Matrix Theory (EE5609) Assignment 12

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Abstract—**This document proves that, each field of the characteristic zero contains a copy of the rational number field.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_12

1 Problem

Consider the system of equations $\mathbf{AX} = 0$ where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix over the field F. Prove the following -

- If every entry of $\mathbf{A}$ is 0, then every pair x_1 and x_2 is a solution of $\mathbf{AX} = 0$.
- If $ad - bc \neq 0$, then the system $\mathbf{AX} = 0$ has only the trivial solution $x_1 = x_2 = 0$
- If $ad - bc = 0$ and some entry of $\mathbf{A}$ is different from 0, then there is a solution x_1^0 and x_2^0 such that x_1 and x_2 is a solution if and only if there is a scalar y such that $x_1 = yx_1^0$ and $x_2 = yx_2^0$

2 Solution

2.1 Solution 1

If every entry of $\mathbf{A}$ is 0 then the equation $\mathbf{AX} = 0$ becomes,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{2.1.1}$$

$$\implies 0.x_1 + 0.x_2 = 0 \qquad \forall x_1, x_2 \in F \tag{2.1.2}$$

Hence proved, every pair x_1 and x_2 is a solution for the equation $\mathbf{AX} = 0$.

2.2 Solution 2

Case 1: Let $a = 0$. Since $ad - bc \neq 0$. As $bc \neq 0$ therefore $b \neq 0$ and $c \neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX}$=0 as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d & 0 \\ 0 & b & 0 \end{pmatrix} \tag{2.2.1}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{b} \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & b & 0 \end{pmatrix} \tag{2.2.2}$$

$$= \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.2.3}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.2.4}$$

Case 2: Let $a, b, c, d \neq 0$. Considering the following case,

$$\mathbf{AX} = \mathbf{u} \tag{2.2.5}$$

$$\implies \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{2.2.6}$$

Row Reducing the augmented matrix of (2.2.6) we get,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & u_1 \\ c & d & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ c & d & u_2 \end{pmatrix} \tag{2.2.7}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & \frac{ad-bc}{a} & \frac{au_2-cu_1}{a} \end{pmatrix} \tag{2.2.8}$$

$$= \begin{pmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \tag{2.2.9}$$

$$= \begin{pmatrix} 1 & 0 & \frac{du_1-bu_2}{ad-bc} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \tag{2.2.10}$$

From (2.2.10) we get,

$$x_1 = \frac{du_1 - bu_2}{ad - bc} \tag{2.2.11}$$

$$x_2 = \frac{au_2 - cu_1}{ad - bc} \tag{2.2.12}$$

Since $u_1 = 0$ and $u_2 = 0$ then from (2.2.11) and (2.2.12),

$$x_1 = 0 \tag{2.2.13}$$

$$x_2 = 0 \tag{2.2.14}$$

Hence we get,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.15}$$

In (2.2.4) and (2.2.15), we can see that $\mathbf{AX} = 0$ has only one trivial solution i.e $x_1 = x_2 = 0$ in all cases. Hence proved, the equation $\mathbf{AX}$=0 has only one trivial solution $x_1 = x_2 = 0$

2.3 Solution 3

Case 1: Let, $a \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \tag{2.3.1}$$

$$= \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{2.3.2}$$

Hence from (2.3.2), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{b}{a} x_2 \qquad [a \neq 0] \tag{2.3.3}$$

Letting $x_1^0 = -\frac{b}{a}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = y x_1^0 \tag{2.3.4}$$

$$x_2 = y x_2^0 \tag{2.3.5}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 2: Let, $b \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, at first we multiply by elementary matrix to change the columns and the we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \tag{2.3.6}$$

Hence using the result obtained from (2.3.2) we can conclude for (2.3.6), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{a}{b} x_1 \qquad [b \neq 0] \tag{2.3.7}$$

Letting $x_2^0 = -\frac{a}{b}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = y x_1^0 \tag{2.3.8}$$

$$x_2 = y x_2^0 \tag{2.3.9}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 3: Let, $c \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \tag{2.3.10}$$

$$= \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ a & b & 0 \end{pmatrix} \tag{2.3.11}$$

$$= \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \tag{2.3.12}$$

Hence from (2.3.12), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{d}{c} x_2 \qquad [a \neq 0] \tag{2.3.13}$$

Letting $x_1^0 = -\frac{d}{c}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = y x_1^0 \tag{2.3.14}$$

$$x_2 = y x_2^0 \tag{2.3.15}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Case 4: Let, $d \neq 0$ for $\mathbf{A}$. Given $ad - bc = 0$, at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \tag{2.3.16}$$

$$= \begin{pmatrix} d & c & 0 \\ b & a & 0 \end{pmatrix} \tag{2.3.17}$$

Hence using the result from (2.3.12) we can conclude for (2.3.17), $\mathbf{AX} = 0$ if and only if

$$x_2 = -\frac{c}{d} x_1 \qquad [a \neq 0] \tag{2.3.18}$$

Letting $x_2^0 = -\frac{c}{d}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{2.3.19}$$

$$x_2 = yx_2^0 \tag{2.3.20}$$

which is a solution of the equation $\mathbf{AX} = 0$.
Hence Proved.