Chapter 5: Scattering Parameters

Review Book: Microwave Engineering by Pozar, Chapter 4.1-4.3

5.1 Introduction

As we have seen, circuits operating at low frequencies can be treated as lumped elements and you can characterize them by measuring the voltage and current across the ports. You have a microwave circuit in a black box with N-ports. You want to characterize its performance in terms of voltage and current so you can know how it will perform in your circuit. We now have a problem because the transmission lines we will need to connect to the circuit will alter the behavior. We need to figure a way out in our measurement techniques on how to handle the transmission line characteristics within the measurements. In this chapter we will see how basic circuit and network concepts can be extended to handle high speed circuits. We do this because while Maxwell's equations provide the most accurate solutions, circuit analysis is easier to apply. Especially for a circuit, we are normally interested in knowing the voltage and current at the ports of the circuit. However, as we have seen before, the measurements of voltage and current at the node is difficult to do. In this chapter, we will learn a method, called Scattering parameters, to experimentally characterize high speed circuits.

5.2 Impedance and Admittance Matrix

Let us start with how we characterize circuits at low frequency. Consider a circuit or a network with N-ports as shown in the figure below. There will be N transmission lines connected to the network at each port. Each of these lines will carry a wave forward and a wave backwards from the network. We will use the convention of wave coming in as positive and a wave coming out of the network as negative. The ultimate goal we are interested in find relations between the forward and backward waves. We first start with first seeing how we can relate the voltages and currents on each port.

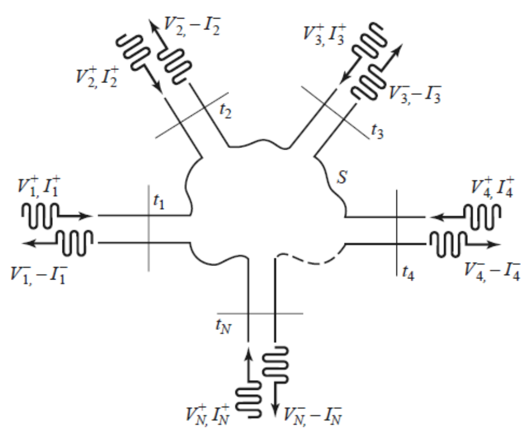


Figure 1. N-port network with input and output waves at each port.

Let us assume that V_i^+ and V_i^- are the forward and backward voltage waves on port i (for $i = 1, 2, ... N$) and corresponding to that we have current waves as I_i^+ and I_i^-. The convention we are using is putting a different z-axis on every transmission line with z=0 at the port. Thus, the voltage V_i and current I_i we will get at every single port will be given as:

$$V_i = V_i^+ + V_i^-$$

$$I_i = I_i^{+} - I_i^{-}$$

We can relate the voltage V_i and current I_i with a matrix as:

$$\begin{bmatrix} V_1 \\ V_2 \\ .. \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \dots & Z_{1N} \\ Z_{21} & Z_{22} & \dots & Z_{2N} \\ \dots & \dots & \dots & \dots \\ Z_{N1} & Z_{N2} & \dots & Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ .. \\ I_N \end{bmatrix}$$

Or in short form as:

$$[V] = [Z][I]$$

$[Z]$ is called the impedance matrix.

What does this mean in terms of measurements? We apply currents to the different ports and measure the voltages. Let us open V_1 up and see how we can measure the various components of $[Z]$.

$$V_1 = Z_{11}I_1 + Z_{12}I_2 + \cdots + Z_{1N}I_N$$

We can write Z_{11} as:

$$Z_{11} = \frac{V_1}{I_1}\bigg]_{I_2, I_3 \dots\dots I_N = 0}$$

To measure Z_{11}, we keep ports I_2, I_3, …. I_N open so no current can pass through those ports and measure the voltage on V_1. While we are doing this, we can also measure the matrix component Z_{i1} by measuring the voltage V_i on the i^{th} port. Thus, in general we can write:

$$Z_{ij} = \frac{V_i}{I_j}\bigg]_{I_k = 0 \; for \; k \neq j}$$

Z_{ij} is the “transfer” impedance between port “*i*” and “*j*” when all other ports are open circuited.

Z_{ij} is the input impedance seen looking into port “*i*” when all other ports are open-circuited.

By doing these measurements and characterizing the impedance matrix, we can know how the circuit will respond to the electrical elements connected outside. We can know everything there is to know about a circuit.

We could also have done the opposite and applied voltage instead of current and then we get an admittance matrix given as:

$$[I] = [Y][V]$$

For calculating admittance matrix, we short the ports (so that voltage on them goes to 0) instead of keeping them open-circuit. Obviously, the two matrices are related by inverse relations such that:

$$[Y] = [Z]^{-1}$$

You should have seen these matrices in your circuits courses, so we are not going to spend much time on these but can we really do the above experiments in high frequency to measure the impedance or admittance matrices? Remember any scopes we use have to be connected to the ports with transmission lines and the values change when we go to different lengths of the transmission lines. We need a new way forward.

5.3 Scattering Matrix

Looking at the network in Figure 1, instead of measuring voltages and currents, we can also measure the forward and backward waves in the ports and relate them by a matrix. Such a matrix is called the Scattering matrix and is written as:

$$\begin{bmatrix} {V_1}^- \\ {V_2}^- \\ .. \\ {V_N}^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1N} \\ S_{21} & S_{22} & \dots & S_{2N} \\ \dots & \dots & \dots & \dots \\ S_{N1} & S_{N2} & \dots & S_{NN} \end{bmatrix} \begin{bmatrix} {V_1}^+ \\ {V_2}^+ \\ .. \\ {V_N}^+ \end{bmatrix}$$

For the time being for the same of developing an understanding, we are measuring these waves right at the port itself. We will correct this later. We are also assuming that we all connecting all ports with transmission lines of same characteristics impedance, Z_0. Now practically this is what will be done (we won't keep different types of cables just for fun, will we) but we can extend the S parameters to generalized form, which will be covered in ECE 373.

Let us see what the scattering parameters, S, imply. Let us open up ${V_1}^-$ and we get:

$${V_1}^- = S_{11}{V_1}^+ + S_{12}{V_2}^+ + \cdots + S_{1N}{V_N}^+$$

What will S_{11} be?

$$S_{11} = \left.\frac{{V_1}^-}{{V_1}^+}\right]_{{V_2}^+, {V_3}^+ \dots {V_N}^+ = 0}$$

It can be considered as the reflection coefficient on port 1 provided there are no waves entering the network at any other port. **Can we achieve this by keeping the other ports open circuit like we did for impedance matrices or closed circuit like we did for admittance matrices.** Absolutely not. Some of ${V_1}^+$ may exit the other ports and if there is an open circuit or short circuit there, will get reflected back and viola, the condition is broken. So how can we achieve no incoming waves at other ports?

There is only one way. We have to match the other ports by terminating them with a load equal to the transmission line characteristic impedance, Z_0. No matter if a wave is transmitted to another port, there will be no reflections at the port and hence, no wave entering the network at that port.

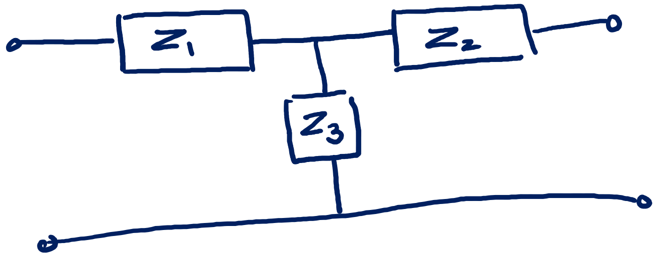
So S_{11} can be defined as the reflection coefficient on port 1 provided all other ports are terminated with a matched load**. This termination is very important and implied in the S-parameter definition**. What is S_{12}. It is the transmission coefficient from port 2 to port 1 when all other ports except for 2 are terminated with a matched load. We apply a source to port 2 and measure how much voltage wave exists on port 1. Remember port 1 has to be matched also.

In general, we can write a S parameters as:

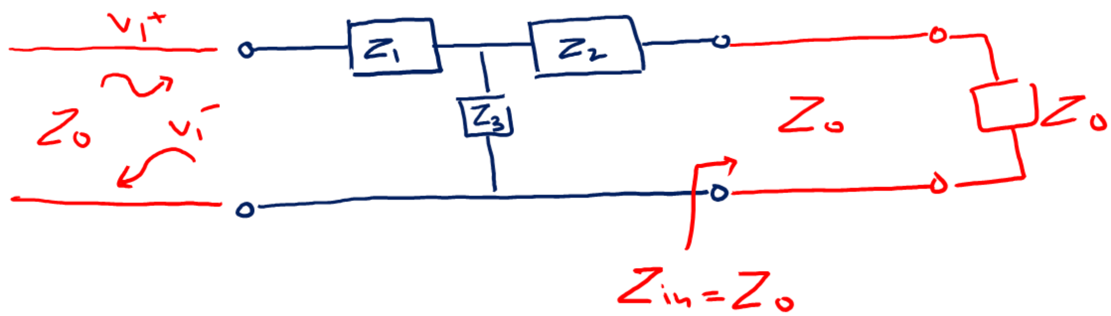
$$S_{ij} = \left.\frac{{V_i}^-}{{V_j}^+}\right]_{all\ ports\ except\ for\ j\ terminated\ with\ a\ matched\ load}$$

In practice, though not necessary, the source impedance on port *j* is also matched to the transmission line. If it is not done so, we will have to set up a calculation for V^+.

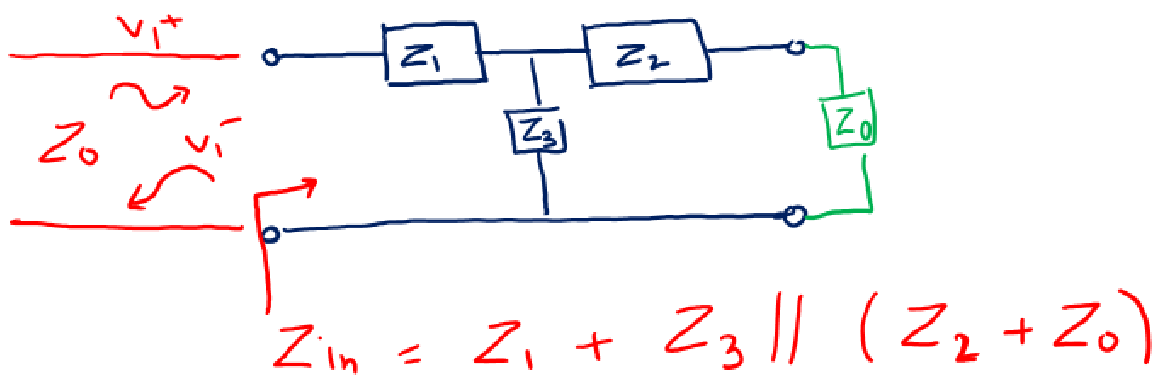
Example 1. Let us calculate the S parameters for the following circuit.



Solution: First question, how many ports does the network have? We have 2 ports and thus, we will have a 2 x 2 matrix consisting of S_{11}, S_{12}, S_{21}, S_{22}. To calculate S_{11} and S_{21}, we apply our signal to port 1 and measure the reflections on port 1 and transmission to port 2. However, we have to make sure, we are have connected a matched load to port 2. Thus, the setup looks like:



We have connected transmission lines to our ports and terminated the transmission line on port 2 with a matched load. Since, port 2 is matched, the input impedance we will see will be Z_0 and the circuit looking from port 1 will look as:



The input impedance we see on port 1 will be Z_2 in series with Z_o the combination of which is parallel with Z_3, and the combination in series with Z_1. This is load which is seen on the transmission line. So we get:

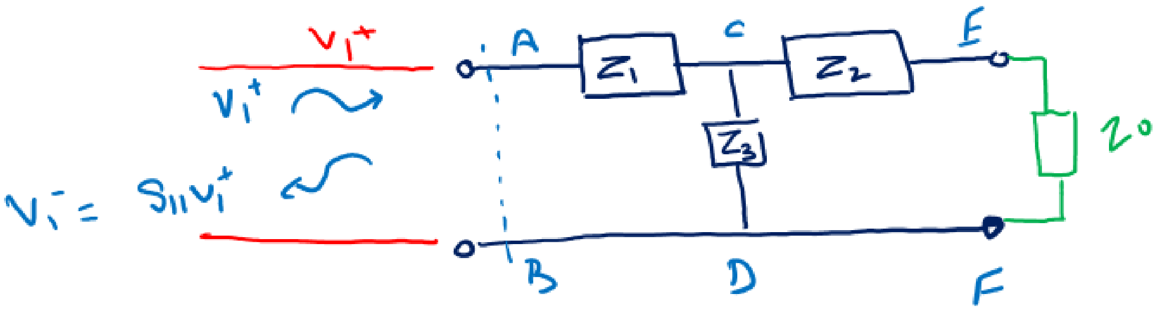
$$Z_{in} = Z_1 + Z_3//(Z_2 + Z_0) = Z_1 + \frac{Z_3(Z_2 + Z_0)}{Z_3 + Z_2 + Z_0}$$

The reflection created at port 1, will be the value of S_{11} and can be written as:

$$S_{11} = \frac{V_1^{-}}{V_1^{+}} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

Thus, S_{11} can be calculated by terminating port 2, and then seeing what equivalent impedance looks at port 1. That impedance acts like a load on the transmission line and thus, reflection coefficient can be calculated.

To calculate S_{21}, we have to calculate the voltage which is dropped on port 2. That voltage will excite a wave on port 2 (remember port 2 is matched and hence, there is no reflected wave). We follow the rules you learned in ECE 140. Let us see the steps and we refer to the figure below:



The voltage across A-B will be:

$$V_{AB} = {V_1}^+ + S_{11}{V_1}^+ = (1 + S_{11}){V_1}^+$$

Part of this voltage drops across CD will be voltage division and will be:

$$V_{CD} = \frac{Z_3//(Z_2 + Z_0)}{Z_3//(Z_2 + Z_0) + Z_1} V_{AB} = \frac{Z_3//(Z_2 + Z_0)}{Z_3//(Z_2 + Z_0) + Z_1}(1 + S_{11}){V_1}^+$$

Voltage on port 2 will be another voltage division between Z_2 and Z_0. We get:

$$V_{EF} = \frac{Z_0}{Z_0 + Z_2} V_{CD} = \frac{Z_0}{Z_0 + Z_2} \frac{Z_3//(Z_2 + Z_0)}{Z_3//(Z_2 + Z_0) + Z_1}(1 + S_{11}){V_1}^+$$

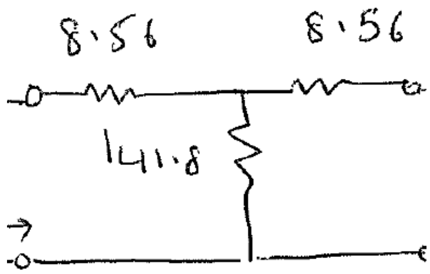
The voltage V_{EF} is going to transmit as ${V_2}^-$ and we can calculate S_{21} as:

$$S_{21} = \frac{{V_2}^-}{{V_1}^+} = \frac{Z_0}{Z_0 + Z_2} \frac{Z_3//(Z_2 + Z_0)}{Z_3//(Z_2 + Z_0) + Z_1}(1 + S_{11})$$

To calculate S_{12} and S_{22}, we repeat the process with matched load on port 1 and wave applied on port 2.

For this course, we will only limit ourselves to calculating S-parameters to 2 port circuits but you should be able to extend it to N port circuits with the same thinking.

Practice the method with the following circuit using a transmission line of characteristics impedance of 50 Ω:



Important question: Will the S-parameters change if the characteristics impedance of the transmission line was 75 Ω?

Remember, S-parameters are defined with respect to the transmission lines used in the test system.

Note: In general, S_{ij} is a complex number and normally written as $|S_{ij}| \angle \theta$

which basically means:

$$|S_{ij}|e^{j\theta}$$

5.4 Relation between S and Z matrix

S and Z matrix are not independent of each other and from one matrix you can get the other. We know that:

$$V_i^- + V_i^+ = V_i$$

and

$$I_i^+ - I_i^- = I_i$$

However, the waves are related to each other by the characteristics impedance of the transmission line.

$$I_i = \frac{V_i^+ - V_i^-}{Z_0}$$

We know that the voltage and current at the ports are related by the characteristics impedance. Thus, we get:

$$[V] = [Z][I]$$

$$[V^+] + [V^-] = [Z]\{[I_i^+] - [I_i^-]\} = \frac{[Z]}{Z_0}\{[V^+] - [V^-]\}$$

To work with matrix, we need to multiply the right hand side terms with the unitary matrix [U].

$$[U][V^+] + [U][V^-] = \frac{[Z]}{Z_0}\{[V^+] - [V^-]\}$$

Reworking the terms, we get:

$$\left\{\frac{[Z]}{Z_0} + [U]\right\}[V^-] = \left\{\frac{[Z]}{Z_0} - [U]\right\}[V^+]$$

So we get:

$$[S] = \left\{\frac{[Z]}{Z_0} + [U]\right\}^{-1}\left\{\frac{[Z]}{Z_0} - [U]\right\}$$

We can do similar work the other way, and we will get:

$$[Z] = Z_0\{[U] + [S]\}\{[U] - [S]\}^{-1}$$

5.5. Reciprocal Network

A reciprocal network is a network where the transmission of a wave from one port to the other does not depend on the direction of propagation and thus, input and output ports are changeable. A reciprocal network does not contain any active circuit elements like amplifiers, diodes etc. and no anisotropic materials like ferrites or plasmas (ferrite materials change how the wave travels depending on direction and we use them to make optical isolators which only allow light to pass through in one direction). In a reciprocal network, if we apply a wave on port *i* and see the transmission on port *j* (with all ports matched), then if we applied the same wave on port *j*, we will get the same value of transmission on port *i*.

The scattering matrix for a reciprocal network is symmetric which basically means that the transpose of the matrix is equal to the matrix itself i.e.

$$[S]^T = [S]$$

Every element S_{ij} should be equal to S_{ji}.

5.6 Loss less network

In a loss less network, no real power is delivered to the network. Thus, all the power which is incident on the ports, also leave the ports. We cannot have any resistors inside the network as resistors will absorb the real power.

For a loss-less network, the scattering matrix is unitary i.e.

$$[S]^T[S]^* = [U]$$

We can reduce this operation to column wise multiplication as:

$$\sum_{k=1}^{N} S_{ki}{S_{ki}}^* = \sum_{k=1}^{N} |S_{ki}|^2 = 1$$

and

$$\sum_{k=1}^{N} S_{ki}{S_{kj}}^* = 0 \text{ for } j \neq i$$

In words, this operation is that we take a column and do a dot product of the column with itself (multiply all a row term with its complex conjugate and add the results), we should get a sum of 1. Every column should do that. But that is not all. When we take a column and take a dot product with another column, we should get a 0. This should happen for every pair of columns we have.

Let us consider a two port network with the following S-matrix:

$$[S] = \begin{bmatrix} 0.15\angle 0^\circ & 0.85\angle -45^\circ \\ 0.85\angle 45^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

Is the network reciprocal and lossless?

If you see carefully, the transpose of the matrix is not the same as itself. S_{12} is not equal to S_{21}. They have the same magnitude but different phases. Phases also matters in determining whether the network is reciprocal or not. The network is not reciprocal.

Let us look at whether it is lossless. We can start with the first column and do a dot product. We take the square of amplitude of each term in the first column and add the terms together. If it is lossless network, the sum should be 1. If it is 1, we then need to move to the next column and make sure the rule is obeyed for all columns. If it is, then we start multiplying terms from two columns and adding them up and that sum should be 0.

Let us look at the first column and do the dot product and we get:

$$|0.15|^2 + |0.85|^2 = 0.745 \neq 1$$

A rule is broken in the first step itself and we do not need to go any further. The network is not loss-less.

5.7 Transmission and Reflection Coefficients for terminated ports

Please remember again that S parameters only represent the reflection coefficients on a port and transmission coefficients between ports when the ports are terminated with matched loads. If we have networks which are terminated with loads other than matched loads, the value of the reflection and transmission coefficients will change. We can use the S-parameters but will have to add other equations at the ports which are terminated. Let us consider an example to understand how. Let us take the network represented by the S parameters given in the matrix above i.e.:

$$[S] = \begin{bmatrix} 0.15\angle 0^\circ & 0.85\angle -45^\circ \\ 0.85\angle 45^\circ & 0.2\angle 0^\circ \end{bmatrix}$$

We terminate port 2 with a short circuit. What is the reflection coefficient seen at port 1?

We start by writing the relations between the reflected waves and the transmitted waves. I am doing it in the general form. These can be written as:

$$V_1{}^- = S_{11}V_1{}^+ + S_{12}V_2{}^+$$

$$V_2{}^- = S_2V_1{}^+ + S_{22}V_2{}^+$$

However, because port 2 is short circuited, the wave $V_2{}^-$ which leaves the network will get reflected back and come into the network as $V_2{}^+$. Thus,

$$V_2{}^+ = -1 * V_2{}^-$$

We can substitute the equation in the second equation above to get:

$$-V_2{}^+ = S_{21}V_1{}^+ + S_{22}V_2{}^+$$

$$(1 + S_{22})V_2{}^+ = -S_{21}V_1{}^+$$

Or

$$V_2{}^+ = \frac{-S_{21}V_1{}^+}{(1 + S_{22})}$$

We can substitute the term in our first equation to get:

$$V_1{}^- = S_{11}V_1{}^+ + S_{12}\left(\frac{-S_{21}V_1{}^+}{(1 + S_{22})}\right) = \left(S_{11} - \frac{S_{12}S_{21}}{1 + S_{22}}\right)V_1{}^+$$

You can substitute the values to get the exact numbers. The reflection coefficient will be:

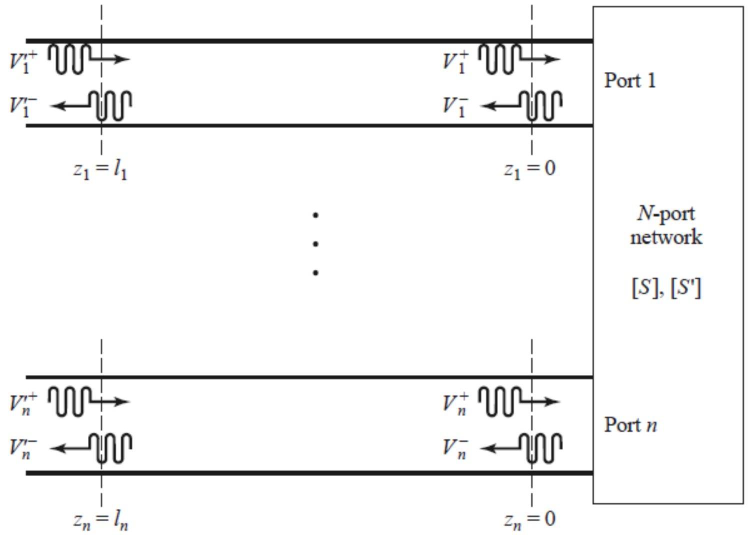
$$\frac{V_1{}^-}{V_1{}^+} = \left(S_{11} - \frac{S_{12}S_{21}}{1 + S_{22}}\right)$$

Let us try to understand what do these terms physically tell us (we are going back to our transient thinking to understand). A wave $V_1{}^+$ hits the network and an immediate reflection of S_{11} is created. That is the first term. Another part of the wave gets transmitted to port 2 with a transmission coefficient of $S_{21.}$ There the wave bounces back and forth between the termination with a coefficient of -1 and the network with a coefficient of S_{22}. As it bounces back and forth, it creates a geometric series of the form 1 + a + a^2 + a^3 + …. where "a" is $\Gamma_L S_{22}, where\ \Gamma_L = -1$ and thus, we get a term which will be $\frac{S_{21}}{1+S_{22}}$, S_{21} due to how much was transmitted and $\frac{1}{1+S_{22}}$ due to the infinite bouncing back and forth in a cavity that is created. Part of this total is transmitted to port 1 with transmission coefficient of S_{12}. That is what the second term is representing.

The network of port 2 is acting like a cavity, like a drum for example and the results we see are the beats of the drum.

5.8 Shift in Reference Plane

Now let us consider what happens when we have transmission lines of lengths added to the ports as shown in the figure below. Instead of measuring right at the port, we will be actually measuring at the ends of the transmission lines.



We can write the relationship between the waves on a transmission line as:

$$V_n'^+ = V_n^+ e^{j\theta_n},$$
$$V_n'^- = V_n^- e^{-j\theta_n},$$

The angle is the electrical length and given as:

$$\theta_n = \beta_n \ell_n$$

What we want to do is relate the S matrix measured right at the network port (as we have been doing) with the S matrix measured at the end of the transmission line. Let us call that S'. We have:

$$[V^-] = [S][V^+],$$
$$[V'^-] = [S'][V'^+],$$

We can substitute the value and write the equation as:

$$\begin{bmatrix} e^{j\theta_1} & & 0 \\ & e^{j\theta_2} & \\ & \ddots & \\ 0 & & e^{j\theta_N} \end{bmatrix} [V'^-] = [S] \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & e^{-j\theta_2} & \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} [V'^+].$$

Multiplying by the inverse of the first matrix we get:

$$[V'^-] = \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & e^{-j\theta_2} & \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} [S] \begin{bmatrix} e^{-j\theta_1} & & 0 \\ & e^{-j\theta_2} & \\ & \ddots & \\ 0 & & e^{-j\theta_N} \end{bmatrix} [V'^+].$$

Thus, the shifted reference plane S' matrix can be written as:

$$[S'] = \begin{bmatrix} e^{-j\theta_1} & & & 0 \\ & e^{-j\theta_2} & & \\ & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix} [S] \begin{bmatrix} e^{-j\theta_1} & & & 0 \\ & e^{-j\theta_2} & & \\ & & \ddots & \\ 0 & & & e^{-j\theta_N} \end{bmatrix}$$

The matrices around [S] account for the phase delays the wave has to take through the transmission line.