Matrix Theory (EE5609) Assignment 13

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***Abstract*—This document proves if AB = I then BA = I given that both A and B are square matrices.**

All the codes for the figure in this document can be found at

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https://github.com/Arko98/EE5609/blob/master/
    Assignment_13
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1 Problem

Let $\mathbf{A}$ and $\mathbf{B}$ be 2×2 matrices such that $\mathbf{AB} = \mathbf{I}$. Prove that $\mathbf{BA} = \mathbf{I}$

2 Solution

2.1 Solution

Let $\mathbf{A}$ and $\mathbf{B}$ such that,

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{2.1.1}$$

$$\mathbf{B} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}. \tag{2.1.2}$$

Since $\mathbf{AB} = \mathbf{I}$ we have,

$$\det(\mathbf{A}) \neq 0 \tag{2.1.3}$$

$$\implies a_1a_4 - a_2a_3 \neq 0 \tag{2.1.4}$$

And,

$$\det(\mathbf{B}) \neq 0 \tag{2.1.5}$$

$$\implies b_1b_4 - b_2b_3 \neq 0 \tag{2.1.6}$$

Now, from $\mathbf{AB} = \mathbf{I}$ we have,

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1.7}$$

Hence we have the following two system of linear equations,

$$a_1b_1 + a_2b_3 = 1 \tag{2.1.8}$$

$$a_3b_1 + a_4b_3 = 0 \tag{2.1.9}$$

And,

$$a_1b_2 + a_2b_4 = 0 \tag{2.1.10}$$

$$a_3b_2 + a_4b_4 = 1 \tag{2.1.11}$$

Solving (2.1.8) and (2.1.9) we get,

$$b_1 = \frac{a_4}{a_1a_4 - a_2a_3} \tag{2.1.12}$$

$$b_3 = -\frac{a_3}{a_1a_4 - a_2a_3} \tag{2.1.13}$$

Again solving (2.1.10) and (2.1.11) we get,

$$b_2 = -\frac{a_2}{a_1a_4 - a_2a_3} \tag{2.1.14}$$

$$b_4 = \frac{a_1}{a_1a_4 - a_2a_3} \tag{2.1.15}$$

Hence we get,

$$\mathbf{BA} = \begin{pmatrix} \frac{a_4}{a_1a_4-a_2a_3} & -\frac{a_2}{a_1a_4-a_2a_3} \\ -\frac{a_3}{a_1a_4-a_2a_3} & \frac{a_1}{a_1a_4-a_2a_3} \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{2.1.16}$$

$$= \begin{pmatrix} \frac{a_1a_4-a_2a_3}{a_1a_4-a_2a_3} & \frac{a_2a_4-a_2a_4}{a_1a_4-a_2a_3} \\ \frac{a_1a_3-a_1a_3}{a_1a_4-a_2a_3} & \frac{a_1a_4-a_2a_3}{a_1a_4-a_2a_3} \end{pmatrix} \tag{2.1.17}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1.18}$$

$$\implies \mathbf{BA} = \mathbf{I} \tag{2.1.19}$$

Hence Proved.

2.2 Alternative Solution

Since $\mathbf{AB} = \mathbf{I}$ we have,

$$\det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{I}) \tag{2.2.1}$$

Hence (2.2.1) implies,

$$\det(\mathbf{A}) \neq 0 \tag{2.2.2}$$

$$\det(\mathbf{B}) \neq 0 \tag{2.2.3}$$

$\mathbf{A}$ and $\mathbf{B}$ both are 2×2 square matrices and from (2.2.2) and (2.2.3), both $\mathbf{A}$ and $\mathbf{B}$ are invertible.

Hence,

$$\mathbf{I} = \mathbf{B}\mathbf{B}^{-1} \tag{2.2.4}$$
$$= \mathbf{B}\mathbf{I}\mathbf{B}^{-1} \tag{2.2.5}$$
$$= \mathbf{B}(\mathbf{A}\mathbf{B})\mathbf{B}^{-1} \tag{2.2.6}$$
$$= \mathbf{B}\mathbf{A}(\mathbf{B}\mathbf{B}^{-1}) \tag{2.2.7}$$
$$= \mathbf{B}\mathbf{A} \tag{2.2.8}$$

Hence Proved.