Matrix Theory (EE5609) Assignment 3

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***Abstract*—This assignment proves that matrix multiplication is not commutative.**

The code for this solution can be found from

https://github.com/Arko98/EE5609/blob/master/Assignment_3/Codes/Solution_3.py

1 Problem Statement

Show that

$$\begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix}$$

2 Solution

Let the two matrices be $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$. From the problem we have to prove the following

$$\mathbf{AB} \neq \mathbf{BA} \tag{2.0.1}$$

At first we compute left hand side of 2.0.1.

$$\mathbf{AB} = \begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \tag{2.0.2}$$

$$\implies \mathbf{AB} = \begin{pmatrix} 5\times 2 - 1\times 3 & 5\times 1 - 1\times 4 \\ 6\times 2 + 7\times 3 & 6\times 1 + 7\times 4 \end{pmatrix} \tag{2.0.3}$$

$$\implies \mathbf{AB} = \begin{pmatrix} 7 & 1 \\ 33 & 34 \end{pmatrix} \tag{2.0.4}$$

Next, we compute right hand side of 2.0.1.

$$\mathbf{BA} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix} \tag{2.0.5}$$

$$\implies \mathbf{BA} = \begin{pmatrix} 2\times 5 + 1\times 6 & 2\times(-1) + 1\times 7 \\ 3\times 5 + 4\times 6 & 3\times(-1) + 4\times 7 \end{pmatrix} \tag{2.0.6}$$

$$\implies \mathbf{BA} = \begin{pmatrix} 16 & 5 \\ 39 & 25 \end{pmatrix} \tag{2.0.7}$$

Clearly we can see from equation 2.0.4 and 2.0.7 that the resultant matrices are not equal. Hence proved,

$$\begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 6 & 7 \end{pmatrix}$$

3 Explanation

Matrix multiplication between two matrices $\mathbf{A}$ and $\mathbf{B}$ is the linear combination of the rows of matrix $\mathbf{B}$ using the elements of matrix $\mathbf{A}$.
If $\mathbf{R_{B1}}$ and $\mathbf{R_{B2}}$ are first row and second row of the matrix $\mathbf{B}$ and $\mathbf{R_{AB1}}$ and $\mathbf{R_{AB2}}$ are first row and second row of the matrix $\mathbf{AB}$ from equation 2.0.2 then rows of $\mathbf{AB}$ is given by

$$\mathbf{R_{AB1}} = 5\mathbf{R_{B1}} - \mathbf{R_{B2}} \tag{3.0.1}$$

$$\mathbf{R_{AB2}} = 6\mathbf{R_{B1}} + 7\mathbf{R_{B2}} \tag{3.0.2}$$

Similarly if $\mathbf{R_{A1}}$ and $\mathbf{R_{A2}}$ are first row and second row of the matrix $\mathbf{A}$ and $\mathbf{R_{BA1}}$ and $\mathbf{R_{BA2}}$ are first row and second row of the matrix $\mathbf{BA}$ from equation 2.0.2 then rows of $\mathbf{BA}$ is given by

$$\mathbf{R_{BA1}} = 2\mathbf{R_{A1}} + \mathbf{R_{A2}} \tag{3.0.3}$$

$$\mathbf{R_{BA2}} = 3\mathbf{R_{A1}} + 4\mathbf{R_{A2}} \tag{3.0.4}$$

Clearly we can see from equations 3.0.1 and 3.0.3 that $\mathbf{R_{AB1}} \neq \mathbf{R_{BA1}}$ and from equations 3.0.2 and 3.0.4 that $\mathbf{R_{AB2}} \neq \mathbf{R_{BA2}}$. Hence matrix multiplication is generally not commutative.
We can define a matrix multiplications $\mathbf{AB}$ and $\mathbf{BA}$ respectively as

$$\mathbf{AB_{ij}} = \mathbf{A_i B_j} = \sum_{k=1}^{n} a_{ki} b_{jk} \tag{3.0.5}$$

$$\mathbf{BA_{ij}} = \mathbf{B_i A_j} = \sum_{k=1}^{n} b_{ki} a_{jk} \tag{3.0.6}$$

where $\mathbf{A_i}$ is ith row of matrix $\mathbf{A}$ and $\mathbf{B_j}$ is jth row of matrix $\mathbf{B}$. Clearly, equations (3.0.5) and (3.0.6) are composition of addition and multiplication. Hence, matrix multiplication can be seen as linear composition of functions. Now for two functions f and g, the composition of the two function is not commutative i.e

$$(f \circ g)(x) = f(g(x)) \neq g \circ f(x) = g(f(x)) \quad (3.0.7)$$

Thus matrix multiplication which is actually a linear combination of functions is generally not commutative.