Chapter 1: Wave Mechanics

The course deals with how Electromagnetic (EM) waves travel in different mediums and how we can create some applications using EM waves. However, before we dive into EM waves, we need to understand what waves are, how can we treat them mathematically and how can we look at a system (both physically and mathematically i.e. a system of equations) and know that it supports a wave. Creating this understanding will allow us to develop intuitions and not be bogged down by math. It is also important that as engineers we are not only able to execute the math but also talk intelligently and intuitively about the topics we are studying.

1.1 What is a wave?

Anyone who has been to the beach has seen a wave. If you can hear or see, you have seen its application. So the question is how will you describe it to someone in an elementary school? Let us consider a simple system of mass-spring oscillators coupled together as shown in the figure below.

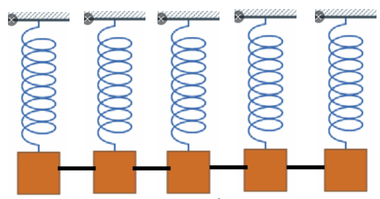

Figure 1.1 Springs and masses coupled together

Each mass and a spring comprise an oscillator. If I disturb a mass by pulling it down, it will start to oscillate around its equilibrium point due to the conversion between kinetic and potential energies. Now we have connected/coupled multiple oscillators together in space. Let us disturb the mass of the left most oscillator by pulling it down. Since, the oscillators are connected, the second mass will be pulled downwards and start to move which in turn will try to pull the third mass downwards and so on. This forward movement will be delayed by some time (technically, we will call it phase) to the second mass, and then to the third mass and so forth. Thus, the oscillation I created in the first mass starts to move forward with time. We created a disturbance and this disturbance we created starts to propagate in space i.e. is move forward in space with time. **So here is another technical word: Propagation which means travel**. Did the mass we disturbed move forward? No. It stayed in its place jumping up and down while the disturbance it created move forward. So we can define a wave as a disturbance which moves forward (or backwards) without the material moving forward (or backwards). However, we will soon see, electromagnetic waves do not even need a material to propagate but every wave needs coupled oscillators. **What is a disturbance?** It is basically a change in energy. The last mass starts to move and thus, have energy when energy was only given to the first oscillator. **So we can define a wave as propagation of energy created by local oscillators which are coupled in space.** Wave propagation is any of the ways in which waves travel.

1.2 Types of Waves

In our example above, we moved the mass up and down and the energy travelled forward. The oscillator is perpendicular to the direction of wave propagation. **Such a wave is called a**

transverse wave. We could also have moved the mass horizontally and then the masses will oscillate in the horizontal direction but the energy will still travel in the horizontal direction. **Such a wave is called a longitudinal wave**. The video here gives a very good illustration: https://www.youtube.com/watch?v=7cDAYFTXq3E

Or we could have a mixture of both. Such waves are called Hybrid waves. Waves you see on the beach are hybrid waves. Water crests and troughs but also comes forward and goes back. Electromagnetic waves consist of oscillating Electric and Magnetic fields. We will see later that Electromagnetic waves in free space (vacuum) or unbound mediums (very large mediums) are always transverse waves i.e. the electric field and magnetic field components are perpendicular to the direction of propagation and in fact, to each other. However, when EM waves are bounded, they can become hybrid in nature. There will always be transverse components of electric and magnetic fields in an EM wave but in some conditions, longitudinal components can exist.

The simple physical model we have designed also tells us that for waves to exist, oscillators (which oscillate/vary with time) must exist, and these oscillators must be coupled in space, somehow. So, to see if waves exists in a system, we will look for whether there are coupled oscillators in a system.

1.3 Mathematical Description

Let us think through functions which will describe the propagation of a wave. Obviously, the function has to have space and time as arguments. So any function which describes a wave should be of the form $f(x,y,z,t)$. For the time being, let us just assume that the wave is propagating in only one direction. We will choose z (out of convention) as our direction of wave propagation. Thus, the wave should have form $f(z,t)$. As you have seen in Fourier, we can represent any function as sum of sinusoid functions. So if we understand how to write a wave in sine/cosine form, we can use Fourier to decompose any wave shape into sum of sine/cosine waves. Let us see how we can write a wave in sine form.

Let first consider the function, sin(ωt). Is this a wave? No, while it changes in time, there is no motion in space. If we plot this function, we get:

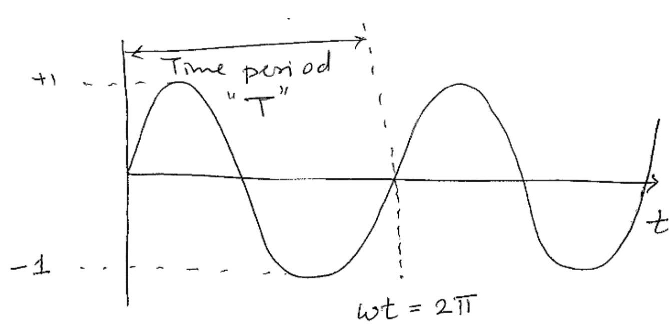

Figure 1.2. Plot of function sin(ωt) with time

ω is called the angular frequency (with units 1/radians). The time it takes for the oscillation to complete and restart is called the time period, T (units, seconds). If we look at the figure above, we see that:

$$\omega T = 2\pi$$

$$\omega = \frac{2\pi}{T}$$

But 1/T is called the frequency (measured in Hertz, Hz). Thus, we get the equation you may have seen in ECE 105.

$$\omega = 2\pi f$$

Now, let us consider another function, *sin(kz)*. I have introduced k as a constant (has nothing to do with the spring constant). Is this function a wave? At least not a wave that moves in space as it does not change with time. If we plot the function, we see it is a sinusoid oscillation in z direction as shown below:

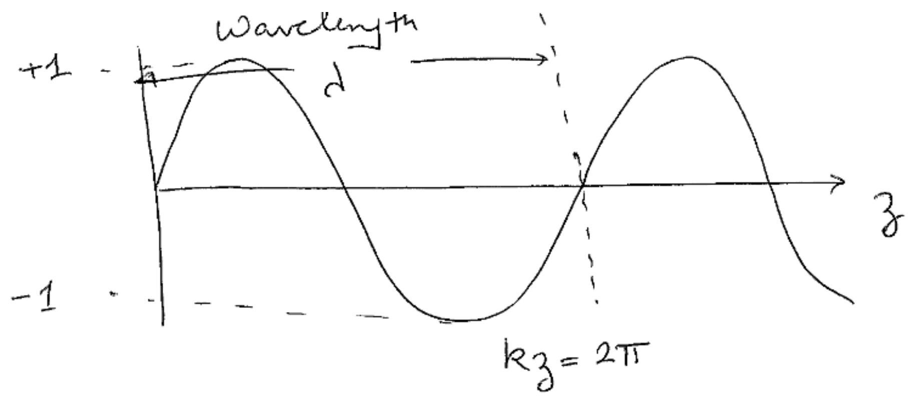

Figure 1.3 Plot of function sin(kz) with changing distance, z.

Let us see what the constant *k,* we introduced represents. The distance over which sinusoid repeats itself is called the wavelength, λ. Thus,

$$k\lambda = 2\pi$$

$$k = \frac{2\pi}{\lambda}$$

Physically if we think about it, k is the number of wavelengths we can fit in a circle represented by an angle of 2π. Thus, k is called the wave number (with units 1/m or many times written as 1/cm). This is the nomenclature which is normally used in physics. In engineering, k is also called the "propagation constant". Why, it will become clear soon.

As you probably know from high school, waves are characterized by frequency, f and wavelength, λ. We have introduced two constants, ω and k and represented then in terms of f and λ. So we are making some progress.

Let us combine the two functions into one function as:

$$\sin(\omega t - kz)$$

Let us see what this function looks like and what happens to it as time changes. At time "t=0", the function is simply sin*(-kz)* = *-sin(kz),* shown in black. At another time, t_1, we get sin $(\omega t_1 - kz)$. That just means the origin of the function has changed from z = 0 to z = $\frac{\omega t_1}{k}$. We can visualize this as following:

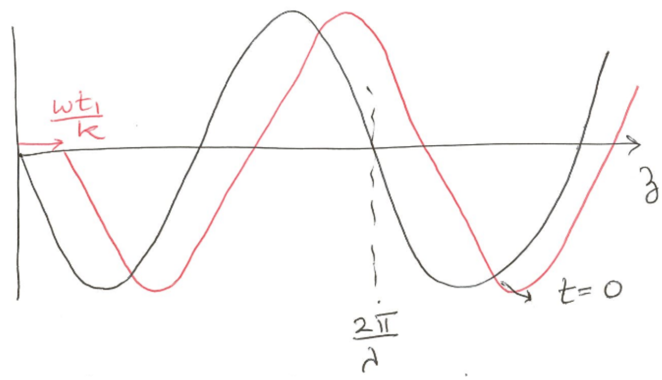

Figure 1.4 Plot of function sin(ωt-kz) with distance z at two different times.

The function sin $(\omega t_1 - kz)$ is represented in the red curve. What do we see? The curve seems to be moving forward with time. Where is the origin of $(\omega t_1 - kz)$)? At point $z = z_1$ such that:

$$\omega t_1 - kz_1 = 0$$

$$z_1 = \frac{\omega}{k} t_1$$

The function has moved by z_1 in the time of t_1. It is representing a wave. The argument of the function *i.e.* $(\omega t - kz)$ is called the phase of the wave. (Let me tell you a secret. We invent lots of terms so when people around us hear us speak, they think we are highly educated. It is a game. I have also learned that most people I respect can explain complicated matters in very simple terms. ***That is the beauty and goal of true education, not to make things complicated but to make things simple***).

What is the velocity with which the argument or phase is moving forward by? It will just be given by:

$$v_p = \frac{z_1}{t_1} = \frac{\omega}{k}$$

where v_p is called the phase velocity. Let us substitute the values of ω and *k* and we get the velocity relation you should have seen in high schools.

$$v_p = \frac{2\pi f}{2\pi/_{\lambda}} = f\lambda$$

Thus, the phase velocity is equal to the frequency multiplied by wavelength. Hopefully, now you understand where it comes from. Since the phase velocity depends on *k*, *k* is also called the propagation constant. That is the term we use in engineering.

Later on we will see energy normally does not travel with phase velocity but another velocity called the group velocity. We will keep that for week 7 of the course.

So we have seen that the function $\sin(\omega t_1 - kz)$ represents a wave of angular frequency ω. However, the functions amplitude is bounded only between -1 to 1. We want the wave function to carry disturbance which can have different values. Let us add, an amplitude term A to the function so it becomes:

$$A\sin(\omega t - kz)$$

A is the amplitude of the wave. The function is now bound between $-A$ to A. A should be related to energy being carried by the wave. A is called the amplitude of the wave.

Now if A is just a constant, there is no fun. All we have is a sine wave with amplitude A coming towards us (or you just keep playing piano on one note). It will be boring (power transmission is one case where A does not change with time). But A can be a function of time. So in that case the wave is carrying a disturbance which is changing with time and we can start to send information. (You will see in ECE 318 that information can also be send if ω is a function of time and in fact is preserved better that way)

A may (will actually be for most cases) be a function of space. Think of a laser beam. While it is propagating forward (we can call that direction z), its energy is also limited in x-y plane. Thus, we can possibly have:

$$A(x, y)\sin(\omega t - kz)$$

We will get to look at all these forms over the time of the course.

1.4 Phase Velocity

There is another way, we could have calculated the phase velocity of the wave. What we calculated before was the average velocity over time period t_1. It is a constant, so we know average velocity is equal to the instantaneous velocity. To calculate instantaneous velocity directly, we can see how an arbitrary point on the wave form moves. To define the same point, the argument or phase of the function should remain constant. Thus,

$$\omega t - kz = constant$$

Let us take the derivative of the function with respect to time.

$$\frac{d}{dt}(\omega t - kz = constant)$$

$$\omega - k\frac{dz}{dt} = 0$$

$$v_p = \frac{dz}{dt} = \frac{\omega}{k}$$

Since, v_p is positive, the wave is travelling in $+z$ direction. **Thus, we see that the function, $A\sin(\omega t - kz)$ is a wave with amplitude A travelling with phase velocity $v_p=\omega/k$ in the +z-direction.** Since, the wave consists of only one frequency and is sinusoid, it is also called a Harmonic wave.

What about the functions, **$A\sin(\omega t + kz)$?** In this case, $v_p = \frac{dz}{dt} = -\frac{\omega}{k}$. The wave will propagate in the $-z$ direction. (Look at the pattern here. – in the argument is $+z$; + in the argument is $-z$). Remember the signs are very important and if we understand them, we can write the wave functions quickly without thinking. Negatives and positives have physical meanings. Whether you pay uWaterloo $17000 to come and study here or whether uWaterloo pays you $17000 to study is only a difference of a negative. But you will agree, it is a big difference.

What about functions? **$A\cos(\omega t \mp kz)$.** They should also be waves propagating in $\pm z$ directions.

1.5 Generalized form

Let us see how we can write the generalized function for describing a wave. Let us start with the function: **$A\sin(\omega t \mp kz)$.** We can write this as:

$$A\sin\left\{\omega\left(t \mp \frac{k}{\omega}z\right)\right\} = A\sin\left\{\omega\left(t \mp \frac{z}{\omega/k}\right)\right\} = A\sin\left\{\omega\left(t \mp \frac{z}{v_p}\right)\right\}$$

Thus, any function of the form: $f\left(t \mp \frac{z}{v_p}\right)$ is a wave moving in $\pm z$ direction with velocity v_p. We sometimes write the functions as $f^+\left(t - \frac{z}{v_p}\right)$ and $f^-\left(t + \frac{z}{v_p}\right)$ where the + and – superscripts represent the fact that the wave is moving in $+z$ and $-z$ direction respectively. Draw an arbitrary function and convince yourselves, that the argument does represent a wave.

1.6 Wave Equation

The next question which arises is what is the equation for which we know the solution has to be a wave. We know that wave is created when there is an oscillator in time connected in space with multiple oscillators. So I should see an oscillator in time and an oscillator in space. Going back to the spring and mass oscillator (or LC oscillator), we know that an oscillator is described by the second order differential equation. So we should have forms $\frac{\partial^2}{\partial t^2}$ and $\frac{\partial^2}{\partial z^2}$. My contention is that any equation of the form where K is a constant:

$$\frac{\partial^2 f}{\partial t^2} = K\frac{\partial^2 f}{\partial z^2}$$

will support a wave with a phase velocity given by a phase velocity given simply by $\sqrt{K}$. Let us make sure that a solution of the above equation is the function $f^+\left(t - \frac{z}{v_p}\right)$ and find the relation between K and v_p.

First, let us substitute the argument: $\zeta = t - \frac{z}{v_p}$. Thus, we get:

$$\frac{\partial \zeta}{\partial t} = 1; \frac{\partial \zeta}{\partial z} = -\frac{1}{v_p}$$

We need to somehow relate $\frac{\partial^2 f}{\partial t^2}$ with $\frac{\partial^2 f}{\partial z^2}$. We can use the variable ζ we have introduced. Thus,

$$\frac{\partial f^+(\zeta)}{\partial t} = \frac{\partial f^+(\zeta)}{\partial \zeta} . \frac{\partial \zeta}{\partial t} = \frac{\partial f^+(\zeta)}{\partial \zeta}$$

and

$$\frac{\partial^2 f^+(\zeta)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f^+(\zeta)}{\partial \zeta}\right) = \frac{\partial^2 f^+(\zeta)}{\partial \zeta^2} . \frac{\partial \zeta}{\partial t} = \frac{\partial^2 f^+(\zeta)}{\partial \zeta^2}$$

Doing the same with the derivative with *z* we get:

$$\frac{\partial f^+(\zeta)}{\partial z} = \frac{\partial f^+(\zeta)}{\partial \zeta} . \frac{\partial \zeta}{\partial z} = -\frac{1}{v_p}\frac{\partial f^+(\zeta)}{\partial \zeta}$$

and

$$\frac{\partial^2 f^+(\zeta)}{\partial z^2} = \frac{\partial}{\partial z}\left(-\frac{1}{v_p}\frac{\partial f^+(\zeta)}{\partial \zeta}\right) = -\frac{1}{v_p}\frac{\partial^2 f^+(\zeta)}{\partial \zeta^2} . \frac{\partial \zeta}{\partial z} = \frac{1}{{v_p}^2}\frac{\partial^2 f^+(\zeta)}{\partial \zeta^2}$$

Substituting this in the proposed wave equation we get:

$$\frac{\partial^2 f}{\partial t^2} = K\frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 f^+(\zeta)}{\partial \zeta^2} = \frac{K}{{v_p}^2}\frac{\partial^2 f^+(\zeta)}{\partial \zeta^2}$$

$$K = {v_p}^2$$

or $v_p = \sqrt{K}$.

Thus, we see that the any arbitrary wave function is a solution of the differential equation we proposed. **So we are going to look for second order differentiation with time related to second order differentiation with space with a constant and when we see that, we know a wave must exist. Further, we know immediately that the square-root of the constant should be equal to the phase velocity.**

1.7 Phasor Form

Is the function $\boldsymbol{A\sin\,(\omega t - kz)}$ really the best function to deal with or can we do better? We know that $e^{j\theta} = \cos(\theta) + jsin(\theta)$. The function has both the sine and cosine form inside it. If we take the real part, we get the *cos* form, if we take the imaginary part, we get the *sine* part. Thus, in one mathematical operation, we can cover both the forms. What does the wave look like in this phasor form? It will simply be:

$$Ae^{j(\omega t-kz)}$$

There is a beauty in this function. We can easily separate the time and space components which is:

$$Ae^{j(\omega t-kz)} = Ae^{j\omega t}e^{-jkz}$$

Many times while the time variation is known (ω will be determined by the oscillations in the source which is under your control), how the wave travels in *z* direction is not clear (e.g. there may be reflections). Using the phasor form, we can use separation of variables and assume that the solution is of the form:

$$Af(z)e^{j\omega t}$$

Substituting the solution in the wave equation, we get:

$$(j\omega)^2 Af(z)e^{j\omega t} = KA\frac{d^2f(z)}{dz^2}e^{j\omega t}$$

which is simply:

$$\frac{d^2f(z)}{dz^2} + \frac{\omega^2}{K} f(z) = 0$$

This is a time independent second-order differential equation only in z and we can find solutions easily for this equation. Further we know, the K = ${v_p}^2$

$$\frac{d^2f(z)}{dz^2} + \frac{\omega^2}{{v_p}^2} f(z) = 0$$

Substituting the value of v_p we get:

$$\frac{d^2f(z)}{dz^2} + k^2 f(z) = 0$$

Now, you again see why *k* is called the propagation constant. Most of the time in the course, we will be looking for solutions of the equation of this form under different conditions.

1.8 What is this course about?

We will look at different physical systems and find what kind of waves are supported in the system. We will find solutions to the wave-equations and try to understand what happens when we have a discontinuity. We will start with transmission lines and building a circuit model based on what you have learned in ECE 106 (don't worry there will be no integrals ☺) and ECE 140 and learn how to analyze high speed signals through the transmission lines. We will learn how we have to be careful of designing our transmission lines in high speed circuits. The presence of waves also complicates the way we can measure our circuits correctly. We will learn how to be able to measure high speed circuit response (called S-parameters). We will design some filters. This will be the first six weeks. After this, we will switch tracks to Maxwell's equations and see how they support EM waves. We will learn how to analyze these waves and find their properties, first in unbound mediums and then when we try to confine the waves in a physical dimension. Finally, we will use what we learned with circuits models to design wave structures like anti-reflection coatings, high reflection coatings, and optical/RF filters. I promise you the course is going to be very interesting and there are times you are going to jump out of your seats and not believe what I say, but because it is a strange world, it is so fun.