

Matrix Theory (EE5609) Assignment 22

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Abstract—This document solves problem on polynomial of a given square matrix.

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_22

1 PROBLEM

Let n be a positive integer and \mathbb{F} be a field. Suppose \mathbf{A} is an $n \times n$ matrix over field \mathbb{F} and \mathbf{P} is an invertible $n \times n$ matrix over field \mathbb{F} . If f is any polynomial over \mathbb{F} , prove that,

$$f(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = \mathbf{P}^{-1}f(\mathbf{A})\mathbf{P}$$

2 SOLUTION

First we observe the following,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^2 = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \quad (2.0.1)$$

$$= \mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} \quad (2.0.2)$$

Let the (2.0.2) be true for a positive integer m i.e.,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^m = \mathbf{P}^{-1}\mathbf{A}^m\mathbf{P} \quad (2.0.3)$$

Now for the integer $m + 1$ we get from (2.0.3),

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^{m+1} = (\mathbf{P}^{-1}\mathbf{A}^m\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \quad (2.0.4)$$

$$= \mathbf{P}^{-1}\mathbf{A}^{m+1}\mathbf{P} \quad (2.0.5)$$

From (2.0.3) and (2.0.5) we get for any positive integer n ,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} \quad (2.0.6)$$

Again we have,

$$\mathbf{P}^{-1}\mathbf{P} = \mathbf{I} \quad (2.0.7)$$

The general form of polynomial $f(\mathbf{A})$ is defined as,

$$f(\mathbf{A}) = a_0 + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n \quad (2.0.8)$$

Now we have,

$$f(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = a_0 + a_1(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) + \cdots + a_n(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n \quad (2.0.9)$$

$$= (\mathbf{P}^{-1}a_0\mathbf{P}) + (\mathbf{P}^{-1}a_1\mathbf{A}\mathbf{P}) + \cdots + (\mathbf{P}^{-1}a_n\mathbf{A}^n\mathbf{P})^n \quad (2.0.10)$$

$$= \mathbf{P}^{-1}(a_0 + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n)\mathbf{P} \quad (2.0.11)$$

$$= \mathbf{P}^{-1}f(\mathbf{A})\mathbf{P} \quad [\text{From (2.0.8)}] \quad (2.0.12)$$

Hence proved (2.0.12)