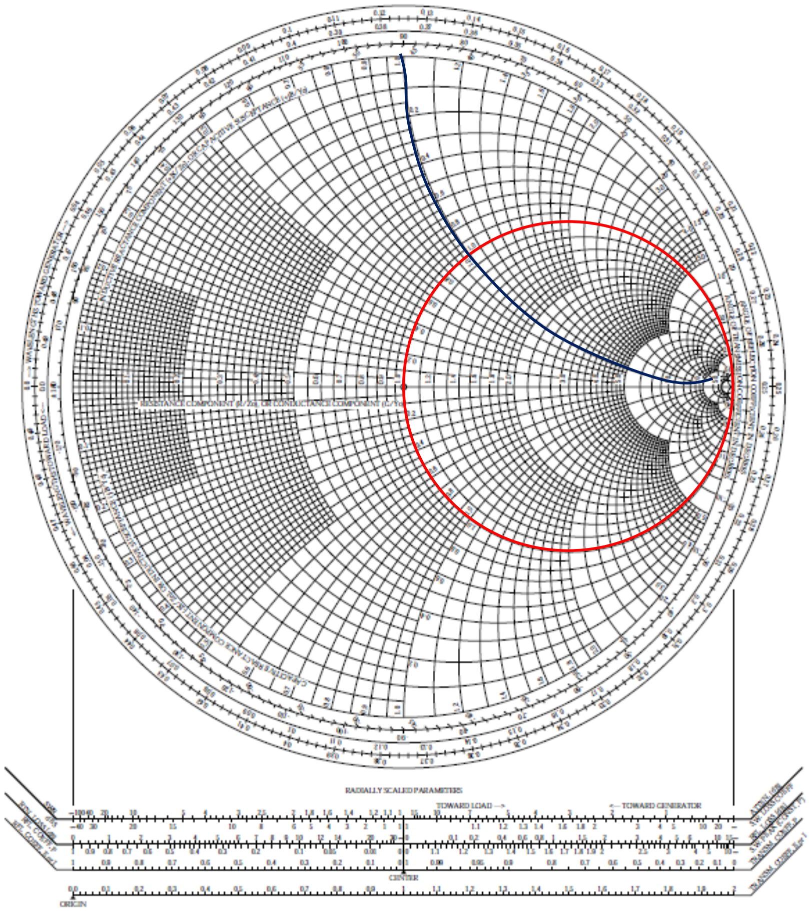
Chapter 4: Smith Chart

Note: To read this chapter, print and keep a Smith Chart with you so you can refer to it. Also you need a compass.

Smith Chart was invented by Phillip H. Smith in second world war and is a graphical calculator to assist solving problems for transmission lines and played an instrumental role in design of radar systems. One may think that in our day and age of high speed computing, Smith Chart has no relevance and it probably is true from the calculation perspective but it still provides a very powerful visual tool and allows one to understand what is happening in Transmission lines. Further, and more importantly, commonly used pieces of laboratory equipment like the virtual network analyzers imitate the Smith Chart on the displays as once you get knowledgeable about them, it is much easier to find faults on your circuits using Smith Charts. In this Chapter, we will learn how to use Smith Charts for the calculations we have been doing and will use Smith Charts for matching the loads in Chapter 7 Let us get started.

4.1 What Does Smith Chart Look Like?

The Smith Chart is shown in the figure below. Looks complex and scary but it is not. In the center, if you notice we have two types of circles. There are full circles which all meet in the right corner but their radius and center change (one such circle is marked in red). Then there are other types of circles which are not full and seem to be going out of the chart (one such circle is marked in black). Around this center region are circles with scales on them and at the bottom part are some straight scales. We will look at each of these features and understand what they mean.



Let us start with the full circles. **I am doing the derivation but if you want you can skip and just go to the end and see what the full circles represent** (put in bold). In Chapter 3, we had seen that the input impedance at any value of z on the transmission line is given as:

$$Z(z) = Z_0 \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

where $\Gamma(z)$ is the generalized reflection coefficient. We want to use the same chart for any value of transmission line characteristic impedance. Thus, it makes sense to normalize all the impedances. In terms of normalized impedance, we will get:

$$\overline{Z(z)} = \frac{Z(z)}{Z_0} = \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

We know that in general $\Gamma(z)$ is a complex number. Let us call it:

$$\Gamma(z) = u + jv$$

Thus, we can write the normalized input impedance as:

$$\overline{Z(z)} = \bar{R} + j\bar{X} = \frac{1+\mathrm{u}+\mathrm{jv}}{1-\mathrm{u}-\mathrm{jv}}$$

Multiplying and dividing RHS by the complex conjugate of the denominator we get:

$$\overline{Z(z)} = \bar{R} + j\bar{X} = \frac{1+\mathrm{u}+\mathrm{jv}}{1-\mathrm{u}-\mathrm{jv}} * \frac{1-\mathrm{u}+\mathrm{jv}}{1-\mathrm{u}+\mathrm{jv}} = \frac{(1-u^2)-v^2}{(1-u)^2+v^2} + j\frac{v(1-u)+v(1+u)}{(1-u)^2+v^2}$$

$$\overline{Z(z)} = \bar{R} + j\bar{X} = \frac{(1-u^2)-v^2}{(1-u)^2+v^2} + j\frac{2v}{(1-u)^2+v^2}$$

Equating the real parts of the equation we get:

$$\bar{R}[(1-u)^2+v^2] = (1-u^2)-v^2$$

This is actually an equation of a circle. Let us arrange it to see how. Expanding we get:

$$(\bar{R}-1) + (\bar{R}+1)u^2 - 2u\bar{R} + (\bar{R}+1)v^2 = 0$$

Let us divide the whole equation by $(\bar{R}+1)$ and we get:

$$\frac{(\bar{R}-1)}{(\bar{R}+1)} + \underbrace{u^2 - \frac{2u\bar{R}}{(\bar{R}+1)}} + v^2 = 0$$

If we look at the terms highlighted by the brackets, they are begging to be converted into a full square. We can do so by adding $\left(\frac{\bar{R}}{(\bar{R}+1)}\right)^2$ term to it. Obviously we will have to subtract it also. So we get:

$$\frac{(\bar{R}-1)}{(\bar{R}+1)}-\left(\frac{\bar{R}}{(\bar{R}+1)}\right)^2+u^2-\frac{2u\bar{R}}{(\bar{R}+1)}+\left(\frac{\bar{R}}{(\bar{R}+1)}\right)^2+v^2=0$$

And we get:

$$\left(u-\frac{\bar{R}}{(\bar{R}+1)}\right)^2+v^2=\left(\frac{1}{1+\bar{R}}\right)^2$$

Viola, a circle shows up in u, v plane with center at $\left(\frac{\bar{R}}{(\bar{R}+1)},0\right)$ and radius equal to $\frac{1}{1+\bar{R}}$. **Full circles represent the real part of the normalized impedance. Let us look at the Smith Chart now. The value of the circle is written next to it as it passes the horizontal axis (u as we have called). For example, the circle with $\bar{R}$= 1 is centered at (1/2, 0) and has a radius of ½. Larger the value of the $\bar{R}$ becomes, the smaller the radius and the center of the circle moves to the right. Thus, in the Smith Chart, the larger the circle, smaller the value of the real part of the normalized impedance.**

Equating the imaginary part in the equation $\bar{R}+j\bar{X}=\frac{(1-u^2)-v^2}{(1-u)^2+v^2}+j\frac{2v}{(1-u)^2+v^2}$ and doing similar algebra as we have above, we also get another equation in $\bar{X}$ as:

$$(u-1)^2+\left(v-\frac{1}{\bar{X}}\right)^2=\left(\frac{1}{\bar{X}}\right)^2$$

This again is an equation of a circle in (u, v) plane with center at (1, $\frac{1}{\bar{X}}$). Now $\bar{X}$ we know can be negative or positive. Thus, we have two possible values for v and two circles for the same magnitude of $\bar{X}$. **The partial circles represent this equation. Let us look at the Smith Chart. The partial circles on the top half are +$\bar{X}$ and thus represent inductive loads and the circles on the bottom half are -$\bar{X}$ and represent capacitive loads. The values of the circles are given along the circumference and also along $\bar{R}=1$ circle. Similarly values of some $\bar{R}$ circles are given along the $\bar{X}=1$ circle.**

The intersection between the two circles gives the value of $\bar{R}+j\bar{X}$. As a practice, locate the normalized impedances 1+j1, 1+j2, 2+j1, 2+j2.

Let us look at the origin of the Smith Chart. It is not 0 but 1. We had seen that $V(z)=V^+e^{-j\beta z}\left[1+\rho e^{j(\psi+2\beta z)}\right]$. The origin of 1 is to represent the vector $1+\rho e^{j(\psi+2\beta z)}$ which changes as we go to different values of z. The graphical representation we had done in Chapter 3 can be done using the origin here.

If we go to the outside scales, we will see, the first scale is a compass with angle in degrees. It has two angles, one on the inside, and the other on the outside. The inner angle represents the phase of the reflection coefficient, ψ as we have defined it. The outside angle is simply $2\pi-\psi$. There are two other scales. The inner one is the distance away from the source (in terms of wavelength) and would have been used if we had put z=0 at the source side. The outer one is the distance away from the load, and this is what we use. Look around the scale and you will see it goes from 0 to 0.5. Now the scales are fixed in the chart, so to use the scale, we will have to use the reference.

The scales on the bottom allow you to calculate the reflection coefficient, ρ, and the standing wave ration, S. There other ways to calculate these values and we will see how we can get these values in multiple ways.

There are some more complex versions which add admittance circles also but as I will show you, the Smith Chart we are using is sufficient to fully do the problems.

Let us start looking at how to use the Smith Chart.

4.2 How to use Smith Chart

Let us illustrate how to use Smith Chart using an example. First let us identify some loads.

Where is the short circuit load?

Where is an open circuit load?

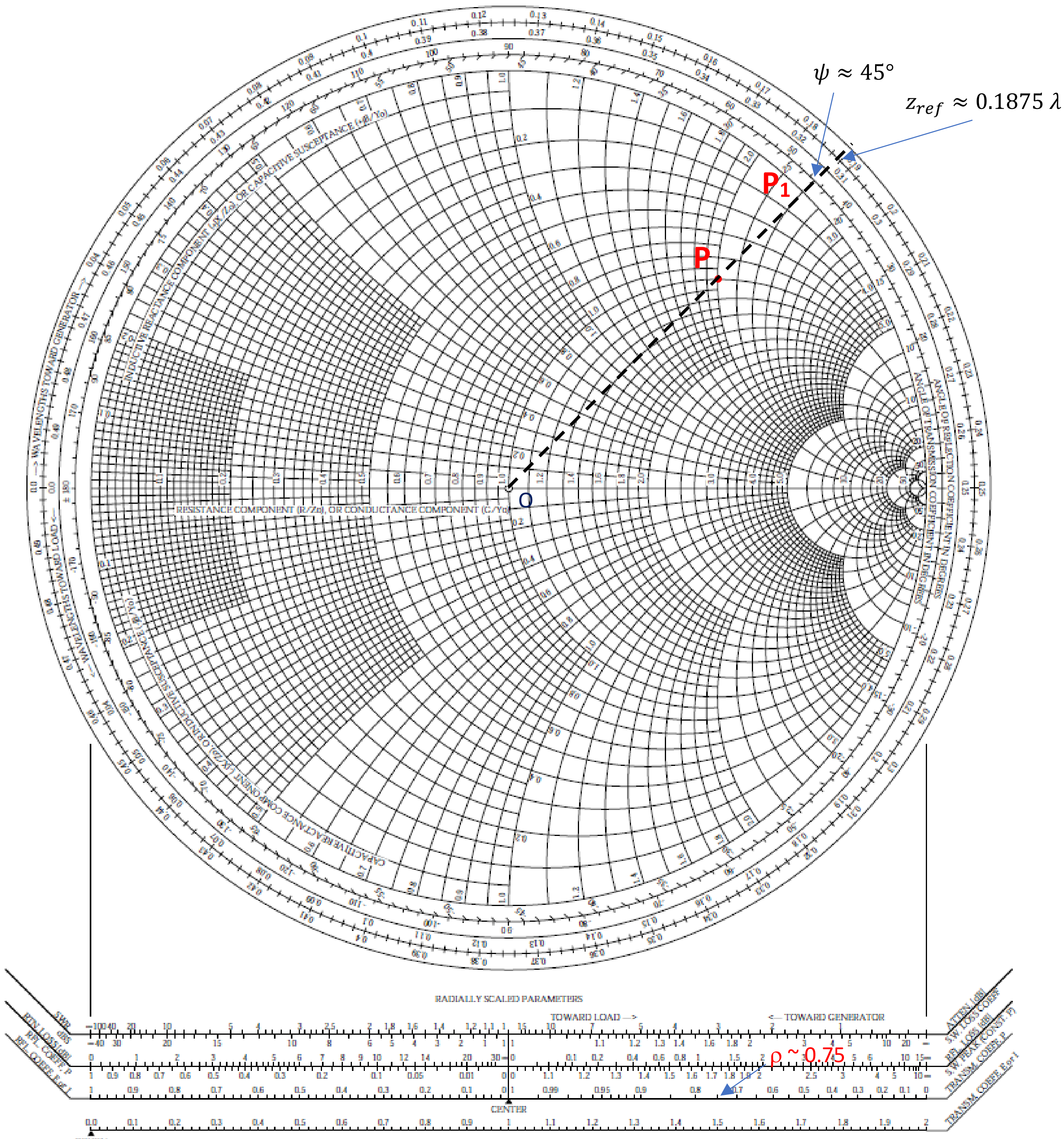
Where is a purely reactive load of the type $\pm jX$?

(see corresponding video for the answers, these are important points on the Smith Chart).

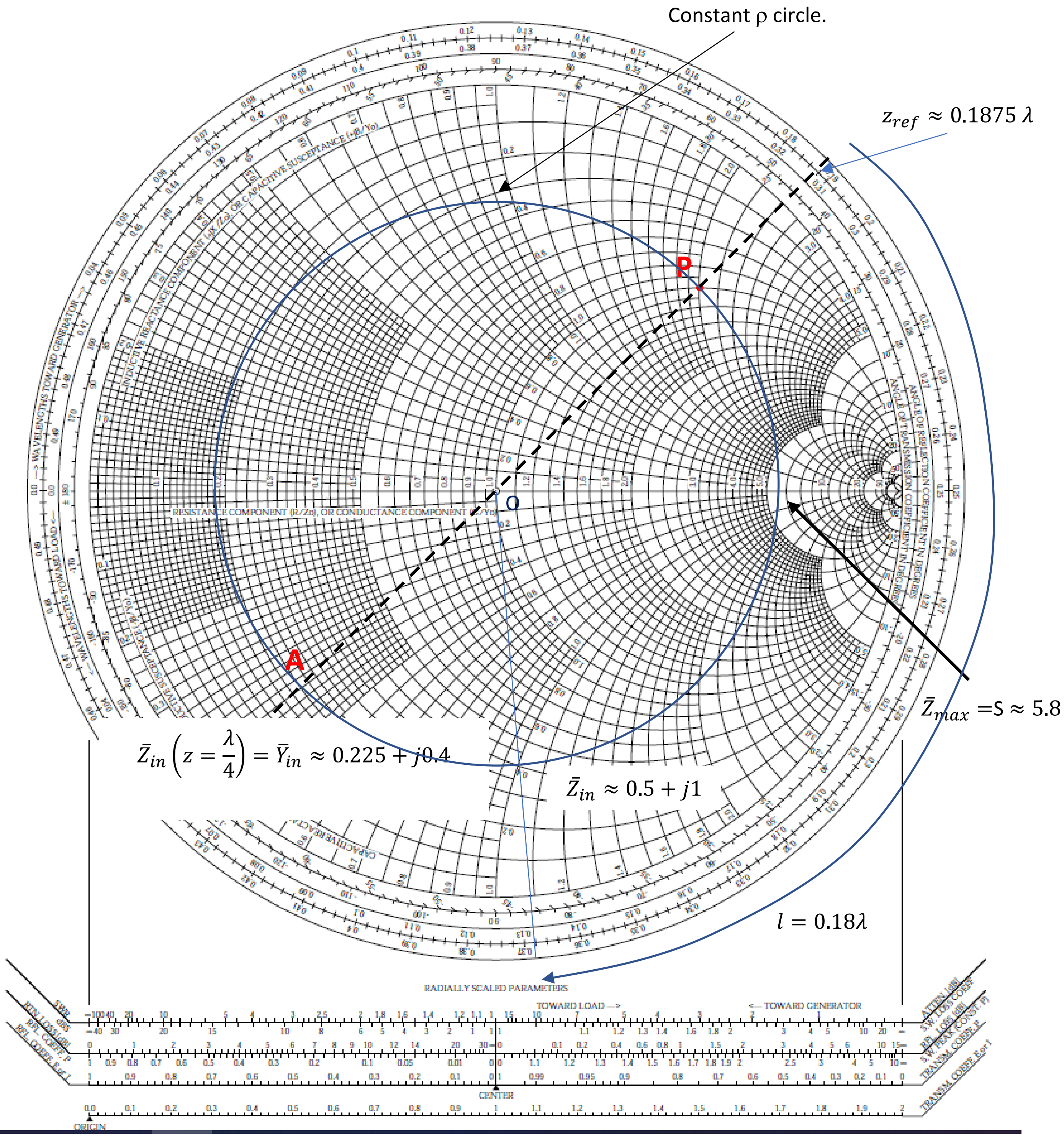
Let us now consider a load 50+j100 Ω connected to the transmission line of Z_0 = 50 Ω. First we normalize the load with the characteristics impedance. Thus, we get:

$$\overline{Z_L} = \frac{Z_L}{Z_0} = 1 + j2$$

We locate the load on the Smith Chart as the intersection between the real circle of 1 and imaginary circle of 2 as shown below and marked as P. To calculate the value of ψ, we connect a straight line from the center to the scale for the load reflection coefficient. This is shown in the chart with a dashed black line. To calculate ρ and S, we can measure the length of OP and use the corresponding scale below. That is one way. But it is a difficult scale to read but it is a difficult scale to read as it is non-linear. **Another way we can measure ρ is by realizing that the value of ρ = 1 for a purely reactive load. A purely reactive load is on the circumference of the chart.** Thus, the length OP_1 on the chart should represent a value of 1. The length for our load is OP. Therefore, ρ can simply be measured as the ratio of the lengths OP/OP_1. Once, we have ρ, it is pretty straightforward to calculate S. There is another way we can get S and then ρ. Any guesses?



Next we draw a circle with radius OP. This circle is the constant S circle we had seen last week. As we go to different values of z, we move on this circle. The dashed line going to the outside tells us where we started at the load. That becomes our reference point, z_{ref}. In this specific case, our z_{ref} is approximately 0.1875 λ.


Constant ρ circle.
$z_{ref} \approx 0.1875\,\lambda$
P
O
A
$\bar{Z}_{max}$ = S ≈ 5.8
$\bar{Z}_{in}\left(z=\frac{\lambda}{4}\right) = \bar{Y}_{in} \approx 0.225 + j0.4$
$\bar{Z}_{in} \approx 0.5 + j1$
$l = 0.18\lambda$
RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
CENTER
ORIGIN

Where is Z_{max} and what is its value? Where the circle cuts through the horizontal axis. The value will be given by the scale on the horizontal axis. Remember, we are measuring the normalized term. Also remember that:

$$Z_{max} = SZ_0$$

Thus, in normalized terms:

$$\overline{Z_{max}} = \frac{Z_{max}}{Z_0} = S$$

Thus, the value we read where the circle intersects the horizontal axis is simply S! To calculate the value of Z_{max}, we just need to multiply this by Z_0. In our specific case, S is approximately equal to 5.8 (the calculation is much more accurate than ρ we calculated from the bottom scale, real S is 5.82 for this problem, hence, I personally don't use the bottom scale) and Z_{max} is 290 Ω.

At what value of z does this happen i.e what is z_{max}? Look at the chart, the maximum is always at 0.25 λ on our outermost chart. The distance from out reference to the place where the maximum happens is value of z_{max}. It simply is $-(0.25 - z_{ref})\lambda$. The negative is just because we have called z=0 on the load. If we were giving the value as a length we go behind, it just is $(0.25 - z_{ref})\lambda$. For this specific case, z_{max} is 0.0625 λ.

Let us say we want to go 0.18 λ behind the load and calculate the input impedance there. So to our z_{ref}, we add 0.18 λ. We move clockwise on the scale till be get to the value we are interested in and mark it. We then draw a straight line from the center to the marked point (dashed red line). The intersection of our line to the constant S circle gives us the normalized input impedance. Multiply the value with Z_0, and we get the actual value of the input impedance.

Let us say, we want to go to 0.4 λ behind the load and want to find the input impedance. When we add 0.4 λ to z_{ref}, we get a number which is larger than the scale. The scale ends at 0.5 λ. Simply subtract 0.5 λ from the number and go to that point. For this case, we will go to (0.4+0.1875)-0.5 = 0.0875 λ. Find the point and see what is the input impedance.

How can we calculate admittance of the load (this will become very useful in the next chapter)? Remember, what we had observed in the last chapter. What happens at quarter wavelength?

$$\bar{Z}_{in}\left(z = -\frac{\lambda}{4}\right) = \frac{1}{\bar{Z}_L} = \bar{Y}_L$$

We don't need to go to our calculators to get the admittance, we just move $\frac{\lambda}{4}$ behind the load and read the input impedance. $\frac{\lambda}{4}$ is just a straight line stretched backwards (remember, semi-circle is $\frac{\lambda}{4}$). This is shown in the figure below, the line stretching to point A. The intersection of the line with the circle gives the value of admittance and also normalized impedance at a quarter wavelength length. For this example, the value is ~$0.225 - j0.4$ in normalized terms. If we had used a calculator, the value would be: 0.2 –j04. We are close. Obviously, we will not be as accurate as a calculator.

Question: To get the admittance value in real terms, do we multiply or divide by Z_0?

This is all there is to the Smith Chart. In next Chapters, we will see how we can engineer designs quickly using the Smith Chart.