Matrix Theory (EE5609) Challenging Problem

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***Abstract*—This document proves that A^TA has positive eigen values.**

Download latex codes from

https://github.com/Arko98/EE5609/tree/master/Challenge_5

1 Problem

Show that the eigen values of $\mathbf{A}^T\mathbf{A}$ are positive.

2 Proof

Let, $\mathbf{A}$ is an arbitrary $m \times n$ matrix. Now consider the matrix $\mathbf{A}^T\mathbf{A}$,

for any n dimensional vector $\mathbf{z}$,

$$\mathbf{z}^T(\mathbf{A}^T\mathbf{A})\mathbf{z} = \mathbf{z}^T\mathbf{A}^T\mathbf{A}\mathbf{z} \tag{2.0.1}$$
$$\implies \mathbf{z}^T(\mathbf{A}^T\mathbf{A})\mathbf{z} = (\mathbf{A}\mathbf{z})^T(\mathbf{A}\mathbf{z}) \tag{2.0.2}$$
$$\implies \mathbf{z}^T(\mathbf{A}^T\mathbf{A})\mathbf{z} = \|\mathbf{A}\mathbf{z}\|^2 \geq 0 \tag{2.0.3}$$

From (2.0.3), if $\mathbf{z} \neq 0$, $\mathbf{A}^T\mathbf{A}$ is positive definite, i.e

$$\|\mathbf{A}\mathbf{z}\|^2 > 0 \tag{2.0.4}$$

Again, $\mathbf{A}^T\mathbf{A}$ is positive semi-definite, if $\mathbf{z} = 0$,

$$\|\mathbf{A}\mathbf{z}\|^2 = 0 \tag{2.0.5}$$

Hence, $\mathbf{A}^T\mathbf{A}$ is positive semi-definite if the columns of $\mathbf{A}$ are linearly dependent and $\mathbf{A}^T\mathbf{A}$ is positive definite if columns of $\mathbf{A}$ are linearly dependent.
Again,

$$(\mathbf{A}^T\mathbf{A})^T = (\mathbf{A}^T)(\mathbf{A}^T)^T = \mathbf{A}^T\mathbf{A} \tag{2.0.6}$$

Hence, $\mathbf{A}^T\mathbf{A}$ is symmetric. As every eigen value of a Hermitian matrix is real and every symmetric matrix is Hermitian then $\mathbf{A}^T\mathbf{A}$ (being a symmetric and hence Hermitian) has real eigen values.
Let λ be a (real) eigenvalue of $\mathbf{B} = \mathbf{A}^T\mathbf{A}$ and let $\mathbf{x}$ be a corresponding real eigen-vector hence,

$$\mathbf{B}\mathbf{x} = \lambda\mathbf{x} \tag{2.0.7}$$

Multiplying $\mathbf{x}^T$ in (2.0.7),

$$\mathbf{x}^T\mathbf{B}\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x} \tag{2.0.8}$$
$$\implies \mathbf{x}^T\mathbf{B}\mathbf{x} = \lambda\,\|\mathbf{x}\|^2 \tag{2.0.9}$$
$$\implies \mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = \lambda\,\|\mathbf{x}\|^2\ [\because \mathbf{B} = \mathbf{A}^T\mathbf{A}] \tag{2.0.10}$$

When $\mathbf{A}^T\mathbf{A}$ is positive definite (i.e columns of A are linearly independent) then, the left hand side of (2.0.10) is positive as $\mathbf{A}^T\mathbf{A}$ is positive-definite and x is a nonzero vector as it is an eigen-vector.
Also $\|\mathbf{x}\|$ cannot be zero if $\mathbf{A}^T\mathbf{A}$ has linearly independent columns because then it will be invertible and hence a non-singular matrix. Since $\|\mathbf{x}\|^2$ is positive, hence all eigen-values must be positive.
Hence proved.