Matrix Theory (EE5609) Assignment 13

Arkadipta De
MTech Artificial Intelligence
AI20MTECH14002

***Abstract*—This document proves if AB = I then BA = I given that both A and B are square matrices.**

All the codes for the figure in this document can be found at

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https://github.com/Arko98/EE5609/blob/master/
    Assignment_13
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1 Problem

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices such that $\mathbf{AB} = \mathbf{I}$. Prove that $\mathbf{BA} = \mathbf{I}$

2 Solution

2.1 Solution 1

Since $\mathbf{AB} = \mathbf{I}$, hence $\mathbf{AB}$ has range equal to the full n - dimensional space. Hence the range of $\mathbf{B}$ is also n - dimensional space. If $\mathbf{B}$ did not have n - dimensional space as it's range then a set of $(n-1)$ vectors would span the range of $\mathbf{B}$, so the range of $\mathbf{AB}$, which is the image under $\mathbf{A}$ of the range of $\mathbf{B}$, would also be spanned by a set of $(n-1)$ vectors, hence would have dimension less than n. Hence we can write,

$$\mathbf{B} = \mathbf{BI} \tag{2.1.1}$$

$$= \mathbf{B}(\mathbf{AB}) \quad [\because \mathbf{AB} = \mathbf{I}] \tag{2.1.2}$$

$$= (\mathbf{BA})\mathbf{B} \tag{2.1.3}$$

Hence from (2.1.3),

$$\mathbf{B} - (\mathbf{BA})\mathbf{B} = \mathbf{0} \tag{2.1.4}$$

$$(\mathbf{I} - \mathbf{BA})\mathbf{B} = \mathbf{0} \quad \text{[Distributive Law]} \tag{2.1.5}$$

Since range of $\mathbf{B}$ is n - dimensional space hence $\mathbf{B} \neq \mathbf{0}$. Thus we can write from (2.1.5),

$$\mathbf{I} - \mathbf{BA} = \mathbf{0} \tag{2.1.6}$$

$$\mathbf{BA} = \mathbf{I} \tag{2.1.7}$$

Hence Proved.

2.2 Solution 2

Let $\mathbf{BX} = 0$ be a system of linear equation with n unknowns and n equations as $\mathbf{B}$ is $n \times n$ matrix. Hence,

$$\mathbf{BX} = 0 \tag{2.2.1}$$

$$\implies \mathbf{A}(\mathbf{BX}) = 0 \tag{2.2.2}$$

$$\implies (\mathbf{AB})\mathbf{X} = 0 \tag{2.2.3}$$

$$\implies \mathbf{IX} = 0 \quad [\because \mathbf{AB} = \mathbf{I}] \tag{2.2.4}$$

$$\implies \mathbf{X} = 0 \tag{2.2.5}$$

From (2.2.5) since $\mathbf{X} = 0$ is the only solution of (2.2.1), hence $rank(\mathbf{B}) = n$. Which implies all columns of $\mathbf{B}$ are linearly independent. Hence $\mathbf{B}$ is invertible. Therefore, every left inverse of $\mathbf{B}$ is also a right inverse of $\mathbf{B}$. Therefore given that,

$$\mathbf{AB} = \mathbf{I} \quad \text{[Where A is the left inverse of B]} \tag{2.2.6}$$

We can conclude from (2.2.6),

$$\mathbf{BA} = \mathbf{I} \quad \text{[Where A is the right inverse of B]} \tag{2.2.7}$$

Hence Proved.