Matrix Theory (EE5609) Assignment 11

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***Abstract*—This document proves that, each field of the characteristic zero contains a copy of the rational number field.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_11

1 Problem

Prove that, each field of the characteristic zero contains a copy of the rational number field.

2 Solution

The characteristic of a field is defined to be the smallest number of times one must use the field's multiplicative identity (1) in a sum to get the additive identity. If this sum never reaches the additive identity (0), then the field is said to have characteristic zero.
Let $\mathbb{Q}$ be the rational number field. Hence,

$$0 \in \mathbb{Q} \qquad \text{[Additive Identity]} \tag{2.0.1}$$
$$1 \in \mathbb{Q} \qquad \text{[Multiplicative Identity]} \tag{2.0.2}$$

As addition is defined on $\mathbb{Q}$ hence we have,

$$1 \neq 0 \tag{2.0.3}$$
$$1 + 1 = 2 \neq 0 \tag{2.0.4}$$

And so on,

$$1 + 1 + \cdots + 1 = n \neq 0 \tag{2.0.5}$$

From the definition of characteristic of a field and from (2.0.3), (2.0.4) and so on upto (2.0.5), the rational number field has characteristic 0.
Now, let F be a field of characteristic 0. Hence we have,

$$0 \in F \qquad \text{[Additive Identity]} \tag{2.0.6}$$
$$1 \in F \qquad \text{[Multiplicative Identity]} \tag{2.0.7}$$

Since the characteristic of F is zero hence,

$$1 \neq 1 + 1 \neq 1 + 1 + \cdots \neq 0 \tag{2.0.8}$$

As F is a field, it is closed under addition. Hence for addition of n number of 1 we have,

$$1 + 1 + \cdots + 1 = n \in F \tag{2.0.9}$$

And,

$$n \neq 0 \tag{2.0.10}$$

If $\mathbb{Z}$ is the set of integers then we have,

$$\mathbb{Z} \subseteq F \tag{2.0.11}$$

As F is a field, every element in F will have a multiplicative inverse, thus,

$$\frac{1}{n} \in F \tag{2.0.12}$$

Also, F is closed under multiplication and thus,

$$\forall m, n \in \mathbb{Z} \quad \text{and } n \neq 0 \tag{2.0.13}$$
$$m \cdot \frac{1}{n} \in F \tag{2.0.14}$$
$$\implies \frac{m}{n} \in F \tag{2.0.15}$$

Hence, if $\mathbb{Q}$ is the rational number field then,

$$\mathbb{Q} \subseteq F \tag{2.0.16}$$

Hence, proved that field F contains a copy of the rational number field.