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Abstract Jones & Pewsey (2009) introduce the 'sinh-arcsinh' transformation and use it to define the sinh-arcsinh family of distributions. When the generating distribution is standard normal, the 'normal sinh-arcsinh' (NSAS) class of distributions is obtained. The four-parameter location-scale extension of this class contains symmetric as well as asymmetric members and allows for tailweights that are heavier or lighter than those of the normal distribution. As will be shown, the NSAS class is highly tractable and has many appealing properties.

Key words: Statistical modelling, Heavy tails, Light tails, Skewness, Transformation

1 Introduction

Proceeding as in Jones & Pewsey (2009), consider a random variable (rv) from a distribution with no parameters other than location and scale. We denote its canonical version, with the location and scale parameters removed, by Z. Now define the rv $X_{\varepsilon,\delta}$ via the so-called *sinh-arcsinh transformation*

$$Z = S_{\varepsilon,\delta}(X_{\varepsilon,\delta}) \equiv \sinh\{\delta \sinh^{-1}(X_{\varepsilon,\delta}) - \varepsilon\}, \tag{1}$$

where $\varepsilon \in \mathbb{R}$ and $\delta > 0$. The four parameter extension of $X_{\varepsilon,\delta}$ is obtained by reinstating the location and scale parameters, $\xi \in \mathbb{R}$ and $\eta > 0$, in the usual way.

Given the normal distribution's prominent role within Statistics, a case of some considerable interest is the *normal sinh-arcsinh* (NSAS) class generated when Z is standard normal. For this class of distributions, ε is a skewness parameter and δ

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controls tailweight. $X_{\varepsilon,\delta}$ is positively (negatively) skew if $\varepsilon > 0$ ($\varepsilon < 0$), and has lighter (heavier) tails than the normal if $\delta > 1$ ($\delta < 1$). Its distribution is symmetric if $\varepsilon = 0$, and has normal tails if $\delta = 1$. Clearly, when $\varepsilon = 0$ and $\delta = 1$, $X_{0,1} = Z$ and its density is that of the standard normal distribution, ϕ.

In Section 2 we present some of the fundamental properties of the NSAS class. In conjunction, they show just how relatively simple, tractable and flexible the NSAS class of distributions is.

2 Properties of the NSAS Class

2.1 Density, Distribution and Quantile Functions, and Simulation

When $Z \sim N(0,1)$ in (1), the resulting NSAS rv, $X_{\varepsilon,\delta}$, has density

$$f_{\varepsilon,\delta}(x) = \{2\pi(1+x^2)\}^{-1/2}\delta C_{\varepsilon,\delta}(x)\exp\left\{-S^2_{\varepsilon,\delta}(x)/2\right\}, \quad -\infty < x < \infty, \qquad (2)$$

where $C_{\varepsilon,\delta}(x) = \cosh\{\delta\sinh^{-1}(x) - \varepsilon\} = \{1 + S^2_{\varepsilon,\delta}(x)\}^{1/2}$. Note that no special functions appear in this density, and that it is always unimodal. Examples of it appear in Fig. 1.

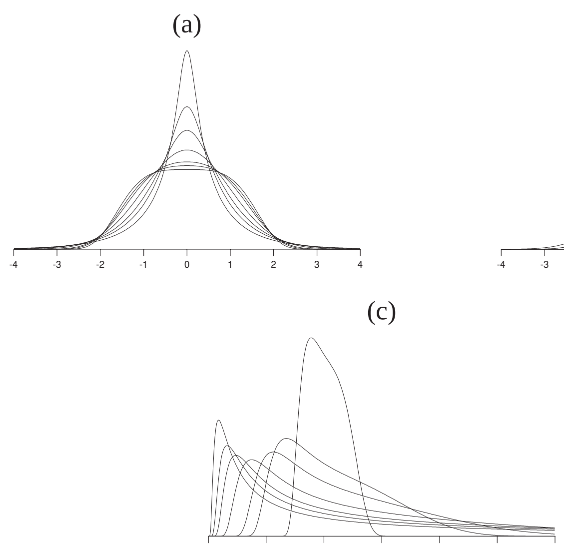

Fig. 1 (a) Scaled NSAS densities with $\varepsilon = 0$ and, in decreasing height, $\delta = 0.5, 0.625, 0.75, 1, 1.5, 2, 5$; (b) standardized NSAS densities with $\delta = 1$ and, in increasing degree of skewness, $\varepsilon = 0, 0.25, 0.5, 0.75, 1$; (c) NSAS densities $f_{\infty,\delta}$ for, reading from left to right, $\delta = 0.5, 0.625, 0.75, 1, 1.5, 2, 5$.

Inverting (1), $X_{\varepsilon,\delta} = S_{-\varepsilon/\delta,1/\delta}(Z) = \sinh[\delta^{-1}\{\sinh^{-1}(Z)+\varepsilon\}]$, so simulation of NSAS variables is trivial. The distribution function of $X_{\varepsilon,\delta}$ is $F_{\varepsilon,\delta}(x) = \Phi\{S_{\varepsilon,\delta}(x)\}$, where Φ is the standard normal distribution function, and its quantile function is $Q_{\varepsilon,\delta}(u) = S_{-\varepsilon/\delta,1/\delta}\{\Phi^{-1}(u)\},\ 0<u<1$. Thus, the distribution and quantile functions of $X_{\varepsilon,\delta}$ effectively have the same levels of computational complexity as those of the standard normal. A particularly simple result is that the median is given by $\sinh(\varepsilon/\delta)$. When $\varepsilon = 0$, the mode $x_0 = 0$, else x_0 has the sign of ε and $0<|x_0|<\sinh(|\varepsilon|/\delta)$.

2.2 Skewness, Tailweight and Moments

First, $f_{-\varepsilon,\delta}(x) = f_{\varepsilon,\delta}(-x)$, and for fixed δ, ε acts as a skewness parameter in the sense of van Zwet's (1964) skewness ordering. It is also possible to identify the limiting densities $f_{\varepsilon,\delta}$ as $\varepsilon \to \pm\infty$. Considering the case of positive skewness, i.e. $\varepsilon > 0$, denote the limiting densities as $f_{\infty,\delta}$. Employing suitable standardization of location and scale, it is found that

$$f_{\infty,\delta}(y) = (2\pi)^{-1/2} y^{-1} \delta \cosh(\delta \log 2y) \exp\left\{-\sinh^2(\delta \log 2y)/2\right\}, \quad y>0.$$

These are the densities of $Y = \exp\{\sinh^{-1}(Z)/\delta\}/2$, where Z is standard normal, and are plotted in Fig. 1(a) for a range of values of δ. The very light-tailed density $f_{\infty,5}$ is not very skew, but the others portrayed there are.

Regarding tailweight, as $|x| \to \infty$, $S_{\varepsilon,\delta}(x) \approx 2^{\delta-1}\mathrm{sgn}(x)\exp\{-\mathrm{sgn}(x)\varepsilon\}|x|^\delta$ and $C_{\varepsilon,\delta}(x) \approx 2^{\delta-1}\exp\{-\mathrm{sgn}(x)\varepsilon\}|x|^\delta$. It follows that

$$f_{\varepsilon,\delta}(|x|) \approx \exp\{-\mathrm{sgn}(x)\varepsilon\}|x|^{\delta-1}\exp\{-e^{-\mathrm{sgn}(x)2\varepsilon}|x|^{2\delta}\}. \quad (3)$$

These are closely related to Weibull and semi-heavy tails for small δ, being heavier than exponentially decaying tails and lighter than tails decreasing as a power of $|x|$.

The moments all exist because of the tail behaviour in (3) and are given by

$$\begin{aligned} E(X^r_{\varepsilon,\delta}) &= \frac{1}{2^r} E\left(\left[e^{\varepsilon/\delta}\left\{Z+(Z^2+1)^{1/2}\right\}^{1/\delta} - e^{-\varepsilon/\delta}\left\{Z+(Z^2+1)^{1/2}\right\}^{-1/\delta}\right]^r\right) \\ &= \frac{1}{2^r}\sum_{i=0}^{r}\binom{r}{i}(-1)^i \exp\left\{(r-2i)\frac{\varepsilon}{\delta}\right\} P_{(r-2i)/\delta} \end{aligned}$$

for integer r, where

$$\begin{aligned} P_q &= E\left[\left\{Z+(Z^2+1)^{1/2}\right\}^q\right] = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\left\{x+(x^2+1)^{1/2}\right\}^q e^{-x^2/2}dx \\ &= \frac{1}{(8\pi)^{1/2}}\int_0^\infty w^q\left(1+\frac{1}{w^2}\right)\exp\left\{-\frac{1}{8}\left(w-\frac{1}{w}\right)^2\right\}dw \end{aligned}$$

$$= \frac{e^{1/4} 8^{(q+1)/2}}{(32\pi)^{1/2}} \int_0^\infty z^{(q-1)/2} \left(1 + \frac{1}{8z}\right) \exp\left\{-\left(z + \frac{1}{64z}\right)\right\} dz$$
$$= \frac{e^{1/4}}{(8\pi)^{1/2}} \left\{K_{(q+1)/2}(1/4) + K_{(q-1)/2}(1/4)\right\},$$

using (3.471.12) of Gradshteyn & Ryzhik (1994). K is the modified Bessel function of the second kind: $K_{-\nu}(z) = K_\nu(z)$ so that $P_{-q} = P_q$, confirming $E(X^r_{0,\delta}) = 0$ for odd r. The first four moments simplify to

$$E(X_{\varepsilon,\delta}) = \sinh(\varepsilon/\delta)P_{1/\delta}, \quad E(X^2_{\varepsilon,\delta}) = \frac{1}{2}\left\{\cosh(2\varepsilon/\delta)P_{2/\delta} - 1\right\},$$
$$E(X^3_{\varepsilon,\delta}) = \frac{1}{4}\left\{\sinh(3\varepsilon/\delta)P_{3/\delta} - 3\sinh(\varepsilon/\delta)P_{1/\delta}\right\},$$
$$E(X^4_{\varepsilon,\delta}) = \frac{1}{8}\left\{\cosh(4\varepsilon/\delta)P_{4/\delta} - 4\cosh(2\varepsilon/\delta)P_{2/\delta} + 3\right\}.$$

Note that the first two odd moments are functions of the median $\sinh(\varepsilon/\delta)$.

2.3 *An Asymmetric Subfamily*

The three-parameter subclass of (2) in which $\delta = 1$ is of some interest. For this case, (1) reduces to $S_{\varepsilon,1}(X) = \cosh(\varepsilon)\, X - \sinh(\varepsilon)\, (1+X^2)^{1/2}$. Distributions with these densities, some of which are depicted in Fig. 1(b), are true skew-normal distributions in the sense that both of their tails are normal-like. They differ from the skew-normal distribution with density $2\phi(x)\Phi(\lambda x)$ (Azzalini, 1985) for which a side-effect of introducing the skewness parameter $\lambda \in \mathbb{R}$ is a change to the weight in one of the tails.

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