Chapter 9: Plane Waves

9.1 Introduction

In the last chapter we had seen the Helmholtz wave equations derived for source free, homogeneous, linear mediums. Helmholtz wave equations are a pair of 2nd order differential equations for electric field and magnetic field in the phasor form and are given as:

$$\nabla^2 \vec{E}(x,y,z) + \omega^2 \mu\epsilon \vec{E}(x,y,z) = 0$$

$$\nabla^2 \vec{H}(x,y,z) + \omega^2 \mu\epsilon \vec{H}(x,y,z) = 0$$

In this chapter, we will look at and understand the solution of the equations for "unbound" mediums. Unbound means that the medium extends everywhere in space i.e., has no boundaries. The solution in such a medium is called a "Plane Wave". The name means something and we will see in due time the significance behind the name but you should realize very quickly that since the medium is unbound, at least in the transverse plane (plane perpendicular to the direction of propagation), we expect the solution to be the same at every point i.e. there should not be any amplitude variation in the transverse plane.

It may so appear that plane wave is a solution of a very theoretical case as no medium will extend beyond a certain boundary. However, that is not the case. Consider an electromagnetic wave emitted by a cell phone tower. It will approximately start with a spherical phase front but will diverge as it propagates. When it hits the antenna on your cell-phone, over the small dimensions of the antenna, the wave will have for all practical purposes the same amplitude in the transverse cross-section. It can be treated as a plane wave. Further, plane wave forms the Fourier basis and any wave with a transverse cross-section i.e. the amplitude varies in the transverse plane (think of a laser pointer, its intensity on the wall changes) can be decomposed as a Fourier sum of plane waves at different angles. Thus, plane waves play a very important role in our knowledge of electromagnetic waves (EM waves). Let dive into the solutions.

9.2 Plane Wave Solution

We are looking for a wave solution propagating in a specific direction in an unbound medium i.e. the medium remains the same at every point in space. Let us choose the direction of propagation of the wave to be *z*. Thus, z becomes the longitudinal direction. The plane perpendicular to it is the transverse plane which in our choice of co-ordinates will become the x-y plane. Now since the medium is unbound, every point in the x-y plane should look the same as another point. Thus, the specific solution we are looking for should not change in x-y plane. Let us try to find the solution for the electric field and hence, we will use the Helmholtz's equation in the Electric field. Since, the solution is the same every where in x-y plane, the solution should be of the form:

$$\vec{E}(x,y,z) = \vec{E}(z)$$

The Helmholtz's equation can be expanded as:

$$\frac{\mathrm{d}^2 \vec{E}(z)}{dx^2} + \frac{\mathrm{d}^2 \vec{E}(z)}{dy^2} + \frac{\mathrm{d}^2 \vec{E}(z)}{dz^2} + \omega^2 \mu\epsilon \vec{E}(x,y,z) = 0$$

Since, the solution is not varying in x-y plane, the derivative with x and y should be 0 and thus, the Helmholtz's equation simplifies to:

$$\frac{\mathrm{d}^2\vec{E}(z)}{dz^2} + \omega^2\mu\epsilon\vec{E}(z) = 0 \qquad (1)$$

The solution $\vec{E}(z)$ is a vector i.e. the field should be pointing in a direction. Let us postulate a solution where the field is pointing in x-direction. It is one of the solutions, not necessarily a unique one. Later on we will see that the only solution which exists have the vector of the Electric field lie in the x-y plane, it cannot point in the longitudinal direction i.e. in z-direction, for a plane wave. For the time being, let us just assume that the direction where the E-field points to is chosen to be the x-direction. So we can postulate that the solution will be of the form:

$$\vec{E}(z) = E(z)\hat{x}$$

Substituting in equation (1), we get:

$$\frac{\mathrm{d}^2E(z)}{dz^2}\hat{x} + \omega^2\mu\epsilon E(z)\hat{x} = 0$$

$$\frac{\mathrm{d}^2E(z)}{dz^2} + \omega^2\mu\epsilon E(z) = 0 \quad (2)$$

Equation (2) is purely a scalar equation. Let us compare this to the equation we saw before in transmission lines i.e.

$$\frac{\mathrm{d}^2V(z)}{dz^2} + \omega^2LCV(z) = 0$$

Only terms have changed, the equations have exactly the same form. The solution should also be the same form. Thus, we can readily write the solutions of $E(z)$ as:

$$E(z) = Ae^{-jkz} + Be^{+jkz}$$

where *k* is given as:

$$k = \omega\sqrt{\mu\epsilon}$$

Remember again the wave is given by $\vec{E}(z,t)$ and can be written as:

$$\vec{E}(z,t) = E(z)\hat{x}e^{j\omega t} = \left(Ae^{j(\omega t-kz)} + Be^{+j(\omega t+kz)}\right)\hat{x}$$

The solutions are looking very similar to what we had seen for V(z,t) for transmission lines where the first term represents a wave traveling in +z direction and the second term represents a wave traveling in -z direction. However, there are certain differences. Remember, both $\mu\ and\ \epsilon$ are in general complex terms and thus, *k* is complex in general. We can write *k* in general as:

$$k = \beta - j\alpha$$

The solution then becomes of the form:

$$\vec{E}(z,t) = Ae^{-\alpha z}e^{j(\omega t-\beta z)}\hat{x} + Be^{\alpha z}e^{j(\omega t-\beta z)}\hat{x} \qquad (3)$$

What does this mean? β represents the propagation part of the wave and is the propagation constant (very similar to *k* we had seen in transmission lines). As the wave is traveling, it amplitude exponentially decays by the term $e^{-\alpha z}$. Thus, α, the loss which the wave is experiencing and is called the attenuation constant. That the amplitude decays by an exponential term is called the Beer-Lambert Law.

It is worth mentioning certain issues here. Photonics and RF share quite a common background especially in Electromagnetics but these fields have been developed by separate group of people and sometimes terms do not match across these fields. In RF, attenuation constant is defined in terms of Electric fields as defined in equation (3). In Photonics, the attenuation constant is defined in terms of power, not electric field. As we will see later power is proportional to the magnitude square of the electric field. Thus, the values of attenuation constants are different in the two fields, though they fundamentally describe the same phenomenon. The relationship between the attenuation constants used in the two fields can be written as:

$$\alpha_{RF} = \frac{\alpha_{Photonics}}{2}$$

Please keep this in mind. I have seen even graduate students make mistakes taking definition from one field, and values from another field and get wrong results. In this course, we will use the attenuation constant from the Electric field definition.

Let us get back to *k*. As we have seen before in transmission lines, *k* could be written as:

$$k = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda} = \beta - j\alpha$$

Let us define a new term called "free-space wavelength" written with symbol, λ_0. Free-space wavelength will be the wavelength of the wave if it was propagating in free-space. The propagation constant will be purely real and given as:

$$k_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{2\pi}{\lambda_0}$$

However, we know that:

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

Therefore, we can write the free-space wavelength as:

$$\lambda_0 = \frac{2\pi}{\omega\sqrt{\mu_0\epsilon_0}} = \frac{2\pi c}{2f\pi}$$

$$\lambda_0 = \frac{c}{f}$$

Where *c* is the speed of light in free-space and *f* is the frequency of the wave. Going back to a general medium, let us define the refractive index, n. We had previously seen the refractive index as:

$$n = \sqrt{\mu_r\epsilon_r}$$

In general as we have said, μ_r and ϵ_r are complex terms, which means the refractive index will be complex too. Now comes another confusing part. The complex refractive index is written as:

$$n \equiv n - jk$$

The k in refractive index is not the k used to describe the propagation constant. They are related but not the same. To avoid this confusion in this course, let us write the refractive index as:

$$n \equiv n_r - jn_i$$

Where n_r represents the real part of the refractive index and n_i represents the imaginary part. We will use these terms in the course but please remember, if you go to see the data sheets (www.refractiveindex.info is a good resource for looking at refractive indices of different materials), you will have *n* and *k* there.

We can write the propagation constant, k, in a medium with complex refractive index as:

$$k = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\mu_r\epsilon_r} = \frac{2\pi}{\lambda_0}(n_r - jn_i)$$

But, we have also defined k as:

$$k = \beta - j\alpha$$

Thus, we can write β, α in terms of the refractive index as:

$$\beta = \frac{2\pi}{\lambda_0}n_r \;\;;\;\; \alpha = \frac{2\pi}{\lambda_0}n_i$$

(Note: in Photonics, the attenuation constant will be given as: $\alpha = \frac{4\pi}{\lambda_0}n_i$)

β represents the propagation constant and depends on the real part of the refractive index while α represents the attenuation constant and is dependent on the imaginary part of the refractive index. Refractive index derives directly from the relative permittivity and relative permeability of the materials and will be frequency dependent. Material datasheets will normally have the real and imaginary part of refractive index as a function of frequency.

So we have seen the electric field of the solution of a wave which can propagate in an unbound medium and looked at the various terms which describe the solution. What about the magnetic field? Let us take a look at it. Let us start with the forward propagating wave (wave propagating in +z direction) i.e.

$$\vec{E}(z) = Ae^{-jkz}\hat{x}$$

Using Faraday's Law in phasor form from Chapter 8, we get:

$$\nabla \times \vec{E}(z) = -j\omega\mu\vec{H}(z)$$

We can calculate $\vec{H}(z)$ from this equation as:

$$\vec{H}(z) = \frac{-1}{j\omega\mu}\left(\nabla \times \vec{E}(z)\right) = \frac{j}{\omega\mu}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \partial/\partial z \\ E(z) & 0 & 0 \end{vmatrix}$$

Here we are using the fact that there is going to be no variation in x and y directions and hence, the derivatives in x and y are going to be 0 (thus, simply replaced the partial derivatives in x and y with zeros).

$$\vec{H}(z) = \frac{j}{\omega\mu}\frac{\partial\vec{E}(z)}{\partial z} = \frac{j}{\omega\mu}(-jk)Ae^{-jkz}\hat{y} = \frac{k}{\omega\mu}Ae^{-jkz}\hat{y}$$

We notice that the amplitude of the magnetic field has modified by $\frac{k}{\omega\mu}$ from the Electric field and the direction is $\hat{y}$. We know that the magnetic field has to be perpendicular to the electric field (because of cross-product involved), and so this makes sense.

We can also calculate the magnetic field for the backward wave and doing the same math, we will get:

$$\vec{H}(z) = \frac{j}{\omega\mu}\frac{\partial\vec{E}(z)}{\partial z} = \frac{j}{\omega\mu}(+jk)Ae^{-jkz}\hat{y} = -\frac{k}{\omega\mu}Ae^{-jkz}\hat{y} = \frac{k}{\omega\mu}Ae^{-jkz}(-\hat{y})$$

The amplitude has again changed by the same term $\frac{k}{\omega\mu}$ but the direction of the magnetic field has flipped!

It is now time to recall the waves in transmission lines. We had kept the voltage polarity the same for the forward and backward wave while the value of the current wave had become negative in the backward wave. The same thing is happening here. We are keeping the direction of Electric field the same for forward and backward wave but the direction of magnetic field in backward wave is becoming negative of the direction of the magnetic field in the forward wave. We can make a strong analogy here which becomes very useful in retaining this knowledge over time. Electric field is equivalent to voltage and magnetic field is equivalent to current. **This will become very powerful over the rest of the course.**

We had previously defined characteristics impedance as the ratio between the voltage and the current in a wave. Using our analogy, we can similarly define **wave impedance** as the ratio of the electric field with the magnetic field. Wave impedance is written as Greek letter, η and is given as:

$$\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}$$

The units of the wave impedance are also ohms (Ω). Magnetic field amplitude in a wave will be related to the amplitude of the electric field in the wave given by the relationship:

$$H = \frac{E}{\eta}$$

The direction of the fields will be perpendicular to each other. There is a beautiful relationship which comes up in direction of the fields too. For a wave propagating in +z-direction, if we chose the direction of the Electric field to be x, then we saw the direction of the magnetic field to be y. Notice that:

$$\hat{x} \times \hat{y} = \hat{z}$$

For the wave propagating in -z-direction, if we chose the direction of the Electric field to be x, then we saw the direction of the magnetic field to be -y. Again notice that:

$$\hat{x} \times (-\hat{y}) = -\hat{z}$$

The cross-product between the direction of the Electric field and the magnetic field always gives the direction of the propagation i.e.

$$\hat{E} \times \hat{H} = direct\ of\ propagation$$

Using these relations, we can quickly write the corresponding field if we know one field for a wave without having to do any math. As an example, let us consider a wave travelling in +z direction with electric field value given as E_0 and pointing in y-direction. Then we can write magnetic field with amplitude E_0/η and pointing in -x direction (as $\hat{y} \times (-\hat{x}) = \hat{z}$).

We have understood the various terms in the plane wave solution in an unbound medium. Now let us try to develop a physical understanding behind these terms. The physical picture of the plane waves is treated in the next section.

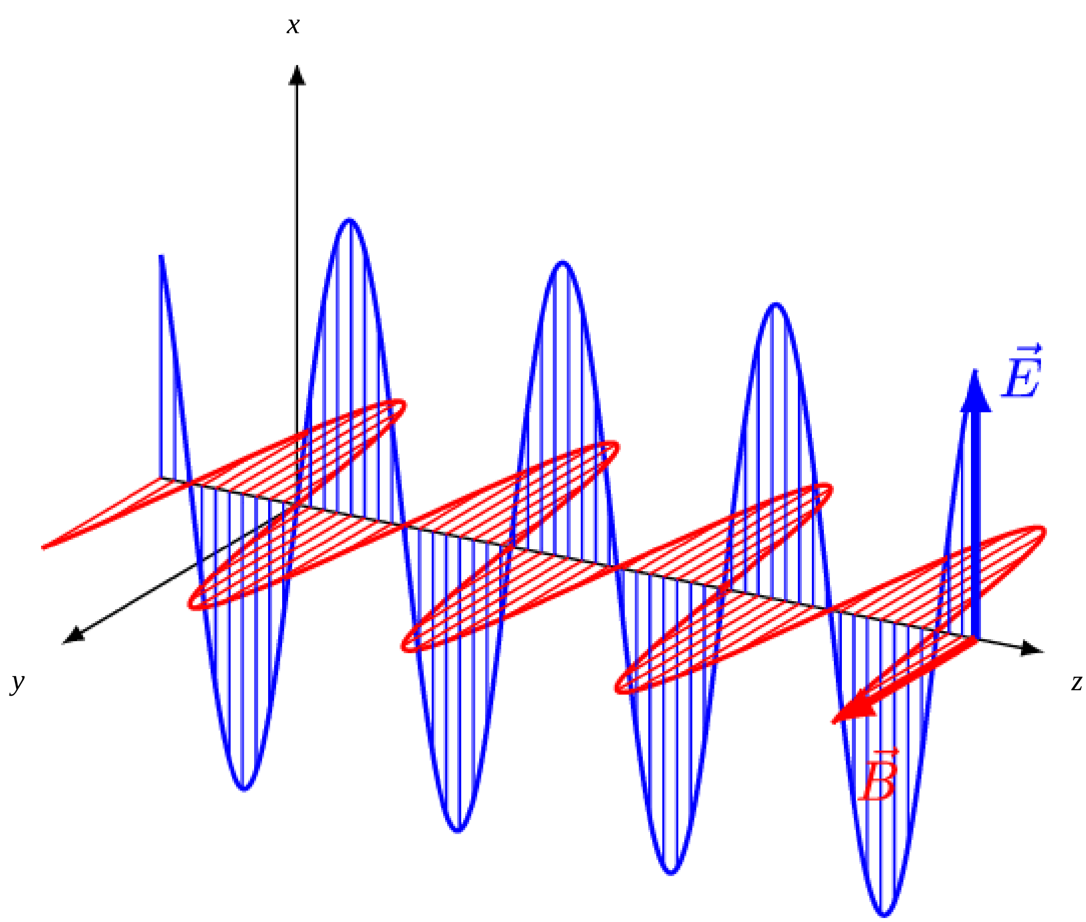
9.3 Physical understanding of Plane Waves

Let us try to build a physical picture of plane waves and also understand why the solution we have looked above is called a "plane wave". Basically, we have seen that if we start oscillating the electric field in an unbound medium, we also start oscillating the magnetic field. Magnetic field is perpendicular to the direction of the electric field and then the energy disturbance we have created will have to start propagating. The direction of propagation will be given by the cross-product of the direction of electric field and the magnetic field. This also means that the direction of propagation will have to be perpendicular to the plane containing the electric field and magnetic field vectors. We have chosen the direction of propagation to the z-axis (we can choose the coordinates to describe a problem) and thus, the plane containing electric field and magnetic field vectors will be the xy plane.

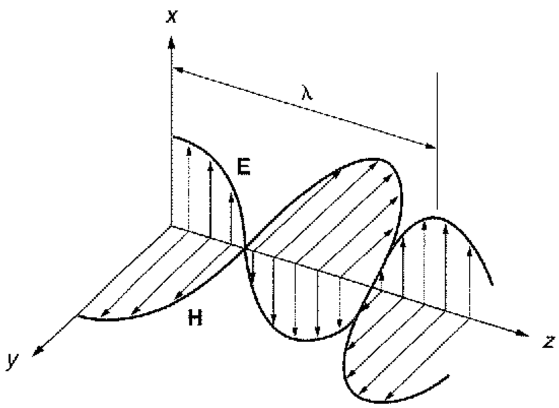
We notice that in plane waves, the electric fields and magnetic fields are both perpendicular to the direction of propagation. **Such EM waves are called Transverse electromagnetic waves (TEM).** The name basically means that both electric and magnetic fields are transverse to the direction of propagation. Further, the amplitude of the electric field and magnetic field do not change at any point in the xy plane. Many online resources and even some introductory books say that electromagnetic waves only have transverse components. That is not true. If the amplitude of the field terms varies in x-y plane, we will have electric or magnetic field components in the longitudinal directions. There will always be transverse components but longitudinal We can easily see this by using Maxwell's equations and there are few examples we will do in the problem set. However, here are certain considerations you can remember (assuming wave is propagating in z-direction).

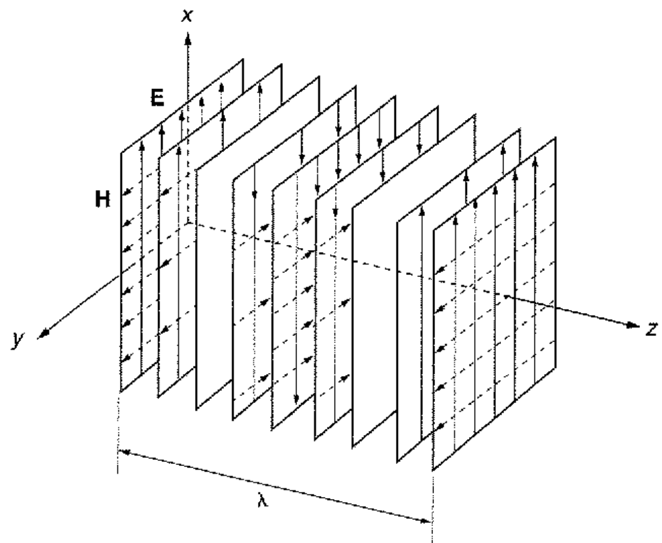
1. If the Electric field points in x-direction and its amplitude varies in y-direction i.e $\vec{E}(x,y,z) = E(y,z)\hat{x}$, then we will have **magnetic field components in y-direction and z-directions**. Thus, electric field is only transverse to the direction of the propagation but magnetic field also has a longitudinal direction. Such a wave is called Transverse Electric or TE wave i.e. only the electric field is transverse to the direction of propagation.
2. If the Electric field points in x-direction and its amplitude varies in x-direction i.e. $\vec{E}(x,y,z) = E(x,z)\hat{x}$, then we have electric field components also in z-direction while the magnetic field components will only be in y-direction. Thus, magnetic field is only transverse to the direction of the propagation but electric field also has a longitudinal direction. Such a wave is called Transverse Magnetic or TM wave.
3. If the Electric field points in x-direction and its amplitude varies in both x and y direction i.e. $\vec{E}(x,y,z) = E(x,y,z)\hat{x}$; then both Electric and magnetic fields will have components in all 3 directions i.e. x, y, and z. Such waves are called Hybrid waves. You take sunlight and focus it with a lens, you end up creating all 6 components of the Electric and magnetic fields.

Let us go back to the plane wave and draw the wave out. We have seen that the peak amplitude of the Electric and magnetic fields are related by the wave impedance, η. Since, $\eta = \sqrt{\frac{\mu}{\epsilon}}$, and μ & ϵ are in general complex terms, the wave impedance can be complex. What that means is that the electric field and magnetic fields will not be in phase with each other. Let us start simple and consider a material where μ & ϵ are real (such a material will be loss-less) and thus, η will be real. So we have Electric field pointing in x-direction, and magnetic field pointing in y-direction. The peak amplitude of the magnetic field is reduced from that of the Electric field by the wave impedance. However, the electric field and the magnetic field are in phase with each other. What that means is that when Electric field is maximum at a point, at that same time magnetic field is also maximum at the same point. Similarly, when the electric field is minimum at a point, the magnetic field is also a minimum. In terms of the picture, I was building with oscillating arms, the two arms are in unison as they oscillate and propagate. The animation below shows the fields oscillating and propagating in such a wave.



Why is such a wave called plane wave? If we go in x-y plane, at any point, we will have the same value for the electric field and the magnetic as any other point in the same plane. At different values of z, i.e. different x-y planes, the value will be different. Thus, we have “planes” in x-y direction moving forward. Thus, the name “plane wave”. Pictorially, we will observe this:





9.4 Plane waves in different types of mediums

Let us consider how plane waves look like in different materials like free space, loss-less dielectric, lossy dielectric and conductors.

9.4.1 Free Space

In free space, $\epsilon = \epsilon_0$ and $\mu = \mu_0$. As we have already seen, the phase velocity is given by the velocity of light, c. Further, the propagation constant has been defined before and will be:

$$k_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{2\pi}{\lambda_0}$$

And the wavelength of the wave will be given by the free-space wavelength as:

$$\lambda_0 = \frac{c}{f}$$

The propagation constant, β, will simply be equal to k_0 and the attenuation constant, α, will be 0. The wave impedance will be given as:

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

Numerically this turns out to be approximately 377 Ω. Both electric field and magnetic fields are in phase with each other. Please again note that it is not if I use an ohm meter, I will measure an impedance of 377 Ω in air (it is ridiculous idea to begin with). What it means is that the ratio of the peak magnitude of electric field and the magnetic field will be 377. It also plays a very important role in system design for wireless communication. You have a circuit feeding an antenna. How do you model the antenna? The antenna is creating an electromagnetic wave at the designed frequency which is propagating in air. At least for the designed frequency, we can simply replace the antenna with a resistance of 377 Ω and use that value to design out feed circuit to the antenna.

9.4.2 Loss-Less Dielectric

For loss less dielectric, $\epsilon = \epsilon_0 \epsilon_r$, and $\mu \approx \mu_0$. Further, the relative permittivity, ϵ_r will be purely real. Thus, the refractive index will be purely real and will be given as:

$$n = n_r = \sqrt{\epsilon_r}$$

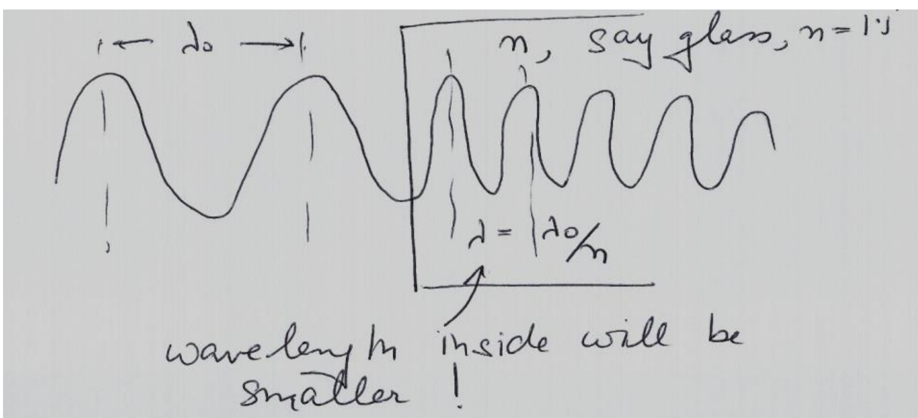
The propagation constant, k, will only be real and the attenuation constant, α, will be 0. We can write them as:

$$k = \beta = \omega\sqrt{\mu_0 \epsilon_0 \epsilon_r} = \frac{2\pi}{\lambda_0} n = k_0 n$$

The propagation constant increases by refractive index as compared to the propagation constant in the medium. Also remember, in general, $\beta = \frac{2\pi}{\lambda}$. Comparing the two equations, we can observe that the wavelength in the medium has compressed from the free space wavelength by the refractive index of the medium. Remember, in linear mediums, the frequency of the wave does not change in the dielectric. However, the wavelength inside the medium will be given as:

$$\lambda = \frac{\lambda_0}{n}$$

We can picture this as following:



Why? Think about the hopping picture I had demonstrated before. The rate of my jumping up and down is determined by the frequency. The length of my hops gives the wavelength. Let us start hopping in free space. As we are jumping up and down, we are also moving forward with speed of light. So we make these hops. The length of the hop will be the distance I will travel forward within the time it takes me to jump forward. Now we go into a medium where the speed has reduced. The time it takes for me to jump up and down still remains the same (that is determined by the frequency). But since I am moving slower forward now, the length of my hop will be shortened. Viola, the wavelength reduces, my hops have compressed.

The wave impedance for the plane wave will be given by:

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0 \epsilon_r}} = \frac{\eta_0}{\sqrt{\epsilon_r}} = \frac{\eta_0}{n}$$

Thus, the wave impedance decreases from the value of the free space wave impedance by the value of the refractive index.

9.4.2 Lossy Dielectric

For lossy dielectric, the relative permittivity is a complex number and can be written as:

$$\epsilon_r = \epsilon' - j\epsilon''$$

Thus, the refractive index becomes complex and will be given by:

$$n \equiv n_r - jn_i = \sqrt{\epsilon' - j\epsilon''}$$

We can write the relationships between the complex parts of the refractive index with those of the relative permittivity as:

$$n_r = \sqrt{\frac{1}{2}\left(\left(\epsilon'^2 + \epsilon''^2\right)^2 + \epsilon'\right)}$$

And

$$n_i = \sqrt{\frac{1}{2}\left(\left(\epsilon'^2 + \epsilon''^2\right)^2 - \epsilon'\right)}$$

The propagation constant will have both the real part and the imaginary part and will be given as:

$$k = \beta - j\alpha = \frac{2\pi}{\lambda_0} n_r - j\frac{2\pi}{\lambda_0} n_i$$

The wavelength inside the medium will be given by:

$$\lambda = \frac{\lambda_0}{n_r}$$

The wave-impedance becomes complex and is equal to:

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0(\epsilon' - j\epsilon'')}} = \frac{\eta_0}{n_r - jn_i}$$

If we write the Electric field for the wave propagating in +z direction, the field will be given by:

$$\vec{E}(z) = E_0 e^{-j\beta z} e^{-\alpha z}\hat{x}$$

Here E_0 is the peak amplitude of the field and will be determined by how the wave is excited by the source. The magnetic field will be given by:

$$\vec{H}(z) = \frac{E_0}{\eta} e^{-j\beta z} e^{-\alpha z}\hat{y} = \frac{E_0}{n_r - jn_i} e^{-j\beta z} e^{-\alpha z}\hat{y} = \frac{E_0}{\sqrt{{n_r}^2 + {n_i}^2}\, e^{-j\theta}} e^{-j\beta z} e^{-\alpha z}\hat{y}$$

Where $\theta = tan^{-1}\left(\frac{n_i}{n_r}\right)$

So we observe that the magnetic field is given by:

$$\vec{H}(z) = \frac{E_0}{\sqrt{{n_r}^2 + {n_i}^2}} e^{j\theta} e^{-j\beta z} e^{-\alpha z} \hat{y} = \frac{E_0}{\sqrt{{n_r}^2 + {n_i}^2}} e^{-j(\beta z - \theta)} e^{-\alpha z} \hat{y}$$

As we can see the magnetic field is behind the electric field by phase angle, θ. If we observe at a point, when the E-field achieves its peak value, the magnetic field still has not and we will have to wait some time before it reaches the peak. The two arms of Electric field and Magnetic field are not synchronized with each other.

9.4.3 Conductors

When we apply Electric field to conductors, we know conduction current will start flowing through it. Thus, the current density $\vec{J}$ will not be equal to 0. We had derived the Helmholtz's wave equation assuming that the current density was 0. So it looks like we may not be able to use our current results to understand how waves travel in conductors. There is a way and we need to revisit the Ampere's Law in phasor form i.e.

$$\nabla \times \vec{H}(x,y,z) = \vec{J} + j\omega\epsilon\vec{E}(x,y,z)$$

Ohm's Law states the relationship between the current density and the applied electric field as:

$$\vec{J} = \sigma\vec{E}(x,y,z)$$

Where σ is the conductivity of the material. Thus, we can write the Ampere's Law as:

$$\nabla \times \vec{H}(x,y,z) = \sigma\vec{E}(x,y,z) + j\omega\epsilon\vec{E}(x,y,z)$$

Rearranging we can write:

$$\nabla \times \vec{H}(x,y,z) = j\omega\epsilon\left(1 + \frac{\sigma}{j\omega\epsilon}\right)\vec{E}(x,y,z)$$

If we define the electric permittivity of the conductor as:

$$\epsilon_c = \epsilon\left(1 + \frac{\sigma}{j\omega\epsilon}\right) = \epsilon\left(1 - \frac{j\sigma}{\omega\epsilon}\right)$$

And substitute in the Equation above, we get:

$$\nabla \times \vec{H}(x,y,z) = j\omega\epsilon_c\vec{E}(x,y,z)$$

Notice this form is the same form as we had for source free mediums when we derived the Helmholtz's wave equation. So if we substitute ϵ with ϵ_c in our results, we should be able to define the properties of the plane waves in conductors!

For perfect conductors, the conductivity σ is infinity and thus, the value of ϵ_c approaches $-j\infty$. There is no EM wave which can propagate through a perfect conductor as the electric field approaches 0. The wave impedance also approaches 0. **A perfect conductor is like a short circuit**.

What about real conductors with a finite value of the conductivity? For good conductors, $\sigma \gg \omega\epsilon$. We can approximate the conductor electric permittivity as:

$$\epsilon_c \approx \epsilon\left(-\frac{j\sigma}{\omega\epsilon}\right) \approx -\frac{j\sigma}{\omega} = \frac{\sigma}{\omega} e^{-j\frac{\pi}{2}}$$

The propagation constant, k, can be written as:

$$k = \omega\sqrt{\mu_0\epsilon_c} = \omega\sqrt{\mu_0\frac{\sigma}{\omega}e^{-j\frac{\pi}{2}}} = \sqrt{\omega\mu_0\sigma}e^{-j\frac{\pi}{4}}$$

However, $e^{-j\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) - jsin\left(\frac{\pi}{4}\right) = \frac{1-j}{\sqrt{2}}$

Thus, we get the propagation constant to be given as:

$$k = \beta - j\alpha = \sqrt{\omega\mu_0\sigma}\frac{1-j}{\sqrt{2}} = \sqrt{\frac{\omega\mu_0\sigma}{2}} - j\sqrt{\frac{\omega\mu_0\sigma}{2}}$$

We notice that for good conductors, $\beta = \alpha = \sqrt{\frac{\omega\mu_0\sigma}{2}}$

Since, σ is large in conductors, both β & α are going to be large. The Electromagnetic wave attenuates quickly inside the conductor and will not penetrate much. Further, note that the phase velocity is given by:

$$v_p = \frac{\omega}{\beta}$$

Since, β is going to be large, the EM wave slows down quite a lot inside the conductor. For example, for copper, the phase velocity of the wave is only approximately 720 m/s as opposed to the value of 3×10^8 m/s in air.

If we look at the decay term $e^{-\alpha z}$, then for a depth of $z = 1/\alpha$, the electric field reduces in amplitude by a ratio of e^{-1} (or becomes 0.367 of the initial starting point). This depth is called the skin depth, δ and is given by:

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$

Larger the conductivity, smaller the skin depth and less the EM wave penetrates into the conductor. Let us see some practical considerations. CPU pins are coated in gold since gold has one of the largest conductivities. However, the bulk of the pin is Aluminum. Ever wondered why such a thin layer of gold works so well. We have seen now that the transmission line made y the pins is really carrying an electromagnetic wave in the same around it. Since the outside surface is gold, the wave only penetrates very little into the metal and the current induced is only on the part where the wave is present. Most of the current is travelling through the outside layer, not through the center of the pin and thus, the resistance the wave sees is mainly from that of the gold. Even a thin layer of gold works!

9.5 Can we have a plane wave with Electric field in direction of propagation?

We previously looked at a specific solution where the Electric field was pointing in x-direction for a plane wave propagating in z-direction. The question which obviously arises is whether we can have a plane wave for which the electric field is pointing in the direction of propagation i.e. z in our choice of coordinates. Basically, the question is whether we can have a field solution of the form:

$$\vec{E}(z) = E(z)\hat{z}$$

Let us see what happens with the magnetic field for such a solution. We can calculate the magnetic field using Faraday's Law as:

$$\vec{H}(z) = \frac{-1}{j\omega\mu}\left(\nabla \times \vec{E}(z)\right) = \frac{j}{\omega\mu}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \partial/\partial z \\ 0 & 0 & E(z) \end{vmatrix} = 0$$

There is no magnetic field existing! Well then, how is Ampere's Law true as we know from Ampere's Law, we need:

$$\vec{E}(z) = \frac{1}{j\omega\epsilon}\left(\nabla \times \vec{H}(z)\right)$$

Since we have concluded $\vec{H}(z)$ should be 0 for the solution, $\vec{E}(z)$ should also be 0. We cannot satisfy both the Faraday's Law and Ampere's Law at the same time for this postulated solution.

Remember for an EM wave to exist, we need both time varying electric field and magnetic field components. For plane waves, the fields will have to be transverse to the direction of the propagation. There is no other way.

9.6 Group Velocity

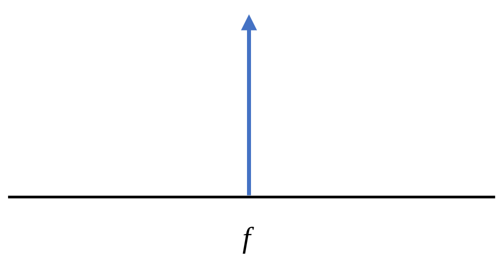
We have considered the plane wave with the Electric field given as:

$$\vec{E}(z,t) = E(z)\hat{x}e^{j\omega t} = \ Ae^{j(\omega t - kz)}\hat{x}$$

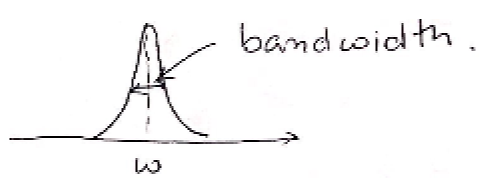
This is a single frequency wave where we found the phase velocity as:

$$v_p = \frac{\omega}{\beta}$$

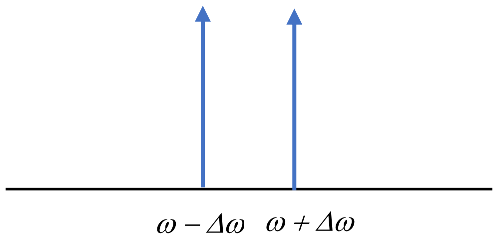
Such a wave is called a monochromatic wave (from single color). In frequency domain, we will get a Dirac-delta function as shown below:



Single frequency sources are made using oscillators (whether it be electrical sources or optical sources e.g. laser is a light oscillator). It is physically impossible to create a single frequency source which only emits as a dirac-delta function as that requires no damping in the oscillator. There is always going to be a spread of energy around the designed frequency *f* as shown in the Figure below:



How does a wave travel with a source where there is a spread in frequency? To understand this, let us approximate the source as two sources with the same amplitude close together in frequency as shown:



As the propagation constant depends on frequency, the two waves will propagate with different values. Let us call then *k*-*Δk* and *k*+*Δk*. Thus, we will have two different waves propagating through the medium and can write the electric field as the sum of the fields in the two waves as:

$$\vec{E}(z,t) = Ae^{j\{(\omega-\Delta\omega)t-(k-\Delta k)z\}}\hat{x} + Ae^{j\{(\omega+\Delta\omega)t-(k+\Delta k)z\}}\hat{x}$$

How do we add two phasors let us say $e^{j\alpha} + e^{j\beta}$? One way is to convert each term into cos and sine form and then use trigonometric relations to do the addition. The other simpler way is to take the average of $\alpha\ and\ \beta$ out. What will happen is as follows:

$$e^{j\alpha} + e^{j\beta} = e^{j\frac{\alpha+\beta}{2}}\left(e^{j\left(\frac{\alpha-\beta}{2}\right)} + e^{j\left(\frac{\beta-\alpha}{2}\right)}\right) = e^{j\frac{\alpha+\beta}{2}}\left(e^{j\left(\frac{\alpha-\beta}{2}\right)} + e^{-j\left(\frac{\alpha-\beta}{2}\right)}\right) = 2\cos\left(\frac{\alpha-\beta}{2}\right)e^{j\frac{\alpha+\beta}{2}}$$

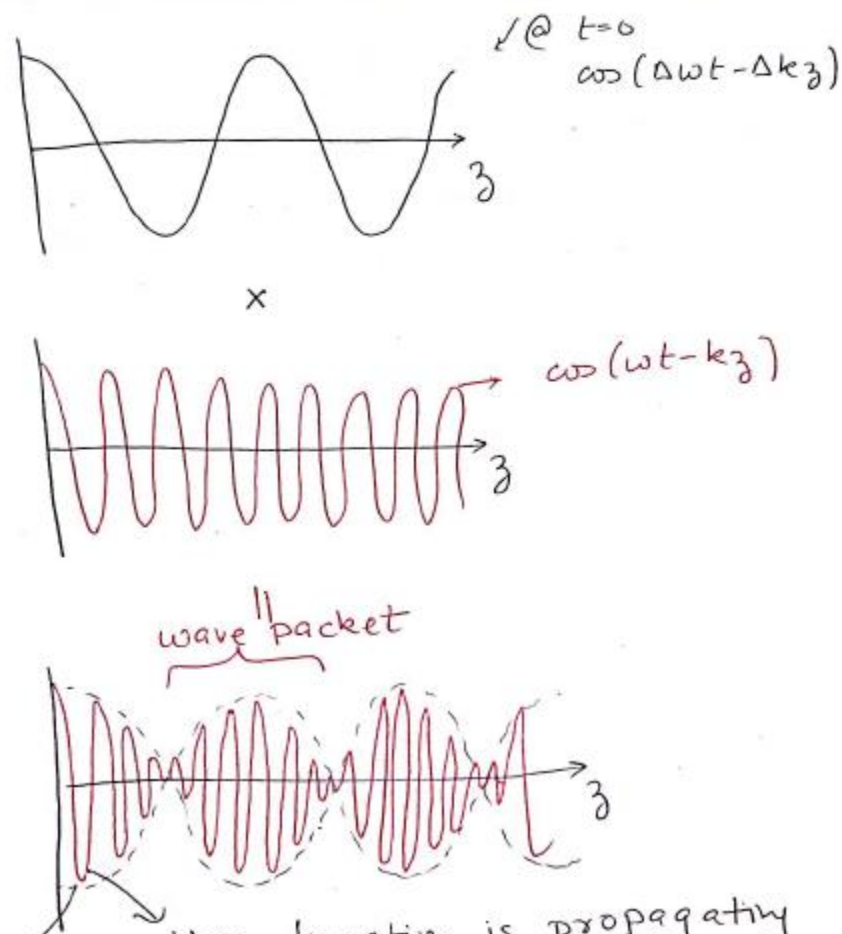
This way, the math is pretty simple. It is a very powerful trick to know and could help you in many other courses also.

In our case, $\alpha = (\omega-\Delta\omega)t - (k-\Delta k)z$ and $\beta = (\omega+\Delta\omega)t - (k+\Delta k)z$. The average of $\alpha, and\ \beta$ is simply $\omega t - kz$. So we can write $\vec{E}(z,t)$ as:

$$\vec{E}(z,t) = \hat{x}Ae^{j(\omega t-kz)}\left(e^{-j(\Delta\omega t-\Delta kz)} + e^{+j(\Delta\omega t-\Delta kz)}\right) = 2Acos(\Delta\omega t - \Delta kz)e^{j(\omega t-kz)}\hat{x}$$

If we look at the expression, we have a term $e^{j(\omega t-kz)}$ which describes a wave with frequency ω and propagation constant, k. We also have a term $cos(\Delta\omega t - \Delta kz)$ which also describes a propagating term but in $\Delta\omega$ and Δk. You should have seen functions like this in ECE 318 course. Let us see what the function tells us.

First we need to remember that $\Delta\omega \ll \omega$ and $\Delta k \ll k$. Thus, $cos(\Delta\omega t - \Delta kz)$ is a much slower varying function than $e^{j(\omega t-kz)}$. $cos(\Delta\omega t - \Delta kz)$ creates an envelope on top of $e^{j(\omega t-kz)}$ function. We can visualize this by the figure shown below:



this function is propagating with phase velocity

$$v_p = \frac{\omega}{k}$$

the envelope is also moving with velocity given by $\frac{\Delta\omega}{\Delta k}$.

Multiplication of the two functions result in a wave packet. The fast-varying function $e^{j(\omega t-kz)}$ is propagating with the phase velocity, $v_p = \frac{\omega}{k}$. However, the envelope is propagating with a different velocity given by $\Delta\omega/\Delta k$. To get the information which the wave is carrying i.e. decide on what the value of the amplitude A is, we need the envelope to reach us. This velocity is called group velocity. We can write group velocity as:

$$v_g = \lim_{\Delta\omega\to 0}\frac{\Delta\omega}{\Delta k} = \lim_{\Delta\omega\to 0}\frac{1}{\Delta k/_{\Delta\omega}} = \frac{1}{dk/_{d\omega}}$$

The information i.e. energy in a wave travels at group velocity for most mediums. Remember again that

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}n_r$$

We can have three different types of materials depending on how the refractive index changes with wavelength.

Case I: Dispersion Less Materials n_r does not change with wavelength. In that case,

$$v_g = \frac{c}{n_r} = v_p$$

In this case, the group velocity is the same as the phase velocity and the information travels at speed of light in these mediums. Natural materials don't exist where dispersion is 0 but we can engineer materials to achieve this, at least small bands of frequency.

Case II: Normal Materials: For normal materials $\frac{dn}{d\omega} > 0$. The group velocity can be written as:

$$v_g = \frac{1}{\frac{d}{d\omega}\left(\frac{\omega}{c}n_r\right)} = \frac{1}{\left(\frac{n_r}{c} + \frac{\omega}{c}\frac{dn_r}{d\omega}\right)} = \frac{c}{n_r + \omega\frac{dn_r}{d\omega}}$$

Since, $\frac{dn}{d\omega} > 0$, the denominator is larger than n_r. The phase velocity is given by c/n_r. Thus, the group velocity is smaller than the phase velocity. This is the oft repeated statement you may have heard that information travels slower than speed of light.

Case III: Anomalous Mediums: For anomalous mediums $\frac{dn}{d\omega} < 0$, the group velocity can be written as:

$$v_g = \frac{1}{\frac{d}{d\omega}\left(\frac{\omega}{c}n_r\right)} = \frac{1}{\left(\frac{n_r}{c} + \frac{\omega}{c}\frac{dn_r}{d\omega}\right)} = \frac{c}{n_r + \omega\frac{dn_r}{d\omega}}$$

Since, $\frac{dn}{d\omega} < 0$, the denominator is smaller than n_r. Thus, the group velocity is larger than the phase velocity. Normally for these mediums, information doesn't travel at group velocity.

9.7 Poynting Vector

Poynting vector defines how energy flows in an electromagnetic wave. In ECE 106, we had seen that the energy density stores in static $\vec{E}$ and $\vec{H}$ fields are given as $\frac{1}{2}\epsilon E^2$ and $\frac{1}{2}\mu H^2$, respectively. When these fields vary with time, the associated stored energies will also vary with time. If we consider a given volume of space, EM energies can be transported in or out of it by EM waves. In addition, EM energy can also be stored in the volume in the form of electric and magnetic fields, and EM power can be dissipated in it in the form of Joule heating. **Poynting's theorem helps us calculate the balance between power flow in and out of a given volume and the rate of change of stored energy and power dissipation. We are doing the derivation to get to our form so you can see where the results are coming from. However, at the end of the day, you need to understand the Poynting vector and what it means. You do not need to memorize the derivation or if you want you can skip below.**

Let us start with Ampere's Law:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

We can rewrite the equation as:

$$\vec{J} = \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t}$$

Let us consider the form:

$$\vec{E}.\vec{J} = \vec{E}.\sigma\vec{E} = \sigma E^2$$

The RHS of the equation should be related to Joule heating. Let us evaluate the LHS by substituting the value of J from Ampere's Law.

$$\vec{E}.\vec{J} = \vec{E}.\left(\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t}\right) \qquad (A)$$

Let us consider this term $\vec{E}.\left(\nabla \times \vec{H}\right)$ and rewrite it using the identity:

$$\nabla.\left(\vec{E} \times \vec{H}\right) = \vec{H}.\left(\nabla \times \vec{\mathrm{E}}\right) - \vec{E}.\left(\nabla \times \vec{\mathrm{H}}\right)$$

Therefore, we can rewrite the term as:

$$\vec{E}.\left(\nabla \times \vec{\mathrm{H}}\right) = \vec{H}.\left(\nabla \times \vec{\mathrm{E}}\right) - \nabla.\left(\vec{E} \times \vec{H}\right)$$

However, from Faraday's Law, we know that:

$$\nabla \times \vec{\mathrm{E}} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$$

Therefore, by substituting we can rewrite:

$$\vec{E}.\left(\nabla \times \vec{\mathrm{H}}\right) = \vec{H}.\left(-\mu \frac{\partial \vec{H}}{\partial t}\right) - \nabla.\left(\vec{E} \times \vec{H}\right) = -\frac{1}{2}\mu \frac{\partial H^2}{\partial t} - \nabla.\left(\vec{E} \times \vec{H}\right)$$

Substituting the value in equation A, we will get:

$$\vec{E}.\vec{J} = -\frac{\mu}{2}\frac{\partial H^2}{\partial t} - \nabla.\left(\vec{E} \times \vec{H}\right) -= -\frac{\mu}{2}\frac{\partial H^2}{\partial t} - \frac{\epsilon}{2}\frac{\partial E^2}{\partial t} - \nabla.\left(\vec{E} \times \vec{H}\right)$$

Let us take a look at the terms. $\vec{E}.\vec{J}$ should be related to Joule heating, i.e. the energy lost in the resistance of the medium. On RHS, we have the term $\frac{\mu}{2}H^2$ and $\frac{\epsilon}{2}E^2$ which are the energy stored in the magnetic and electric fields respectively. And we get a new term $\vec{E} \times \vec{H}$ which we have never seen before. But remember that previously, we had seen that $\hat{E} \times \hat{H}$ is the direction of the propagation of the wave. So maybe, $\vec{E} \times \vec{H}$ has something to do with energy being propagated by the wave.

Let us integrate the above equation over some arbitrary volume:

$$\int \vec{E}.\vec{J}\,dv = -\frac{\partial}{\partial t}\int \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right) dv - \int \nabla.\left(\vec{E} \times \vec{H}\right) dv$$

Using divergence theorem, we can change the last term in the equation and write as:

$$\int \vec{E}.\vec{J}\, dv = -\frac{\partial}{\partial t}\int \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right) dv - \oint_S \left(\vec{E}\times\vec{H}\right).\overrightarrow{dS}$$

The integral $\int \vec{E}.\vec{J}\, dv$ is the Joule heating (I^2R loss in ECE 140) and is equal to the instantaneous power dissipated in the volume due to the resistive loss in the medium. When you study antennas in ECE 475, then the integral $\int \vec{E}.\vec{J}\, dv$ will be negative and represent the power radiating out of the antenna.

$-\frac{\partial}{\partial t}\int \left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right) dv$ terms is the sum of the energy stored in magnetic and electric fields in the volume.

We know energy is conserved. The last term is an integral over a closed surface. Thus, from conservation of energy principles, the last term $-\oint_S \left(\vec{E}\times\vec{H}\right).\overrightarrow{dS}$ must represent the energy flowing in and out of a surface.

(If you skipped the derivation, this is where you need to start reading again).

$\vec{E}\times\vec{H}$ is called the Poynting Vector denoted by $\vec{\mathcal{P}}$ and it gives the **instantaneous power density** carried by the EM wave at a point. This is true for all EM waves, not just plane waves. Integrating it over the transverse surface will give us the total power at a plane. Thus,

$$\vec{\mathcal{P}}(x,y,z,t) = \vec{E}(x,y,z,t)\times\vec{H}(x,y,z,t)$$

Normally, we are not interested in the instantaneous power density but the average power density over one frequency cycle.

Let us consider a plane wave propagating in +z direction where:

$$\vec{E}(x,y,z,t) = \hat{x}E_0 e^{-\alpha z}\cos\left(\omega t - \beta z\right)$$

Thus, the magnetic field will be given by:

$$\vec{H}(x,y,z,t) = \hat{y}\frac{E_0}{\eta}e^{-\alpha z}\cos\left(\omega t - \beta z\right)$$

In general, the wave impedance is a complex number. Let us write $\eta = |\eta|e^{j\phi}$.

We can then write the magnetic field as:

$$\vec{H}(x,y,z,t) = \hat{y}\frac{E_0}{|\eta|}e^{-\alpha z}\cos\left(\omega t - \beta z - \phi\right)$$

Thus, we can write the Poynting vector as:

$$\vec{\mathcal{P}}(x,y,z,t) = (\hat{x}\times\hat{y})\frac{{E_0}^2}{|\eta|}e^{-2\alpha z}\cos\left(\omega t - \beta z\right)\cos\left(\omega t - \beta z - \phi\right)$$

Using $\cos(A)\cos(B) = \frac{1}{2}\cos(A+B) - \frac{1}{2}\cos(A-B)$, we can write the Poynting vector as:

$$\vec{\mathcal{P}}(x,y,z,t) = (\hat{x}\times\hat{y})\frac{{E_0}^2}{2|\eta|}e^{-2\alpha z}\left[\cos(\phi) + \cos\left(2\omega t - 2\beta z - \phi\right)\right]$$

If we take the time average of the Poynting vector i.e. do the operation: $\frac{1}{T}\int_0^T \vec{\mathcal{P}}(x,y,z,t)dt$, (where T is the time period), the integral of the second term in the bracket, $\cos(2\omega t - 2\beta z - \phi)$, will be 0 and only the integral of the first term will be 0.

Thus, the time averaged Poynting vector will be given as:

$$\frac{1}{T}\int_0^T \vec{\mathcal{P}}(x,y,z,t)dt = (\hat{x}\times\hat{y})\frac{{E_0}^2}{2[\eta]}e^{-2\alpha z}\cos(\phi)$$

Remember for AC circuits from ECE 140, the time averaged power is given as: $\frac{V^2}{2[Z]}\cos(\theta)$ where $\cos(\theta)$ is the power factor of the load impedance, Z. Notice that there are similarities in the results.

I want to come up with a better form for calculating time averaged Poynting vector, so we don't have to do the integral every single time. Let us consider two vectors which could be complex: $\vec{A}, and\ \vec{B}$. We can write the real part of the vectors as:

$$Re\{\vec{A}\} = \frac{1}{2}(\vec{A}+\vec{A}^*) \text{ and } Re\{\vec{B}\} = \frac{1}{2}(\vec{B}+\vec{B}^*)$$

Where $\vec{A}^*$ and $\vec{B}^*$ are the complex conjugate of the vectors.

Let us consider:

$$Re\{\vec{A}\}\times Re\{\vec{B}\} = \frac{1}{2}(\vec{A}+\vec{A}^*)\times\frac{1}{2}(\vec{B}+\vec{B}^*) = \frac{1}{4}(\vec{A}\times\vec{B}+\vec{A}\times\vec{B}^*+\vec{A}^*\times\vec{B}+\vec{A}^*\times\vec{B}^*)$$

But we can rearrange these terms as:

$$Re\{\vec{A}\}\times Re\{\vec{B}\} = \frac{1}{4}\left(\vec{A}\times\vec{B}+(\vec{A}\times\vec{B})^*+\vec{A}\times\vec{B}^*+(\vec{A}\times\vec{B}^*)^*\right)`$$

We can combine the first two terms as: $\vec{A}\times\vec{B}+(\vec{A}\times\vec{B})^* = 2\,Re(\vec{A}\times\vec{B})$ and the second term as: $\vec{A}\times\vec{B}^*+(\vec{A}\times\vec{B}^*)^* = 2Re(\vec{A}\times\vec{B}^*)$. Thus,

$$Re\{\vec{A}\}\times Re\{\vec{B}\} = \frac{1}{2}\left[Re(\vec{A}\times\vec{B})+Re(\vec{A}\times\vec{B}^*)\right]$$

We know that the Poynting vector for the plane wave is:

$$\vec{\mathcal{P}}(z,t) = Re\{E_0 e^{j\omega t}\hat{x}\}\times Re\left\{\frac{E_0}{\eta}e^{j\omega t}\hat{y}\right\}$$

Using our relationship above, we can write our instantaneous Poynting vector as:

$$\vec{\mathcal{P}}(z,t) = \frac{1}{2}\left[Re\left\{E_0 e^{j\omega t}\hat{x}\times\frac{E_0}{\eta}e^{j\omega t}\hat{y}\right\}+Re\left\{E_0 e^{j\omega t}\hat{x}\times\left(\frac{E_0}{\eta}e^{j\omega t}\hat{y}\right)^*\right\}\right]$$

$$\vec{\mathcal{P}}(z,t) = \frac{1}{2}\left[Re\left\{E_0 e^{j\omega t}\hat{x}\times\frac{E_0}{\eta}e^{j\omega t}\hat{y}\right\}+Re\left\{E_0\hat{x}\times\left(\frac{E_0}{\eta}\hat{y}\right)^*\right\}\right]$$

When we do the time averaging then, the integral of the first term vanishes and only second term survives. Thus, the time average Poynting vector is given as:

$$\vec{\mathcal{P}}_{av}(z) = \frac{1}{2}\, Re\left\{E_0\hat{x} \times \left(\frac{E_0}{\eta}\hat{y}\right)^*\right\}$$

Or we can write the time averaged Poynting vector as:

$$\vec{\mathcal{P}}_{av}(z) = \frac{1}{2}\, Re\left\{\overrightarrow{E(z)} \times \left(\overrightarrow{H(z)}\right)^*\right\}$$

In general, the time averaged Poynting vector is given by:

$$\vec{\mathcal{P}}_{av}(x, y, z) = \frac{1}{2}\, Re\left\{\vec{E}(x, y, z) \times \left(\vec{H}(x, y, z)\right)^*\right\}$$

Please note, it only the spatial component of the field which goes in the time averages Poynting vector.