

# Matrix Theory (EE5609) Assignment 8

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**Abstract**—This document finds what conic section a given second degree equation represent.

All the codes for the figure in this document can be found at

[https://github.com/Arko98/EE5609/blob/master/Assignment\\_8](https://github.com/Arko98/EE5609/blob/master/Assignment_8)

## 1 PROBLEM

What conic does the following equation represent.

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$$

Find the center.

## 2 SOLUTION

The general second degree equation can be expressed as follows,

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 13 & -9 \\ -9 & 37 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \quad (2.0.3)$$

$$f = -2 \quad (2.0.4)$$

Expanding the determinant of  $\mathbf{V}$  we observe,

$$\begin{vmatrix} 13 & -9 \\ -9 & 37 \end{vmatrix} = 400 > 0 \quad (2.0.5)$$

Hence from (2.0.5) we conclude that given equation is an ellipse. The characteristic equation of  $\mathbf{V}$  is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 13 & 9 \\ 9 & \lambda - 37 \end{vmatrix} = 0 \quad (2.0.6)$$

$$\Rightarrow \lambda^2 - 50\lambda + 400 = 0 \quad (2.0.7)$$

Hence the characteristic equation of  $\mathbf{V}$  is given by (2.0.7). The roots of (2.0.7) i.e the eigenvalues are given by

$$\lambda_1 = 10, \lambda_2 = 40 \quad (2.0.8)$$

The eigen vector  $\mathbf{p}$  is defined as,

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.0.9)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V})\mathbf{p} = 0 \quad (2.0.10)$$

for  $\lambda_1 = 10$ ,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & 9 \\ 9 & -27 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = R_2 + 3R_1} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \quad (2.0.11)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad (2.0.12)$$

Again, for  $\lambda_2 = 40$ ,

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 27 & 9 \\ 9 & 3 \end{pmatrix} \xrightarrow[R_1 = \frac{1}{27}R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \quad (2.0.13)$$

$$\Rightarrow \mathbf{p}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad (2.0.14)$$

Again, Hence from the equation

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \quad (2.0.15)$$

Where  $\mathbf{D}$  is a diagonal matrix, we get,

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \quad (2.0.16)$$

$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \quad (2.0.17)$$

Now (2.0.1) can be written as,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |\mathbf{V}| \neq 0 \quad (2.0.18)$$

And,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (2.0.19)$$

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \quad (2.0.20)$$

The centre/vertex of the conic section in (2.0.1) is given by  $\mathbf{c}$  in (2.0.19). We compute  $\mathbf{V}^{-1}$  as follows,

$$\begin{pmatrix} 13 & -9 & 1 & 0 \\ -9 & 37 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 = \frac{13}{400}R_2]{R_2 = R_2 + \frac{9}{13}R_1} \begin{pmatrix} 13 & -9 & 1 & 0 \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.0.21)$$

$$\xrightarrow[R_1 = R_1 + \frac{9}{13}R_2]{R_1 = \frac{1}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{37}{400} & \frac{9}{400} \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.0.22)$$

Hence  $\mathbf{V}^{-1}$  is given by,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{37}{400} & \frac{9}{400} \\ \frac{9}{400} & \frac{13}{400} \end{pmatrix} \quad (2.0.23)$$

Now  $\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}$  is given by,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} = \frac{1}{400} \begin{pmatrix} 1 & 7 \end{pmatrix} \begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} = 2 \quad (2.0.24)$$

And,  $\mathbf{V}^{-1}\mathbf{u}$  is given by,

$$\mathbf{V}^{-1}\mathbf{u} = \frac{1}{400} \begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.25)$$

By putting the value of (2.0.25), the center of the ellipse is given by (2.0.19) as follows,

$$\mathbf{c} = -\frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \quad (2.0.26)$$

Also the semi-major axis ( $a$ ) and semi-minor axis ( $b$ ) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \frac{\sqrt{10}}{5} \quad (2.0.27)$$

$$b = \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \frac{\sqrt{10}}{10} \quad (2.0.28)$$

Finally from (2.0.18), the equation of ellipse is given by,

$$\mathbf{y}^T \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \mathbf{y} = 4 \quad (2.0.29)$$

The following figure is the graphical representation of the ellipse in (2.0.29),

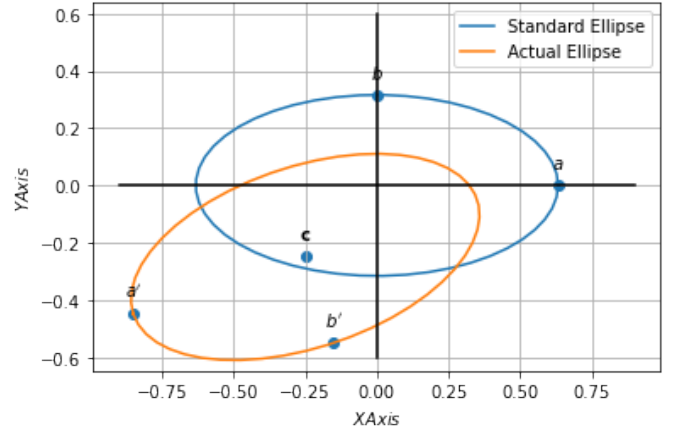


Fig. 1: Graphical representation of the ellipse