

Matrix Theory (EE5609) Challenging Problem

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Abstract—This document proves that $\mathbf{A}^T\mathbf{A}$ has positive eigen values.

Download latex codes from

https://github.com/Arko98/EE5609/tree/master/Challenge_5

1 PROBLEM

Show that the eigen values of $\mathbf{A}^T\mathbf{A}$ are positive.

2 PROOF

Let, \mathbf{A} is an arbitrary $m \times n$ matrix. Now consider the matrix $\mathbf{A}^T\mathbf{A}$,

for any n dimensional vector \mathbf{z} ,

$$\mathbf{z}^T(\mathbf{A}^T\mathbf{A})\mathbf{z} = \mathbf{z}^T\mathbf{A}^T\mathbf{A}\mathbf{z} \quad (2.0.1)$$

$$\Rightarrow \mathbf{z}^T(\mathbf{A}^T\mathbf{A})\mathbf{z} = (\mathbf{A}\mathbf{z})^T(\mathbf{A}\mathbf{z}) \quad (2.0.2)$$

$$\Rightarrow \mathbf{z}^T(\mathbf{A}^T\mathbf{A})\mathbf{z} = \|\mathbf{A}\mathbf{z}\|^2 \geq 0 \quad (2.0.3)$$

From (2.0.3), if $\mathbf{z} \neq 0$, $\mathbf{A}^T\mathbf{A}$ is positive definite, i.e

$$\|\mathbf{A}\mathbf{z}\|^2 > 0 \quad (2.0.4)$$

Again, $\mathbf{A}^T\mathbf{A}$ is positive semi-definite, if $\mathbf{z} = 0$,

$$\|\mathbf{A}\mathbf{z}\|^2 = 0 \quad (2.0.5)$$

Hence, $\mathbf{A}^T\mathbf{A}$ is positive semi-definite if the columns of \mathbf{A} are linearly dependent and $\mathbf{A}^T\mathbf{A}$ is positive definite if columns of \mathbf{A} are linearly independent.

Again,

$$(\mathbf{A}^T\mathbf{A})^T = (\mathbf{A}^T)(\mathbf{A}^T)^T = \mathbf{A}^T\mathbf{A} \quad (2.0.6)$$

Hence, $\mathbf{A}^T\mathbf{A}$ is symmetric. As every eigen value of a Hermitian matrix is real and every symmetric matrix is Hermitian then $\mathbf{A}^T\mathbf{A}$ (being a symmetric and hence Hermitian) has real eigen values.

Let λ be a (real) eigenvalue of $\mathbf{B} = \mathbf{A}^T\mathbf{A}$ and let \mathbf{x} be a corresponding real eigen-vector hence,

$$\mathbf{B}\mathbf{x} = \lambda\mathbf{x} \quad (2.0.7)$$

Multiplying \mathbf{x}^T in (2.0.7),

$$\mathbf{x}^T\mathbf{B}\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x} \quad (2.0.8)$$

$$\Rightarrow \mathbf{x}^T\mathbf{B}\mathbf{x} = \lambda \|\mathbf{x}\|^2 \quad (2.0.9)$$

$$\Rightarrow \mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = \lambda \|\mathbf{x}\|^2 \quad [\because \mathbf{B} = \mathbf{A}^T\mathbf{A}] \quad (2.0.10)$$

When $\mathbf{A}^T\mathbf{A}$ is positive definite (i.e columns of \mathbf{A} are linearly independent) then, the left hand side of (2.0.10) is positive as $\mathbf{A}^T\mathbf{A}$ is positive-definite and \mathbf{x} is a nonzero vector as it is an eigen-vector. Since $\|\mathbf{x}\|^2$ is positive, hence all eigen-values must be positive. Hence proved.