

Matrix Theory (EE5609) Assignment 12

Arkadipta De
MTech Artificial Intelligence
AI20MTECH14002

Abstract—This document proves that, each field of the characteristic zero contains a copy of the rational number field.

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_12

1 PROBLEM

Consider the system of equations $\mathbf{AX} = 0$ where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix over the field F . Prove the following

- If every entry of \mathbf{A} is 0, then every pair (x_1, x_2) is a solution of $\mathbf{AX} = 0$.
- If $ad - bc \neq 0$, then the system $\mathbf{AX} = 0$ has only the trivial solution $x_1 = x_2 = 0$
- If $ad - bc = 0$ and some entry of \mathbf{A} is different from 0, then there is a solution (x_1^0, x_2^0) such that (x_1, x_2) is a solution if and only if there is a scalar y such that $x_1 = yx_1^0$ and $x_2 = yx_2^0$

2 SOLUTION

2.1 Solution 1

If every entry of \mathbf{A} is 0 then the equation $\mathbf{AX} = 0$ becomes,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.1.1)$$

$$\Rightarrow 0.x_1 + 0.x_2 = 0 \quad \forall x_1, x_2 \in F \quad (2.1.2)$$

Hence proved, every pair (x_1, x_2) is a solution for the equation $\mathbf{AX} = 0$.

2.2 Solution 2

Case 1: Let $a = 0$. Since $ad - bc \neq 0$. As $bc \neq 0$ therefore $b \neq 0$ and $c \neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} 0 & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d & 0 \\ 0 & b & 0 \end{pmatrix} \quad (2.2.1)$$

$$\xleftrightarrow{\begin{matrix} R_1 = \frac{1}{c}R_1 \\ R_2 = \frac{1}{b}R_2 \end{matrix}} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.2.2)$$

$$\xleftrightarrow{R_1 = R_1 - \frac{d}{c}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.2.3)$$

Case 2: Let $a \neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \quad (2.2.4)$$

$$\xleftrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & \frac{ad-bc}{a} & 0 \end{pmatrix} \quad (2.2.5)$$

$$\xleftrightarrow{R_2 = \frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.2.6)$$

$$\xleftrightarrow{R_1 = R_1 - \frac{b}{a}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.2.7)$$

Case 3: Let $a, b, c, d \neq 0$. Hence, we can perform row reduction on the augmented matrix of equation $\mathbf{AX} = 0$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \quad (2.2.8)$$

$$\xleftrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & \frac{ad-bc}{a} & 0 \end{pmatrix} \quad (2.2.9)$$

$$\xleftrightarrow{R_2 = \frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.2.10)$$

$$\xleftrightarrow{R_1 = R_1 - \frac{b}{a}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.2.11)$$

Case 4: Considering the following case,

$$\mathbf{AX} = \mathbf{u} \neq \mathbf{0} \quad (2.2.12)$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \neq \mathbf{0} \quad (2.2.13)$$

Comparing (2.2.13) with $\mathbf{AX} = \mathbf{0}$, the equation will be satisfied if and only if,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.2.14)$$

In (2.2.3), (2.2.7), (2.2.11) and (2.2.14), we can see that it has only one trivial solution $x_1 = x_2 = 0$. Hence proved, the equation $\mathbf{AX} = \mathbf{0}$ has only one trivial solution $x_1 = x_2 = 0$

2.3 Solution 3

Case 1: Let, $a \neq 0$ for \mathbf{A} . Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = \mathbf{0}$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \quad (2.3.1)$$

$$\xrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (2.3.2)$$

Hence from (2.3.19), $\mathbf{AX} = \mathbf{0}$ if and only if

$$x_1 = -\frac{b}{a}x_2 \quad [a \neq 0] \quad (2.3.3)$$

Letting $x_1^0 = -\frac{b}{a}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \quad (2.3.4)$$

$$x_2 = yx_2^0 \quad (2.3.5)$$

which is a solution of the equation $\mathbf{AX} = \mathbf{0}$.

Case 2: Let, $b \neq 0$ for \mathbf{A} . Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = \mathbf{0}$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xrightarrow{R_1 = \frac{1}{b}R_1} \begin{pmatrix} \frac{a}{b} & 1 & 0 \\ c & d & 0 \end{pmatrix} \quad (2.3.6)$$

$$\xrightarrow{R_2 = R_2 - dR_1} \begin{pmatrix} \frac{a}{b} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (2.3.7)$$

Hence from (2.3.19), $\mathbf{AX} = \mathbf{0}$ if and only if

$$x_2 = -\frac{a}{b}x_1 \quad [b \neq 0] \quad (2.3.8)$$

Letting $x_2^0 = -\frac{a}{b}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \quad (2.3.9)$$

$$x_2 = yx_2^0 \quad (2.3.10)$$

which is a solution of the equation $\mathbf{AX} = \mathbf{0}$.

Case 3: Let, $c \neq 0$ for \mathbf{A} . Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = \mathbf{0}$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \quad (2.3.11)$$

$$\xrightarrow{R_1 = \frac{1}{c}R_1} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ a & b & 0 \end{pmatrix} \quad (2.3.12)$$

$$\xrightarrow{R_2 = R_2 - aR_1} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (2.3.13)$$

Hence from (2.3.19), $\mathbf{AX} = \mathbf{0}$ if and only if

$$x_1 = -\frac{d}{c}x_2 \quad [a \neq 0] \quad (2.3.14)$$

Letting $x_1^0 = -\frac{d}{c}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \quad (2.3.15)$$

$$x_2 = yx_2^0 \quad (2.3.16)$$

which is a solution of the equation $\mathbf{AX} = \mathbf{0}$.

Case 4: Let, $d \neq 0$ for \mathbf{A} . Given $ad - bc = 0$, we can perform row reduction on augmented matrix of equation $\mathbf{AX} = \mathbf{0}$ as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \quad (2.3.17)$$

$$\xrightarrow{R_1 = \frac{1}{d}R_1} \begin{pmatrix} \frac{c}{d} & 1 & 0 \\ a & b & 0 \end{pmatrix} \quad (2.3.18)$$

$$\xrightarrow{R_2 = R_2 - bR_1} \begin{pmatrix} \frac{c}{d} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (2.3.19)$$

Hence from (2.3.19), $\mathbf{AX} = \mathbf{0}$ if and only if

$$x_2 = -\frac{d}{c}x_1 \quad [a \neq 0] \quad (2.3.20)$$

Letting $x_2^0 = -\frac{d}{c}$ and $x_1^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \quad (2.3.21)$$

$$x_2 = yx_2^0 \quad (2.3.22)$$

which is a solution of the equation $\mathbf{AX} = \mathbf{0}$.

Hence Proved.