

Matrix Theory (EE5609) Assignment 19

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Abstract—This document solves a problem on a functional.

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_19

1 PROBLEM

Let \mathbb{V} be the vector space of all 2×2 matrices over the field of real numbers, and let

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$$

Let \mathbb{W} be the subspace of \mathbb{V} consisting of all \mathbf{A} such that $\mathbf{AB} = \mathbf{0}$. Let f be a linear functional on \mathbb{V} which is in the annihilator of \mathbb{W} . Suppose that $f(\mathbf{I}) = 0$ and $f(\mathbf{C}) = 3$, where \mathbf{I} is the 2×2 identity matrix and

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Find $f(\mathbf{B})$

2 SOLUTION

Let,

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \forall \mathbf{A} \in \mathbb{W} \quad (2.0.1)$$

From $\mathbf{AB} = \mathbf{0}$ we have,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.2)$$

From (2.0.2) we get,

$$y = 2x \quad (2.0.3)$$

$$w = 2z \quad (2.0.4)$$

Hence, using (2.0.3) and (2.0.5) we conclude that \mathbb{W} consists of all the matrices of the following form,

$$\mathbf{A} = \begin{pmatrix} x & 2x \\ z & 2z \end{pmatrix} \quad \forall \mathbf{A} \in \mathbb{W} \quad (2.0.5)$$

Hence from (2.0.5) we get,

$$\mathbf{A} \in \left\{ a \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\} \quad \forall a, b \in \mathbb{R} \quad (2.0.6)$$

$$\Rightarrow \text{span}(\mathbb{W}) = \left\{ \mathbf{e}_1 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\} \quad (2.0.7)$$

Hence, \mathbf{e}_1 and \mathbf{e}_2 are basis of \mathbb{W} . Since $f \in \mathbb{W}^0$, it follows that,

$$f(\mathbf{e}_1) = 0 \quad (2.0.8)$$

$$f(\mathbf{e}_2) = 0 \quad (2.0.9)$$

Again given,

$$f(\mathbf{I}) = 0 \quad (2.0.10)$$

$$f(\mathbf{C}) = 0 \quad (2.0.11)$$

From (2.0.8), (2.0.9), (2.0.10) and (2.0.11) we conclude that $\mathbf{I}, \mathbf{C}, \mathbf{e}_1, \mathbf{e}_2$ are linearly independent. We prove the statement as follows,

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.12)$$

$$\Rightarrow \begin{pmatrix} c_1 + c_3 & 2c_3 \\ c_4 & c_1 + c_2 + 2c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.13)$$

$$\Rightarrow c_1 = c_2 = c_3 = c_4 = 0 \quad (2.0.14)$$

Since $\dim(\mathbb{V}) = 2 \times 2 = 4$, hence the set $\{\mathbf{I}, \mathbf{C}, \mathbf{e}_1, \mathbf{e}_2\}$ forms a basis for the vector-space \mathbb{V} which is the vector-space of all 2×2 matrices.

Since $\mathbf{B} \in \mathbb{V}$ then there exists c_1, c_2, c_3, c_4 such that,

$$c_1 \mathbf{I} + c_2 \mathbf{C} + c_3 \mathbf{e}_1 + c_4 \mathbf{e}_2 = \mathbf{B} \quad (2.0.15)$$

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \quad (2.0.16)$$

$$= \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \quad (2.0.17)$$

$$\begin{pmatrix} c_1 + c_3 & 2c_3 \\ c_4 & c_1 + c_2 + 2c_4 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} \quad (2.0.18)$$

$$(2.0.19)$$

From (2.0.18) we get,

$$c_1 = 3 \quad (2.0.20)$$

$$c_2 = 0 \quad (2.0.21)$$

$$c_3 = -1 \quad (2.0.22)$$

$$c_4 = -1 \quad (2.0.23)$$

From (2.0.20), (2.0.21), (2.0.22) and (2.0.23) we get,

$$f(\mathbf{B}) = f(3\mathbf{I} - \mathbf{e}_1 - \mathbf{e}_2) \quad (2.0.24)$$

$$= 3f(\mathbf{I}) - f(\mathbf{e}_1) - f(\mathbf{e}_2) \quad (2.0.25)$$

$$= 0 \quad (2.0.26)$$

The (2.0.26) is the required answer.