Chapter 6. Matching Circuits

Most of the sources at high speeds are very sensitive to reflections back into them and can easily blow up (there is a power amplifies which generally suffers) in presence of reflections. It is important to make sure loads do not send reflections back to sources. Further, in most applications, it is desirable to reduce the reflections and standing waves in transmission lines as they jeopardize the power handing capabilities of the lines and distort the information being carried by the transmission lines. Reducing reflections also reduces the amplitude and phase errors. In this chapter we will look at different ways for matching an arbitrary load, Z_L to transmission lines.

6.1 Matching using Shunt or Series Stub

6.1.1. Shunt Stub: We have previously seen that we can make any value of inductance or capacitance by adjusting the length of short circuit and open circuit transmission lines from 0 to λ\2. These lengths are small and look like "stubs" on the main transmission line. Hence, the method for using short circuit or open circuit lines to match the load to the transmission line is called "Stub Matching". The way we do the matching with a "shunt" or parallel stub is shown in Figure 6.1 below. Let us first understand how the matching is achieved physically before we do any math.

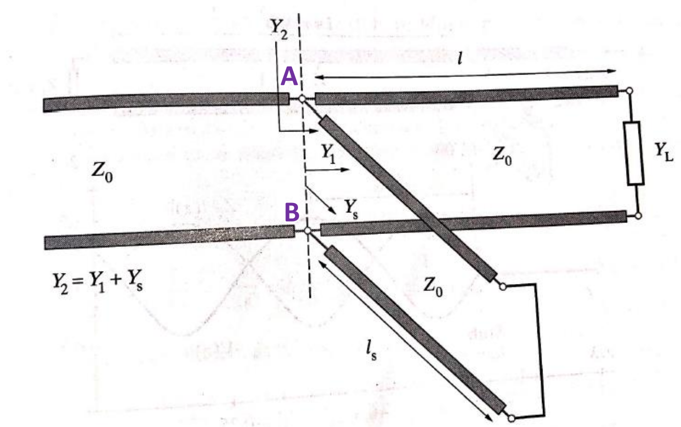

Figure 6.1 Schematic of the shunt matching circuit. We control the length "l" where the shunt is attached and the length, ls, of the shunt. The goal of the design is to make Y_2 equal to Y_0.

What we are trying to do is have the input impedance at points AB becomes equal to the characteristic impedance of the transmission line. Since, we are placing the stub (it could be short circuit line or open circuit line); it is easier to do our calculations with admittance. So we are trying to change the admittance Y_L to Y_0 (where Y_0 is equal to $1/Z_0$). Y_L in general will be complex and have a real part and an imaginary. So there are two terms we need to change, the real part needs to become equal to Y_0 and the imaginary part needs to become 0. To change two things, we need two controls. **The stub (whether it is short circuit or open circuit) only has imaginary part to it, so we cannot do anything to the real part with it.** This is how the stub matching works. From the load, we move behind on the transmission line till the real part of the input admittance becomes equal to Y_0. We change the real part of the input impedance by choosing the right "*l*". We get whatever value of the imaginary part is projected back. However, we do not need to worry. We can get any value of imaginary part with the short circuit or open circuit stub. We cancel the imaginary part of the input admittance using the admittance from the stub (which is only imaginary. Also remember admittances add up). Thus, the two controls we have are the lengths "*l*" (which determines the position of the stub on the line) and "l_s" which allows us to cancel the imaginary part.

Let us now do the math and derive the values for "*l*" and "l_s". We will do the calculations in terms of normalized values. In normalized terms, the input admittance at any "z" on the transmission line will be given as:

$$\bar{Y}(z) = \frac{Y(z)}{Y_0} = \frac{1 - \Gamma_L e^{j2\beta z}}{1 + \Gamma_L e^{j2\beta z}}$$

We know that $\Gamma_L = \rho e^{j\psi}$. We can write the normalized input admittance as:

$$\bar{Y}(z) = \frac{1 - \rho e^{j\psi} e^{j2\beta z}}{1 + \rho e^{j\psi} e^{j2\beta z}}$$

Let us call the term: $\psi + 2\beta z = \theta$, we will get:

$$\bar{Y}(z) = \frac{1 - \rho e^{j\theta}}{1 + \rho e^{j\theta}}$$

We want the real part to be equal to 1 (Y_0/Y_0 in normalized terms and cancel the imaginary part with the stub). Let us find the real part and imaginary part, by multiplying with the complex conjugate and we get:

$$\bar{Y}(z) = \frac{1 - \rho e^{j\theta}}{1 + \rho e^{j\theta}} \frac{1 + \rho e^{-j\theta}}{1 + \rho e^{-j\theta}} = \frac{1 - \rho^2}{1 + \rho^2 + 2\rho\cos(\theta)} - j\frac{2\rho\sin(\theta)}{1 + \rho^2 + 2\rho\cos(\theta)}$$

As we have said before, the real part needs to be 1 for matching. So

$$\bar{Y}(z) = \frac{1 - \rho^2}{1 + \rho^2 + 2\rho\cos(\theta)} - j\frac{2\rho\sin(\theta)}{1 + \rho^2 + 2\rho\cos(\theta)} = 1 - j\bar{B}$$

So we get:

$$\frac{1 - \rho^2}{1 + \rho^2 + 2\rho\cos(\theta)} = 1$$

$$\text{Or } \cos(\theta) = -\rho$$

Thus we get:

$$\psi + 2\beta z = cos^{-1}(-\rho)$$

Remember, z = -*l*. So we will get:

$$l = \frac{\psi - cos^{-1}(-\rho)}{2\beta} = \frac{\lambda}{4\pi}[\psi - cos^{-1}(-\rho)]$$

Also remember, $cos^{-1}(-\rho)$ will have two values one in the range of $\pi/2<\theta<\pi$ and another in the range from $-\pi<\theta<-\pi/2$. For both these ranges, cos function is negative but sine function (which shows up in the imaginary part) is different. For $\pi/2<\theta<\pi$, the sine function is positive while for $-\pi<\theta<-\pi/2$, the sine function is negative. **Thus, we have two solutions for "*l*"**. At one point, the input admittance is inductive and at another point, the input admittance is capacitive. Now, when you do the calculations with a complex calculator, $cos^{-1}(-\rho)$ only comes between $\pi/2<\theta<\pi$. **Use radians and have the second solution of *l* by adding a negative to the calculated value. Thus, we have:**

$$l = \frac{\lambda}{4\pi}[\psi - \pm cos^{-1}(-\rho)] = \frac{\lambda}{4\pi}[\psi \mp cos^{-1}(-\rho)]$$

What if the value of "*l*" turns out to be negative? Remember it is a physical distance and hence, should always be positive. It happens due to how calculators calculate ψ. Just add a $\lambda/2$ to the length as we know everything repeats every $\lambda/2$

The imaginary part of the input admittance becomes:

$$\bar{B} = \frac{2\rho\sin(\theta)}{1+\rho^2+2\rho\cos(\theta)} = \frac{2\rho(\pm\sqrt{1-\rho^2})}{1-\rho^2} = \pm\frac{2\rho}{\sqrt{1-\rho^2}}$$

The + value is when we use the positive value for $cos^{-1}(-\rho)$ (remember $\sin(\theta)$ is positive in that range); and the – value is when we use the negative value of the $cos^{-1}(-\rho)$.

Thus, we have two pairs for the solution:

$$l = \frac{\lambda}{4\pi}[\psi - cos^{-1}(-\rho)] \text{ and } \bar{B} = \frac{2\rho}{\sqrt{1-\rho^2}}$$

and

$$l = \frac{\lambda}{4\pi}[\psi + cos^{-1}(-\rho)] \text{ and } \bar{B} = -\frac{2\rho}{\sqrt{1-\rho^2}}$$

We have changed the real part. Now all we have to do is cancel the imaginary part. If the admittance of the stub is $j\bar{Y}_S$, then the total input impedance we will have will be given as:

$$1 - j\bar{B} + \bar{Y}_S$$

We need this value to be 1. Thus, we simply need to make $\bar{Y}_S = j\bar{B}$.

For short circuit line, we have $\bar{Z}_{in} = jtan(\beta l_s)$. Thus,

$$\bar{Y}_S = \frac{1}{\mathrm{j}\tan(\beta l_s)} = j\bar{B}$$

Or

$$\tan(\beta l_s) = -\frac{1}{\bar{B}} = -\frac{\pm\sqrt{1-\rho^2}}{2\rho}$$

For open circuit line, we have $\bar{Z}_{in} = -jcot(\beta l_s)$. Thus,

$$\bar{Y}_S = \frac{1}{-\text{jcot}\,(\beta l_s)} = j\bar{B}$$

Or

$$\tan(\beta l_s) = \bar{B}$$

Matching using Smith Chart:

We can also use the Smith Chart to match the circuit. Let us discuss the general method here and we will see its implantation in the next example.

Step 1: We locate the normalized load resistance on the Smith chart by finding the intersection between the real and imaginary circles.

Step 2: Draw the constant S circle.

Step 3: We locate the admittance by going to the opposite side of the load by drawing a straight line from the load impedance through the other side till we intersect the circle. This point will give us the value of normalized load admittance. We extend the line to the outer scale to mark the z_{ref}.

Step 4. We travel clockwise on the constant S circle till we intersect the R = 1 circle. At that point we have the real part of input admittance converted to 1. We note down the value of the imaginary part; let us call it X. The difference between the z and zref gives us one value of *l*.

Step 5. We go to the short circuit admittance point if we are using a short circuit stub or to the open circuit admittance point if we are using an open circuit stub and go to the value –X. This gives us the length of ls which corresponds with l in step 5.

Step 6: To find the second set of solutions, we continue on the main constant S circle till we hit the second intersection with R= 1 and repeat steps 4 and 5.

Question: Can you single stub match a purely reactive load e.g. an inductor or a capacitor?

Let us do an example to understand the single stub matching we have.

Example 6.1 Design a single shunt stub system to match a load consisting of a resistance, R_L = 200 Ω in parallel with an inductance L_L= 200/π nH to a transmission line with characteristics impedance Z_0 = 100 Ω and operating at 500 MHz.

Solution:

Since, the load resistance and inductance are in parallel, it is easier for us to calculate the load admittance rather than the load impedance. We can calculate the load admittance as:

$$Y_L = \frac{1}{R_L} - j\frac{1}{\omega L} = \frac{1}{200} - j\frac{1}{2\pi(500 \times 10^6)\left(\frac{200}{\pi} \times 10^{-9}\right)} = 0.005 - j0.005\, S$$

Let us see how we can rewrite the load reflection coefficient in terms of admittances. We know that:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

We can take $Z_L Z_0$ out as common in the numerator and the denominator. We thus, get:

$$\Gamma_L = \frac{Z_L Z_0}{Z_L Z_0} \frac{\frac{1}{Z_0} - \frac{1}{Z_L}}{\frac{1}{Z_0} + \frac{1}{Z_L}} = \frac{Y_0 - Y_L}{Y_0 + Y_L}$$

Thus, the load reflection coefficient in admittance is:

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} = \frac{1/_{100} - (0.005 - j0.005)}{1/_{100} + (0.005 - j0.005)} = \frac{1 + j}{3 - j} \approx 0.447e^{j63.4°}$$

Thus, $\rho \approx 0.447$ and $\psi \approx 63.4°$ or 1.11 radians

The length, *l*, at which we can attach the shunt stub will be given as:

$$l = \frac{\lambda}{4\pi}[\psi \mp cos^{-1}(-\rho)] = \frac{\lambda}{4\pi}[1.11 \mp cos^{-1}(-0.447)] = \frac{\lambda}{4\pi}[1.11 \mp 2.034]$$

Thus, we get two values of position where we can place the stub, one when $cos^{-1}(-\rho)$ is between π/2 to π and another when it is between –π to π/2. Let us call the two solutions as l_1 and l_2. We get the values:

$$l_1 \approx -0.073\lambda \text{ and } l_2 \approx 0.25\lambda$$

Since, l_1 is negative we need to add 0.5λ to it and thus, it becomes:

$$l_1 \approx -0.073\lambda + 0.5\lambda \approx 0.426\lambda$$

For l_1, sin(θ) is positive and thus for calculating the stub length, we should use the positive value of the square-too.

For the position l_1, we get: $\bar{B} = \frac{2\rho}{\sqrt{1-\rho^2}} = \frac{2*0.447}{\sqrt{1-0.447^2}} \approx 1$

If we use a short circuit stub, then the length of the stub we need to put at location l_1 will be:

$$l_{s1} = \frac{\lambda}{2\pi} tan^{-1}\left(-\frac{1}{\bar{B}}\right) = -0.125\lambda = -0.125\lambda + 0.5\lambda = 0.375\lambda$$

Again, because we got a negative value, we had to add 0.5λ to the length.

For the position l_2, sin(θ) is negative and thus, for calculating the stub length, we need to use the negative value of the square root. For position l_2, we get: $\bar{B} = -\frac{2\rho}{\sqrt{1-\rho^2}} = -\frac{2*0.447}{\sqrt{1-0.447^2}} \approx -1$

The short circuit stub at this position needs to have a length of:

$$l_{s2} = \frac{\lambda}{2\pi} tan^{-1}\left(-\frac{1}{\bar{B}}\right) = 0.125\lambda$$

Both of these solutions are shown in the Figure 6.2 below.

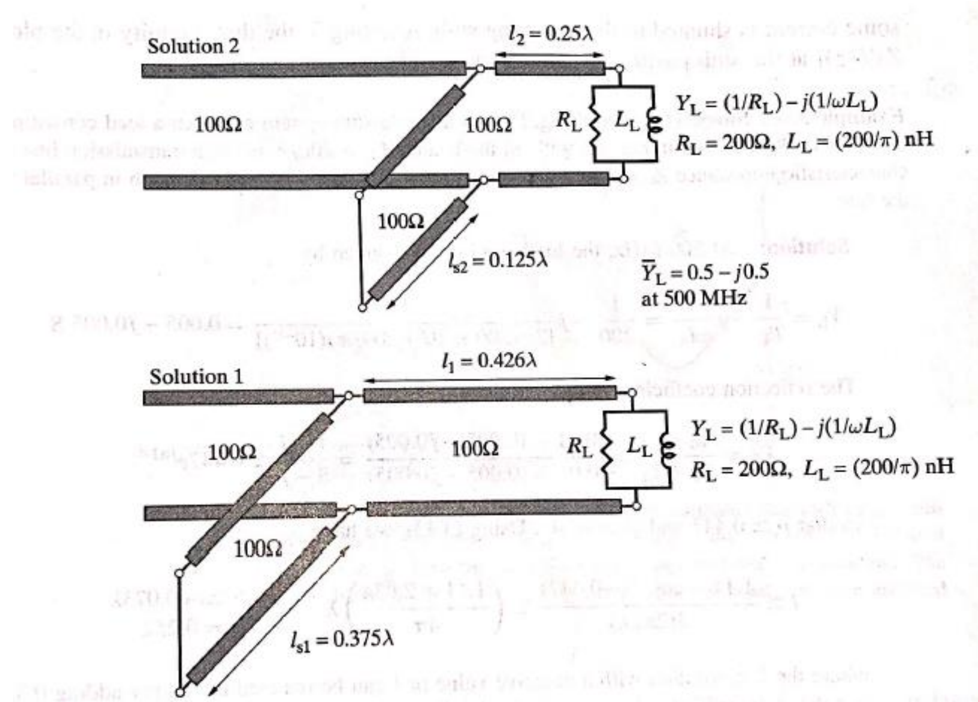

Figure 6.2 The two solutions for the matching circuit we calculated.

We need to remember that the design only works for the frequency designed and if the frequency is changed, then there will be reflections again. **Normally, the choice among the two designs is made based on the frequency response.** The figure below shows the frequency response of the two designs, where the standing wave ratio on the main line is plotted with frequency (will be related to S_{11} for the matching circuit). We can see that the bandwidth of the two designs are dramatically different. If the application requires wider bandwidth, then the second solution is optimal as it provides matching over a wider bandwidth.

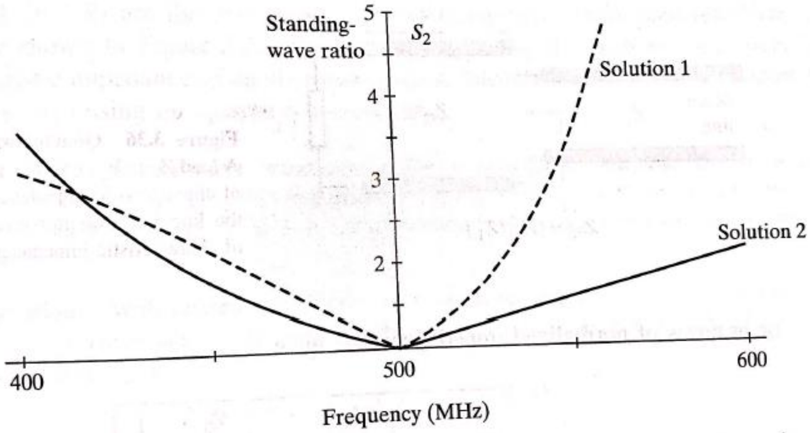

Figure 6.3 Simulated frequency response for S ratio for the two solutions. Solution 2 provides a wider bandwidth.

Example 6.2: Matching using a Smith Chart

We need to match a load impedance $Z_L = 160 - j80\ \Omega$ to a transmission line of characteristic impedance $Z_0 = 80\ \Omega$.

Solution: The various steps are shown in the Figure 6.3 below. We start by calculating the normalized load impedance and marking it on the Smith Chart. The normalized load impedance will be given as:

$$\bar{Z}_L = \frac{160 - j80}{80} = 2 - j1$$

We locate it on the Smith Chart and draw a constant S circle. We draw a line going through $\bar{Z}_L$ and origin. The intersection of the line with the circle on the other side gives us $\bar{Y}_L$. We extend the line to the outside scale and mark that as the starting reference point. From there we travel backwards on the constant S circle, till be hit the intersection with R=1 circle. The amount that we need to move by determines the stub position, l. For any problem, there will be two intersection points.

At the first point (called Solution 1), $l \approx 0.126\lambda$. The input admittance seen at this location is approximately 1+j1. Thus, the stub admittance should be –j1. To determine the length of the short circuit stub, we start from the short circuit admittance (on the right hand side where R = ∞; opposite of the impedance point). We move clockwise till we clockwise until we intersect the X=1 circle. This determines the length of the stub to be $l_s = 0.125\lambda$.

At the second intersection point (called Solution 2), $l \approx 0.302\lambda$. The input admittance seen at this location is approximately 1-j1 (it will always be the complex conjugate of the admittance seen in Solution 1). Now we need the stub admittance to be +j1. We again start from the short circuit admittance point and move clockwise until we intersect the X=1 circle. This determines the length of the stub to be $l_s = 0.375\lambda$.

We could have also used open circuit stubs. Want to try out what the corresponding lengths of the open circuit stubs be?

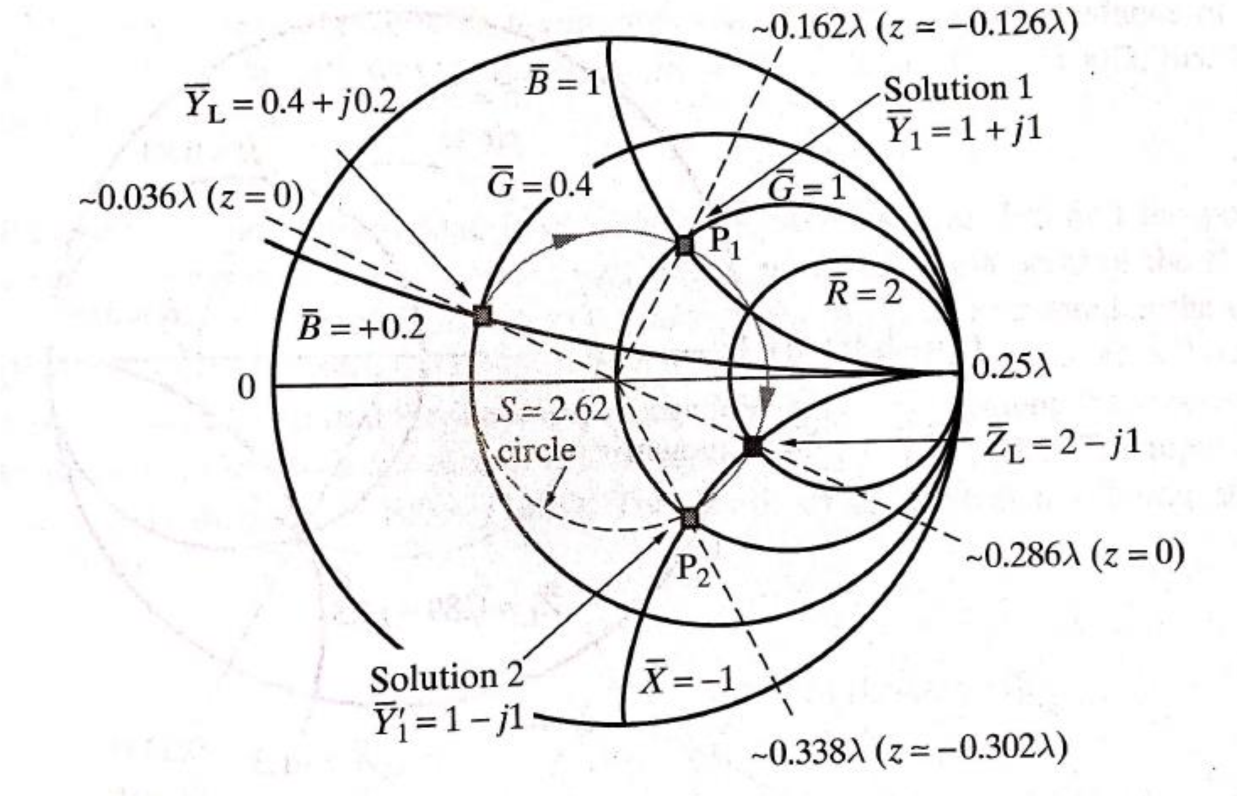

Figure 6.3. Different steps in the Smith Chart.

Note: Instead of using stubs, we can also use discrete components like inductors and capacitors to do the matching. The procedure remains the same except we just need to have the impedance of the discrete component equal to $\bar{B}$.

6.1.2 Series Stub

Instead of putting the stub in parallel we can also place the stub in series as shown in the Figure 6.4 below. The procedure for calculation remains roughly the same as for the shunt stub we saw above except, we now do the calculations in terms of impedance instead of admittance we did before**. It will be a good practice to do the derivations for "*l*" and "*l_s*"**. In terms of Smith Chart, the only difference is that we put our reference z at the load impedance, not the load admittance and for the stubs, we start from their impedance points, not their admittance points.

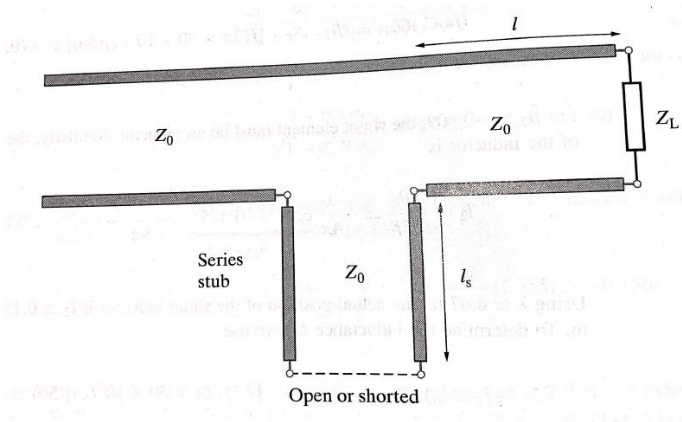

Figure 6.4 Series Stub matching

6.2 Quarter Wave Transformer Matching

Another very powerful way for matching a given load impedance is the quarter wave transformer matching. This method is used in optics also. In fact, if you wear glasses which have an anti-reflection coating, you are using a quarter wave transformer. We will revisit this when we do EM waves. For now let us understand it how it works for circuit theory.

The quarter wave transformer uses the fact that $Z_{in}(z = -\frac{\lambda}{4}) = \frac{{Z_0}^2}{Z_L}$. A quarter wave transformer uses another transmission line with a different characteristic impedance, Z_Q to transform the load to characteristic impedance, Z_0 of the main line. Let us see how first with purely resistive loads and then with complex loads.

6.2.1 Purely resistive loads

Consider the Figure 6.5 below. We attach a λ/4 long transmission line with characteristic impedance, Z_Q to the load. At the input of the transmission line, the input impedance we will see will be given by:

$$Z_{in} = \frac{{Z_Q}^2}{R_L}$$

To match, we need $Z_{in} = Z_0$. Thus, we get:

$$\frac{{Z_Q}^2}{R_L} = Z_0$$

or

$$Z_Q = \sqrt{Z_0 R_L}$$

If we design the quarter wave matching transmission line to have characteristics impedance of Z_Q which is the geometric mean of Z_0 and R_L, then we achieve matching (store this in memory as this will become powerful later on).

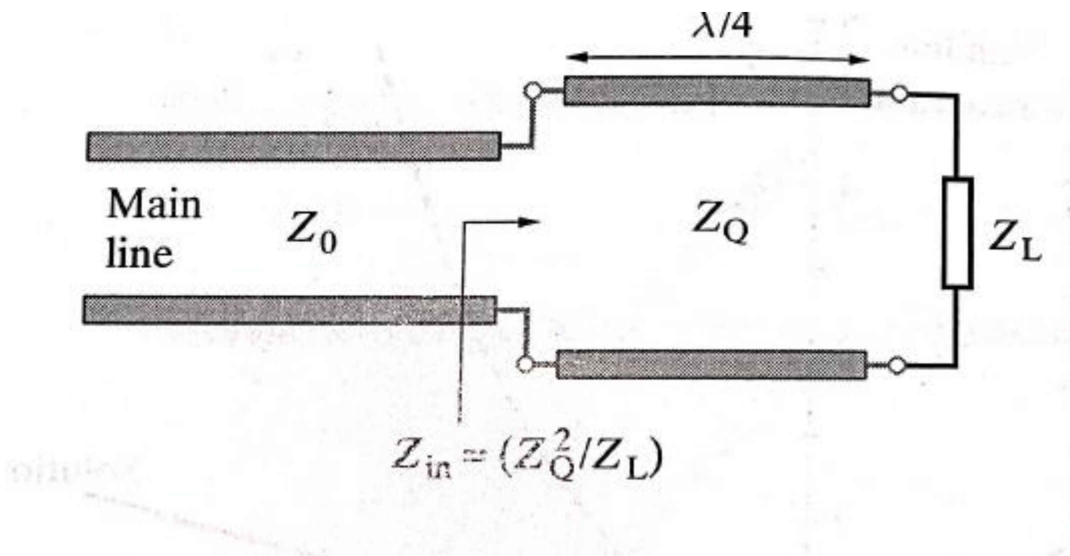

Figure 6.5 Quarter wave matching for a resistive load

6.2.2 Complex load impedances.

Will the idea in 6.2.1 work if we had a complex load? We see that:

$$Z_{in} = \frac{{Z_Q}^2}{R_L + jX_L}$$

We see that Z_{in} is going to be complex term. Thus, the quarter wave transformer can only transform the real impedances to match Z_0. Is there a solution for complex loads? Are there points on transmission lines which are purely resistive? Yes, when we have Z_{max} or Z_{min}. **These values are purely resistive and are equal to SZ_0 and Z_0/S respectively**. Instead of attaching Z_Q right at the load, we can go to a point where we get a maximum or a minimum and attached Z_Q there. If we go to the maximum point, then

$$Z_Q = \sqrt{Z_0(SZ_o)} = Z_0\sqrt{S}$$

If we attach it at a minimum point, then

$$Z_Q = \sqrt{Z_0\left(\frac{Z_o}{S}\right)} = \frac{Z_0}{\sqrt{S}}$$

Let us consider an example to see how this matching can be achieved. We want to match a load

$Z_L = 73 + j42.5\,\Omega$ to a transmission line of characteristics impedance 100 Ω. We start by calculating the reflection coefficient for the load as:

$$\Gamma_L = \rho e^{j\psi} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{73 + j42.5 - 100}{72 + j42.5 + 100} \approx 0.283 e^{j109^{\circ}}$$

We can calculate the standing wave ratio as:

$$S = \frac{1+\rho}{1-\rho} \approx 1.79$$

We go behind to a place where we see the first voltage maximum, i.e. at z_{max} which will be given by:

$$z_{max} = -\frac{\psi}{2\beta} \approx -0.151\lambda$$

We insert the quarter wave section at $z_{max} \approx -0.151\lambda$. Since, we are inserting at the maximum point,

$$Z_Q = \sqrt{S} Z_0 \approx 133.7\ \Omega.$$

The arrangement is shown in the figure 6.6 below.

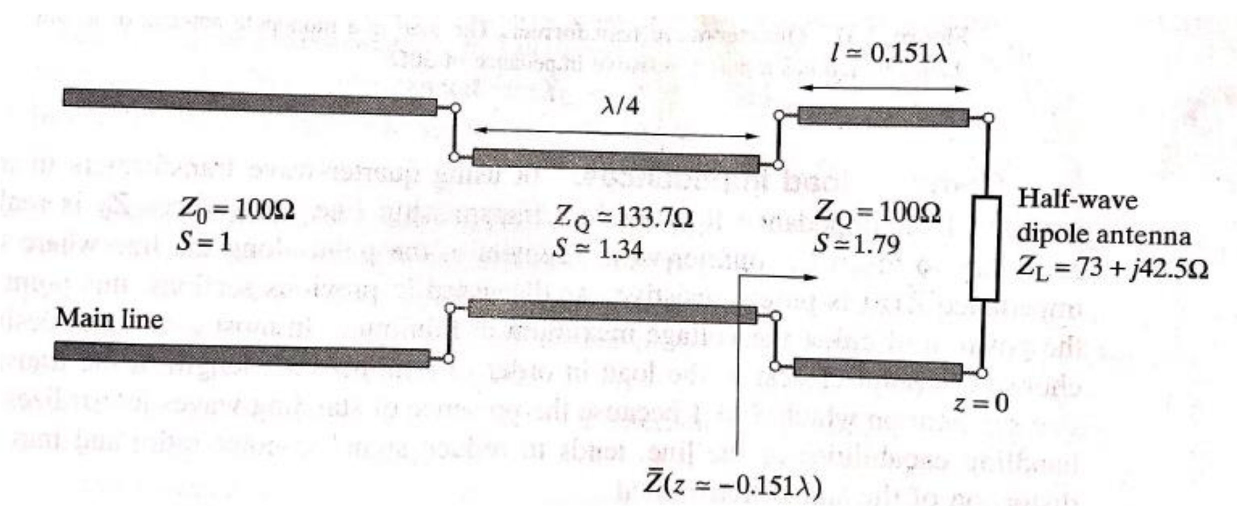

Figure 6.6 Quarter wave matching for a complex load.