

Matrix Theory (EE5609) Assignment 13

Arkadipta De
MTech Artificial Intelligence
AI20MTECH14002

Abstract—This document proves if $\mathbf{AB} = \mathbf{I}$ then $\mathbf{BA} = \mathbf{I}$ And, given that both \mathbf{A} and \mathbf{B} are square matrices.

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_13

1 PROBLEM

Let \mathbf{A} and \mathbf{B} be 2×2 matrices such that $\mathbf{AB} = \mathbf{I}$. Prove that $\mathbf{BA} = \mathbf{I}$

2 SOLUTION

2.1 Solution

Let \mathbf{A} and \mathbf{B} such that,

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad (2.1.1)$$

$$\mathbf{B} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}. \quad (2.1.2)$$

Since $\mathbf{AB} = \mathbf{I}$ we have,

$$\det(\mathbf{A}) \neq 0 \quad (2.1.3)$$

$$\implies a_1a_4 - a_2a_3 \neq 0 \quad (2.1.4)$$

And,

$$\det(\mathbf{B}) \neq 0 \quad (2.1.5)$$

$$\implies b_1b_4 - b_2b_3 \neq 0 \quad (2.1.6)$$

Now, from $\mathbf{AB} = \mathbf{I}$ we have,

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.1.7)$$

Hence we have the following two system of linear equations,

$$a_1b_1 + a_2b_3 = 1 \quad (2.1.8)$$

$$a_3b_1 + a_4b_3 = 0 \quad (2.1.9)$$

$$a_1b_2 + a_2b_4 = 0 \quad (2.1.10)$$

$$a_3b_2 + a_4b_4 = 1 \quad (2.1.11)$$

Solving (2.1.8) and (2.1.9) we get,

$$b_1 = \frac{a_4}{a_1a_4 - a_2a_3} \quad (2.1.12)$$

$$b_3 = -\frac{a_3}{a_1a_4 - a_2a_3} \quad (2.1.13)$$

Again solving (2.1.10) and (2.1.11) we get,

$$b_2 = -\frac{a_2}{a_1a_4 - a_2a_3} \quad (2.1.14)$$

$$b_4 = \frac{a_1}{a_1a_4 - a_2a_3} \quad (2.1.15)$$

Hence we get,

$$\mathbf{BA} = \begin{pmatrix} \frac{a_4}{a_1a_4 - a_2a_3} & -\frac{a_2}{a_1a_4 - a_2a_3} \\ -\frac{a_3}{a_1a_4 - a_2a_3} & \frac{a_1}{a_1a_4 - a_2a_3} \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad (2.1.16)$$

$$= \begin{pmatrix} \frac{a_1a_4 - a_2a_3}{a_1a_4 - a_2a_3} & -\frac{a_2a_4 - a_2a_4}{a_1a_4 - a_2a_3} \\ -\frac{a_1a_3 - a_1a_3}{a_1a_4 - a_2a_3} & \frac{a_1a_4 - a_2a_3}{a_1a_4 - a_2a_3} \end{pmatrix} \quad (2.1.17)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.1.18)$$

$$\implies \mathbf{BA} = \mathbf{I} \quad (2.1.19)$$

Hence Proved.

2.2 Alternative Solution

Since $\mathbf{AB} = \mathbf{I}$ we have,

$$\det(\mathbf{A}) \det(\mathbf{B}) = \det(\mathbf{I}) \quad (2.2.1)$$

Hence (2.2.1) implies,

$$\det(\mathbf{A}) \neq 0 \quad (2.2.2)$$

$$\det(\mathbf{B}) \neq 0 \quad (2.2.3)$$

\mathbf{A} and \mathbf{B} both are 2×2 square matrices and from (2.2.2) and (2.2.3), both \mathbf{A} and \mathbf{B} are invertible.

Hence,

$$\mathbf{I} = \mathbf{B}\mathbf{B}^{-1} \quad (2.2.4)$$

$$= \mathbf{B}\mathbf{I}\mathbf{B}^{-1} \quad (2.2.5)$$

$$= \mathbf{B}(\mathbf{A}\mathbf{B})\mathbf{B}^{-1} \quad (2.2.6)$$

$$= \mathbf{B}\mathbf{A}(\mathbf{B}\mathbf{B}^{-1}) \quad (2.2.7)$$

$$= \mathbf{B}\mathbf{A} \quad (2.2.8)$$

Hence Proved.