A Study on Two Graph Problems: Upward Pointset Embeddability Testing and Minimum Consistent Subset

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***Abstract*—We study two combinatorial problems on graphs, namely the upward pointset embeddings (UPSEs) problem and the minimum consistent subsets (MCSs) problem. Let G be a directed planar graph and let $\mathcal{S} \subset \mathbb{R}^2$ be a pointset with $|\mathcal{S}| = |V(G)|$. A *UPSE of G on $\mathcal{S}$* is an upward planar straight-line drawing of G that maps the vertices of G to the points of $\mathcal{S}$. We consider the problem of testing the existence of an UPSE of G on S (UPSE Testing), and prove that UPSE Testing is NP-complete even for maximal outerplanar graphs. In the *Minimum Consistent Subset (MCS)* problem, we are given a simple connected undirected graph $G = (V, E)$ in which each vertex is colored by one of the possible colors $\{c_1, c_2, \ldots, c_k\}$. The goal is to find a subset $C \subseteq V$ of minimum cardinality such that, for every vertex $v \in V$, at least one of its *nearest neighbors*—that is, a vertex in C at minimum hop-distance from v—has the same color as v. We extend this line of research to cycle graphs, a critical model in network design and periodic systems.**

***Index Terms*—Upward Pointset Embedding (UPSE), Maximal Outerplanar Graph, Minimum Consistent Subset (MCS), Cycle Graph**

I. Introduction

Our thesis is divided into two parts. The first part examines *Upward Pointset Embeddability (UPSE) Testing*, which determines if a directed graph G can be drawn upward-planar on a given pointset S. In particular, we focus on *directed maximal outerplanar graphs*. Pointset embedding questions form a classic area in Graph Drawing, encompassing both undirected and directed versions. Given an n-vertex upward planar graph G and a set S of n points in the plane, an *upward pointset embedding (UPSE) of G on S* is an upward planar drawing of G with vertices mapped to points in S (with no two points sharing the same horizontal line) and edges as straight-line segments. The UPSE Testing problem asks whether G admits such an embedding on S. with edges strictly increasing in the y-direction. UPSE Testing is known to be NP-complete for planar st-graphs [1] and polynomial-time solvable when S is in convex position [1]. In this paper, we extend the intractability to *maximal outerplanar* graphs, proving NP-completeness under these conditions.

The second part deals with the *Minimum Consistent Subset (MCS)* problem: given a connected simple undirected graph G, each vertex colored from $\{c_1, \ldots, c_k\}$, we seek a smallest subset $C \subseteq V$ ensuring that every vertex v has a nearest neighbor in C (by hop-distance) with the same color as v. Although the decision version of MCS is NP-complete even for planar graphs, recent research has produced polynomial-time algorithms on certain structured graphs (e.g., paths, trees, spiders, and combs) as shown by Dey *et al.* [2] and Biniaz *et al.* [3]. We extend these to *cycle graphs*, modeling cyclic dependencies via an *overlay graph* that reduces MCS to a shortest s–t path. This leverages the cycle's symmetry and avoids dynamic programming, broadening the algorithmic approaches to classification and clustering problems.

II. Preliminaries

An *outerplanar graph* is a planar graph that admits a drawing in the plane such that all vertices lie on the boundary of a single, unbounded face. Equivalently, an outerplanar graph is one that does not contain K_4 (the complete graph on four vertices) or $K_{2,3}$ (the complete bipartite graph on two and three vertices) as a minor. A *maximal outerplanar graph* is an outerplanar graph to which no additional edge can be inserted while preserving outerplanarity. Every face of such graphs is a triangle.

A drawing of a directed graph is *straight-line* if each edge is represented by a straight-line segment, it is *planar* if no two edges cross, and it is *upward* if every edge is drawn as a Jordan arc monotonically increasing in the y-direction from tail to head. A digraph that admits an upward planar drawing is an *upward planar graph*. An *Upward Pointset Embedding (UPSE)* of an upward planar graph G on a pointset S is an upward planar straight-line drawing of G that maps each vertex of G to a point in S.

A *cycle* is a graph where each vertex has degree 2, forming a closed path, and we define *runs* as maximal sequences of consecutive vertices sharing the same color. A *valid pair* is formed by one vertex from each of two neighboring runs so as to maintain consistency for the entire section between them, including the chosen vertices. To handle class transitions and cyclic dependencies, we introduce an *overlay graph*, which augments G with edges or vertices reflecting possible color shifts around the cycle. From these notions, we observe that a cycle whose vertices all share the same color has an MCS of size 1 (Observation 1). Moreover, $|\text{MCS}| \geq |\text{Runs}|$ because each run is necessarily bounded by runs of different colors (Observation 2).

III. The NP-Completeness of UPSE Testing for Maximal Outerplanar Graph

The membership in NP is obvious, since one can guess a vertex-to-point assignment and check it in polynomial time. So we start by showing a reduction from 3-PARTITION to UPSE on a directed maximal outerplanar graph. An instance of 3-PARTITION has $3b$ integers $A = \{a_1, \ldots, a_{3b}\}$ where

$$\sum_{i=1}^{3b} a_i \;=\; bB \quad \text{and} \quad \tfrac{B}{4} \;\leq\; a_i \;\leq\; \tfrac{B}{2} \quad (1 \;\leq\; i \;\leq\; 3b).$$

The 3-PARTITION problem asks whether A can be partitioned into b subsets $A_1, \ldots, A_b$, each with three integers, so that the sum of the integers in each set A_i is B. Since 3-PARTITION is strongly NP-hard [**?**], we may assume that B is bounded by a polynomial function of b. Given an instance A of 3-PARTITION, we show how to construct in polynomial time, specifically $O(b \cdot B)$, an equivalent instance (G, S) of *UPSE Testing*.

We begin by selecting an initial reference point p_0 at the origin $(0, 0)$. From p_0, we draw $2b + 2$ upward rays $\rho_1, \rho_2, \ldots, \rho_{2b+2}$, each forming angles $\alpha_1, \ldots, \alpha_{2b+2}$ with the x-axis so that $\frac{3\pi}{4} > \alpha_1 > \alpha_2 > \cdots > \alpha_{2b+2} > \frac{\pi}{4}$. For every $1 \leq i < b+1$, the angular gap between ρ_{2i} and ρ_{2i-1} is smaller than that between ρ_{2i} and ρ_{2i+1}. A line ℓ of positive slope, less than $\pi/4$, intersects all these rays at distinct points $p_1, p_2, \ldots, p_{2b+2}$. Next, for each even index $2i$ with $1 \leq i \leq b+1$, we place

$$t_i \;=\; \left(\tfrac{x(p_{2i}) + x(p_{2i-1})}{2},\; 100 \cdot y\big(p_{2i}\big)\right),$$

and join p_{2i-1} to t_i and p_{2i} to t_i by straight segments. Then, for every $i = 1, \ldots, b+1$, we place six points $q_{i,1}, q_{i,2}, \ldots, q_{i,6}$ along a non-horizontal, positively sloped segment s_i entirely contained within the quadrilateral bounded by $t_i, t_{i+1}, p_{2i+1}, p_{2i}$, ensuring that no two points on different segments s_i and s_k share the same y-coordinate. Finally, for each $j = 1, \ldots, b$, we place B distinct points along another non-horizontal segment ℓ_j, which also lies inside the quadrilateral bounded by $t_j, t_{j+1}, p_{2j+1}, p_{2j}$, keeping its points strictly above all points on s_j and avoiding any shared y-coordinates across the segments ℓ_i and ℓ_k. Altogether, these placements yield a set S of cardinality 1 + (2b + 2) + 6b + (B · b) + (b + 1) = n.

This reduction is the main ingredient of the following theorem, whose proof is omitted in this extended abstract.

Theorem 1. *UPSE Testing is NP-hard even for directed maximal outerplanar graphs.*

IV. MCS of a Cycle Graph

To find the MCS of a cycle, we construct an *overlay graph* $G' = (V', E')$, which is directed and weighted. Specifically, V' contains all the original vertices of V, plus d dummy vertices, where d equals the number of runs. For each vertex, we add directed edges of weight 0 to all *valid pair* vertices in the neighboring run (in the clockwise direction). Additionally,

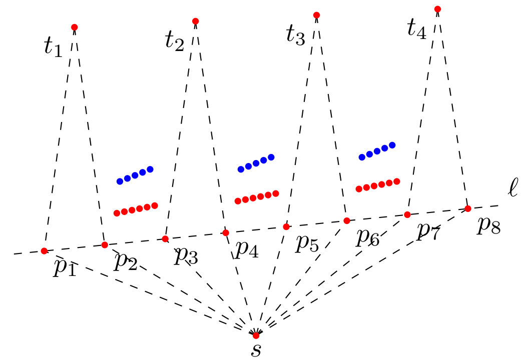
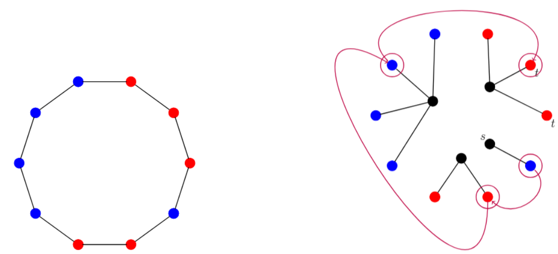


Fig. 1: An illustration of the pointset S for the UPSE of a maximal outerplanar graph of Theorem 1.

each dummy vertex is connected by bi-directed edges of weight 1 to the corresponding run vertices.



(a) (b)

Fig. 2: (a) A cycle graph with 2 colors, and (b) an MCS of the cycle based on the overlay graph.

Next, we designate the dummy vertex of a run as *source* (s) and mark a starting vertex in that run. For the marked vertex, we locate its valid pair vertices in the *neighboring run* in the counterclockwise direction, label them as t, and then run a shortest s–t path algorithm, which operates in $O(n)$ time because of the edge-weight structure. From the resulting path, we extract the desired subset, which constitutes an MCS of the cycle.

This leads us to the following theorem, whose proof and detailed analysis of correctness and time complexity is omitted in this extended abstract.

Theorem 2. *A Minimum Consistent Subset (MCS) of a cycle can be determined in $O(n^2)$ time complexity.*

References

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