Matrix Theory (EE5609) Assignment 12

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Abstract—**This document proves that, each field of the characteristic zero contains a copy of the rational number field.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_12

1 Problem

Consider the system of equations $\mathbf{AX} = 0$ where

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix over the field F. Prove the following -

- If every entry of $\mathbf{A}$ is 0, then every pair $(x1, x2)$ is a solution of $\mathbf{AX} = 0$.
- If $ad - bc \neq 0$, then the system $\mathbf{AX} = 0$ has only the trivial solution $x1 = x2 = 0$
- If $ad - bc = 0$ and some entry of $\mathbf{A}$ is different from 0, then there is a solution (x_1^0, x_2^0) such that $(x1, x2)$ is a solution if and only if there is a scalar y such that $x_1 = yx_1^0$ and $x_2 = yx_2^0$

2 Solution

2.1 Solution 1

If every entry of $\mathbf{A}$ is 0 then the equation $\mathbf{AX} = 0$ becomes,

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{2.1.1}$$

$$\implies 0.x_1 + 0.x_2 = 0 \qquad \forall x_1, x_2 \in F \tag{2.1.2}$$

Hence proved, every pair (x_1, x_2) is a solution for the equation $\mathbf{AX} = 0$.

2.2 Solution 2

Given $ad - bc \neq 0$, we can perform row reduction on the augmented matrix of equation $\mathbf{AX}$=0 as follows,

$$\begin{pmatrix} a & b & x_1 \\ c & d & x_2 \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{x_1}{a} \\ c & d & x_2 \end{pmatrix} \tag{2.2.1}$$

$$\xleftrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{x_1}{a} \\ 0 & \frac{ad-bc}{a} & \frac{ax_2 - cx_1}{a} \end{pmatrix} \tag{2.2.2}$$

$$\xleftrightarrow{R_2 = \frac{a}{ad-bc}R_2} \begin{pmatrix} 1 & \frac{b}{a} & \frac{x_1}{a} \\ 0 & 1 & \frac{ax_2 - cx_1}{ad-bc} \end{pmatrix} \tag{2.2.3}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{b}{a}R_2} \begin{pmatrix} 1 & 0 & \frac{dx_1 - bx_2}{ad-bc} \\ 0 & 1 & \frac{ax_2 - cx_1}{ad-bc} \end{pmatrix} \tag{2.2.4}$$

Hence we get,

$$\frac{dx_1 - bx_2}{ad - bc} = 0 \tag{2.2.5}$$

$$\frac{ax_2 - cx_1}{ad - bc} = 0 \tag{2.2.6}$$

As $ad - bc \neq 0$ hence,

$$dx_1 - bx_2 = 0 \tag{2.2.7}$$

$$ax_2 - cx_1 = 0 \tag{2.2.8}$$

Solving (2.2.7) and (2.2.8) we get $x_1 = 0$ and $x_2 = 0$. Hence proved, the equation $\mathbf{AX}$=0 has only one trivial solution $x_1 = x_2 = 0$

2.3 Solution 3

Let, a is the nonzero entry of $\mathbf{A}$. Given $ad - bc = 0$, we can perform row reduction on $\mathbf{A}$ as follows,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \tag{2.3.1}$$

$$\xleftrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 0 \end{pmatrix} \qquad [\because ad - bc = 0] \tag{2.3.2}$$

Hence from (2.3.2), $\mathbf{AX} = 0$ if and only if

$$x_1 = -\frac{b}{a}x_2 \qquad [a \neq 0] \tag{2.3.3}$$

Letting $x_1^0 = -\frac{b}{a}$ and $x_2^0 = 1$ we get for $y = 1$,

$$x_1 = yx_1^0 \tag{2.3.4}$$

$$x_2 = yx_2^0 \tag{2.3.5}$$

which is a solution of the equation $\mathbf{AX} = 0$. An analogous argument can be given if we assume any other of the entries to be different from 0. Hence proved.