Chapter 3: Steady State Waves on Transmission Lines

In digital circuits, and computer communications, it is important to understand the transient response of the transmission lines, which is what we did in Chapter 2. However, in many engineering applications, it is also important to understand the steady state response of the transmission lines especially due to sinusoidal excitations. Power transmission and communication links (you will see this in ECE 318) are transmitted in sinusoids or modified sinusoids. Other periodic signals can just be considered as a superposition of sinusoids (Fourier!) of different frequencies. So if we understand how transmission lines respond to sinusoid signals, we can know what will happen for any periodic function.

When a sinusoid signal is initially turned on, there will be a transient response. We are not interested in that. **We are interested in after the signal has stabilized, and the transmission line is being driven constantly by a sinusoid signal.** Thus, the voltage and current at any point on the transmission line will vary with time determined by the frequency of the source (at least in the linear systems we are considering). Thus, we can use the phasor form to calculate the transmission line response.

3.1 Wave Solutions using Phasors

In phasor form, we can write the voltage and current waves as:

$$V(z,t) = V(z)e^{j\omega t} \text{ and } I(z,t) = I(z)e^{j\omega t}$$

In general *V(z)* and *I(z)* can be complex terms. To find these solutions, we need to substitute them in the wave equations i.e.:

$$\frac{\partial^2 V(z,t)}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 V(z,t)}{\partial z^2}$$

and

$$\frac{\partial^2 I(z,t)}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 I(z,t)}{\partial z^2}$$

The beauty of phasor form is that it is very easy to differentiate. The exponent just keeps coming out. Substituting the terms, we get:

$$\frac{1}{LC}\frac{d^2 V(z)}{dz^2} - (j\omega)^2 V(z) = 0$$

or

$$\frac{d^2 V(z)}{dz^2} + \omega^2 LCV(z) = 0$$

Similarly, we get an equation in I(z) as

$$\frac{d^2 I(z)}{dz^2} + \omega^2 LCI(z) = 0$$

Both of the above equations are time independent wave equations and provide the solution for V(z) and I(z). Let us look at the form of the equations. What they are saying is that the second order time derivative of the function is related to the function itself by some constants ($\omega^2 LC$). Note the equation is true for any value of z. There are only two ways we can come to a solution. One is either the second order derivative goes to 0 and the constant is 0 i.e. is ω is 0 or DC case. That is definitely not a solution we are looking for. The second way we can solve the equation is such that when we take the derivative twice, the function still comes out and we can take it the function out as common i.e.

$$\frac{d^2V(z)}{dz^2} = KV(z) \text{ and } (K + \omega^2 LC)V(z) = 0$$

where K is a constant. We can equate the sum of constants to be 0.

There are two functions where after second order derivative, the function comes out. One is the sinusoid function and the other is the exponential function. But we know using phasors, we can represent both the functions as exponential functions (for exponential function, the argument in the phasor will simply be imaginary). So we can postulate that the solution of V(z) is going to be of the form:

$$V(z) = V^+e^{-j\beta z} + V^-e^{+j\beta z}$$

We have introduced β as a constant. The wave form will be given as:

$$V(z,t) = V(z)e^{j\omega t} = \left(V^+e^{-j\beta z} + V^-e^{+j\beta z}\right)e^{j\omega t} = V^+e^{j(\omega t-\beta z)} + V^-e^{j(\omega+t\beta z)}$$

The term $V^+e^{j(\omega t-\beta z)}$ represents a wave travelling in +z direction and $V^-e^{j(\omega+t\beta z)}$ represents a wave travelling in –z direction. Thus, I had put the constants V^+ and V^- as the amplitudes to remind us V^+ term is for the forward wave and V^- term is for the backward wave.

In Transient analysis, we had treated each forward and backward wave individually. In steady state analysis, effectively what we are doing is calling all forward waves as one forward wave $V^+e^{j(\omega t-\beta z)}$ and all backward waves as one single backward wave $V^-e^{j(\omega+t\beta z)}$.

Similar to what we saw in Transient's, if we choose the value for V(z), I(z) takes can only take a specific form and it will be:

$$I(z) = I^+e^{-j\beta z} - I^-e^{+j\beta z} = \frac{1}{Z_0}\left(V^+e^{-j\beta z} - V^-e^{+j\beta z}\right)$$

Where Z_0 is the characteristics impedance and given by: $Z_0 = \sqrt{\frac{L}{C}}$

If we substitute the solution in the time independent wave equation, we get $K = (-j\beta)^2 = -\beta^2$ and thus, we get:

$$\beta = \omega\sqrt{LC}$$

and the phase velocity still remains the same as:

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

There is a reason I am using β, instead of k which will only become clear when we bring the loss terms in at a later time.

The goal is to calculate the values of V(z) and I(z) for different loads.

3.2 Lines terminated in an arbitrary impedance

Let us consider the circuit shown below where a source with impedance Z_s is connected to a load Z_L using a transmission line of length *l*. In general, Z_s and Z_L are reactive (with the form $R+jX$). We want to find the solution of V(z) along the line.

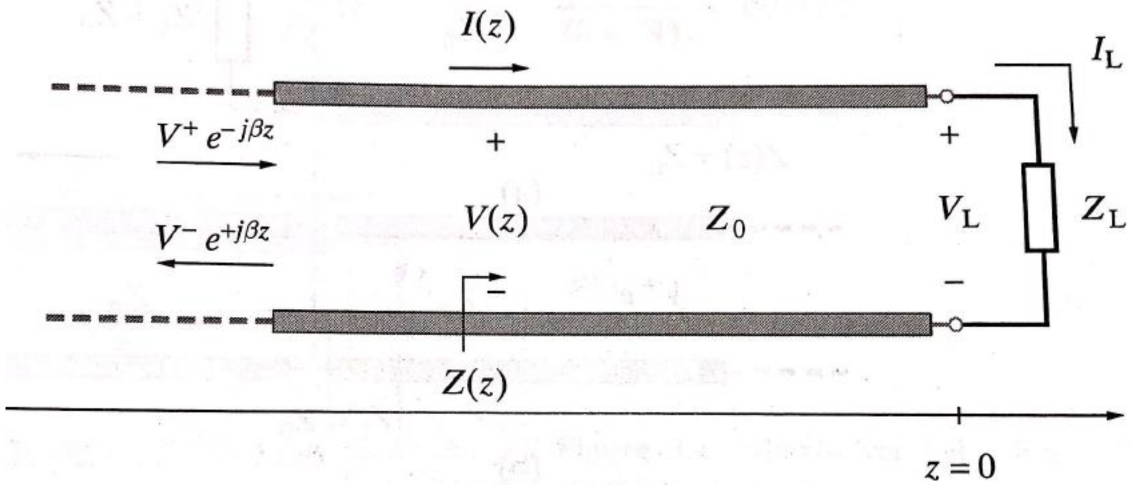


Figure 3.1. Steady-state Voltage waves in a load terminated transmission line.

The first question which arises is where to put the origin *z*=0. We have two obvious choices, one at the source (at the start of the transmission line) and another one at the load (at the end of the transmission line). Let us see what happens if we put the origin at the load as shown in the Figure 3.1. We still have the *z*-axis going in the forward direction (after all we have defined waves as forward and backward and have no choice left). Thus, when we go behind the load on the transmission line towards the source, *z* will be negative.

Thus, our solution on the transmission line looks like what is shown in the figure below. We have a wave $V^+e^{-j\beta z}$ going in the forward direction and wave $V^-e^{+j\beta z}$ going in the reverse direction. Associated with the wave is current going forward as $I^+e^{-j\beta z}$ and current going backwards as $I^-e^{+j\beta z}$. At the load, z=0, and the waves just become V^+, V^-, I^+, and I^-. This is the same values we had seen in transients. KVL and KCL hold and the voltage across the terminal should be the same and current going into node should be equal to the current coming out of the node. Thus:

$$V^+ + V^- = V_L$$

and

$$I^+ - I^- = I_L$$

These expressions again look the same as we had seen in the transients and we can use these equations to calculate the load voltage reflection coefficient as (see the derivations in transients, it is the exact same equations):

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

By placing the z=0 at the load, we keep the consistency with transients and have less to remember. If we had placed the origin at the source, then the waves at the load would have been $V^+e^{-j\beta z}, V^-e^{+j\beta z}, I^+e^{-j\beta z}, I^-e^{+j\beta z}$. We will have to put these terms in voltage and current and we will have to derive a totally new term for the load reflection coefficient. We can do that but why?

Let us now spend time understanding various parameters by which we can characterize the waves in this section and then we will consider different types of termination in the next sections and try to create a physical understanding.

3.2.1 Reflection coefficient

We see that

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

In general, Z_L could be reactive of the form R_L+jX_L. Thus, Γ_L will be a complex number. We can write in the phasor form as:

$$\Gamma_L = \frac{R_L + jX_L - Z_0}{R_L + jX_L + Z_0} = \rho e^{j\psi}$$

ρ is the amplitude of the reflection coefficient and ψ is the phase. We can calculate these terms as:

$$\rho = \sqrt{\frac{(R_L - Z_0)^2 + {X_L}^2}{(R_L + Z_0)^2 + {X_L}^2}}$$

and $$\psi = tan^{-1}\left(\frac{X_L}{R_L - Z_0}\right) - tan^{-1}\left(\frac{X_L}{R_L + Z_0}\right)$$

We know from our previous discussions that $-1 \leq \Gamma_L \leq 1$. Thus ρ is restricted in the range:

$$0 \leq \rho \leq 1$$

For example when $\Gamma_L = -1$, we can write it as $\Gamma_L = 1e^{j\pi}$, thus, $\rho = 1$ and $\psi = \pi$.

We had previously seen that $|\Gamma_L| = 1$ for an open circuit and a short circuit line. However, $|\Gamma_L| = 1$ is also true for another type of load. Can you guess it? Let us look at the equation for ρ. The numerator and denominator has to be equal for it to be 1. That will also happen for R_L =0 but X_L exists i.e. a purely reactive load like a capacitor or an inductor. For these loads also, $\rho = 1$. However, ψ will be different than 0 or π.

Let us see what happens to the voltage and current phasors in the transmission lines. The voltage and current phasors will be given by the form:

$$V(z) = V^+e^{-j\beta z} + V^-e^{+j\beta z} = V^+e^{-j\beta z} + \rho e^{j\psi} V^+ e^{+j\beta z}$$

$$I(z) = I^+e^{-j\beta z} - I^-e^{+j\beta z} = \frac{1}{Z_0}\left[V^+e^{-j\beta z} - \rho e^{j\psi} V^+ e^{+j\beta z}\right]$$

Let us take $V^+e^{-j\beta z}$ term out of the equation as a common term, we will get:

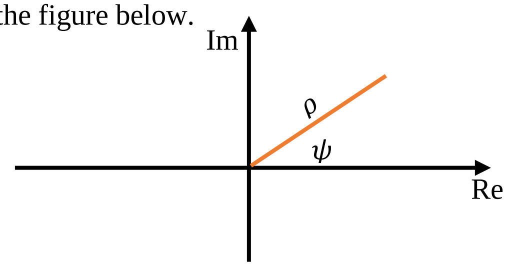
$$V(z) = V^+e^{-j\beta z}\left[1 + \rho e^{j\psi} e^{+j2\beta z}\right]$$

$$I(z) = \frac{V^+ e^{-j\beta z}}{Z_0}\left[1 - \rho e^{j\psi} e^{+j2\beta z}\right]$$

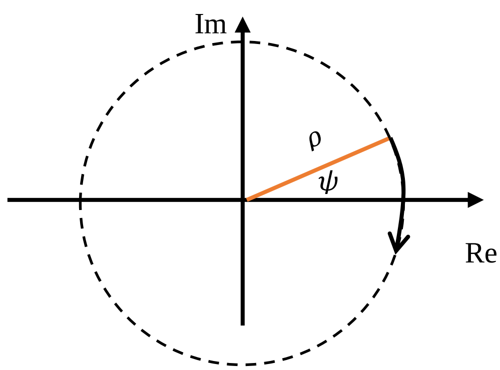
When we go to a distance *z* behind the load, it looks like to us that the reflection coefficient has changed. If you were sitting there as an observer, to you it will look like as if there is forward wave of $V^+e^{-j\beta z}$ and a backward wave of $\rho e^{j\psi}e^{+j2\beta z}V^+e^{-j\beta z}$. Since you can only see where you are sitting, you will think the *voltage reflection coefficient* is:

$$\Gamma(z) = \rho e^{j\psi} e^{+j2\beta z}$$

Remember again *z* is always negative. Thus, the argument $2\beta z$ will always be negative. Let us visualize this as visualization helps us to understand things better. $\rho e^{j\psi}$ can be treated as a vector with amplitude ρ and phase angle ψ as shown in the figure below.



As we go behind the load, magnitude of z increases. Let us say we go to a distance z = -*l*, the phase becomes $\psi - 2\beta l$. So what is happening is the line rotates about in a circle going clockwise (the angle is decreasing, clockwise is negative angle). We will see:



$\Gamma(z)$ can be considered as a generalized reflection coefficient at any arbitrary z on the transmission line.

3.2.2 Input Impedance

Let us again think that you are an observer sitting at some point *z* (again z will be always negative in our reference) on the transmission lines and measuring the voltage and current amplitudes. The amplitudes will be V(z) and I(z). Now coming from circuits point of view, to you it will look like that you have an impedance of Z(z). This is called the input impedance of the transmission line. If you attach a source at this value of z, what it will see is as if it is connected to Z(z) not Z_L. What does Z(z) look like? Lets evaluate it.

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V^+e^{-j\beta z} + \Gamma_L V^+ e^{+j\beta z}}{\frac{1}{Z_0}\left[V^+e^{-j\beta z} - \Gamma_L V^+ e^{+j\beta z}\right]} = Z_0 \frac{e^{-j\beta z} + \frac{Z_L - Z_0}{Z_L + Z_0} e^{+j\beta z}}{e^{-j\beta z} - \frac{Z_L - Z_0}{Z_L + Z_0} e^{+j\beta z}}$$

$$Z(z) = Z_0 \frac{(Z_L + Z_0)e^{-j\beta z} + (Z_L - Z_0)e^{+j\beta z}}{(Z_L + Z_0)e^{-j\beta z} - (Z_L - Z_0)e^{+j\beta z}} = Z_0 \frac{Z_L\left(e^{-j\beta z} + e^{+j\beta z}\right) + Z_0\left(e^{-j\beta z} - e^{+j\beta z}\right)}{Z_L\left(e^{-j\beta z} - e^{+j\beta z}\right) + Z_0\left(e^{-j\beta z} + e^{+j\beta z}\right)}$$

$$Z(z) = Z_0 \frac{Z_L 2\cos(\beta z) - Z_0 2j\sin(\beta z)}{Z_0 2\cos(\beta z) - Z_L 2j\sin(\beta z)}$$

Let us take $2\cos(\beta z)$ as common term out, and we get:

$$Z(z) = Z_0 \frac{Z_L - jZ_0\tan(\beta z)}{Z_0 - jZ_L\tan(\beta z)}$$

We also write Z(z) in another form as:

$$Z(z) = \frac{V(z)}{Z(z)} = \frac{V^+ e^{-j\beta z}\left[1 + \rho e^{j\psi} e^{+j2\beta z}\right]}{\frac{V^+ e^{-j\beta z}}{Z_0}\left[1 - \rho e^{j\psi} e^{+j2\beta z}\right]} = Z_0 \frac{1 + \rho e^{j\psi} e^{+j2\beta z}}{1 - \rho e^{j\psi} e^{+j2\beta z}}$$

When we see some practical examples, we will see where the second form comes in handy. Both expressions show that the impedance in transmission lines change as we go to different length. How, let's wait to understand the behavior till we look at a few more parameters.

3.2.3 Normalized Input Impedance

We normalize the input impedance with the characteristic impedance of the transmission line: $\overline{Z(z)}$.

$$\overline{Z(z)} = \frac{Z(z)}{Z_0} = \frac{Z_L - jZ_0\tan(\beta z)}{Z_0 - jZ_L\tan(\beta z)} = \frac{1 + \rho e^{j\psi} e^{+j2\beta z}}{1 - \rho e^{j\psi} e^{+j2\beta z}}$$

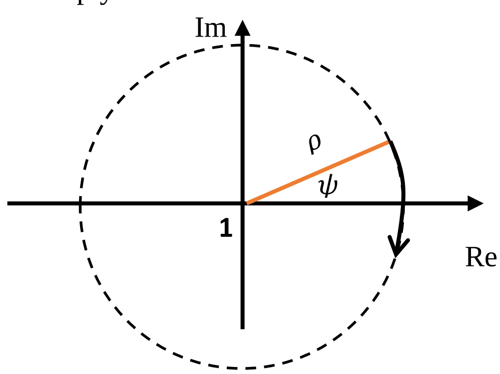
We will see its use a lot when we learn a graphical method to calculate transmission line parameters.

3.2.4 Voltage and Current Standing-Wave Patterns

Let us consider what happens to V(z) and I(z) as we go different values of "z". We have seen V(z) can be written as:

$$V(z) = V^+ e^{-j\beta z}\left[1 + \rho e^{j\psi} e^{+j2\beta z}\right]$$

The argument $\left[1 + \rho e^{j\psi} e^{+j2\beta z}\right]$ is the rotating vector $\Gamma(z)$ we have seen which has been shifted to a new origin of +1. Thus, this argument simply looks like:



You can visualize and see very quickly and clearly that as the red line rotates, it comes back to the original position when it makes a complete circle. Things repeat every full circle. Whatever I measure at point, if I go full circle behind it, I will measure the same thing again. So at what length differentials does this full circle happen? When:

$$2\beta z = -2\pi, -4\pi, -6\pi = -2m\pi, where\ m\ is\ an\ integer$$

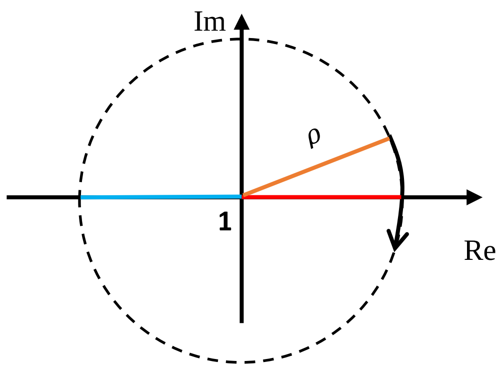
We are using a negative because we are going clockwise.

But, $\beta = \frac{2\pi}{\lambda}$. So the z for which things repeat are:

$$z = -\frac{m\lambda}{2}$$

Remember again the way we choose the origin z has to be always negative. **In terms of physical length, *l*, (z=-*l*), every λ/2 things repeat (not λ, why?).**

As $\rho e^{j\psi} e^{+j2\beta z}$ rotates with changing values of z, the function $\left[1 + \rho e^{j\psi} e^{+j2\beta z}\right]$ will go through a maximum and a minimum. The maximum and minimum value this function will take is when the line lies on the real axis (if you are not convinced, multiply the function with its complex conjugate). Maximum is the red line orientation and minimum is the blue line orientation.



V(z) when the $\rho e^{j\psi} e^{+j2\beta z}$ is going to be maximum for the red line orientation and its value is going to be:

$$V(z)_{max} = V^{+} e^{-j\beta z}(1 + \rho)$$

What happens to I(z) at this place? I(z) is given as:

$$I(z) = \frac{V^{+} e^{-j\beta z}}{Z_0}\left[1 - \rho e^{j\psi} e^{+j2\beta z}\right] = \frac{V^{+} e^{-j\beta z}}{Z_0}[1 - \rho]$$

When we have $V(z)_{max}$, I(z) will be minimum value it can take!

At what point on the transmission line does it happen. When $\psi + 2\beta z = 0$ or z_{max} will be:

$$z_{max} = -\frac{\psi}{2\beta} = -\frac{\lambda\psi}{4\pi}$$

z_{max} is the place where V(z) is maximum and I(z) is minimum. This will also repeat every λ/2 after that.

Practical note: z_{max} should always be negative. However, the way our calculators work, ψ can sometimes be calculated as negative resulting in z_{max} to be positive. **What should you do then?** You can correct ψ by adding 2π to it. Or we know that everything repeats $\lambda/2$. **Simply subtract $\lambda/2$ from the from the positive z_{max} you get.**

Going back to the figure above, we will get the minimum V(z) when the rotating $\rho e^{j\psi}e^{+j2\beta z}$ gets to the blue orientation. When the value of V(z) is minimum, the value of I(z) will be maximum and will be given as:

$$V(z)_{min} = V^+e^{-j\beta z}(1-\rho)$$

$$I(z)_{min} = \frac{V^+e^{-j\beta z}}{Z_0}\left[1+\rho e^{j\psi}e^{+j2\beta z}\right] = \frac{V^+e^{-j\beta z}}{Z_0}[1+\rho]$$

Where does this happen? When $\psi + 2\beta z = -\pi$

When:

$$z_{min} = -\frac{\pi+\psi}{2\beta} = -\frac{\lambda(\pi+\psi)}{4\pi}$$

z_{min} is the place on the transmission line where V(z) is minimum and I(z) is maximum.

What is happening? We have interference between the forward wave and the backward wave creating nodes and antinodes. Where they interfere constructively, we get a maximum voltage and where they interfere destructively we get a minimum voltage. A standing wave is created on the transmission line and the maximum value a wave can take as time changes is given by V(z) at that point. A standing wave is shown in the figure below for a short circuit line. When we have a short circuit, the backward reflected wave has the same amplitude as the forward wave. Hence, the two waves cancel out perfectly where they are π out of phase with each other (we will do calculations laters). **See the video to visualize the standing waves in transmission lines.**

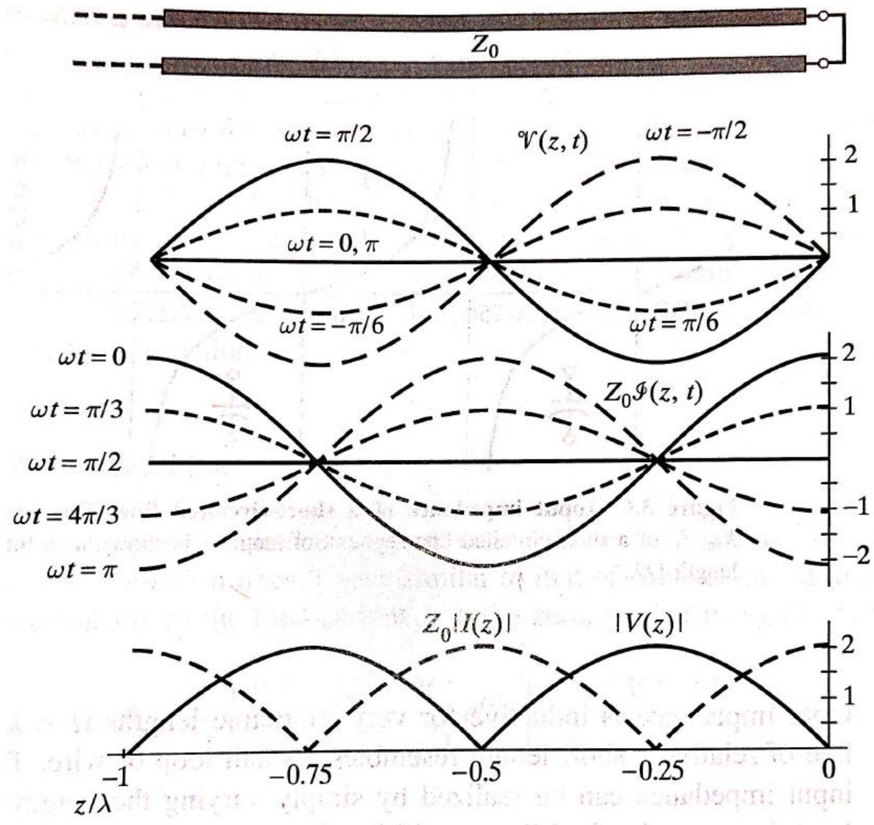


Figure. Voltage, and current standing waves for a short circuit load. The magnitude of V(z), and I(z) are also plotted. You can see that they are out of phase with each other.

We define a term called voltage standing wave ratio (VSWR, or simply S) which is the ratio between maximum V(z) and minimum V(z). Please remember, that the maximum value for the standing wave and the minimum of the standing wave happen at different places on the transmission line as shown in the figure above. To measure S, we will have to probe the transmission line at different points and find the maximum and the minimum voltage amplitudes and take the ratio.

$$S = \frac{V(z)_{max}}{V(z)_{min}} = \frac{V^{+}e^{-j\beta z}(1+\rho)}{V^{+}e^{-j\beta z}(1-\rho)} = \frac{1+\rho}{1-\rho}$$

What happens to Z(z)? Let us see:

$$Z(z) = \frac{V(z)_{max}}{I(z)_{min}}$$

Denominator takes a maximum value it can take, while numerator takes the minimum value it can take; that Z(z) will be the maximum it can be, let us call it Z_{max}. Not only that, there is another interesting observation. Let us calculate Z(z).

$$Z_{max} = \frac{V^{+}e^{-j\beta z}(1+\rho)}{\frac{V^{+}e^{-j\beta z}}{Z_0}[1-\rho]} = Z_0\frac{1+\rho}{1-\rho} = SZ_0$$

Every element in the term is real. No matter whether the load is complex or not, Z_{max} is purely resistive! If I connect a source at this z, then, it will think it is connected to a resistor with a value of SZ_0.

Similarly when we have $V(z)_{min}$ and $I(z)_{maz}$, Z(z) will be the minimum it can be and will be:

$$Z_{min} = Z_0 \frac{1-\rho}{1+\rho} = \frac{Z_0}{S}$$

Again it is purely resistive!

We have learned all the terms we need to understand transmission line characteristics and can now look at practical cases and improve our understanding even further.

3.3 What happens at $z = -\lambda/_4$?

Let us look at the input impedance. We have:

$$Z(z) = Z_0 \frac{Z_L - jZ_0 \tan{(\beta z)}}{Z_0 - jZ_L \tan{(\beta z)}}$$

at $z = -\lambda/_4$, $\beta z = -\frac{2\pi}{\lambda}\frac{\lambda}{4} = -\frac{\pi}{2}$. Hence, $\tan\left(-\pi/_2\right) = -\infty$

$$Z(z) = Z_0 \frac{Z_L + jZ_0 \infty}{Z_0 + jZ_L \infty} = Z_0 \frac{\infty}{\infty} \frac{Z_L/_\infty + jZ_0}{Z_0/_\infty + jZ_L} = Z_0 \frac{jZ_0}{jZ_L} = \frac{{Z_0}^2}{Z_L}$$

Thus, we get:

$$Z(z) = \frac{{Z_0}^2}{Z_L}$$

In terms of normalized impedance, we get:

$$\overline{Z(z)} = \frac{Z(z)}{Z_0} = \frac{Z_0}{Z_L} = \frac{1}{\overline{Z_L}}$$

where $\overline{Z_L}$ is the normalized load.

In terms of normalized impedances, things completely reverse. An inductive load will look like a capacitive load while a capacitive load will behave like an inductive load. A short circuit line will behave like….?

3.4 Short Circuit Line

For the short circuit line, $\Gamma_L = -1 = 1e^{j\pi}$. Thus, the reflected wave has the same amplitude as the forward wave. Maximum V(z) will be twice the V^+ when the peaks of the two waves align and the minimum V(z) will be 0 where the peak of the forward wave aligns with the trough of the backward wave. Thus, S will be ∞.

Let us consider a short circuit (s.c.) line as shown in the figure below and see what Z(z) we see as we go to different lengths behind. For s.c. line, we have the load value to be 0.

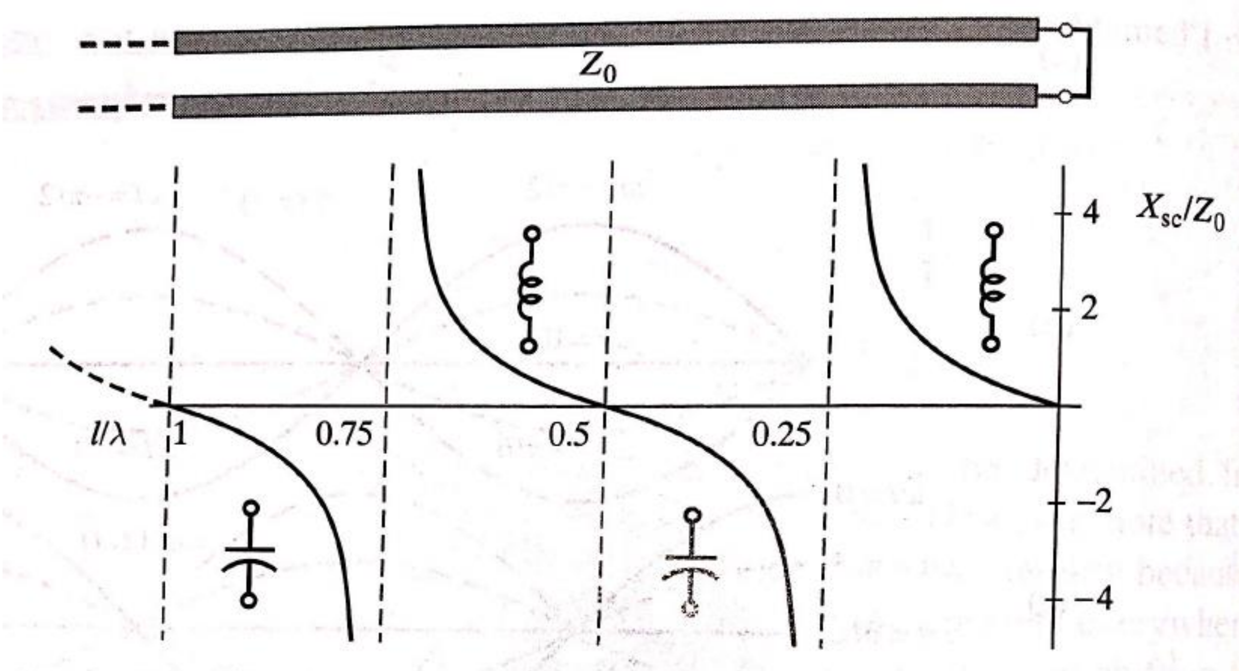
$$Z(z) = Z_0 \frac{Z_L - jZ_0 \tan(\beta z)}{Z_0 - jZ_L \tan(\beta z)} = Z_0 \frac{0 - jZ_0 \tan(\beta z)}{Z_0 - j0 \tan(\beta z)} = -jZ_0 \tan(\beta z)$$

Since z is negative, at a length *l* behind the short circuit, z = -*l* and thus, the input impedance we see is:

$$Z_{sc} = Z(z) = -jZ_0 \tan(-\beta l) = jZ_0 \tan(\beta l)$$

Let us plot Z_{SC} on the z-axis as shown below. We see something totally unexpected but powerful.

The length element is normally plotted in term of wavelength, rather than physical dimensions. Sometimes is also plotted in radians in which case it the value βl *(remember $e^{-j\beta l}$ is like $e^{-j\theta}$)*. These terms are called the electrical length of the line.



For the length of short circuit line with length between 0< *l* < λ/4, the short circuit line behaves like an inductor. Not only that I can design an inductor of any value from 0 to infinity. I want an inductor, I can just a short transmission line with a short circuit on it.

At *l* = λ/4, Z_{in} is infinity. It behaves like an open circuit line. If you think carefully and connect it to ECE 240, it is actually a parallel capacitor and inductor circuit operating at resonance.

From λ/4 < *l* < λ/2, the short circuit line behaves like a capacitor and again by changing the length of the line, we can get any capacitance we want. A source connected at this length will think it is connected to a capacitor and thus, I can make any capacitance by just a transmission line of specified length which is short circuited.

What is the value of S? As we have seen before, ρ is 1 for short circuit line. Thus, S will be ∞ . Why? The reflected wave has the same magnitude as the forward wave. They cancel out perfectly at destructive interference points and thus, V_{min} becomes 0.

3.5 Open Circuit Line

Again for open circuit line, the reflected wave has the same amplitude as the forward wave and V_{min} will be 0 resulting in S to be ∞.

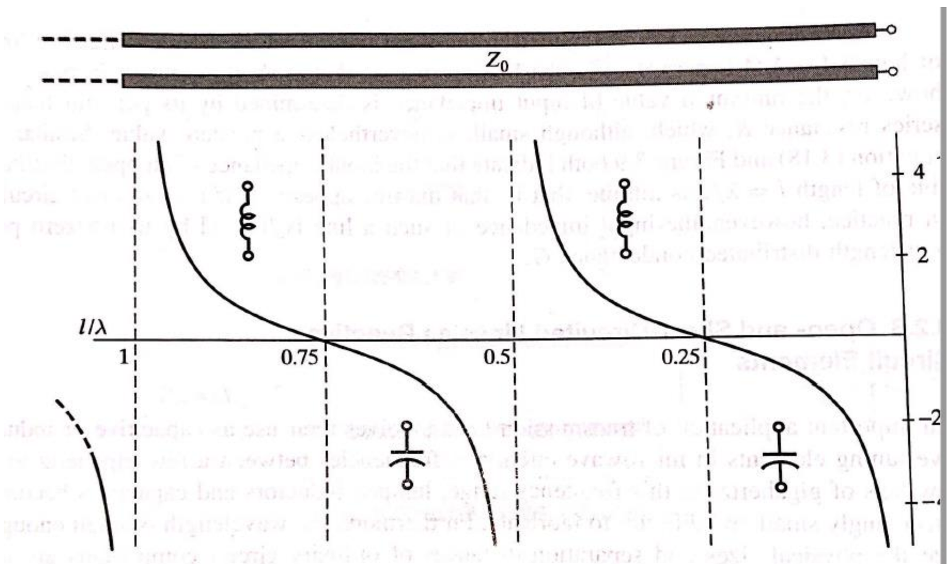
Z(z) for an open circuit line will be:

$$Z(z) = Z_0 \frac{\infty - jZ_0 \tan(\beta z)}{Z_0 - j\infty \tan(\beta z)} = Z_0 \frac{\infty}{\infty} \frac{1 - \left({}^{jZ_0}/_{\infty}\right) \tan(\beta z)}{{}^{Z_0}/_{\infty} - j\tan(\beta z)} = jZ_0 \cot(\beta z)$$

For z = -*l*, we get:

$$Z_{oc} = jZ_0 \cot(-\beta l) = -jZ_0 \cot(\beta l)$$

Let us draw it and as expected we get exactly the opposite behavior of a short circuit line.



For the length of open circuit line with length between 0< *l* < λ/4, the line behaves like a capacitor. Not only that we can design an capacitor of any value from 0 to infinity.

At *l* = λ/4, Z_{in} is 0. It behaves like a short circuit line like an inductor and capacitor in series at resonance.

From λ/4 < *l* < λ/2, the open circuit line behaves like an inductor.

So both short circuit line and open circuit line, we have a powerful design elements which will allow us to design inductors and capacitors in a high speed circuit.

Lumped element models of different lengths of open circuit and short circuit line segments are shown in the figure on the next page.

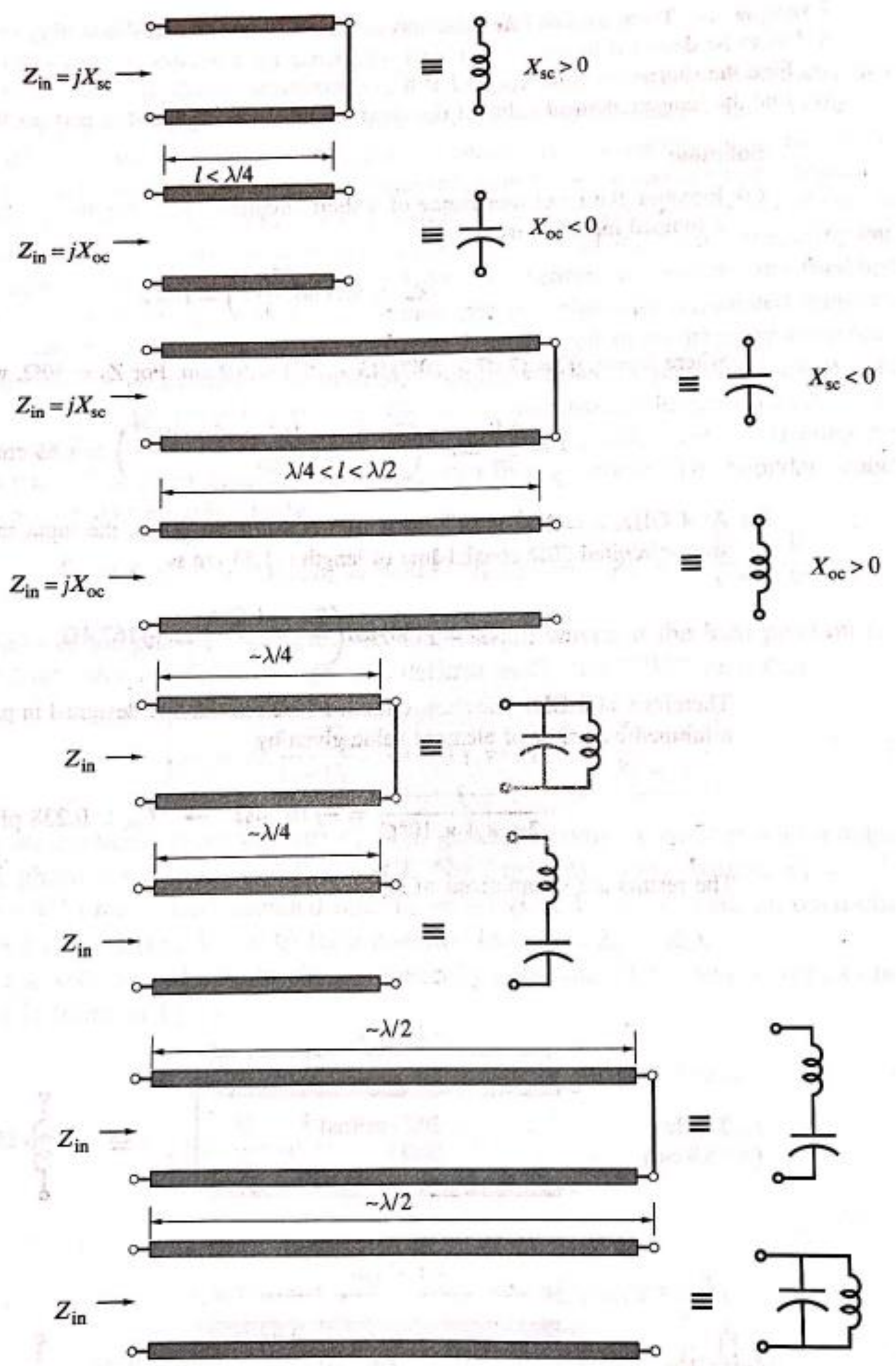
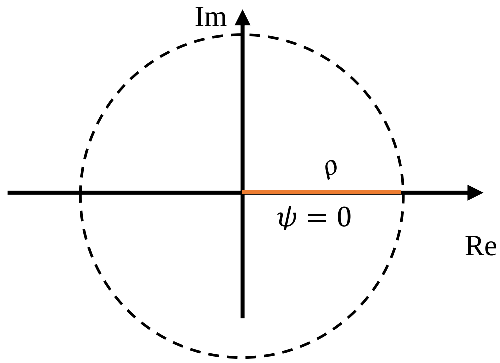


Figure 3.2. Lumped circuit models for open circuit and short circuit lines.

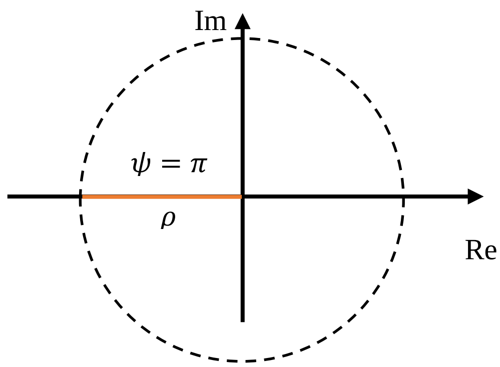
3.6 Resistive Load

For, $R_L > Z_0$, reflection, Γ_L, will be a positive number and thus, equal to ρ ($\psi = 0$). In terms of visualization Γ_L will lie on the Re axis on the positive sides as:

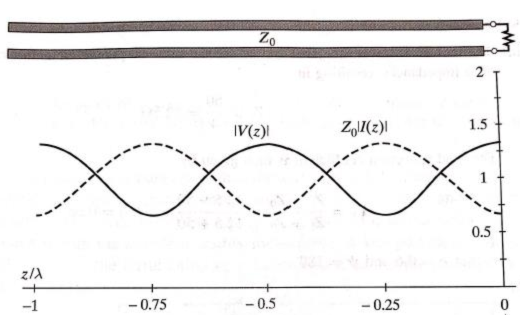


We will see a maximum voltage and minimum current right at the load itself. Move λ/4 back and we see a minimum voltage, maximum current, minimum Z(z). Further, Z_{max} is going to R_L itself and Z_{min} is going to $\frac{{Z_o}^2}{R_L}$.

For $R_L < Z_0$, Γ_L, will be a negative number and thus, $\Gamma_L = -\rho = \rho e^{j\pi}$. In terms of visualization, we see that

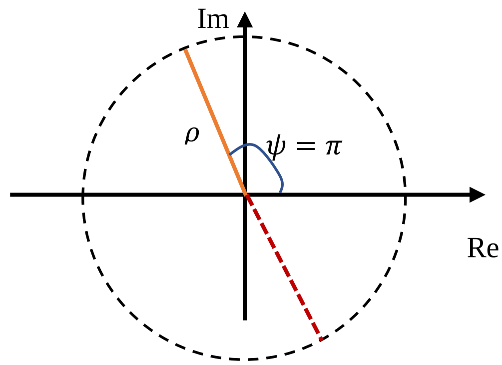


We will see a minimum voltage and maximum current right at the load itself. Move λ/4 back and we see a maximum voltage and minimum current. Further, Z_{max} is going to R_L itself. An example of V(z) is shown below.

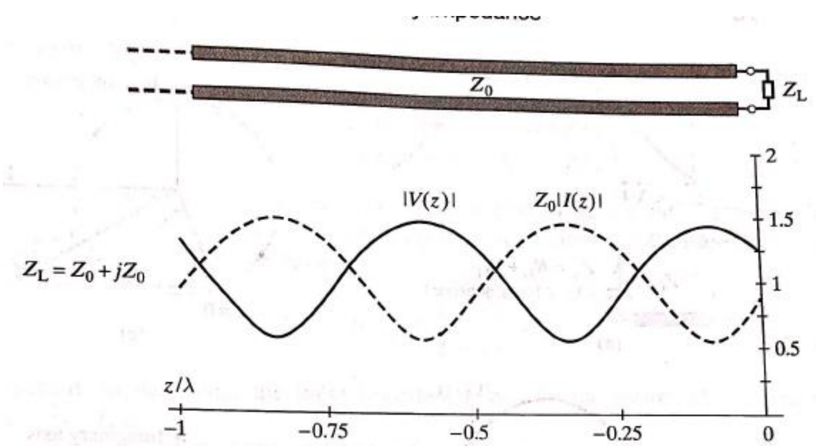


3.7 Inductive Load

For an inductive load, we have the phase of the reflection obey: $0 \leq \psi \leq \pi$. Thus, on our visualization diagram, we observe:

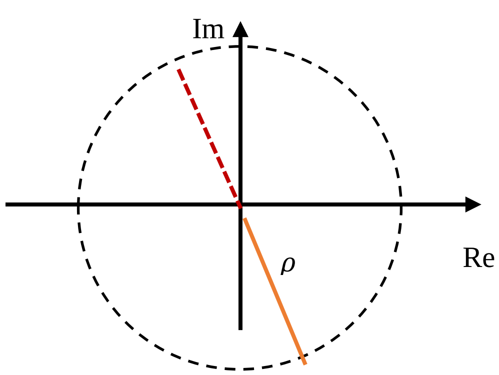


As we go behind from the load, the orange line rotates clockwise. The dashed red line is where the reflection coefficient will be when the line is λ/4 long. So as we go behind an inductive load, we cross a voltage maximum before λ/4 and then cross a voltage minimum between λ/4 and λ/2. A typical V(z) on an inductive load is shown in the figure below.



3.8 Capacitive Load

For capacitive load, the phase of the reflection ψ will be negative and our visualization diagram will look like:



Thus, as we go behind on the transmission line, we will pass through a voltage minimum before the length of λ/4 and then a voltage maximum before a length of λ/2.

These observations will become important when we design matching circuits in another week or so.

3.9 Finding an unknown load

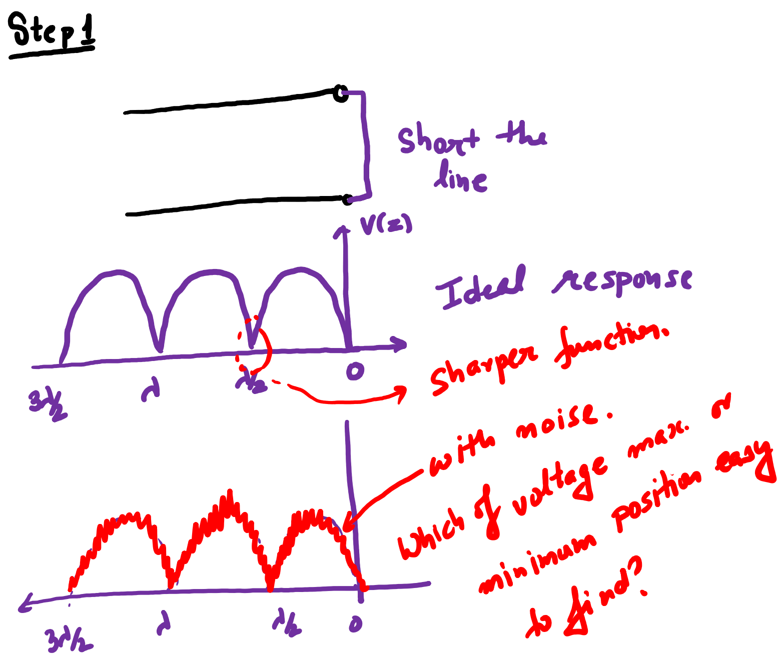
We have a load in a black box we do not know anything about. We want to find its value. Let us design an experiment to find its value. We will use a transmission line to connect to the load and probe it. However, before we can do any measurements, we need to know the wavelength on the transmission line and mark some reference points.

Step 1: We short circuit the transmission line (we can keep is open circuit but short circuit is a better choice, why?). With a probe we measure the voltage amplitude along the length. We should observe voltage distribution as shown below. Obviously there is going to experimental noise.

The length between two minimums should be λ/2 (everything repeats λ/2) and thus, by measuring the distance, we get the wavelength of the wave on the transmission line.

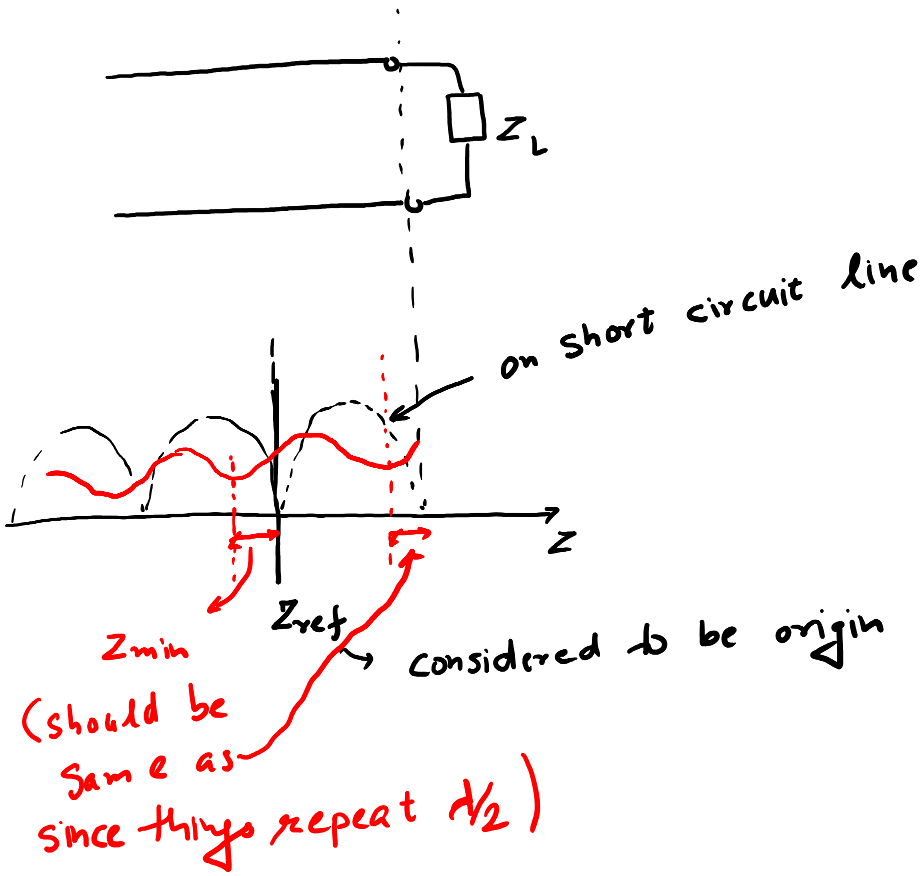
Question: Is voltage minimum or voltage maximum a good reference point?

Look at the shape of the curve. At voltage minimum, we have a very sharp curve whereas at voltage maximum we have a slowly varying curve. With noise added, it will be much more accurate to locate the minimum than the maximum. A voltage minimum serves as a good reference point.



Why do we need a reference point? When we connect the load, we may not always be able to probe the first minimum due to the connectors (the minimum may be inside the connector). Look at the voltage standing wave on the short circuit transmission line. Since, everything repeats at λ/2, a voltage minimum behaves exactly as the short circuit load. Putting a reference there is same as starting from the load. (**Now do you get why it is better to use a short circuit line as compared to open circuit line**).

Step 2: So we choose one of the points as a reference point. Now we connect the load. Starting from the reference point, we go backwards and locate the position of the first minimum. The length we travel back from the reference is going to be l_{min} for the load. If $l_{min} < \lambda/4$, then we have a minimum before a maximum and thus, the load should be capacitive. If $l_{min} > \lambda/4$, then we have a maximum before a minimum and thus, the load should be inductive. We won't need this information to calculate, but it is always good to check whether the answer makes sense.



The next thing we measure is the VSWR, S on the line by measuring the maximum and minimum voltage amplitude along the different distances. If we know S, we know Z_{min} which is nothing but Z_0/S. We can use this information. Let us see how:

$$Z_{min} = \frac{Z_0}{S} = Z_0 \frac{Z_L - jZ_0 \tan(\beta(-l_{min}))}{Z_0 - jZ_L \tan(\beta(-l_{min}))} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l_{min})}{Z_0 + jZ_L \tan(\beta l_{min})}$$

We have substituted z = $-l_{min}$ as the location of the first minimum. We can algebraically rewrite this to get the load in terms of parameters we have measured as:

$$Z_L = Z_0 \frac{1 - jStan(\beta l_{min})}{S - jtan(\beta l_{min})}$$

This becomes a powerful method to find an unknown load.

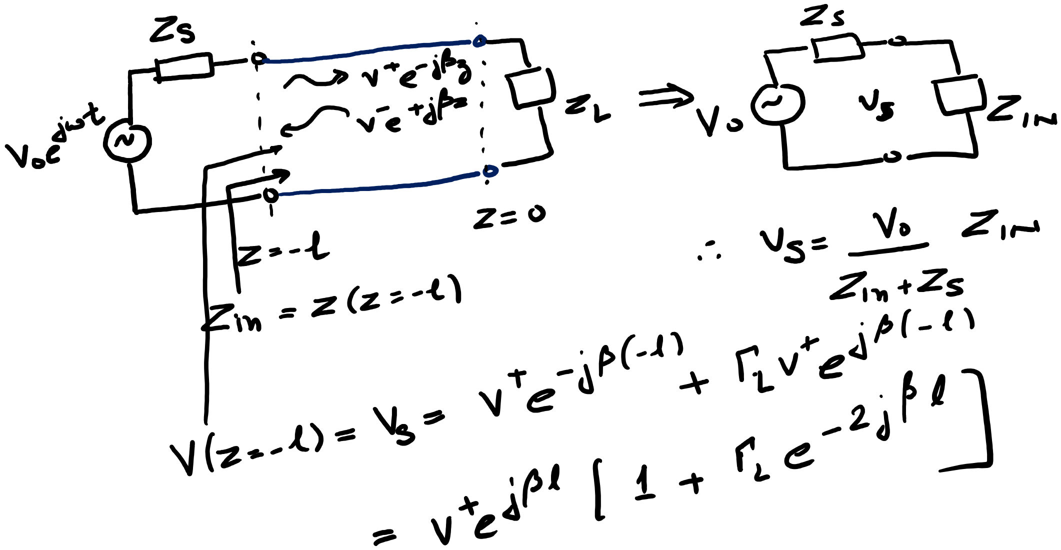
3.10 Calculating V^+

To calculate V^+, we need to go back to the source i.e. z = -*l* where l is the length of the transmission line. At the source, the input impedance will be:

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

The source thinks that it is connected to a load of Zin as shown in the circuit below and thus, voltage across the transmission line, V_s will be:

$$V_s = \frac{Z_{in}}{Z_{in} + Z_s} V_0$$



However, this voltage must be equal to the voltage we observe as the sum of the forward and backward waves, i.e. V(z=-*l*). Thus, Vs is also equal to:

$$V_s = V^+ e^{j\beta l}\left[1 + \Gamma_L e^{-j2\beta l}\right]$$

By equating the two terms we can calculate the value of V^+.

$$V^+ = \frac{Z_{in} V_0}{(Z_{in} + Z_s) e^{j\beta l}\left[1 + \Gamma_L e^{-j2\beta l}\right]}$$

3.11 Power Transmission

When you are designing a circuit, in principle your goal is to maximize the time averaged power deliver to the load. We know that in phasor form, the time averaged power can be written as $\frac{1}{2} Re[VI^*]$. We just have to replace V and I with *V(z)* and *I(z)*. Thus, the time averaged power along the transmission line will be:

$$P_{av}(z) = \frac{1}{2} Re[V(z) I(z)^*]$$

We know that

$V(z) = V^+e^{-j\beta z} + \Gamma_L V^+e^{-j\beta z}$ and $I(z) = \frac{V^+e^{-j\beta z}}{Z_0} - \frac{\Gamma_L V^+e^{-j\beta z}}{Z_0}$

The power carried by the forward wave P^+, can be calculated as:

$$P^+ = \frac{1}{2}Re\left[V^+e^{-j\beta z}\frac{\left(V^+e^{-j\beta z}\right)^*}{Z_0}\right] = \frac{1}{2}\frac{|V^+|^2}{Z_0}$$

Similarly, the power carried by the backward wave will be:

$$P^- = \frac{1}{2}Re\left[\Gamma_L V^+e^{-j\beta z}\frac{\left(-\Gamma_L V^+e^{-j\beta z}\right)^*}{Z_0}\right] = -\frac{1}{2}\frac{|\Gamma_L V^+|^2}{Z_0} = -\frac{1}{2}\frac{\rho^2|V^+|^2}{Z_0}$$

P^- being negative simply indicates that the power is travelling backwards.

The power delivered to the load will be:

$$P_L = P^+ + P^- = \frac{1}{2}\frac{|V^+|^2}{Z_0} - \frac{1}{2}\frac{\rho^2|V^+|^2}{Z_0} = \frac{1}{2}\frac{|V^+|^2}{Z_0}[1-\rho^2]$$

The ratio of the power delivered to the load as compared to the incident power will be:

$$\frac{P_L}{P^+} = [1-\rho^2]$$

Substituting ρ in terms of S (practically we measure S and get ρ from there) , we get:

$$\frac{P_L}{P^+} = \frac{4S}{(1+S)^2}$$

The degree of mismatch between the load and the transmission line is called the return loss and is normally written in decibel scale.

$$Return\ Loss = -20\log(\rho) = 20log\left(\frac{S+1}{S-1}\right)$$

If the load is perfectly matched to the transmission line, then ρ = 0, the return loss is infinite which simply means that there is no reflected power. Larger the return loss, better matched the load is to the transmission line. In practice, a well matched system has a return loss of over 15 dB.

3.12 When do we need to use transmission line theory?

As we have seen, if the length is λ/4, then the response of the load is complete opposite. A good rule of thumb is if the transmission line is longer than λ/10, then we definitely need to consider the transmission line effects. The velocity of the wave is approximately half the speed of light (we will see that when we start doing EM waves). Let us consider what λ/10 will be for different frequencies and transmission systems.

Frequency	Wavelength (λ)	λ/10
60 Hz (power transmission)	2500 km	250 km
100 MHz (FM Radio)	1.5 m	15 cm
2.4 GHz (cell phones etc.)	6.25 cm	0.625 cm
4.0 GHz (CPUs)	3.75 cm	0.375 cm

As you can see higher the frequency you are operating at, the effects of waves has to be accounted for. As circuits are becoming faster and faster, you have to account for transmission line effects.

3.13 Cascaded Transmission Lines:

How can we calculate transmission lines which are cascaded in series or parallel? We always go to the last place on the circuit and trace our steps back. So we go to the furthermost load, and calculate its projected impedance on the junction using the input impedance calculations. If there are more than one line in parallel at the junction, we do this independently for each line. At the junction, we may now have multiple input impedances and we calculate the effective impedances using circuit theory. This equivalent impedance becomes the load for the subsequent transmission line and we can now trace our steps further back. Remember, at a node/junction, standard circuit theory applies. What the transmission line does is transform the load on its end into an input impedance on its input.