Matrix Theory (EE5609) Assignment 22

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***Abstract*—This document solves problem on polynomial of a given square matrix.**

All the codes for the figure in this document can be found at

https://github.com/Arko98/EE5609/blob/master/Assignment_22

1 Problem

Let n be a positive integer and $\mathbb{F}$ be a field. Suppose $\mathbf{A}$ is an $n \times n$ matrix over field $\mathbb{F}$ and $\mathbf{P}$ is an invertible $n \times n$ matrix over field $\mathbb{F}$. If f is any polynomial over $\mathbb{F}$, prove that,

$$f(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = \mathbf{P}^{-1}f(\mathbf{A})\mathbf{P}$$

2 Solution

First we observe the following,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^2 = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \tag{2.0.1}$$
$$= \mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} \tag{2.0.2}$$

Let the (2.0.2) be true for a positive integer m i.e,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^m = \mathbf{P}^{-1}\mathbf{A}^m\mathbf{P} \tag{2.0.3}$$

Now for the integer $m+1$ we get from (2.0.3),

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^{m+1} = (\mathbf{P}^{-1}\mathbf{A}^m\mathbf{P})(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) \tag{2.0.4}$$
$$= \mathbf{P}^{-1}\mathbf{A}^{m+1}\mathbf{P} \tag{2.0.5}$$

From (2.0.3) and (2.0.5) we get for any positive integer n,

$$(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} \tag{2.0.6}$$

Again we have,

$$\mathbf{P}^{-1}\mathbf{P} = \mathbf{I} \tag{2.0.7}$$

The general form of polynomial $f(\mathbf{A})$ is defined as,

$$f(\mathbf{A}) = a_0 + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n \tag{2.0.8}$$

Now we have,

$$f(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) = a_0 + a_1(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}) + \cdots + a_n(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n \tag{2.0.9}$$
$$= (\mathbf{P}^{-1}a_0\mathbf{P}) + (\mathbf{P}^{-1}a_1\mathbf{A}\mathbf{P}) + \cdots + (\mathbf{P}^{-1}a_n\mathbf{A}\mathbf{P})^n \tag{2.0.10}$$
$$= (\mathbf{P}^{-1}a_0\mathbf{P}) + (\mathbf{P}^{-1}a_1\mathbf{A}\mathbf{P}) + \cdots + (\mathbf{P}^{-1}a_n\mathbf{A}^n\mathbf{P}) \tag{2.0.11}$$
$$= \mathbf{P}^{-1}(a_0 + a_1\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_n\mathbf{A}^n)\mathbf{P} \tag{2.0.12}$$
$$= \mathbf{P}^{-1}f(\mathbf{A})\mathbf{P} \quad \text{[From (2.0.8)]} \tag{2.0.13}$$

Hence proved (2.0.13)