Chapter 2 - Outline

2.1 Context-Free Grammars

- Formal definition of a context-free grammar
- Examples of context-free grammars
- Designing context-free grammars
- Ambiguity
- Chomsky normal form

2.2 Pushdown Automata

- Formal definition of a pushdown automaton
- Examples of pushdown automata
- Equivalence with context-free grammars

2.3 Non-Context-Free Languages

- The pumping lemma for context-free languages

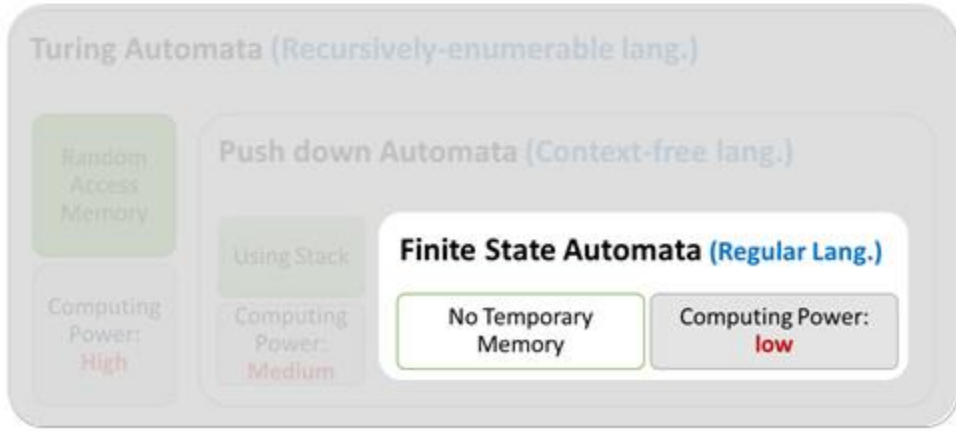
Chapter 2

Introduction

In chapter 1 :

- We learned that
  - we can describe many **languages** (regular languages) using two equivalent methods:
    - finite automata
    - regular expressions

- We found out
  - There are some **languages** that cannot be described using the methods above
    - Non-regular languages
    - Example: $\{0^n1^n \mid n \geq 0\}$



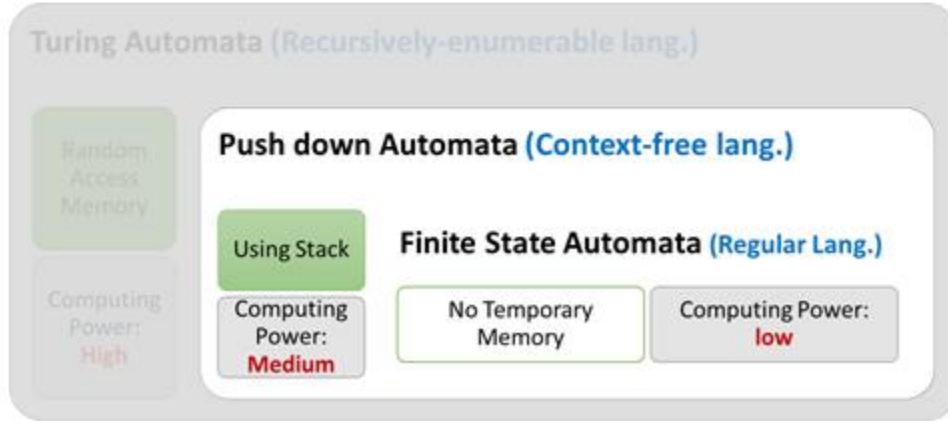
Chapter 2

Introduction

In chapter 2 :

- We present a **more powerful method** of describing languages:
  - called context-free grammars (CFG)

- the collection of **languages** associated with CFG are called the context-free languages (CFL).
  - They can describe languages with a recursive structure,

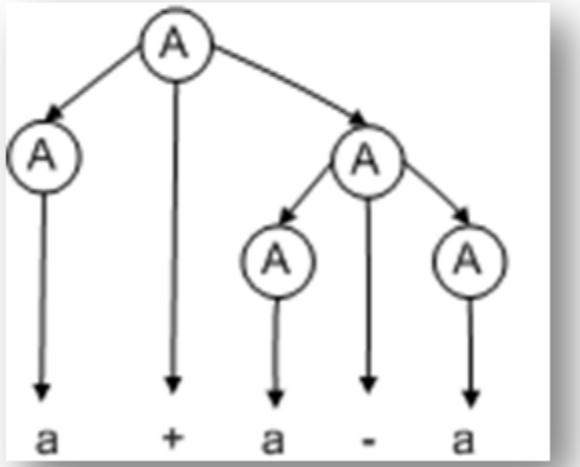


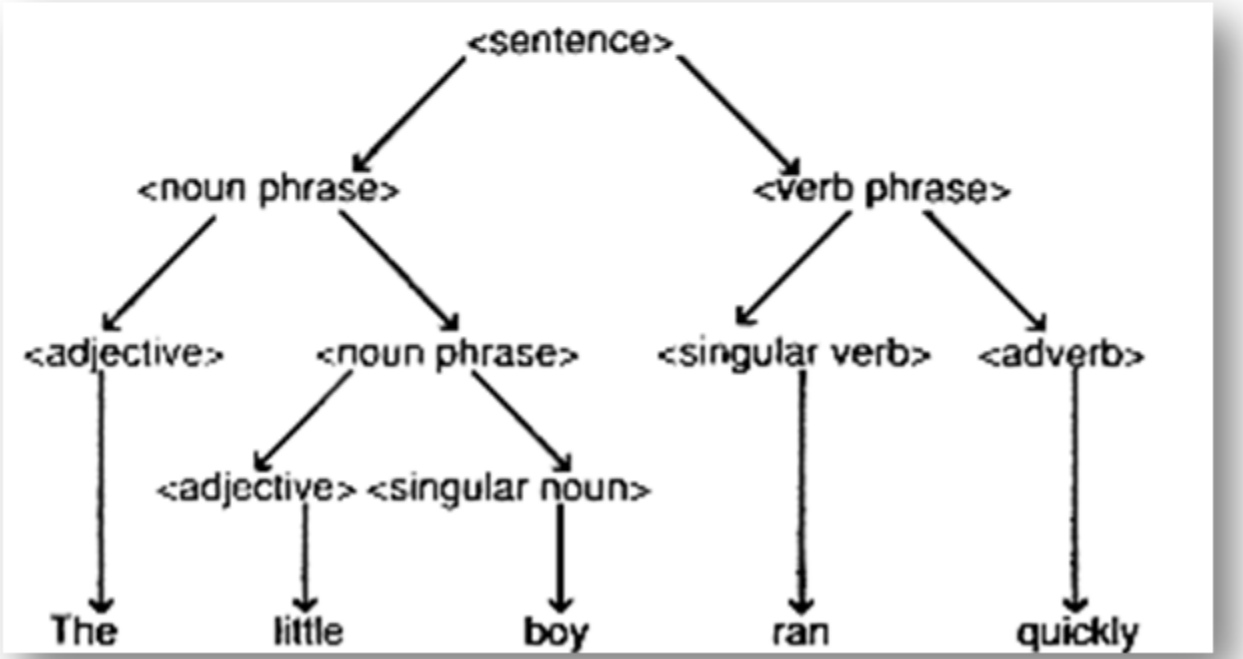
Introduction

context-free grammars:

- first **used** in the study of **human languages**
  - To organize and understand the **relationship of terms** such as noun, verb, and preposition and their respective phrases.

- another **application**: **compilers** and **interpreters** for programming languages using **parser**

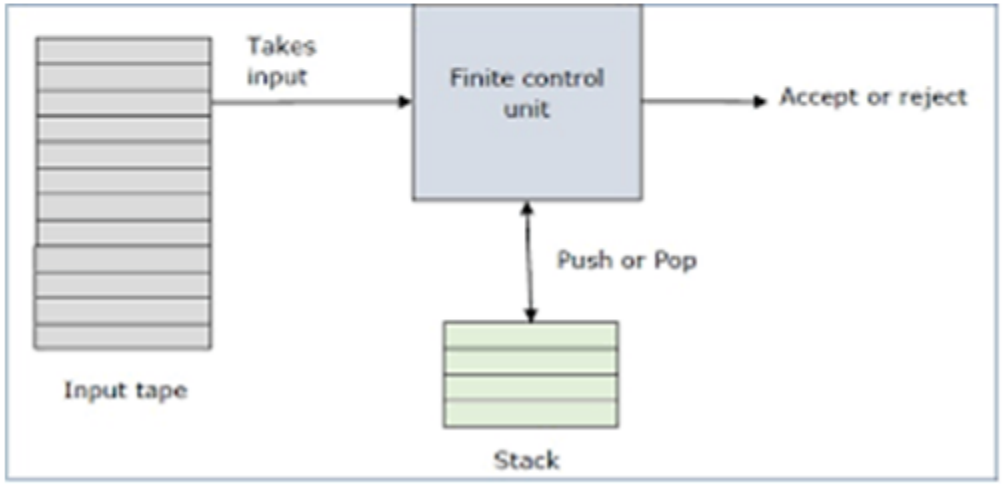




Introduction

pushdown automata:

- We also introduce pushdown automata, a class of machines recognizing the context-free languages



Chapter 2

2.1 Context-Free Grammars

CONTEXT-FREE GRAMMARS

Example : G_1 is a CFG.

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

CONTEXT-FREE GRAMMARS

Example : G_1 is a CFG.

substitution rules (productions)

$$\left.\begin{array}{l} A \to 0A1 \\ A \to B \\ B \to \# \end{array}\right\} \quad A \to 0A1 \mid B$$

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  - each rule appears as a line in the grammar

CONTEXT-FREE GRAMMARS

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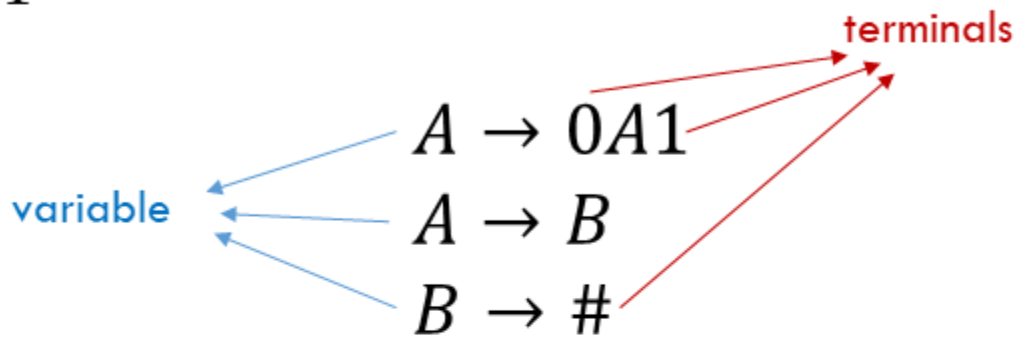

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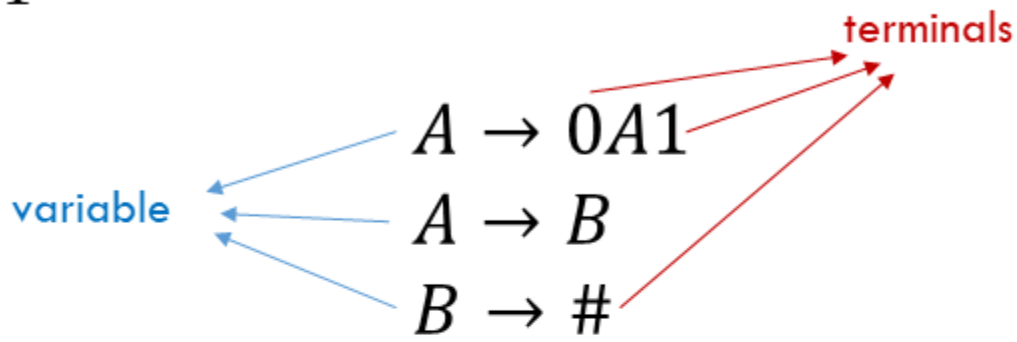

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- **variable symbols:** often are represented by capital letters.
- **terminals** : are analogous to the input alphabet and often are represented by lowercase letters, numbers, or special symbols.

CONTEXT-FREE GRAMMARS

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start variable ← $A \rightarrow 0A1$

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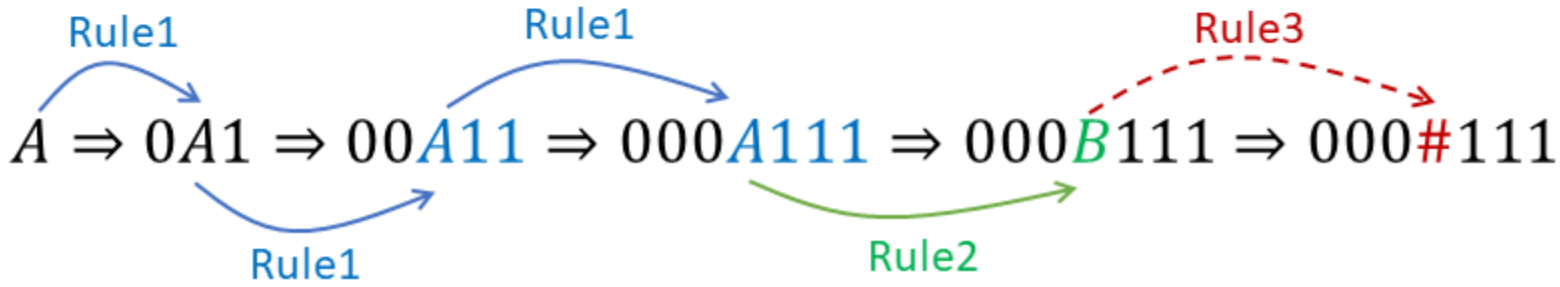
- **variable symbols:** often are represented by capital letters.
- **terminals** : are analogous to the input alphabet and often are represented by lowercase letters, numbers, or special symbols.
- One variable is designated as the start variable.
  - It usually occurs on the left-hand side of the topmost rule.

DERIVATION

A **grammar** describes a language by generating each string of that language in the following manner.

1. Write down the **start variable**.
2. Find a variable that is written down and a rule that starts with that variable. Replace the written down variable with the right-hand side of that rule.
3. Repeat step 2 until no variables remain.



G1

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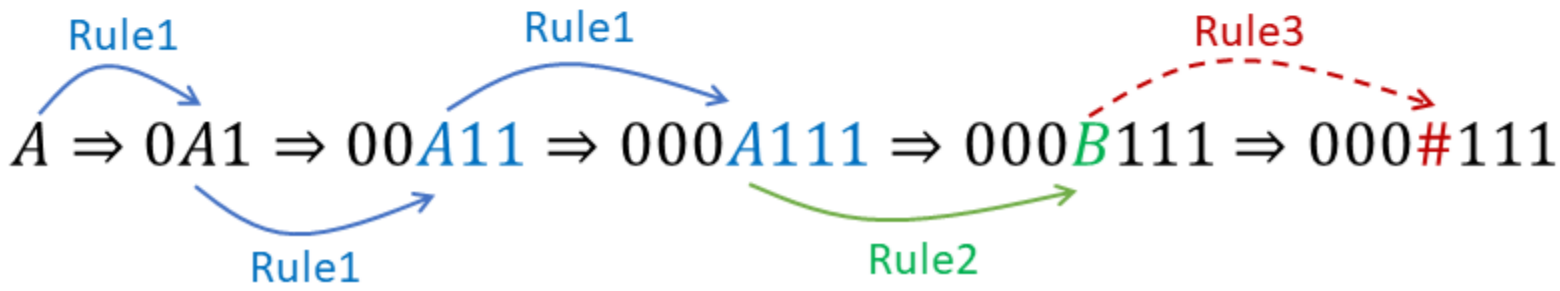
- The **sequence of substitutions** to obtain a string is called a derivation.
- A derivation of string 000#111 in grammar G1 is



G1

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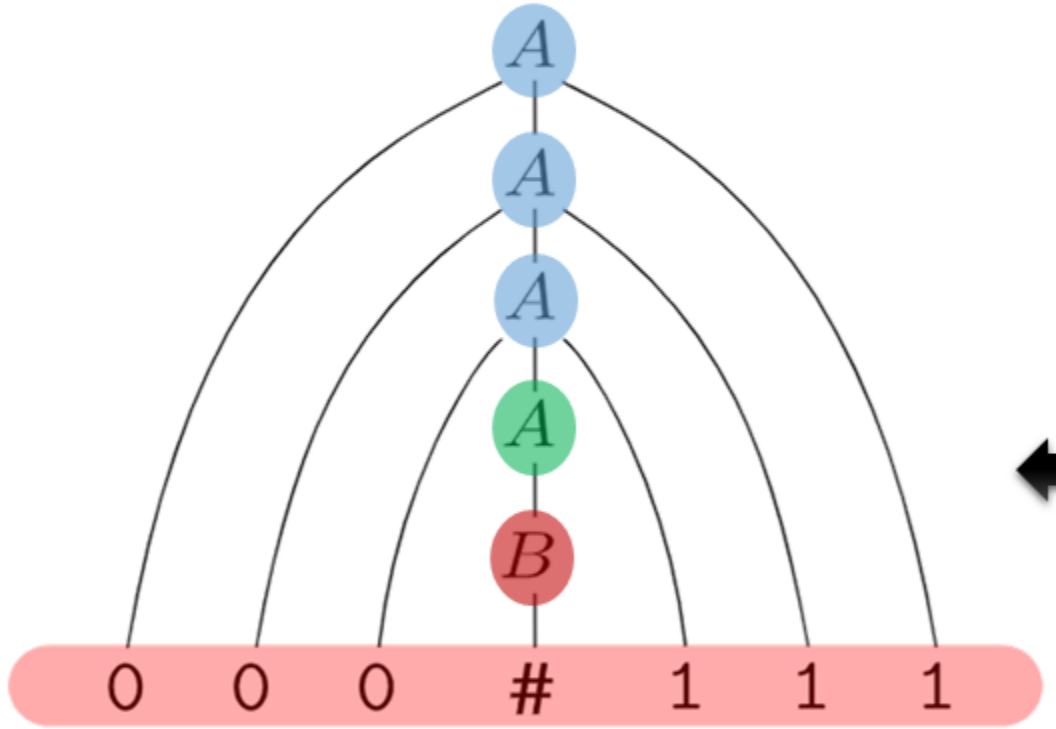
PARSE TREE

You may also represent the same information pictorially with a parse tree.



G1

$A \rightarrow 0A1$

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Rule1 Rule1 Rule3

$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$

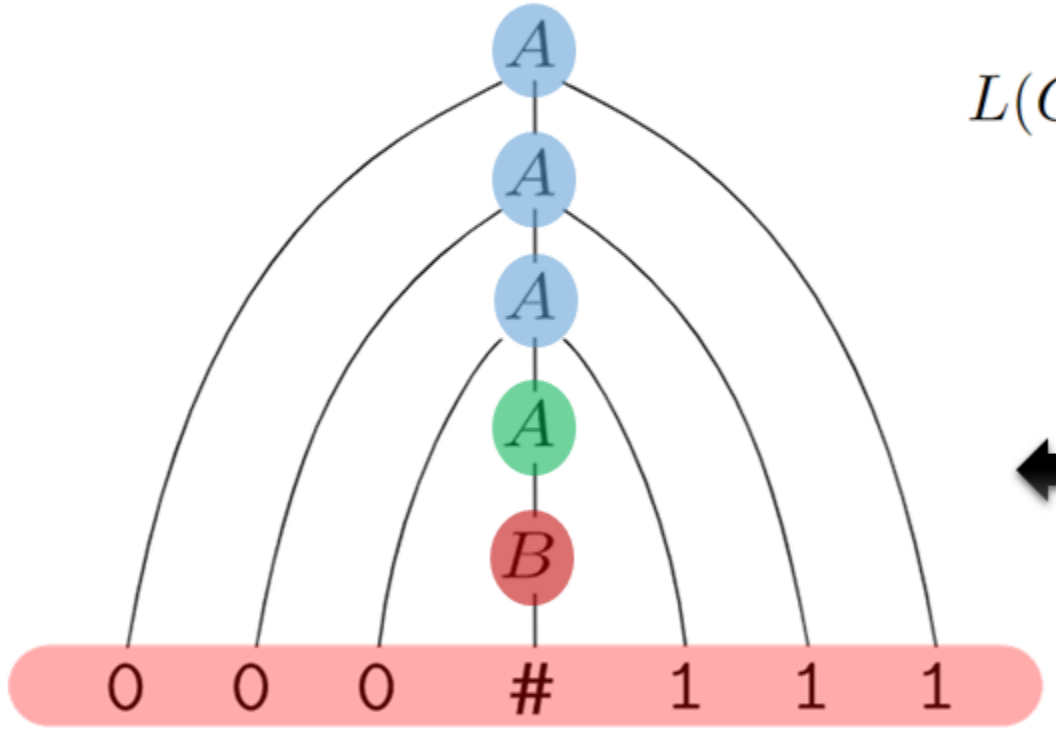
Rule1 Rule2

PARSE TREE

You may also represent the same information pictorially with a parse tree.

Example Sentences

- A
- 0A1
- 00A11
- 000A111
- 0000A1111
- 00000A11111
- 000000A111111
- 0000000A1111111
- 00000000A11111111
- 000000000A111111111



$L(G_1)$ is $\{0^n\#1^n \mid n \geq 0\}$

G1

$A \rightarrow 0A1$

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Rule1 Rule1 Rule3

$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$

Rule1 Rule2

DERIVATION

Example 2 : G_2 which describes a fragment of the English language.

$$
\begin{aligned}
\langle \text{SENTENCE} \rangle &\rightarrow \langle \text{NOUN-PHRASE} \rangle \langle \text{VERB-PHRASE} \rangle \\
\langle \text{NOUN-PHRASE} \rangle &\rightarrow \langle \text{CMPLX-NOUN} \rangle \mid \langle \text{CMPLX-NOUN} \rangle \langle \text{PREP-PHRASE} \rangle \\
\langle \text{VERB-PHRASE} \rangle &\rightarrow \langle \text{CMPLX-VERB} \rangle \mid \langle \text{CMPLX-VERB} \rangle \langle \text{PREP-PHRASE} \rangle \\
\langle \text{PREP-PHRASE} \rangle &\rightarrow \langle \text{PREP} \rangle \langle \text{CMPLX-NOUN} \rangle \\
\langle \text{CMPLX-NOUN} \rangle &\rightarrow \langle \text{ARTICLE} \rangle \langle \text{NOUN} \rangle \\
\langle \text{CMPLX-VERB} \rangle &\rightarrow \langle \text{VERB} \rangle \mid \langle \text{VERB} \rangle \langle \text{NOUN-PHRASE} \rangle \\
\langle \text{ARTICLE} \rangle &\rightarrow \texttt{a} \mid \texttt{the} \\
\langle \text{NOUN} \rangle &\rightarrow \texttt{boy} \mid \texttt{girl} \mid \texttt{flower} \\
\langle \text{VERB} \rangle &\rightarrow \texttt{touches} \mid \texttt{likes} \mid \texttt{sees} \\
\langle \text{PREP} \rangle &\rightarrow \texttt{with}
\end{aligned}
$$

DERIVATION

Example 2 : G_2 which describes a fragment of the English language.

10 variables

$$
\begin{aligned}
\langle \text{SENTENCE} \rangle &\rightarrow \langle \text{NOUN-PHRASE} \rangle \langle \text{VERB-PHRASE} \rangle \\
\langle \text{NOUN-PHRASE} \rangle &\rightarrow \langle \text{CMPLX-NOUN} \rangle \mid \langle \text{CMPLX-NOUN} \rangle \langle \text{PREP-PHRASE} \rangle \\
\langle \text{VERB-PHRASE} \rangle &\rightarrow \langle \text{CMPLX-VERB} \rangle \mid \langle \text{CMPLX-VERB} \rangle \langle \text{PREP-PHRASE} \rangle \\
\langle \text{PREP-PHRASE} \rangle &\rightarrow \langle \text{PREP} \rangle \langle \text{CMPLX-NOUN} \rangle \\
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\end{aligned}
$$

- 27 terminals
- the standard **English alphabet** plus a **space character**.

18 rules

DERIVATION

Example 2 : G_2 which describes a fragment of the English language.

- Strings in L(G2) include:
  - a boy sees
  - the boy sees a flower
  - a girl with a flower likes the boy

⟨SENTENCE⟩ → ⟨NOUN-PHRASE⟩⟨VERB-PHRASE⟩
⟨NOUN-PHRASE⟩ → ⟨CMPLX-NOUN⟩ | ⟨CMPLX-NOUN⟩⟨PREP-PHRASE⟩
⟨VERB-PHRASE⟩ → ⟨CMPLX-VERB⟩ | ⟨CMPLX-VERB⟩⟨PREP-PHRASE⟩
⟨PREP-PHRASE⟩ → ⟨PREP⟩⟨CMPLX-NOUN⟩
⟨CMPLX-NOUN⟩ → ⟨ARTICLE⟩⟨NOUN⟩
⟨CMPLX-VERB⟩ → ⟨VERB⟩ | ⟨VERB⟩⟨NOUN-PHRASE⟩
⟨ARTICLE⟩ → a | the
⟨NOUN⟩ → boy | girl | flower
⟨VERB⟩ → touches | likes | sees
⟨PREP⟩ → with

$CFG\ G_2$

DERIVATION

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(a derivation)

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&\Rightarrow \texttt{a}\ \langle \text{NOUN} \rangle \langle \text{VERB-PHRASE} \rangle \\
&\Rightarrow \texttt{a boy}\ \langle \text{VERB-PHRASE} \rangle \\
&\Rightarrow \texttt{a boy}\ \langle \text{CMPLX-VERB} \rangle \\
&\Rightarrow \texttt{a boy}\ \langle \text{VERB} \rangle \\
&\Rightarrow \texttt{a boy sees}
\end{aligned}$$

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$CFG\ G_2$

FORMAL DEFINITION OF A CONTEXT-FREE GRAMMAR

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

G1

$A \rightarrow 0A1$

$A \rightarrow B$

$B \rightarrow \#$

In grammar G1

- $V = \{A, B\}$
- $\Sigma = \{0, 1, \#\}$
- $S = A$

DERIVATION

- If $u, v,$ and w are strings of **variables and terminals**, and $A \rightarrow w$ is a rule of the grammar, we say that uAv yields uwv ,written $uAv \Rightarrow uwv$.

DERIVATION

- If $u, v,$ and w are strings of **variables and terminals**, and $A \to w$ is a rule of the grammar, we say that uAv yields uwv ,written $uAv \Rightarrow uwv$.

- Say that u derives v, written $u \overset{*}{\Rightarrow} v$,
  - if $u = v$ or
  - if a sequence $u_1, u_2, \ldots, u_k$ exists for k≥0 and

$$u \Rightarrow u1 \Rightarrow u2 \Rightarrow \cdots \Rightarrow uk \Rightarrow v.$$

- The **language of the grammar** is $\{w \in \Sigma^* | S \overset{*}{\Rightarrow} w\}$.

EXAMPLES OF CONTEXT-FREE GRAMMARS

- Consider grammar $G_3 = (\{S\}, \{a, b\}, R, S)$.
- The set of rules, R, is $S \rightarrow aSb|SS|\varepsilon$.

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    - $S \Rightarrow SS \Rightarrow aSbS \Rightarrow aSbaSb \Rightarrow a\varepsilon baSb \Rightarrow aba\varepsilon b \Rightarrow abab$
    - $S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaa\varepsilon bbb \Rightarrow aaabbb$
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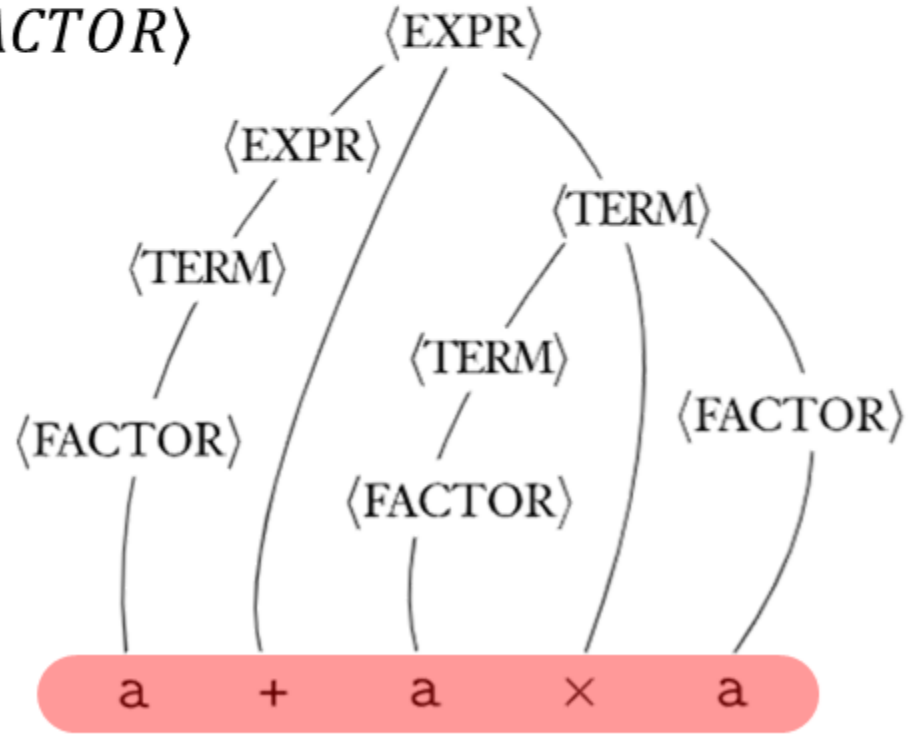
  - if you think of a as a left parenthes is "("and b as a right parenthes is ")"
    - Then L(G_3) is the language of all strings of properly nested parentheses.

EXAMPLES OF CONTEXT-FREE GRAMMARS

- Consider grammar $G_4 = (V, \Sigma, R, \langle EXPR \rangle)$
- $V = \{\langle EXPR \rangle, \langle TERM \rangle, \langle FACTOR \rangle\}$
- $\Sigma = \{a, +, \times, (,)\}$.
- The **rules** are
  - $\langle \boldsymbol{EXPR} \rangle \rightarrow \langle EXPR \rangle + \langle TERM \rangle | \langle TERM \rangle$
  - $\langle \boldsymbol{TERM} \rangle \rightarrow \langle TERM \rangle \times \langle FACTOR \rangle | \langle FACTOR \rangle$
  - $\langle \boldsymbol{FACTOR} \rangle \rightarrow (\langle EXPR \rangle) | a$

- **Grammar G_4** describes :
  - a fragment of a programming language concerned with arithmetic expressions.



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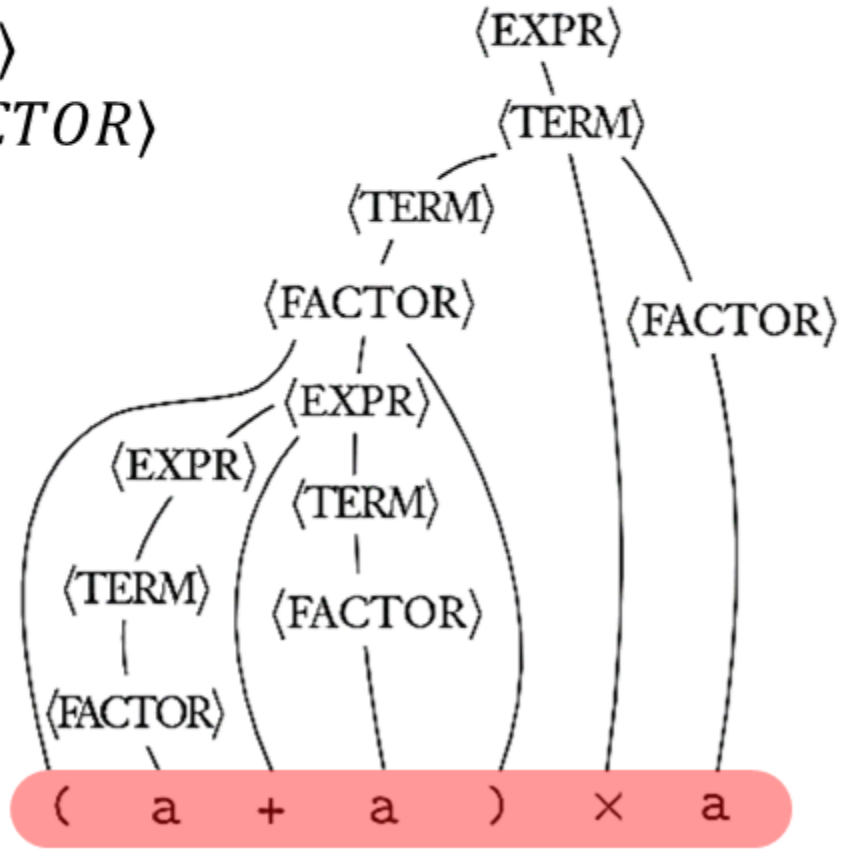

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Helpful techniques

Method 1: Based on this Fact: Many CFLs are the union of simpler CFLs.

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Helpful techniques

Method 1: Based on this Fact: Many CFLs are the union of simpler CFLs.

- break a CFL into simpler pieces
- and then construct individual grammars for each piece
- merge the individual grammars into a grammar for the original language
  - By combining their rules and then adding the new rule

$$S \to S_1 | S_2 | \cdots | S_k,$$

Where the variables S_i are the start variables for the individual grammars.

DESIGNING CONTEXT-FREE GRAMMARS

- Example: Construct a grammar for the language

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DESIGNING CONTEXT-FREE GRAMMARS

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- $L_2 = \{1^n 0^n | n \geq 0\}$ Rules: $S_2 \rightarrow 1S_2 0 | \varepsilon$

- $L_1 \cup L_2$ Rules: $S \rightarrow S1 | S2$
  $S_1 \rightarrow 0S_1 1 | \varepsilon$
  $S_2 \rightarrow 1S_2 0 | \varepsilon$

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Method 2: constructing a CFG for a regular language :

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Method 2: constructing a CFG for a regular language :

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- convert the DFA into an equivalent CFG as follows.

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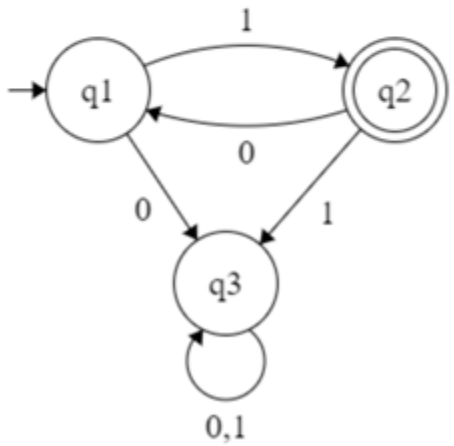

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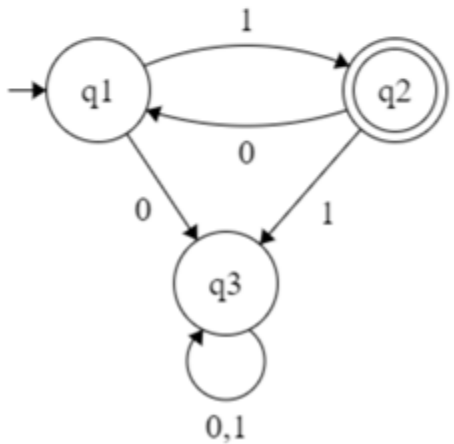

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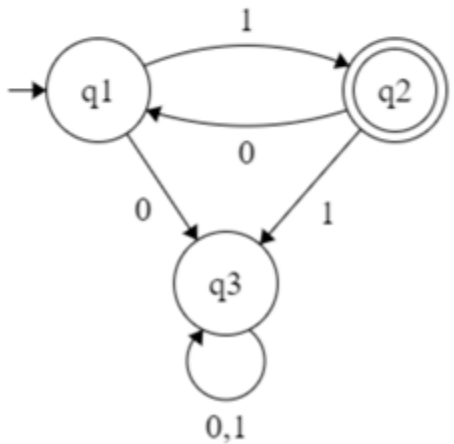

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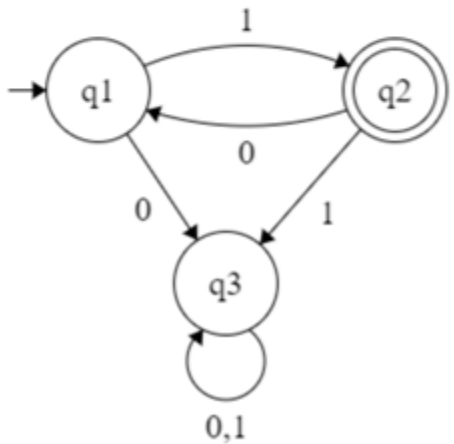

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  - Add the rule $R_i \rightarrow \varepsilon$ if q_i is an accept state of the DFA.
  - Make R_0 the start variable of the grammar, where q_0 is the start state of the machine.
  - the resulting CFG generates the same language that the DFA recognizes. (Verify it on your own)

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- strings with two substrings that are "linked"
  - need to remember an unbounded amount of information about one of the substrings
    - to verify that it corresponds properly to the other substring.
  - **Example**: $\{0^n 1^n | n \geq 0\}$

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- **construct a CFG:** by using a rule of the form $R \rightarrow uRv$

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Helpful techniques

Method 4: constructing a CFG for a language with a recursive structure:

- **Example**:

- Consider grammar $G_4 = (V, \Sigma, R, \langle EXPR \rangle)$
- $V = \{\langle EXPR \rangle, \langle TERM \rangle, \langle FACTOR \rangle\}$
- $\Sigma = \{a, +, \times, (,)\}$.
- The **rules** are
  - $\langle \boldsymbol{EXPR} \rangle \rightarrow \langle EXPR \rangle + \langle TERM \rangle | \langle TERM \rangle$
  - $\langle \boldsymbol{TERM} \rangle \rightarrow \langle TERM \rangle \times \langle FACTOR \rangle | \langle FACTOR \rangle$
  - $\langle \boldsymbol{FACTOR} \rangle \rightarrow (\langle EXPR \rangle) | a$

AMBIGUITY

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- Ambiguity is because of
  - several different parse trees (not two different derivations.)
  - thus several different meanings
  - **undesirable** for certain applications, such as programming languages
    - They need a unique interpretation.

AMBIGUITY

Example: consider grammarG5:

$$\langle \text{EXPR} \rangle \rightarrow \langle \text{EXPR} \rangle + \langle \text{EXPR} \rangle | \langle \text{EXPR} \rangle \times \langle \text{EXPR} \rangle | (\langle \text{EXPR} \rangle) | \text{a}$$

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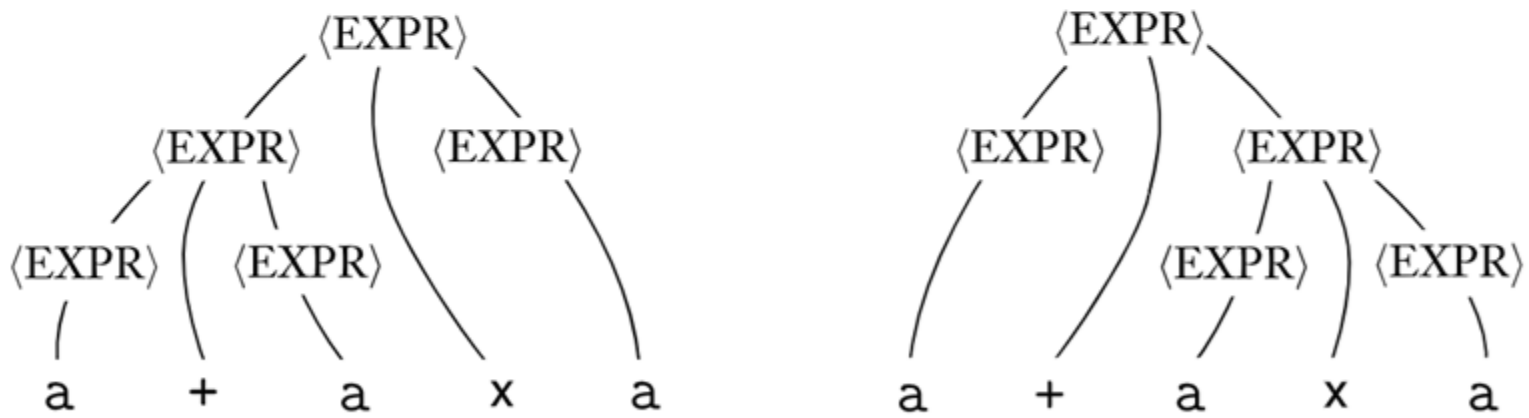
This grammar generates the string a + a × a ambiguously.



LEFTMOST AND RIGHTMOST DERIVATION

- A derivation of a string w in a grammar G is a leftmost derivation
  - if **at every step** the leftmost remaining variable is the one replaced.

- A derivation of a string w in a grammar G is a rightmost derivation
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LEFTMOST AND RIGHTMOST DERIVATION

Example: Let any set of production rules in a CFG be

$$X \rightarrow X + X \mid X \times X \mid |X| \; a$$

over an alphabet $\{a\}$. We want to generate $a + a \times a$.

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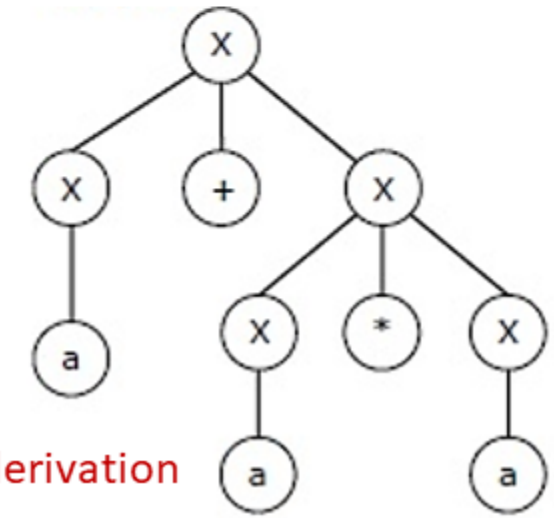
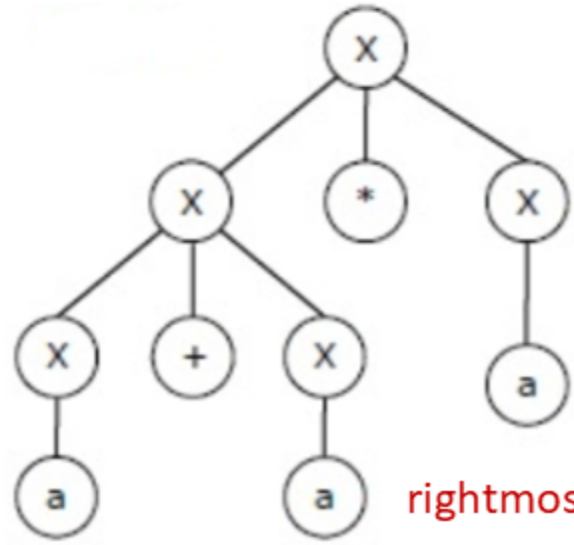


leftmost derivation



rightmost derivation

AMBIGUITY AND LEFTMOST AND RIGHTMOST DERIVATION

- A string w is derived ambiguously in context-free grammar G if it has two or more different leftmost derivations.

- Grammar G is ambiguous if it generates some string ambiguously.

AMBIGUITY AND LEFTMOST AND RIGHTMOST DERIVATION

- A string *w* is derived ambiguously in context-free grammar G if it has two or more different leftmost derivations.
- Grammar G is ambiguous if it generates some string ambiguously.
- Some context-free languages, can be generated
  - **using an ambiguous grammar and an unambiguous grammar**.
- Some context-free languages, however, can be generated
  - only by ambiguous grammars.
  - Such languages are called inherently ambiguous.

Parse tree and deviation

- For every parse tree, there is a unique leftmost, and a unique rightmost derivation.

- The pars tree of an unambiguous grammar is unique.
  - every generated string in a CFG has a unique parse tree.

CHOMSKY NORMAL FORM

- One of the simplest and most useful forms of context-free grammars is called Chomsky normal form.

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A context-free grammar is in ***Chomsky normal form*** if every rule is of the form

$$A \to BC$$
$$A \to a$$

where a is any terminal and A, B, and C are any variables—except that B and C may not be the start variable. In addition, we permit the rule $S \to \varepsilon$, where S is the start variable.

CHOMSKY NORMAL FORM

- Theorem: Any context-free language is generated by a context-free grammar in Chomsky normal form.

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PROOF:

1. **Eliminate the start symbol from right-hand sides**
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$$A \rightarrow B \mid S$$
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CHOMSKY NORMAL FORM

$$A \rightarrow BC$$
$$A \rightarrow a$$

CHOMSKY NORMAL FORM

PROOF: (cont.)

2. Removing ε-rules.

- We remove an ε-rule $A \to \varepsilon$, where ***A* is not the start variable**.
- Then for each occurrence of an A on the right-hand side of a rule,
  - we add a new rule with that occurrence deleted.
  - In other words, if $R \to uAv$ is a rule in which u and v are strings of variables and terminals, we add rule $R \to uv$.

$$
\begin{aligned}
S0 &\to S \\
S &\to ASA|aB \\
A &\to B|S \\
B &\to b|\varepsilon
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
S0 &\to S \\
S &\to ASA|aB|a \\
A &\to B|S|\varepsilon \\
B &\to b|\varepsilon
\end{aligned}
$$

CHOMSKY NORMAL FORM

PROOF: (cont.)

2. **Removing ε-rules.** (cont.)
   - We do so **for each occurrence of an A**,
   - **Example**:
     - $A \to \varepsilon,\ R \to uAvAw \quad \Rightarrow\ R \to uvAw \mid uAvw \mid uvw.$
     - $A \to \varepsilon,\ R \to A \quad \Rightarrow\ R \to A \mid \varepsilon$ (unless we had previously removed it)

   - We **repeat these steps** until we eliminate all ε-rules not involving the start variable.

CHOMSKY NORMAL FORM

PROOF: (cont.)

3. we handle all unit rules.

- We remove a unit rule $A \to B$.
- Then, whenever a rule $B \to u$ appears,
  - we add the rule $A \to u$ unless this was a unit rule previously removed.
  - As before, u is a string of variables and terminals.
- We repeat these steps until we eliminate all unit rules.

$$\begin{matrix} A \to B \\ B \to CDEF \end{matrix} \quad \Rightarrow \quad A \to CDEF$$

CHOMSKY NORMAL FORM

PROOF: (cont.)

4. **Finally, we convert all remaining rules into the proper form.**
   - (Eliminate right-hand sides with more than 2 nonterminals)

CHOMSKY NORMAL FORM

PROOF: (cont.)

4. Finally, we convert all remaining rules into the proper form.

- (Eliminate right-hand sides with more than 2 nonterminals)
- We replace each rule $A \to u_1 u_2 \cdots u_k$,
  - where k≥3 and each u_i is a variable or terminal symbol,
- with the rules $A \to u_1 A_1$, $A_1 \to u_2 A_2$, $A_2 \to u_3 A_3$,...,and $A_{k-2} \to u_{k-1} u_k$.
  - The A_i's are new variables.

CHOMSKY NORMAL FORM

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CHOMSKY NORMAL FORM

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$$S0 \to ASA|aB|a|SA|AS$$
$$S \to ASA|aB|a|SA|AS$$
$$A \to S|b|ASA|aB|a|SA|AS$$
$$B \to b$$

$$S0 \to AA_1|UB|a|SA|AS$$
$$S \to AA_1|UB|a|SA|AS$$
$$A \to b|AA_1|UB|a|SA|AS$$
$$A_1 \to SA$$
$$U \to a$$
$$B \to b$$

CHOMSKY NORMAL FORM

Example: Convert the following CFG to CNF

$$S \rightarrow ASA|aB$$
$$A \rightarrow B|S$$
$$B \rightarrow b|\varepsilon$$

- **Step1:** Eliminate the start symbol from right-hand sides and add **a new start symbol**

$$S0 \rightarrow S$$
$$S \rightarrow ASA|aB$$
$$A \rightarrow B|S$$
$$B \rightarrow b|\varepsilon$$

CHOMSKY NORMAL FORM

Example: (Cont.)

- **Step2:** Eliminate all ε-rules

$$S0 \rightarrow S$$
$$S \rightarrow ASA|aB$$
$$A \rightarrow B|S$$
$$B \rightarrow b|\varepsilon$$

Previous rules

$$S0 \rightarrow S$$
$$S \rightarrow ASA|aB|a$$
$$A \rightarrow B|S|\varepsilon$$
$$B \rightarrow b|\varepsilon$$

$$S0 \rightarrow S$$
$$S \rightarrow ASA|aB|a|SA|AS|S$$
$$A \rightarrow B|S|\varepsilon$$
$$B \rightarrow b$$

CHOMSKY NORMAL FORM

Example: (Cont.)

- **Step3a:** Eliminate all unit rules; $S \rightarrow S$ and $S0 \rightarrow S$

$S0 \rightarrow S$
$S \rightarrow ASA|aB|a|SA|AS|S$
$A \rightarrow B|S|\varepsilon$
$B \rightarrow b$

Previous rules

$S0 \rightarrow S$
$S \rightarrow ASA|aB|a|SA|AS|S$
$A \rightarrow B|S$
$B \rightarrow b$

$S0 \rightarrow ASA|aB|a|SA|AS$
$S \rightarrow ASA|aB|a|SA|AS$
$A \rightarrow B|S$
$B \rightarrow b$

CHOMSKY NORMAL FORM

Example: (Cont.)

- **Step3b:** Eliminate all unit rules; $A \rightarrow B$ and $A \rightarrow S$

$$S0 \rightarrow S|ASA|aB|a|SA|AS$$
$$S \rightarrow ASA|aB|a|SA|AS$$
$$A \rightarrow B|S$$
$$B \rightarrow b$$

Previous rules

$$S0 \rightarrow ASA|aB|a|SA|AS$$
$$S \rightarrow ASA|aB|a|SA|AS$$
$$A \rightarrow B|S|b$$
$$B \rightarrow b$$

$$S0 \rightarrow ASA|aB|a|SA|AS$$
$$S \rightarrow ASA|aB|a|SA|AS$$
$$A \rightarrow S|b|ASA|aB|a|SA|AS$$
$$B \rightarrow b$$

CHOMSKY NORMAL FORM

Example: (Cont.)

- **Step4:** Convert the remaining rules into the proper form by adding additional variables and rules.

$$S0 \rightarrow ASA|aB|a|SA|AS$$
$$S \rightarrow ASA|aB|a|SA|AS$$
$$A \rightarrow S|b|ASA|aB|a|SA|AS$$
$$B \rightarrow b$$

Previous rules

$$S0 \rightarrow AA_1|UB|a|SA|AS$$
$$S \rightarrow AA_1|UB|a|SA|AS$$
$$A \rightarrow b|AA_1|UB|a|SA|AS$$
$$A_1 \rightarrow SA$$
$$U \rightarrow a$$
$$B \rightarrow b$$

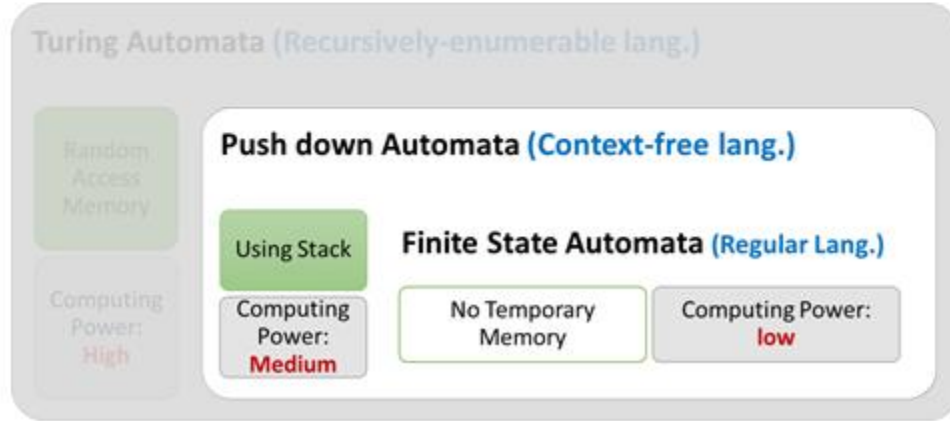
$$A \rightarrow BC$$
$$A \rightarrow a$$

Chapter 2

2.2 Pushdown Automata

PUSHDOWN AUTOMATA (PDA)

- PDAs are like nondeterministic finite automata but have an extra component called a stack.
  - provides additional memory beyond the finite amount available in the control.
  - allows pushdown automata to recognize some nonregular languages.



Chapter 2

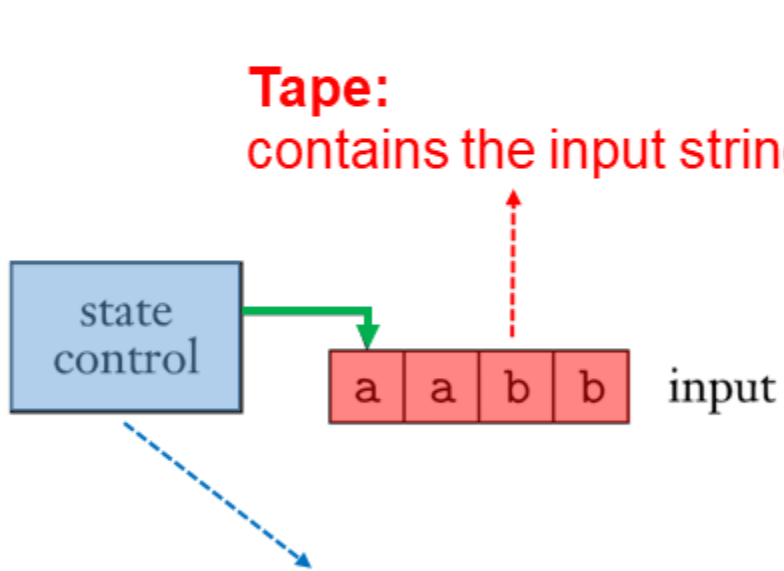
PUSHDOWN AUTOMATA (PDA)

- Pushdown automata are equivalent in power to context-free grammars.

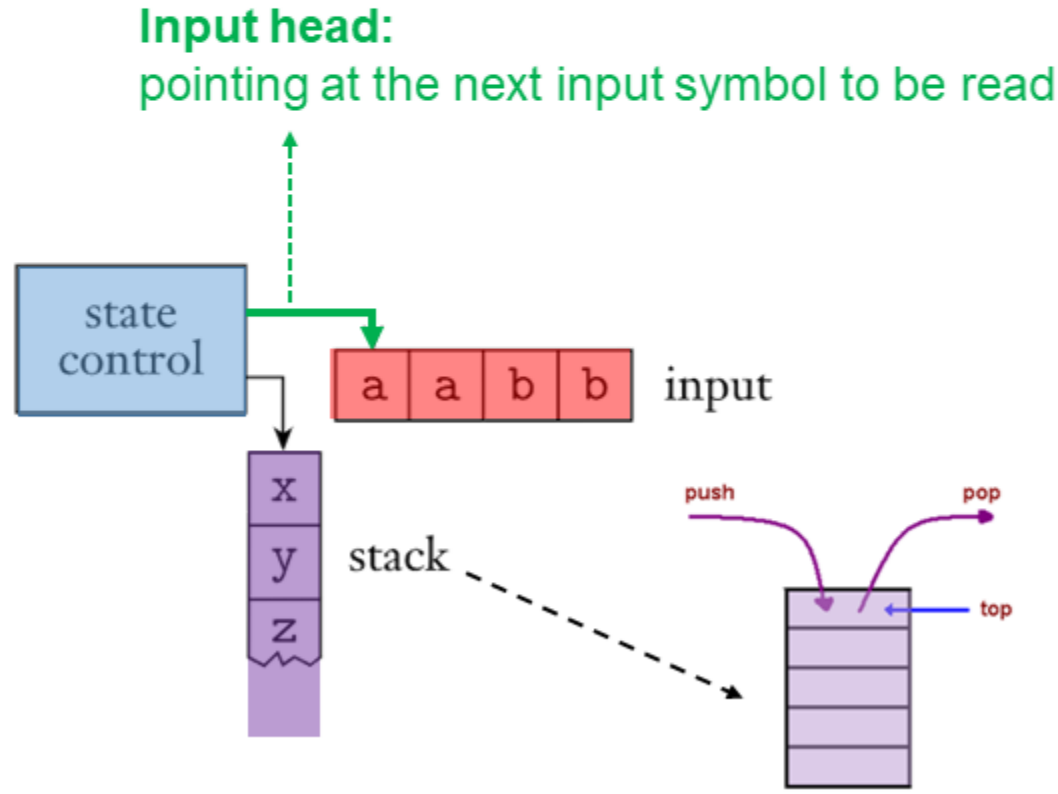
- A language is context free if
  - a **context-free grammar** generating it or
  - a **push-down automaton** recognizing it.

PUSHDOWN AUTOMATA (PDA)

- A PDA can write symbols on the stack and read them back later.
  - Pushing : writing a symbol on the stack
  - Popping : removing a symbol from the stack.
  - stack is a "last in, first out" storage device.



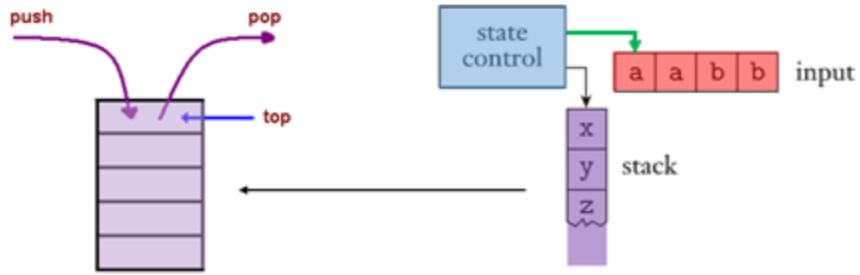
Finite automata



Pushdown automata

PUSHDOWN AUTOMATA (PDA)

- Stack is of infinite size.
- Stack is a "last in, first out" storage device.
- **Example:** How a PDA recognizes the nonregular language $\{0^n 1^n | n \geq 0\}$.



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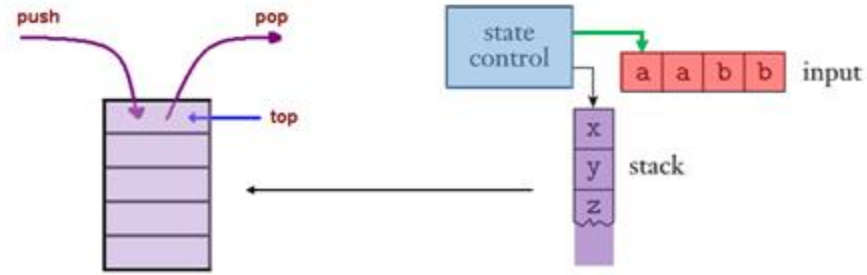

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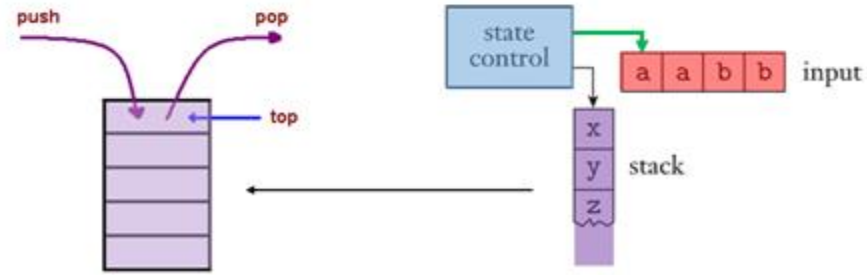
- **Example:** How a PDA recognizes the nonregular language $\{0^n 1^n | n \geq 0\}$.
- Can use its stack to store the number of 0's it has seen.

> - As each 0 is read, push it onto the stack
> - As soon as 1's are read, pop a 0 off the stack
> - If reading the input is finished exactly when the stack is empty, accept the input else reject the input



DETERMINISTIC AND NONDETERMINISTIC (PDA)

- **Deterministic** and **nondeterministic** pushdown automata are not equivalent in power.
  - Nondeterministic pushdown automata recognize certain languages that no deterministic pushdown automata can recognize.

- Note: nondeterministic PDAs are equivalent in power to context-free grammars.

FORMAL DEFINITION OF A PUSHDOWN AUTOMATON

A ***pushdown automaton*** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta: Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

Where $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$ and $\Gamma_\varepsilon = \Gamma \cup \{\varepsilon\}$

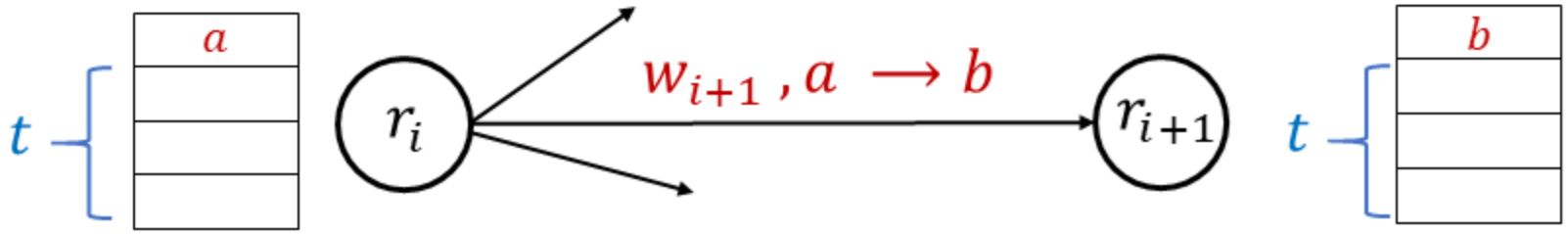
COMPUTAIONS OF A PUSHDOWN AUTOMATON

- A PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ computes as follows.
- It accepts input $w = w_1 w_2 \cdots w_m$, where each $w_i \in \Sigma_\varepsilon$ and sequences of states $r_0, r_1, \cdots, r_m \in Q$ and strings $s_0, s_1, \cdots, s_m \in \Gamma^*$ exist that satisfy the following conditions.

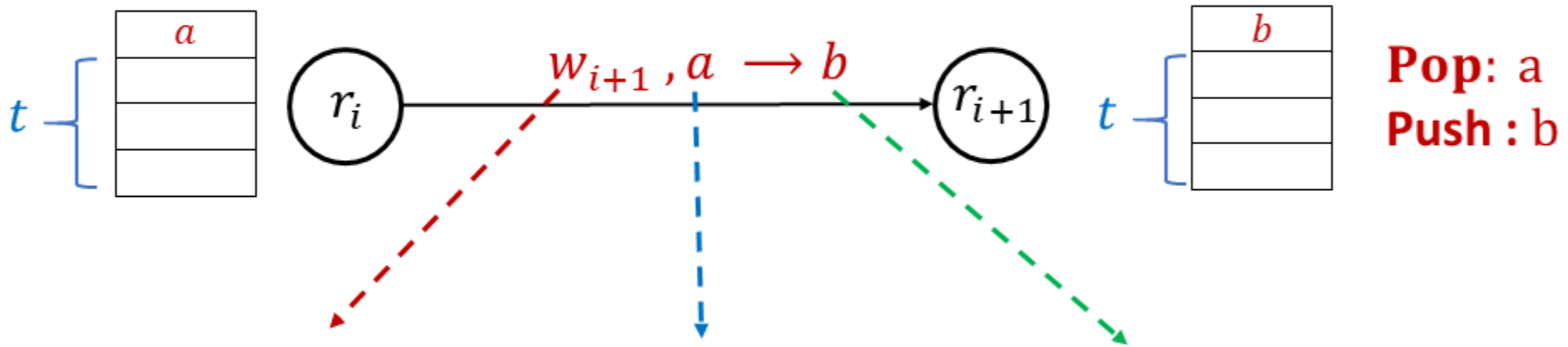
1. $r_0 = q_0$ and $s_0 = \varepsilon$.

2. For $i = 0, \ldots, m-1$, we have $(r_{i+1}, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$. This condition states that M moves properly according to the state, stack, and next input symbol.

3. $r_m \in F$. This condition states that an accept state occurs at the input end.



COMPUTAIONS OF A PUSHDOWN AUTOMATON



Input symbol
Maybe ε

Symbol on top of the stack.
This symbol is popped.
Maybe ε : means the stack is
Neither read nor popped

This symbol is pushed onto the stack.
This symbol is popped.
Maybe ε : means nothing is pushed.

EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 1:** The formal description of the PDA that recognizes the language $\{0^n 1^n | n \geq 0\}$.

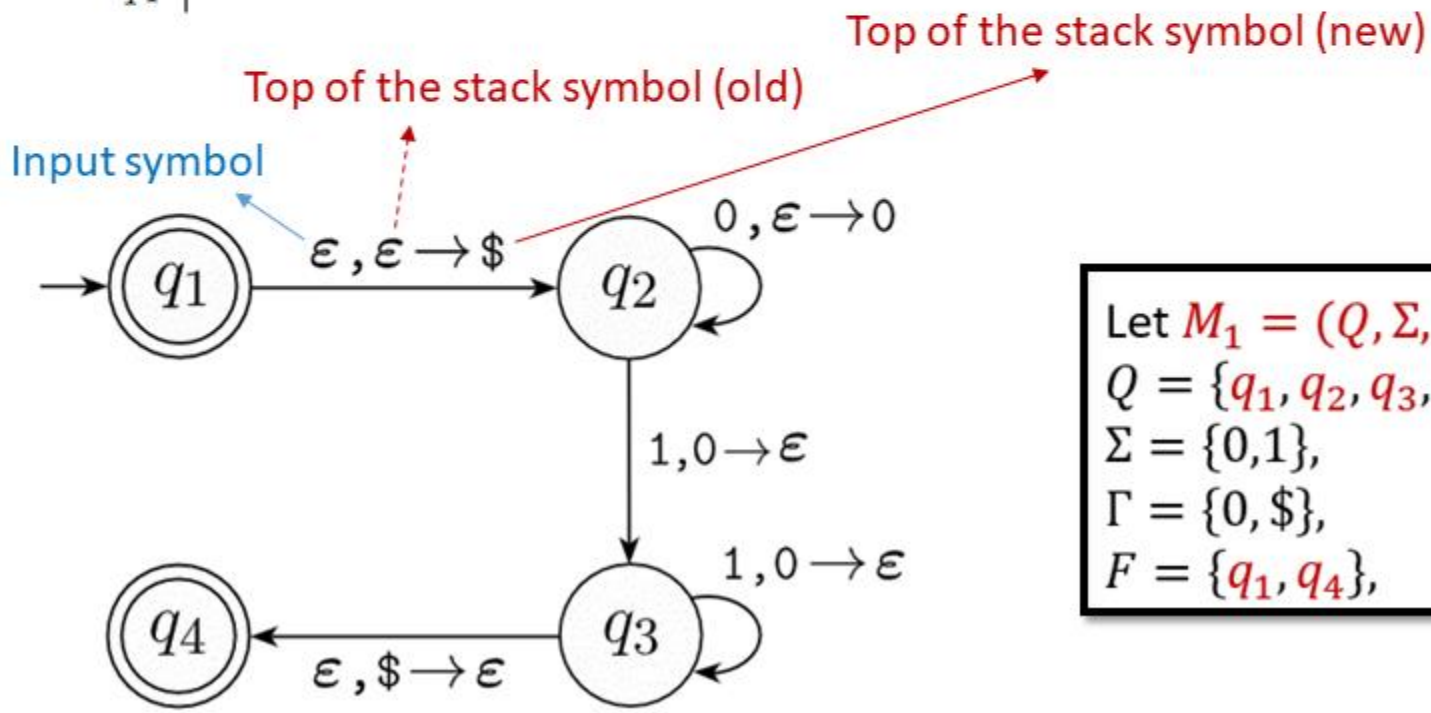
- Let $M_1 = (Q, \Sigma, \Gamma, \delta, q_1, F)$, where
- $Q = \{q_1, q_2, q_3, q_4\}$,
- $\Sigma = \{0,1\}$,
- $\Gamma = \{0, \$\}$,
- $F = \{q_1, q_4\}$,
- And δ is given by the following table, wherein blank entries signify Ø.

Input:	0			1			ε		
Stack:	0	$	ε	0	$	ε	0	$	ε
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$					
q_3				$\{(q_3, \varepsilon)\}$				$\{(q_4, \varepsilon)\}$	
q_4									

EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 1:** (cont.)

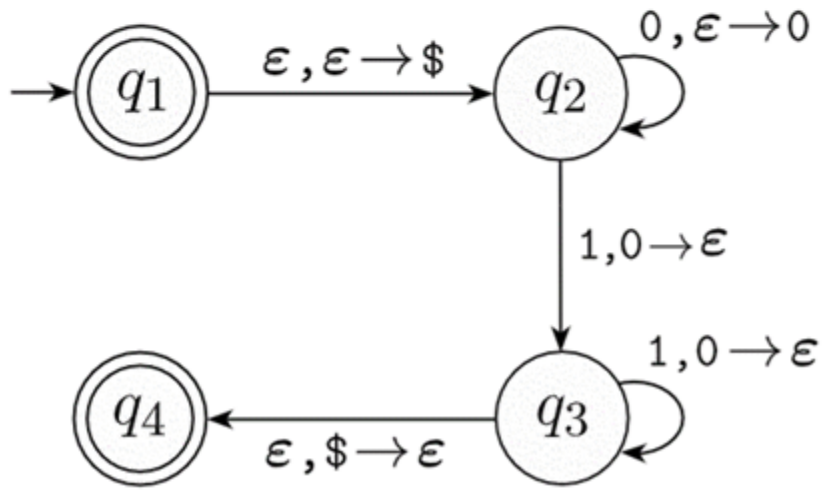
Input:	0			1			ε		
Stack:	0	$	ε	0	$	ε	0	$	ε
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$					
q_3				$\{(q_3, \varepsilon)\}$				$\{(q_4, \varepsilon)\}$	
q_4									



Let $M_1 = (Q, \Sigma, \Gamma, \delta, q_1, F)$, where
$Q = \{q_1, q_2, q_3, q_4\}$,
$\Sigma = \{0,1\}$,
$\Gamma = \{0, \$\}$,
$F = \{q_1, q_4\}$,

EXAMPLES OF PUSHDOWN AUTOMATA

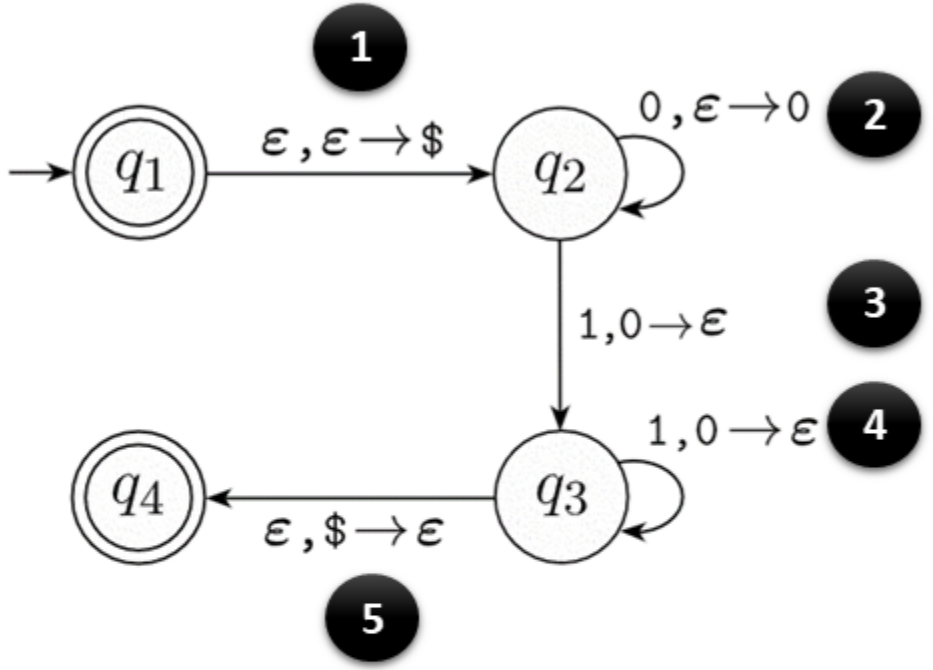
- **Example 1:** (cont.)
- "$a, b \rightarrow c$" : to signify that when the machine is reading an a from the input, it may replace the symbol b on the top of the stack with a c.
- $a = \varepsilon$: the machine may make this transition without reading any symbol from the input.
- $b = \varepsilon$: the machine may make this transition without reading and popping any symbol from the stack.
- $c = \varepsilon$: the machine does not write any symbol on the stack when going along this transition.



EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 1:** (cont.)

- no explicit mechanism to allow the PDA to test for an empty stack.
- This PDA is able to get the same effect by initially placing a special **symbol $** on the stack.



- $b = \varepsilon$: the machine may make this transition without
the st

- $c =$
on the

1. Creating an empty stack
2. If input is 0, then push a 0 onto the stack
3,4. If input is 1, then pop a 0 from the stack and no push
5. The last input symbol and stack is empty → accept the string

EXAMPLES OF PUSHDOWN AUTOMATA

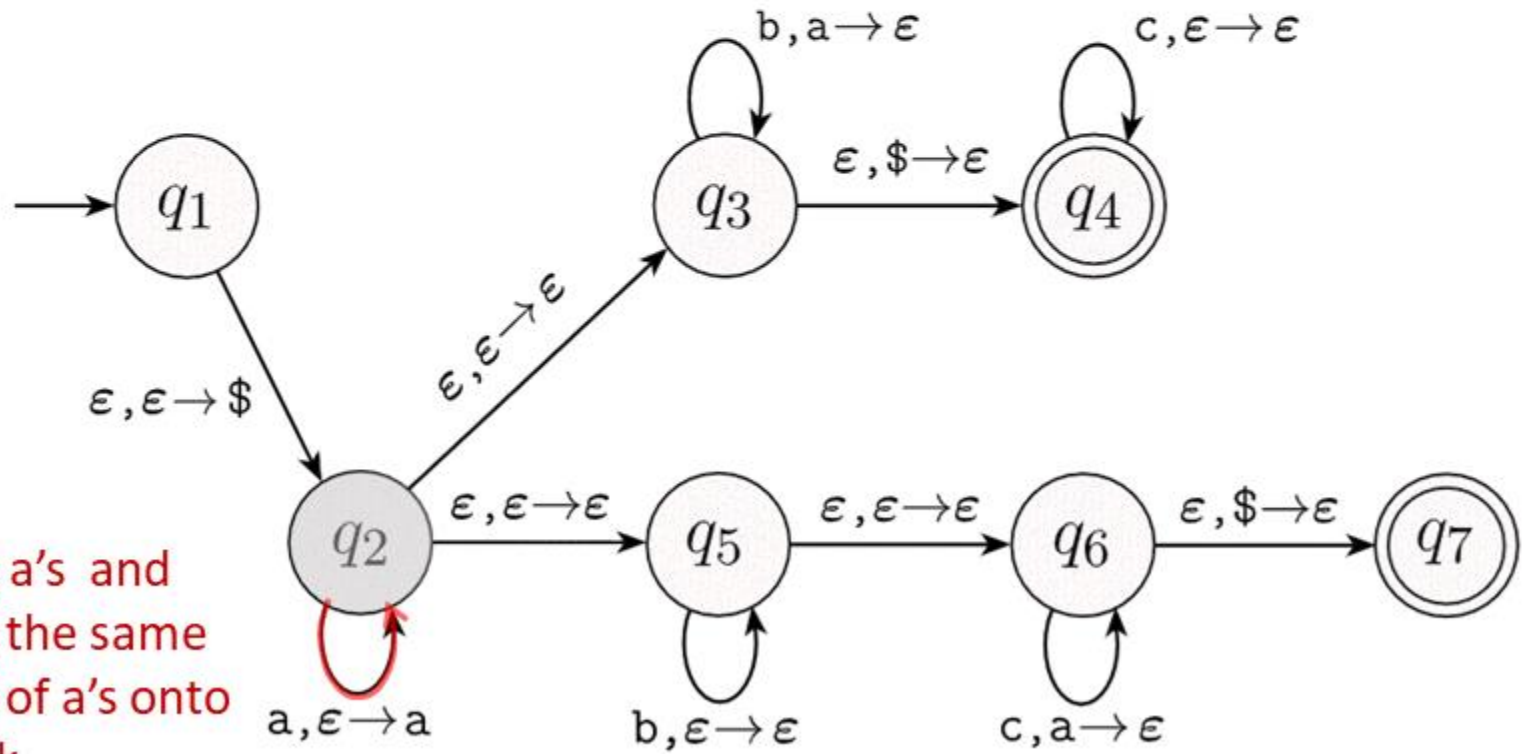
- **Example 3:** Find the PDA that recognizes the language

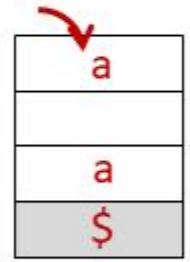
$$\{\mathrm{a}^i\mathrm{b}^j\mathrm{c}^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}.$$

EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 2:** Find the PDA that recognizes the language

$$\{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}.$$



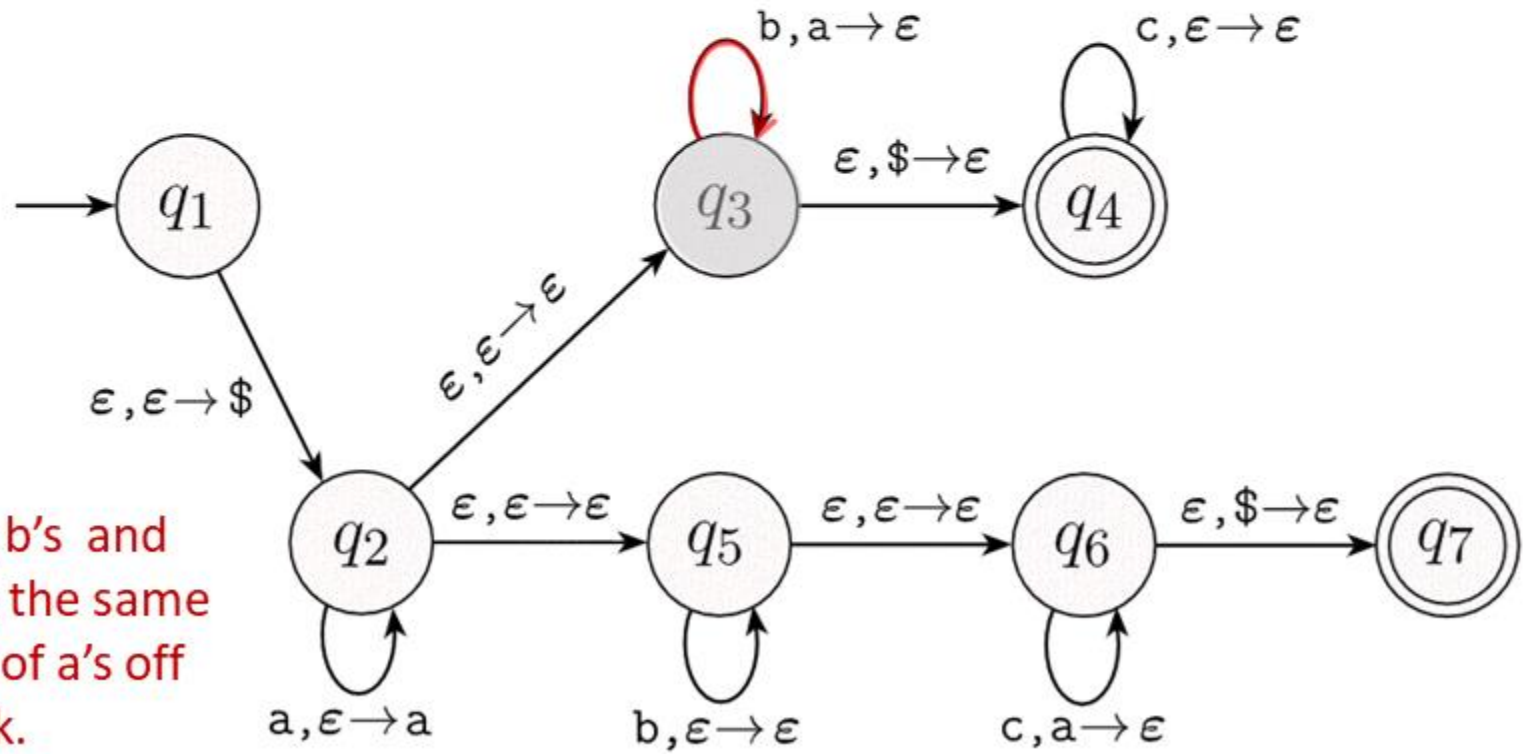


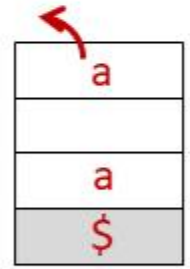
- Reading a's and pushing the same amount of a's onto the stack.

EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 2:** Find the PDA that recognizes the language

$$\{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}.$$



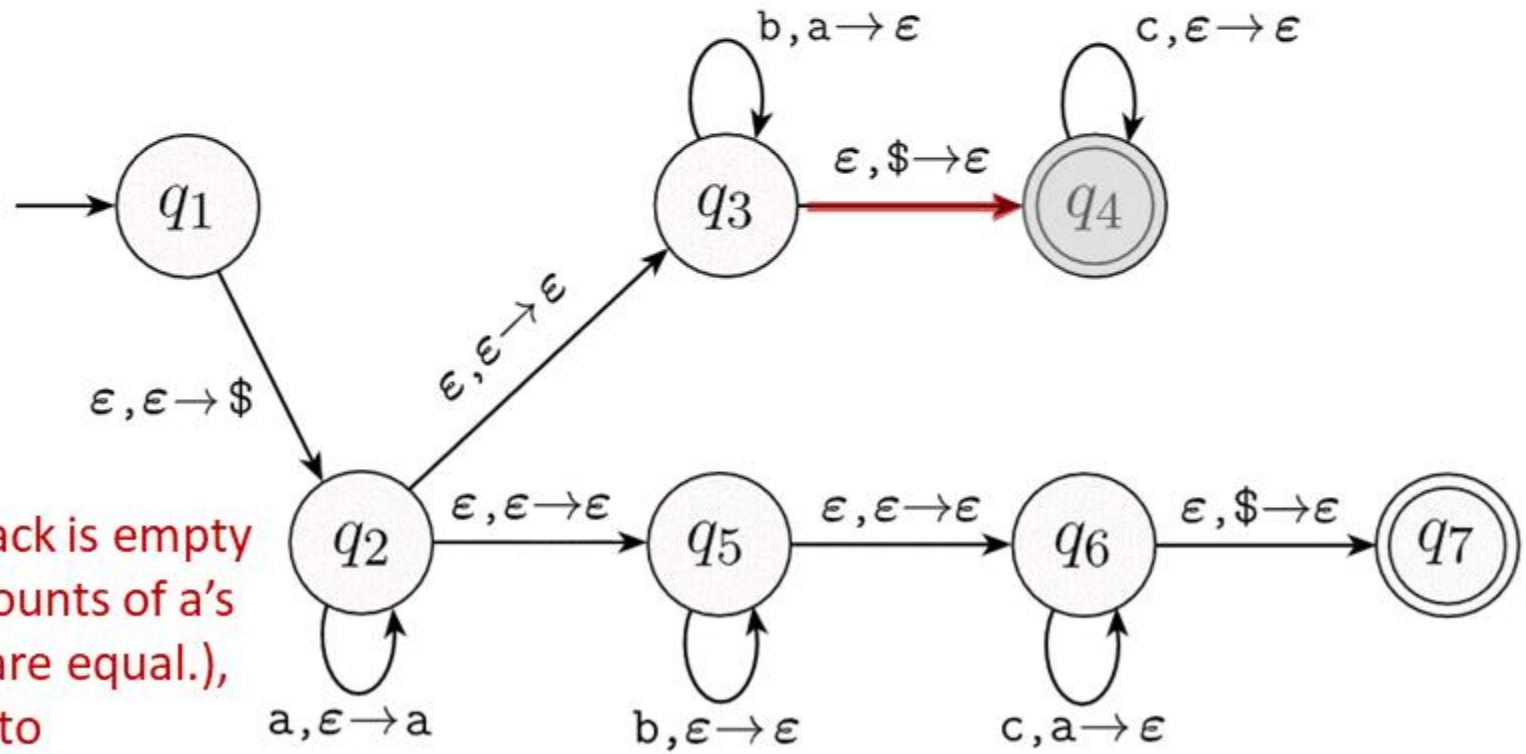


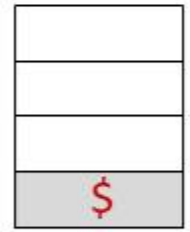
- Reading b's and popping the same amount of a's off the stack.

EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 2:** Find the PDA that recognizes the language

$$\{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}.$$



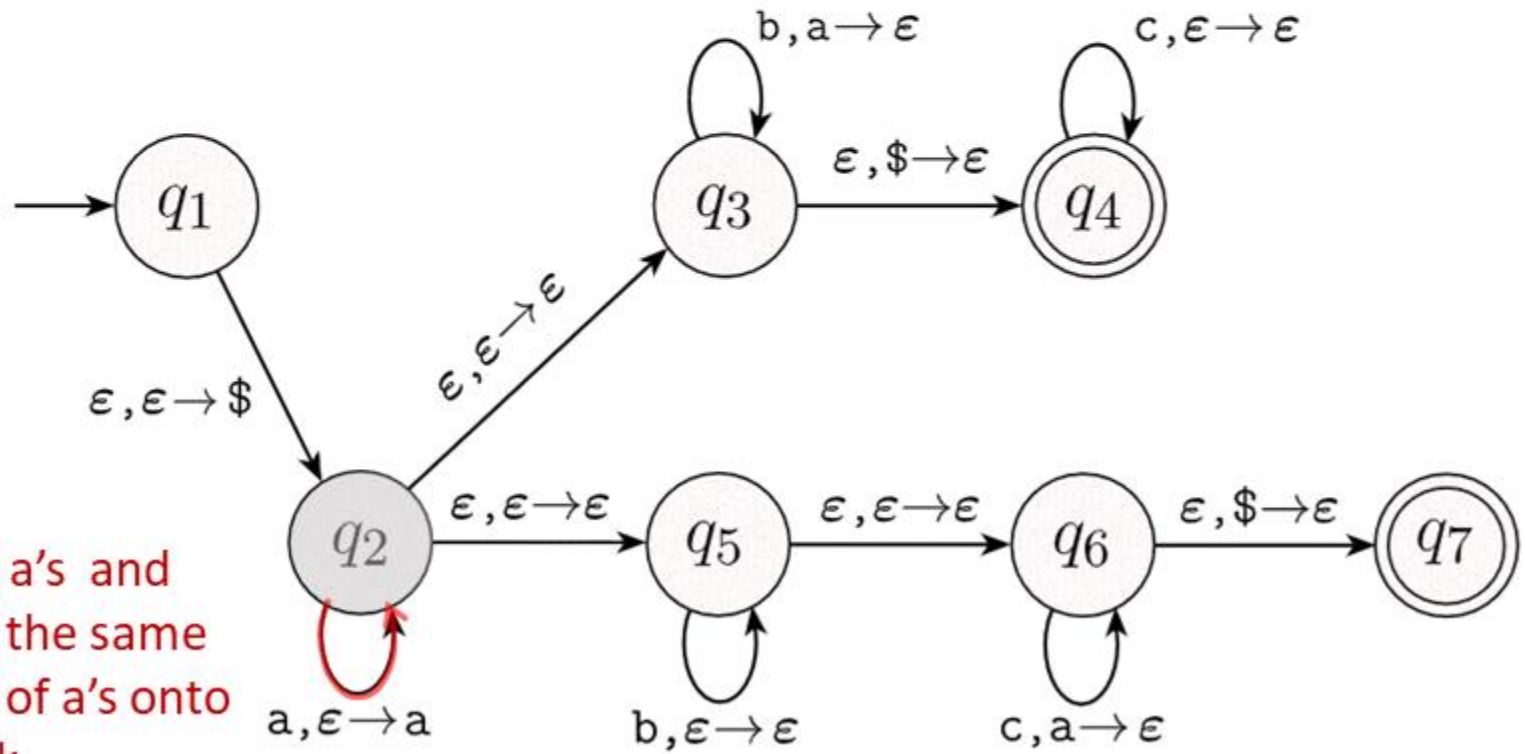


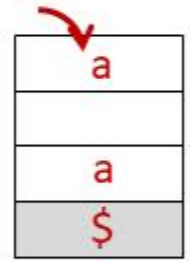
- If the stack is empty (the amounts of a's and b's are equal.), then go to the accept state.

EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 2:** Find the PDA that recognizes the language

$$\{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}.$$





- Reading a's and pushing the same amount of a's onto the stack.

EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 2:** Find the PDA that recognizes the language

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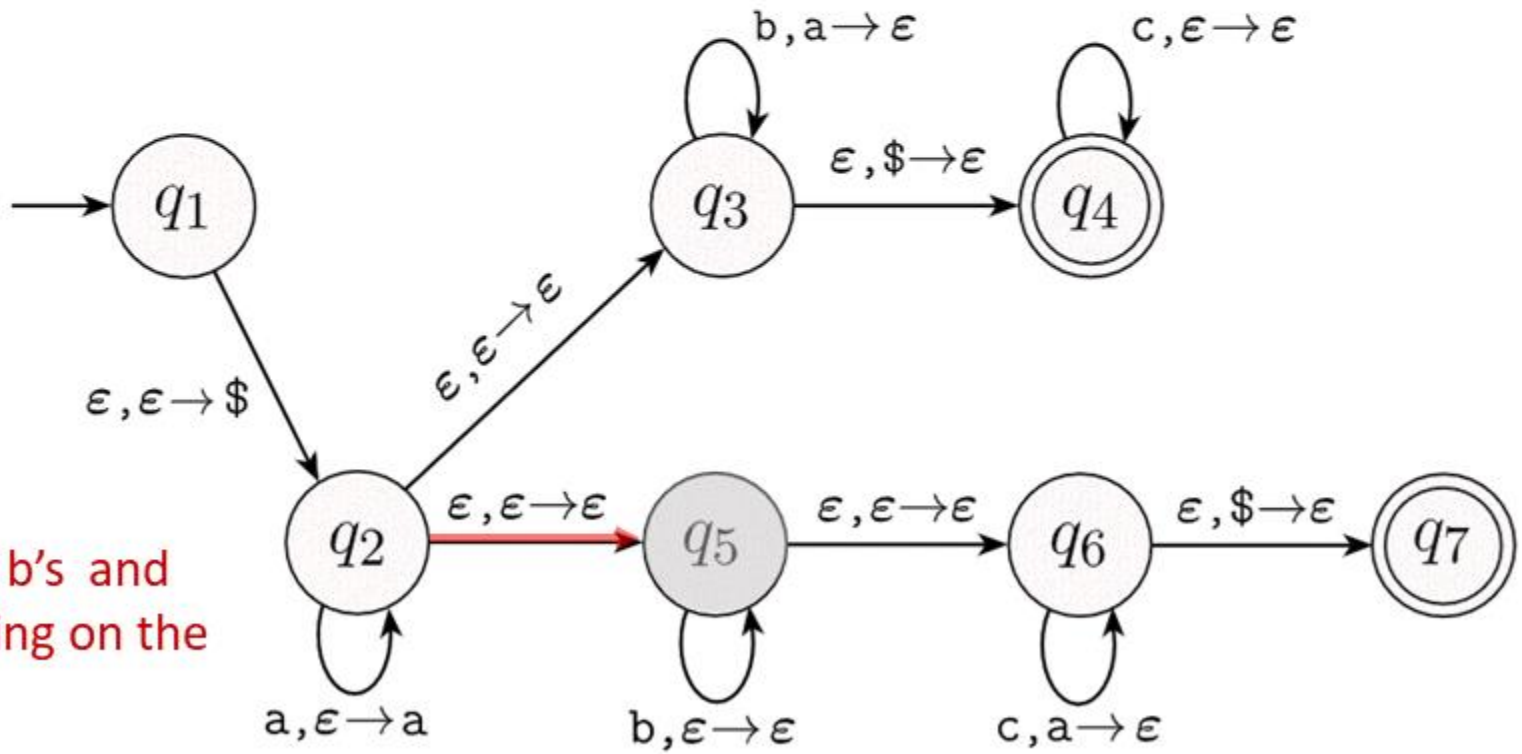
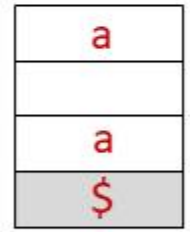


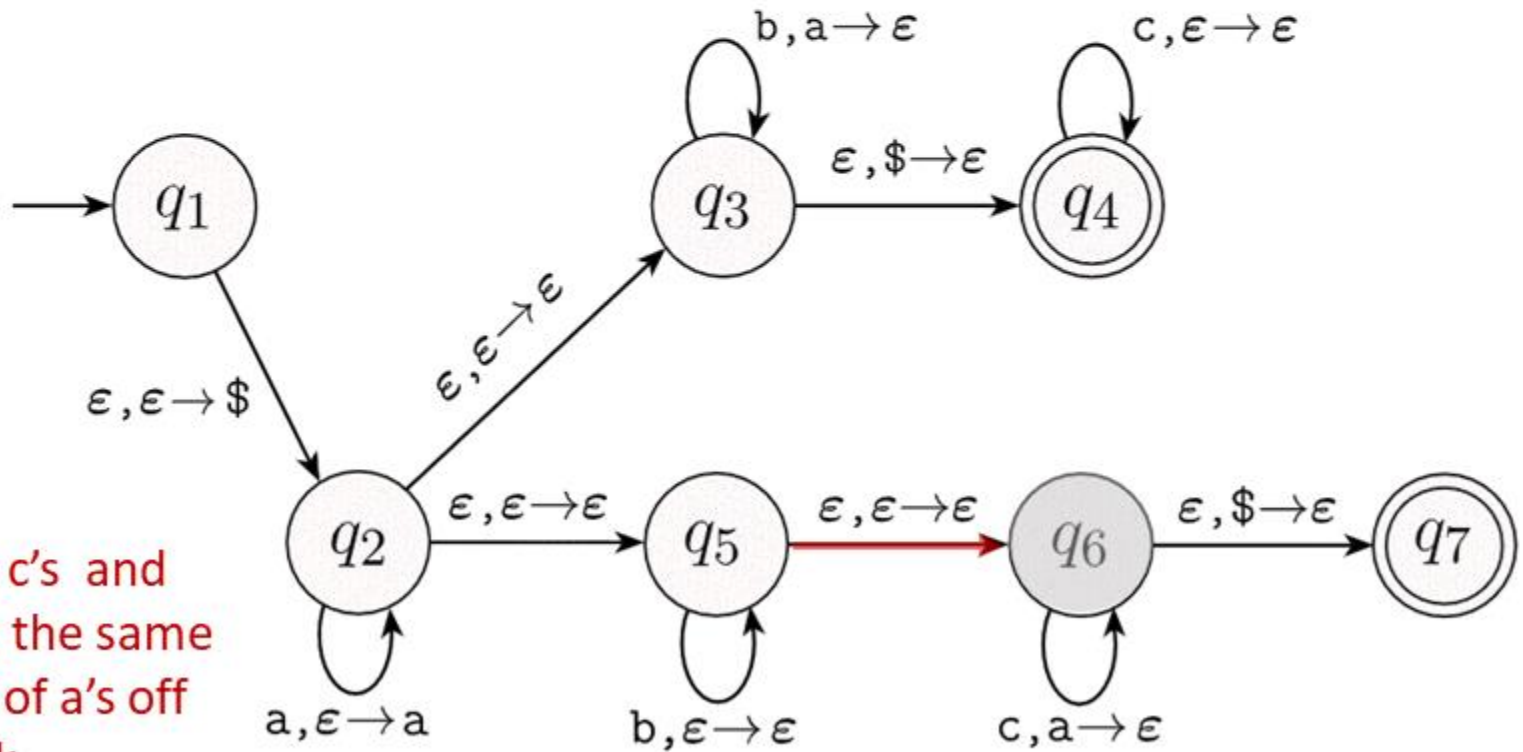


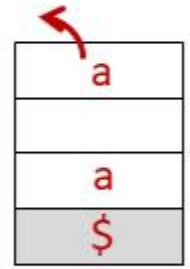
- Reading b's and do nothing on the stack.

EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 2:** Find the PDA that recognizes the language

$$\{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}.$$





- Reading c's and popping the same amount of a's off the stack.

EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 2:** Find the PDA that recognizes the language

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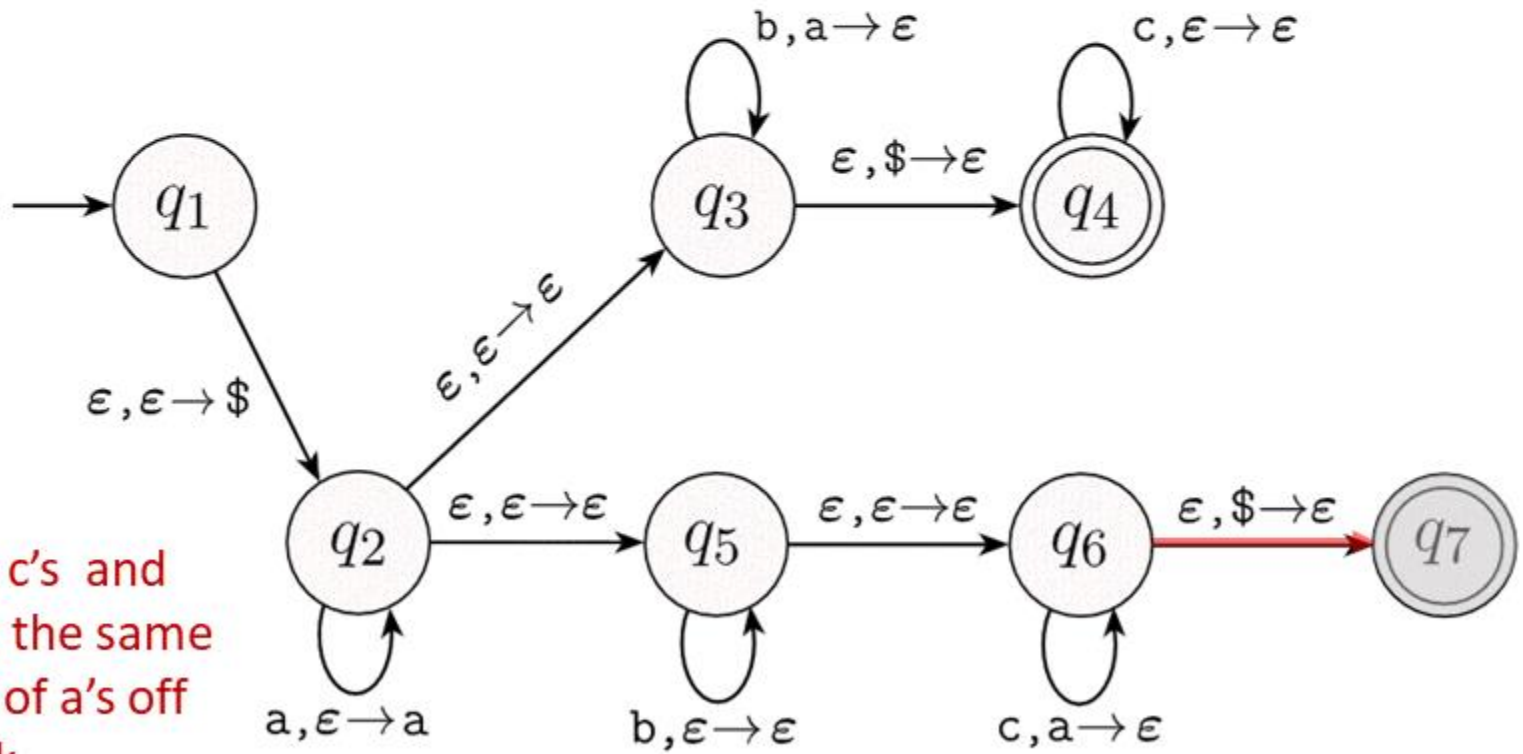
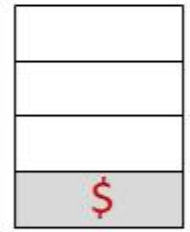




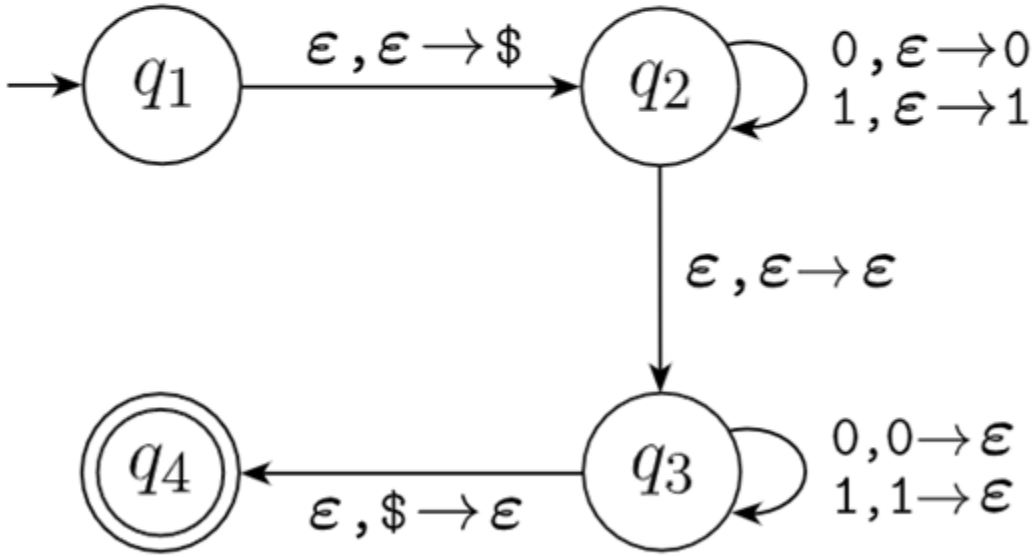
- Reading c's and popping the same amount of a's off the stack.

EXAMPLES OF PUSHDOWN AUTOMATA

- **Example 3:** Find the PDA that recognizes the language

$$\{ww^{\mathcal{R}} | w \in \{0,1\}^*\}$$

NOLEMON | NOMELON



EQUIVALENCE OF PDA WITH CFG

- THEOREM: A language is context free if and only if some pushdown automaton recognizes it.

- Part1: If a language is context free, then some pushdown automaton recognizes it.

- Part2: If a pushdown automaton recognizes some language, then it is context free.

EQUIVALENCE OF PDA WITH CFG

- Part1: If a language is context free, then some pushdown automaton recognizes it.

- PROOF IDEA:
- Let A be a CFL, Then A has a CFG **G**, generating it.
- Using the following example we show how to convert **G** into an equivalent PDA, **P**.

- **Example:** Grammar **G**

$$S \rightarrow BS|A \quad : R1, R2$$
$$A \rightarrow 2A|\varepsilon \quad : R3, R4$$
$$B \rightarrow BB1|0 \quad : R5, R6$$

Using **left-most derivation** for generating **001** will be :

$$S \Rightarrow BS \Rightarrow BB1S \Rightarrow 0B1S \Rightarrow 001S \Rightarrow 001A \Rightarrow 001$$

R1 R5 R6 R6 R2 R4

EQUIVALENCE OF PDA WITH CFG

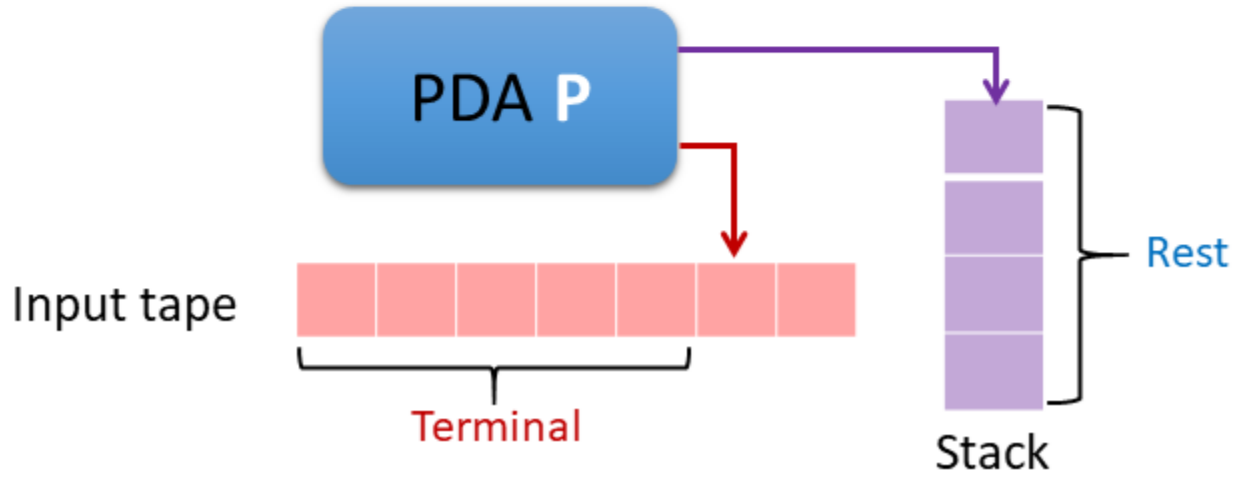
- PROOF IDEA: (cont.)

- In general form all of intermediate string in the derivation for generating a string will be in form

$$0001100101\underbrace{01}_{\text{Terminal}}\underbrace{SA01SSA00SSAA0}_{\text{Rest}}$$

(contains terminals and nonterminals)

- The **PDA** P will work by accepting its input w, if G generates that input,
  - by determining whether there is a derivation for w. (left-most derv.)

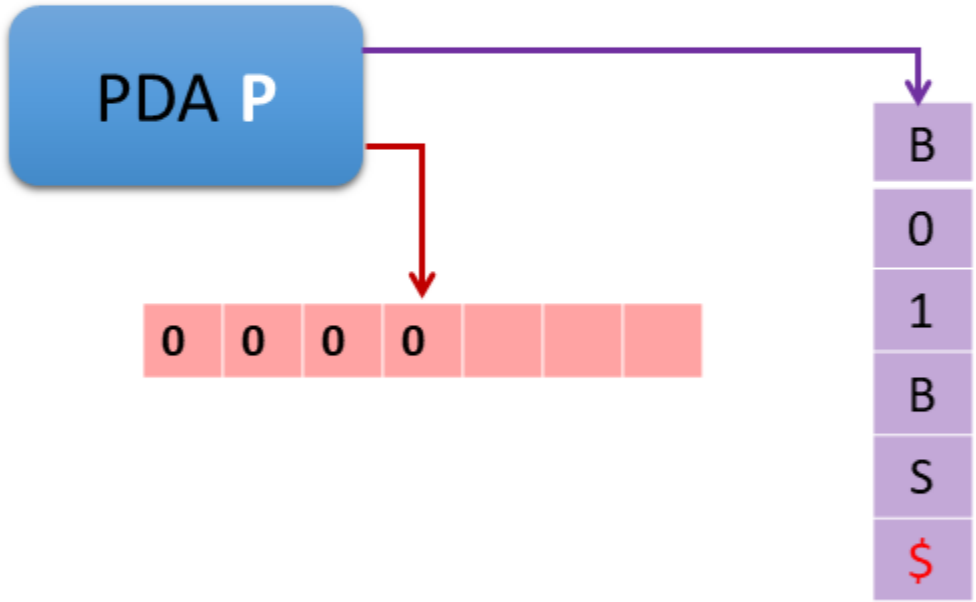


EQUIVALENCE OF PDA WITH CFG

- PROOF IDEA: (cont.)

- Let's see how we can store a an **intermediate string** (sentential form) in the derivation in a pushdown automaton.

$$S \overset{*}{\Rightarrow} 0000B01BS \Rightarrow \cdots$$



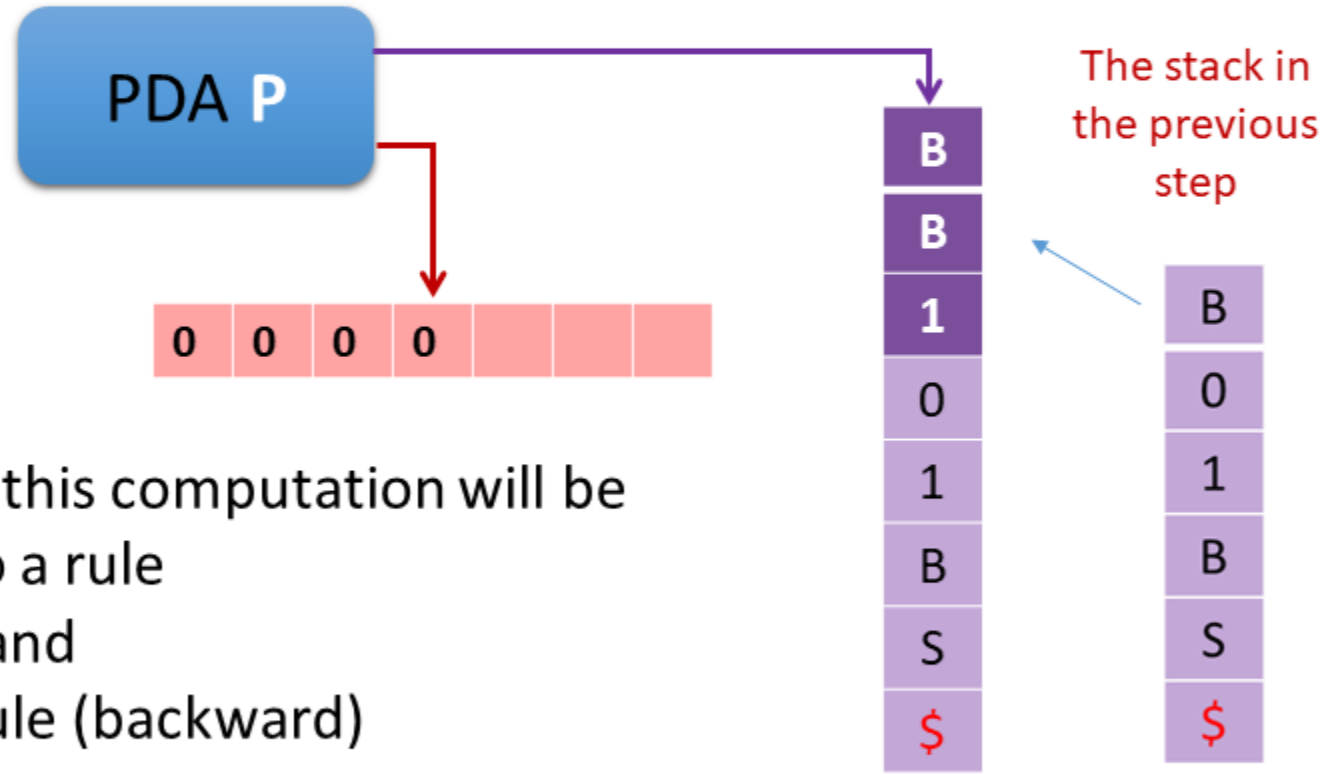
EQUIVALENCE OF PDA WITH CFG

- PROOF IDEA: (cont.)

- At each step of the derivation, we expand the leftmost nonterminal using the rules.

$$S \overset{*}{\Rightarrow} 0000B01BS \Rightarrow \cdots$$

Rule: $\mathrm{B} \longrightarrow \mathrm{BB1}$



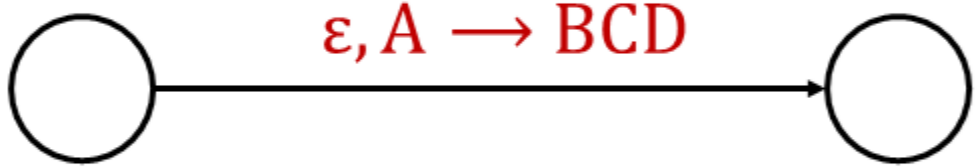
- so the steps that we follow in this computation will be
  - We match the stack top to a rule
  - Pop the top of the stacks and
  - push the left side of the rule (backward)

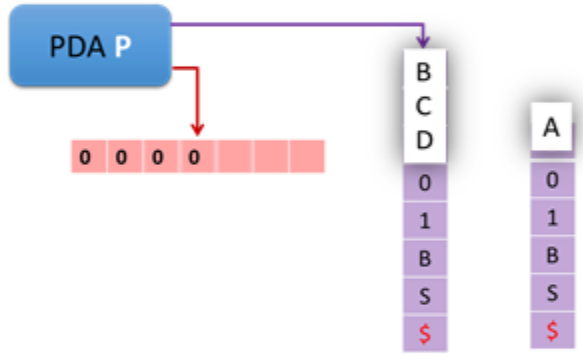
- Since PDA is nondeterministic, all rules at the same time will be examined.

EQUIVALENCE OF PDA WITH CFG

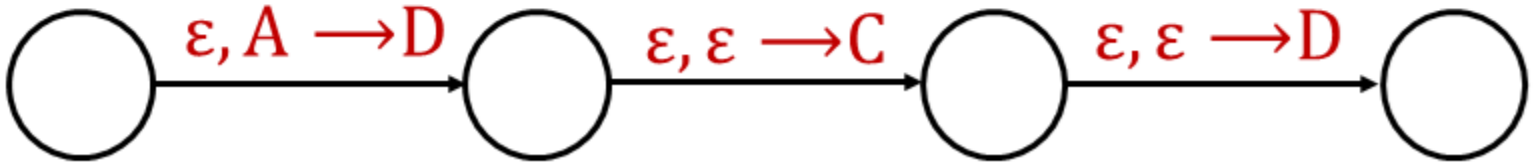
- PROOF IDEA: (cont.)

- How can we add the following rule in a PDA?
- **Rule**: $A \longrightarrow BCD$ where A,B,C, and D are nonterminals





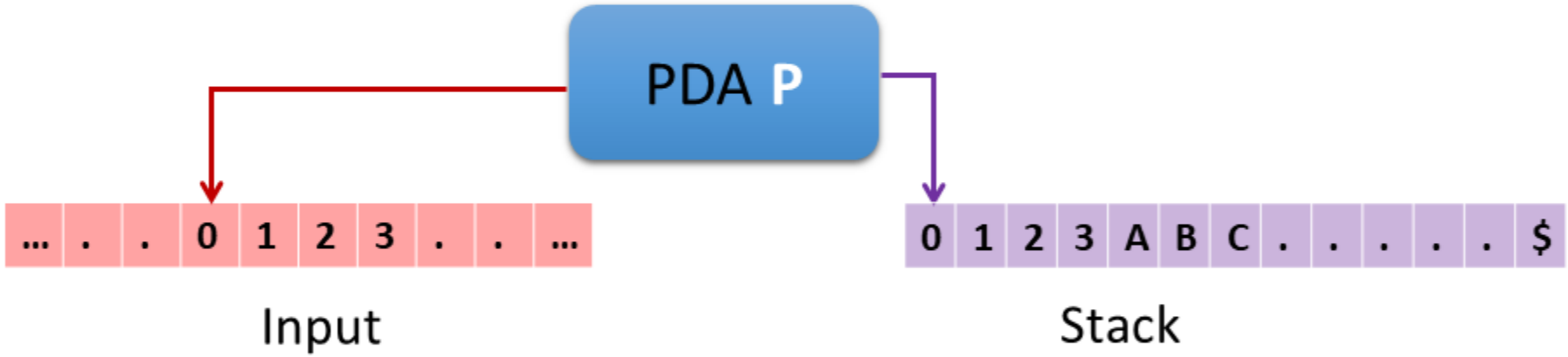
- We are not allowed to push multiple symbols.

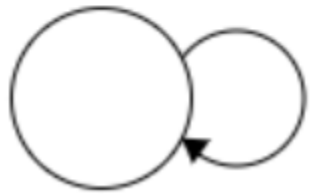


EQUIVALENCE OF PDA WITH CFG

- PROOF IDEA: (cont.)

- What will be the **transition** if there is a **terminal** on top of the stack?





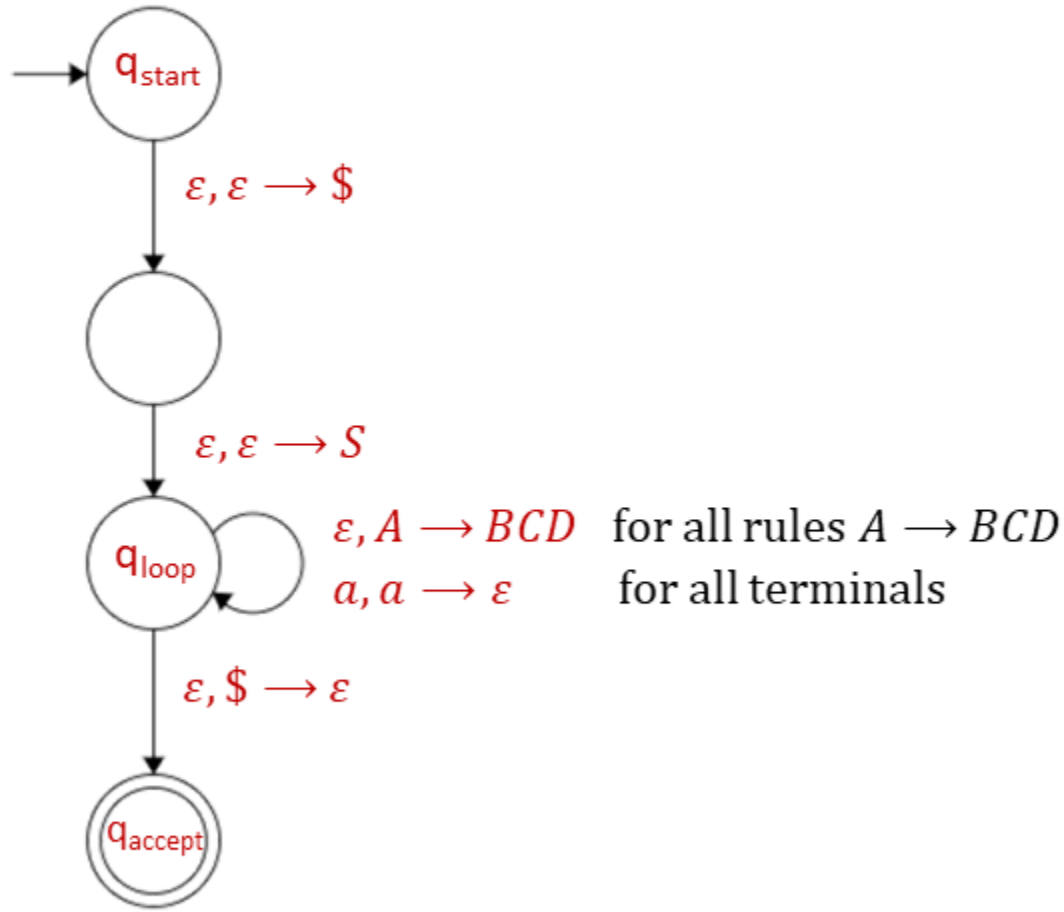
$0,0 \longrightarrow \varepsilon$
$1,1 \longrightarrow \varepsilon$
$2,2 \longrightarrow \varepsilon$
$3,3 \longrightarrow \varepsilon$

For all symbols in alphabet.
(nonterminals)

EQUIVALENCE OF PDA WITH CFG

- PROOF IDEA: (cont.)

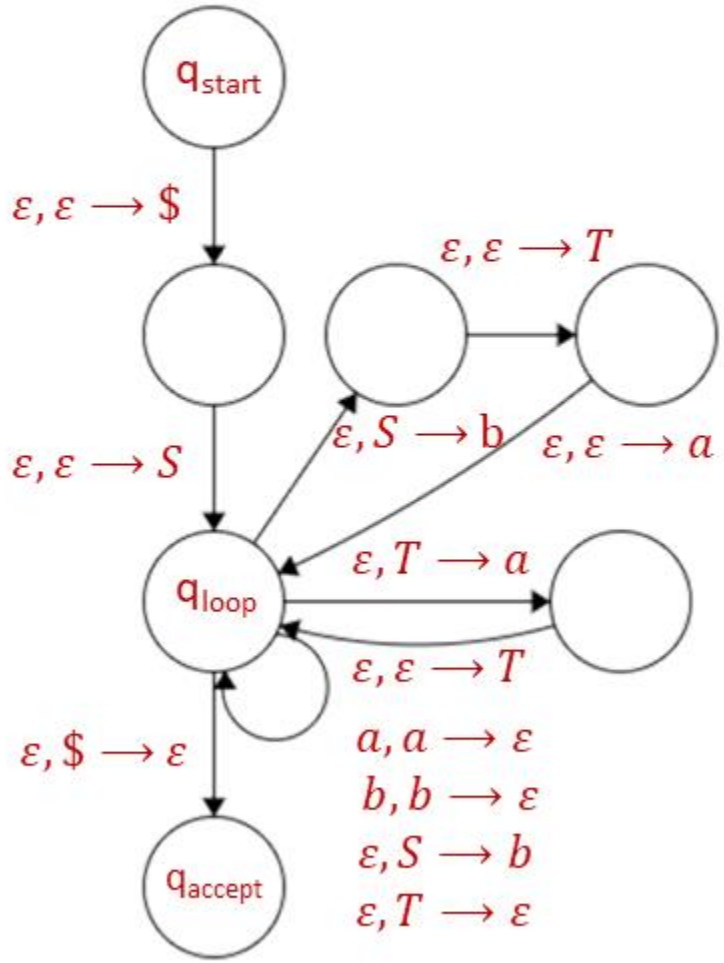
- The Final PDA



EQUIVALENCE OF PDA WITH CFG

- Example: construct aPDA P1 from the following CFG G.

$S \rightarrow aTb|b$
$T \rightarrow Ta|\varepsilon$



EQUIVALENCE OF PDA WITH CFG

- Part2: If a pushdown automaton recognizes some language, then it is context free.

EQUIVALENCE OF PDA WITH CFG

- Let's see one example that shows how we can convert an NFA to a CFG.
  - (**Note:** Any Regular language has CFG.)

EQUIVALENCE OF PDA WITH CFG

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  - (**Note:** Any Regular language has CFG.)

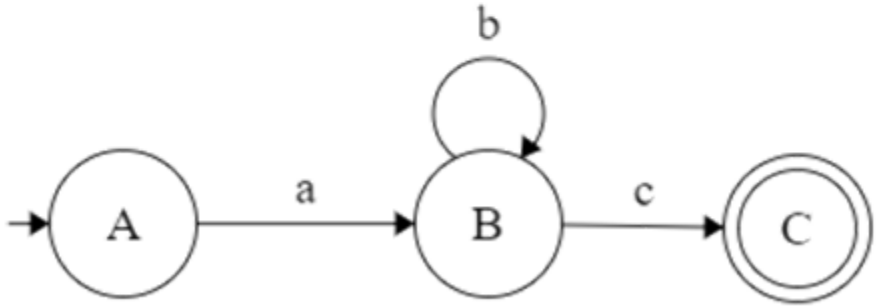
- Follow the following steps:
  - Make a variable for each state.
  - Make the variable for the starting state the **starting variable**.
  - Make a rule for each edge.
  - Add an ε-rule for each accept state.

EQUIVALENCE OF PDA WITH CFG

- Let's see one example that shows how we can convert an NFA to a CFG.
  - (**Note:** Any Regular language has CFG.)

- Follow the following steps:
  - Make a variable for each state.
  - Make the variable for the starting state the **starting variable**.
  - Make a rule for each edge.
  - Add an ε-rule for each accept state.

- **Example:** Find a CFG for the following Machine.



NFA M1

EQUIVALENCE OF PDA WITH CFG

- Let's see one example that shows how we can convert an NFA to a CFG.
  - (**Note:** Any Regular language has CFG.)

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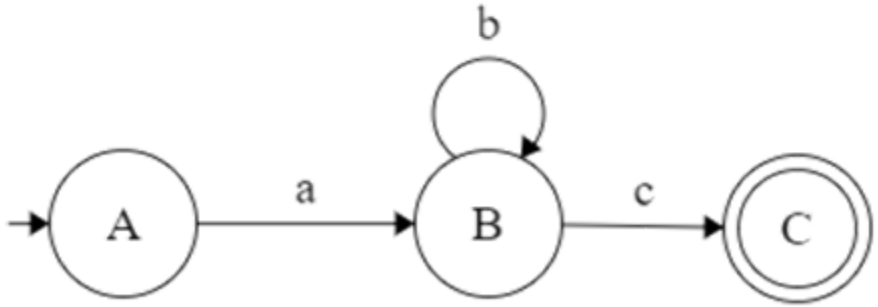
  - Make a variable for each state.
  - Make the variable for the starting state the **starting variable**.
  - Make a rule for each edge.
  - Add an ε-rule for each accept state.

- **Example:** Find a CFG for the following Machine.



NFA M1

- $A \rightarrow aB$
- $B \rightarrow bB \mid cC$
- $C \rightarrow \varepsilon$

CFG G1

$$A \rightarrow aB \rightarrow abB \rightarrow abcC \rightarrow abc\varepsilon = abc$$

EQUIVALENCE OF PDA WITH CFG

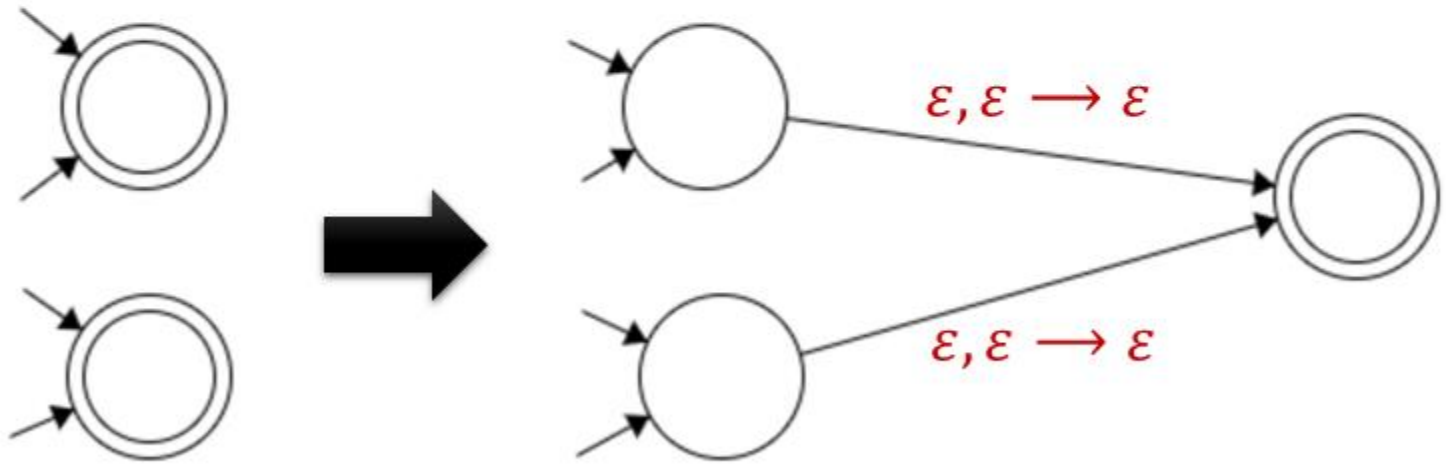
- Part2: If a pushdown automaton recognizes some language, then it is context free.

- Proof idea:
- We have a PDA P, and we want to make a CFG G that generates all the strings that P accepts.
- G should **generate a string** if that string causes the PDA to go from its start state to an accept state.

- **Steps:**
  - **Step 1:** simplify the PDA
  - **Step2:** Building CFG

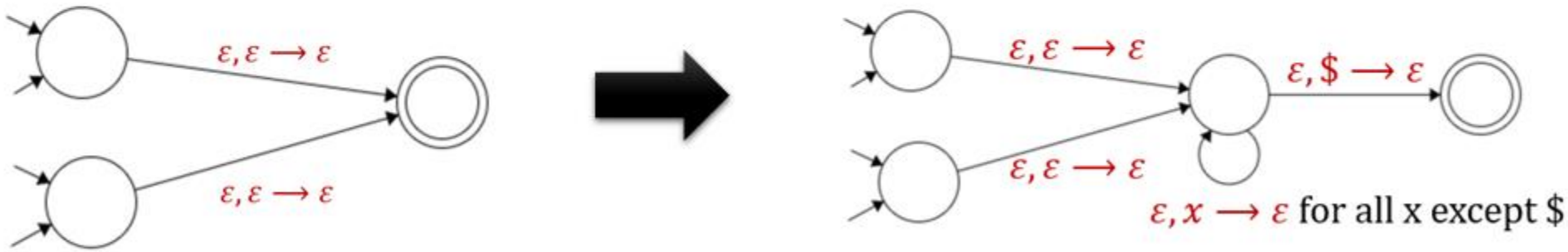
EQUIVALENCE OF PDA WITH CFG

- Proof idea: (cont.)
- **Step1:** Simplification by modifying P as below to give it the following three features:
  1. It has a single accept state, q_{accept}.



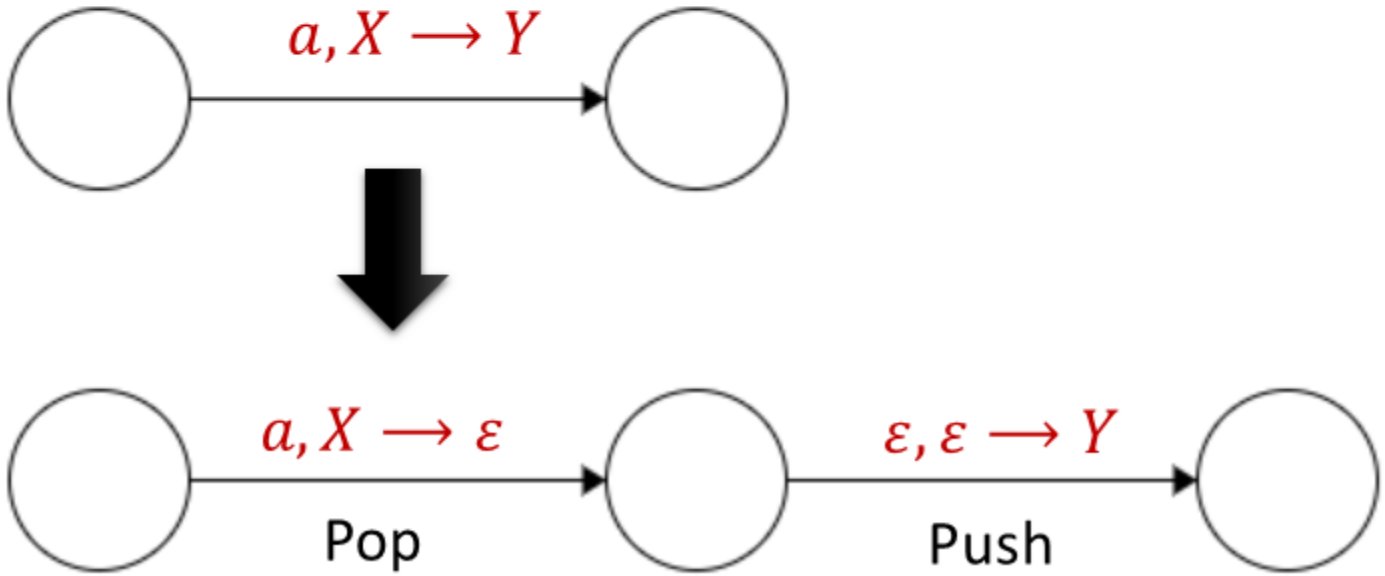
EQUIVALENCE OF PDA WITH CFG

- Proof idea: (cont.)
- **Step1:** Simplification by modifying P as below to give it the following three features:
  2. It empties its stack before accepting.



EQUIVALENCE OF PDA WITH CFG

- Proof idea: (cont.)
- **Step1:** Simplification by modifying P as below to give it the following three features:
  3. Each transition either pushes a symbol onto the stack (a push move) or pops one off the stack (a pop move), but it does not do both at the same time.



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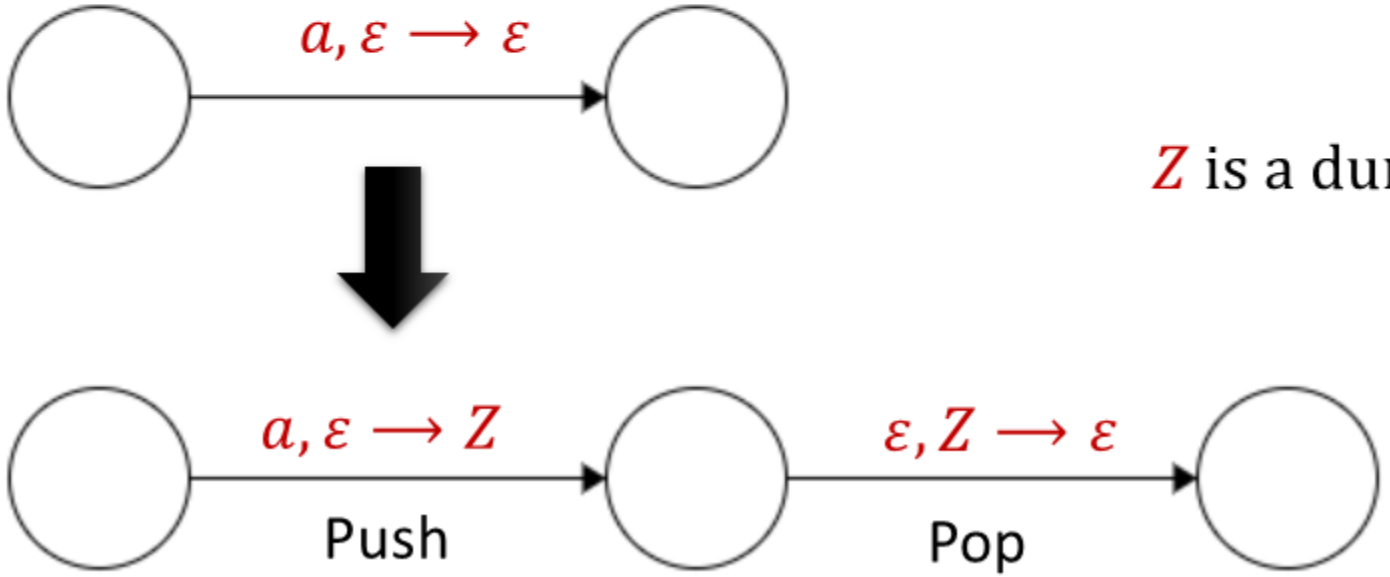
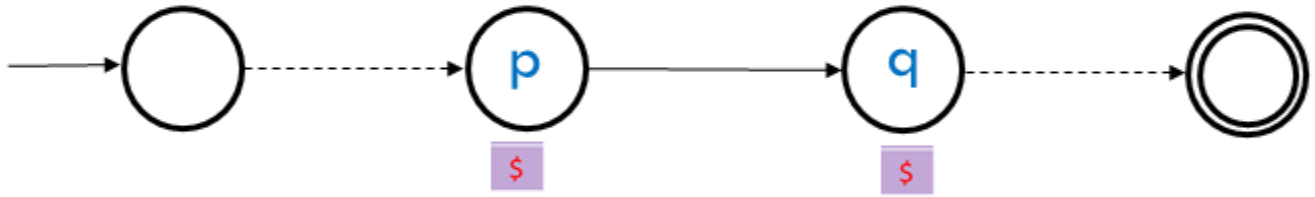


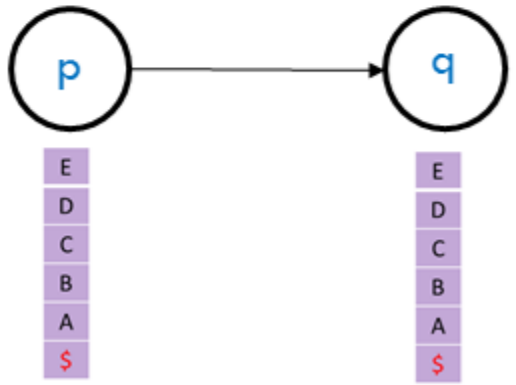
Z is a dummy symbol.

EQUIVALENCE OF PDA WITH CFG

- Proof idea: (Cont.)
- **Step2: Building CFG**
- For **each pair of states p and q** in P, the grammar will have a **variable** A_{pq}.
  - This variable **generates all the strings** that can take P from p with an empty stack to q with an empty stack.



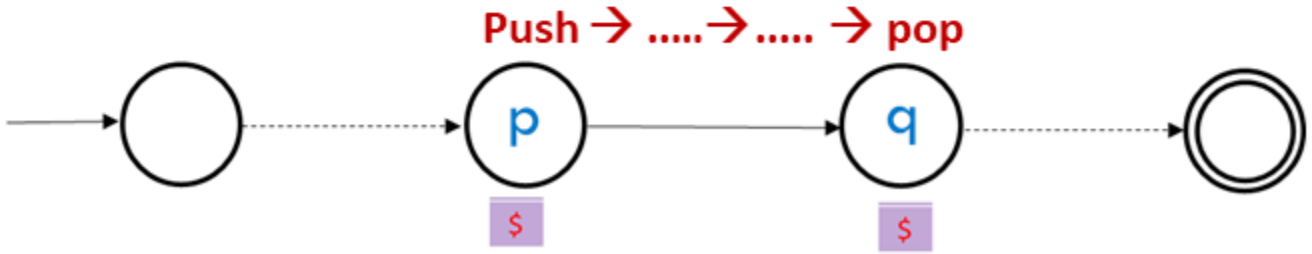
  - such strings can also take P from p to q, regardless of the stack contents at p, leaving the stack at q in the same condition as it was at p.



EQUIVALENCE OF PDA WITH CFG

- Proof idea: (Cont.)
- **Step2:**

- To design G, we must understand **how P operates** on the strings generated by A_{pq}.
  - For any such string x, P's **first move** on x must be a **push**,
    - Since **stack is empty**.
  - Similarly, the **last move** on x must be a **pop**.
    - Since the stack should be empty.

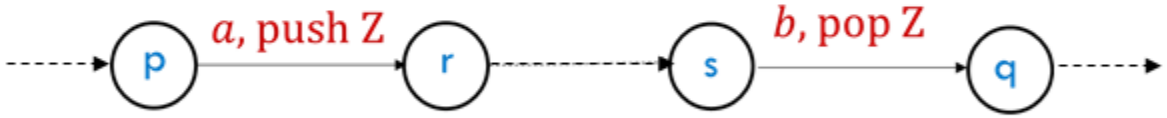


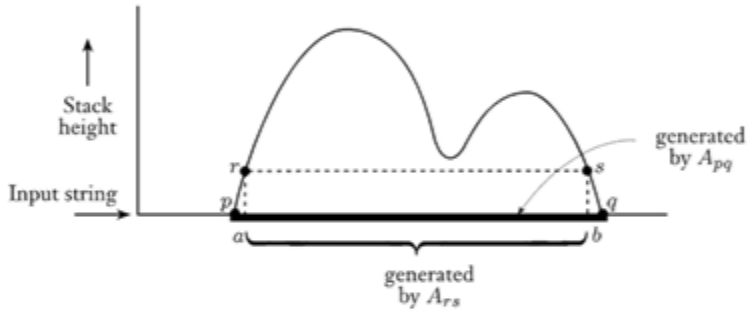
EQUIVALENCE OF PDA WITH CFG

- Proof idea: (Cont.)
- **Step2:**

- Two possibilities occur during P's computation on x.

1. the **symbol popped at the end is** the **symbol that was pushed at the beginning**.





$A_{pq} \longrightarrow \mathrm{a}\ A_{\mathrm{rs}}\ \mathrm{b}$

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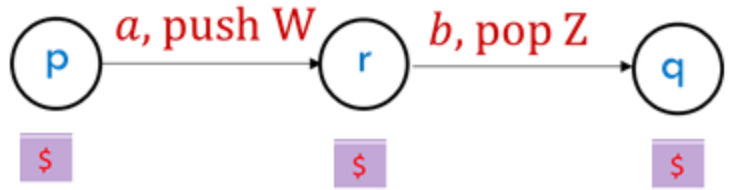
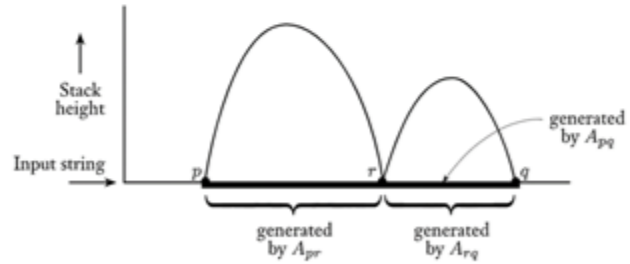




$A_{pq} \longrightarrow A_{\mathrm{pr}}\ A_{\mathrm{rq}}$

EQUIVALENCE OF PDA WITH CFG

- Part2: If a pushdown automaton recognizes some language, then it is context free.

- Proof :
- Say that $P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{accept}\})$ and construct G. The variables of G are $\{A_{pq} | p, q \in Q\}$. The start variable is $A_{q_0, q_{accept}}$. Now we describe G's rules in three parts.

1. For each $p, q, r, s \in Q$, $u \in \Gamma$, and $a, b \in \Sigma_\varepsilon$, if $\delta(p, a, \varepsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ε), put the rule $A_{pq} \to aA_{rs}b$ in G.
2. For each $p, q, r \in Q$, put the rule $A_{pq} \to A_{pr}A_{rq}$ in G.
3. Finally, for each $p \in Q$, put the rule $A_{pp} \to \varepsilon$ in G.

Chapter 2

2.2 Non-Context-Free Grammars

Introduction

- We have shown that language $A = \{a^n b^n | n \geq 0\}$ is a CFL.
- How about $B = \{a^n b^n c^n | n \geq 0\}$?

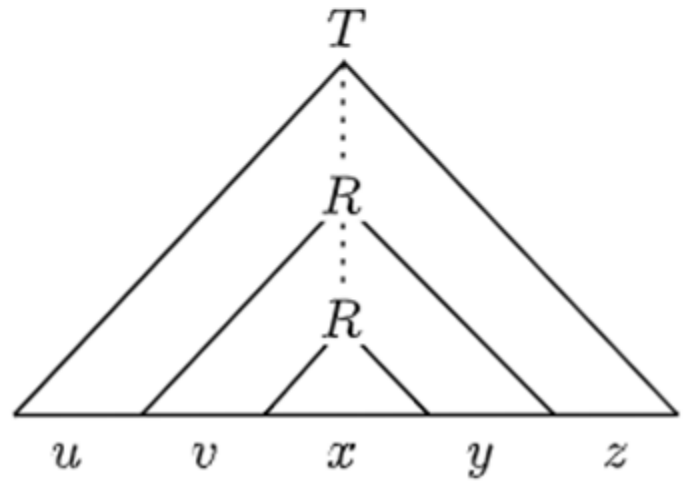
In this section:

- A similar pumping lemma for context-free languages is presented.
  - It states that **every context-free language** has a special value called the pumping length such that **all longer strings** in the language can be "pumped."

Pumping Lemma for CFL

Theorem: **Pumping lemma for context-free languages :**

- If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into **five piece** $s = uvxyz$ satisfying the conditions



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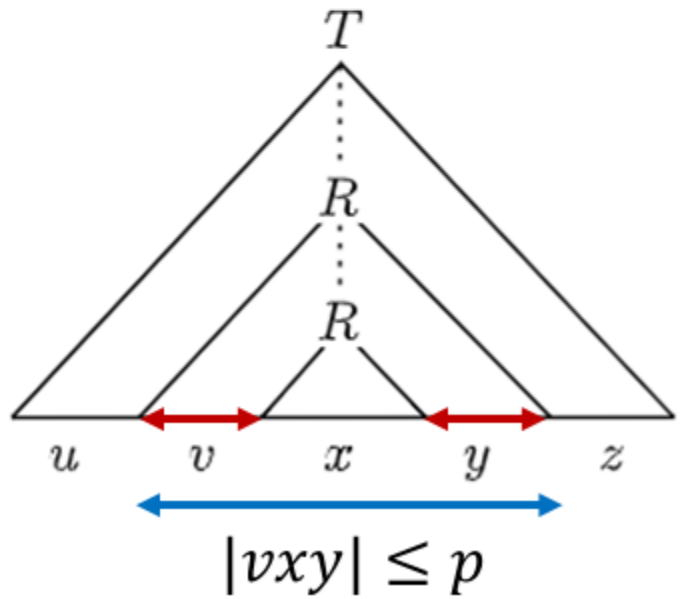
1. for each $i \geq 0, uv^i xy^i z \in A$,
2. $|vy| > 0$, and → either v or y is not the empty string.
3. $|vxy| \leq p$. → the pieces v, x, and y together have length at most p.



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Proof idea:

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- The parse tree for s must be very tall
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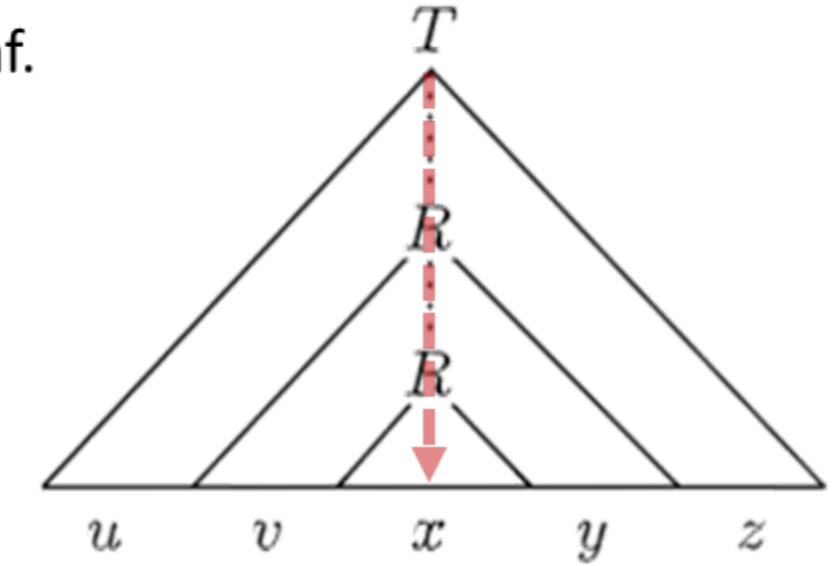

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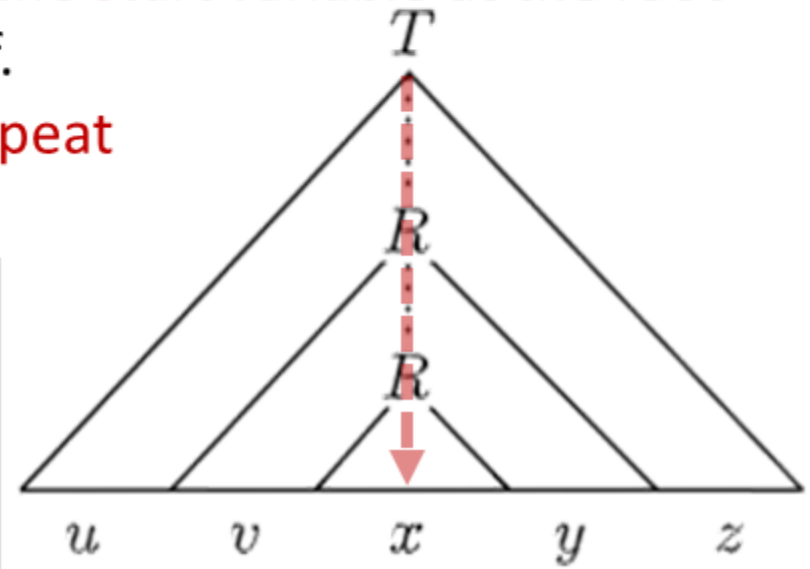
- The parse tree must contain some long path from the start variable at the root of the tree to one of the terminal symbols at a leaf.
- On this **long path**, some variable symbol R must repeat
  - because of the pigeonhole principle.

$T \to R \quad R \to aR\text{b}|\text{B} \quad B \to c$

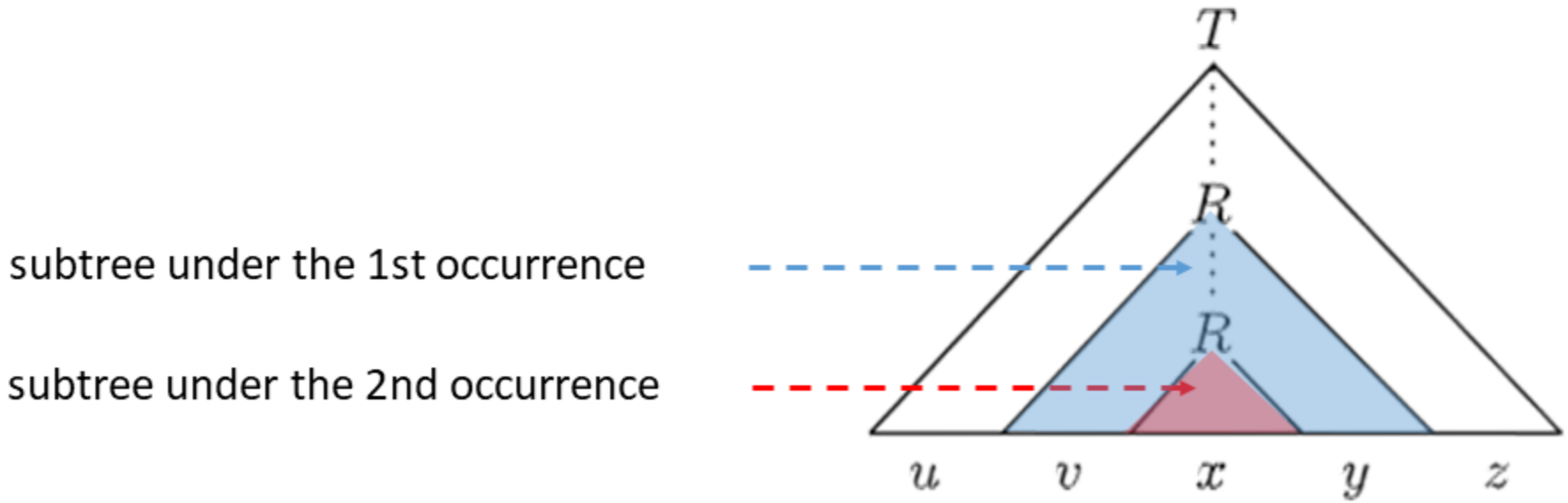
$T \to R \to aR\text{b} \to aB\text{b} \to ac\text{b}$



Theorem: Pumping Lemma for CFL

Proof idea: (cont.)

- This repetition allows us to replace the subtree under the second occurrence of R with the subtree under the first occurrence of R and still get a legal parse tree.



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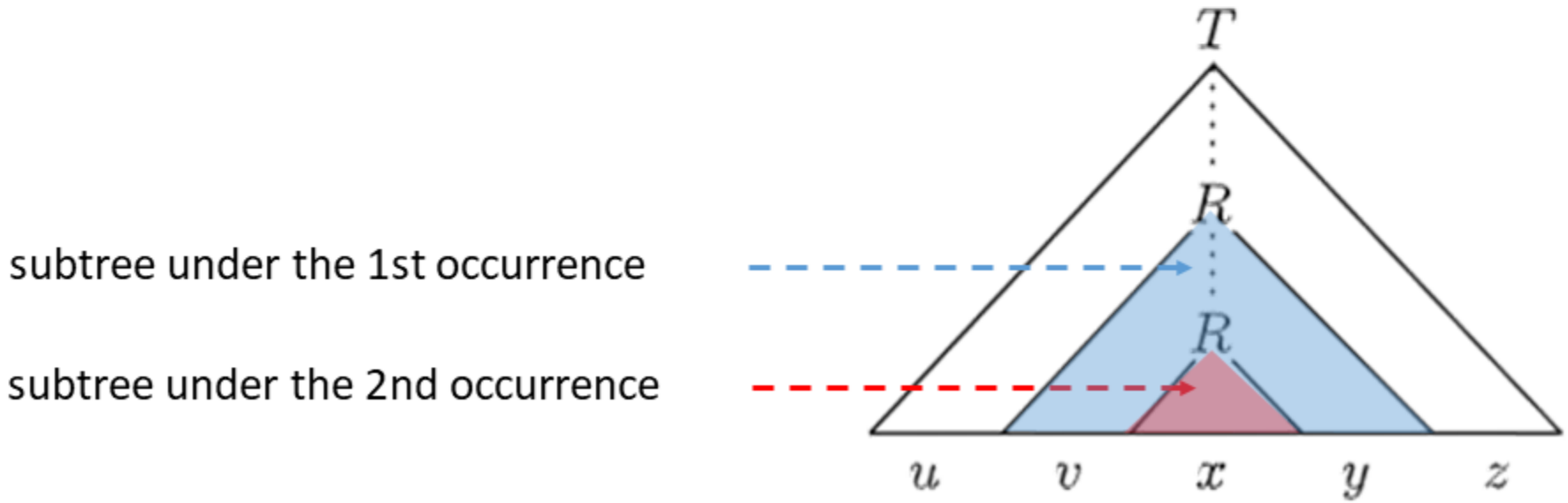
- Therefore, we may cut s into five pieces $uvxyz$ as the figure indicates.
- And we may repeat the second and fourth pieces
  - and obtain a string still in the language.



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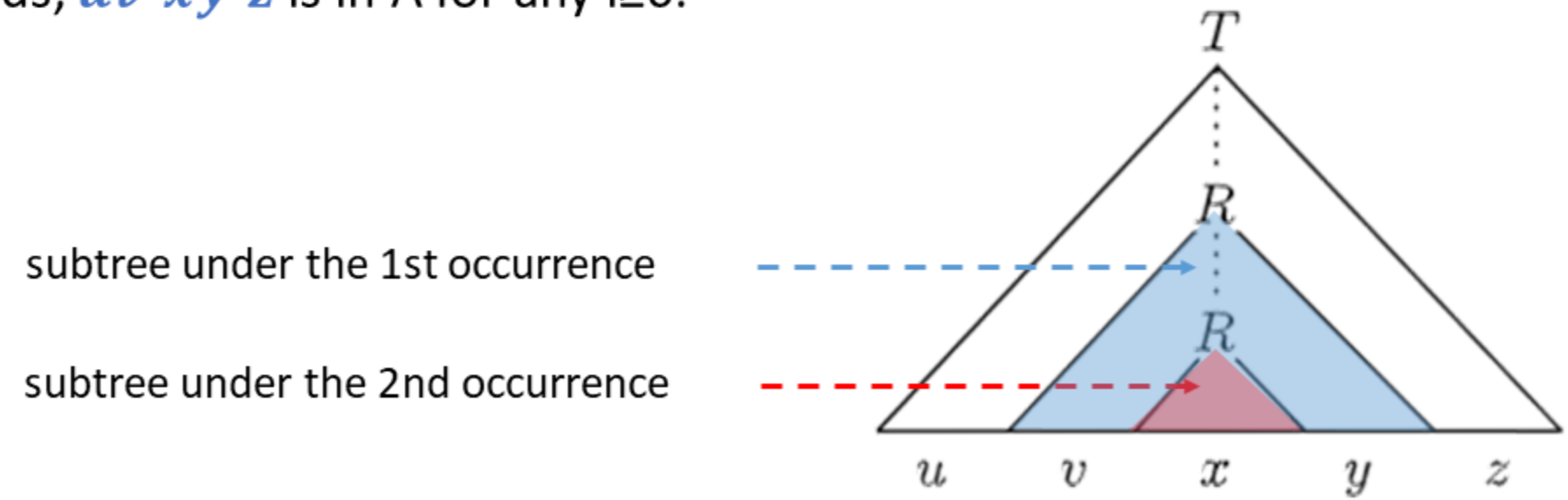
- In other words, uv^ixy^iz is in A for any i≥0.



Introduction

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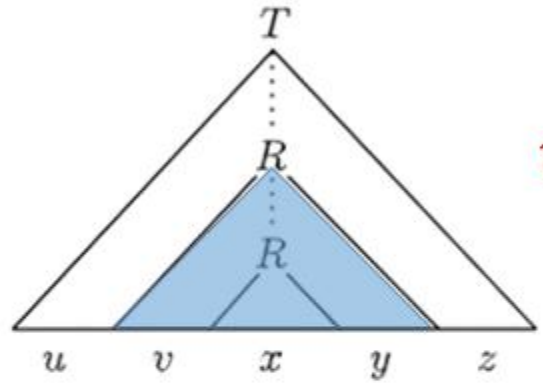
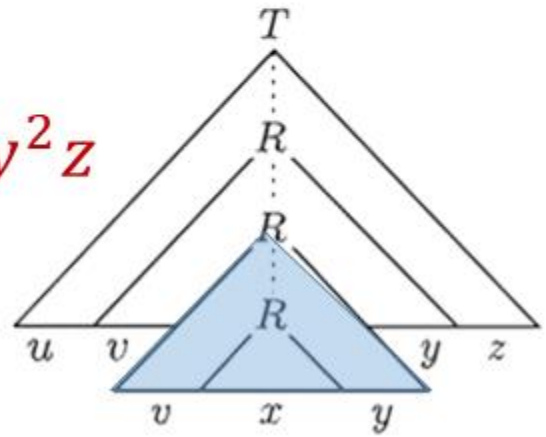
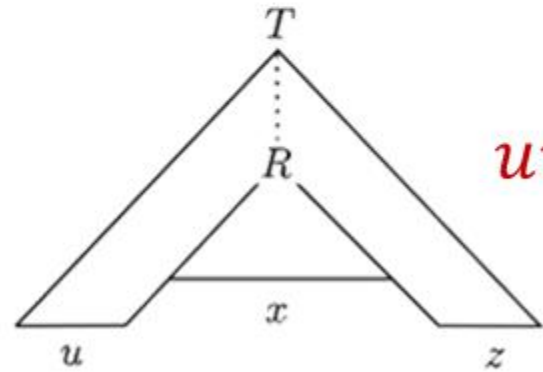


uv^1xy^1z



uv^2xy^2z



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- Use the pumping lemma to show that the language $B = \{a^n b^n c^n | n \geq 0\}$ is not context free.

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Proof by Contradiction: (Cont.)

- First, **condition 2** stipulates that either v or y is nonempty. $(|vy| > 0)$
- Then we consider one of two cases:
  1. **When both v and y contain only one type of alphabet symbol**
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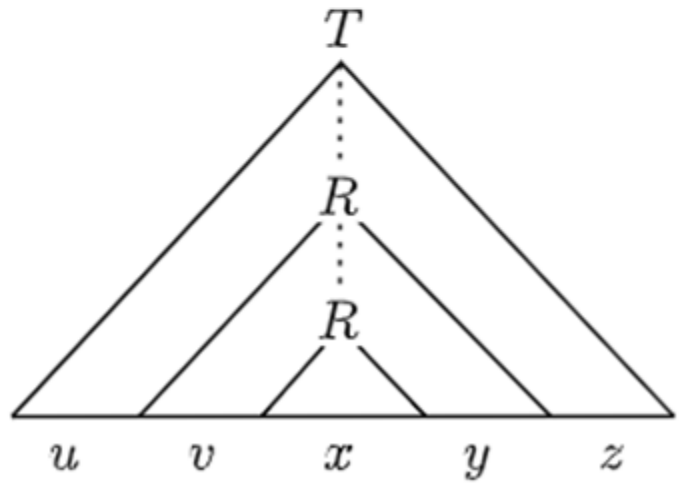
$$B = \{a^n b^n c^n | n \geq 0\}$$



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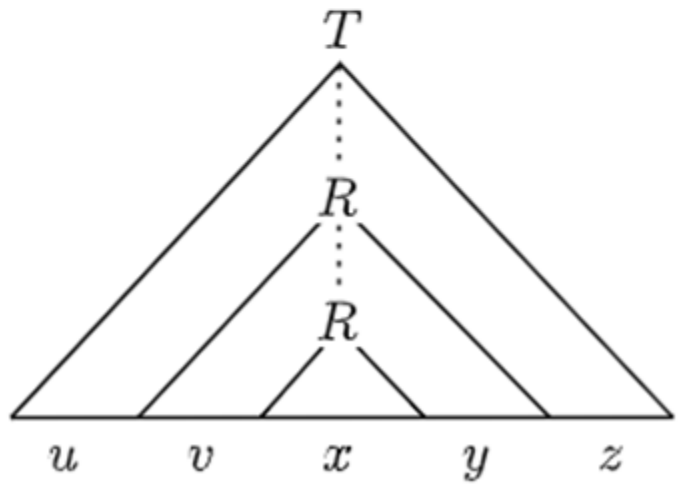

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     - Therefore, it cannot be a member of B.
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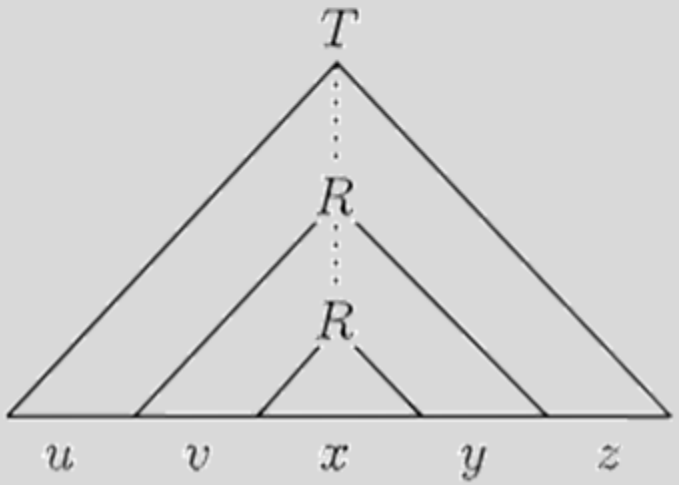

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     - Therefore, it cannot be a member of B.
     - That violates condition 1 of the lemma and is thus a contradiction.

- Example: aaaaa bbbbb ccccc or aaaaa bbbbb ccccc
  v y v y=ε

Proof by Contradiction: (Cont.)

2. **When either v or y contains more than one type of symbol,**
   - uv^2xy^2z may contain equal numbers of the three alphabet symbols
     - but not in the correct order.
   - Hence it cannot be a member of B and a contradiction occurs.

   - Example: aaaa**ab**bbb**b**ccccc → aaaa**abab**bbb**bb**ccccc
     v y
     uv^2xy^2z

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- Let s = $uvxyz$.
- Then we consider two cases:
  1. **When both v and y contain only one type of alphabet symbol,**
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Proof by Contradiction: (Cont.)

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     - Because v and y contain only one type of alphabet symbol, one of the symbols a, b, or c **doesn't appear** in v or y.
     - We further subdivide this case into three subcases according to **which symbol does not appear**.

Proof by Contradiction: (Cont.)

- Let s = uvxyz.
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  1. **When both v and y contain only one type of alphabet symbol,**
     - v **does not contain** both a's and b's or both b's and c's, and the same holds for y.
       - **The a's do not appear.** Then we try **pumping down** to obtain the string $uv^0xy^0z = uxz$.
       - **Example:** aaaaa bbbbb ccccc → v=bb , y=cc

$$C = \{a^i b^j c^k | 0 \leq i \leq j \leq k\}$$

Proof by Contradiction: (Cont.)

- Let s = uvxyz.
- Then we consider two cases:
  1. **When both v and y contain only one type of alphabet symbol,**
     - v **does not contain** both a's and b's or both b's and c's, and the same holds for y.
       - **The a's do not appear.** Then we try pumping down to obtain the string $uv^0xy^0z = uxz$.
       - **Example:** aaaaa bbbbb ccccc → v=bb , y=cc
       - That contain s the same number of a's as s does,
         - but it contains fewer b's or fewer c's.
       - Therefore, it is not a member of C,
         - and a **contradiction** occurs.

Proof by Contradiction: (Cont.)

- Let s = uvxyz.
- Then we consider two cases:
  1. **When both v and y contain only one type of alphabet symbol,**
     - **The b's do not appear.** Then either **a's** or **c's** must appear in v or y because both can't be the empty string.
     - **Example:** aaaaa bbbbb ccccc → v=aa , y=cc

$$C = \{a^i b^j c^k | 0 \leq i \leq j \leq k\}$$

Proof by Contradiction: (Cont.)

- Let s = uvxyz.
- Then we consider two cases:
  1. **When both v and y contain only one type of alphabet symbol,**
     - **The b's do not appear.** Then either **a's** or **c's** must appear in v or y because both can't be the empty string.
     - **Example:** aaaaa bbbbb ccccc → v=aa , y=cc
     - If **a's appear**,
       - the string uv^2xy^2z contains more a's than b's,
       - so it is **not in C**.
     - If **c's appear**,
       - the string uv^0xy^0z contains more b's than c's,
       - so it is **not in C**.
     - Either way, a **contradiction** occurs.

Proof by Contradiction: (Cont.)

- Let s = uvxyz.
- Then we consider two cases:

1. **When both v and y contain only one type of alphabet symbol,**
   - **The b's do not appear.** Then either **a's** or **c's** must appear in v or y because both can't be the empty string.
   - **Example:** aaaaa bbbbb ccccc → v=aa , y=cc
   - If **a's appear**,
     - the string uv^2xy^2z contains more a's than b's,
     - so it is **not in C**.

$$C = \{a^i b^j c^k | 0 \leq i \leq j \leq k\}$$

Proof by Contradiction: (Cont.)

- Let s = uvxyz.
- Then we consider two cases:
  1. **When both v and y contain only one type of alphabet symbol,**
     - **The c's do not appear.** Then the string uv^2xy^2z contains more a's or more b's than c's,
       - so it is **not in C**, and a **contradiction** occurs.

Proof by Contradiction: (Cont.)

- Let s = uvxyz.
- Then we consider two cases:
  2. **When either v or y contains more than one type of symbol**
     - Then the string uv^2xy^2z will not contain the symbols in the correct order.
       - so it is **not in C**, and a **contradiction** occurs.

Example 3:

- Use the pumping lemma to show that the language $D = \{ww | w \in \{0,1\}*\}$ is not context free.

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- We assume that D is a CFL and obtain a contradiction.
- Let p be the pumping length for D that is guaranteed to exist by the pumping lemma.
- Select the string $s = 0^p 1^p 0^p 1^p$.

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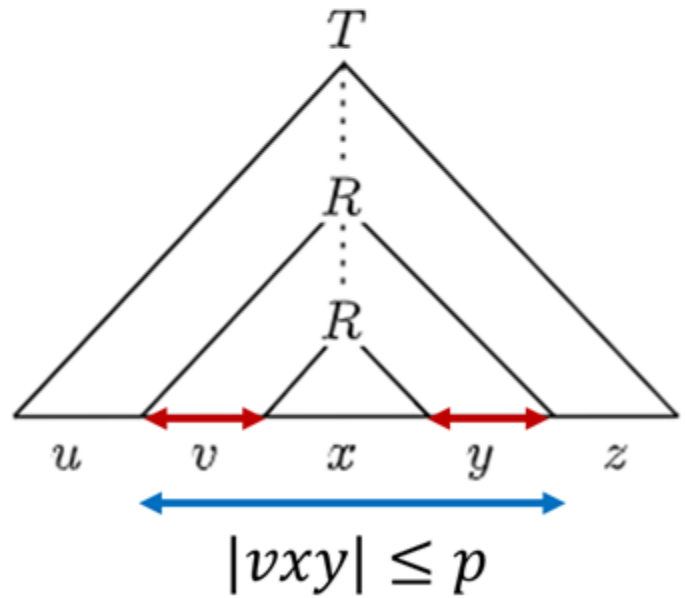
- We assume that D is a CFL and obtain a contradiction.
- Let p be the pumping length for D that is guaranteed to exist by the pumping lemma.
- Select the string $s = 0^p 1^p 0^p 1^p$.
- Clearly s is a member of D and of length at least p.
- The **pumping lemma** states that s can be **pumped**,
  - but we show that it **cannot**.

- In other words, we show that **no matter how we divide s** into $uvxyz$,
  - **one of the three conditions** of the lemma is violated

Proof by Contradiction (Cont.):

- We use **condition 3** of the pumping lemma
  - It says that we can pump s by dividing $s = uvxyz$, where $|vxy| \leq p$.

- We show that if **condition 3** holds, then the **other conditions** must fail.



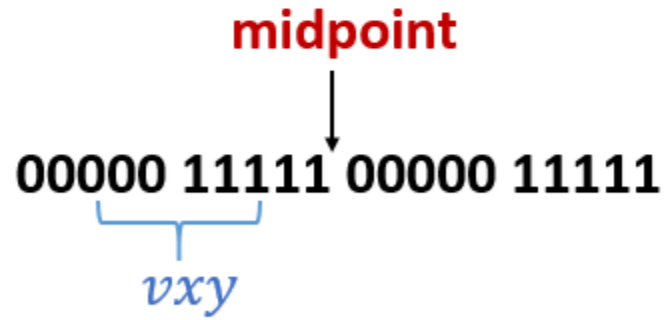
Proof by Contradiction (Cont.):

- We use **condition 3** of the pumping lemma
  - It says that we can pump s by dividing $s = uvxyz$, where $|vxy| \leq p$.

- We show that if condition 3 holds, then the other conditions must fail.

- **Case 1:** the **substring occurs only in the first half** of s, pumping s up to uv^2xy^2z moves a **1** into the first position of the second half,
  - and so it cannot be of the form ww.

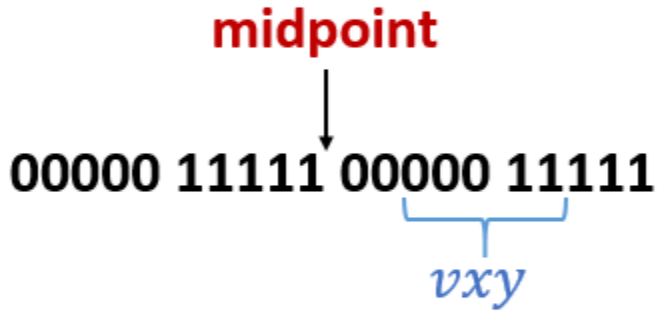
$$s = 0^p 1^p | 0^p 1^p$$



Proof by Contradiction (Cont.):

- **Case 2:** similarly, if vxy occurs in the second half of s, pumping s up to uv^2xy^2z moves a **0** into the last position of the first half, and so it cannot be of the form ww.

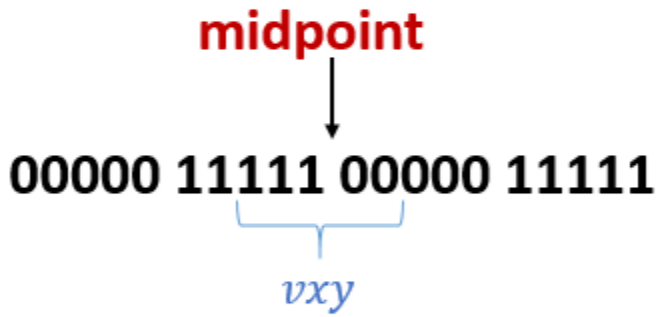
$$s = 0^p 1^p | 0^p 1^p$$



Proof by Contradiction (Cont.):

- **Case3:** the substring vxy straddles the midpoint of s,
  - when we try to pump s down to uxz it has the form $0^p1^i0^j1^p$, where i and j cannot both be p.
  - This string is not of the form ww.

$$s = 0^p1^p|0^p1^p$$



Then this string fails the conditions of pumping lemma.