Matrix Theory (EE5609) Challenging Problem

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Abstract—**This document proves that $\mathbf{A^T A}$ has positive eigen values.**

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1 PROBLEM

Show that the eigen values of $\mathbf{A^T A}$ are positive.

2 PROOF

Let, $\mathbf{A}$ is an arbitrary $m \times n$ matrix. Now consider the symmetric matrix $\mathbf{A^T A}$, Let λ be a (real) eigen-value of $\mathbf{B} = \mathbf{A^T A}$ and let $\mathbf{x}$ be a corresponding real eigen-vector hence,

$$\mathbf{Bx} = \lambda \mathbf{x} \tag{2.0.1}$$

Multiplying $\mathbf{x^T}$ in (2.0.1),

$$\mathbf{x^T Bx} = \lambda \mathbf{x^T x} \tag{2.0.2}$$
$$\implies \mathbf{x^T Bx} = \lambda \|\mathbf{x}\|^2 \tag{2.0.3}$$
$$\implies \mathbf{x^T (A^T A)x} = \lambda \|\mathbf{x}\|^2 \, [\because \mathbf{B} = \mathbf{A^T A}] \tag{2.0.4}$$
$$\implies \mathbf{(x^T A^T)Ax} = \lambda \|\mathbf{x}\|^2 \tag{2.0.5}$$
$$\implies \mathbf{(Ax)^T (Ax)} = \lambda \|\mathbf{x}\|^2 \tag{2.0.6}$$
$$\implies \|\mathbf{Ax}\|^2 = \lambda \|\mathbf{x}\|^2 \tag{2.0.7}$$

In (2.0.7) as $\|\mathbf{x}\|^2 > 0$ then,

$$\lambda > 0 \tag{2.0.8}$$

Hence proved that all eigen values of $\mathbf{A^T A}$ are positive.