NMC Problem Set #9

OneShot Math Group

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Welcome!

This is a selection of interesting problems derived from curious thoughts, curated so you can nibble on them throughout the week! The point of this document is to introduce you to fun puzzles that require thinking. We recommend you try the ones that you find interesting! Feel free to work on them with others (even us teachers!). Harder problems are marked with chilies (🌶), in case you want to challenge yourself.

Have fun! *Note: New variants on these problems may be released throughout the week. Remember to check back once in a while!*

§1 Algebra

A1. **Reprise of Fibonacci**
Define F_n as the nth Fibonacci number, with the sequence starting at $F_1 = 1$, $F_2 = 1$, $F_3 = 2$... Show that

$$F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}.$$

A2. **Machin-like Formulas**
The *Machin-like formula* is defined as any equation of the form,

$$k\frac{\pi}{4} = \sum c_n \arctan \frac{a_n}{b_n},$$

with positive integer k and non-negative integers $(a_n), (b_n), (c_n)$. The classic such formula is as follows:
$$\frac{\pi}{4} = 4\arctan \frac{1}{5} - \arctan \frac{1}{239}.$$

a) Prove that the classic Machin-like formula holds true.

b) Determine k in the following equation,

$$\frac{\pi}{4} = 2\arctan \frac{1}{3} + \arctan k.$$

c) Create your own!

§2 Combinatorics

C1. **Alternating Sequences**

Call a permutation $(a_1, a_2, a_3, \ldots, a_n)$ of $\{1, 2, 3, \ldots, n\}$ *alternating* if

$$a_1 > a_2 < a_3 > a_4 < a_5 > \ldots.$$

Define E_n as the number of alternating permutations of $\{1, 2, 3, \ldots, n\}$.

a) (🌶) We see that $E_0 = E_1 = 1$. Create a combinatorial argument that

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}$$

for $n \geq 1$.

b) (🌶 × 2 [1]) Show that

$$\sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x.$$

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[1] calculus is handy here

§3 Geometry

G1. **The Kakeya Needle Problem**

This problem asks whether there is a minimum area D in which a needle of unit length can be rotated 360 degrees.

a) Suppose we take a circle with diameter 1. Naturally, we can fully rotate a needle of unit length inside. What's the minimum area of an equilateral triangle that allows for a needle to fully rotate?

b) (🌶) Is the shape above the minimal achievable area? If so, provide an example of a shape that allows for the needle to rotate fully with less area than our equilateral triangle above.

§4 Number Theory

N1. **Arithmetic functions**

Arithmetic functions refer to any function that takes natural numbers as inputs and outputs a subset of complex numbers (which includes the reals). Below, we list some identities on <u>additive</u> and <u>multiplicative</u> arithmetic functions.

a) Given a prime p, we can define a function $\nu_p(n)$ that tells us the exponent of p in the factorization of n. For example, $\nu_3(18) = 2$ because $18 = 2 \cdot 3^2$.
Convince yourself that ν_p is additive, that is, for all numbers a, b this equation holds:

$$\nu_p(ab) = \nu_p(a) + \nu_p(b)$$

b) With $d(n)$, we denote the number of divisors of n. For example, $d(12) = 6$ because there are 6 numbers that divide 12: $1, 2, 3, 4, 6, 12$.
Show that the function d is multiplicative. This means that if m, n are co-prime integers, then

$$d(mn) = d(m)d(n).$$

c) This is Euler's totient function, $\varphi(n)$. It calculates how many numbers from $1, ..., n$ are co-prime with n. Show that φ is multiplicative.

d) (🌶) Suppose f is a multiplicative function. If $f^*(n) = f(1) + \cdots + f(n)$, ranging over the divisors d of n, show that f^* is also multiplicative.