Pigeonhole Principle: Torture Packet

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> Everybody wants happiness, nobody wants pain, but you can't have a rainbow without a little rain. Now suffer.
>
> — Lawrence

Figure 1: 9 pigeons fit nicely into 9 holes but... where does the 10^{th} guy go? ;w;

§1 Definition

Suppose we have n pigeons and k holes, or just n objects and k boxes. If $n > k$, then there exists a hole/box with 2 or more pigeons/objects. In general, it is guaranteed that at least one box has $\left\lfloor \frac{n}{k} \right\rfloor$ objects or more.

§2 Problems

Remember the steps: identify the pigeons, identify the holes. Figure out how to apply the pigeonhole principle. The problems are ordered from easy to difficult, labeled with a difficulty rating from 0 to 9 as "D#". Goal: Solve at problems 1 to 5, then pick 3 more to solve as a table.

1. (D0.5) There are 13 pigeons and 6 holes. The hole with the most pigeons has n of them. What's the minimum value of n?
2. (D1: Handshake Theorem [1]) Prove that in any graph with two or more vertices, there are two vertices with the same degree.
3. (D1) Prove that if you pick 17 integers, you can always choose 5 such that the sum will be divisible by 5.
4. (D1.5) Prove that if you put 9 points in a diamond (parallelogram with equal sides) of side length n, then there exists a pair of points such that they are equal to or less than length $n/2$ apart.
5. (D2: IMO '72, P1) Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.
6. (D2: Folklore) In any group of 6 people, there will always be a group of 3 mutual friends or 3 mutual strangers.
7. (D2.5) Suppose that all the integer coordinates on the Cartesian plane ($\mathbb{Z}^2$) have been conquered by n players. Prove that someone owns 4 points that form a rectangle.
   a) (D5+) Prove *or* disprove that someone owns 4 points that form a square.
8. (D3: Putnam '02, A2) Prove that if we put 5 points on a sphere, then at least 4 points lie on the same closed hemisphere.
9. (D3) Prove that every rational number has an eventually repeating decimal expansion.
10. (D3.5) What is the size of the largest subset $\mathcal{A}$ of $\{1, 2, \dots, 2n\}$ such that for all $a, b \in \mathcal{A}$, $a \nmid b$.
11. (D3.5: Japan 1997) Prove that among any 10 points in a circle of diameter 5 there exist at least two points at a distance less than 2 from each other.
12. (D4) The prime factorizations of $n + 1$ positive integers $x_1, x_2, \dots, x_{n+1}$ only involve n primes $p_1, p_2, \dots, p_n$. Prove that there exists a nonempty subset of $\{x_1, x_2, \dots, x_{n+1}\}$ whose elements multiply to a perfect square.

[1] who knows, idk... either way it's a classic

13. (D4: Personal Problem Collection) Pick a set S_n of n integers. Prove that there exists a nonempty subset such that the sum of all elements in it is divisible by n.

14. (D5: Dirichlet's Approximation Theorem) Let a be irrational. Show that there exists infinitely many rationals $r = p/q$ where $(p,q) = 1$ such that $|a - r| < q^{-2}$.

15. (D5) Use Dirichlet approximation to prove that there are infinitely many $n \in \mathbb{Z}$ where
$$\frac{1}{n^2 |\sin n|^2} > \frac{1}{(1+\pi)^2}.$$

16. (D5) Prove that for any positive integer n, there exists a fibonacci number divisible by n.
    a) (D7) Corollary: Prove that for primes p, $f_{p-\left(\frac{p}{5}\right)} \equiv 0 \mod p$. Note: $\left(\frac{p}{5}\right)$ refers to the Legendre symbol, and f_n is the nth fibonacci number.

17. (D5: Erdós-Szekeres) For all r, s, any sequence of distinct real numbers with at least length $(r-1)(s-1)+1$, then it contains a monotonically increasing subsequence of length r or monotonically decreasing subsequence of length s.

18. (D6: Bolzano-Weierstrass) If (x_n) is a sequence of real numbers lying in a bounded interval, then (x_n) contains a Cauchy subsequence.

19. (D6.5: MathSE Noud) Let $x_1, x_2, \dots, x_7 \in \mathbb{R}$. Then there are distinct positive integers $i, j \le 7$ such that
$$0 \le \frac{x_i - x_j}{1 + x_i x_j} \le \frac{1}{\sqrt{3}}.$$

20. (D8: Ramsey's Theorem) For all $c, m \ge 2$, there exists $n \ge m$ such that every c-coloring of K_n has a monochromatic K_m.

21. (D8.5: Van der Waerden's Theorem) For any $r, k \in \mathbb{Z}^+$, there is some number N such that if each of the numbers $\{1, 2, 3, \dots, N\}$ are assigned one of r distinct colors, then there are at least k integers in arithmetic progression whose elements are of the same color.

§3 Solutions

1. Directly from the pigeonhole principle - answer is $\boxed{3}$.

2. Suppose there are n vertices. If one of the vertices has a degree of 0, then that implies that the maximum degree possible is $n-2$. By the pigeonhole principle, $n-1$ vertices have a degree of 1 to $n-2$. Here, the pigeons are the vertices and the pigeonholes are the possible degrees. This implies that at least two vertices have the same degree. On the other hand, if all of the vertices are connected to at least one other vertex, we apply the pigeonhole principle again to see that the degrees range from 1 to $n-1$. By the same way above, the problem is proved. We do not need to consider the case if 2 or more points are completely disconnected, as that simply satisfies the theorem.

3. 17 is not a tight bound! Actually, I'm pretty sure 9 is. Consider the 17 integers modulo 5. Notice that $0+1+2+3+4 = 10 \equiv 0 \mod 5$. We want to avoid this! The notion of the pigeonhole principle is similar to the idea of "overflowing" the maximum capacity of sets. Suppose we have 4 numbers that are equivalent to $0, 1, 2, 3$ modulo 5. Then we see that it is impossible to pick a 17th integer such that there does not exist a subset of 5 integers that has elements that sum to a multiple of 5.

4. The 9 points are the pigeons, the pigeonholes are if you split the diamond into 8 equilateral triangles. It follows from there.

5. There are a total of $2^{10} - 2$ subsets of the set of ten distinct two-digit numbers that we might choose to consider. Naturally, this excludes the $\emptyset$ set its complement. However, the possible sums of elements of these subsets ranges from 10 to 855 (sum of integers from 91 to 99). This implies that there are more pigeons (sets) than pigeonholes (sums of elements). From this, we see that we can select some subsets P, Q such that they have the same elements sum. To make them disjoint, simply subtract $P \cap Q$ from P and Q to obtain the desired result.

6. If we wanted to do this WITHOUT pigeonhole, consider the complete graph K_6. Color all the edges either red or blue. Then there exists a blue or red triangle, by Ramsey's Theorem. If we want to do this with pigeonhole, consider any point. Let's draw K_6 again, and color all edges connected to this particular point red or blue. Red represents a friendship. Blue represents that they do not know each other. By the pigeonhole principle, there has to be 3 red or 3 blue (or more). Suppose that our point has 3 red edges. Let's trace to any of 3 points that share a red edge with it. If any of these 3 points share a red edge, then we have a red triangle (friendship of 3 people). If they don't, then we have a blue triangle (mutual strangers of 3 people). The same holds the other way around, if our original selected point has 3 or more blue edges instead.

7. Consider the set of points $\{(0,0), (1,0), \ldots, (n,0)\}$. Here, there are $n+1$ points. By the pigeonhole principle, at least one of the points in that set are owned by the same person. If we consider the entire columns where $x = 0, 1, \ldots, n$, we see once more by the pigeonhole principle that there are at least some two identical rows. This implies that someone owns 4 points that form a rectangle.

   a) Actually, I haven't solved this yet! I'm just predicting that i'll be around a $D5$ or above tbh... but feel free to try!

8. Take the center of the sphere and any two points. Let there be a plane passing through these 3 points. This cuts the sphere in half. Now, there are 3 points and 2 halves. By the pigeonhole principle, one of the halves has at least 2 points. Thus, $2 + 2 = 4 \Rightarrow$ there is some way to pick a closed hemisphere that contains 4 points.

9. [KSU Solution] Consider the $m, n \in \mathbb{Z}$. WLOG, let n be positive. Apply the division algorithm to get $m = qn + r_0$ with $0 \leq r_0 \leq n - 1$. Here, q is the integer part of the rational m/n. To compute the tenth place digit a_1, use the division algorithm $r_0 \times 10 = a_1 n + r_1$ and division algorithm $r_1 \times 10 = a_i n + r_i$. When $i = n$, then the $n + 1$ remainders $r_0, r_1, \ldots, r_n$ have values ranging from 0 to $n - 1$. Thus, by the pigeonhole principle, there must be two equal remainders, say $r_i = r_j$ with $i < j$. Let j be the smallest positive number such that $r_i = r_i$ for some $i < j$. Then $a_{j+1} = a_{i+1}$ with $r_{i+1} = r_{j+1}$ and, inductively, $a_{i+k} = a_{j+k}$ and $a_i = a_{i+(j-i)} = a_{i+2(j-1)} = \ldots$. This shows that the decimal is repeating with the repeating part $a_i a_{i+1} \ldots a_{j-1}$.

10. Don't have a proof on me right now, but this shouldn't be too hard to intuitively solve...

11. Consider a circle in the center with diameter 2 and the outer ring partitioned into 8 parts. This creates 9 pigeonholes with minimum crossing distance greater or equal to 2. By pigeonhole, the problem is done.

12. The prime factorization of a square only contains even powers. Consider the combinations. Then pigeonhole (I haven't written a formal proof for this yet sorry).

13. Pick a non-empty set of one element of S_n, S_1. Add elements one by one to obtain the series $S_1, S_2, \ldots, S_n$ such that $S_1 \subset S_2 \subset \cdots \subset S_n$. Let $s_1, s_2, \ldots, s_n$ denote the sums of the elements of $S_1, S_2, \ldots, S_n$ respectively, mod n. If any of $(s_n) = 0$, then there exists a set with an element sum of a multiple of n. If no, then $s_1, s_2, \ldots, s_n$ are equal to some numbers 1 to $n - 1$. By the pigeonhole principle, there are n sets and $n - 1$ holes. This means there is a pair of sets, call them S_p, S_q such that $p < q$, such that $s_p = s_q$. Then we know that $S_q \setminus S_p$ has a sum of elements divisible by n.