

Matrix Theory (EE5609) Assignment 17

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Abstract—This document proves the invertibility of a certain linear operator.

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_17

1 PROBLEM

Let T be a linear operator on the finite-dimensional space \mathbb{V} . Suppose there is a linear operator U on \mathbb{V} such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when \mathbb{V} is not finite-dimensional.

2 SOLUTION

2.1 Proof

Let $T : \mathbb{V} \rightarrow \mathbb{V}$ be a linear operator, where \mathbb{V} is a finite dimensional vectors space and $U : \mathbb{V} \rightarrow \mathbb{V}$ is also a linear operator such that,

$$TU = I \quad (2.1.1)$$

Where, I is an identity transformation. Now we know that linear transformations are functions. Hence,

$$TU = I \text{ is a function} \quad (2.1.2)$$

$$\Rightarrow I : \mathbb{V} \rightarrow \mathbb{V} \quad (2.1.3)$$

Such that $T(V) = V$. Defining $TU : \mathbb{V} \rightarrow \mathbb{V}$ to be a linear operator, we have

$$T[U(V_i)] = V_i \quad [V_i \in \mathbb{V}] \quad (2.1.4)$$

Let $\mathbf{V}_1, \mathbf{V}_2 \in \mathbb{V}$ then,

If $\mathbf{V}_1 \neq \mathbf{V}_2$ then, $T[U(\mathbf{V}_1)] \neq T[U(\mathbf{V}_2)]$. Hence,

$$T \text{ must be one-one function} \quad (2.1.5)$$

Again, T is linear operator on finite dimensional vector space. Hence,

$$T \text{ must be onto function} \quad (2.1.6)$$

From (2.1.5) and (2.1.6) we get,

$$T \text{ is invertible function} \quad (2.1.7)$$

From (2.1.7) we know,

$$TT^{-1} = I \quad (2.1.8)$$

Where T^{-1} is an inverse function of linear operator T . Hence,

$$TT^{-1} = I = TU \quad (2.1.9)$$

$$\Rightarrow T^{-1}(TT^{-1}) = T^{-1}(TU) \quad (2.1.10)$$

$$\Rightarrow T^{-1}(I) = IU \quad (2.1.11)$$

$$\Rightarrow T^{-1} = U \quad (2.1.12)$$

Hence from (2.1.7) and (2.1.12) it is proven that T is invertible and $T^{-1} = U$

2.2 Example

Let D be the differential operator $D : \mathbb{V} \rightarrow \mathbb{V}$ where \mathbb{V} is a space of polynomial functions in one variable over \mathbb{R} . Hence,

$$D(c_0 + c_1x + \cdots + c_nx^n) = c_1 + c_2'x + \cdots + c_n'x^{n-1} \quad (2.2.1)$$

And, $U : \mathbb{V} \rightarrow \mathbb{V}$ is another linear operator such that,

$$U(c_0 + c_1x + \cdots + c_nx^n) = c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1} \quad (2.2.2)$$

Then $UD : \mathbb{V} \rightarrow \mathbb{V}$ is a linear operator such that,

$$UD(c_0 + c_1x + \cdots + c_nx^n) \quad (2.2.3)$$

$$= U[D(c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1})] \quad (2.2.4)$$

$$= U[c_1 + c'_2x + \cdots + c'_nx^{n-1}] \quad (2.2.5)$$

$$= c_1x + c_2\frac{x^2}{2} + \cdots + c_n\frac{x^n}{n} \quad (2.2.6)$$

Hence, from (2.2.6),

$$UD \neq I \quad (2.2.7)$$

Again, $DU : \mathbb{V} \rightarrow \mathbb{V}$ is a linear operator such that,

$$DU(c_0 + c_1x + \cdots + c_nx^n) \quad (2.2.8)$$

$$= D[U(c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1})] \quad (2.2.9)$$

$$= D[c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1}] \quad (2.2.10)$$

$$= c_0 + c_1\frac{2x}{2} + \cdots + c_n\frac{(n+1)x^n}{n+1} \quad (2.2.11)$$

$$= c_0 + c_1x + \cdots + c_nx^n \quad (2.2.12)$$

Hence, from (2.2.12),

$$DU = I \quad (2.2.13)$$

Hence, from (2.2.7) and (2.2.13), D is not invertible.