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ALAFF Midterm-2

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1. a) Cholesky using Bordered Algorithm

step 1. Derivation

Using notation from 5.5.1.1, let's consider the loop invariant for Cholesky:

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} (L\backslash L^H)_{TL} & \hat{A}_{TR} \\ \hline \hat{A}_{BL} & \hat{A}_{BR} \end{array}\right) \wedge L_{TL} L_{TL}^H = \hat{A}_{TL}$$

meaning principal submatrix A_{TL} has been overwritten with it's Cholesky factors & rest have not been touched.

After repartitioning, A contains:

$$\left(\begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{01}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{c|cc} (L\backslash L^H)_{00} & \hat{a}_{01} & \hat{A}_{02} \\ \hline \hat{a}_{01}^T & \hat{\alpha}_{11} & \hat{a}_{12}^T \\ \hat{A}_{20} & \hat{a}_{21} & \hat{A}_{22} \end{array}\right)$$

$$\wedge L_{00} L_{00}^H = \hat{A}_{00}$$

where, A is symmetric, we have substituted $a_{10}^T = a_{01}^T$

After updating, to maintain the invariant, A should contain –

$$\left(\begin{array}{cc|c} A_{00} & a_{01} & A_{02} \\ a_{01}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right) = \left(\begin{array}{cc|c} L\backslash \hat{L}_{00} & l_{10} & \hat{A}_{02} \\ l_{10}^T & \lambda_{11} & \hat{a}_{12}^T \\ \hline \hat{A}_{20} & \hat{a}_{21} & \hat{A}_{22} \end{array}\right)$$

$$\wedge \begin{pmatrix} L_{00} & 0 \\ l_{10}^T & \lambda_{11} \end{pmatrix} \begin{pmatrix} L_{00}^T & l_{10} \\ 0 & \lambda_{11} \end{pmatrix} = \begin{pmatrix} \hat{A}_{00} & a_{01} \\ a_{01}^T & \alpha_{11} \end{pmatrix}$$

expanding, we get:

$$L_{00} \cdot l_{10} = a_{01}$$
$$\lambda_{11}^2 = \alpha_{11}$$

We then solve for λ_{11} & l_{10}:

$$\lambda_{11} = +\sqrt{\alpha_{11}}$$
$$l_{10} = \text{sol}^n \text{ to } L_{00} \cdot l_{10} = a_{01}$$

Then we update

$$a_{01} := l_{10}$$
$$\alpha_{11} := \lambda_{11}$$

Note: This works because $\alpha_{11} \geq 0$ (H.W. 5.4.1.2)

and

$$A = \begin{pmatrix} \alpha_{11} & \star^T \\ a_{21} & A_{22} \end{pmatrix}$$ then A_{22} is HPD (H.W. 5.4.1.3)

Full algorithm

$A = LL\text{-}bordered(A)$

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$$

A_{TL} is 0×0

while $n(A_{TL}) < n(A)$:

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{01}^T & \alpha_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{array}\right)$$

1. Solve $L_{00} \cdot l_{10} = a_{01}$, overwrite a_{01}

$a_{01} := l_{10}$

2. $\alpha_{11} := +\sqrt{\alpha_{11}}$

1. b) Proof: Bordered Cholesky is well defined for an SPD matrix

We use induction on above algorithm.

Base Case: if A is size 1×1, $A_{00} > 0$ & $L = \sqrt{A_{01}}$

n: At a step of the algo, we assume

$$A_{TL} = (L \backslash L)_{TL} \wedge L_{TL} L_{TL}^H = \hat{A}_{TL}$$

Induction step:

From H.W. 5.4.1.2, we know $\alpha_{11} > 0$

① $\Rightarrow \lambda_{11} = +\sqrt{\alpha_{11}}$ is well defined

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(2) From H.W. 5.4.1.3, we know that $\cancel{a_{01} = a_{10}^T}$ A_{BR} is also HPD, hence the algorithm can continue.

(3) Also, for solving

$$L_{00} \cdot l_{10} = a_{01}$$

we note that L_{00} has all diag. elements +ve ($= \sqrt{\alpha_{11}}$ for each step), hence L_{00} is a full rank.

$\Rightarrow$ $L_{00} \cdot l_{10} = a_{01}$ has a definite solⁿ.

Given, (1), (2) & (3), we see that induction step of algorithm is well defined.

Hence, proved!