NMC Problem Set #14

OneShot Math Group

Nov. 20, 2022

Welcome!

This is a selection of interesting problems derived from curious thoughts, curated so you can nibble on them throughout the week! The point of this document is to introduce you to fun puzzles that require thinking. We recommend you try the ones that you find interesting! Feel free to work on them with others (even us teachers!). Harder problems are marked with chilies (🌶), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. Fussy Polynomial
Suppose that we have $P(x)$, a polynomial whose coefficients are all ± 1 and whose roots are all real. Show that $P(x)$ has degree of at most 3.

A2. (🌶) Social Distancing
Suppose we place n points $p_1, p_2, p_3, \ldots, p_n$ on the unit sphere,

$$\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

Prove that the sum of the squares of their mutual distances is at most n^2.

§2 Combinatorics

C1. Despicable Puzzle
Niko despises Rubik's Cubes. That's all.

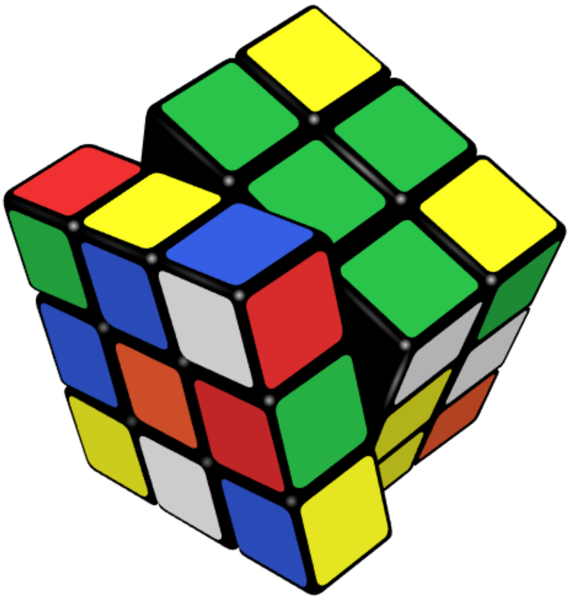

Figure 1: Thanks to Wikipedia Commons!

a) How many possible configurations of the 3×3 Rubik's Cube are there?

b) (🌶) In the context of a 3×3 cube, we say the puzzle is *solved* if each face has 9 tiles of the same color. Prove that if a corner has been twisted, the puzzle cannot be solved. Consider group theory.

c) (🌶 $\times$ 2) Let the *twisting number* of a Rubik's Cube be $C = 0$, where we add 1 for each clockwise corner twist and subtract 1 for each counterclockwise corner twist. Prove that if C is odd, the puzzle cannot be solved.

§3 Geometry

G1. (🌶) **Dimension-travelling Circle**

Prove that the largest possible radius of a circle contained in an n-dimensional unit hypercube is

$$\frac{1}{2}\sqrt{\frac{n}{2}}.$$

It may be helpful to consider the generalized parametric equation for a circle,

$$\overrightarrow{C} = \overrightarrow{a} + \overrightarrow{u}\cos t + \overrightarrow{v}\sin t,$$

where a, u, v are vectors satisfying $||\overrightarrow{u}|| = ||\overrightarrow{v}||$ and $u \cdot v = 0$ (both vectors are perpendicular).

§4 Number Theory

N1. Recurring Divisibility
Prove that the smallest integer greater than $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1} for all naturals n.

N2. Collatz Conjecture Numero Duo (Aliquot Sequences)
Let $s(n) = \sigma(n) - n$, where $\sigma(n)$ denotes the sum of the factors of n. Here, we have 3 possible outcomes for what $s(n)$ could be.

$$\text{If } s(n) \begin{cases} < & n \text{ then } n \text{ is } \textit{deficient.} \\ = & n \text{ then } n \text{ is } \textit{perfect.} \\ > & n \text{ then } n \text{ is } \textit{abundant.} \end{cases}$$

a) If n is a perfect number, we see that

$$s(n) = s(s(n)) = s(s(s(n))) = \dots.$$

How many such n can you find?

b) ( ) Suppose a, b are relatively prime. Prove that $s(a)s(b) \le s(ab)$.

c) ($\times$ Open) Does there exist some n such that the entire sequence below

$$s(n) < s(s(n)) < s(s(s(n))) < \dots$$

is abundant?