Matrix Theory (EE5609) Assignment 25

Arkadipta De
MTech Artificial Intelligence
AI20MTECH14002

***Abstract*—This document solves a problem on Jordan form of a matrix.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_25

1 Problem

The differentiation operator on the space of polynomials of degree less than or equal to 3 is represented in the natural ordered basis by the matrix,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

What is the Jordan form of this matrix? ($\mathbb{F}$ a subfield of the complex numbers.)

2 Solution

First, we find the characteristic polynomial of $\mathbf{A}$,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{2.0.1}$$

$$\implies \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 2 & 0 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0 \tag{2.0.2}$$

$$\implies \lambda^4 = 0 \tag{2.0.3}$$

(2.0.3) is the required characteristic equation and $\lambda_1 = 0$ is the only eigen value of $\mathbf{A}$. Hence the characteristic polynomial of $\mathbf{A}$ is,

$$f(\lambda) = \lambda^4 \tag{2.0.4}$$

Again we observe that for $k = 4$ we have,

$$\mathbf{A}^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.5}$$

$$\implies f(\mathbf{A}) = 0 \tag{2.0.6}$$

And for $k = 3$ we also have,

$$\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.7}$$

$$\implies \mathbf{A}^3 \neq 0 \tag{2.0.8}$$

From (2.0.6) and (2.0.8) we conclude that the minimal polynomial of $\mathbf{A}$ is,

$$g(\lambda) = \lambda^4 \tag{2.0.9}$$

Hence, the Jordan form of $\mathbf{A}$ is a 4×4 matrix consisting of only one block with principal diagonal values as $\lambda_1 = 0$ and super diagonal of the matrix (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. Hence the required Jordan form of $\mathbf{A}$ is,

$$\mathbf{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.10}$$

(2.0.10) is the required Jordan form of $\mathbf{A}$.