Matrix Theory (EE5609) Assignment 21

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Abstract—**This document solves a problem on a functional and linear transformation.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_21

1 Problem

Let $\mathbb{F}$ be a subfield of the complex numbers and let $\mathbf{A}$ be the following 2×2 matrix over $\mathbb{F}$,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

For the following polynomial f over $\mathbb{F}$,

$$f = x^3 - 1$$

compute $f(\mathbf{A})$

2 Solution

2.1 Method 1 (Using Diagonalization of Matrix)

We first find the eigen values of the $\mathbf{A}$. We get the characteristic equation of $\mathbf{A}$ as follows,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{2.1.1}$$

$$\implies \lambda^2 - 5\lambda + 7 = 0 \tag{2.1.2}$$

From (2.1.2) we get the eigen values of $\mathbf{A}$ as follows,

$$\lambda_1 = \frac{1}{2}(5 + i\sqrt{3}) \tag{2.1.3}$$

$$\lambda_2 = \frac{1}{2}(5 - i\sqrt{3}) \tag{2.1.4}$$

And corresponding eigen vectors are as follows,

$$\mathbf{e_1} = \begin{pmatrix} \frac{1}{2}(1 - i\sqrt{3}) & 1 \end{pmatrix} \tag{2.1.5}$$

$$\mathbf{e_2} = \begin{pmatrix} \frac{1}{2}(1 + i\sqrt{3}) & 1 \end{pmatrix} \tag{2.1.6}$$

From the eigen values in (2.1.3),(2.1.4) and eigen vectors (2.1.5) and (2.1.6) we get the eigenvalue diagonalization of $\mathbf{A}$ as follows,

$$\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1} \tag{2.1.7}$$

Where,

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2}(1 - i\sqrt{3}) & \frac{1}{2}(1 + i\sqrt{3}) \\ 1 & 1 \end{pmatrix} \tag{2.1.8}$$

$$\mathbf{\Lambda} = \begin{pmatrix} \frac{1}{2}(5 + i\sqrt{3}) & 0 \\ 0 & \frac{1}{2}(5 - i\sqrt{3}) \end{pmatrix} \tag{2.1.9}$$

$$\mathbf{P}^{-1} = \begin{pmatrix} -\frac{i}{\sqrt{3}} & \frac{1}{6}(3 + i\sqrt{3}) \\ \frac{i}{\sqrt{3}} & \frac{1}{6}(3 - i\sqrt{3}) \end{pmatrix} \tag{2.1.10}$$

Hence,

$$\mathbf{A}^3 = \mathbf{P}\mathbf{\Lambda}^3\mathbf{P}^{-1} \tag{2.1.11}$$

$$\implies \mathbf{A}^3 = \begin{pmatrix} 0 & 18 \\ -18 & 18 \end{pmatrix} \tag{2.1.12}$$

Using (2.1.12) in $f(\mathbf{A})$ we get,

$$f(\mathbf{A}) = \mathbf{A}^3 - \mathbf{I} \quad \text{[Where } \mathbf{I} \text{ is } 2 \times 2 \text{ Identity matrix]} \tag{2.1.13}$$

$$= \begin{pmatrix} 1 & 18 \\ -18 & 19 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1.14}$$

$$= \begin{pmatrix} 0 & 18 \\ -18 & 18 \end{pmatrix} \tag{2.1.15}$$

Here, (2.1.15) is the required answer.

2.2 Method 2 (Direct Method)

From the equation of polynomial we get,

$$f(\mathbf{A}) = \mathbf{A}^3 - \mathbf{I} \quad \text{[Where } \mathbf{I} \text{ is } 2 \times 2 \text{ Identity matrix]} \tag{2.2.1}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}^3 - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2.2}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2.3}$$

$$= \begin{pmatrix} 1 & 18 \\ -18 & 19 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2.4}$$

$$= \begin{pmatrix} 0 & 18 \\ -18 & 18 \end{pmatrix} \tag{2.2.5}$$

Here, (2.2.5) is the required answer.