Induction Worksheet

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The goal of this worksheet is to help us gain intuition for induction problems and how to do them. I don't expect you guys to solve all of them, but give them a look to see how induction should be applied. All problems in this packet are to be solved with induction proof only - easier methods exist, but induction is the key here. Problems with ($\star$) are especially difficult.

— Arkyter, 5/5/2022

Induction is the process of proving a base case then showing through a domino-like toppling effect that every other case follows in an induction step. Let's practice some problems!

§1 Problems (Ascending Difficulty)

1. $$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

2. $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. $$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)$$

4. Prove that $2^n + 1$ is divisible by 3 for all odd integers n.

5. Prove that $n^3 + 2n$ is divisible by 3 for all integers n.

6. Prove that $n! > 2^n$ for integers $n \geq 4$.

7. Prove that
$$\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$$

8. Prove Bernoulli's Inequality
$$(1+x)^n \geq 1 + nx$$
for all naturals n and $x \geq -1$.

9. Let f_n be the n^{th} Fibonacci number. Prove
$$f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1$$

10. Prove that a convex n-gon has $\dfrac{n(n-3)}{2}$ diagonals.

11. The binomial coefficients are given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Prove that all binomial coefficients are integers.

12. Show that

$$\binom{2n}{n} < 2^{2n-2}$$

for all $n \geq 5$.

13. Prove deMoivre's formula with induction: For any $x \in \mathbb{C}$ and $n \in \mathbb{Z}$,

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

14. Prove Binet's Formula:

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

15. Prove that a closed knight's tour on a $4 \times n$ chessboard does not exist for $n = 1, 2, 3$ and $n \geq 5$. A closed knight's tour is a series of paths a chess knight can take with valid knight moves on a chessboard such that it steps on every tile.

16. Consider the following proof that every horse is of the same color. In the base case, we have only one horse. Trivially, we see every horse in this group of one horse is of the same color. Assume that n horses have the same color. Consider a group containing $n + 1$ horses. First, exclude one horse and look at the other n horses: these all must be of the same color since n horses always have the same color. Now, let's exclude a different horse. The new group of n horses must always have the same color as well! Therefore, we've proven that if n horses are of the same color, $n + 1$ horses are of the same color as well. By induction, all horses are of the same color. What is wrong about this proof?

17. ($\star$) Hence conclude

$$\sum_{n=1}^{\infty} \frac{nf_n}{2^n} = 10.$$

18. ($\star$) Prove that for naturals $n \geq 2$,

$$\sum_{k=2}^{n} \left(1 - \frac{1}{\sqrt{k}}\right) < \frac{2}{n^2}$$

19. ($\star$) Prove that if $\sin x \neq 0$ and $n \in \mathbb{N}$,

$$\cos x \cdot \cos 2x \cdot \cdots \cdot \cos 2^{n-1}x = \frac{\sin 2^n x}{2^n \sin x}$$