Chapter 7. Maxwell's Equations

Review of ECE 106

In ECE 106, we saw the four Maxwell's equations. In differential form, they are written as:

$$\nabla.\vec{D} = \rho \qquad\qquad \text{Gauss law} \quad (1.1)$$

$$\nabla.\vec{B} = 0 \qquad\qquad \text{No magnetic monopoles} \quad (1.2)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad\qquad \text{Ampere's law} \quad (1.3)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad\qquad \text{Faraday's law} \quad (1.4)$$

These equations consists of four field terms, $\vec{D}$, $\vec{E}$,(related to electric forces) $\vec{B}$, and $\vec{H}$ (related to magnetic forces); two source terms ρ and $\vec{J}$; and hidden within these equations are two material properties. These material properties relate $\vec{E}$ with $\vec{D}$ and $\vec{B}$ with $\vec{H}$. In this chapter, we will do a quick review of the Maxwell's equations, the various field terms and the material properties.

7.1 Electric field, $\vec{E}$

Electric field was defined in terms of Electrostatics and is the force per unit positive test charge at a point P. A charge distribution creates a field around and when we bring in a new charge into the field, the field interacts with the charge and places a force on it. For a point charge, Q, the Electric field at a point P which is $\vec{r}$ away from Q is given as:

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q}{q} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \quad [\text{N/C}]$$

If the charge is distributed over a volume as shown in the figure below, then we can use the superposition principle (all EM is based on superposition) and write the Electric field as:

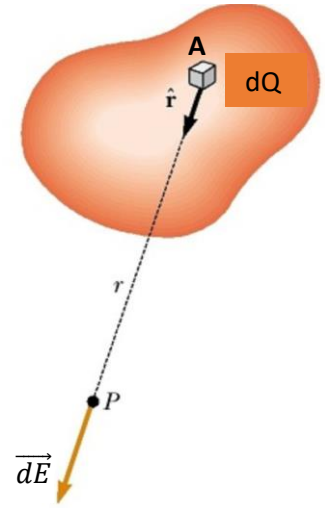$$\vec{E} = \int_{vol} \overrightarrow{dE} = \int_{vol} \frac{k\rho_v dv}{r^2}\hat{r}$$

Figure 1. The differential Electric field at point P due to a differential charge dQ at an arbitrary point in the volume where charge is present. The Electric field at point P due to a distributed charge is the vectorial sum of the differential Electric field due to each small differential element, dv.

7.3 Gauss Law

Gauss Law is nothing but a mathematical restatement of the square law in the equation above but provides a new perspective of looking at fields. Mathematically, Gauss Law states that:

$$\oint \vec{D}.\overrightarrow{dS} = \oint \epsilon_0 \vec{E}.\overrightarrow{dS} = Q_{enclosed}.$$

For the time being, we can write it as:

$$\oint \vec{E}.\overrightarrow{dS} = \frac{Q_{enclosed}}{\epsilon_0}.$$

As you have seen in Calculus III, using the divergence theorem, the law can be written in a point form as:

$$\nabla . \vec{E} = \frac{\rho_v}{\epsilon_0}$$

where ρ_v is the volume charge density. It is just a mathematical statement of what we have said before. What the equation says is that if we calculate Electric flux over a closed surface, then the Electric flux will always be equal to the enclosed charge within the surface. **The circle around the integral sign reminds us that the Law is only applicable on a closed surface i.e. the surface encompasses a three dimensional volume which is completely enclosed by the surface.** Gauss Law is always true and is basically based on the $1/r^2$ nature of the force law. If you can find an exception to Gauss Law, the complete 20th century physics will collapse and you will have a Nobel prize in physics waiting for you (or most probably the Ig Nobel award).

The integral form of the Gauss Law is applied over 3 dimensional space while the differential form it describes the behavior of Electric fields due to charge densities at any specific point in space. Conceptually, Gauss Law states that the charge density ρ_v acts as a source for the Electric field and the field diverges out of the point if charge density is present.

7.4 Polarization vector, $\vec{P}$

When we apply an external electric field to an insulator, the electrons in the atom still feel the force and are displaced towards one side as compared to the nucleus as shown in the figure below.
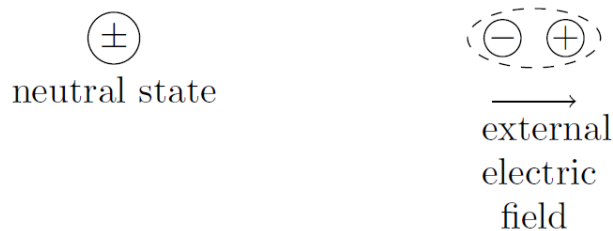
Figure 2. Dipole being induced in an atom due to an external electric field

The insulator is said to be "polarized" and when we consider the polarizability of the insulator, we refer to them as "Dielectrics". The name "dielectric" was coined by Faraday who extensively studied their properties when placed between two conductors.

Polarization of dielectrics occurs due to the displacement of one or more electrons per atom over subatomic or microscopic distance as shown in Figure 1. The atom of the dielectric materials has their outermost electron shell nearly full resulting in the electrons becoming tightly bound to the atom. When a dielectric material is placed in an external electric field, the electrons feel a force and respond by shifting to one side as compared to the nuclei. The displacement, though very small as compared to conductors, nevertheless polarize the material and create electric dipoles. There will be an induced electric field inside the dielectric which will modify the total electric field in the system as shown in Figure 3. This will also induce charges on the surface and the volume of the material.

A dipole is characterized by a dipole moment, $\vec{p}$, given as $\vec{p} = q\vec{d}$ where q is the value of the charge and d is the distance between –q and +q charge and the direction of the vector is always from –q charge to +q charge. $\vec{p}$ is a microscopic quantum mechanical property. In Engineering and classical physics, we are interested in knowing how the material responds as an ensemble to external electric fields. Any dimensions of interest will have hundreds of thousands of these dipoles. So we are interested in figuring out how the induced dipoles effects the material macroscopically. We define a macroscopic term called Polarization vector, $\vec{P}$. Mathematically, it is defined as:

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\sum_{i=0}^{n\Delta v} \vec{p_i}}{\Delta v}$$

where n is the number of atoms per unit volume. The numerator represents the vector sum of the induced dipole moments contained in a very small volume Δv. **The polarization vector, $\vec{P}$ is a smoothed point function and is the volume density of the electric dipole moment**. Without going into the derivations, the induced charge on the surface of the dielectric is given as:

$$\rho_s = \vec{P}.\hat{n}$$

where $\hat{n}$ is the normal unit vector to the surface pointing out of the dielectric. The induced charge in the volume is given by the divergence of $\vec{P}$ as:

$$\rho_v = -\nabla.\vec{P}$$

Divergence of a vector field A at a point is the net outward flux of A per unit volume as the volume about the point tends to zero and is given as:

$$div \, \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}.\overrightarrow{dS}}{\Delta v}$$

In Cartesian coordinates, the divergence is simply:

$$\text{div A} = \nabla.A = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \right).\left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The physical significance of divergence is the rate at which density exits a given region of space e.g. you go on top of a hill and through water; how fast would water flow away from the point.

Polarization vector, $\vec{P}$, is a material property and defines how the material is going to respond to the Electric fields.

7.5 Displace Vector, $\vec{D}$, and Dielectric permittivity

A polarized dielectric gives rise to a volume charge density, ρ_v. Thus, we expect the electric field intensity within the material to be different from that in free space. Let us see how we can include this effect. We start from Gauss Law:

$$\nabla.\vec{E} = \frac{\rho}{\epsilon_0}$$

The charge density, ρ includes both the free charges (charges places in the system) and the induced charges (charges created due to polarization).

In a dielectric material, the induced charges will be related to the polarization vector, $\vec{P}$. Thus, we have:

$$\nabla.\vec{E} = \frac{\rho^{free} + \rho_v}{\epsilon_0} = \frac{\rho^{free} - \nabla.\vec{P}}{\epsilon_0}$$

Rearranging we get:

$$\nabla.\epsilon_0\vec{E} + \nabla.\vec{P} = \rho^{free}$$

Leading to:

$$\nabla.(\epsilon_0\vec{E} + \vec{P}) = \rho^{free}$$

We define a new fundamental field quantity called electric flux density or electric flux displacement vector, $\vec{D}$ given as: $\vec{D} = \epsilon_0\vec{E} + \vec{P}$. The use of $\vec{D}$ enables us to write the divergence relation between the electric field and the distribution of free charges in any medium without the necessity of dealing explicitly with the polarization vector $\vec{P}$ or the polarization charge density. This is very powerful. Unlike what we were doing in conductors, we don't have to worry about induced charges in our calculations! The units of $\vec{D}$ are C/m². Thus, the new Gauss Law in point form is:

$$\nabla.\vec{D} = \rho^{free}$$

Converting it back into integral form we get the equation:

$$\oint_S \vec{D}.\overrightarrow{dS} = Q_{enc}^{free}$$

The set is the 1st Maxwell's Equations in point form and integral form, respectively.

When the dielectric properties of the medium are linear and isotropic (doesn't change with direction), the polarization vector will be directly proportional to the electric field intensity. We can write:

$$\vec{P} = \epsilon_0\chi_e\vec{E}$$

where χ_e is a dimensionless quantity called electric susceptibility. A dielectric medium is linear if χ_e is independent of $\vec{E}$ and homogeneous if χ_e is independent of space co-ordinates. Substituting we get:

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} = \epsilon_0(1 + \chi_e)\vec{E}$$

We define $1 + \chi_e$ as **relative permittivity or dielectric constant**. We write:

$$\epsilon_r = 1 + \chi_e$$

ϵ_r is a dimensionless quantity. We thus, have:

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$$

The coefficient $\epsilon = \epsilon_0 \epsilon_r$ is the absolute dielectric permittivity with units of Farads/m (F/m). Thus, the relative permittivity ϵ_r measures how strongly does the material polarize with respect to free space.

7.6 Physical Understanding of Relative Permittivity, ϵ_r

We will be using ϵ_r a lot in defining electromagnetic waves. We apply Electric field and a dipole is created. As the charges spread, a restoring force is going to be created. Thus, the system is acting like a spring which is being stretched.
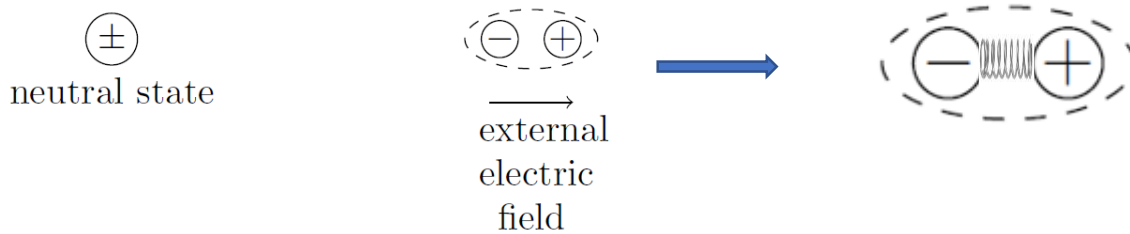
How much will the spring stretch? It should be proportional to the force we are putting on it, at least to the first order. As the spring stretches, the distance between the two charges increases and thus, the dipole strength starts to increase. This increases the polarization vector, $\vec{P}$. That is the linear relationship we put in between $\vec{P}$ and $\vec{E}$. Besides, the separation, the values of the dipole moments will also depend on the atomic structure and how the electrons surround the nucleus. The proportionality constant is what we called electric susceptibility. Relative permittivity is related to the susceptibility by a simple addition of 1. Thus, the relative permittivity depends on the atomic structure of the atoms for the dielectric and corresponds to the strength of the dipoles that will be created inside the material when we apply Electric field to it.

What happens, if we apply a time varying Electric field, $E_o e^{j\omega t}$? This would be like driving this oscillator. There is going to be a resonance frequency. At resonance frequency, the amplitude of the oscillation is going to appreciable increase. Higher than the resonance frequency, the amplitude of the oscillation is going to fall off very quickly. Thus, we expect ϵ_r to be a function of ω. This will lead to **chromatic dispersion** in a wave travelling through the medium. You have seen its effects since you were a child. Rainbows are created due to the frequency variation of ϵ_r of water in the droplets. This effect can be useful to tailor devices (an optical multiplexer which combines different wavelengths of light in a single beam allowing us to send multiple wavelengths in a single fiber dramatically increasing the transmission capacity) or harmful in other areas (different frequencies of the signal reach the destination at different times leading to spreading of the pulse and interference).

Any spring system will also have damping in it. Damping absorbs the energy. Similarly, in the system above, we will have dampling. ϵ_r will have a real part and an imaginary part to it and can be written as:

$$\epsilon_r = \epsilon' + j\epsilon''$$

We will later see, that when we consider waves, this will result in propagation of the wave and a loss term in the wave.

7.7 Boundary Conditions

Electromagnetic problems often involve media with different physical properties and require the knowledge of the relations of the field quantities at an interface between the two media. We want to understand how $\vec{E}$ and $\vec{D}$ fields change in crossing an interface. Consider the boundary between two dielectrics with absolute permittivities of ϵ_1 and ϵ_2 as shown in the figure below.
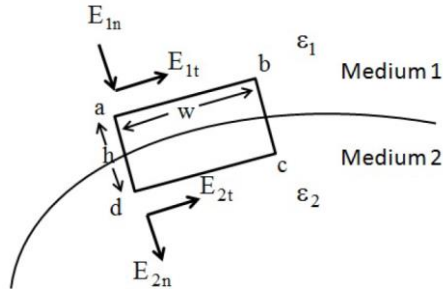
We want to calculate E_{2t}, E_{2n} given E_{1n}, E_{1t} as we go from medium 1 to medium 2.

Tangential Boundary condition: Let us first look at what happens to the tangential i.e. parallel field components. Like we did in conductors, let us make a loop around the boundary.
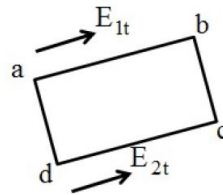
If we calculate the potential difference along the path starting from a and coming back to a, it should be 0. Thus,
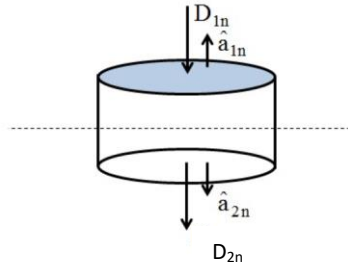
$$-\oint_{abcda} \vec{E}.\overrightarrow{dl} = 0$$

Thus, $-E_{1t}w + E_{2t}w = 0$ which means that:

$$E_{1t} = E_{2t}$$

The tangential components of the electric fields have to be equal across the boundary.

Normal Boundary Conditions: To find the relation between the normal components of the field, let us apply Gauss Law across the boundary as shown in the figure below.

Applying Gauss Law (the new form) we get:

$$D_{2n}A - D_{1n}A = \rho_s^{free}A$$

where A is the cross sectional area of the pill box and ρ_s^{free} is any free charge density which may be on the boundary. Thus, we have

$$\boldsymbol{D_{2n} - D_{1n} = \rho_s^{free}}$$

Remember this relationship is true as we have drawn in the figure and is specific to the directions in the figure.

If $\rho_s^{free} = 0$ i.e. the boundary has no free charge on it, then

$$D_{2n} = D_{1n}$$

Which means $\epsilon_2 E_{2n} = \epsilon_1 E_{1n}$. There is a discontinuity in the normal component of the Electric fields across the boundary. If $\epsilon_2 > \epsilon_1$, then $E_{1n} > E_{2n}$.

These rules are very, very important as they will allow us to solve for transitions between different boundary (same as we were using KVL and KCL for transmission lines to find reflection and transmission coefficients).

To be continued……