Matrix Theory (EE5609) Assignment 17

Arkadipta De
MTech Artificial Intelligence
AI20MTECH14002

***Abstract*—This document proves the invertibility of a certain linear operator.**

All the codes for the figure in this document can be found at

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https://github.com/Arko98/EE5609/blob/master/
    Assignment_17
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1 Problem

Let T be a linear operator on the finite-dimensional space $\mathbb{V}$. Suppose there is a linear operator U on $\mathbb{V}$ such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when $\mathbb{V}$ is not finite-dimensional.

2 Solution

2.1 Proof

Let $T : \mathbb{V} \to \mathbb{V}$ be a linear operator, where $\mathbb{V}$ is a finite dimensional vectors space and $U : \mathbb{V} \to \mathbb{V}$ is also a linear operator such that,

$$TU = I \tag{2.1.1}$$

Where, I is an identity transformation. Now we know that linear transformations are functions. Hence,

$$TU = I \quad \text{is a function} \tag{2.1.2}$$

$$\implies I : \mathbb{V} \to \mathbb{V} \tag{2.1.3}$$

Such that $T(V) = V$. Defining $TU : \mathbb{V} \to \mathbb{V}$ to be a linear operator, we have

$$T[U(V_i)] = V_i \qquad [V_i \in \mathbb{V}] \tag{2.1.4}$$

Now we show in the below Table that T is one-one and onto as follows,

Proof	Conclusion
Let $\mathbf{V_1}, \mathbf{V_2} \in \mathbb{V}$ then, If $\mathbf{V_1} \neq \mathbf{V_2}$ then, $T[U(\mathbf{V_1})] \neq T[U(\mathbf{V_2})]$	T is one-one function
T is linear operator on finite dimensional vector space	T is onto function

TABLE 1: Proof of Invertibility of transformation

Hence we get from Table 1 that, T is invertible. Hence we get the following,

$$TT^{-1} = I \tag{2.1.5}$$

Where T^{-1} is an inverse function of linear operator T. Hence,

$$TT^{-1} = I = TU \tag{2.1.6}$$

$$\implies T^{-1}(TT^{-1}) = T^{-1}(TU) \tag{2.1.7}$$

$$\implies T^{-1}(I) = IU \tag{2.1.8}$$

$$\implies T^{-1} = U \tag{2.1.9}$$

Hence from (2.1.9) it is proven that T is invertible and $T^{-1} = U$

2.2 Example

Let D be the differential operator $D : \mathbb{V} \to \mathbb{V}$ where $\mathbb{V}$ is a space of polynomial functions in one variable x over $\mathbb{R}$ as follows,

$$D(c_0 + c_1 x + \cdots + c_n x^n) = c_1 + c_2' x + \cdots + c_n' x^{n-1} \tag{2.2.1}$$

We first prove that the vector space $\mathbb{V}$ is infinite dimensional.
Suppose to the contrary that $\mathbb{V}$ is finite dimensional vector space and is given by the span of k polynomials in $\mathbb{V}$ as follows,

$$span(\mathbb{V}) = \{p_1, p_2, \ldots, p_k\} \tag{2.2.2}$$

Also let m be the maximum of the degree of these k polynomials in (2.2.2). Now let an element of the vector space $\mathbb{V}$ be,

$$cx^{m+1} \in \mathbb{V} \tag{2.2.3}$$

As maximum degree of the basis of $\mathbb{V}$ is m hence cx^{m+1} cannot be represented by any linear combination of the basis of $\mathbb{V}$. If $\mathbb{F}$ is field corresponding to $\mathbb{V}$ then we have,

$$cx^{m+1} \neq \sum_{i=1}^{k} \alpha_i p_i \quad [\alpha_i \in \mathbb{F}\ \forall i] \tag{2.2.4}$$

Hence, cx^{m+1} is not in the span of $p_1, p_2, \ldots, p_k$. Hence, $\mathbb{V}$ is infinite dimensional vector space.
Next we prove that D is not one-one operator. Let, two different elements from the vector space $\mathbb{V}$ be as follows,

$$c_1 + x^m \in \mathbb{V} \tag{2.2.5}$$
$$c_2 + x^m \in \mathbb{V} \tag{2.2.6}$$

From definition (2.2.1) of operator D we have,

$$D(c_1 + x^m) = mx^{m-1} \tag{2.2.7}$$
$$D(c_2 + x^m) = mx^{m-1} \tag{2.2.8}$$

From (2.2.7) and (2.2.8),

$$c_1 + x^m \neq c_2 + x^m \tag{2.2.9}$$

But

$$D(c_1 + x^m) = D(c_2 + x^m) \tag{2.2.10}$$

Hence from (2.2.10) we see that D is not One-One operator.
And, $U : \mathbb{V} \to \mathbb{V}$ is another linear operator such that,

$$U(c_0 + c_1x + \cdots + c_nx^n) = c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1} \tag{2.2.11}$$

Now, $DU : \mathbb{V} \to \mathbb{V}$ is a linear operator such that,

$$DU(c_0 + c_1x + \cdots + c_nx^n) \tag{2.2.12}$$
$$= D[U(c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1})] \tag{2.2.13}$$
$$= D[c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1}] \tag{2.2.14}$$
$$= c_0 + c_1\frac{2x}{2} + \cdots + c_n\frac{(n+1)x^n}{n+1} \tag{2.2.15}$$
$$= c_0 + c_1x + \cdots + c_nx^n \tag{2.2.16}$$

Hence, from (2.2.16),

$$DU = I \tag{2.2.17}$$

Again $UD : \mathbb{V} \to \mathbb{V}$ is a linear operator such that,

$$UD(c_0 + c_1x + \cdots + c_nx^n) \tag{2.2.18}$$
$$= U[D(c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1})] \tag{2.2.19}$$
$$= U[c_1 + c_2'x + \cdots + c_n'x^{n-1}] \tag{2.2.20}$$
$$= c_1x + c_2\frac{x^2}{2} + \cdots + c_n\frac{x^n}{n} \tag{2.2.21}$$

Hence, from (2.2.21),

$$UD \neq I \tag{2.2.22}$$

Hence, from (2.2.17) and (2.2.22), D is not invertible.