

Matrix Theory (EE5609) Assignment 14

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Abstract—This document proves the existence of inverse of Hilbert Matrix.

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_14

1 PROBLEM

Prove that the following matrix is invertible and \mathbf{A}^{-1} has integer entries.

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

2 SOLUTION

Let \mathbf{A}_3 be 3×3 matrix i.e

$$\mathbf{A}_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \quad (2.0.1)$$

Now we find the inverse of the matrix \mathbf{A}_3 as follows,

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{pmatrix} \quad (2.0.2)$$

$$\begin{matrix} \xleftarrow{R_2=R_2-\frac{1}{2}R_1} \\ \xleftarrow{R_3=R_3-\frac{1}{3}R_1} \end{matrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{1}{45} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \quad (2.0.3)$$

$$\xleftarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{pmatrix} \quad (2.0.4)$$

$$\begin{matrix} \xleftarrow{R_2=12R_2} \\ \xleftarrow{R_3=180R_3} \end{matrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (2.0.5)$$

$$\begin{matrix} \xleftarrow{R_2=R_2-R_3} \\ \xleftarrow{R_1=R_1-R_3} \end{matrix} \begin{pmatrix} 1 & \frac{1}{2} & 0 & -9 & 60 & -60 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (2.0.6)$$

$$\xleftarrow{R_1=R_1-\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (2.0.7)$$

Hence we see that \mathbf{A}_3 is invertible and the inverse contains integer entries and \mathbf{A}_3^{-1} is given by,

$$\mathbf{A}_3^{-1} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \quad (2.0.8)$$

Let, \mathbf{A}_4 be 4×4 matrix as follows,

$$\mathbf{A}_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \quad (2.0.9)$$

Now, expressing \mathbf{A}_4 using \mathbf{A}_3 we get,

$$\mathbf{A}_4 = \begin{pmatrix} \mathbf{A}_3 & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (2.0.10)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \quad (2.0.11)$$

$$d = \frac{1}{7} \quad (2.0.12)$$

Now assuming \mathbf{A}_4 has an inverse, then from (2.0.10), the inverse of \mathbf{A}_4 can be written using block matrix inversion as follows,

$$\mathbf{A}_4^{-1} = \begin{pmatrix} \mathbf{A}_3^{-1} + \mathbf{A}_3^{-1}\mathbf{u}x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} & -\mathbf{A}_3^{-1}\mathbf{u}x^{-1} \\ -x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} & x^{-1} \end{pmatrix} \quad (2.0.13)$$

where,

$$x = d - \mathbf{u}^T\mathbf{A}_3^{-1}\mathbf{u} \quad (2.0.14)$$

Now, the assumption of \mathbf{A}_4 being invertible will hold if and only if \mathbf{A}_3 is invertible, which has been proved in (2.0.8) and x from (2.0.14) is invertible or x is a nonzero scalar. We now prove that x is invertible as follows,

$$x = \frac{1}{7} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \quad (2.0.15)$$

$$\Rightarrow x = \frac{1}{2800} \quad (2.0.16)$$

Hence, x is a scalar, hence x^{-1} exists and is given by,

$$x^{-1} = 2800 \quad (2.0.17)$$

Hence, \mathbf{A}_4 is invertible. Now putting the values of \mathbf{A}_3^{-1} , x^{-1} and \mathbf{u} we get,

$$\mathbf{A}_3^{-1} + \mathbf{A}_3^{-1}\mathbf{u}x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} = \begin{pmatrix} 16 & -120 & 240 \\ -120 & 1200 & -2700 \\ 240 & -2700 & 6480 \end{pmatrix} \quad (2.0.18)$$

$$-\mathbf{A}_3^{-1}\mathbf{u}x^{-1} = \begin{pmatrix} -140 \\ 1680 \\ -4200 \end{pmatrix} \quad (2.0.19)$$

$$x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} = \begin{pmatrix} -140 & 1680 & -4200 \end{pmatrix} \quad (2.0.20)$$

$$x^{-1} = 2800 \quad (2.0.21)$$

Putting values from (2.0.18), (2.0.19), (2.0.20) and (2.0.21) into (2.0.13) we get,

$$\mathbf{A}_4^{-1} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix} \quad (2.0.22)$$

Hence, from (2.0.22) we prove that, \mathbf{A}_4 is invertible and has integer entries.

Let \mathbf{A}_{n-1} be invertible with integer entries. Then we can represent \mathbf{A}_n as follows,

$$\mathbf{A}_n = \begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (2.0.23)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (2.0.24)$$

$$d = \frac{1}{2n-1} \quad (2.0.25)$$

Now assuming \mathbf{A}_n has an inverse, then from (2.0.23), the inverse of \mathbf{A}_n can be written using block matrix inversion as follows,

$$\mathbf{A}_n^{-1} = \begin{pmatrix} \mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1}\mathbf{u}x^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & -\mathbf{A}_{n-1}^{-1}\mathbf{u}x^{-1} \\ -x^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & x^{-1} \end{pmatrix} \quad (2.0.26)$$

$$= x^{-1} \begin{pmatrix} x\mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1} - \mathbf{1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & -\mathbf{A}_{n-1}^{-1}\mathbf{u} \\ \mathbf{u}^T\mathbf{A}_{n-1}^{-1} & 1 \end{pmatrix} \quad (2.0.27)$$

where,

$$x = d - \mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u} \quad (2.0.28)$$

Now, the assumption of \mathbf{A}_n being invertible will hold if and only if \mathbf{A}_{n-1} is invertible, which has been assumed and x from (2.0.28) is invertible or x is a nonzero scalar. We now prove that x is invertible as follows,

$$x = \frac{1}{2n-1} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{2n-2} \end{pmatrix} \mathbf{A}_{n-1}^{-1} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (2.0.29)$$

In equation (2.0.29) \mathbf{u} contains no negative or zero entries, again \mathbf{A}_{n-1}^{-1} has non zero integer entries, hence $\mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u}$ is a non zero scalar. Moreover d is

not equal to $\mathbf{u}^T \mathbf{A}_{n-1}^{-1} \mathbf{u}$ hence in (2.0.29) x is non-zero scalar and invertible and hence it has an inverse. Hence \mathbf{A}_n is invertible, proved.