Chapter 8: Electromagnetic Waves

8.1 Introduction: Last week, we have summarized the 4 Maxwell's equations and reviewed how they came about through experiments in electrostatics and magnetostatics. Now we will understand, how these same equations leas to our understanding of how electromagnetic waves propagate in different mediums. The four Maxwell's equations in the differential form are:

$$\nabla.\vec{D} = \rho \qquad \text{Gauss law} \quad (1.1)$$

$$\nabla.\vec{B} = 0 \qquad \text{No magnetic monopoles} \quad (1.2)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{Ampere's law} \quad (1.3)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{Faraday's law} \quad (1.4)$$

The equations consist of 4 field terms, viz. $\vec{D}, \vec{E}, \vec{B}, and\ \vec{H}$. $\vec{D}$, & $\vec{E}$ and $\vec{B}$, & $\vec{H}$ are related to each other by the material/medium properties, electric permittivity, ϵ and magnetic permeability, μ, respectively as:

$$\vec{D} = \epsilon\vec{E} \text{ and } \vec{B} = \mu\vec{H}$$

There are also two source terms i.e. terms which generate the fields. These terms are the charge density, ρ, and current density, $\vec{J}$. Further, as Maxwell predicted, displacement current, ${\partial\vec{D}}/_{\partial t}$, is also a source for magnetic field.

Let us consider the last two equations i.e. the Faraday's Law and Ampere's Law. Time varying magnetic field creates an electric field which curls around it i.e. it creates a spatially varying Electric field. This will get coupled into equation 1.3 as time varying electric field which in turn creates a magnetic field which curls around it i.e. it creates a spatially varying magnetic field. If we look at these equations, it is quite apparent that these two equations are a pair of coupled first order differential equations and there is a relationship between time variation and space variation. Let us recall the telegrapher's equations in transmission lines i.e.

$$\frac{\partial V(z,t)}{\partial z} = -\left(R + L\frac{\partial}{\partial t}\right) I(z,t)$$

$$\frac{\partial I(z,t)}{\partial z} = -\left(G + C\frac{\partial}{\partial t}\right) V(z,t)$$

We had seen similar coupling in the telegrapher's equations where time varying voltage created current which varied in space; which then got coupled into the other equation to create voltage which varied in space. And we realized the solution was a wave which propagates through the transmission lines. This is the same behavior we are observing between the Faraday's Law and Ampere's Law. We should have a wave resulting from these equations also. Let us see how we can get to the wave equation.

8.2 Maxwell's Equations in Source Free, Homogeneous Medium

For this course, we will restrict ourselves to source free and homogeneous mediums. Source free means that there is no free charge or current source we have placed in the medium. Thus, $\rho = 0$ and $\vec{J} = 0$. Homogeneous mediums means that the properties of the medium remain the same every where i.e. the medium is uniform. Thus, the material properties ϵ and μ are independent of space i.e. they are constants in the equation. The Maxwell's equations then become:

$$\nabla.\epsilon\vec{E} = 0 \equiv \epsilon\nabla.\vec{E} = 0 \equiv \nabla.\vec{E} = 0$$

$$\nabla.\vec{B} = 0 \equiv \mu\nabla.\vec{H} = 0 \equiv \nabla.\vec{H} = 0$$

$$\nabla \times \vec{H} = \frac{\partial(\epsilon\vec{E})}{\partial t} \equiv \nabla \times \vec{H} = \epsilon\frac{\partial\vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial(\mu\vec{H})}{\partial t} \equiv \nabla \times \vec{E} = -\mu\frac{\partial(\vec{H})}{\partial t}$$

Thus, we can write these equations as:

$$\nabla.\vec{E} = 0 \qquad 8.1$$

$$\nabla.\vec{H} = 0 \qquad 8.2$$

$$\nabla \times \vec{H} = \epsilon\frac{\partial\vec{E}}{\partial t} \qquad 8.3$$

$$\nabla \times \vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \qquad 8.4$$

To find the wave equations, we need to convert the coupled one-dimensional partial differential equation into a second ordered differential equation only in one of the field terms, $\vec{E}$ or $\vec{H}$. Let us take another curl on equation 8.4 i.e.

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla \times \left(\mu\frac{\partial\vec{H}}{\partial t}\right)$$

On the RHS of the equation, space and time differentials are independent of each other and thus, we can move them around. Also, for homogeneous mediums, μ is a constant and can be taken out of the differential. Thus, we get:

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\mu\frac{\partial\left(\nabla \times \vec{H}\right)}{\partial t}$$

We can replace equation 8.3 on the RHS to get:

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\mu\frac{\partial\left(\epsilon\frac{\partial\vec{E}}{\partial t}\right)}{\partial t}$$

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

On the LHS term, we get:

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla.\vec{\mathrm{E}}) - \nabla^2 \vec{E} \qquad 8.5$$

However, for source free mediums, we know from equation 8.1 that $\nabla.\vec{\mathrm{E}} = 0$. Substituting 8.5 above we get:

$$-\nabla^2 \vec{E} = -\mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Or

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad 8.6$$

Viola, we have a second order differential equation only in terms of one field term, $\vec{E}$. LHS is second order differentiation in space e.g. in cartesian coordinates the LHS of the equation can be written as:

$$\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial x^2} + \frac{\partial^2 \vec{E}}{\partial y^2} + \frac{\partial^2 \vec{E}}{\partial z^2}$$

RHS of (8.6) is second order differentiation in time. Equation (8.6) is a second order differentiation of a function with time related to its second order differentiation with space by a constant (the material properties). Rewriting it in the form:

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu\epsilon} \nabla^2 \vec{E}$$

This is of the form $\frac{\partial^2 f}{\partial t^2} = K \frac{\partial^2 f}{\partial z^2}$, we had seen before. So we can immediately conclude that the equation should support a wave and the phase velocity of the wave should be given by $\frac{1}{\sqrt{\mu\epsilon}}$. We will talk more about it in the next section.

We could have also started with curl on equation 8.3 instead of 8.4 and would have resulted in a similar equation in magnetic field as:

$$\frac{\partial^2 \vec{H}}{\partial t^2} = \frac{1}{\mu\epsilon} \nabla^2 \vec{H}$$

These, two equations, viz.

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\mu\epsilon} \nabla^2 \vec{E}$$

$$\frac{\partial^2 \vec{H}}{\partial t^2} = \frac{1}{\mu\epsilon} \nabla^2 \vec{H}$$

These two equations are known as "vectorial wave equations". Vectorial because the solution is are time propagating fields which are vectors. The goal will be to find the specific solutions of these equations and understand what the field terms, $\vec{E}$ and $\vec{H}$ (we will start looking at specific solutions and their properties in the next chapter). However, for the time being let us try to build a physical picture.

First we notice that $\vec{E}$ and $\vec{H}$ are related to each other by a cross-product in space. This means that for any wave solution, $\vec{E}$ and $\vec{H}$ will always be perpendicular to each other. The wave will have two arms: an arm of $\vec{E}$ and an arm of $\vec{H}$. Please note, there are not two wave, a wave in $\vec{E}$ and a wave in $\vec{H}$. There is going to be one wave, which will have two vector fields in $\vec{E}$ and $\vec{H}$ like we have two arms. Further, these arms are always going to be perpendicular to one another. Since, we have these two arms, the propagating waves are called ***Electromagnetic waves***.

What we see is that if we start to oscillate one arm, say $\vec{H}$ (time varying current will create $\vec{H}$ which changes with time), then $\vec{E}$ will also oscillate. Further, the energy carried by these fields will have to propagate with phase velocity given by $\frac{1}{\sqrt{\mu\epsilon}}$. Suddenly out of equations which were derived from experiments in electrostatics, magnetostatics and very slowly varying terms (Faraday's Laws), out pops an equation which describes how waves can travel with time. Let us spend some time, understanding the phase velocity.

8.3 Phase Velocity

We have seen that the phase velocity of the wave which will propagate will be:

$$v_p = \frac{1}{\sqrt{\mu\epsilon}}$$

The electric permittivity and magnetic permeability can be written in relative terms as:

$$\epsilon = \epsilon_0\epsilon_r \ \ and \ \mu = \mu_0\mu_r$$

Thus, the phase velocity becomes:

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon_0}\sqrt{\mu_r\epsilon_r}}$$

For free space (devoid of any medium) or vacuum, $\epsilon_r = \mu_r = 1$, and the phase velocity become:

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

If we substitute the value of μ_0 & ϵ_0, the phase velocity turns out to be exactly the speed of light i.e.

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \approx 3 \times 10^8 \ m/s$$

Think about it. ϵ_0, μ_0 were proportionality constants in force equations i.e. how much force was created by charges or moving charges. These same universal constants are also related to how light travels in free space! Nature works in wonderfully mysterious ways. Jokingly it is also said that Maxwell wrote his equations and then God said let there be light (from genesis). But seriously, it was a major achievement in the history of science as it led to the unification of light with electricity and magnetism (the second major unification in physics, the first being Newton's Laws and Kepler's Laws). The same fundamental equations could describe physical phenomena in all these areas.

In a medium defined by ϵ_r, μ_r, the velocity of the propagating Electromagnetic wave will be given by:

$$v_p = \frac{c}{\sqrt{\mu_r\epsilon_r}}$$

For the time being we are assuming, $\epsilon_r, and\ \mu_r$ are real i.e. the material is lossless (We will consider what happens in lossy mediums in the next chapter). Thus, in a medium the velocity is reduced from the velocity of light in free space by $\sqrt{\mu_r \epsilon_r}$. From your high schools, the ratio with which the velocity of light is reduced in a medium is called the refractive index, n. Thus, the refractive index is dependent on how the material polarizes and magnetizes in presence of Electric and magnetic fields, respectively and is given by:

$$n = \sqrt{\mu_r \epsilon_r}$$

For most dielectric material which we work with, $\mu_r \approx 1$. Such materials are sometimes also called non-magnetic. For these materials, the speed of the electromagnetic wave going through it will be given by:

$$v_p = \frac{c}{n} = \frac{c}{\sqrt{\epsilon_r}}$$

The transmission lines we have studied previously also really have EM waves going through them, and thus, the speed of the wave going through them is given by $\frac{c}{\sqrt{\epsilon_r}}$ where ϵ_r is the relative permittivity of the medium between the conductors of the transmission lines.

8.4 Phasor Form for Maxwell Equations

Just like in transmission line, we will be looking for steady state solutions for sinusoid variations of the electric field and magnetic field. The field terms $\vec{E}\ and\ \vec{H}$, will have both space and time variables and in cartesian coordinates will be of the form: $\vec{E}(x,y,z,t) and\ \vec{H}(x,y,z,t)$. The time variation will come from the oscillations in the source. Just like we saw in steady states for transmission lines, for linear mediums, the time variation will be independent of the variations in the space. For single frequency source with angular frequency, ω, we can write the field terms in phasor form as:

$$\vec{E}(x,y,z,t) = \vec{E}(x,y,z)e^{j\omega t}$$

$$\vec{H}(x,y,z) = \vec{H}(x,y,z)e^{j\omega t}$$

Substituting these solutions in Maxwell's equations for source free, homogeneous mediums, the equations take the form:

$$\nabla . \overrightarrow{E(x,y,z)} = 0$$

$$\nabla . \vec{H}(x,y,z) = 0$$

$$\nabla \times \vec{H}(x,y,z) = j\omega\epsilon\vec{E}(x,y,z)$$

$$\nabla \times \vec{E}(x,y,z) = -j\omega\mu\vec{H}(x,y,z)$$

Note these equations are time independent. The effect of time is included in the angular frequency parameter, ω.

Substituting the solution in the vectorial wave equation (8.6), we get:

$$\nabla^2\vec{E}(x,y,z,t) - \mu\epsilon\frac{\partial^2\vec{E}(x,y,z,t)}{\partial t^2} = 0$$

$$\nabla^2\vec{E}(x,y,z)e^{j\omega t} - \mu\epsilon\vec{E}(x,y,z)\frac{\partial^2 e^{j\omega t}}{\partial t^2} = 0$$

$$\nabla^2 \vec{E}(x,y,z)e^{j\omega t} - (j\omega)^2 \mu\epsilon \vec{E}(x,y,z)e^{j\omega t} = 0$$

$$\nabla^2 \vec{E}(x,y,z) + \omega^2 \mu\epsilon \vec{E}(x,y,z) = 0$$

Similarly if we started with the vectorial wave equation for the magnetic field, we will get:

$$\nabla^2 \vec{H}(x,y,z) + \omega^2 \mu\epsilon \vec{H}(x,y,z) = 0$$

These two equations:

$$\nabla^2 \vec{E}(x,y,z) + \omega^2 \mu\epsilon \vec{E}(x,y,z) = 0$$

$$\nabla^2 \vec{H}(x,y,z) + \omega^2 \mu\epsilon \vec{H}(x,y,z) = 0$$

are called ***Helmholtz's wave equations***. Please note these equations are only true for homogeneous, linear mediums. Also note the equations are time independent and we can look for solutions to get how the electric field and magnetic fields will behave in an electromagnetic wave propagating through a medium with frequency, ω. Also note that once we fix a solution for (let's say) Electric field, then magnetic field cannot take an arbitrary value. It will be related to the electric field through the Maxwell's equation (Faraday's Law). Thus, the solutions of $\vec{E}(x,y,z)$ and $\vec{H}(x,y,z)$ are going to be dependent on each other, the same way we saw solutions for voltage and current waves are related to each other (through characteristics impedance) for a transmission line.

For the rest of the course, we will look for solutions of the Helmholtz's wave equation under different conditions starting with an unbound medium (the medium is the same all the way to infinity in all directions). One thing to keep in mind will be create equivalency between $\vec{E}$, $\vec{H}$ and voltage and current we saw in transmission lines. These equivalencies will allow us to relate to what we have already learned in first 6 weeks and allow us to solidify our knowledge quickly. Please note that as we look at the properties of electromagnetic waves, $\vec{E}$ will be equivalent to voltage, V; and $\vec{H}$ will be equivalent to current, I. Keep this in mind as we look for solutions and understand their properties in next chapter.