NMC Problem Set #24

OneShot Math Group

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Welcome!

This is a selection of interesting problems derived from curious thoughts, curated so you can nibble on them throughout the week! The point of this document is to introduce you to fun puzzles that require thinking. We recommend you try the ones that you find interesting! Feel free to work on them with others (even us teachers!). Harder problems are marked with chilies (🌶), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. Fibonacci Obsession[1]

Prove that the infinite sum,

$$\sum_{n=1}^{\infty} \frac{F_n}{10^n} = 0.1 + 0.01 + 0.002 + 0.0003 + 0.00005 + \dots,$$

is rational.

a) Suppose further that we create a new infinite sum with some positive integer z,

$$\sum_{n\geq 1} \frac{F_n}{z^n} = \frac{F_1}{z} + \frac{F_2}{z^2} + \frac{F_3}{z^3} + \frac{F_4}{z^4} + \frac{F_5}{z^5} + \dots.$$

Prove that this sum is also rational.

[1] i kinda am obsessed ngl... are genfuns algebra? i think so.

§2 Combinatorics

C1. Eulerian Numbers

The Eulerian Numbers, $A(n,m)$ or $\genfrac{\langle}{\rangle}{0pt}{}{n}{m}$, are defined as the number of permutations of the numbers 1 to n with exactly m elements being greater than the previous element (permutations with m "ascents").

a) From the recurrence

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle,$$

prove Worpitzky's identity,

$$x^n = \sum_k \left\langle {n \atop k} \right\rangle \binom{x+k}{n},$$

for integers $n \geq 0$.

b) (🌶️)[2] Furthermore, for a function $f : \mathbb{R} \to \mathbb{C}$ integrable over $(0,n)$, explain

$$\int_0^1 \cdots \int_0^1 f(\lfloor x_1 + x_2 + \cdots + x_n \rfloor)\, dx_1 \ldots dx_n = \sum_k \left\langle {n \atop k} \right\rangle \frac{f(k)}{n!}$$

with a combinatorial argument.

[2] concrete mathematics 6.65. cool stuff

§3 Geometry

G1. "Arbitrary" Square Root

Given a segment AB of arbitrary length and the ability to construct any arbitrary segment of unit length, perform a straightedge and compass construction of a line with length $\sqrt{AB}$.

a) Is $\sqrt[3]{AB}$ constructible? What about with origami?

§4 Number Theory

N1. Carefully Collecting Squares

Suppose the positive integers x, y satisfy $2x^2 + x = 3y^2 + y$. Show that $x - y, 2x + 2y + 1, 3x + 3y + 1$ are all perfect squares.

N2. ( ) Imaginary Product turned Real[3]

Is the product,

$$\prod_{k=1}^{\infty} (k^2 + i),$$

ever real?

[3] lucid dream