Matrix Theory (EE5609) Assignment 19

Arkadipta De
MTech Artificial Intelligence
AI20MTECH14002

***Abstract*—This document solves a problem on a functional.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_19

1 Problem

Let $\mathbb{V}$ be the vector space of all 2×2 matrices over the field of real numbers, and let

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$$

Let $\mathbb{W}$ be the subspace of $\mathbb{V}$ consisting of all $\mathbf{A}$ such that $\mathbf{AB} = 0$. Let f be a linear functional on $\mathbb{V}$ which is in the annihilator of $\mathbb{W}$. Suppose that $f(\mathbf{I}) = 0$ and $f(\mathbf{C}) = 3$, where $\mathbf{I}$ is the 2×2 identity matrix and

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Find $f(\mathbf{B})$

2 Solution

The general linear functional f on vector space $\mathbb{V}$ is of the form,

$$f(\mathbf{A}) = \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \tag{2.0.1}$$

Where,

$$\mathbf{A} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \quad \forall\ \mathbf{A} \in \mathbb{W} \tag{2.0.2}$$

$$a, b, c, d \in \mathbb{R} \tag{2.0.3}$$

From $\mathbf{AB} = 0$ we have,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.4}$$

From (2.0.4) we get,

$$y = 2x \tag{2.0.5}$$

$$w = 2z \tag{2.0.6}$$

Hence, using (2.0.5) and (2.0.7) we conclude that $\mathbb{W}$ consists of all the matrices of the following form,

$$\mathbf{A} = \begin{pmatrix} x & 2x \\ z & 2z \end{pmatrix} \quad \forall\ \mathbf{A} \in \mathbb{W} \tag{2.0.7}$$

Hence from (2.0.7) we get,

$$f\left(\begin{pmatrix} x & 2x \\ z & 2z \end{pmatrix}\right) = 0 \quad \forall\ x, z \in \mathbb{R} \tag{2.0.8}$$

$$\implies \begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} x \\ 2x \\ z \\ 2z \end{pmatrix} = 0 \quad [\text{From (2.0.1)}] \tag{2.0.9}$$

From (2.0.9) we get,

$$b = -\frac{1}{2}a \tag{2.0.10}$$

$$d = -\frac{1}{2}c \tag{2.0.11}$$

Hence, from (2.0.10), (2.0.11) and (2.0.1), the general form of the functional f on vector space $\mathbb{V}$ becomes,

$$f(\mathbf{A}) = \begin{pmatrix} a & -\frac{1}{2}a & c & -\frac{1}{2}c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \quad \forall\ \mathbf{A} \in \mathbb{W} \tag{2.0.12}$$

Now,

$$f(\mathbf{C}) = 3 \tag{2.0.13}$$

$$\implies \begin{pmatrix} a & -\frac{1}{2}a & c & -\frac{1}{2}c \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 3 \tag{2.0.14}$$

$$\implies c = -6 \tag{2.0.15}$$

Again,

$$f(\mathbf{I}) = 0 \tag{2.0.16}$$

$$\implies \begin{pmatrix} a & -\frac{1}{2}a & c & -\frac{1}{2}c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0 \tag{2.0.17}$$

$$\implies a - \frac{1}{2}c = 0 \tag{2.0.18}$$

$$\implies a = -3 \quad \text{[Using (2.0.15)]} \tag{2.0.19}$$

Hence, using (2.0.15) and (2.0.19) the general form of f in (2.0.12) becomes,

$$f(\mathbf{A}) = \begin{pmatrix} -3 & \frac{3}{2} & -6 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \quad \forall\ \mathbf{A} \in \mathbb{W} \tag{2.0.20}$$

Now for given $\mathbf{B}$, from (2.0.20) we get,

$$f(\mathbf{B}) = \begin{pmatrix} -3 & \frac{3}{2} & -6 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix} \tag{2.0.21}$$

$$\implies f(\mathbf{B}) = 0 \tag{2.0.22}$$

(2.0.22) is the required answer.