Matrix Theory (EE5609) Assignment 10

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***Abstract*—This finds the distance between a given point and a plane using Singular Value Decomposition.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_10

1 Problem

Find the distance of the point $\begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ from the plane $\begin{pmatrix} 6 & -3 & 2 \end{pmatrix}\mathbf{x} = 4$

2 Solution

First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{2.0.1}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = 0 \tag{2.0.2}$$

$$\implies 6a - 3b + 2c = 0 \tag{2.0.3}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \tag{2.0.4}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{3}{2} \end{pmatrix} \tag{2.0.5}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.6}$$

Putting values in (2.0.6),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & \frac{3}{2} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \tag{2.0.7}$$

Now, to solve (2.0.7), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.8}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix} \tag{2.0.9}$$

$$\mathbf{MM}^T = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & \frac{3}{2} \\ 3 & \frac{3}{2} & \frac{45}{4} \end{pmatrix} \tag{2.0.10}$$

From (2.0.6) putting (2.0.8) we get,

$$\mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{2.0.11}$$

$$\implies \mathbf{x} = \mathbf{VS_+U^Tb} \tag{2.0.12}$$

Where $\mathbf{S_+}$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{MM}^T$,

$$\left|\mathbf{MM}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & \frac{3}{2} \\ 3 & \frac{3}{2} & \frac{45}{4}-\lambda \end{pmatrix} = 0 \tag{2.0.14}$$

$$\implies \lambda^3 - \frac{53}{4}\lambda^2 + \frac{49}{4}\lambda = 0 \tag{2.0.15}$$

Hence eigen values of $\mathbf{MM}^T$ are,

$$\lambda_1 = \frac{49}{4} \tag{2.0.16}$$

$$\lambda_2 = 1 \tag{2.0.17}$$

$$\lambda_3 = 0 \tag{2.0.18}$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{4}{15} \\ \frac{2}{15} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -3 \\ -\frac{3}{2} \\ 1 \end{pmatrix} \tag{2.0.19}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{4}{7\sqrt{5}} \\ \frac{2}{7\sqrt{5}} \\ \frac{3\sqrt{5}}{7} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{6}{7} \\ -\frac{3}{7} \\ \frac{2}{7} \end{pmatrix} \tag{2.0.20}$$

Hence we obtain $\mathbf{U}$ of (2.0.8) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7} \\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7} \\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7} \end{pmatrix} \tag{2.0.21}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (2.0.8) as follows,

$$\mathbf{S} = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.22}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.23}$$

$$\implies \begin{pmatrix} 10-\lambda & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4}-\lambda \end{pmatrix} = 0 \tag{2.0.24}$$

$$\implies \lambda^2 - \frac{53}{4}\lambda + \frac{49}{4} = 0 \tag{2.0.25}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{49}{4} \tag{2.0.26}$$

$$\lambda_5 = 1 \tag{2.0.27}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \tag{2.0.28}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.0.29}$$

Hence we obtain $\mathbf{V}$ of (2.0.8) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \tag{2.0.30}$$

Finally from (2.0.8) we get the Singualr Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{4}{7\sqrt{5}} & -\frac{1}{\sqrt{5}} & -\frac{6}{7} \\ \frac{2}{7\sqrt{5}} & \frac{2}{\sqrt{5}} & -\frac{3}{7} \\ \frac{3\sqrt{5}}{7} & 0 & \frac{2}{7} \end{pmatrix} \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T \tag{2.0.31}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{2}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.32}$$

From (2.0.12) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{27}{7\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \\ -\frac{33}{7} \end{pmatrix} \tag{2.0.33}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{54}{49\sqrt{5}} \\ \frac{8}{7\sqrt{5}} \end{pmatrix} \tag{2.0.34}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \tag{2.0.35}$$

Verifying the solution of (2.0.35) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.36}$$

Evaluating the R.H.S in (2.0.36) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.37}$$

$$\implies \begin{pmatrix} 10 & \frac{9}{2} \\ \frac{9}{2} & \frac{13}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.38}$$

Solving the augmented matrix of (2.0.38) we get,

$$\begin{pmatrix} 10 & \frac{9}{2} & -7 \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \xleftrightarrow{R_1=\frac{1}{10}R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ \frac{9}{2} & \frac{13}{4} & \frac{1}{2} \end{pmatrix} \tag{2.0.39}$$

$$\xleftrightarrow{R_2=R_2-\frac{9}{2}R_1} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & \frac{49}{40} & \frac{73}{20} \end{pmatrix} \tag{2.0.40}$$

$$\xleftrightarrow{R_2=\frac{40}{49}R_2} \begin{pmatrix} 1 & \frac{9}{20} & -\frac{7}{10} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \tag{2.0.41}$$

$$\xleftrightarrow{R_1=R_1-\frac{9}{20}R_1} \begin{pmatrix} 1 & 0 & -\frac{100}{49} \\ 0 & 1 & \frac{146}{49} \end{pmatrix} \tag{2.0.42}$$

Hence, Solution of (2.0.36) is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{100}{49} \\ \frac{146}{49} \end{pmatrix} \tag{2.0.43}$$

Comparing results of $\mathbf{x}$ from (2.0.35) and (2.0.43) we conclude that the solution is verified.