Matrix Theory (EE5609) Challenging Problem

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Abstract—**This document proves some properties of matrices.**

Download latex codes from

https://github.com/Arko98/EE5609/tree/master/Challenge_7

1 Problem

Given two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ prove the following using determinant properties,

- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$
- $\det(\mathbf{A^T}) = \det(\mathbf{A})$
- If $\mathbf{A}$ is a matrix with orthonormal columns then $|\det(\mathbf{A})| = 1$

2 Proof

2.1 Proof 1

Case 1: If $\mathbf{A}$ is not invertible, then $\mathbf{AB}$ is not invertible. Hence,

$$\det(\mathbf{AB}) = 0 \quad [\because \mathbf{AB} \text{ is not invertible}] \tag{2.1.1}$$
$$\det(\mathbf{A}) = 0 \quad [\because \mathbf{A} \text{ is not invertible}] \tag{2.1.2}$$

Hence,

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) = 0 \tag{2.1.3}$$

Case 2: If $\mathbf{A}$ is invertible then, there exists a series of elementary row operations $\mathbf{E_k}, \mathbf{E_{k-1}}, \dots \mathbf{E_1}$ such that,

$$\mathbf{A} = \mathbf{E_k}\mathbf{E_{k-1}}\dots\mathbf{E_1} \tag{2.1.4}$$

Now, the determinant of an elementary matrix $\mathbf{E}$ is given as follows -

If $\mathbf{E}$ interchanges two rows

$$\det(\mathbf{E}) = -1 \tag{2.1.5}$$

If $\mathbf{E}$ multiplies a row with nonzero constant c

$$\det(\mathbf{E}) = c \tag{2.1.6}$$

If $\mathbf{E}$ multiplies Row i by nonzero constant c and adds to Row j

$$\det(\mathbf{E}) = 1 \tag{2.1.7}$$

Now, if $\mathbf{A}$ interchanges two rows and $\mathbf{E}$ is the corresponding elementary matrix then,

$$\det(\mathbf{EA}) = -\det(\mathbf{A}) \tag{2.1.8}$$
$$\implies \det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A}) \quad [\text{From (2.1.5)}] \tag{2.1.9}$$

If we multiply the ith row of $\mathbf{A}$ by nonzero constant c and $\mathbf{E}$ is the corresponding elementary matrix then,

$$\det(\mathbf{EA}) = c\det(\mathbf{A}) \quad [\text{By property}] \tag{2.1.10}$$
$$\implies \det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A}) \quad [\text{From (2.1.6)}] \tag{2.1.11}$$

Lastly, if we multiply the ith row of $\mathbf{A}$ by nonzero constant c and add it to jth row of $\mathbf{A}$ and $\mathbf{E}$ is the corresponding elementary matrix then,

$$\det(\mathbf{EA}) = \det(\mathbf{A}) \quad [\text{By property}] \tag{2.1.12}$$
$$\implies \det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A}) \quad [\text{From (2.1.7)}] \tag{2.1.13}$$

Hence, from (2.1.9), (2.1.11) and (2.1.13) we get, if elementary matrix $\mathbf{E}$ represents an elementary row operation and A is a $n \times n$ matrix then,

$$\det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A}) \tag{2.1.14}$$

Hence from (2.1.4) and (2.1.14) we can write,

$$\det(\mathbf{AB}) = \det(\mathbf{E_k}\dots\mathbf{E_1}\mathbf{B}) \tag{2.1.15}$$
$$= \det(\mathbf{E_k})\det(\mathbf{E_{k-1}}\dots\mathbf{E_1}\mathbf{B}) \tag{2.1.16}$$
$$= \det(\mathbf{E_k})\det(\mathbf{E_{k-1}})\dots\det(\mathbf{E_1})\det(\mathbf{B}) \tag{2.1.17}$$
$$= \det(\mathbf{E_k}\mathbf{E_{k-1}}\dots\mathbf{E_1})\det(\mathbf{B}) \tag{2.1.18}$$
$$= \det(\mathbf{A})\det(\mathbf{B}) \tag{2.1.19}$$

Hence from (2.1.3) and (2.1.19) proved.

2.2 Proof 2

Case 1: If $rank(\mathbf{A}) < n$ then A is not invertible and $\det(\mathbf{A}) = 0$. Since the row rank and the column rank are equal, it follows that $\det(\mathbf{A^T}) = 0$, hence for Case 1,

$$\det(\mathbf{A}) = \det(\mathbf{A^T}) \tag{2.2.1}$$

Case 2: $\mathbf{A}$ is invertible. By Gauss elimination $\mathbf{A}$ can be reduced to the identity matrix, $\mathbf{I}$ by elementary row operations. Thus $\mathbf{A}$ is a product of elementary matrices. Hence,

$$\mathbf{A} = \mathbf{E_1 E_2 \ldots E_r} \tag{2.2.2}$$

Now, each elementary matrix is either symmetric or lower-triangular or upper-triangular and by properties of matrices, determinant of a lower-triangular or upper-triangular matrix is product of it's diagonal elements, hence for every elementary matrix $\mathbf{E_i}$ we have,

$$\det(\mathbf{E_i}) = \det(\mathbf{E_i^T}) \tag{2.2.3}$$

Hence from (2.2.2) we can write,

$$\begin{aligned}
\det(\mathbf{A}) &= \prod_{i=1}^{r} \det(\mathbf{E_i}) && (2.2.4)\\
&= \prod_{i=1}^{r} \det(\mathbf{E_i^T}) && (2.2.5)\\
&= \prod_{i=1}^{r} \det(\mathbf{E_{r-i}^T}) && (2.2.6)\\
&= \det(\mathbf{A}) && (2.2.7)
\end{aligned}$$

Hence, from (2.2.1) and (2.2.7), Proved.

2.3 Proof 3

If $\mathbf{A}$ has orthonormal columns then,

$$\mathbf{A^T A} = \mathbf{I} \tag{2.3.1}$$

Again from properties of determinant we have,

$$\det(\mathbf{I}) = 1 \tag{2.3.2}$$

Hence, we can write from (2.3.1) and (2.3.2),

$$\begin{aligned}
&\det(\mathbf{I}) = 1 && (2.3.3)\\
\implies &\det(\mathbf{A^T A}) = 1 && (2.3.4)\\
\implies &\det(\mathbf{A^T})\det(\mathbf{A}) = 1 \quad \text{[By Proof 1]} && (2.3.5)\\
\implies &\det(\mathbf{A})\det(\mathbf{A}) = 1 \quad \text{[By Proof 2]} && (2.3.6)\\
\implies &|\det(\mathbf{A})| = 1 && (2.3.7)
\end{aligned}$$

Hence from (2.3.7) Proved.