Matrix Theory (EE5609) Challenging Problem

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Abstract—This document proves that $A^{T}A$ has positive Multiplying x^{T} in (2.0.7), eigen values.

Download latex codes from

https://github.com/Arko98/EE5609/tree/master/ Challenge 5

1 Problem

Show that the eigen values of $A^{T}A$ are positive.

2 Proof

Let, **A** is an arbitrary $m \times n$ matrix. Now consider the matrix $A^{T}A$,

for any n dimensional vector z,

$$\mathbf{z}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{z} = \mathbf{z}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{z}$$
 (2.0.1)

$$\implies \mathbf{z}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{z} = (\mathbf{A}\mathbf{z})^{\mathrm{T}}(\mathbf{A}\mathbf{z})$$
 (2.0.2)

$$\implies \mathbf{z}^{\mathbf{T}}(\mathbf{A}^{\mathbf{T}}\mathbf{A})\mathbf{z} = ||\mathbf{A}\mathbf{z}||^2 \ge 0$$
 (2.0.3)

From (2.0.3), if $\mathbf{z} \neq 0$, $\mathbf{A}^{T}\mathbf{A}$ is positive definite, i.e

$$\|\mathbf{A}\mathbf{z}\|^2 > 0 \tag{2.0.4}$$

Again, $A^{T}A$ is positive semi-definite, if z = 0,

$$\|\mathbf{A}\mathbf{z}\|^2 = 0 \tag{2.0.5}$$

Hence, A^TA is positive semi-definite if the columns of A are linearly dependent and $A^{T}A$ is positive definite if columns of A are linearly dependent. Again,

$$(\mathbf{A}^{\mathsf{T}}\mathbf{A})^{\mathsf{T}} = (\mathbf{A}^{\mathsf{T}})(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}}\mathbf{A}$$
(2.0.6)

Hence, $A^{T}A$ is symmetric. As every eigen value of a Hermitian matrix is real and every symmetric matrix is Hermitian then $A^{T}A$ (being a symmetric and hence Hermitian) has real eigen values.

Let λ be a (real) eigenvalue of $\mathbf{B} = \mathbf{A}^{T}\mathbf{A}$ and let \mathbf{x} be a corresponding real eigen-vector hence,

$$\mathbf{B}\mathbf{x} = \lambda \mathbf{x} \tag{2.0.7}$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{B}\mathbf{x} = \lambda \mathbf{x}^{\mathbf{T}}\mathbf{x} \tag{2.0.8}$$

1

$$\implies \mathbf{x}^{\mathsf{T}}\mathbf{B}\mathbf{x} = \lambda \|\mathbf{x}\|^2 \tag{2.0.9}$$

$$\implies \mathbf{x}^{\mathbf{T}}(\mathbf{A}^{\mathbf{T}}\mathbf{A})\mathbf{x} = \lambda \|\mathbf{x}\|^2 [:: \mathbf{B} = \mathbf{A}^{\mathbf{T}}\mathbf{A}] \quad (2.0.10)$$

When $A^{T}A$ is positive definite (i.e columns of A are linearly independent) then, the left hand side of (2.0.10) is positive as $A^{T}A$ is positive-definite and x is a nonzero vector as it is an eigen-vector.

Also if A has linearly independent columns then A^TA will be invertible and hence a non-singular matrix, so $\|\mathbf{x}\|$ cannot be zero in (2.0.10). Since $\|\mathbf{x}\|^2$ is positive, hence all eigen-values must be positive. Hence proved.