Matrix Theory (EE5609) Assignment 8

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***Abstract*—This document finds what conic section a given second degree equation represent.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_8

1 Problem

What conic does the following equation represent.

$$13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$$

Find the center.

2 Solution

The general second degree equation can be expressed as follows,

$$\mathbf{x^T V x} + 2\mathbf{u^T x} + f = 0 \tag{2.0.1}$$

From the given second degree equation we get,

$$\mathbf{V} = \begin{pmatrix} 13 & -9 \\ -9 & 37 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 7 \end{pmatrix} \tag{2.0.3}$$

$$f = -2 \tag{2.0.4}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 13 & -9 \\ -9 & 37 \end{vmatrix} = 400 > 0 \tag{2.0.5}$$

Hence from (2.0.5) we conclude that given equation is an ellipse. The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 13 & 9 \\ 9 & \lambda - 37 \end{vmatrix} = 0 \tag{2.0.6}$$

$$\implies \lambda^2 - 50\lambda + 400 = 0 \tag{2.0.7}$$

Hence the characteristic equation of $\mathbf{V}$ is given by (2.0.7). The roots of (2.0.7) i.e the eigenvalues are given by

$$\lambda_1 = 10, \lambda_2 = 40 \tag{2.0.8}$$

The eigen vector $\mathbf{p}$ is defined as,

$$\mathbf{Vp} = \lambda\mathbf{p} \tag{2.0.9}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{2.0.10}$$

for $\lambda_1 = 10$,

$$(\lambda_1\mathbf{I} - \mathbf{V}) = \begin{pmatrix} -3 & 9 \\ 9 & -27 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{3}R_1]{R_2 = R_2 + 3R_1} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \tag{2.0.11}$$

$$\implies \mathbf{p_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{2.0.12}$$

Again, for $\lambda_2 = 40$,

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 27 & 9 \\ 9 & 3 \end{pmatrix} \xleftrightarrow[R_1 = \frac{1}{27}R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \tag{2.0.13}$$

$$\implies \mathbf{p_2} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \tag{2.0.14}$$

Again, Hence from the equation

$$\mathbf{V} = \mathbf{PDP^{-1}} \tag{2.0.15}$$

Where $\mathbf{D}$ is a diagonal matrix, we get,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \tag{2.0.16}$$

$$\mathbf{D} = \begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix} \tag{2.0.17}$$

Now (2.0.1) can be written as,

$$\mathbf{y^T D y} = \mathbf{u^T V^{-1} u} - f \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.0.18}$$

And,

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad \left|\mathbf{V}\right| \neq 0 \tag{2.0.19}$$

$$\mathbf{y} = \mathbf{P}^{\mathbf{T}}\left(\mathbf{x} - \mathbf{c}\right) \tag{2.0.20}$$

The centre/vertex of the conic section in (2.0.1) is given by $\mathbf{c}$ in (2.0.19). We compute $\mathbf{V}^{-1}$ as follows,

$$\begin{pmatrix} 13 & -9 & 1 & 0 \\ -9 & 37 & 0 & 1 \end{pmatrix} \xleftrightarrow[R_2 = \frac{13}{400}R_2]{R_2 = R_2 + \frac{9}{13}R_1} \begin{pmatrix} 13 & -9 & 1 & 0 \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{2.0.21}$$

$$\xleftrightarrow[R_1 = R_1 + \frac{9}{13}R_2]{R_1 = \frac{1}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{37}{400} & \frac{9}{400} \\ 0 & 1 & \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{2.0.22}$$

Hence $\mathbf{V}^{-1}$ is given by,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{37}{400} & \frac{9}{400} \\ \frac{9}{400} & \frac{13}{400} \end{pmatrix} \tag{2.0.23}$$

Now $\mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u}$ is given by,

$$\mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u} = \frac{1}{400}\begin{pmatrix} 1 & 7 \end{pmatrix}\begin{pmatrix} 37 & 9 \\ 9 & 13 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \end{pmatrix} = 2 \tag{2.0.24}$$

And, $\mathbf{V}^{-1}\mathbf{u}$ is given by,

$$\mathbf{V}^{-1}\mathbf{u} = \frac{1}{400}\begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.25}$$

By putting the value of (2.0.25), the center of the ellipse is given by (2.0.19) as follows,

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \tag{2.0.26}$$

Also the semi-major axis (a) and semi-minor axis (b) of the ellipse are given by,

$$a = \sqrt{\frac{\mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \frac{\sqrt{10}}{5} \tag{2.0.27}$$

$$b = \sqrt{\frac{\mathbf{u}^{\mathbf{T}}\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \frac{\sqrt{10}}{10} \tag{2.0.28}$$

Again, $\mathbf{y}$ from (2.0.20) is given by,

$$\mathbf{y} = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x + \frac{1}{4} \\ y + \frac{1}{4} \end{pmatrix} \tag{2.0.29}$$

Finally putting values from (2.0.24) and (2.0.29) in (2.0.18), the equation of ellipse is given by,

$$\mathbf{y}^{\mathbf{T}}\begin{pmatrix} 10 & 0 \\ 0 & 40 \end{pmatrix}\mathbf{y} = 4 \tag{2.0.30}$$

The following figure is the graphical representation of the ellipse in (2.0.30),

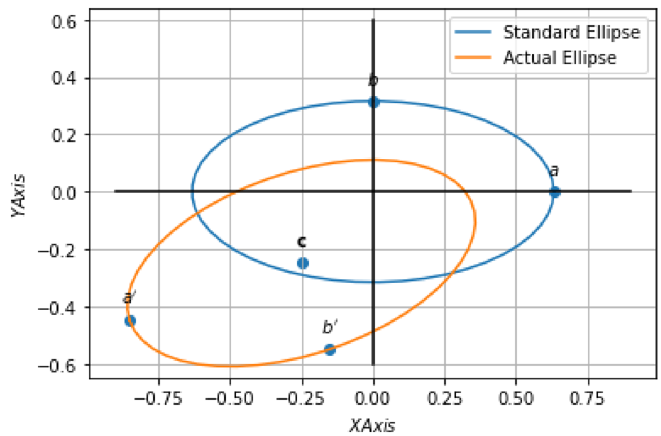

Fig. 1: Graphical representation of the ellipse