

# Matrix Theory (EE5609) Assignment 12

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**Abstract**—This document proves that, each field of the characteristic zero contains a copy of the rational number field.

All the codes for the figure in this document can be found at

[https://github.com/Arko98/EE5609/blob/master/Assignment\\_12](https://github.com/Arko98/EE5609/blob/master/Assignment_12)

## 1 PROBLEM

Consider the system of equations  $\mathbf{AX} = 0$  where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a  $2 \times 2$  matrix over the field  $F$ . Prove the following

- If every entry of  $\mathbf{A}$  is 0, then every pair  $x_1$  and  $x_2$  is a solution of  $\mathbf{AX} = 0$ .
- If  $ad - bc \neq 0$ , then the system  $\mathbf{AX} = 0$  has only the trivial solution  $x_1 = x_2 = 0$
- If  $ad - bc = 0$  and some entry of  $\mathbf{A}$  is different from 0, then there is a solution  $x_1^0$  and  $x_2^0$  such that  $x_1$  and  $x_2$  is a solution if and only if there is a scalar  $y$  such that  $x_1 = yx_1^0$  and  $x_2 = yx_2^0$

## 2 SOLUTION

### 2.1 Solution 1

If every entry of  $\mathbf{A}$  is 0 then the equation  $\mathbf{AX} = 0$  becomes,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.1.1)$$

$$\Rightarrow 0.x_1 + 0.x_2 = 0 \quad \forall x_1, x_2 \in F \quad (2.1.2)$$

Hence proved, every pair  $x_1$  and  $x_2$  is a solution for the equation  $\mathbf{AX} = 0$ .

### 2.2 Solution 2

**Case 1:** Let  $a = 0$ . Since  $ad - bc \neq 0$ . As  $bc \neq 0$  therefore  $b \neq 0$  and  $c \neq 0$ . Hence, we can perform row reduction on the augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} 0 & b & 0 \\ c & d & 0 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d & 0 \\ 0 & b & 0 \end{pmatrix} \quad (2.2.1)$$

$$\xleftrightarrow[R_2 = \frac{1}{b}R_2]{R_1 = \frac{1}{c}R_1} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.2.2)$$

$$\xleftrightarrow{R_1 = R_1 - \frac{d}{c}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.2.3)$$

**Case 2:** Let  $a, b, c, d \neq 0$ . Considering the following case,

$$\mathbf{AX} = \mathbf{u} \quad (2.2.4)$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (2.2.5)$$

Row Reducing the augmented matrix of (2.2.5) we get,

$$\begin{pmatrix} a & b & u_1 \\ c & d & u_2 \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ c & d & u_2 \end{pmatrix} \quad (2.2.6)$$

$$\xleftrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & \frac{ad-bc}{a} & \frac{au_2-cu_1}{a} \end{pmatrix} \quad (2.2.7)$$

$$\xleftrightarrow{R_2 = \frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} & \frac{u_1}{a} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \quad (2.2.8)$$

$$\xleftrightarrow{R_1 = R_1 - \frac{b}{a}R_2} \begin{pmatrix} 1 & 0 & \frac{du_1-bu_2}{ad-bc} \\ 0 & 1 & \frac{au_2-cu_1}{ad-bc} \end{pmatrix} \quad (2.2.9)$$

From (2.2.9) we get,

$$x_1 = \frac{du_1 - bu_2}{ad - bc} \quad (2.2.10)$$

$$x_2 = \frac{au_2 - cu_1}{ad - bc} \quad (2.2.11)$$

Since  $u_1 = 0$  and  $u_2 = 0$  then from (2.2.10) and (2.2.11),

$$x_1 = 0 \quad (2.2.12)$$

$$x_2 = 0 \quad (2.2.13)$$

Hence we get,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.2.14)$$

In (2.2.3) and (2.2.14), we can see that  $\mathbf{AX} = 0$  has only one trivial solution i.e  $x_1 = x_2 = 0$  in all cases. Hence proved, the equation  $\mathbf{AX} = 0$  has only one trivial solution  $x_1 = x_2 = 0$

### 2.3 Solution 3

**Case 1:** Let,  $a \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ c & d & 0 \end{pmatrix} \quad (2.3.1)$$

$$\xrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} & 0 \\ 0 & \frac{d}{a} & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (2.3.2)$$

Hence from (2.3.21),  $\mathbf{AX} = 0$  if and only if

$$x_1 = -\frac{b}{a}x_2 \quad [a \neq 0] \quad (2.3.3)$$

Letting  $x_1^0 = -\frac{b}{a}$  and  $x_2^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (2.3.4)$$

$$x_2 = yx_2^0 \quad (2.3.5)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

**Case 2:** Let,  $b \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \quad (2.3.6)$$

$$\xrightarrow{R_1 = \frac{1}{b}R_1} \begin{pmatrix} 1 & \frac{a}{b} & 0 \\ d & c & 0 \end{pmatrix} \quad (2.3.7)$$

$$\xrightarrow{R_2 = R_2 - dR_1} \begin{pmatrix} 1 & \frac{a}{b} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (2.3.8)$$

Hence from (2.3.21),  $\mathbf{AX} = 0$  if and only if

$$x_2 = -\frac{a}{b}x_1 \quad [b \neq 0] \quad (2.3.9)$$

Letting  $x_2^0 = -\frac{a}{b}$  and  $x_1^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (2.3.10)$$

$$x_2 = yx_2^0 \quad (2.3.11)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

**Case 3:** Let,  $c \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d & 0 \\ a & b & 0 \end{pmatrix} \quad (2.3.12)$$

$$\xrightarrow{R_1 = \frac{1}{c}R_1} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ a & b & 0 \end{pmatrix} \quad (2.3.13)$$

$$\xrightarrow{R_2 = R_2 - aR_1} \begin{pmatrix} 1 & \frac{d}{c} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (2.3.14)$$

Hence from (2.3.21),  $\mathbf{AX} = 0$  if and only if

$$x_1 = -\frac{d}{c}x_2 \quad [c \neq 0] \quad (2.3.15)$$

Letting  $x_1^0 = -\frac{d}{c}$  and  $x_2^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (2.3.16)$$

$$x_2 = yx_2^0 \quad (2.3.17)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

**Case 4:** Let,  $d \neq 0$  for  $\mathbf{A}$ . Given  $ad - bc = 0$ , at first we multiply by elementary matrix to change the columns and then we can perform row reduction on augmented matrix of equation  $\mathbf{AX} = 0$  as follows,

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} b & a & 0 \\ d & c & 0 \end{pmatrix} \quad (2.3.18)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} d & c & 0 \\ b & a & 0 \end{pmatrix} \quad (2.3.19)$$

$$\xrightarrow{R_1 = \frac{1}{d}R_1} \begin{pmatrix} 1 & \frac{c}{d} & 0 \\ b & a & 0 \end{pmatrix} \quad (2.3.20)$$

$$\xrightarrow{R_2 = R_2 - bR_1} \begin{pmatrix} 1 & \frac{c}{d} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\because ad - bc = 0] \quad (2.3.21)$$

Hence from (2.3.21),  $\mathbf{AX} = 0$  if and only if

$$x_2 = -\frac{c}{d}x_1 \quad [d \neq 0] \quad (2.3.22)$$

Letting  $x_2^0 = -\frac{c}{d}$  and  $x_1^0 = 1$  we get for  $y = 1$ ,

$$x_1 = yx_1^0 \quad (2.3.23)$$

$$x_2 = yx_2^0 \quad (2.3.24)$$

which is a solution of the equation  $\mathbf{AX} = 0$ .

Hence Proved.