Matrix Theory (EE5609) Challenging Problem 1

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***Abstract*—This document explains the concept of finding the closest points on two skew lines in 3-Dimensions.**

The code for the solution of this problem can be found at

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https://github.com/Arko98/EE5609/blob/master/
    Challenge_1/Codes/Figure.py
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1 Problem

Find the points on two skew lines that are closest to each other in 3-Dimensions.

2 Explanation

Let, skew line, $\mathbf{L_1}$ is passing through the point $A(a_1, b_1, c_1)$ with direction vector $(D_1(l_1, m_1, n_1)$ and skew line, $\mathbf{L_2}$ is passing through the point $B(a_2, b_2, c_2)$ with direction vector $(D_2(l_2, m_2, n_2)$. The equations of skew lines are given by,

$$\mathbf{L_1} : \mathbf{r_1} = A + k_1(\mathbf{D_1}) \tag{2.0.1}$$

$$\mathbf{L_2} : \mathbf{r_2} = B + k_2(\mathbf{D_2}) \tag{2.0.2}$$

Where $r_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $r_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ be two arbitrary points on skew lines $\mathbf{L_1}$ and $\mathbf{L_2}$, respectively.
Let, the closest points on skew lines $\mathbf{L_1}$ and $\mathbf{L_2}$ be $\mathbf{P}$ and $\mathbf{Q}$, respectively. Hence $\mathbf{P}$ and $\mathbf{Q}$ can be expressed in terms of equation (2.0.1) and (2.0.2),

$$\mathbf{P} = \begin{pmatrix} a_1 + k_1 l_1 \\ b_1 + k_1 m_1 \\ c_1 + k_1 n_1 \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{Q} = \begin{pmatrix} a_2 + k_2 l_2 \\ b_2 + k_2 m_2 \\ c_2 + k_2 n_2 \end{pmatrix} \tag{2.0.4}$$

So, the position vector from $\mathbf{P}$ to $\mathbf{Q}$ i.e $\mathbf{Q} - \mathbf{P}$ is given by,

$$(\mathbf{Q} - \mathbf{P}) = \begin{pmatrix} a_2 + k_2 l_2 - (a_1 + k_1 l_1) \\ b_2 + k_2 m_2 - (b_1 + k_1 m_1) \\ c_2 + k_2 n_2 - (c_1 + k_1 n_1) \end{pmatrix} \tag{2.0.5}$$

Since the points $\mathbf{P}$ and $\mathbf{Q}$ are closest points, position vector $\mathbf{PQ}$ will be perpendicular to both the skew lines $\mathbf{L_1}$ and $\mathbf{L_2}$ or will be perpendicular to both the direction vectors $\mathbf{D_1}$ and $\mathbf{D_2}$.
Therefore,

$$(\mathbf{Q} - \mathbf{P}) \cdot \mathbf{D_1} = 0 \tag{2.0.6}$$

$$\implies (\mathbf{Q} - \mathbf{P})^T \mathbf{D_1} = 0 \tag{2.0.7}$$

And,

$$\mathbf{Q} - \mathbf{P}) \cdot \mathbf{D_2} = 0 \tag{2.0.8}$$

$$\implies (\mathbf{Q} - \mathbf{P})^T \mathbf{D_2} = 0 \tag{2.0.9}$$

By solving equations 2.0.7 and 2.0.9 we will get k_1 and k_2. Substituting the obtained values of k_1 and k_2 in equation 2.0.3 and 2.0.4 gives the closest points $\mathbf{P}$ and $\mathbf{Q}$.

3 solution

Let us illustrate the above approach using an example. Let, the equations of skew lines are given by,

$$\mathbf{L_1} : \mathbf{r_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + k_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{L_2} : \mathbf{r_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{3.0.2}$$

So, if $\mathbf{P}$ and $\mathbf{Q}$ are the points on the skew lines $\mathbf{L_1}$ and $\mathbf{L_2}$, then from equation 2.0.5 the position vector $\mathbf{Q} - \mathbf{P}$ is given by,

$$\mathbf{Q} - \mathbf{P} = \begin{pmatrix} 4 + 2k_2 - (1 + k_1) \\ 5 + 3k_2 - (2 - 3k_1) \\ 6 + k_2 - (3 + 2k_1) \end{pmatrix} \tag{3.0.3}$$

$$\implies \mathbf{Q} - \mathbf{P} = \begin{pmatrix} -k_1 + 2k_2 + 3 \\ 3k_1 + 3k_2 + 3 \\ -2k_1 + k_2 + 3 \end{pmatrix} \tag{3.0.4}$$

Now from equation 2.0.7 and 2.0.9, we get the two equations for k_1 and k_2 i.e

$$\begin{pmatrix} 14 & 5 \\ 5 & 14 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \end{pmatrix} \tag{3.0.5}$$

Solving the system of equations 3.0.5, we get

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \frac{10}{19} \\ \frac{28}{19} \end{pmatrix} \tag{3.0.6}$$

Thus putting the value of k_1 in equation 2.0.3 and putting the value of k_2 in equation 2.0.4 we get **P** and **Q** points as follows,

$$\mathbf{P} = \begin{pmatrix} \frac{29}{19} \\ \frac{8}{19} \\ \frac{77}{19} \end{pmatrix} = \begin{pmatrix} 1.52 \\ 0.42 \\ 4.05 \end{pmatrix} \tag{3.0.7}$$

$$\mathbf{Q} = \begin{pmatrix} \frac{20}{19} \\ \frac{11}{19} \\ \frac{86}{19} \end{pmatrix} = \begin{pmatrix} 1.05 \\ 0.57 \\ 4.52 \end{pmatrix} \tag{3.0.8}$$

Below is the figure corresponding to the solution.

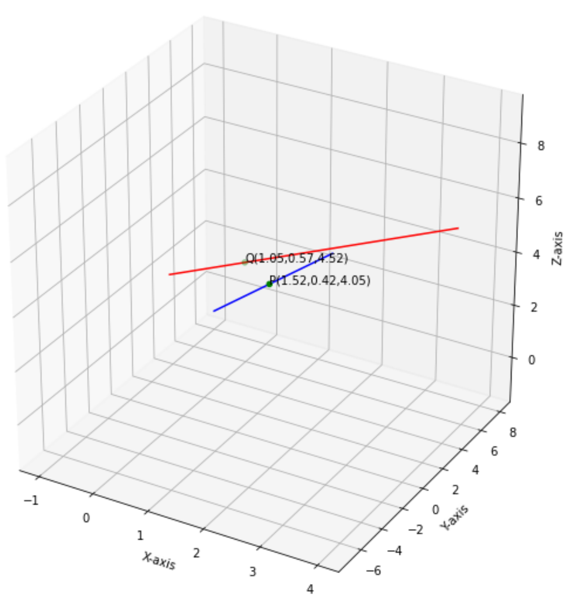

Fig. 1: Closest Points on Skew Lines