Matrix Theory (EE5609) Assignment 17

Arkadipta De
MTech Artificial Intelligence
AI20MTECH14002

***Abstract*—This document proves the invertibility of a certain linear operator.**

All the codes for the figure in this document can be found at

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https://github.com/Arko98/EE5609/blob/master/
    Assignment_17
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1 Problem

Let T be a linear operator on the finite-dimensional space $\mathbb{V}$. Suppose there is a linear operator U on $\mathbb{V}$ such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when $\mathbb{V}$ is not finite-dimensional.

2 Solution

2.1 Proof

Let $T : \mathbb{V} \to \mathbb{V}$ be a linear operator, where $\mathbb{V}$ is a finite dimensional vectors space and $U : \mathbb{V} \to \mathbb{V}$ is also a linear operator such that,

$$TU = I \tag{2.1.1}$$

Where, I is an identity transformation. Now we know that linear transformations are functions. Hence,

$$TU = I \quad \text{is a function} \tag{2.1.2}$$
$$\implies I : \mathbb{V} \to \mathbb{V} \tag{2.1.3}$$

Such that $T(V) = V$. Defining $TU : \mathbb{V} \to \mathbb{V}$ to be a linear operator, we have

$$T[U(V_i)] = V_i \qquad [V_i \in \mathbb{V}] \tag{2.1.4}$$

Let $\mathbf{V_1}, \mathbf{V_2} \in \mathbb{V}$ then,

If $\mathbf{V_1} \neq \mathbf{V_2}$ then, $T[U(\mathbf{V_1})] \neq T[U(\mathbf{V_2})]$. Hence,

$$T \quad \text{must be one-one function} \tag{2.1.5}$$

Again, T is linear operator on finite dimensional vector space. Hence,

$$T \quad \text{must be onto function} \tag{2.1.6}$$

From (2.1.5) and (2.1.6) we get,

$$T \quad \text{is invertible function} \tag{2.1.7}$$

From (2.1.7) we know,

$$TT^{-1} = I \tag{2.1.8}$$

Where T^{-1} is an inverse function of linear operator T. Hence,

$$TT^{-1} = I = TU \tag{2.1.9}$$
$$\implies T^{-1}(TT^{-1}) = T^{-1}(TU) \tag{2.1.10}$$
$$\implies T^{-1}(I) = IU \tag{2.1.11}$$
$$\implies T^{-1} = U \tag{2.1.12}$$

Hence from (2.1.7) and (2.1.12) it is proven that T is invertible and $T^{-1} = U$

2.2 Example

Let D be the differential operator $D : \mathbb{V} \to \mathbb{V}$ where $\mathbb{V}$ is a space of polynomial functions in one variable over $\mathbb{R}$. Hence,

$$D(c_0 + c_1x + \cdots + c_nx^n) = c_1 + c_2'x + \cdots + c_n'x^{n-1} \tag{2.2.1}$$

And, $U : \mathbb{V} \to \mathbb{V}$ is another linear operator such that,

$$U(c_0 + c_1x + \cdots + c_nx^n) = c_0x + c_1\frac{x^2}{2} + \cdots + c_n\frac{x^{n+1}}{n+1} \tag{2.2.2}$$

Then $UD : \mathbb{V} \to \mathbb{V}$ is a linear operator such that,

$$UD(c_0 + c_1 x + \cdots + c_n x^n) \tag{2.2.3}$$

$$= U[D(c_0 x + c_1 \frac{x^2}{2} + \cdots + c_n \frac{x^{n+1}}{n+1})] \tag{2.2.4}$$

$$= U[c_1 + c_2' x + \cdots + c_n' x^{n-1}] \tag{2.2.5}$$

$$= c_1 x + c_2 \frac{x^2}{2} + \cdots + c_n \frac{x^n}{n} \tag{2.2.6}$$

Hence, from (2.2.6),

$$UD \neq I \tag{2.2.7}$$

Again, $DU : \mathbb{V} \to \mathbb{V}$ is a linear operator such that,

$$DU(c_0 + c_1 x + \cdots + c_n x^n) \tag{2.2.8}$$

$$= D[U(c_0 x + c_1 \frac{x^2}{2} + \cdots + c_n \frac{x^{n+1}}{n+1})] \tag{2.2.9}$$

$$= D[c_0 x + c_1 \frac{x^2}{2} + \cdots + c_n \frac{x^{n+1}}{n+1}] \tag{2.2.10}$$

$$= c_0 + c_1 \frac{2x^2}{2} + \cdots + c_n \frac{(n+1)x^n}{n+1} \tag{2.2.11}$$

$$= c_0 + c_1 x + \cdots + c_n x^n \tag{2.2.12}$$

Hence, from (2.2.12),

$$DU = I \tag{2.2.13}$$

Hence, from (2.2.7) and (2.2.13), D is not invertible.