Matrix Theory (EE5609) Assignment 4

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Abstract—**This document solves an equation on matrix.**

The code for the solution of this problem can be found at

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https://github.com/Arko98/EE5609/blob/master/
    Assignment_4/Codes/Solution.py
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1 Problem

If $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$, prove that $\mathbf{A^3}-6\mathbf{A^2}+7\mathbf{A}+2\mathbf{I}=0$

2 Solution

The equation in the problem can be modified as follows

$$\mathbf{A^3} - 6\mathbf{A^2} + 7\mathbf{A} + 2\mathbf{I} = 0 \tag{2.0.1}$$

$$\implies \mathbf{A^2}(\mathbf{A} - \mathbf{6I}) + \mathbf{7A} + \mathbf{2I} = 0 \tag{2.0.2}$$

So we need to prove equation 2.0.2. Now, at first we calculate the value of $\mathbf{A^2}$ as follows

$$\mathbf{A^2} = \mathbf{A} \cdot \mathbf{A} \tag{2.0.3}$$

$$\implies \mathbf{A^2} = \mathbf{A^T}\mathbf{A} \tag{2.0.4}$$

$$\implies \mathbf{A^2} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} \tag{2.0.5}$$

$$\implies \mathbf{A^2} = \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix} \tag{2.0.6}$$

Next we calculate, $\mathbf{A}-6\mathbf{I}$ where I is identity matrix of order 3, as follows

$$\mathbf{A} - 6\mathbf{I} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.7}$$

$$\implies \mathbf{A} - 6\mathbf{I} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} \tag{2.0.8}$$

$$\implies \mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -5 & 0 & 2 \\ 0 & -4 & 1 \\ 2 & 0 & -3 \end{pmatrix} \tag{2.0.9}$$

Now we compute $\mathbf{A^2}(\mathbf{A} - 6\mathbf{I})$ by putting values of $\mathbf{A^2}$ from equation 2.0.6 and value of $\mathbf{A} - 6\mathbf{I}$ from 2.0.9 as follows

$$\mathbf{A^2} \cdot (\mathbf{A} - 6\mathbf{I}) = (\mathbf{A^2})^{\mathbf{T}}(\mathbf{A} - 6\mathbf{I}) \tag{2.0.10}$$

$$\implies \mathbf{A^2} \cdot (\mathbf{A} - 6\mathbf{I}) = \begin{pmatrix} 5 & 2 & 8 \\ 0 & 4 & 0 \\ 8 & 5 & 13 \end{pmatrix}\begin{pmatrix} -5 & 0 & 2 \\ 0 & -4 & 1 \\ 2 & 0 & -3 \end{pmatrix} \tag{2.0.11}$$

$$\implies \mathbf{A^2} \cdot (\mathbf{A} - 6\mathbf{I}) = \begin{pmatrix} -9 & 0 & -14 \\ 0 & -16 & -7 \\ -14 & 0 & -23 \end{pmatrix} \tag{2.0.12}$$

Next we compute $7\mathbf{A} + 2\mathbf{I}$ as follows

$$7\mathbf{A} + 2\mathbf{I} = 7\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.0.13}$$

$$\implies 7\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 9 & 0 & 14 \\ 0 & 16 & 7 \\ 14 & 0 & 23 \end{pmatrix} \tag{2.0.14}$$

Now putting the value of $\mathbf{A^2}(\mathbf{A} - 6\mathbf{I})$ from equation 2.0.12 and value of $7\mathbf{A} + 2\mathbf{I}$ from equation 2.0.14

into the left hand side of equation 2.0.2 we get

$$\mathbf{A^2(A-6I)+7A+2I}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \quad (2.0.15)$$

$$\implies \mathbf{A^2(A-6I)+7A+2I}=0 \quad (2.0.16)$$

Thus from equation 2.0.16 we arrive at the right hand side of equation 2.0.2, Hence proved.

3 Problem 2

Find the characteristic equation of the matrix $\mathbf{A}$ $=\begin{pmatrix}1&0&2\\0&2&1\\2&0&3\end{pmatrix}$

4 Solution

For a general order k square matrix $\mathbf{A}$, the characteristic equation in variable λ is defined by

$$\det(A-\lambda I)=0 \quad (4.0.1)$$

where $\mathbf{I}$ is identity matrix of order k. Now we compute $\lambda\mathbf{I}$ as follows

$$\lambda\mathbf{I}=\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad (4.0.2)$$

$$\implies \lambda\mathbf{I}=\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix} \quad (4.0.3)$$

$$\implies \lambda\det I=\begin{vmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{vmatrix} \quad (4.0.4)$$

So in equation 4.0.1, putting the values from 4.0.4 we get,

$$\det(A-\lambda I)=0 \quad (4.0.5)$$

$$\implies \det A-\lambda\det I=0 \quad (4.0.6)$$

$$\implies \begin{vmatrix}1&0&2\\0&2&1\\2&0&3\end{vmatrix}-\begin{vmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{vmatrix}=0 \quad (4.0.7)$$

$$\implies \begin{vmatrix}1-\lambda&0&2\\0&2-\lambda&1\\2&0&3-\lambda\end{vmatrix}=0 \quad (4.0.8)$$

Hence, equation 4.0.8 is the required characteristic equation of matrix $\mathbf{A}$. Further expanding the determinant from the equation 4.0.8 we get the following polynomial equation of λ

$$(1-\lambda)((2-\lambda)(3-\lambda)-1)-4(2-\lambda)=0 \quad (4.0.9)$$

$$\implies \lambda^3-6\lambda^2+6\lambda+3=0 \quad (4.0.10)$$

Hence, equation 4.0.10 is the required characteristic equation of matrix $\mathbf{A}$ for λ.