Matrix Theory (EE5609) Challenging Problem

Arkadipta De
MTech Artificial Intelligence
AI20MTECH14002

***Abstract*—This document proves that $\mathbf{A^TA}$ has positive eigen values.**

Download latex codes from

https://github.com/Arko98/EE5609/tree/master/Challenge_5

1 Problem

Show that the eigen values of $\mathbf{A^TA}$ are positive.

2 Proof

Let, $\mathbf{A}$ is an arbitrary $m \times n$ matrix. Now consider the symmetric matrix $\mathbf{A^TA}$, Let λ be a (real) eigen-value of $\mathbf{B} = \mathbf{A^TA}$ and let $\mathbf{x}$ be a corresponding real eigen-vector hence,

$$\mathbf{Bx} = \lambda\mathbf{x} \tag{2.0.1}$$

Multiplying $\mathbf{x^T}$ in (2.0.1),

$$\mathbf{x^TBx} = \lambda\mathbf{x^Tx} \tag{2.0.2}$$
$$\implies \mathbf{x^TBx} = \lambda\left\|\mathbf{x}\right\|^2 \tag{2.0.3}$$
$$\implies \mathbf{x^T(A^TA)x} = \lambda\left\|\mathbf{x}\right\|^2 \left[\because \mathbf{B} = \mathbf{A^TA}\right] \tag{2.0.4}$$
$$\implies \mathbf{(x^TA^T)Ax} = \lambda\left\|\mathbf{x}\right\|^2 \tag{2.0.5}$$
$$\implies \mathbf{(Ax)^T(Ax)} = \lambda\left\|\mathbf{x}\right\|^2 \tag{2.0.6}$$
$$\implies \left\|\mathbf{Ax}\right\|^2 = \lambda\left\|\mathbf{x}\right\|^2 \tag{2.0.7}$$

In (2.0.7) as $\left\|\mathbf{Ax}\right\|^2 > 0$ then,

$$\lambda > 0 \tag{2.0.8}$$

Hence proved that all eigen values of $\mathbf{A^TA}$ are positive.