# Matrix Theory (EE5609) Challenging Problem

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*Abstract*—**This document proves that $\mathbf{A^T A}$ has positive eigen values.**

## 1 PROBLEM

Show that the eigen values of $\mathbf{A^T A}$ are positive.

## 2 PROOF

Let, $\mathbf{A}$ is an arbitrary $m \times n$ matrix. Now consider the matrix $\mathbf{A^T A}$,

for any n dimensional vector $\mathbf{z}$,

$$\mathbf{z^T(A^T A)z = z^T A^T A z} \tag{2.0.1}$$

$$\implies \mathbf{z^T(A^T A)z = (Az)^T(Az)} \tag{2.0.2}$$

$$\implies \mathbf{z^T(A^T A)z = \|Az\|}^2 \geq 0 \tag{2.0.3}$$

From (2.0.3), if $\mathbf{z} \neq 0$, $\mathbf{A^T A}$ is positive definite, i.e

$$\mathbf{\|Az\|}^2 > 0 \tag{2.0.4}$$

Again, $\mathbf{A^T A}$ is positive semi-definite, if $\mathbf{z} = 0$,

$$\mathbf{\|Az\|}^2 = 0 \tag{2.0.5}$$

Hence, $\mathbf{A^T A}$ is positive semi-definite if the columns of $\mathbf{A}$ are linearly dependent and $\mathbf{A^T A}$ is positive definite if columns of $\mathbf{A}$ are linearly dependent.
Again,

$$\mathbf{(A^T A)^T = (A^T)(A^T)^T = A^T A} \tag{2.0.6}$$

Hence, $\mathbf{A^T A}$ is symmetric. As every eigen value of a Hermitian matrix is real and every symmetric matrix is Hermitian then $\mathbf{A^T A}$ (being a symmetric and hence Hermitian) has real eigen values.
Let $\lambda$ be a (real) eigenvalue of $\mathbf{B = A^T A}$ and let $\mathbf{x}$ be a corresponding real eigen-vector hence,

$$\mathbf{Bx} = \lambda\mathbf{x} \tag{2.0.7}$$

Multiplying $\mathbf{x^T}$ in (2.0.7),

$$\mathbf{x^T Bx} = \lambda\mathbf{x^T x} \tag{2.0.8}$$

$$\implies \mathbf{x^T Bx} = \lambda \mathbf{\|x\|}^2 \tag{2.0.9}$$

$$\implies \mathbf{x^T(A^T A)x} = \lambda \mathbf{\|x\|}^2 \,[\because \mathbf{B = A^T A}] \tag{2.0.10}$$

When $\mathbf{A^T A}$ is positive definite (i.e columns of $\mathbf{A}$ are linearly independent) then, the left hand side of (2.0.10) is positive as $\mathbf{A^T A}$ is positive-definite and $\mathbf{x}$ is a nonzero vector as it is an eigen-vector.
Also if $\mathbf{A}$ has linearly independent columns then $\mathbf{A^T A}$ will be invertible and hence a non-singular matrix, so $\|\mathbf{x}\|$ cannot be zero in (2.0.10). Since $\|\mathbf{x}\|^2$ is positive, hence all eigen-values must be positive. Hence proved.