

Matrix Theory (EE5609) Assignment 14

Arkadipta De
MTech Artificial Intelligence
AI20MTECH14002

Abstract—This document proves the existence of inverse of Hilbert Matrix.

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_14

1 PROBLEM

Prove that the following matrix is invertible and \mathbf{A}^{-1} has integer entries.

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

2 SOLUTION

Let \mathbf{A}_3 be 3×3 matrix i.e

$$\mathbf{A}_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \quad (2.0.1)$$

Now we find the inverse of the matrix \mathbf{A}_3 as follows,

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{pmatrix} \quad (2.0.2)$$

$$\begin{matrix} \xleftarrow{R_2=R_2-\frac{1}{2}R_1} \\ \xleftarrow{R_3=R_3-\frac{1}{3}R_1} \end{matrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{12} & \frac{1}{45} & -\frac{1}{3} & 0 & 1 \end{pmatrix} \quad (2.0.3)$$

$$\xleftarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{180} & \frac{1}{6} & -1 & 1 \end{pmatrix} \quad (2.0.4)$$

$$\begin{matrix} \xleftarrow{R_2=12R_2} \\ \xleftarrow{R_3=180R_3} \end{matrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 1 & 1 & -6 & 12 & 0 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (2.0.5)$$

$$\begin{matrix} \xleftarrow{R_2=R_2-R_3} \\ \xleftarrow{R_1=R_1-R_3} \end{matrix} \begin{pmatrix} 1 & \frac{1}{2} & 0 & -9 & 60 & -60 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (2.0.6)$$

$$\xleftarrow{R_1=R_1-\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{pmatrix} \quad (2.0.7)$$

Hence we see that \mathbf{A}_3 is invertible and the inverse contains integer entries and \mathbf{A}_3^{-1} is given by,

$$\mathbf{A}_3^{-1} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \quad (2.0.8)$$

Let, \mathbf{A}_4 be 4×4 matrix as follows,

$$\mathbf{A}_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \quad (2.0.9)$$

Now, expressing \mathbf{A}_4 using \mathbf{A}_3 we get,

$$\mathbf{A}_4 = \begin{pmatrix} \mathbf{A}_3 & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (2.0.10)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \quad (2.0.11)$$

$$d = \frac{1}{7} \quad (2.0.12)$$

Now assuming \mathbf{A}_4 has an inverse, then from (2.0.10), the inverse of \mathbf{A}_4 can be written using block matrix inversion as follows,

$$\mathbf{A}_4^{-1} = \begin{pmatrix} \mathbf{A}_3^{-1} + \mathbf{A}_3^{-1}\mathbf{u}x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} & -\mathbf{A}_3^{-1}\mathbf{u}x^{-1} \\ -x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} & x^{-1} \end{pmatrix} \quad (2.0.13)$$

where,

$$x = d - \mathbf{u}^T\mathbf{A}_3^{-1}\mathbf{u} \quad (2.0.14)$$

Now, the assumption of \mathbf{A}_4 being invertible will hold if and only if \mathbf{A}_3 is invertible, which has been proved in (2.0.8) and x from (2.0.14) is invertible or x is a nonzero scalar. We now prove that x is invertible as follows,

$$x = \frac{1}{7} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \end{pmatrix} \quad (2.0.15)$$

$$\Rightarrow x = \frac{1}{2800} \quad (2.0.16)$$

Hence, x is a scalar, hence x^{-1} exists and is given by,

$$x^{-1} = 2800 \quad (2.0.17)$$

Hence, \mathbf{A}_4 is invertible. Now putting the values of \mathbf{A}_3^{-1} , x^{-1} and \mathbf{u} we get,

$$\mathbf{A}_3^{-1} + \mathbf{A}_3^{-1}\mathbf{u}x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} = \begin{pmatrix} 16 & -120 & 240 \\ -120 & 1200 & -2700 \\ 240 & -2700 & 6480 \end{pmatrix} \quad (2.0.18)$$

$$-\mathbf{A}_3^{-1}\mathbf{u}x^{-1} = \begin{pmatrix} -140 \\ 1680 \\ -4200 \end{pmatrix} \quad (2.0.19)$$

$$x^{-1}\mathbf{u}^T\mathbf{A}_3^{-1} = \begin{pmatrix} -140 & 1680 & -4200 \end{pmatrix} \quad (2.0.20)$$

$$x^{-1} = 2800 \quad (2.0.21)$$

Putting values from (2.0.18), (2.0.19), (2.0.20) and (2.0.21) into (2.0.13) we get,

$$\mathbf{A}_4^{-1} = \begin{pmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{pmatrix} \quad (2.0.22)$$

Hence, from (2.0.22) we prove that, \mathbf{A}_4 is invertible and has integer entries.

Let \mathbf{A}_{n-1} be invertible with integer entries. Then we can represent \mathbf{A}_n as follows,

$$\mathbf{A}_n = \begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{u} \\ \mathbf{u}^T & d \end{pmatrix} \quad (2.0.23)$$

where,

$$\mathbf{u} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (2.0.24)$$

$$d = \frac{1}{2n-1} \quad (2.0.25)$$

Now assuming \mathbf{A}_n has an inverse, then from (2.0.23), the inverse of \mathbf{A}_n can be written using block matrix inversion as follows,

$$\mathbf{A}_n^{-1} = \begin{pmatrix} \mathbf{A}_{n-1}^{-1} + \mathbf{A}_{n-1}^{-1}\mathbf{u}x^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & -\mathbf{A}_{n-1}^{-1}\mathbf{u}x^{-1} \\ -x^{-1}\mathbf{u}^T\mathbf{A}_{n-1}^{-1} & x^{-1} \end{pmatrix} \quad (2.0.26)$$

where,

$$x = d - \mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u} \quad (2.0.27)$$

Now, the assumption of \mathbf{A}_n being invertible will hold if and only if \mathbf{A}_{n-1} is invertible, which has been assumed and x from (2.0.27) is invertible or x is a nonzero scalar. We now prove that x is invertible as follows,

$$x = \frac{1}{2n-1} - \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \dots & \frac{1}{2n-2} \end{pmatrix} \mathbf{A}_{n-1}^{-1} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2n-2} \end{pmatrix} \quad (2.0.28)$$

In equation (2.0.28) \mathbf{u} contains no negative or zero entries, again \mathbf{A}_{n-1}^{-1} has non zero integer entries, hence $\mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u}$ is a non zero scalar. Moreover d is not equal to $\mathbf{u}^T\mathbf{A}_{n-1}^{-1}\mathbf{u}$ hence in (2.0.28) x is non-zero scalar and invertible and hence it has an inverse. Hence \mathbf{A}_n is invertible, proved.