

Matrix Theory (EE5609) Assignment 7

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Abstract—This finds whether a given second degree equation represents a pair of straight lines or not.

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_7

1 PROBLEM

Find the value of k so that the following equation may represent a pair of straight lines -

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0$$

2 THEORY

The general equation of second degree is given by,

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

(2.0.1) can be written as,

$$(2.0.2)$$

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3)$$

where,

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} d \\ e \end{pmatrix} \quad (2.0.5)$$

(2.0.3) represents a pair of straight lines if,

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \quad (2.0.6)$$

Otherwise, (2.0.3) represents a conic section.

3 SOLUTION

The given second degree equation is,

$$6x^2 + xy + ky^2 - 11x + 43y - 35 = 0 \quad (3.0.1)$$

Comparing coefficients of (3.0.1) with (2.0.1) we get,

$$\mathbf{V} = \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & k \end{pmatrix} \quad (3.0.2)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (3.0.3)$$

$$f = -35 \quad (3.0.4)$$

From (2.0.6) the given second degree equation (3.0.1) will represent a pair of straight line if,

$$\begin{vmatrix} 6 & \frac{1}{2} & -\frac{11}{2} \\ \frac{1}{2} & k & \frac{43}{2} \\ -\frac{11}{2} & \frac{43}{2} & -35 \end{vmatrix} = 0 \quad (3.0.5)$$

Expanding the determinant,

$$k + 12 = 0 \quad (3.0.6)$$

$$\implies k = -12 \quad (3.0.7)$$

Hence, from (3.0.7) we find that for $k = -12$, the given second degree equation (3.0.1) represents pair of straight lines. For the appropriate value of k , (3.0.1) becomes,

$$6x^2 + xy - 12y^2 - 11x + 43y - 35 = 0 \quad (3.0.8)$$

4 GRAPHICAL ILLUSTRATION

Let the pair of straight lines in vector form is given by

$$\mathbf{n}_1^T \mathbf{x} = c_1 \quad (4.0.1)$$

$$\mathbf{n}_2^T \mathbf{x} = c_2 \quad (4.0.2)$$

The pair of straight lines is given by,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (4.0.3)$$

Putting the values of \mathbf{V} and \mathbf{u} we get,

$$\mathbf{x}^T \begin{pmatrix} 6 & \frac{1}{2} \\ \frac{1}{2} & -12 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -\frac{11}{2} & \frac{43}{2} \end{pmatrix} \mathbf{x} - 35 = 0 \quad (4.0.4)$$

Hence, from (4.0.4) we get,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} 6 \\ 1 \\ -12 \end{pmatrix} \quad (4.0.5)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (4.0.6)$$

$$c_1 c_2 = -35 \quad (4.0.7)$$

The slopes of the pair of straight lines are given by the roots of the polynomial,

$$cm^2 + 2bm + a = 0 \quad (4.0.8)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\det(V)}}{c} \quad (4.0.9)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (4.0.10)$$

Substituting the values in above equations (4.0.8) we get,

$$-12m^2 + m + 6 = 0 \quad (4.0.11)$$

$$\Rightarrow m_i = \frac{-\frac{1}{2} \pm \sqrt{-(-\frac{289}{4})}}{-12} \quad (4.0.12)$$

Solving equation (4.0.12) we get ,

$$m_1 = -\frac{2}{3} \quad (4.0.13)$$

$$m_2 = \frac{3}{4} \quad (4.0.14)$$

Hence putting the values of m_1 and m_2 in (4.0.10) we get

$$\mathbf{n}_1 = k_1 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad (4.0.15)$$

$$\mathbf{n}_2 = k_2 \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (4.0.16)$$

Substituting equations (4.0.15), (4.0.16) in equation (4.0.5) we get

$$k_1 k_2 = -12 \quad (4.0.17)$$

Possible combinations of (k_1, k_2) are (6,-2), (-6,2), (3,-4), (-3,4) Lets assume $k_1 = 3$, $k_2 = -4$, then we get,

$$\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (4.0.18)$$

$$\mathbf{n}_2 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad (4.0.19)$$

From equation (4.0.6) we get

$$\begin{pmatrix} \mathbf{n}_1 & \mathbf{n}_2 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (4.0.20)$$

$$\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2 \begin{pmatrix} -\frac{11}{2} \\ \frac{43}{2} \end{pmatrix} \quad (4.0.21)$$

Hence we get the following equations,

$$2c_2 + 3c_1 = 11 \quad (4.0.22)$$

$$3c_2 - 4c_1 = -43 \quad (4.0.23)$$

Solving equations (4.0.22) ,(4.0.23) we get

$$c_1 = -5 \quad (4.0.24)$$

$$c_2 = 7 \quad (4.0.25)$$

Hence (4.0.1), (4.0.2) can be modified as follows,

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \mathbf{x} = -5 \quad (4.0.26)$$

$$\begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 7 \quad (4.0.27)$$

The figure below corresponds to the pair of straight lines represented by (4.0.26) and (4.0.27).

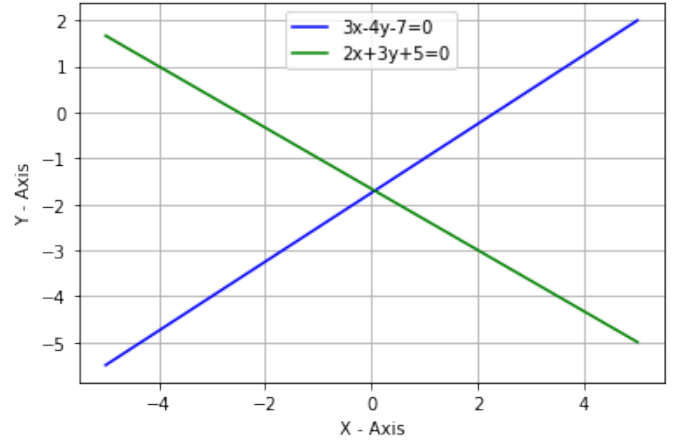


Fig. 1: Pair of Straight Lines