

# Matrix Theory (EE5609) Challenging Problem 1

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**Abstract**—This document explains the concept of finding the closest points on two skew lines in 3-Dimensions.

The code for the solution of this problem can be found at

[https://github.com/Arko98/EE5609/blob/master/Challenge\\_1/Codes/Figure.py](https://github.com/Arko98/EE5609/blob/master/Challenge_1/Codes/Figure.py)

## 1 PROBLEM

Find the points on two skew lines that are closest to each other in 3-Dimensions.

## 2 EXPLANATION

Let, skew line, **L1** is passing through the point  $A(a_1, b_1, c_1)$  with direction vector  $(D_1(l_1, m_1, n_1))$  and skew line, **L2** is passing through the point  $B(a_2, b_2, c_2)$  with direction vector  $(D_2(l_2, m_2, n_2))$ . The equations of skew lines are given by,

$$\mathbf{L1} : \mathbf{r}_1 = A + k_1(\mathbf{D}_1) \quad (2.0.1)$$

$$\mathbf{L2} : \mathbf{r}_2 = B + k_2(\mathbf{D}_2) \quad (2.0.2)$$

Where  $r_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$  and  $r_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$  be two arbitrary points on skew lines **L1** and **L2**, respectively.

Let, the closest points on skew lines **L1** and **L2** be **P** and **Q**, respectively. Hence **P** and **Q** can be expressed in terms of equation (2.0.1) and (2.0.2),

$$\mathbf{P} = \begin{pmatrix} a_1 + k_1 l_1 \\ b_1 + k_1 m_1 \\ c_1 + k_1 n_1 \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{Q} = \begin{pmatrix} a_2 + k_2 l_2 \\ b_2 + k_2 m_2 \\ c_2 + k_2 n_2 \end{pmatrix} \quad (2.0.4)$$

So, the position vector from **P** to **Q** i.e  $\mathbf{Q} - \mathbf{P}$  is given by,

$$(\mathbf{Q} - \mathbf{P}) = \begin{pmatrix} a_2 + k_2 l_2 - (a_1 + k_1 l_1) \\ b_2 + k_2 m_2 - (b_1 + k_1 m_1) \\ c_2 + k_2 n_2 - (c_1 + k_1 n_1) \end{pmatrix} \quad (2.0.5)$$

Since the points **P** and **Q** are closest points, position vector **PQ** will be perpendicular to both the skew lines **L1** and **L2** or will be perpendicular to both the direction vectors **D1** and **D2**.

Therefore,

$$(\mathbf{Q} - \mathbf{P}) \cdot \mathbf{D}_1 = 0 \quad (2.0.6)$$

$$\Rightarrow (\mathbf{Q} - \mathbf{P})^T \mathbf{D}_1 = 0 \quad (2.0.7)$$

And,

$$(\mathbf{Q} - \mathbf{P}) \cdot \mathbf{D}_2 = 0 \quad (2.0.8)$$

$$\Rightarrow (\mathbf{Q} - \mathbf{P})^T \mathbf{D}_2 = 0 \quad (2.0.9)$$

By solving equations 2.0.7 and 2.0.9 we will get  $k_1$  and  $k_2$ . Substituting the obtained values of  $k_1$  and  $k_2$  in equation 2.0.3 and 2.0.4 gives the closest points **P** and **Q**.

## 3 SOLUTION

Let us illustrate the above approach using an example. Let, the equations of skew lines are given by,

$$\mathbf{L1} : \mathbf{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{L2} : \mathbf{r}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad (3.0.2)$$

So, if **P** and **Q** are the points on the skew lines **L1** and **L2**, then from equation 2.0.5 the position vector  $\mathbf{Q} - \mathbf{P}$  is given by,

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} 4 + 2k_2 - (1 + k_1) \\ 5 + 3k_2 - (2 - 3k_1) \\ 6 + k_2 - (3 + 2k_1) \end{pmatrix} \quad (3.0.3)$$

$$\Rightarrow \mathbf{Q} - \mathbf{P} = \begin{pmatrix} -k_1 + 2k_2 + 3 \\ 3k_1 + 3k_2 + 3 \\ -2k_1 + k_2 + 3 \end{pmatrix} \quad (3.0.4)$$

Now from equation 2.0.7 and 2.0.9, we get the two equations for  $k_1$  and  $k_2$  i.e

$$\begin{pmatrix} 14 & 5 \\ 5 & 14 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \end{pmatrix} \quad (3.0.5)$$

Solving the system of equations 3.0.5, we get

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \frac{10}{19} \\ \frac{28}{19} \end{pmatrix} \quad (3.0.6)$$

Thus putting the value of  $k_1$  in equation 2.0.3 and putting the value of  $k_2$  in equation 2.0.4 we get **P** and **Q** points as follows,

$$\mathbf{P} = \begin{pmatrix} \frac{29}{19} \\ \frac{8}{19} \\ \frac{19}{77} \\ \frac{19}{19} \end{pmatrix} = \begin{pmatrix} 1.52 \\ 0.42 \\ 4.05 \end{pmatrix} \quad (3.0.7)$$

$$\mathbf{Q} = \begin{pmatrix} \frac{20}{19} \\ \frac{11}{19} \\ \frac{19}{86} \\ \frac{19}{19} \end{pmatrix} = \begin{pmatrix} 1.05 \\ 0.57 \\ 4.52 \end{pmatrix} \quad (3.0.8)$$

Below is the figure corresponding to the solution.

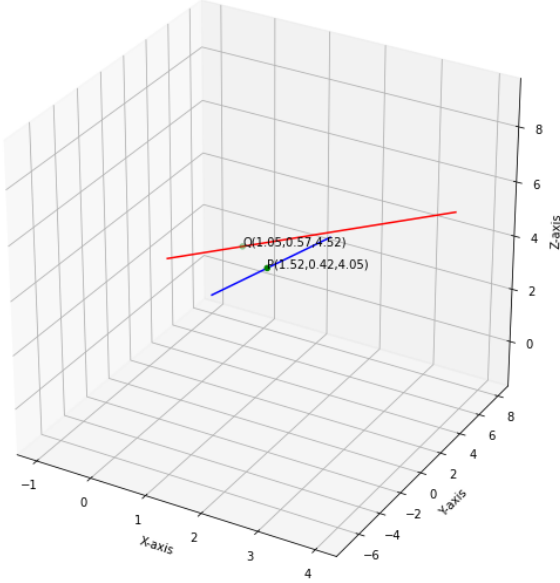


Fig. 1: Closest Points on Skew Lines