Chapter 2: Transmission Lines: Circuit Model and Transient Response

2.1 What is a Transmission Line?

As you have seen in your circuits courses and labs, we need conductors (wires, cables, microstrip lines) to carry the electrical signal from the source to the load. So far you have been neglecting the effect of these wires and model the circuit as if the wires were not even present. However, in ECE 106, you saw that when you have two conductors carrying charges you create electric fields and magnetic fields around the conductors, the effect of which was summarized by the capacitance and inductance which is created between the conductors. These capacitances and inductances will play a role in how a circuit behaves and you probably have observed the effects while doing labs but did not understand where they came from. For example, if you send a pulse through a circuit, you probably observed the behavior shown in the figure below, called the ringing. Ringing is unwanted oscillations in voltage or current response.

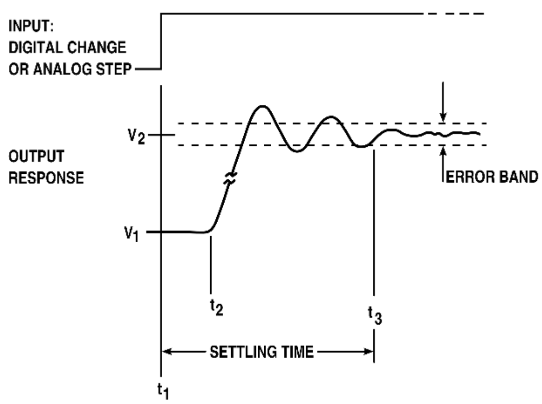


Figure 2.1. Ringing due to “parasitic” capacitance and inductance in a circuit.

So far in circuits you may have attributed it to “parasitic” capacitance and inductance: capacitance and inductance which is not part of the design but due to the byproducts of the circuit design. The issue is that wires used to carry the signal are an important part of the circuit and become more important as the frequency at which the circuit is operating at increases. Thus, we need to learn how to understand the effect of wires in a circuit.

Transmission lines is a pair of conductors used to carry the voltage and the current from the source to the conductors. Transmission lines may consist of pair of parallel wires (this is how we will draw it), coaxial cable (consisting of an inner conductor surrounded by a ground conductors, the two conductors being separated by a dielectric, you have been using these in your labs) or any two conductors separated by an insulating material or vacuum. On a printed circuit board, transmission lines may consist of microstrip lines or coplanar lines. Some examples of transmission lines are shown in the Figure 2.2 below.

The most accurate description of transmission lines is through Maxwell’s equations. However, as you have seen in ECE 106, calculating fields in space becomes quite complex and mathematically involved. In ECE 106, we had also summarized the effect of electric and magnetic fields in two discrete components: capacitances and inductances. Capacitance and inductance was a way for us

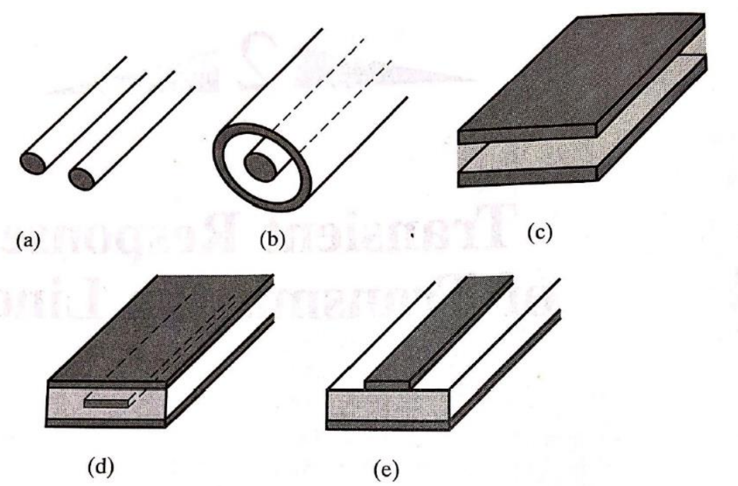


Figure 2.2 Examples of some transmission lines including (a) parallel lines; (b) coaxial cable; (c) parallel plates; (d) strip line; (e) microstrip line.

to take three-dimensional fields and represent it by two circuit parameters. We can use that to treat transmission lines as a circuit element and build a circuit model.

2.2 Circuit Model of Transmission Lines

Figure 2.3 shows a circuit model for two conductor transmission line. Due to the generation of electric and magnetic fields, inductance and capacitance are created in the line. We represent these inductances (L) and capacitance (C) as per unit length (remember ECE 106). As you seen before, the values depend on the geometrical parameters like the shape of the conductors, how they are placed with respect to each other, the properties of the dielectrics between the conductors etc. The conductors will also have some resistance which we can again represent it as resistance per unit length, R. Further, the dielectric between the conductors may leak some charges (no dielectric will be a perfect insulator). We represent this will a leakage conductance, G. **L, C, R and G are all defined per unit length.**

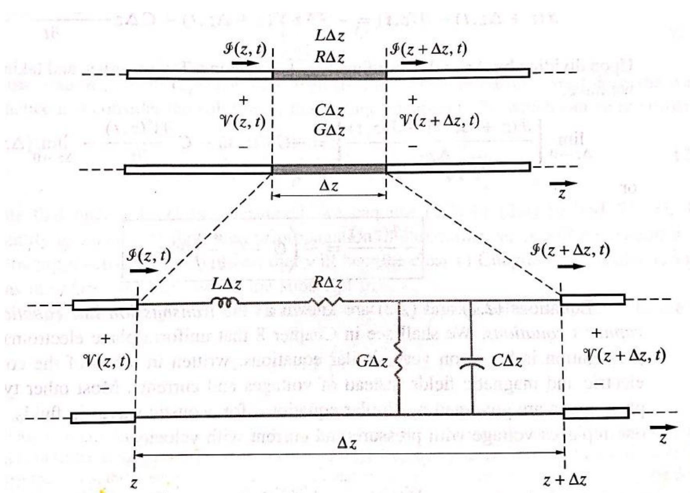


Figure 2.3 Circuit model of a transmission line

Let us consider a very small section, Δz of the transmission line some distance "z" away from a selected point of reference, say the source. We want to treat this section as a circuit model and find the relationship between the voltage and current at z i.e. V(z,t) and I(z,t) with the voltage and current at z+Δz i.e. V(z+ Δz, t) and I(z Δz, t). Using Kirchhoff's voltage law (KVL), we can write the difference in V(z+ Δz, t) and V(z, t) as the voltage drop across the resistance RΔz and inductance LΔz. The equation will be given as:

$$V(z+\Delta z,t)-V(z,t) = -R\Delta z I(z,t)-L\Delta z\frac{\partial I(z,t)}{\partial t}$$

If we considered the section Δz to be as short as possible so that the lumped circuit model of the segment can be represented by the actual distributed line, then we have, Δz→0. This can be written as:

$$\lim_{\Delta z\to 0}\frac{V(z+\Delta z,t)-V(z,t)}{\Delta z} = -RI(z,t)-L\frac{\partial I(z,t)}{\partial t}$$

which simply becomes:

$$\frac{\partial V(z,t)}{\partial z}=-RI(z,t)-L\frac{\partial I(z,t)}{\partial t}=-\left(R+L\frac{\partial}{\partial t}\right)I(z,t)$$

Similarly using Kirchhoff's current law, we can write the difference between the current at the output and input and the current which will leak through the parallel elements GΔz and CΔz. We will thus, have:

$$I(z+\Delta z,t)-I(z,t)=-G\Delta zV(z+\Delta z,t)-C\Delta z\frac{\partial V(z+\Delta z,t)}{\partial t}$$

When we consider $\Delta z\to 0$, we can open V(z+Δz,t) in Taylor series as V(z,t) and ignore the higher order terms. We thus, get:

$$\lim_{\Delta z\to 0}\frac{I(z+\Delta z,t)-I(z,t)}{\Delta z}=-GV(z,t)-C\frac{\partial V(z,t)}{\partial t}$$

which become:

$$\frac{\partial I(z,t)}{\partial z}=-\left(G+C\frac{\partial}{\partial t}\right)V(z,t)$$

We have thus, achieved two first order partial differential equations which are coupled with each other. **These equations are called the Transmission Line Equations or Telegrapher's Equations.** The first equation tells us that the spatial variation of voltage depends on the temporal variation of current. The second equation tells us that the spatial variation of current depends on the temporal variation of voltage. Further, these equations are coupled to each other. Should a wave exist in this system?

If we look at the circuit model, we see that we have LC oscillators in Δz which are connected to LC oscillators in the next subsection. We have oscillators connected together in space. R and G are the damping constants in the oscillators, they will absorb energy as the wave travels forward and will result in the loss of the wave amplitude. So we see when we have two coupled first order differential equations in space and time, we should get a wave.

Transmission line equations or Telegrapher's Equations:

$$\frac{\partial V(z,t)}{\partial z} = -\left(R + L\frac{\partial}{\partial t}\right) I(z,t)$$

$$\frac{\partial I(z,t)}{\partial z} = -\left(G + C\frac{\partial}{\partial t}\right) V(z,t)$$

2.3 Travelling Wave Solution for Lossless Lines

R and G are the dampers in the oscillator and lead to the loss of energy as the wave propagates. For the time being, let us assume that the transmission lines are ideal and loss less. In many applications, R and G are very small and the loss terms can be neglected. When R and G are both 0, the transmission line will be "lossless". The Transmission line equations become:

$$\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t}$$

We can solve these equations by taking another derivative with z on one of the equations and substituting the second one. Let us do this with the first equation and see what happens:

$$\frac{\partial}{\partial z}\left(\frac{\partial V(z,t)}{\partial z}\right) = -L\frac{\partial}{\partial z}\left(\frac{\partial I(z,t)}{\partial t}\right)$$

On RHS, the derivatives with space and time can be moved around in order as they are derivatives with respect to different variables. Thus, we get:

$$\frac{\partial^2 V(z,t)}{\partial z^2} = -L\frac{\partial}{\partial t}\left(\frac{\partial I(z,t)}{\partial t}\right)$$

Substituting the second equation, we get:

$$\frac{\partial^2 V(z,t)}{\partial z^2} = -L\frac{\partial}{\partial t}\left(-C\frac{\partial V(z,t)}{\partial t}\right)$$

resulting in:

$$\frac{\partial^2 V(z,t)}{\partial z^2} = LC\frac{\partial^2 V(z,t)}{\partial t^2}$$

Or:

$$\frac{\partial^2 V(z,t)}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 V(z,t)}{\partial z^2}$$

This is a second order differential equation which related the variation of V with time to variation of V with space by a constant given by 1/LC. If we had started with the second equation, we would have achieved a similar second order differential equation in current as:

$$\frac{\partial^2 I(z,t)}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 I(z,t)}{\partial z^2}$$

The two equations are known as the **wave equations** for voltage and current, respectively. Do these equations look like the form:

$$\frac{\partial^2 f(z,t)}{\partial t^2} = K\frac{\partial^2 f(z,t)}{\partial z^2}$$

Viola, we should have waves supported in the transmission lines. Further, the wave should have phase velocity given by:

$$v_p = \sqrt{K} = \frac{1}{\sqrt{LC}}$$

In general, the solution for the voltage wave equation should be of the form:

$$V(z,t) = f^+\left(t - \frac{z}{v_p}\right) + f^-\left(t + \frac{z}{v_p}\right)$$

where $f^+\left(t - \frac{z}{v_p}\right)$ is a wave carrying voltage disturbance forward and $f^-\left(t + \frac{z}{v_p}\right)$ is a wave carrying voltage disturbance backwards. In general, f^+ and f^- could be completely different functions.

If we assume that there is a wave carrying voltage disturbance *V(z,t)*, can current take a completely independent form. No! the voltage and the current wave will be related to each other. To find what the solution for *I(z,t)*, we can substitute the solution for V(z,t) in the Transmission line equation. Let us again put the argument

$$\zeta_1 = t - \frac{z}{v_p} \text{ and } \zeta_2 = t + \frac{z}{v_p}$$

and

$$\frac{\partial \zeta_1}{\partial t} = \frac{\partial \zeta_2}{\partial t} = 1$$

$$\frac{\partial \zeta_1}{\partial z} = -\frac{1}{v_p} \text{ and } \frac{\partial \zeta_2}{\partial z} = \frac{1}{v_p}$$

Thus, we get:

$$V(z,t) = f^+(\zeta_1) + f^-(\zeta_2)$$

$$\frac{\partial V}{\partial z} = \frac{\partial\left(f^{+}(\zeta_1)\right)}{\partial z} + \frac{\partial\left(f^{-}(\zeta_2)\right)}{\partial z} = \frac{\partial\left(f^{+}(\zeta_1)\right)}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial z} + \frac{\partial\left(f^{-}(\zeta_2)\right)}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial z} = -\frac{1}{v_p}\frac{\partial\left(f^{+}(\zeta_1)\right)}{\partial \zeta_1} + \frac{1}{v_p}\frac{\partial\left(f^{-}(\zeta_2)\right)}{\partial \zeta_2}$$

Let us for the time being assume, the current wave will be given as:

$$I(z,t) = I^{+}\left(t - \frac{z}{v_p}\right) + I^{-}\left(t + \frac{z}{v_p}\right) = I^{+}(\zeta_1) + I^{-}(\zeta_2)$$

Thus,

$$\frac{\partial I}{\partial t} = \frac{\partial(I^{+}(\zeta_1))}{\partial t} + \frac{\partial(I^{-}(\zeta_1))}{\partial t} = \frac{\partial(I^{+}(\zeta_1))}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial t} + \frac{\partial(I^{-}(\zeta_2))}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial t} = \frac{\partial(I^{+}(\zeta_1))}{\partial \zeta_1} + \frac{\partial(I^{-}(\zeta_2))}{\partial \zeta_2}$$

Substituting the terms in Transmission line equation, we get:

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$

$$-\frac{1}{v_p}\frac{\partial\left(f^{+}(\zeta_1)\right)}{\partial \zeta_1} + \frac{1}{v_p}\frac{\partial\left(f^{-}(\zeta_2)\right)}{\partial \zeta_2} = -L\left(\frac{\partial(I^{+}(\zeta_1))}{\partial \zeta_1} + \frac{\partial(I^{-}(\zeta_2))}{\partial \zeta_2}\right)$$

Comparing the common terms, we get:

$$\frac{\partial(I^{+}(\zeta_1))}{\partial \zeta_1} = \frac{1}{Lv_p}\frac{\partial\left(f^{+}(\zeta_1)\right)}{\partial \zeta_1}$$

Thus,

$$I^{+}(\zeta_1) = \frac{1}{L.{}^{1}\!/\!\sqrt{LC}}f^{+}(\zeta_1) = \frac{1}{\sqrt{L/C}}f^{+}(\zeta_1)$$

$f^{+}(\zeta_1)$ represents how the voltage wave travels forwards. The equation is saying the magnitude of the forward current wave is going to be related by voltage wave through the term $\sqrt{L/C}$. Considering the voltage and current relationships in an impedance (Z), i.e. I = V/Z; to us if we measure the voltage wave and current wave, the transmission line is exhibiting an impedance given by $\sqrt{L/C}$.

Similarly, for the backward propagating wave we get:

$$I^{-}(\zeta_2) = -\frac{1}{L.{}^{1}\!/\!\sqrt{LC}}f^{-}(\zeta_1) = -\frac{1}{\sqrt{L/C}}f^{-}(\zeta_2)$$

We get the same impedance. However, we get a negative. What does that physically mean? If we call the current going forward as +ve, what will be current which is going backwards?

The negative in the reflected current wave represents change in direction of current. For a voltage wave given by:

$$V(z,t) = f^{+}\left(t - \frac{z}{v_p}\right) + f^{-}\left(t + \frac{z}{v_p}\right)$$

The general solution for I(z,t) will depend on V(z,t) and will be given by:

$$I(z,t) = \frac{1}{Z_o}\left[f^{+}\left(t - \frac{z}{v_p}\right) - f^{-}\left(t + \frac{z}{v_p}\right)\right]$$

where $Z_o = \sqrt{\frac{L}{C}}$ is called the ***characteristic impedance*** of the transmission line. It is a very important parameter as we will see later. Look at coaxial cables which you may have in your homes. You may see specifications printed on the cable with a characteristic impedance of 75 Ω. **It is not impedance you will measure with the ohmmeter but relates to how the voltage wave and current wave amplitudes are related to each other inside the cable. As we have seen in ECE 106; L, C depends on the geometric parameters of the conductors and the material properties of the dielectric surrounding the conductors, Z_0 depends on the transverse physical dimensions. Also remember L and C are per unit length. Z_0 does not depend on the length.** No matter what the length of the transmission line is, the amplitude of the voltage and current wave are going to be related by Z_0.

A common misconception which students make is that voltage and current waves are two different waves. Conceptually, the voltage and the current are two arms of the same wave which is travelling in the transmission line. They are travelling together and their magnitudes at any point are related by Z_o.

2.4 Transient Response of Transmission Lines

Transient wave occurs due to sudden disturbance at the source or elsewhere on the transmission line e.g. you turn on a switch. A wave will be created at the disturbance and that wave will start to propagate through the transmission lines. Many important electrical and computer engineering applications involve transients. Digital signals consist of a sequence of pulses which can be treated as superposition of successive step-like changes. Think of cars waiting at a red traffic light. When the light turns green, the cars will not start moving all at the same time. The first car will move first, then the car behind it and so forth. Similar things will happen when we turn on the switch to the source on the transmission line. Let us consider and see what will happen.

2.4.1 Step response of an infinite long loss line:

Consider a very, very long transmission line connected to a source with a step response as shown in the figure 2.4 below. (You have two very, very long wires connected to a battery and a switch, you turn the switch on what happens). If I ask you based on your circuits knowledge, will there be any current flowing in the circuit? You will answer no, because the source is open circuit.

However, when we consider the transmission line behavior, then at time t = 0, to the source it looks like as if it is connected to an impedance of Z_0 as shown in the figure 2.4 (b). Thus, a voltage is

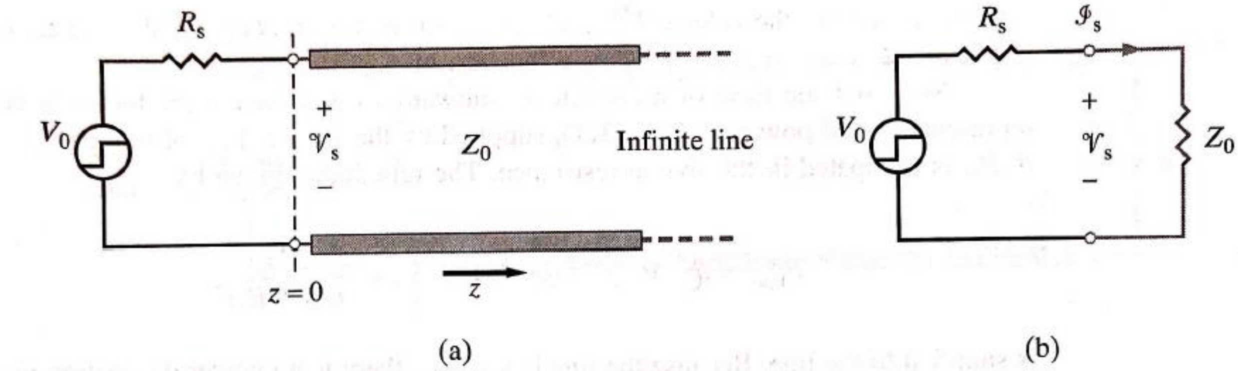


Figure 2.4 (a) An infinitely long transmission line connected to a step voltage source. (b) What the circuit looks like to the source at time t=0.

created on the transmission line terminal which will not be V_0 but $\frac{V_0 Z_0}{Z_0+R_S}$. For time $t=0^-$, the voltage on the transmission line is 0, now it sees a voltage of $\frac{V_0 Z_0}{Z_0+R_S}$. Thus, a disturbance is created which starts to move forward with velocity, v_p. The wave which is created will be given as:

$$V^+(z=0,t) = V_s^{\,t=0^+} - V_s^{\,t=0^-} = \frac{V_0 Z_0}{Z_0 + R_s}$$

Corresponding to the voltage wave, we will also get a current wave which will be given by the voltage wave divided by characteristics impedance as:

$$I^+(z=0,t) = \frac{V^+(z=0,t)}{Z_0} = \frac{V_0}{Z_0 + R_s}$$

There is current being generated at the source! The wave propagation can be visualized in a graph shown in the figure below.

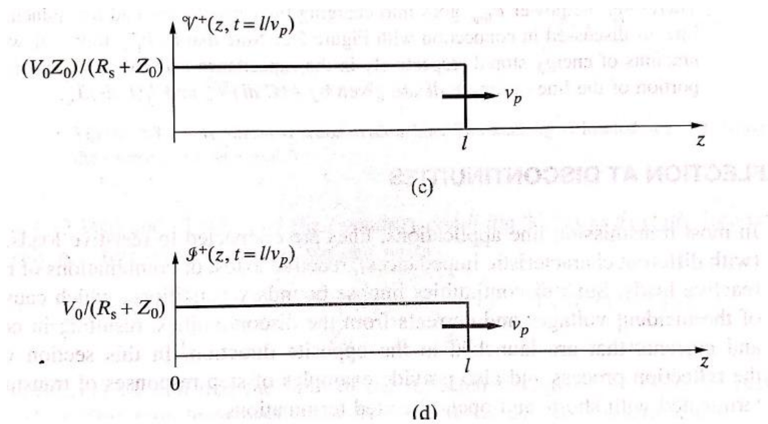


Figure 2.5 How the current and voltage moves in z with time.

2.4.2 Reflection at Discontinuities

What happens if the Transmission line is terminated with a load? At time t=0, a wave consisting of components V^+ and I^+ (with amplitudes same as section 2.4.1) is created which starts to travel forward with phase velocity, v_p. At time, $t_d = l/ v_p$, where l is the length of the transmission line, the wave reaches the load. The voltage $V^+ = I^+ Z_0$. However, the load has its own strict rules i.e. $V_L = I_L Z_L$. The two rules are not compatible with each other, so a backward wave will be created which will start to travel in the reverse direction. Thus, a wave will be created with voltage V^- and current I^- which again should be related by Z_0. Using KVL and KCL at the terminals of the load, we will get:

$$V_L(t) = V^+(t) + V^-(t)$$

$$I_L(t) = I^+(t) - I^-(t) = \frac{V^+(t)}{Z_0} - \frac{V^-(t)}{Z_0}$$

However, $V_L(t) = I_L(t) Z_L$. Thus, we get:

$$V^+(t) + V^-(t) = \frac{Z_L}{Z_0}\left(V^+(t) - V^-(t)\right)$$

Rearranging we can write as:

$$\frac{V^-(t)}{V^+(t)} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

The ratio of the backward wave with the forward wave is called the load voltage reflection coefficient, Γ_L. Thus, the load reflection coefficient is given as:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

The reflection coefficient is one of the most important parameters in transmission line analysis. The simple expression should be memorized. What is the range for Γ_L? It can vary from $-1 \leq \Gamma_L \leq +1$. When the load is a short circuit, you get Γ_L = -1. When the load is open circuit, you get Γ_L = +1. What does negative Γ_L mean? Voltage is the potential difference across the two conductors. With negative reflection, it simply means that the polarity of the voltage changes compared to what was on the incident wave. If the potential of the top conductor was higher than the potential of the bottom conductor in the incident wave, then for reflected wave with negative reflection coefficient, the bottom conductor will become higher potential as compared to the top one.

When $Z_L = Z_0$, there is no reflection created. This is a "matched load". In general Γ_L can be a complex number. We will see how to deal with complex loads in Chapter 3. For the rest of the discussion in this chapter, we will assume the load is purely resistive i.e. $Z_L = R_L$.

Now the wave which is reflected back, reaches the source at time t = $2t_d$. It sees an impedance of the source, in general, Z_s. So a reflection is created at the source with a voltage reflection coefficient given by:

$$\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0}$$

This bouncing back and forth continues indefinitely. Very soon we will have many waves within the transmission line and keeping track of them becomes difficult. So we create a graphical method to keep track which is called as the Bounce Diagram.

2.4.3 The Bounce Diagram

A bounce diagram, also called a reflection diagram or lattice diagram, is a distance-time plot used to illustrate successive reflections in the transmission line. The distance along the line is plotted on the horizontal axis and the time is plotted along vertical direction. Waves are represented by straight lines with arrows showing the direction the wave is propagating in. A bounce diagram for a finite length line is shown in the figure below. We follow two rules in determining the voltage and current in the lines. The rules are:

1. Once a wave reaches a point, the amplitude carried by the wave remains at the point for eternity. When a new wave reaches the point, the total voltage amplitude changes by what was there before plus the amplitude of the new wave reaching the point.

2. Reflections are instantaneous. Place where the reflections happen, the voltage changes by the incident wave and the reflected wave.

We can use the Bounce diagram to calculate V(z,t) at any point along the transmission line and at the source and the load.

Thus, if we want to see how the voltage changes on the load with time, we will have to consider the following:

At t=0, a voltage wave V_1^+ is created and seen before V_1^+ will have the magnitude of $\frac{Z_0}{Z_0+Z_S}V_0$

For 0 <t<t_d, no wave has reached the load. Hence, V_L = 0.

At $t = t_d$, V_1^+, reaches the load and immediately a reflection equal to $\Gamma_L V_1^+$ is created. Thus, V_L becomes $V_1^+ + \Gamma_L V_1^+ = (1+\Gamma_L)V_1^+$

There is no new wave till time $t = 3t_d$, when the reflection from the source reaches load. The reflected wave has an amplitude of $\Gamma_L\Gamma_S V_1^+$ and as soon as it hits the load, a new reflection is created which will have amplitude of $\Gamma_L(\Gamma_L\Gamma_S V_1^+) = \Gamma_L^2\Gamma_S V_1^+$. Thus, the voltage at the load becomes the value it was before $3t_d$, and the two new waves which are created at the load. It will be: $(1+\Gamma_L)V_1^+ + \Gamma_L\Gamma_S V_1^+ + \Gamma_L^2\Gamma_S V_1^+ = (1+\Gamma_L+\Gamma_L\Gamma_S+\Gamma_L^2\Gamma_S)V_1^+$. This remains the voltage till t=$5t_d$ when a new wave comes and we can do the similar additions again.

You can see that the amplitude of the wave is reducing with every reflection, so subsequent waves may not be as strong as the first few ones and the circuit will eventually approach steady state values.

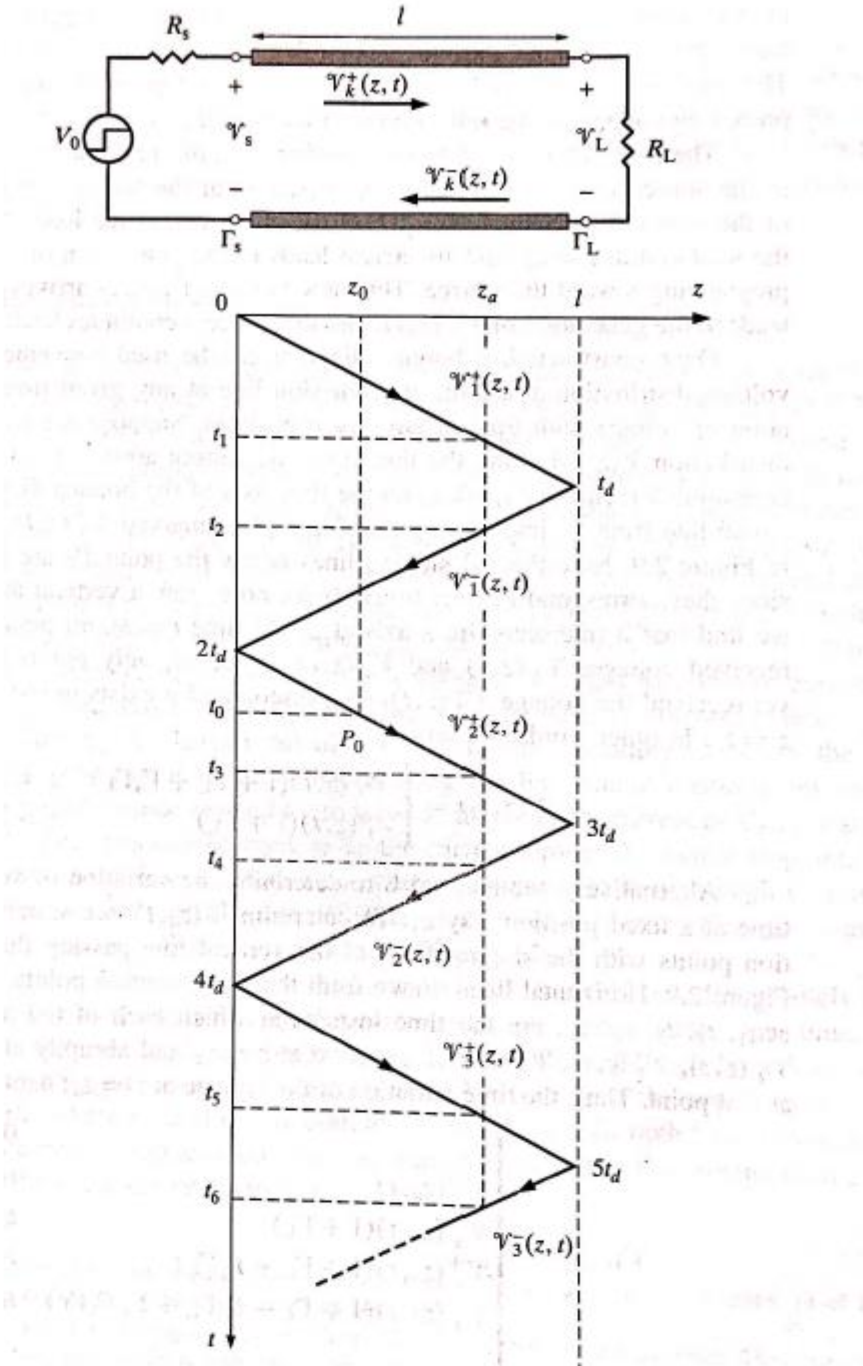


Figure 2.6 Bounce diagram for a transmission line terminated with a load.

2.4.4 What happens if we wait long enough?

What would the voltage on the load be if we wait long time? We just need to keep on adding the terms. Thus, the voltage at time equal to infinity will be:

$$V_L = \left(1 + \Gamma_L + \Gamma_L\Gamma_S + {\Gamma_L}^2\Gamma_S + {\Gamma_L}^2{\Gamma_S}^2 + {\Gamma_L}^3{\Gamma_S}^2 + {\Gamma_L}^3{\Gamma_S}^3 \ldots\right){V_1}^+$$

$$V_L = \left(\left(1 + \Gamma_L\Gamma_S + {\Gamma_L}^2{\Gamma_S}^2 + \cdots\right) + \Gamma_L\left(1 + \Gamma_L\Gamma_S + {\Gamma_L}^2{\Gamma_S}^2 + \cdots\right)\right){V_1}^+$$

$$V_L = \left[\frac{1}{1-\Gamma_L\Gamma_S} + \frac{\Gamma_L}{1-\Gamma_L\Gamma_S}\right]{V_1}^+ = \left[\frac{1+\Gamma_L}{1-\Gamma_L\Gamma_S}\right]{V_1}^+$$

If we substitute the values of $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$, $\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0}$, and ${V_1}^+ = \frac{Z_0}{Z_0 + Z_S} V_0$, we will get:

$$V_L = \frac{Z_L}{Z_L + Z_S} V_0$$

This is as if the load was connected directly to the source and what you expect from circuit theory. If we wait long enough, the voltage and the current will approach that given in steady state. Thus, you see the ringing in pulse signals slowly approaching what you expect from your designed circuit.

2.4.5 Junction between Transmission Lines

What happens when transmission line of characteristics impedance of Z_{01} is cascaded with a transmission line of characteristics impedance of Z_{02}. When a wave reaches the junction, to the wave it looks like you have a load of Z_{02}. Thus, a reflection will be created which will be given as:

$$\Gamma_{21} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}}$$

What will the transmission from line 1 to 2 be? Many students believe it will be 1-Γ_{21}. You are forgetting simple KVL. The voltage on the left hand side of the junction should be equal to the voltage on the right hand side of the junction. On the left hand side you have the original wave and the reflected wave. Thus, on the right hand side, you will get the sum of the waves (not subtraction). Thus, the transmission, *t* will be

$$t = 1 + \Gamma_{21} = \frac{2Z_{02}}{Z_{02} + Z_{01}}$$

Please note, I reiterate than many students assume the transmission coefficient to be $t = 1 - \Gamma_{21}$. That is very wrong. Voltage must be the same across a terminal and the reflection coefficient we have defined is in terms of voltage, not power. So, how does this comply with energy conservation? Remember voltage is not energy but the work you need to do to move a unit positive charge across two points. Power is given by voltage multiplied by the current. The incident power P_1 is given by ${V_1}^+{I_1}^+ = \frac{({V_1}^+)^2}{Z_0}$. Reflected power will be given by ${V_1}^-{I_1}^- = (\Gamma_{21})^2\frac{({V_1}^+)^2}{Z_0}$. While the transmitted voltage is $(1+\Gamma_{21})\ V_1{}^+$, the transmitted current wave is $(1-\Gamma_{21})\ I_1{}^+$. Remember, we had found that the reflected current wave was negative current. Thus, the transmitted power will be given by:

$$(1+\Gamma_{21})\ V_1{}^+(1-\Gamma_{21})\ I_1{}^+ = \left(1-\Gamma_{21}{}^2\right)V_1{}^+I_1{}^+ = \left(1-\Gamma_{21}{}^2\right)\frac{\left(V_1{}^+\right)^2}{Z_0} = P_1 - P_r$$

where P_r is the reflected power. Thus, transmitted power is indeed incident power minus the reflected power. Everything is kosher here.

2.4.6 Why don't we use current reflections?

Sometimes it is easier to think what is happening to the current (e.g. switch opens, we know the current has to go to 0). However, we normally still do bounce diagrams with voltage waves. If we know the current wave, the voltage wave is straightforward to calculate. The reason we don't use current waves is because at every reflection the direction of the current changes, so we will have to deal with another negative. We will have to keep on switching the negatives and will have to be extra careful which is not needed for the voltage.

Examples:

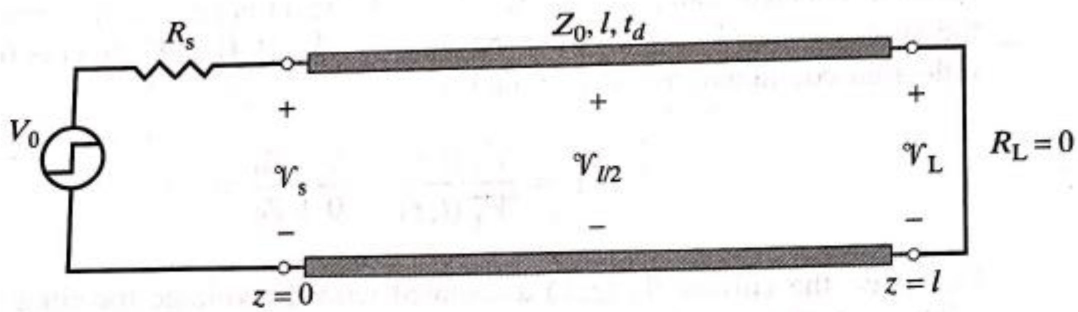
Ex. 1: Short circuited line

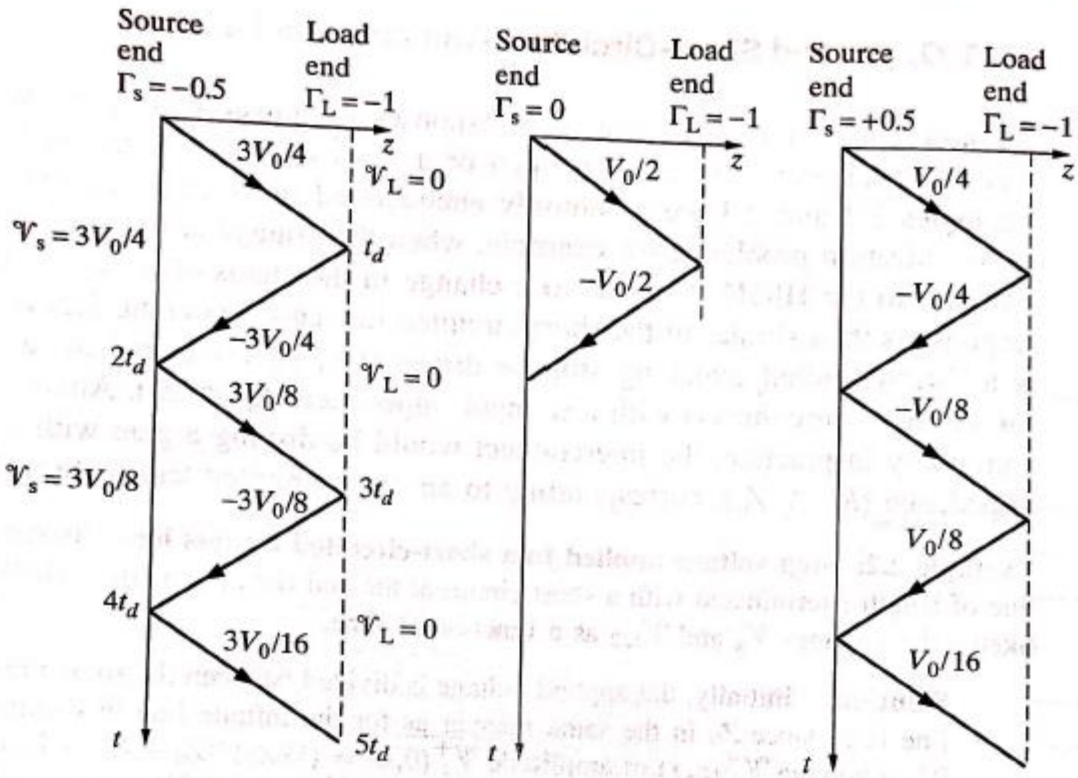
A transmission line with characteristics impedance of Z_0 of length *l* is terminated with a short circuit on the end as shown in the figure. Calculate the voltage V_s as a function of time. Let us calculate the values for $Z_s = {}^{Z_0}\!/_3\,; Z_s = Z_0; and\ Z_s = 3Z_0$.

Solution: First we need to calculate the load and source reflection coefficients.

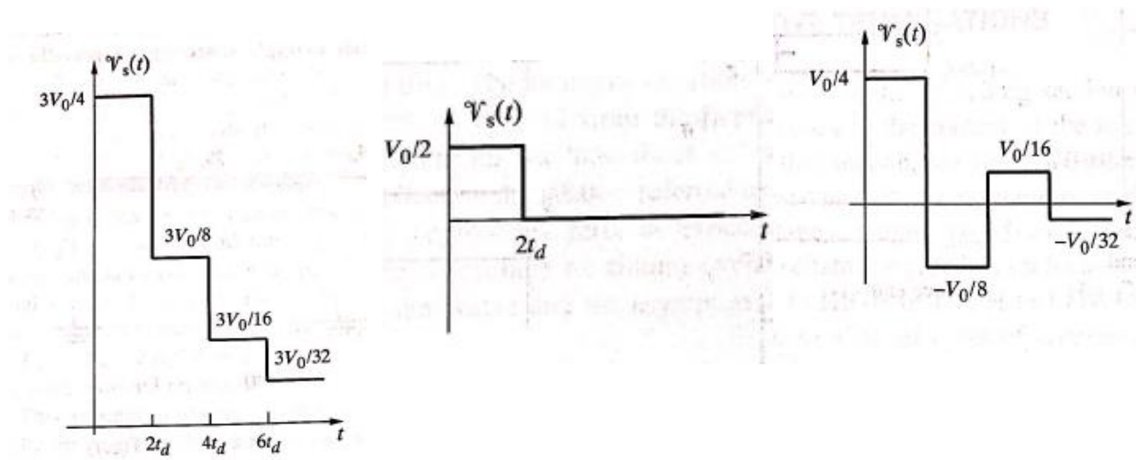
$\Gamma_L = \frac{0-Z_0}{0+Z_0} = -1$; Γ_S = -0.5, 0, and 0.5 for the three source impedances respectively. Using these we can draw the bounce diagrams as shown below. Once, we have the bounce diagram, we just have to add the waves at different times. The source voltages we will see are shown in the figure below for the three circuits. What do we observe, when $Z_s = Z_0$, the voltage stabilizes the fastest We learn an important lesson. We should match our transmission lines to our source.

Short circuited lines are observed in computer communications (you should see this in ECE 358). In an interconnect when the switch switches to High state, the resultant response is similar to a short circuited line when the interconnect drives a subsequent logic gate with low input impedance. If the interconnect drives a subsequent logic gate with high input impedance, then the response is that of an open circuit line.





(a) $R_s = Z_0/3$ (b) $R_s = Z_0$ (c) $R_s = 3Z_0$



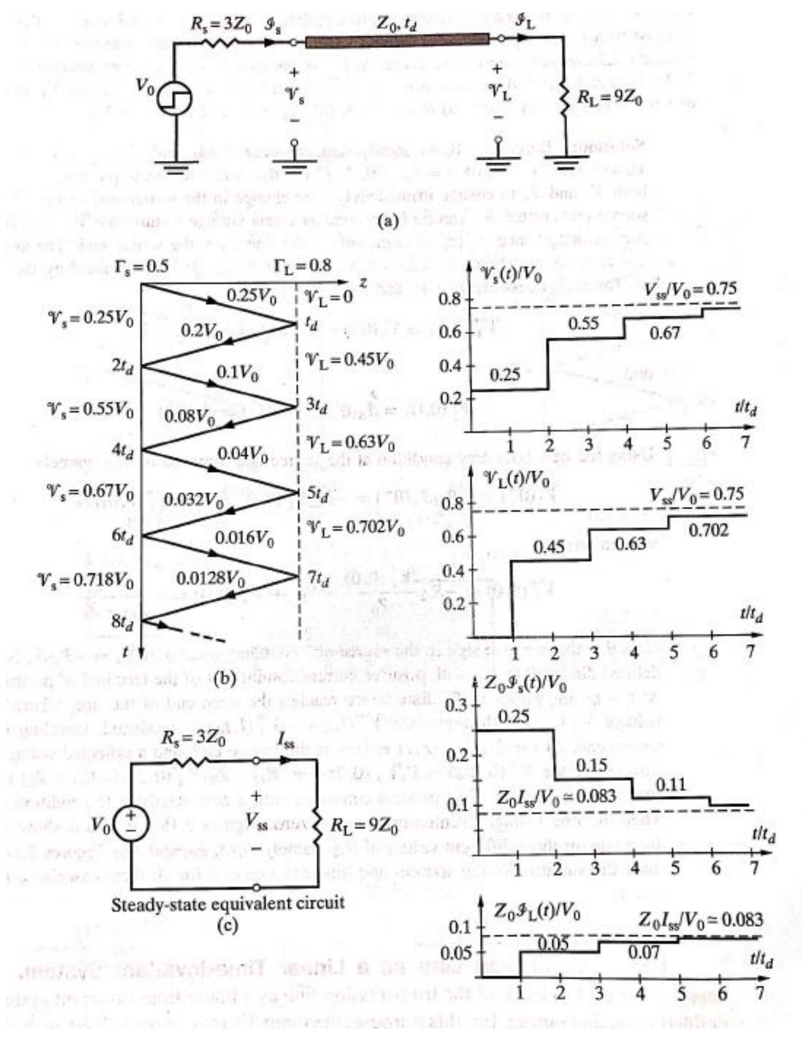
Ex 2: Step response of a resistively terminated lossless line.

Consider the circuit shown below. Sketch V_L, I_L, V_S, and I_S as a function of time.

Solution:

In the first step we calculate Γ_L and Γ_S. As a habit, I put the values on the Bounce diagrams themselves. Then we calculate, the amplitude of ${V_1}^+$ as $\frac{Z_0 V_0}{Z_0+Z_S}$ which in this case will be 0.25 V_0.

We can now draw the Bounce diagram and simply need to add the waves to plot the voltages. To calculate the current we just have to remember that the forward waves are positive currents and the backward waves are negative current. Bounce diagram and the resultant plots are shown in the figure below.



Ex 3: Cascaded transmission lines:

Consider the transmission line system shown in the figure below. A step voltage is applied at the source. Draw the Bounce diagram and calculate the voltage at the source and load with time.

Solution:

We calculate Γ_L, and Γ_s. The amplitude of ${V_{1A}}^+$ is calculated as $\frac{Z_{0A}V_0}{Z_{0A}+Z_S}$. In addition we need to calculate the reflection coefficient from transmission line A to B and also B to A. They will just be negative of each other. The various coefficients are also shown in the figure. Once we have the coefficients, we can start drawing the bounce diagram. We have to take care that waves will be transmitted from A to B and from B to A. Once we have the diagram, we have to just add the waves as they arrive to get voltage curves. Bounce diagrams and the voltage curves are shown in the figure below.