefore $b \neq 0$ and $c \neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX}$=0 as follows,

$$\begin{pmatrix} 0 & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d & 0 \\ 0 & b & 0 \end{pmatrix} \tag{2.2.1}$$

$$\xleftrightarrow[R_2=\frac{1}{b}R_2]{R_1=\frac{1}{c}R_1} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.2.2}$$

$$\xleftrightarrow{R_1=R_1-\frac{d}{c}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.2.3}$$

Case 2: Let $a \neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX}$=0 as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \tag{2.2.4}$$

$$\xleftrightarrow{R_2=R_2-cR_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & \frac{ad-bc}{a} & 0 \end{pmatrix} \tag{2.2.5}$$

$$\xleftrightarrow{R_2=\frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.2.6}$$

$$\xleftrightarrow{R_1=R_1-\frac{b}{a}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.2.7}$$

Case 3: Let $a, b, c, d \neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX}$=0 as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \tag{2.2.8}$$

$$\xleftrightarrow{R_2=R_2-cR_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & \frac{ad-bc}{a} & 0 \end{pmatrix} \tag{2.2.9}$$

$$\xleftrightarrow{R_2=\frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.2.10}$$

$$\xleftrightarrow{R_1=R_1-\frac{b}{a}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.2.11}$$

Case 4: Let $a, b, c, d \neq 0$. Considering the following case,

$$\mathbf{AX} = \mathbf{u} \neq 0 \tag{2.2.12}$$

$$\implies \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \mathbf{0} \tag{2.2.13}$$

Row Reducing the augmented matrix of (2.2.13) we get,

$$\begin{pmatrix} a & b & u_1 \\ c & d & u_2 \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ c & d & u_2 \end{pmatrix} \tag{2.2.14}$$

$$\xleftrightarrow{R_2=R_2-cR_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & \frac{ad-bc}{a} & \frac{au_2-cu_1}{a} \end{pmatrix} \tag{2.2.15}$$

$$\xleftrightarrow{R_2=\frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \tag{2.2.16}$$

$$\xleftrightarrow{R_1=R_1-\frac{b}{a}R_2} \begin{pmatrix} 1 & 0 & \frac{du_1-bu_2}{ad-bc} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \tag{2.2.17}$$

From (2.2.17) we get,

$$du_1 - bu_2 = 0 \tag{2.2.18}$$

$$au_2 - cu_1 = 0 \tag{2.2.19}$$

Solving (2.2.18) and (2.2.19),

$$u_1 = 0 \tag{2.2.20}$$

$$u_2 = 0 \tag{2.2.21}$$

Hence we get,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.22}$$

Thus $\mathbf{AX} = \mathbf{u}$ holds true if and only if $\mathbf{u} = 0$.
In (2.2.3), (2.2.7), (2.2.11) and (2.2.22), we can see that $\mathbf{AX} = 0$ has only one trivial solution i.e $x_1 = x_2 = 0$ in all cases. Hence proved, the equation $\mathbf{AX}$=0 has only one trivial solution $x_1 = x_2 = 0$

2.3 Solution 3

Case 1: Let, $a \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \tag{2.3.1}$$

$$\xleftrightarrow{R_2=R_2-cR_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{2.3.2}$$

Hence from (2.3.19), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{b}{a}x_2 \qquad [a \neq 0] \tag{2.3.3}$$

Letting $x_1^0 = -\frac{b}{a}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{2.3.4}$$

$$x_2 = yx_2^0 \tag{2.3.5}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 2: Let, $b \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{b}R_1} \begin{pmatrix} \frac{a}{b} & 1 & 0 \\ c & d & 0 \end{pmatrix} \tag{2.3.6}$$

$$\xleftrightarrow{R_2=R_2-dR_1} \begin{pmatrix} \frac{a}{b} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{2.3.7}$$

Hence from (2.3.19), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{a}{b}x_1 \qquad [b \neq 0] \tag{2.3.8}$$

Letting $x_2^0 = -\frac{a}{b}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{2.3.9}$$

$$x_2 = yx_2^0 \tag{2.3.10}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 3: Let, $c \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1\leftrightarrow R_2} \begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \tag{2.3.11}$$

$$\xleftrightarrow{R_1=\frac{1}{c}R_1} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ a & b & 0 \end{pmatrix} \tag{2.3.12}$$

$$\xleftrightarrow{R_2=R_2-aR_1} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{2.3.13}$$

Hence from (2.3.19), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{d}{c}x_2 \qquad [a \neq 0] \tag{2.3.14}$$

Letting $x_1^0 = -\frac{d}{c}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{2.3.15}$$
$$x_2 = yx_2^0 \tag{2.3.16}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 4: Let, $d \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \tag{2.3.17}$$
$$\xleftrightarrow{R_1 = \frac{1}{d}R_1} \begin{pmatrix} \frac{c}{d} & 1 & 0 \\ a & b & 0 \end{pmatrix} \tag{2.3.18}$$
$$\xleftrightarrow{R_2 = R_2 - bR_1} \begin{pmatrix} \frac{c}{d} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{2.3.19}$$

Hence from (2.3.19), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{d}{c}x_1 \qquad [a \neq 0] \tag{2.3.20}$$

Letting $x_2^0 = -\frac{d}{c}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{2.3.21}$$
$$x_2 = yx_2^0 \tag{2.3.22}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Hence Proved.