Matrix Theory (EE5609) Assignment 19

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***Abstract*—This document solves a problem on a functional.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_19

1 Problem

Let $\mathbb{V}$ be the vector space of all 2×2 matrices over the field of real numbers, and let

$$\mathbf{B} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$$

Let $\mathbb{W}$ be the subspace of $\mathbb{V}$ consisting of all $\mathbf{A}$ such that $\mathbf{AB} = 0$. Let f be a linear functional on $\mathbb{V}$ which is in the annihilator of $\mathbb{W}$. Suppose that $f(\mathbf{I}) = 0$ and $f(\mathbf{C}) = 3$, where $\mathbf{I}$ is the 2×2 identity matrix and

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Find $f(\mathbf{B})$

2 Solution

The general linear functional f on vector space $\mathbb{V}$ is of the form,

$$f(\mathbf{A}) = ax + by + cz + dw \tag{2.0.1}$$

Where,

$$\mathbf{A} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \quad \forall\ \mathbf{A} \in \mathbb{W} \tag{2.0.2}$$

$$a, b, c, d \in \mathbb{R} \tag{2.0.3}$$

From $\mathbf{AB} = 0$ we have,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.0.4}$$

From (2.0.4) we get,

$$y = 2x \tag{2.0.5}$$

$$w = 2z \tag{2.0.6}$$

Hence, using (2.0.5) and (2.0.7) we conclude that $\mathbb{W}$ consists of all the matrices of the following form,

$$\mathbf{A} = \begin{pmatrix} x & 2x \\ z & 2z \end{pmatrix} \quad \forall\ \mathbf{A} \in \mathbb{W} \tag{2.0.7}$$

Hence from (2.0.7) we get,

$$f\left(\begin{pmatrix} x & 2x \\ z & 2z \end{pmatrix}\right) = 0 \quad \forall\ x, z \in \mathbb{R} \tag{2.0.8}$$

$$\implies ax + 2bx + cz + 2dz = 0 \quad \text{[From (2.0.1)]} \tag{2.0.9}$$

$$\implies (a + 2b)x + (c + 2d)z = 0 \quad \forall\ x, z \in \mathbb{R} \tag{2.0.10}$$

From (2.0.10) we get,

$$b = -\frac{1}{2}a \tag{2.0.11}$$

$$d = -\frac{1}{2}c \tag{2.0.12}$$

Hence, from (2.0.11), (2.0.12) and (2.0.1), the general form of the functional f on vector space $\mathbb{V}$ becomes,

$$f(\mathbf{A}) = ax - \frac{1}{2}ay + cz - \frac{1}{2}cw \quad \forall\ \mathbf{A} \in \mathbb{W} \tag{2.0.13}$$

Now,

$$f(\mathbf{C}) = 3 \tag{2.0.14}$$

$$\implies -\frac{1}{2}c = 3 \quad \text{[From (2.0.13)]} \tag{2.0.15}$$

$$\implies c = -6 \tag{2.0.16}$$

Again,

$$f(\mathbf{I}) = 0 \tag{2.0.17}$$

$$\implies a - \frac{1}{2}c = 0 \quad \text{[From (2.0.13)]} \tag{2.0.18}$$

$$\implies a = -3 \quad \text{[Using (2.0.16)]} \tag{2.0.19}$$

Hence, using (2.0.16) and (2.0.19) the general form of f in (2.0.13) becomes,

$$f(\mathbf{A}) = -3x + \frac{3}{2}y - 6z + 3w \quad \forall\ \mathbf{A} \in \mathbb{W} \tag{2.0.20}$$

Now for given B, from (2.0.20) we get,

$$f(\mathbf{B}) = -3(2) + \frac{3}{2}(-2) - 6(-1) + 3(1) \tag{2.0.21}$$

$$\implies f(\mathbf{B}) = 0 \tag{2.0.22}$$

(2.0.22) is the required answer.