Matrix Theory (EE5609) Assignment 14

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***Abstract*—This document proves the existence of inverse of Hilbert Matrix.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_14

1 Problem

Prove that the following matrix is invertible and $\mathbf{A^{-1}}$ has integer entries.

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} & \dots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n+1} \\ \vdots & \vdots & \dots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \dots & \frac{1}{2n-1} \end{pmatrix}$$

2 Solution

Let $\mathbf{A_3}$ be 3×3 matrix i.e

$$\mathbf{A_3} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \tag{2.0.1}$$

Now we find the inverse of the matrix $\mathbf{A_3}$ as follows,

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{pmatrix} \tag{2.0.2}$$

$$\xleftrightarrow[R_3=R_3-\frac{1}{3}R_1]{R_2=R_2-\frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{1}{45} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \tag{2.0.3}$$

$$\xleftrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{pmatrix} \tag{2.0.4}$$

$$\xleftrightarrow[R_3=180R_3]{R_2=12R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \tag{2.0.5}$$

$$\xleftrightarrow[R_1=R_1-R_3]{R_2=R_2-R3} \begin{pmatrix} 1 & \frac{1}{2} & 0 & -9 & 60 & -60 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \tag{2.0.6}$$

$$\xleftrightarrow{R_1=R_1-\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \tag{2.0.7}$$

Hence we see that $\mathbf{A_3}$ is invertible and the inverse contains integer entries and $\mathbf{A_3^{-1}}$ is given by,

$$\mathbf{A_3^{-1}} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \tag{2.0.8}$$

Let, $\mathbf{A_4}$ be 4×4 matrix as follows,

$$\mathbf{A_4} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \tag{2.0.9}$$

Now, expressing $\mathbf{A_4}$ using $\mathbf{A_3}$ we get,

$$\mathbf{A_4} = \begin{pmatrix} \mathbf{A_3} & \mathbf{u} \\ \mathbf{u^T} & \mathbf{d} \end{pmatrix} \tag{2.0.10}$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \tag{2.0.11}$$

$$\mathbf{d} = \begin{pmatrix} \frac{1}{7} \end{pmatrix} \tag{2.0.12}$$

Now assuming $\mathbf{A_4}$ has an inverse, then from (2.0.10), the inverse of $\mathbf{A_4}$ can be written using block matrix inversion as follows,

$$\mathbf{A_4^{-1}} = \begin{pmatrix} \mathbf{A_3^{-1} + A_3^{-1}uX^{-1}u^TA_3^{-1}} & \mathbf{-A_3^{-1}uX^{-1}} \\ \mathbf{-X^{-1}u^TA_3^{-1}} & \mathbf{X^{-1}} \end{pmatrix} \tag{2.0.13}$$

where,

$$\mathbf{X = d - u^TA_3^{-1}u} \tag{2.0.14}$$

Now, the assumption of $\mathbf{A_4}$ being invertible will hold if and only if $\mathbf{A_3}$ is invertible, which has been proved in (2.0.8) and $\mathbf{X}$ from (2.0.14) is invertible. We now prove that $\mathbf{X}$ is invertible as follows,

$$\mathbf{X} = \begin{pmatrix} \frac{1}{7} \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \tag{2.0.15}$$

$$\implies \mathbf{X} = \begin{pmatrix} \frac{1}{2800} \end{pmatrix} \tag{2.0.16}$$

Hence, $\mathbf{X}$ is a square invertible matrix as it has linearly independent columns, hence $\mathbf{X^{-1}}$ is given by,

$$\mathbf{X^{-1}} = \begin{pmatrix} 2800 \end{pmatrix} \tag{2.0.17}$$

Hence, $\mathbf{A_4}$ is invertible. Now putting the values of $\mathbf{A_3^{-1}}$, $\mathbf{X^{-1}}$ and $\mathbf{u}$ we get,

$$\mathbf{A_3^{-1} + A_3^{-1}uX^{-1}u^TA_3^{-1}} = \begin{pmatrix} 16 & -120 & 240 \\ -120 & 1200 & -2700 \\ 240 & -2700 & 6480 \end{pmatrix} \tag{2.0.18}$$

$$\mathbf{-A_3^{-1}uX^{-1}} = \begin{pmatrix} -140 \\ 1680 \\ -4200 \end{pmatrix} \tag{2.0.19}$$

$$\mathbf{-X^{-1}u^TA_3^{-1}} = \begin{pmatrix} -140 & 1680 & -4200 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{X^{-1}} = \begin{pmatrix} 2800 \end{pmatrix} \tag{2.0.21}$$

Putting values from (2.0.18), (2.0.19), (2.0.20) and (2.0.21) into (2.0.13) we get,

$$\mathbf{A_4^{-1}} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix} \tag{2.0.22}$$

Hence, from (2.0.22) we prove that, $\mathbf{A_4}$ is invertible and has integer entries.
Let $\mathbf{A_{n-1}}$ be invertible with integer entries. Then we can represent $\mathbf{A_n}$ as follows,

$$\mathbf{A_n} = \begin{pmatrix} \mathbf{A_{n-1}} & \mathbf{u} \\ \mathbf{u^T} & \mathbf{d} \end{pmatrix} \tag{2.0.23}$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \tag{2.0.24}$$

$$\mathbf{d} = \begin{pmatrix} \frac{1}{2n-1} \end{pmatrix} \tag{2.0.25}$$

Now assuming $\mathbf{A_n}$ has an inverse, then from (2.0.23), the inverse of $\mathbf{A_n}$ can be written using block matrix inversion as follows,

$$\mathbf{A_n}^{-1} = \begin{pmatrix} \mathbf{A_{n-1}^{-1} + A_{n-1}^{-1}uX^{-1}u^TA_{n-1}^{-1}} & \mathbf{-A_{n-1}^{-1}uX^{-1}} \\ \mathbf{-X^{-1}u^TA_{n-1}^{-1}} & \mathbf{X^{-1}} \end{pmatrix} \tag{2.0.26}$$

where,

$$\mathbf{X = d - u^TA_{n-1}^{-1}u} \tag{2.0.27}$$

Now, the assumption of $\mathbf{A_n}$ being invertible will hold if and only if $\mathbf{A_{n-1}}$ is invertible, which has been assumed and $\mathbf{X}$ from (2.0.27) is invertible. We now prove that $\mathbf{X}$ is invertible as follows,

$$\mathbf{X} = \begin{pmatrix} \frac{1}{2n-1} \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \dots & \frac{1}{2n-2} \end{pmatrix} \mathbf{A_{n-1}^{-1}} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \dots \\ \frac{1}{2n-2} \end{pmatrix} \tag{2.0.28}$$

In equation (2.0.28) $\mathbf{u}$ contains no negative or zero entries, again $\mathbf{A_{n-1}^{-1}}$ has non zero integer entries, hence $\mathbf{u^TA_{n-1}^{-1}u}$ is a non zero matrix with size 1×1. Moreover $\mathbf{d}$ is not equal to $\mathbf{u^TA_{n-1}^{-1}u}$ hence in (2.0.28) $\mathbf{X}$ is non-zero matrix and hence it has an inverse. Hence $\mathbf{A_n}$ is invertible, proved.