Matrix Theory (EE5609) Assignment 19

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Abstract—**This document solves a problem on a functional.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_19

1 Problem

Let $\mathbb{V}$ be the vector space of all 2×2 matrices over the field of real numbers, and let

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$$

Let $\mathbb{W}$ be the subspace of $\mathbb{V}$ consisting of all $\mathbf{A}$ such that $\mathbf{AB} = 0$. Let f be a linear functional on $\mathbb{V}$ which is in the annihilator of $\mathbb{W}$. Suppose that $f(\mathbf{I}) = 0$ and $f(\mathbf{C}) = 3$, where $\mathbf{I}$ is the 2×2 identity matrix and

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Find $f(\mathbf{B})$

2 Solution

Let,

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \forall\ \mathbf{A} \in \mathbb{W} \tag{2.0.1}$$

From $\mathbf{AB} = 0$ we have,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.2}$$

From (2.0.2) we get,

$$y = 2x \tag{2.0.3}$$
$$w = 2z \tag{2.0.4}$$

Hence, using (2.0.3) and (2.0.4) we conclude that $\mathbb{W}$ consists of all the matrices of the following form,

$$\mathbf{A} = \begin{pmatrix} x & 2x \\ z & 2z \end{pmatrix} \quad \forall\ \mathbf{A} \in \mathbb{W} \tag{2.0.5}$$

Hence from (2.0.5) we get,

$$A \in \left\{ a\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\} \quad \forall\ a, c \in \mathbb{R} \tag{2.0.6}$$

$$\implies span(\mathbb{W}) = \left\{ \mathbf{e_1} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\} \tag{2.0.7}$$

Hence, $\mathbf{e_1}$ and $\mathbf{e_2}$ are basis of $\mathbb{W}$. Since $f \in \mathbb{W}^0$, it follows that,

$$f(\mathbf{e_1}) = 0 \tag{2.0.8}$$
$$f(\mathbf{e_2}) = 0 \tag{2.0.9}$$

Again given,

$$f(\mathbf{I}) = 0 \tag{2.0.10}$$
$$f(\mathbf{C}) = 0 \tag{2.0.11}$$

From (2.0.8), (2.0.9), (2.0.10) and (2.0.11) we conclude that $\mathbf{I}, \mathbf{C}, \mathbf{e_1}, \mathbf{e_2}$ are linearly independent. We prove the statement as follows,

$$c_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + c_4\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.12}$$

$$\implies \begin{pmatrix} c_1 + c_3 & 2c_3 \\ c_4 & c_1 + c_2 + 2c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\implies c_1 = c_2 = c_3 = c_4 = 0 \tag{2.0.14}$$

Since $dim(\mathbb{V}) = 2 \times 2 = 4$, hence the set $\{\mathbf{I}, \mathbf{C}, \mathbf{e_1}, \mathbf{e_2}\}$ forms a basis for the vector-space $\mathbb{V}$ which is the vector-space of all 2×2 matrices.

Since $\mathbf{B} \in \mathbb{V}$ then there exists c_1, c_2, c_3, c_4 such that,

$$c_1\mathbf{I} + c_2\mathbf{C} + c_3\mathbf{e_1} + c_4\mathbf{e_2} = \mathbf{B} \tag{2.0.15}$$

$$c_1\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + c_2\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} + c_3\begin{pmatrix}1 & 2\\0 & 0\end{pmatrix} + c_4\begin{pmatrix}0 & 0\\1 & 2\end{pmatrix} \tag{2.0.16}$$

$$= \begin{pmatrix}2 & -2\\-1 & 1\end{pmatrix} \tag{2.0.17}$$

$$\begin{pmatrix}c_1 + c_3 & 2c_3\\c_4 & c_1 + c_2 + 2c_4\end{pmatrix} = \begin{pmatrix}2 & -2\\-1 & 1\end{pmatrix} \tag{2.0.18}$$

From (2.0.18) we get,

$$c_1 = 3 \tag{2.0.19}$$

$$c_2 = 0 \tag{2.0.20}$$

$$c_3 = -1 \tag{2.0.21}$$

$$c_4 = -1 \tag{2.0.22}$$

From (2.0.19), (2.0.20), (2.0.21) and (2.0.22) we get,

$$f(\mathbf{B}) = f(3\mathbf{I} - \mathbf{e_1} - \mathbf{e_2}) \tag{2.0.23}$$

$$= 3f(\mathbf{I}) - f(\mathbf{e_1}) - f(\mathbf{e_2}) \tag{2.0.24}$$

$$= 0 \tag{2.0.25}$$

The (2.0.25) is the required answer.