2. Backward error analysis for bordered L-U

Key steps:

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ a_{10}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{pmatrix}$$

$L_{00} \cdot u_{01} = a_{01}$ — (1)

$l_{10}^T \cdot U_{00} = a_{10}^T$ — (2)

$\alpha_{11} := v_{11} = \alpha_{11} - a_{10}^T \cdot a_{01}$ — (3)

a) $n = 1$

For $n=1$, There's only one step:

$$\alpha_{11} := \alpha_{11} - 0^T \cdot 0 = \alpha_{11}$$

& LU factorization is

$$(\alpha_{11}) = \underset{L}{(\alpha_{11})} \; \underset{U}{(1)}$$

$$\Rightarrow \boxed{\Delta \alpha_{11} = 0 \leq \gamma_1 \cdot |\alpha_{11}| \, |1|}$$

Hence proved for $n=1$

(ii) Inductive step:

To prove:

$$\left|\begin{pmatrix} \Delta A_{00} & \delta a_{01} \\ \delta a_{10}^T & \delta \alpha_{11} \end{pmatrix}\right| \leq \gamma_{n+1} \left(\left| \begin{pmatrix} \check{L}_{00} & 0 \\ \check{l}_{10}^T & 1 \end{pmatrix} \right| \left| \begin{pmatrix} \check{U}_{00} & \check{u}_{01} \\ 0 & \check{v}_{11} \end{pmatrix} \right| \right)$$

We open up the right side & comparewith each element on left:

$$\left|\begin{pmatrix} \Delta A_{00} & \delta a_{01} \\ \delta a_{10}^T & \delta \alpha_{11} \end{pmatrix}\right| \leq \gamma_{n+1} \left| \begin{pmatrix} \check{L}_{00} \check{U}_{00} & \check{L}_{00} \check{u}_{01} \\ \check{l}_{10} \check{U}_{00} & \check{v}_{11} \end{pmatrix} \right|$$

We're given:

$$|\Delta A_{00}| \leq \gamma_n |\check{L}_{00}| |\check{U}_{00}| \leq \gamma_{n+1} |\check{L}_{00}| |\check{U}_{00}|$$

(Lemma 6.6.2.8)

— Eq^n (4)

From (1),

~~a_{01}~~

$$a_{01} + \Delta a_{01} = \check{L}_{00} \cdot \check{u}_{01}$$

$$\Rightarrow |\Delta a_{01}| \leq \gamma_n \cdot |\check{L}_{00}| |\check{u}_{01}|$$

[Theorem 6.6.2.11, R-1F]

$$\Rightarrow \boxed{|\delta a_{01}| \leq \gamma_{n+1} |\check{L}_{00}| |\check{u}_{01}|}$$ [6.6.2.8]

(5)

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From ②,

$$l_{10}^T \cdot U_{00} = a_{10}^T$$

$$\Rightarrow \quad a_{10}^T + |\delta a_{10}^T| = \check{l}_{10}^T \cdot \check{U}_{00}$$

$$\Rightarrow \quad |\delta a_{10}^T| \leq \gamma_n \cdot |\check{l}_{10}^T||\check{U}_{00}| \quad \begin{bmatrix} 6.6.2.11 \\ R\text{-}1F \end{bmatrix}$$

$$\leq \gamma_{n+1} |\check{l}_{10}^T||\check{U}_{00}| \quad [6.6.2.8]$$

$$— ⑥$$

From ③,

$$v_{11} = \alpha_{11} - a_{10}^T \cdot a_{01} = \check{\alpha}_{11}$$

$$\Rightarrow \quad \delta\alpha_{11} \leq \gamma_n \cdot \underbrace{|\check{a}_{10}^T||\check{a}_{01}|}$$

multiplication ~~introdu~~

introduces additional error

$1 + \epsilon$

$$\Rightarrow |\delta\alpha_{11}| \leq \gamma_n \cdot (1+\epsilon) \, |\check{a}_{10}^T \cdot \check{a}_{01}|$$

$$\Rightarrow \quad \boxed{|\delta\alpha_{11}| \leq \gamma_{n+1} \cdot \check{v}_{11}} \quad — ⑦$$

Combining ④, ⑤, ⑥ & ⑦ into 1 matrix:

$$\left| \begin{pmatrix} \Delta A_{00} & \delta a_{01} \\ \delta a_{10}^T & \delta\alpha_{11} \end{pmatrix} \right| \leq \gamma_{n+1} \left(\left| \begin{matrix} \check{L}_{00}\check{U}_{00} & \check{L}_{00}\check{u}_{01} \\ \check{l}_{10}^T \check{U}_{00} & \check{v}_{11} \end{matrix} \right| \right)$$

which is same as the opened form.

Hence, proved.